

Chapter 18

Temperature, Heat, and the First Law of Thermodynamics

First Edition

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Preface

Thermal physics asks deceptively simple questions. What does a thermometer actually measure? Why can two bodies have the same temperature while containing very different amounts of internal energy? How can a gas reach the same final equilibrium state by two processes that transfer different amounts of work and heat? Why does a hot component sometimes cool almost exponentially and sometimes refuse to follow that simple law?

This chapter builds a macroscopic answer in three connected blocks. First, it defines thermal equilibrium and temperature, then studies thermal expansion and calorimetry. Second, it establishes the system boundary, thermodynamic state variables, boundary work, and the first law of thermodynamics using one explicit sign convention. Third, it introduces restricted engineering models for conduction, convection, radiation, and transient cooling. The aim is not to collect formulas, but to learn how each formula is tied to a physical boundary, geometry, process, and range of validity.

The chapter deliberately stops short of microscopic kinetic theory, molecular heat-capacity models, the reversible adiabatic power law, entropy, and heat-engine limits. Those topics belong to Chapters 19 and 20. An empirical ideal-gas equation is used here only as an equation of state linking equilibrium variables. Whenever a result from later theory is temporarily invoked, it is identified explicitly rather than smuggled into the core argument.

All numerical datasets identified as synthetic are constructed for teaching and verification. Material properties in worked examples are stated as model inputs rather than claimed measurements of a specific commercial object. The figures are editable representations generated inside this source; construction-method credits are provided at the end of the chapter.

The Guideline

Start every thermal problem by drawing or naming the system boundary. Decide which quantities are properties of the system and which are transfers across that boundary. Then state the process model before using a formula. A correct sign with the wrong boundary is still a wrong thermodynamic solution.

Pedagogical boundary

Chapter 18 is macroscopic. The ideal-gas equation $pV = nRT$ may be used empirically, but its molecular derivation, ideal-gas heat-capacity theory, and the reversible adiabatic relation are deferred to Chapter 19. Entropy and reversible-performance limits are deferred to Chapter 20. In this chapter, *adiabatic* means $Q = 0$; it does not automatically mean isentropic or reversible.

Conventions and notation

Thermodynamic temperature is denoted by T and expressed in kelvins. Celsius temperature is denoted by t_C . The present SI defines the kelvin by fixing the Boltzmann constant to exactly $k = 1.380649 \times 10^{-23} \text{ J K}^{-1}$; the Celsius temperature is related to thermodynamic temperature by $t_C/^{\circ}\text{C} = T/\text{K} - 273.15$ [3]. Consequently, a temperature interval has the same numerical size in kelvins and degrees Celsius, but absolute ratios must use kelvins.

For thermal energy accounting we use the book-wide work-by convention:

$$Q > 0 \text{ into the system,} \quad W_{\text{by}} > 0 \text{ by the system on the surroundings,} \quad \Delta E_{\text{int}} = Q - W_{\text{by}}.$$

In this chapter Q includes net thermal radiation as well as conductive and convective thermal exchange. This is a bookkeeping choice shared by the transfer models below; a reference that names radiation as a separate transfer channel must include that channel explicitly when translating the ledger.

If the same interaction is described using work done *on* the system, then $W_{\text{on}} = -W_{\text{by}}$ and $\Delta E_{\text{int}} = Q + W_{\text{on}}$. These are two notations for the same energy transfer, not two different physical laws.

Symbol	Meaning	SI unit
T	thermodynamic temperature	K
t_C	Celsius temperature	$^{\circ}\text{C}$
p	equilibrium pressure when defined	Pa
V	system volume	m^3
n	amount of substance	mol
R	molar gas constant	$\text{J mol}^{-1} \text{ K}^{-1}$
m	mass	kg
Q	heat transferred into the chosen system when positive	J
W_{by}	work done by the chosen system when positive	J
E_{int}	internal energy	J
C	total heat capacity	J K^{-1}
c	specific heat capacity	$\text{J kg}^{-1} \text{ K}^{-1}$
L_{phase}	specific latent heat for a stated transition	J kg^{-1}
α_L	linear thermal-expansion coefficient	K^{-1}
E	Young's modulus in the stated linear-elastic model	Pa
σ	axial normal stress (tension positive)	Pa
τ	lumped thermal time constant	s
L_c	characteristic length used in Bi	m
κ	thermal conductivity	$\text{W m}^{-1} \text{ K}^{-1}$
h	convection heat-transfer coefficient	$\text{W m}^{-2} \text{ K}^{-1}$
ε_{rad}	hemispherical emissivity in the stated grey-surface model	1
σ_{SB}	Stefan–Boltzmann constant	$\text{W m}^{-2} \text{ K}^{-4}$
R_{th}	thermal resistance for a specified geometry	K W^{-1}
Bi	Biot number, hL_c/κ	1

We use $\sigma_{\text{SB}} = 5.670374419 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$; the value follows from SI defining constants and is tabulated by NIST [4]. Unless explicitly stated otherwise, quoted material properties and transfer coefficients in worked examples are stipulated model inputs rather than measurements.

Chapter 18

Temperature, Heat, and the First Law of Thermodynamics

18.1 Thermal equilibrium and temperature

A thermometer reading seems simple only after its measurement model has become reliable. Put a cold sensor into a warm sample: energy generally enters the sensor, the sample changes, and neither initial temperature is immediately represented by the display. A stable number eventually appears, but interpreting that number requires three separate arguments. The sensor and sample must approach an appropriate thermal equilibrium; the sensor's measured property must have a known calibration; and the disturbance of the sample must either be small or be corrected with a justified model. This section develops those arguments before using temperature in an expansion or energy calculation.

Touch supplies a useful warning, but a poor temperature measurement. A metal bench and a wooden bench in the same room can feel different because they draw energy from the skin at different rates. The sensation depends on the thermal interaction as well as the initial temperature. Likewise, a temperature sensor in sunlight can respond to absorbed radiation as well as to the air around it. Specifying what the instrument contacts, how it exchanges energy, and what reading is intended is part of specifying the measurement.

Learning goals

The core of this chapter develops a connected set of modeling skills. You will distinguish temperature, heat, work, and internal energy; calibrate a thermometer and distinguish readings from intervals; calculate free expansion and qualified constrained stress; choose and check calorimetry phase branches; and apply the first law with an explicit system boundary. You will then calculate work for specified processes, analyze conduction and radiation balances, and decide when a lumped cooling model is supported. Absolute temperatures are expressed in kelvins; Celsius readings are denoted by t_C .

18.1.1 Thermal equilibrium and the zeroth law

Two systems are in thermal equilibrium when there is no net energy transfer driven by their temperature difference under the specified thermal-contact conditions. For ordinary macroscopic equilibrium systems, the zeroth law states that if A is separately in thermal equilibrium with a third system C, and B is separately in thermal equilibrium with C, then A and B are in thermal equilibrium with each other. A reproducible property shared by the three systems labels their equilibrium class: temperature [1, 2].

Definition 18.1: Temperature as an equilibrium state variable

Temperature is a state variable used to label thermal equilibrium. Equal temperatures do not imply equal masses, equal internal energies, or identical materials. Heat is energy transferred through a thermal interaction; it is not a substance stored inside a body.

Figure 18.1 shows the logical use of a thermometer. The same calibrated reading in the first two panels supports the equilibrium conclusion in the third. This argument assumes the readings refer to the states being compared. If a probe substantially cools A during its measurement and substantially heats B during its measurement, agreement of the *disturbed* readings does not establish equality of A's and B's initial temperatures. We return to this issue quantitatively below.

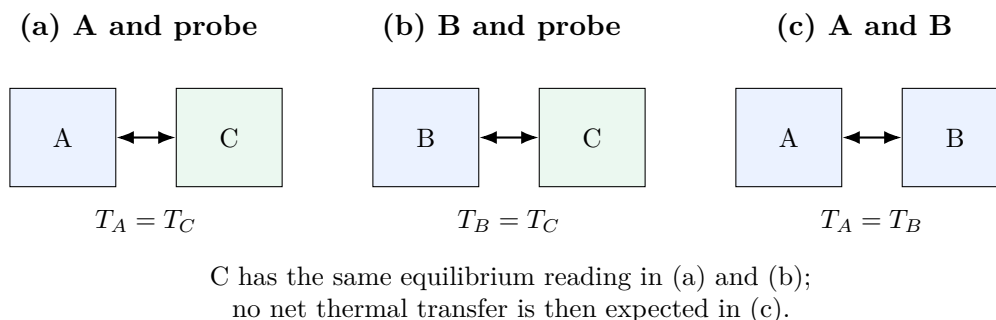


Figure 18.1: The zeroth-law inference uses equilibrium states and a reproducible thermometer. These are schematic contact diagrams; double arrows indicate the possibility of exchange, not a continuing net heat flow at equilibrium. Probe disturbance must be negligible or corrected when inferring the bodies' initial states.

A steady sensor reading alone is weaker evidence than thermal equilibrium. A sensor could be losing energy to a cool support while gaining the same amount from a warm sample. Its temperature would remain constant although energy continues to cross its boundary. In a uniform insulated sample with good contact, sufficient waiting may justify equilibrium; with competing contacts or persistent gradients, it may not. Temperature can still be defined locally when local equilibrium is a suitable approximation, but one sensor then cannot represent an entire nonuniform object.

18.1.2 Absolute temperature, offset scales, and intervals

The SI unit of thermodynamic temperature is the kelvin, written K without a degree symbol. The present SI fixes the numerical value of the Boltzmann constant, rather than defining the unit by a particular water cell [3]. Historical gas-thermometer and triple-point constructions remain useful ways to explain calibration, but should not be presented as the current defining convention. We need no microscopic temperature formula to use the absolute scale here.

The Celsius reading and absolute temperature satisfy the exact relation

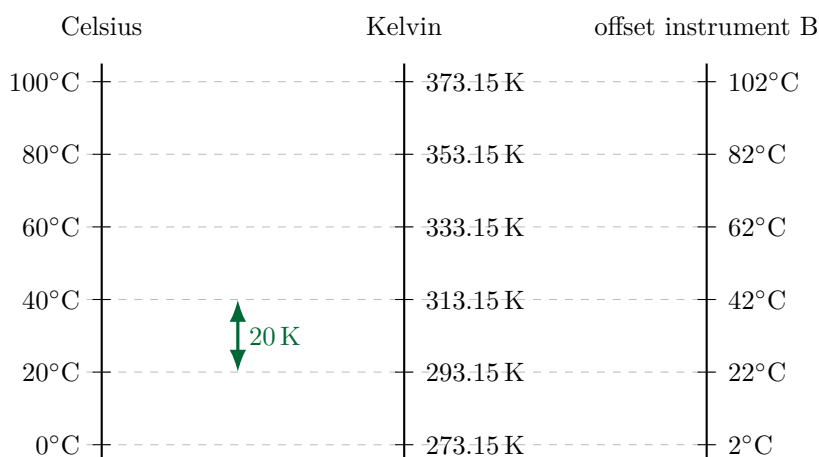
$$\frac{t_C}{^\circ\text{C}} = \frac{T}{\text{K}} - 273.15. \quad (18.1)$$

This equation relates numerical readings expressed in their indicated units. Subtracting two states cancels the offset:

$$\frac{\Delta t_C}{^\circ\text{C}} = \frac{\Delta T}{\text{K}}. \quad (18.2)$$

Thus a temperature rise of 20 K is a rise of 20°C, although a state at 20°C has $T = 293.15$ K. An absolute temperature ratio, an ideal-gas equation, and a radiation fourth power require kelvins. A heat-capacity term involving only a temperature *difference* can use either interval unit consistently.

For completeness, the Fahrenheit reading t_F satisfies $t_F/(^\circ\text{F}) = (9/5)t_C/(^\circ\text{C}) + 32$. Its interval size therefore differs: $\Delta t_F/(^\circ\text{F}) = (9/5)\Delta T/(\text{K})$. An offset affects readings; a scale factor affects both readings and intervals. The parallel scales in Fig. 18.2 make that distinction visible.



Each horizontal guide is one physical state. Equal vertical intervals here have equal physical size.

Figure 18.2: Celsius and kelvin have equal interval units and different zeros. Instrument B has a constant $+2^\circ\text{C}$ bias relative to the Celsius reading. The vertical coordinate is the Celsius numerical reading, so the plotted interval geometry is quantitative.

18.1.3 A calibration model and its uncertainty

A thermoscope is an instrument with a temperature-dependent observable s , such as resistance, length, or pressure. Calibration supplies a model relating s to temperature. If a linear relation is supported over a stated range, two reference states (s_1, t_1) and (s_2, t_2) determine

$$t_C = t_1 + (t_2 - t_1) \frac{s - s_1}{s_2 - s_1}. \quad (18.3)$$

In this expression t_C, t_1, t_2 are numerical Celsius readings in a common unit. The fraction is dimensionless. Calibration endpoints are not proof of linearity between them; a third reference or a validated sensor model is needed to investigate curvature. Extrapolating beyond both references is especially sensitive to a wrong slope or an unrecognized change in sensor response.

In an uncertainty calculation, a standard uncertainty is an estimated one-standard-deviation scale for an input. For independent small input uncertainties and a smooth function $f(x_1, \dots, x_n)$, linearization gives

$$u_f^2 \simeq \sum_j \left(\frac{\partial f}{\partial x_j} \right)^2 u_{x_j}^2. \quad (18.4)$$

The derivative describes how strongly the result responds to an input change; the square removes the sign because variances add. This approximation estimates uncertainty *within* the chosen model. It does not quantify an omitted nonlinear response or an unknown heat leak. Correlated inputs require covariance terms and cannot automatically be treated as independent.

Example 18.1: Calibrating a sensor and correcting an offset

An illustrative sensor produces signal $s_1 = 2.00 \text{ mV}$ at $t_1 = 20.00^\circ\text{C}$ and $s_2 = 8.00 \text{ mV}$ at $t_2 = 80.00^\circ\text{C}$. Treat the reference temperatures as exact stipulated values. A sample gives $s = 5.15 \text{ mV}$. Assume a linear response over this range and independent standard signal uncertainties $u_s = 0.03 \text{ mV}$ and $u_{s_1} = u_{s_2} = 0.02 \text{ mV}$. Determine (a) the calibrated temperature and its standard uncertainty; (b) the corrected reading and absolute temperature if a different thermometer B reads 62.0°C with a known constant $+2.0^\circ\text{C}$ offset; and (c) the thermodynamic ratio of temperatures at 40.0°C and 20.0°C .

Model and strategy. Each sensor is assumed to have equilibrated with its sample, with negligible sample disturbance. The signal uncertainties apply to independent readings, and the known bias of B is corrected rather than treated as unexplained scatter. The signal experiment is synthetic teaching data, not an actual laboratory record.

(a) **Calibration.** The slope is $(80 - 20)/(8 - 2) = 10.0 \text{ K mV}^{-1}$. Equation (18.3) yields

$$t_C = 20 + 60 \frac{5.15 - 2.00}{6.00} = 51.50^\circ\text{C}.$$

This signal is between both reference signals, as the temperature must be for the monotonic model. Differentiation at this state gives

$$\frac{\partial t_C}{\partial s} = 10.0, \quad \frac{\partial t_C}{\partial s_1} = -4.75, \quad \frac{\partial t_C}{\partial s_2} = -5.25 \quad (\text{K mV}^{-1}).$$

For example, the first negative derivative reflects the fact that raising the lower reference signal while holding the other inputs fixed shifts the inferred sample temperature downward. The standard uncertainty is

$$u_t = \sqrt{(10.0 \times 0.03)^2 + (4.75 \times 0.02)^2 + (5.25 \times 0.02)^2} = 0.332 \text{ K}.$$

The model result is $t_C = (51.50 \pm 0.33)^\circ\text{C}$, where the reported uncertainty is standard uncertainty, not a guaranteed bound.

(b) **Offset correction.** B's corrected reading is $62.0 - 2.0 = 60.0^\circ\text{C}$, so $T = 333.15 \text{ K}$. A constant bias correction changes the reading but leaves temperature differences unchanged.

(c) **Absolute ratio.** The physical ratio is

$$\frac{T_2}{T_1} = \frac{313.15 \text{ K}}{293.15 \text{ K}} = 1.06822.$$

The numerical quotient $40/20 = 2$ compares offset-scale labels and has no such thermodynamic interpretation.

Checks and interpretation. Substituting s_1 and s_2 into the calibration recovers both reference temperatures. The signal slope carries the units needed to convert mV uncertainty into K uncertainty. The largest stated uncertainty contribution is the sample signal itself, but a nonlinear response or poor contact could produce an additional systematic discrepancy even if repeated readings scatter little.

18.1.4 The finite heat capacity of a thermometer

A thermometer has a thermal response because its state changes when energy enters it. That change must be paid for by the sample or by the surroundings. Suppose a uniform sample of total heat capacity C_s starts at $t_{s,i}$ and a probe of total heat capacity C_p starts at $t_{p,i}$. Take the combined sample–probe boundary to be insulated, neglect mechanical work and phase change, and treat both capacities as constant. Internal exchanges cancel in the combined energy balance:

$$C_s(t_f - t_{s,i}) + C_p(t_f - t_{p,i}) = 0.$$

Solving gives

$$t_f = \frac{C_s t_{s,i} + C_p t_{p,i}}{C_s + C_p}, \quad t_f - t_{s,i} = \frac{r}{1+r} (t_{p,i} - t_{s,i}), \quad r = \frac{C_p}{C_s}. \quad (18.5)$$

Celsius readings are allowed because only differences enter the derivation. The first expression is a heat-capacity-weighted mean, so the final state lies between both initial states. A small probe

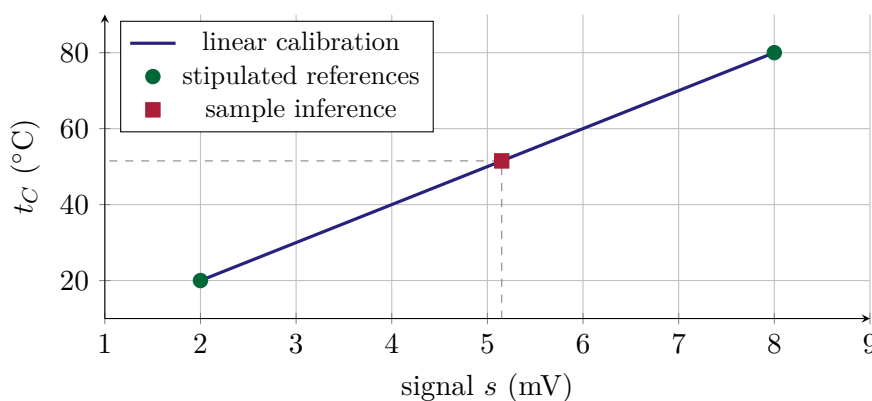


Figure 18.3: The synthetic two-reference calibration of Example 18.1. The straight line is a modeling assumption tested at its endpoints, not evidence that every sensor is linear. The sample signal is interpolated within the reference range; the small stated uncertainties are calculated in the example rather than enlarged in the drawing.

capacity, or a probe initially near the sample temperature, reduces disturbance. A heavy probe initially far from the sample temperature can dominate the measurement.

Figure 18.4 distinguishes the *final equilibrium* from the *initial sample* temperature. It does not describe how fast equilibration occurs. A time model requires additional heat-transfer information, developed later in the chapter.

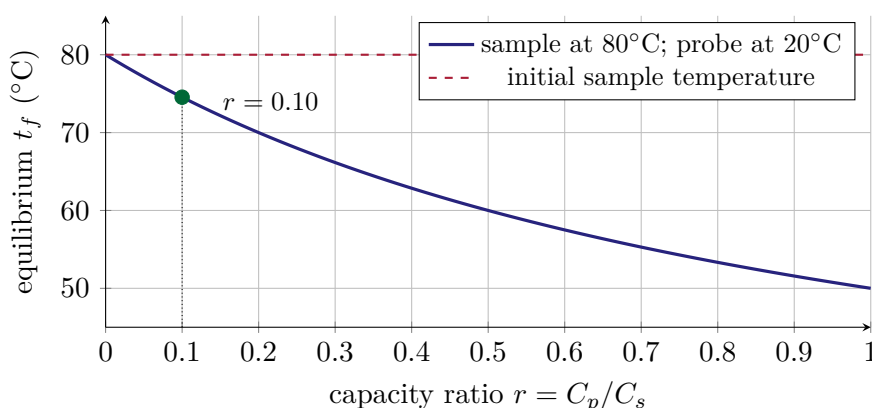


Figure 18.4: Quantitative disturbance caused by an initially cooler probe in an insulated two-capacity model. The curve approaches the original sample temperature as $r \rightarrow 0$. At $r = 1$, equal capacities yield the arithmetic mean 50°C . The graph concerns equilibrium states, not response times.

Example 18.2: A small sample measured by a finite probe

A sample has $C_s = 20.0 \text{ J K}^{-1}$ and initial temperature 80.0°C . A probe with $C_p = 2.00 \text{ J K}^{-1}$ starts at 20.0°C . The combined system is insulated, capacities remain constant, and neither body changes phase. Find (a) the stable reading and the energy gained by the probe; (b) the largest allowed probe capacity if disturbance of the sample must not exceed 1.00 K ; and (c) the original sample temperature inferred from a measured stable reading when the capacities and initial probe temperature are known.

Strategy. A calibration correction alone cannot recover the initial sample state: the sample actually cools. First conserve energy for the two bodies; then use the disturbance equation to impose the design requirement.

(a) **Equilibrium.** Since $r = 0.100$, Eq. (18.5) gives

$$t_f = \frac{20.0(80.0) + 2.00(20.0)}{22.0} = 74.545^\circ\text{C}.$$

The sample disturbance is -5.455 K . The probe gains

$$Q_p = C_p(t_f - t_{p,i}) = 2.00(74.545 - 20.0) = 109.09\text{ J},$$

while the sample loses $20.0(74.545 - 80.0) = -109.09\text{ J}$. The cancellation checks the chosen combined boundary.

(b) **Capacity requirement.** The initial difference is 60.0 K , so

$$\frac{60.0r}{1+r} \leq 1.00 \implies r \leq \frac{1}{59.0} = 0.016949.$$

Consequently $C_p \leq 20.0/59.0 = 0.339\text{ J K}^{-1}$. Saying only that the probe must be “small” would miss this quantitative constraint.

(c) **Inverse correction.** Rearranging the balance gives

$$t_{s,i} = t_f + \frac{C_p}{C_s}(t_f - t_{p,i}).$$

For the reading in (a), $74.545 + 0.100(74.545 - 20.0) = 80.000^\circ\text{C}$. This correction is justified only if the capacities and insulation assumptions describe the actual experiment.

Interpretation. Prewarming the probe close to the sample temperature would also reduce disturbance, but doing so requires prior information about the sample. The inverse calculation can amplify uncertainty in capacities or the initial probe state. It should not be used to explain away an uncontrolled heat leak.

Guided checks

Intervals. Warming from 15°C to 35°C is a 20 K rise because the offset cancels on subtraction.

Ratios. A body at 40°C does not have twice the absolute temperature of one at 20°C ; the kelvin ratio is about 1.068.

Equilibrium evidence. Equal stable readings support a zeroth-law comparison only when the readings represent the compared states. A finite probe can alter those states, and a stable reading under opposing heat flows need not represent equilibrium.

18.2 Thermal expansion and material response

A temperature change can alter a component’s dimensions even when the amount of material remains fixed. The macroscopic description uses an expansion coefficient measured for the material, temperature interval, and mechanical conditions. At the microscopic level, expansion involves changes in average separations and structure; an exactly symmetric harmonic potential by itself would not shift an oscillator’s mean separation when the amplitude increases. A detailed material theory is unnecessary for the engineering models here, but avoids the misleading claim that stronger vibration automatically implies expansion.

A length measurement also presupposes a thermal reference state. If an uncorrected metal ruler expands, its marked intervals grow. Measuring a dimension with that ruler can change the numerical reading even if the object changes less than the ruler. Expansion belongs to both the specimen and the measurement apparatus. In a precision alignment assembly or an optical mount, the relevant

prediction is often *differential* expansion between parts, not the length change of just one part.

18.2.1 Small-strain linear, area, and volume expansion

For a homogeneous unconstrained member with a uniform temperature change, the leading approximation is

$$\Delta L \simeq \alpha_L L_0 \Delta T, \quad \varepsilon_{\text{th}} \equiv \frac{\Delta L}{L_0} \simeq \alpha_L \Delta T. \quad (18.6)$$

Here L_0 refers to the initial state, and the stated coefficient is an average over the interval if it is taken constant. The dimensionless product $\alpha_L \Delta T$ is the thermal strain. A negative temperature change produces contraction for positive α_L . Some materials or crystallographic directions can have negative coefficients; positive expansion is a property of the chosen material and range rather than a universal law.

For isotropic expansion, all length directions share a factor $\lambda \simeq 1 + \alpha_L \Delta T$. A rectangular area then has factor λ^2 and a volume factor λ^3 . If $\epsilon = \alpha_L \Delta T$, direct multiplication gives

$$\frac{A_f - A_0}{A_0} = 2\epsilon + \epsilon^2, \quad \frac{V_f - V_0}{V_0} = 3\epsilon + 3\epsilon^2 + \epsilon^3.$$

Keeping only first-order strain terms yields

$$\Delta A \simeq 2\alpha_L A_0 \Delta T, \quad \Delta V \simeq \beta V_0 \Delta T, \quad \beta \simeq 3\alpha_L. \quad (18.7)$$

These finite-interval approximations require isotropy and small strain. An anisotropic block with three directional coefficients instead has first-order volumetric coefficient $\alpha_x + \alpha_y + \alpha_z$; no single factor of three is implied. Liquids possess a volume response but no persistent solid shape, so a linear coefficient should not be assigned to them by analogy with a rod.

18.2.2 Holes, cavities, and differential fits

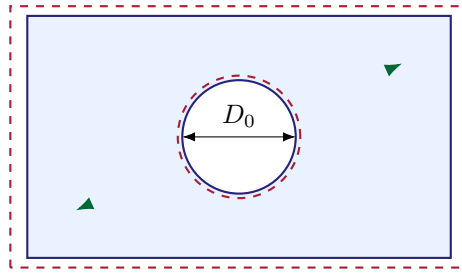
A uniformly heated free plate expands geometrically like an enlarged image. Choose a point as the expansion origin and map each material coordinate to λ times that coordinate. An interior edge follows that mapping just as an exterior edge does. A hole's diameter therefore increases for positive expansion. Imagining a plug made from the same material makes the argument especially clear: the filled plate expands uniformly; remove the expanded imagined plug and the enlarged hole remains. This reasoning would fail if nonuniform heating or a constraint distorted the plate.

Figure 18.5 shows corresponding inner and outer edges. The enlarged geometry is intentionally exaggerated. The actual strain in the example is only 0.00230, far too small to display as a large visible gap at this page scale.

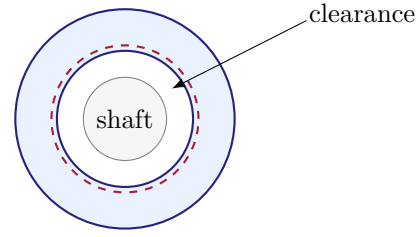
Example 18.3: An expanding hole and a differential fit

An unconstrained isotropic plate initially has exterior dimensions 80.0 mm $\times$ 120.0 mm and a circular hole of diameter $D_{h,0} = 20.000$ mm. Stipulate $\alpha_h = 23.0 \times 10^{-6} \text{ K}^{-1}$. A separate shaft has initial diameter $D_{s,0} = 20.030$ mm and coefficient $\alpha_s = 12.0 \times 10^{-6} \text{ K}^{-1}$. Both begin at 20.0°C. Find (a) the hole diameter and exterior area change if the plate alone warms by 100.0 K; (b) whether the shaft fits if both parts warm by that amount; and (c) the temperature increase giving zero diametric interference if both are heated together. Assume constant stipulated coefficients and small free strains throughout.

Representation. The plate's inner and outer boundaries share the same expansion factor, but the separate shaft has a different one. Define diametric clearance $g = D_h - D_s$; a negative value is interference.



Solid: initial plate. Dashed: heated plate.
Display factor 1.08 exaggerates the effect.



A separately heated shaft has
its own expansion factor. Hole
growth alone does not establish a fit.

Figure 18.5: Uniform isotropic free expansion enlarges a plate's hole as well as its exterior dimensions. In a fit problem, clearance is the hole diameter minus the shaft diameter; the two parts may have different coefficients and temperatures. The drawings are schematic, with expansion exaggerated and a generic visible clearance rather than the numerical interference of the example.

(a) Hole and area. The plate strain is $\epsilon_h = (23.0 \times 10^{-6})(100.0) = 0.00230$, giving

$$D_h = 20.000(1.00230) = 20.046 \text{ mm.}$$

The outer rectangular area initially is $A_0 = 9600 \text{ mm}^2$. Multiplying the two side factors gives

$$\Delta A = A_0[(1.00230)^2 - 1] = 44.2108 \text{ mm}^2.$$

The first-order estimate is $2\epsilon_h A_0 = 44.16 \text{ mm}^2$; their difference is the retained product term $A_0\epsilon_h^2 = 0.0508 \text{ mm}^2$. This algebraic comparison does not establish that the measured coefficient remains constant to that precision. If the hole is subtracted to obtain the area occupied by material, the initial area is $9600 - \pi(10.000)^2 = 9285.84 \text{ mm}^2$ and its corresponding increase is 42.7640 mm^2 .

(b) Both parts heated. The shaft strain is 0.00120, so

$$D_s = 20.030(1.00120) = 20.054036 \text{ mm,} \quad g = 20.046 - 20.054036 = -0.008036 \text{ mm.}$$

The hole grows, but the shaft still does not fit in the no-contact geometric model. If only the plate were heated by 100 K, it would have 0.016 mm clearance over the unheated shaft.

(c) Zero interference. The condition is

$$20.000(1 + \alpha_h \Delta T) = 20.030(1 + \alpha_s \Delta T),$$

which yields

$$\Delta T = \frac{20.030 - 20.000}{20.000\alpha_h - 20.030\alpha_s} = 136.587 \text{ K.}$$

The common temperature is about 156.6°C . Actual insertion needs positive clearance, so this limiting equality is not a manufacturing allowance.

Checks and limitations. The denominator is positive, so the plate can eventually overcome the initial interference. If it were zero or negative, this shared-heating model would predict no positive crossing. At the computed crossing the larger strain is about 0.00314, consistent with a small-strain estimate. Surface tolerances, temperature gradients, and contact stresses are additional fit-design inputs.

18.2.3 Temperature-dependent coefficients

The meaning of a coefficient must be specified before integrating it. An *instantaneous* linear coefficient at a stated stress condition is

$$\alpha_L(T) = \frac{1}{L(T)} \frac{dL}{dT}.$$

Dividing by L gives the differential fractional change per kelvin. Integration from T_0 to T gives

$$\ln \frac{L(T)}{L_0} = \int_{T_0}^T \alpha_L(\vartheta) d\vartheta, \quad L(T) = L_0 \exp \left[\int_{T_0}^T \alpha_L(\vartheta) d\vartheta \right]. \quad (18.8)$$

The dummy variable ϑ runs over the temperature interval. If the integral is small, $e^x \simeq 1 + x$ gives $\Delta L/L_0 \simeq \int \alpha_L dT$. A tabulated finite-interval average $\bar{\alpha}_L = \Delta L/(L_0 \Delta T)$ is already defined relative to L_0 and must not silently be treated as the exact instantaneous coefficient. The distinction is usually numerically modest at small strain; stating it prevents inconsistent claims of exactness.

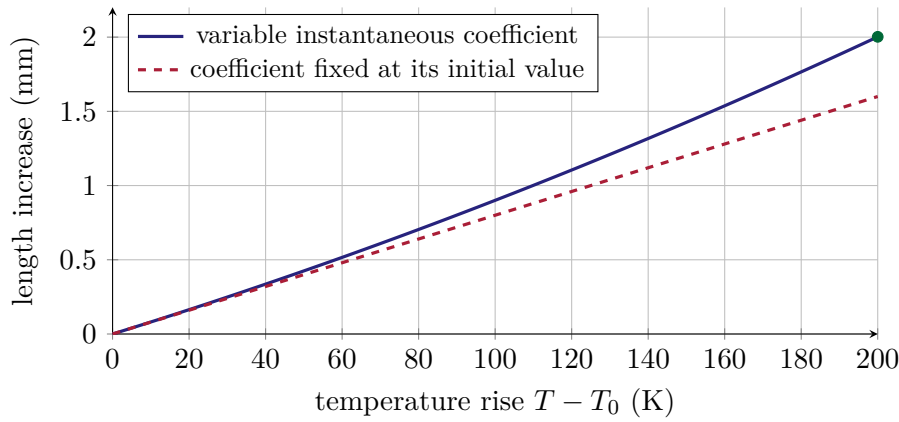


Figure 18.6: A one-metre member with stipulated $\alpha_L = a + b(T - T_0)$, $a = 8.0 \times 10^{-6} \text{ K}^{-1}$, and $b = 2.0 \times 10^{-8} \text{ K}^{-2}$. Integrating the instantaneous coefficient produces upward curvature. The comparison uses the same axes and a constant initial coefficient, so the difference reflects a material-model choice rather than a rescaled plot.

Example 18.4: A coefficient that changes over the working range

A freely expanding 1.000 m member starts at $T_0 = 293.15 \text{ K}$. Over $0 \leq T - T_0 \leq 200 \text{ K}$, stipulate the instantaneous coefficient

$$\alpha_L(T) = a + b(T - T_0), \quad a = 8.0 \times 10^{-6} \text{ K}^{-1}, \quad b = 2.0 \times 10^{-8} \text{ K}^{-2}.$$

For a 200 K rise, determine (a) the length increase from this model; (b) the error caused by keeping the coefficient at a in the first-order formula; and (c) the small-strain approximation using the integrated coefficient.

Model. Temperature is uniform, the member is unconstrained, and the coefficient law is stipulated rather than a claim about a named alloy. Equation (18.8) applies because the coefficient is explicitly instantaneous.

(a) Integration. With $x = T - T_0$,

$$I = \int_0^{200} (a + bx) dx = a(200) + \frac{1}{2}b(200)^2 = 0.00160 + 0.00040 = 0.00200.$$

Thus

$$\Delta L = L_0(e^I - 1) = 0.002002001 \text{ m} = 2.00200 \text{ mm}.$$

The final length is 1.00200200 m under the stipulated model.

(b) Initial-coefficient approximation. The estimate $L_0 \alpha \Delta T = 1.600$ mm is lower by 0.40200 mm, or 20.08% of the integrated-model increase. A small strain does not make a wrong average coefficient accurate: neglecting coefficient variation and neglecting higher strain terms are separate approximations.

(c) Integrated small-strain approximation. Taking $\Delta L \simeq L_0 I$ gives 2.000 mm, only 0.002001 mm below the exponential result. The leading omitted contribution is $L_0 I^2/2 = 0.002000$ mm, which independently explains the difference.

Interpretation. Figure 18.6 shows the growing discrepancy over the range. The derivative $dL/dT = L\alpha_L$ is positive and increases, as expected from the positive increasing coefficient. In a real design, property uncertainty could exceed the small exponential correction; reporting many digits here documents the mathematical comparison rather than achievable material accuracy.

18.2.4 Constrained expansion and elastic stress

Free thermal strain need not imply a force. Stress arises when a constraint prevents some of the natural deformation. For a one-dimensional member in the linear elastic regime, with tensile stress positive, add the mechanical strain to the thermal strain:

$$\varepsilon_{\text{tot}} = \frac{\sigma}{E} + \alpha_L \Delta T.$$

Perfect axial restraint imposes $\varepsilon_{\text{tot}} = 0$, so

$$\sigma = -E\alpha_L \Delta T. \quad (18.9)$$

Heating produces compression, not tension. Cooling under the same ideal constraint produces tension. This is a uniaxial model with lateral freedom; fully constraining transverse strains requires an appropriate three-dimensional elastic law. The formula also assumes uniform temperature, an initially unstressed reference state, and material properties applicable over the range.

A support need not be rigid. Model its outward displacement δ by a spring of stiffness k_s , so its restoring force has magnitude $k_s \delta$. The rod's axial stiffness is $k_r = EA/L_0$. Its actual extension is its free thermal extension minus the shortening due to compression:

$$\delta = \delta_{\text{free}} - \frac{k_s \delta}{k_r}.$$

Therefore

$$\delta = \frac{k_r}{k_r + k_s} \delta_{\text{free}}, \quad \sigma = -E\alpha_L \Delta T \frac{k_s}{k_r + k_s}. \quad (18.10)$$

The factor approaches zero for an unresisting support and one for a rigid support. The member's boundary condition, rather than its thermal coefficient alone, determines its mechanical outcome.

Example 18.5: Rigid supports, compliant supports, and a stress check

A uniform initially unstressed rod has $L_0 = 2.00$ m, cross-sectional area $A = 100$ mm², $\alpha_L = 12.0 \times 10^{-6}$ K⁻¹, and $E = 200$ GPa. Its temperature increases uniformly by 50.0 K. Find (a) its free extension; (b) its axial stress and compressive force under perfect axial restraint; and (c) its extension, stress, and force when one end is fixed and the other end acts against an initially unstressed support spring with $k_s = 10.0$ MN m⁻¹. Use tension-positive stress, linear elasticity, and no buckling.

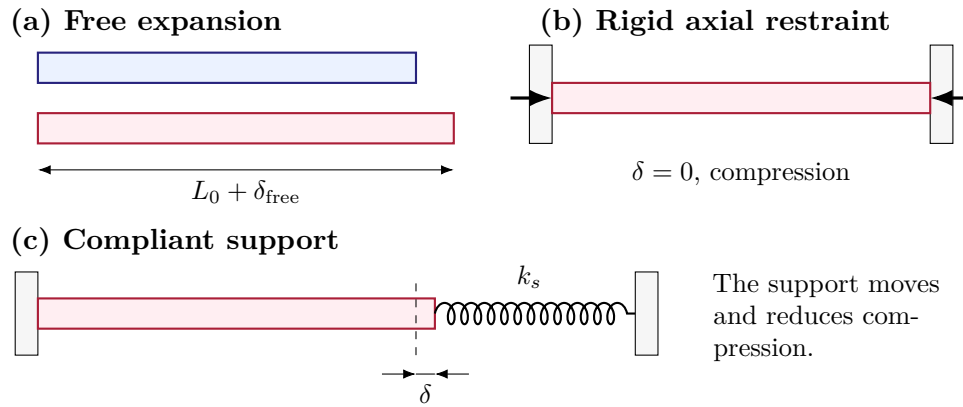


Figure 18.7: Free expansion, rigid restraint, and compliant restraint produce distinct mechanical outcomes. Heating is common to all panels; displacements are exaggerated. In (b), arrows are support forces on the rod. In (c), the initially unstressed support resists its displacement according to the stiffness model in Eq. (18.10). None of these straight-rod schematics establishes resistance to buckling.

(a) Free extension. The thermal strain is $\alpha_L \Delta T = 6.00 \times 10^{-4}$, so

$$\delta_{\text{free}} = (2.00)(6.00 \times 10^{-4}) = 1.20 \times 10^{-3} \text{ m} = 1.20 \text{ mm}.$$

The freely expanding rod needs no axial stress in this model.

(b) Rigid restraint. Its length remains fixed, so mechanical strain cancels thermal strain:

$$\sigma = -(200 \times 10^9)(6.00 \times 10^{-4}) = -120 \text{ MPa}.$$

Since $100 \text{ mm}^2 = 1.00 \times 10^{-4} \text{ m}^2$, the force magnitude is $|\sigma|A = 12.0 \text{ kN}$. The negative stress means compression.

(c) Compliant restraint. The rod stiffness is

$$k_r = \frac{EA}{L_0} = \frac{(200 \times 10^9)(1.00 \times 10^{-4})}{2.00} = 10.0 \text{ MN m}^{-1}.$$

Here $k_s = k_r$, so Eq. (18.10) gives $\delta = 0.600 \text{ mm}$ and $\sigma = -60.0 \text{ MPa}$. The support force is $k_s \delta = (10.0 \times 10^6)(0.600 \times 10^{-3}) = 6.00 \text{ kN}$, matching $|\sigma|A$.

Checks. The mechanical shortening relative to the free state is $F/k_r = 0.600 \text{ mm}$; subtracting it from 1.20 mm recovers the actual extension. Units of E are Pa and strain is dimensionless. As $k_s \rightarrow \infty$, the extension tends to zero and the rigid stress is recovered.

Engineering interpretation. A real design must compare the predicted stress with the material's elastic limit and examine support stiffness, tolerances, and buckling stability. The computed axial stress alone is not a certificate that a slender rod stays straight. If the rod buckles, the assumed axial displacement relation and straight-rod force response must be revised.

A useful consequence is that thermal gaps should be specified using the thermal reference state of all connected components. A gap that exactly accommodates an isolated rod's expansion may not accommodate movement of the supports, a different heated length, or a thermal gradient. Conversely, calculating the perfectly restrained stress in a visibly flexible assembly can greatly overestimate axial compression. The right question is which displacements the actual geometry and stiffness permit.

Guided checks

Hole size. Uniform free heating of a positive-expansion plate enlarges a hole, because the inner boundary follows the same coordinate scaling as the exterior.

Volume factor. $\beta \simeq 3\alpha_L$ is the finite small-strain isotropic approximation; directional coefficients or temperature-dependent properties require their stated model.

Stress sign. Preventing free expansion during heating requires compressive mechanical strain, giving negative stress in the adopted tension-positive convention. A compliant support permits some extension and reduces its magnitude.

18.3 Heat capacities, calorimetry, and phase changes

A temperature difference tells us that thermal exchange can occur; it does not tell us the amount of energy involved. Two samples can undergo the same temperature change while absorbing very different energies. Calorimetry relates those energies to mass, material response, initial state, and phase. Its central task is to define a system boundary and account for every included part before solving for the final state.

The sign convention throughout this chapter is positive heat entering the chosen system. A cooling body's thermal transfer is therefore negative. For several interacting bodies inside an insulated boundary, the energy given up by one body enters another, and those internal transfers cancel. In the negligible-work approximation used in this section, the final balance is $\sum_j Q_j = 0$. This compact equation is a consequence of energy conservation under stated conditions, not an assertion that each individual body's heat transfer vanishes.

18.3.1 Heat capacity as a process-qualified response

For a sample undergoing a small sensible temperature change under specified mechanical conditions, a total heat capacity C relates the thermal transfer to the temperature increment. When the response can be treated constant over the interval,

$$Q \simeq C\Delta T = mc\Delta T, \quad [C] = \text{J K}^{-1}, \quad [c] = \text{J kg}^{-1}\text{K}^{-1}. \quad (18.11)$$

Here m is mass and c is specific heat capacity. A molar heat capacity divides C by amount of substance rather than by mass. The term “capacity” does not mean a maximum amount of heat that a body can store. A body has internal energy; heat describes a transfer during a process.

The mechanical conditions matter. A sample heated at constant pressure can expand and transfer some energy as work, whereas a rigidly confined sample cannot do that boundary work. For solids and liquids in many introductory calorimetry experiments the associated work is small enough to neglect at the intended accuracy. Gas heat capacities depend substantially on the process, so Eq. (18.11) must not be applied to a gas without specifying the relevant capacity. Microscopic derivations of gas capacities belong to the next chapter. Here we use empirical solid/liquid responses with their conditions stated.

When the response varies with temperature, divide the process into sufficiently small increments $\delta Q = mc(T) dT$ and add them by integration:

$$Q = m \int_{T_i}^{T_f} c(T) dT. \quad (18.12)$$

This formula describes a single phase under the specified process. It cannot cross a phase transition without a separate phase contribution. It gives negative transfer for cooling when $c(T) > 0$, because the integral has reversed limits. If c is constant it reduces to $mc(T_f - T_i)$.

Example 18.6: Heating a sample with temperature-dependent capacity

A 0.500 kg sample is heated from 20.0°C to 100.0°C without phase change. Stipulate the specific-capacity model

$$c(T) = 700 + 1.50(T - 293.15 \text{ K}) \quad \text{J kg}^{-1} \text{K}^{-1},$$

where the numerical temperature difference is measured in K. A 400 W source delivers a stipulated constant fraction 0.800 of its power to the sample, with the remainder lost elsewhere. Neglect the container capacity and mechanical work. Find (a) the energy transferred into the sample; (b) the heating duration in the model; and (c) the error caused by replacing $c(T)$ with its initial value.

Model and strategy. The stipulated material law applies to this single-phase interval. The effective power entering the sample is constant, but its increasing heat capacity makes the temperature-rise rate decrease. The delivered fraction is part of the teaching model, not a measured efficiency claim.

(a) Energy. Let $x = T - 293.15 \text{ K}$, so the interval runs from $x = 0$ to 80.0 K. Then

$$Q = m \int_0^{80} (700 + 1.50x) dx = 0.500[700(80) + 0.750(80)^2] = 30\,400 \text{ J}.$$

The average capacity over this linear model is $(700 + 820)/2 = 760 \text{ J kg}^{-1} \text{K}^{-1}$. Multiplying $0.500(760)(80)$ independently recovers the same energy.

(b) Duration. The net source power is $P_{\text{in}} = 0.800(400) = 320 \text{ W}$. Conservation over the heating interval gives $P_{\text{in}}\tau = Q$, hence

$$\tau = 30\,400/320 = 95.0 \text{ s}.$$

At the start, $dT/dt = P_{\text{in}}/(mc) = 0.914 \text{ K s}^{-1}$; at the end it is 0.780 K s^{-1} . The temperature does not rise at a constant rate, despite the constant delivered power.

(c) Constant initial capacity. The estimate is $Q_0 = 0.500(700)(80) = 28\,000 \text{ J}$ and $\tau_0 = 87.5 \text{ s}$. It underestimates the model energy and time by $2400/30\,400 = 7.89\%$. The energy difference is exactly the integral of the extra capacity term, $0.500(0.750)(80)^2 = 2400 \text{ J}$.

Interpretation. A short heating estimate can be biased by using the wrong average capacity even when no phase change occurs. A real apparatus would also need evidence for the delivered power, container response, and thermal uniformity. A constant efficiency factor should not substitute for a heat-loss model when losses vary strongly with temperature.

18.3.2 Choosing the calorimetry boundary

Consider a hot sample dropped into cooler water. A boundary around only the sample has heat leaving it. A boundary around only the water has heat entering it. A boundary around sample, water, and calorimeter has no external thermal transfer in the insulated idealization, although substantial exchange occurs internally. That distinction makes the signs unambiguous. Write each contribution using its own initial temperature and the common final equilibrium temperature:

$$\sum_j m_j c_j (t_f - t_{j,i}) + C_{\text{cal}}(t_f - t_{\text{cal},i}) = 0. \quad (18.13)$$

Temperature differences in this equation can use Celsius readings because their offsets cancel. The container's initial temperature may differ from the water's; they may be combined only if their initial temperatures are equal.

The container, probe, stirrer, and any water adhering to a sample can matter. Omitting a component is a modeling decision that needs an energy-scale estimate. A container heat capacity of 40 J K^{-1}

may be small beside a kilogram of water but important beside a few grams. Figure 18.8 emphasizes that the wall belongs inside the chosen insulated system rather than being an unnamed source or sink.

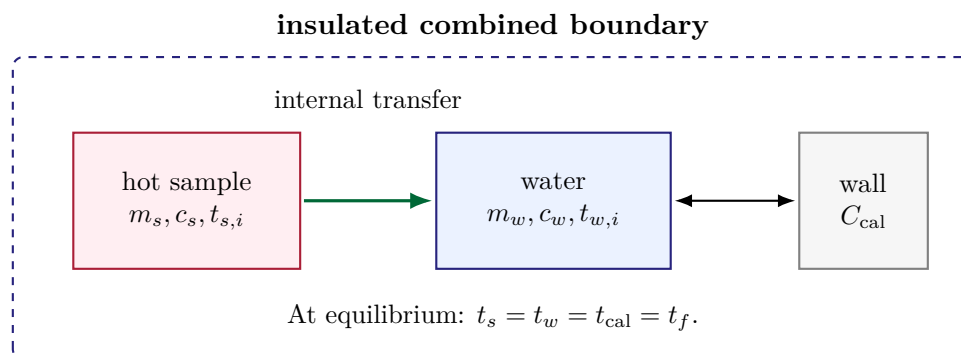


Figure 18.8: Calorimeter bookkeeping for a hot sample and cooler water. The dashed boundary includes the container wall, whose response must be accounted for. The hot-to-cold arrow describes the initial net thermal transfer; the wall exchange depends on its temperature. The geometry and arrow lengths are schematic. At final equilibrium no net thermal exchange continues between these parts.

For constant positive capacities and no phase change, solving the balance gives a heat-capacity-weighted mean of the initial temperatures. The result must lie between their minimum and maximum. If a proposed solution lies outside that range, it signals a sign error, an omitted energy transfer, or a model that includes more than passive mixing. This simple bound is often a stronger check than agreement with a calculator.

Example 18.7: Inferring a specific capacity from a mixing experiment

In a synthetic calorimetry record, a 0.1000 kg sample initially at 100.0°C is placed in 0.2000 kg of water initially at 20.0°C. The calorimeter starts at the water temperature and has $C_{\text{cal}} = 30.0 \text{ J K}^{-1}$. The final common reading is 26.80°C. Stipulate $c_w = 4180 \text{ J kg}^{-1} \text{ K}^{-1}$, insulation, no phase change, and negligible work. Determine (a) the inferred sample specific heat capacity; (b) the result if the container is wrongly omitted; and (c) what an unknown outward heat leak would do to the inference.

System and strategy. The combined boundary includes sample, water, and calorimeter. The sample cools while water and container warm. For the sample capacity c_s ,

$$m_s c_s (t_f - t_{s,i}) + (m_w c_w + C_{\text{cal}})(t_f - t_{w,i}) = 0.$$

Keeping the hot-sample difference negative prevents the most common sign mistake.

(a) Correct inference. Solving symbolically gives

$$c_s = \frac{(m_w c_w + C_{\text{cal}})(t_f - t_{w,i})}{m_s (t_{s,i} - t_f)} = \frac{(836 + 30)(6.80)}{0.1000(73.20)} = 804.481 \text{ J kg}^{-1} \text{ K}^{-1}.$$

The water gains $836(6.80) = 5684.8 \text{ J}$, the wall gains $30(6.80) = 204.0 \text{ J}$, and the sample loses 5888.8 J. Their algebraic sum is zero.

(b) Container omitted. Setting $C_{\text{cal}} = 0$ yields

$$c_s^{\text{omitted}} = \frac{836(6.80)}{0.1000(73.20)} = 776.612 \text{ J kg}^{-1} \text{ K}^{-1}.$$

The inferred capacity is 3.46% too low relative to the model result, because some sample energy went into the wall rather than into the water alone.

(c) Unknown heat leak. If a positive amount Q_{loss} leaves the combined system, the sample loss is the water-and-wall gain *plus* Q_{loss} . At the same measured temperatures, the correct inverse relation becomes

$$c_s = \frac{(m_w c_w + C_{\text{cal}})(t_f - t_{w,i}) + Q_{\text{loss}}}{m_s(t_{s,i} - t_f)}.$$

Neglecting outward leakage therefore underestimates the sample capacity. With neither leakage nor capacity independently known, this one endpoint measurement cannot determine both.

Interpretation. The computed value is an inference within the stipulated constant-capacity model, not an identification of a particular material. A reproducible endpoint could still contain systematic transfer losses. Improving insulation and measuring the empty-calorimeter response address different sources of bias.

18.3.3 Latent transfer and a quantitative heating curve

At the coexistence conditions of a pure material, energy can change the relative amounts of two phases while the equilibrium temperature stays approximately constant. Let L_f denote a positive specific latent heat of fusion. Melting a mass m requires $Q = +mL_f$; freezing it gives $Q = -mL_f$. Similarly, vaporization requires a positive mL_v transfer and condensation releases the corresponding energy. Pressure, purity, and the thermodynamic conditions must be specified: a horizontal temperature segment is not an unconditional property of every transition in every experiment.

In the ordinary solid/liquid calorimetry approximation with negligible expansion work, a convenient energy reference is ice at 0°C . Per unit mass, define the branch energy function

$$e_i(t_C) = c_i t_C \quad (t_C \leq 0), \quad e_w(t_C) = L_f + c_w t_C \quad (t_C \geq 0). \quad (18.14)$$

Numerical Celsius readings are used in these branch formulas relative to the chosen zero. The liquid branch carries the latent offset L_f . That offset represents an energy difference, not a temperature shift. With pressure held near the stipulated coexistence condition and negligible solid/liquid volume work, it provides the energy bookkeeping needed for the examples. Where pressure-volume work is important, the appropriate enthalpy or full first-law accounting must be used instead.

Figure 18.9 has a true energy scale rather than an arbitrary plateau length. A 0.100 kg sample begins as ice at -10°C . Warming to the phase boundary requires 2.10 kJ; melting requires a further 33.40 kJ; warming all the meltwater to 20°C requires 8.36 kJ. The visible long plateau correctly reflects the energy scale of melting in this model.

The slope of a sensible-heating segment is $dt_C/dQ = 1/(mc)$, so a larger capacity makes that segment flatter. On the plateau the phase fraction supplies the additional state information: the temperature alone does not specify the sample's energy. A final state at 0°C can be all ice, all liquid, or any equilibrium mixture between them depending on the total energy. Thus “the temperature has stopped rising” does not prove that no energy is entering.

18.3.4 Choosing the final phase before solving for temperature

For initial water mass m_w at $t_w > 0$ and initial ice mass m_i at $t_i < 0$, let the container capacity be C_{cal} and its initial temperature be $t_{\text{cal},i}$. The total reference energy in the approximation of Eq. (18.14) is

$$H_* = m_w(L_f + c_w t_w) + m_i c_i t_i + C_{\text{cal}} t_{\text{cal},i}. \quad (18.15)$$

The symbol H_* denotes this bookkeeping function; it is not a general definition of enthalpy. Let $M = m_w + m_i$. Three regimes follow by comparing H_* with the energies of all-solid and all-liquid

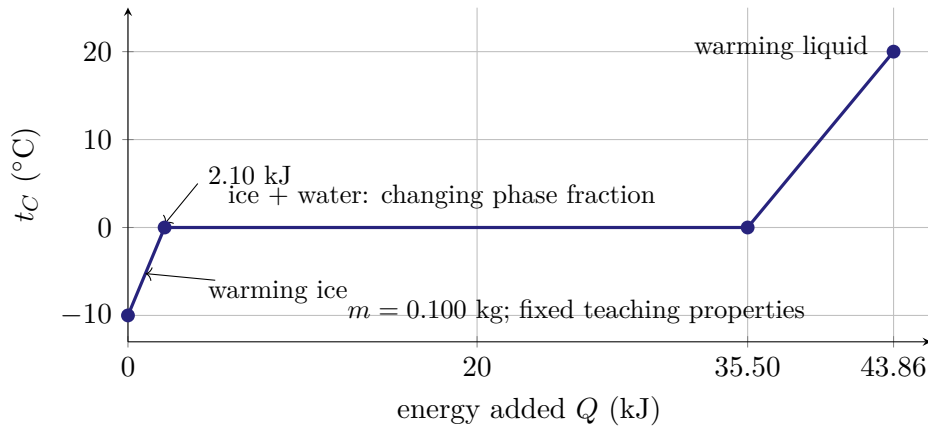


Figure 18.9: Quantitative single-pressure heating curve for stipulated $c_i = 2100 \text{ J kg}^{-1} \text{ K}^{-1}$, $c_w = 4180 \text{ J kg}^{-1} \text{ K}^{-1}$, and $L_f = 334 \text{ kJ kg}^{-1}$. At the horizontal segment, energy changes the mass fraction of liquid rather than the common equilibrium temperature. These are model values, not measured data; pressure shifts, supercooling, and heat losses are excluded.

states at the phase boundary:

$$\begin{aligned}
 H_* < 0 : \quad t_f &= \frac{H_*}{Mc_i + C_{\text{cal}}} < 0, \quad \text{all ice;} \\
 0 \leq H_* \leq ML_f : \quad t_f &= 0, \quad m_{\text{liquid},f} = \frac{H_*}{L_f}; \\
 H_* > ML_f : \quad t_f &= \frac{H_* - ML_f}{Mc_w + C_{\text{cal}}} > 0, \quad \text{all liquid.}
 \end{aligned} \tag{18.16}$$

Within the middle regime, the final ice mass is $M - H_*/L_f$. Some original water may freeze if the ice warming demand is sufficiently large. It is therefore not enough to say “not all the ice melts” when the available warm-water cooling energy is insufficient to bring the cold ice to zero. Freezing water releases latent energy and may allow a mixture at zero; for an even larger cold demand, all the water freezes and the final state is below zero.

This formulation provides branch selection before any assumed all-liquid temperature is calculated. It also supplies checks at both boundaries: the temperature is continuous as H_* reaches 0 or ML_f , and the phase fraction remains between zero and one. The formulas assume no vapor is present in the final state, no mass escapes, the container undergoes no transition, and the stipulated properties are adequate. They should not be extrapolated to a boiling experiment without further branches and boundary analysis.

Example 18.8: Ice calorimetry with a container and uncertainty

An insulated calorimeter contains $m_w = 0.300 \text{ kg}$ of water at $t_w = 30.0^\circ\text{C}$; its heat capacity is $C_{\text{cal}} = 40 \text{ J K}^{-1}$ and its initial temperature is also 30.0°C . Add $m_i = 0.0500 \text{ kg}$ ice at $t_i = -10.0^\circ\text{C}$. Use stipulated constant teaching values $c_w = 4180 \text{ J kg}^{-1} \text{ K}^{-1}$, $c_i = 2100 \text{ J kg}^{-1} \text{ K}^{-1}$, and $L_f = 334 \text{ kJ kg}^{-1}$. Find (a) the final phase and temperature; (b) its standard temperature uncertainty for independent input standard uncertainties $u_{m_w} = 0.001 \text{ kg}$, $u_{m_i} = 0.0005 \text{ kg}$, $u_{t_w} = u_{t_i} = 0.2 \text{ K}$, and $u_C = 5 \text{ J K}^{-1}$; and (c) which inputs dominate this uncertainty. The container and water share the same initial-temperature input.

System and assumptions. Include water, ice, and container. Neglect external heat transfer, evaporation, retained surface water, and mechanical work. The material inputs are treated exact within the stated model, so the uncertainty budget will describe the listed measured quantities rather than property or model uncertainty.

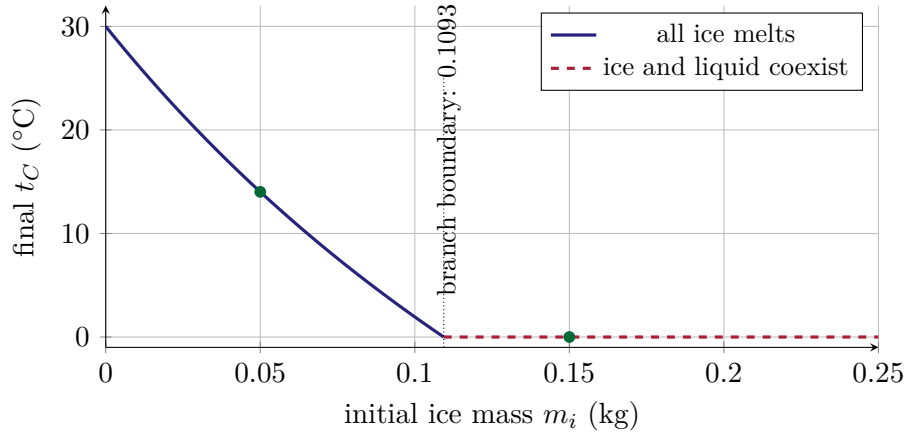


Figure 18.10: Final temperature versus initial ice mass for 0.300 kg water and a 40 J K^{-1} calorimeter initially at 30°C , with ice at -10°C and the stated capacities/latent heat. The curve reaches zero at $m_i = 0.10935 \text{ kg}$. Further ice addition over the displayed range changes phase fractions rather than temperature. The all-solid regime occurs beyond this plotted range and is evaluated explicitly in Example 18.9.

(a) Phase test. Cooling water and container to zero releases

$$Q_{\text{avail}} = (m_w c_w + C_{\text{cal}}) t_w = (1254 + 40)(30.0) = 38.82 \text{ kJ}.$$

Bringing the ice to zero requires $m_i c_i (0 - t_i) = 1.05 \text{ kJ}$ and melting it requires $m_i L_f = 16.70 \text{ kJ}$. Since $38.82 > 17.75 \text{ kJ}$, all ice melts and there is energy remaining to warm the combined liquid above zero. The appropriate balance is

$$(m_w c_w + C_{\text{cal}})(t_w - t_f) = m_i c_i (0 - t_i) + m_i L_f + m_i c_w t_f.$$

Solving gives

$$t_f = \frac{(m_w c_w + C_{\text{cal}}) t_w + m_i c_i t_i - m_i L_f}{(m_w + m_i) c_w + C_{\text{cal}}} = \frac{21\,070}{1503} = 14.0186^\circ\text{C}. \quad (18.17)$$

The numerator is the residual energy after bringing all material to liquid at zero; the denominator is the final liquid-plus-container heat capacity.

(b) Propagation. Write $D = (m_w + m_i) c_w + C_{\text{cal}} = 1503 \text{ J K}^{-1}$. Differentiating the quotient and simplifying with its own temperature result gives

$$\begin{aligned} \frac{\partial t_f}{\partial m_w} &= \frac{c_w(t_w - t_f)}{D} = 44.4459 \text{ K kg}^{-1}, \\ \frac{\partial t_f}{\partial m_i} &= \frac{c_i t_i - L_f - c_w t_f}{D} = -275.182 \text{ K kg}^{-1}, \\ \frac{\partial t_f}{\partial t_w} &= \frac{m_w c_w + C_{\text{cal}}}{D} = 0.860945, \\ \frac{\partial t_f}{\partial t_i} &= \frac{m_i c_i}{D} = 0.0698603, \\ \frac{\partial t_f}{\partial C_{\text{cal}}} &= \frac{t_w - t_f}{D} = 0.0106330 \text{ K}^2 \text{ J}^{-1}. \end{aligned}$$

Multiplying by the corresponding standard uncertainties gives absolute temperature contributions 0.04445, 0.13759, 0.17219, 0.01397, and 0.05316 K. Equation (18.4) then gives

$$u_{t_f} = \sqrt{0.04445^2 + 0.13759^2 + 0.17219^2 + 0.01397^2 + 0.05316^2} = 0.23147 \text{ K}.$$

Thus the result can be reported as $t_f = (14.02 \pm 0.23)^\circ\text{C}$ with standard uncertainty.

(c) Interpretation and checks. The initial warm-water/container temperature and the ice mass supply the largest variance contributions, shown in Fig. 18.11. The final temperature is between zero and the warm-water initial temperature, as required. The complete-melting margin is 21.07 kJ, so the stated small input uncertainties do not put the nominal result close to a branch switch. Near such a switch, differentiating a single branch would be inadequate. A boundary-energy check gives a warm-water-plus-container loss of $1294(30 - 14.0186) = 20.6799$ kJ. Ice warming plus melting uses 17.7500 kJ and the meltwater warms using $0.0500(4180)(14.0186) = 2.92989$ kJ, reproducing that loss. Ambient leakage, incomplete mixing, uncertain latent heat, and liquid attached to the ice could bias a real result; they are excluded from this deliberately specified uncertainty budget.

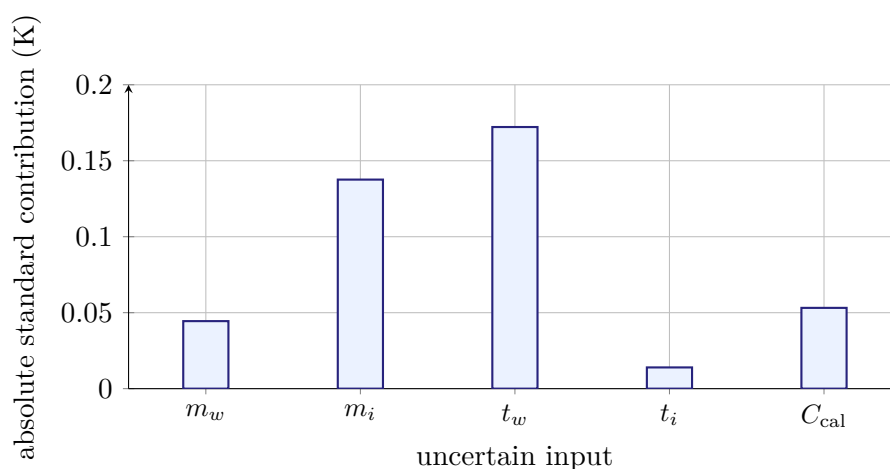


Figure 18.11: The five $|\partial t_f / \partial x| u_x$ contributions in Example 18.8. They combine in quadrature, not by adding bar heights. The temperature input for warm water and calorimeter is shared as specified; treating their initial readings as independent would be a different uncertainty model. Property uncertainty and heat-leak bias are excluded.

18.3.5 Incomplete melting and freezing of initial water

A trial all-liquid calculation that gives a negative Celsius temperature is inconsistent with that branch. It does not describe liquid equilibrium below the assumed freezing point in this phase-equilibrium model. Conversely, stopping at zero after all ice has melted discards the residual energy that raises the liquid temperature. Neither mistake is repaired by changing a sign after the calculation; the phase branch must be reconsidered.

Example 18.9: Selecting a mixture branch and an all-solid branch

Use the same 0.300 kg water, 40 J K^{-1} container, initial 30.0°C warm state, initial -10.0°C ice state, and stipulated properties as Example 18.8. Determine (a) the final state if 0.150 kg ice is added; (b) the greatest initial ice mass that can melt completely; and (c) the final state for the deliberately extreme addition of 8.00 kg ice. Assume the apparatus can hold these masses and that insulation and uniform equilibrium remain valid.

Strategy. Use one reference-energy balance for all three cases. It will distinguish partial melting from freezing of original water. With initial ice mass m_i in kg,

$$H_* = m_w L_f + (m_w c_w + C_{\text{cal}}) t_w + m_i c_i t_i = 139\,020 - 21\,000 m_i \quad \text{J.}$$

The total mass is $M = 0.300 + m_i$ and the all-liquid-zero threshold is ML_f .

(a) Incomplete melting. For $m_i = 0.150$, $H_* = 135\,870\text{ J}$ whereas $ML_f = 0.450(334\,000) = 150\,300\text{ J}$. Because $0 < H_* < ML_f$, the final state is an ice–water mixture at 0°C . Its liquid mass is

$$m_{\text{liquid},f} = H_*/L_f = 0.406796\text{ kg}.$$

Thus 0.106796 kg of the added ice melts and 0.0432036 kg ice remains. Independently, warming all the ice costs $0.150(2100)(10) = 3.15\text{ kJ}$, leaving $38.82 - 3.15 = 35.67\text{ kJ}$ for melting; dividing by 334 kJ kg^{-1} recovers the melted mass.

(b) Complete-melting boundary. Setting $H_* = ML_f$ gives

$$38\,820 = m_i(21\,000 + 334\,000), \quad m_{i,\text{crit}} = 0.109352\text{ kg}.$$

At this limit the sample is all liquid at zero. Less ice leaves positive energy to warm the liquid; more ice leads initially into the coexistence branch in Fig. 18.10.

(c) All-solid branch. For $m_i = 8.00$, the reference energy is $H_* = 139\,020 - 168\,000 = -28\,980\text{ J} < 0$. Even freezing *all* the initial water cannot produce an equilibrium mixture at zero. The final system contains 8.30 kg ice and satisfies

$$(Mc_i + C_{\text{cal}})t_f = H_* \quad \Rightarrow \quad t_f = \frac{-28\,980}{8.30(2100) + 40} = -1.65884^\circ\text{C}.$$

The original water first cools, freezes, and then cools as ice. Ignoring its freezing would miss a $0.300(334\,000) = 100.2\text{ kJ}$ transfer and give the wrong branch.

Checks and interpretation. Final masses add to the original total in every case. The coexistence solution has a liquid fraction between zero and one. In (c), the final temperature lies between the initial -10°C and 30°C and uses the solid capacity for *both* initial masses. The large ice addition is a controlled mathematical comparison, not a proposed small-calorimeter measurement.

Near a phase boundary, output uncertainty can change its character. Inside the coexistence region, the equilibrium temperature remains zero in the ideal model while uncertain input energy changes the liquid fraction. A conventional thermometer uncertainty is still present in an observation, but the physical state uncertainty is largely a mass-fraction uncertainty. At a boundary, uncertain inputs can populate two different branches, and one Gaussian temperature error bar from a smooth derivative need not summarize the result well. A justified calculation then evaluates branch selection for each possible input state or gives explicit branch probabilities from a specified uncertainty distribution. It should not claim that a plateau eliminates uncertainty in the final state.

Guided checks

Container. Omitting a warming calorimeter underestimates the inferred capacity of a cooling sample, because part of the sample's energy enters the wall.

Latent energy. A zero temperature slope during ideal coexistence permits nonzero energy transfer: the phase fraction changes.

Branch consistency. A negative temperature from an assumed all-liquid branch means that branch failed. Test a mixture, including possible freezing of initial water, and then the all-solid branch if necessary.

Uncertainty. Standard input contributions add in quadrature only under the independent linearized model. Unknown losses and property bias require additional evidence; they are not removed by averaging display readings.

The temperature, expansion, and calorimetry models all depend on stating what can change and

what is constrained. A thermometer may disturb its sample, an expanding rod may move a support or develop stress, and a calorimeter may change phase fractions without changing temperature. The next sections generalize this bookkeeping to arbitrary combinations of heat and work while keeping the distinction between a state change and the transfers used to produce it.

18.4 Systems, state variables and the ideal-gas bridge

Before writing an energy equation, identify the material being followed and draw its boundary. The gas in a sealed cylinder, the gas together with its piston, and the complete cylinder together with a heater are three different systems. An interaction crossing one boundary can be internal to another. This is why a description such as “the heater supplies energy” is incomplete until the system has been specified.

A *closed system* exchanges no mass with its surroundings. Its boundary may move and energy may cross it. An *isolated system* exchanges neither mass nor energy. An *open system* permits mass flow; a fixed region around a tank with an outlet is one example. Energy carried by the moving mass must then be included in a control-volume balance. We retain closed systems for the first-law calculations in this chapter.

These definitions concern the chosen boundary, rather than the name of an object. A sealed container is closed. Thermal insulation motivates neglecting heat transfer during a specified interval. A rigid wall excludes work from motion of that wall. None of these statements alone excludes every energy transfer. In particular, an insulated rigid vessel may receive electrical work through leads, or mechanical work through a rotating shaft.

Figure 18.12 separates the boundary tests. Its arrows identify possible interactions; they do not prescribe their magnitudes.

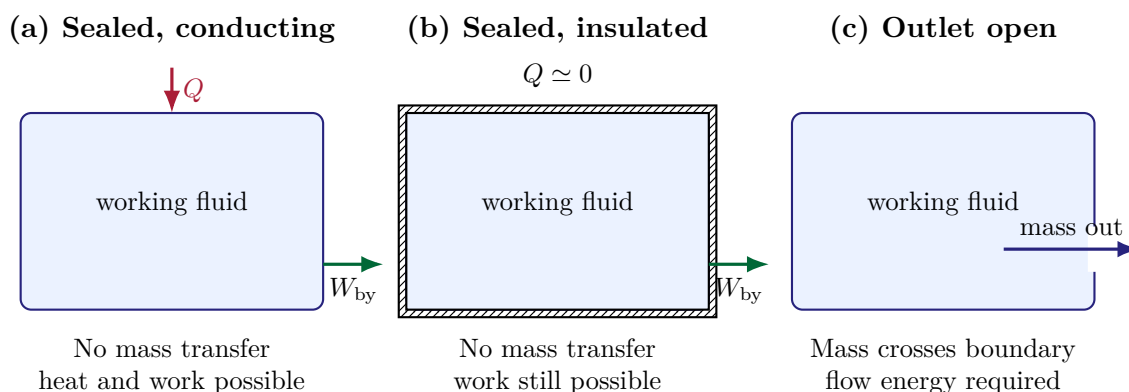


Figure 18.12: Three schematic system boundaries. Sealed and insulated impose different restrictions. The insulated middle system is isolated only if every work interaction is also excluded. The open outlet requires mass and energy-flow bookkeeping beyond the closed-system model.

Example 18.10: Diagnose three system boundaries

For the three systems in Fig. 18.12, determine whether the system is closed, open, or isolated. Explain what can be inferred about heat and work without being given a process history.

(a) Sealed, conducting vessel. No mass crosses, so the system is closed. The conducting wall allows heat transfer when the surroundings and system have different temperatures. A piston or shaft could allow work. Conducting does not prove that heat actually crosses during a particular interval: the temperatures might coincide. This vessel is not constrained to be isolated.

(b) Sealed, insulated vessel. The system is closed. Effective insulation justifies the process approximation $Q \simeq 0$. It does not stop a piston, electrical leads, or a shaft from transferring work. The middle drawing therefore does not establish isolation. If the boundary is also rigid and every other work interaction is excluded, the idealized system is isolated.

(c) Vessel with an open outlet. The chosen region is open because mass leaves. Applying $\Delta E_{\text{int}} = Q - W_{\text{by}}$ to the changing tank contents without a flow-energy term would omit an interaction. Alternatively, following a specified fixed amount of gas, including the portion that leaves, defines a different closed material system; its boundary moves with that material.

Check and interpretation. The classification is consistent with three separate tests: mass crossing, heat crossing, and work crossing. “Sealed”, “insulated”, and “rigid” answer different tests and cannot be interchanged.

18.4.1 Equilibrium states and independent coordinates

An equilibrium state has sufficiently well-defined macroscopic quantities to permit a reproducible description. For a simple gas these include pressure p , volume V , absolute temperature T , amount n , and internal energy E_{int} . Internal energy includes microscopic energy stored in the material; it excludes the kinetic energy of the system’s centre of mass and its external gravitational potential energy. The same letter U is often used for internal energy in other books. Here E_{int} avoids confusion with mechanical potential energy.

State variables are linked by material relations and need not be independent. For a fixed amount of a single-phase ideal gas, specifying two of p, V, T fixes the third. Heat Q and work W_{by} have a different role: they count transfers *during a process*. There is no amount of “heat contained in the gas” to be read from an equilibrium state. The gas contains energy; heat is one way of transferring energy across its boundary.

A nonequilibrium gas can contain temperature and pressure gradients. A single thermometer reading or a pressure at one wall need not then represent the whole gas. Local fields can be used in more advanced models, but they are not the single equilibrium coordinates of a simple p – V diagram. We must first allow the gas to settle before assigning an equilibrium endpoint.

18.4.2 An empirical equation of state

For a sufficiently dilute gas in a regime where the ideal-gas approximation is adequate, the measured equilibrium variables satisfy

$$pV = nRT. \quad (18.18)$$

The molar gas constant is $R = 8.314462618 \dots \text{J mol}^{-1}\text{K}^{-1}$. We use this equation as an empirical bridge between macroscopic states. Chapter 19 develops the molecular interpretation and the additional property that ideal-gas internal energy depends only on temperature. Equation (18.18) alone does not establish that property, nor does it determine a heat capacity.

Absolute pressure and kelvin temperature are required in the gas equation. Gauge pressure omits atmospheric pressure; Celsius temperature has an additive offset. Either substitution generally changes the physical equation. For the same fixed amount of gas at two equilibrium states,

$$\frac{p_f V_f}{p_i V_i} = \frac{T_f}{T_i}. \quad (18.19)$$

This relates the endpoints but gives no work until a process has been specified.

At fixed n and T , Eq. (18.18) gives $p = nRT/V$. The curves in Fig. 18.13 are equilibrium isotherms, each labelled by its kelvin temperature. Moving to a higher curve at a fixed volume means a larger pressure. Doubling volume on one curve halves its pressure.

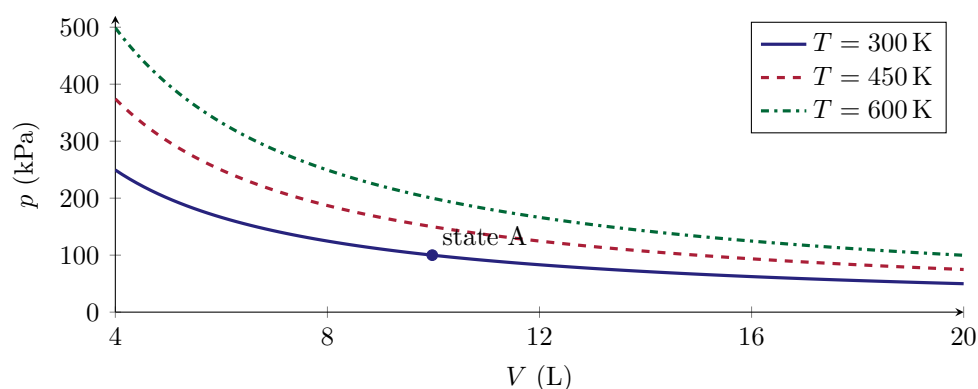


Figure 18.13: Analytical equilibrium isotherms for $n = 0.400$ mol in the ideal-gas model. The axes use $1 \text{ kPa L} = 1 \text{ J}$. The marked state A belongs to the 300 K curve. These curves are allowable equilibrium states, not evidence that a particular experiment followed one.

Example 18.11: Infer a gas state without inventing a process

A sealed sample contains 0.400 mol of gas modelled as ideal. At state A its absolute pressure is 100 kPa and its temperature is 300 K. Find its volume. At a later equilibrium state B the volume has doubled and the absolute pressure is 75.0 kPa. Find T_B and state whether the work can be calculated from this information.

Model and strategy. The sealed sample has fixed n . Both endpoints are equilibrium states. First use $p_A V_A = nRT_A$; then compare the two state equations. No path constraint has been supplied.

$$V_A = \frac{(0.400)(8.314462618)(300)}{100\,000} = 9.97736 \times 10^{-3} \text{ m}^3 = 9.98 \text{ L}.$$

Since $V_B = 2V_A$, the temperature ratio is

$$\frac{T_B}{T_A} = \frac{p_B V_B}{p_A V_A} = \frac{75.0}{100} \times 2 = 1.50, \quad T_B = 450 \text{ K}.$$

Work inference. Work cannot be uniquely determined. An expansion at one prescribed external pressure, a sequence of different pressures, and an expansion into vacuum are different interactions. The equilibrium endpoints do not specify which occurred. In particular, inserting a guessed curve into $\int p dV$ would supply information that was not in the statement.

Check. The endpoint product increases from $p_A V_A = 997.736 \text{ J}$ to $p_B V_B = 1496.60 \text{ J}$, exactly the factor 1.50 required by the temperature ratio. State A also lies on the appropriate curve in Fig. 18.13.

18.5 Work and thermodynamic paths

18.5.1 From force and displacement to boundary work

Consider a gas behind a piston of area A , as in Fig. 18.14. Let x increase outward, so an outward movement dx increases gas volume by $dV = A dx$. For a frictionless piston whose kinetic-energy change is negligible, the resisting external force is $F_{\text{ext}} = p_{\text{ext}} A$. Work delivered by the gas through the moving boundary is therefore the force times its signed displacement:

$$\delta W_{\text{by}} = F_{\text{ext}} dx = p_{\text{ext}} dV, \quad W_{\text{by}} = \int_{V_i}^{V_f} p_{\text{ext}} dV. \quad (18.20)$$

The symbol δW_{by} means a small work transfer. It is not the differential of a state function. The integration sums many small transfers when the resisting pressure varies.

Here p_{ext} denotes the resisting pressure transmitted at the moving system boundary, rather than a guessed average gas pressure. In a piston calculation, the atmosphere, an added load, friction, and appreciable piston acceleration must be treated consistently. The simple frictionless model excludes the last two. An arbitrary imposed load cannot be equated to uniform gas pressure while simultaneously neglecting the unbalanced piston force.

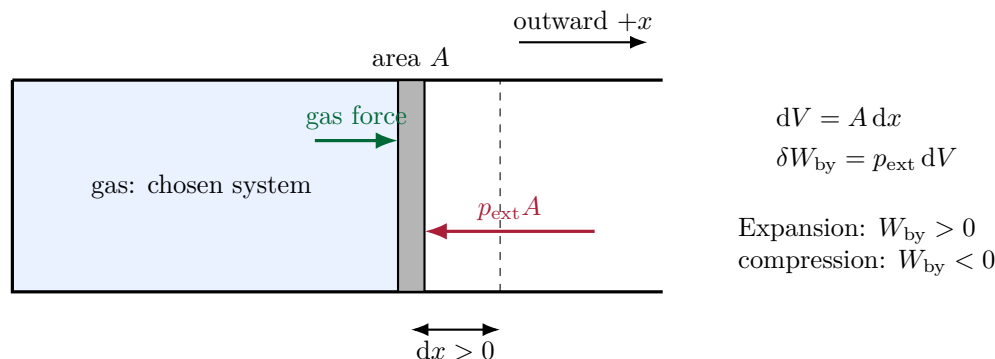


Figure 18.14: Schematic piston geometry for boundary work. The gas boundary moves outward through dx against the external resisting pressure. The displayed force balance assumes negligible piston acceleration and friction. Arrow lengths are schematic; the signed volume change determines the work sign.

Positive pressure with expansion gives positive work-by; energy leaves the gas through the mechanical interaction. Compression has $V_f < V_i$ and gives negative work-by. The surroundings then do positive work on the gas:

$$W_{\text{on}} = -W_{\text{by}}. \quad (18.21)$$

The negative sign is a convention translation, not an energy loss.

The units provide a useful check: $\text{Pa m}^3 = \text{N m} = \text{J}$, $\text{kPa m}^3 = \text{kJ}$, and $\text{kPa L} = \text{J}$. Keeping the volume unit visible prevents a factor-of-one-thousand error when reading a plotted area.

18.5.2 Quasistatic paths and common endpoints

A quasistatic process proceeds slowly enough relative to the system's internal relaxation that intermediate states are close to equilibrium. For the mechanical quasi-equilibrium approximation, the equilibrium gas pressure p equals the boundary resisting pressure to the required accuracy. Only under this condition may the work be evaluated as $\int p dV$ using the equilibrium p - V curve. Slow operation by itself does not remove friction or finite temperature differences and does not establish reversibility.

On such a diagram a horizontal segment changes volume at constant pressure; its signed rectangular area is work. A vertical segment changes pressure at constant volume and contributes no boundary work. A sloping segment requires the integral of its actual pressure function. Figure 18.15 compares three equilibrium paths with common endpoints. Its direction arrows are essential: the same geometric curve traversed in reverse has the opposite work sign.

Example 18.12: Three paths between the same gas states

An ideal-gas sample moves from A, $V_A = 0.020 \text{ m}^3$, $p_A = 400 \text{ kPa}$, to B, $V_B = 0.040 \text{ m}^3$, $p_B = 200 \text{ kPa}$. Find the work for the upper, lower, and isothermal quasistatic paths in Fig. 18.15. Given the ideal-gas property $E_{\text{int}} = E_{\text{int}}(T)$, compare the net heats for these three paths and the work for reversal of the upper path.

Endpoint check. Both products are $p_A V_A = p_B V_B = 8.00 \text{ kJ}$. Fixed n and $pV = nRT$ therefore give $T_A = T_B$. With the stated additional ideal-gas energy property, $\Delta E_{\text{int}} = 0$ between these endpoints.

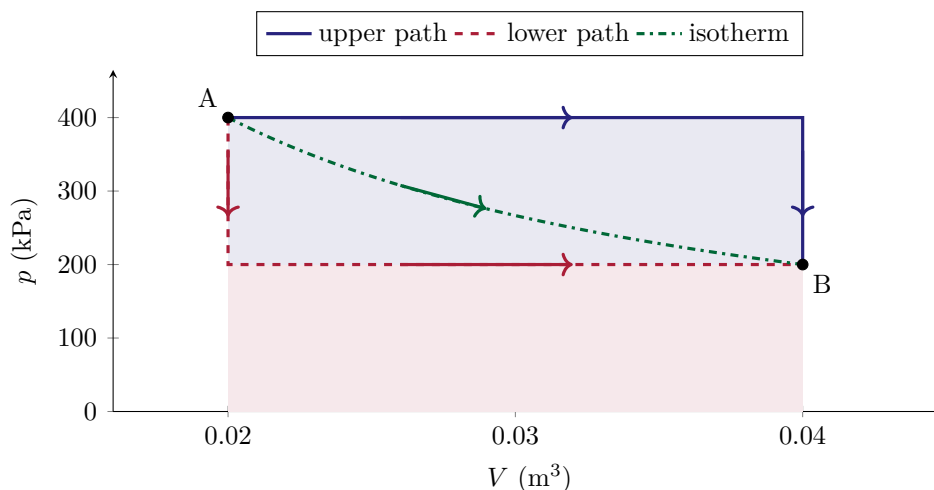


Figure 18.15: Three mechanically quasistatic paths from A to B. The upper and lower piecewise paths give different rectangular work areas; the ideal-gas isotherm $pV = 8.00 \text{ kJ}$ gives an intermediate area. Equal endpoints do not fix work, even when they have equal ideal-gas temperatures.

Upper path. Expansion occurs at 400 kPa, followed by a pressure decrease at fixed V_B :

$$W_{\text{by}}^{(u)} = (400 \text{ kPa})(0.040 - 0.020) \text{ m}^3 = 8.00 \text{ kJ}.$$

Lower path. The pressure first decreases at fixed V_A ; expansion then occurs at 200 kPa:

$$W_{\text{by}}^{(l)} = (200 \text{ kPa})(0.040 - 0.020) \text{ m}^3 = 4.00 \text{ kJ}.$$

The fixed-volume legs contribute zero boundary work in both cases.

Isothermal path. Along this particular path $p = C/V$, where $C = p_A V_A = 8.00 \text{ kJ}$. The antiderivative of $1/V$ is $\ln V$, so

$$W_{\text{by}}^{(T)} = C \int_{V_A}^{V_B} \frac{dV}{V} = C \ln\left(\frac{V_B}{V_A}\right) = (8.00 \text{ kJ}) \ln 2 = 5.54518 \text{ kJ}.$$

The ratio inside the logarithm is dimensionless. The result lies between 4.00 and 8.00 kJ, as the graph requires.

Heat and reversal. The first law with $\Delta E_{\text{int}} = 0$ gives $Q_u = 8.00 \text{ kJ}$, $Q_l = 4.00 \text{ kJ}$, and $Q_T = 5.55 \text{ kJ}$. The gas receives net heat and delivers the same net energy as work along each path. Reversing the upper path reverses the integration limits, giving $W_{\text{by}} = -8.00 \text{ kJ}$ and $Q = -8.00 \text{ kJ}$. Equality of endpoint temperatures does not mean that the piecewise upper and lower paths are isothermal at intermediate states.

18.5.3 A graph may encode an integral, not an average endpoint

For a straight pressure segment between (V_i, p_i) and (V_f, p_f) , the mean pressure over volume is $(p_i + p_f)/2$. The work is the signed trapezoidal area,

$$W_{\text{by}} = \frac{p_i + p_f}{2} (V_f - V_i) \quad \text{for a pressure linear in volume.} \quad (18.22)$$

This endpoint average is exact for a straight line. It is not generally exact for a curved path. For the isotherm above it would give 6.00 kJ, instead of 5.54518 kJ. The area under a decreasing convex hyperbola lies below its endpoint chord, explaining the direction of that error without calculation.

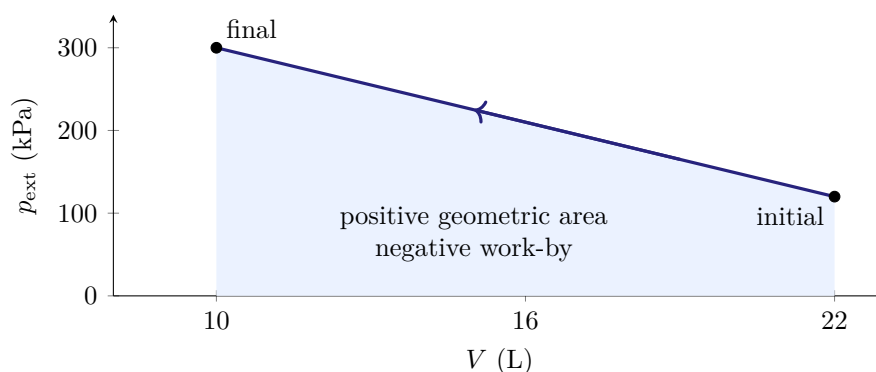


Figure 18.16: A prescribed linear pressure-volume compression. The arrow points from larger to smaller volume. Its geometric area is positive, but integrating with compression limits gives negative work-by. This is a boundary-pressure plot; it is also an equilibrium-state path when mechanical quasi-equilibrium holds.

Example 18.13: Signed compression and an energy check

A closed gas is compressed quasistatically from 22.0 L at 120 kPa to 10.0 L at 300 kPa. Pressure varies linearly with volume as in Fig. 18.16. During compression the gas loses 0.800 kJ as heat. Find work-by, work-on, and the internal-energy change. Changes of bulk kinetic and potential energy are negligible.

Strategy. A signed graph area supplies the boundary work. Heat loss has $Q = -0.800$ kJ in the book convention.

$$W_{\text{by}} = \frac{120 + 300}{2} \text{ kPa} (10.0 - 22.0) \text{ L} = -2520 \text{ J} = -2.520 \text{ kJ}.$$

Thus $W_{\text{on}} = +2.520$ kJ and

$$\Delta E_{\text{int}} = Q - W_{\text{by}} = -0.800 - (-2.520) = +1.720 \text{ kJ}.$$

Independent area check. The magnitude is the sum of a rectangle $120 \times 12 = 1440$ J and a triangle $\frac{1}{2}(300 - 120) \times 12 = 1080$ J: total 2520 J. The compression arrow fixes the negative work-by sign.

Interpretation. Mechanical input exceeds heat loss, so stored internal energy increases. A gas can gain internal energy while losing heat; heat and internal-energy change are not interchangeable quantities.

18.5.4 Quasistatic motion with friction

Reversibility imposes more than slowness. To reverse a process while restoring both system and surroundings, dissipation must be absent and driving differences must be arbitrarily small. A piston can move very slowly while rubbing against a seal. The gas may remain nearly uniform, yet friction still converts mechanical energy into internal energy outside the gas.

Consider a piston of area A under a constant ideal load represented by pressure p_0 . A sliding seal exerts a constant friction force magnitude F_f . Piston inertia is negligible. On outward motion the gas must overcome load and friction; on inward motion friction assists the gas in resisting the load:

$$p_{\text{out}} = p_0 + \frac{F_f}{A}, \quad p_{\text{in}} = p_0 - \frac{F_f}{A}. \quad (18.23)$$

These are the total resisting pressures transmitted at the gas boundary for the respective directions. They are not both equal to the ideal load pressure p_0 . The gas supplies more work during outward motion than the ideal load returns during inward motion. The difference is dissipated by the seal.

Figure 18.17 shows the two motion branches. Vertical joins represent pressure changes with the piston held fixed before its direction is reversed. With appropriate thermal control, these joins close the gas-state cycle. The finite pressure separation persists as the motion is slowed.

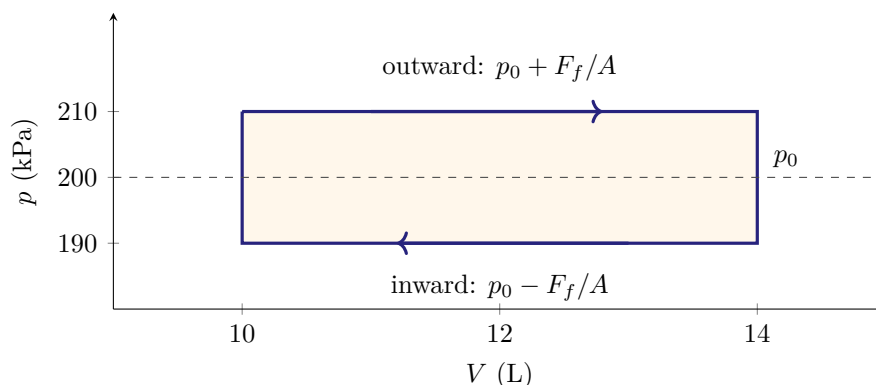


Figure 18.17: An idealized slow piston cycle with sliding friction: $p_0 = 200$ kPa and $F_f/A = 10.0$ kPa. The separation between the motion branches is hysteresis. The cycle area is energy delivered by the gas and dissipated at the seal; the ideal load receives zero net work over the round trip.

Example 18.14: Slow motion that is still irreversible

A piston of area $A = 0.0100 \text{ m}^2$ moves slowly against an ideal constant load with $p_0 = 200$ kPa. The seal friction magnitude is $F_f = 100$ N. The gas expands from 10.0 to 14.0 L, then returns along the lower motion branch in Fig. 18.17. Pressure is changed at fixed volume at each end to complete the cycle. Find the gas work, ideal-load work, and friction dissipation. Assume thermal control returns the gas to its initial state and frictional heating remains in the piston and surroundings.

Force balance. The friction pressure is $F_f/A = 10\,000 \text{ Pa} = 10.0$ kPa. Thus the outward and inward motion pressures are 210 and 190 kPa.

$$W_{\text{by,out}} = (210 \text{ kPa})(4.00 \text{ L}) = 840 \text{ J}, \quad W_{\text{by,in}} = (190 \text{ kPa})(-4.00 \text{ L}) = -760 \text{ J}.$$

The fixed-volume joins have zero boundary work, giving $W_{\text{by,net}} = +80.0 \text{ J}$.

Load and dissipation. The ideal load receives $p_0 \Delta V = 800 \text{ J}$ outward and returns the same amount inward; its net work is zero. The piston travels $\Delta x = \Delta V/A = 0.00400/0.0100 = 0.400 \text{ m}$ each way. Friction dissipates

$$E_{\text{diss}} = F_f(2\Delta x) = (100)(0.800) = 80.0 \text{ J},$$

equal to the pressure-loop area $[2F_f/A]\Delta V = (20.0 \text{ kPa})(4.00 \text{ L})$.

First-law check. The gas has completed a state cycle, so $\Delta E_{\text{int}} = 0$ and net heat entering the gas must be $+80.0 \text{ J}$. Its supplied energy ends in the dissipative surroundings. Making the piston speed smaller does not remove the 80.0 J in this constant-friction model. Quasistatic motion therefore does not establish reversibility.

18.6 The first law and basic processes

18.6.1 The general closed-system ledger

Conservation of energy for a closed material system is

$$\Delta(E_{\text{int}} + K_{\text{bulk}} + E_{\text{pot}}) = Q - W_{\text{by}}, \quad (18.24)$$

where K_{bulk} is macroscopic kinetic energy and E_{pot} is external potential energy. Work-by here includes *all* work interactions crossing the chosen boundary, not only piston work. The equation applies between specified endpoints even if the intermediate process is irreversible. For a stationary system whose macroscopic kinetic and potential energies do not change appreciably, it reduces to

$$\Delta E_{\text{int}} = Q - W_{\text{by}}. \quad (18.25)$$

Internal energy is a state function. Its change is fixed by the initial and final states; neither heat nor work separately is. The combination $Q - W_{\text{by}}$ must have the same value for every process between the same endpoints. Figure 18.18 makes the bookkeeping direction explicit. A negative value of Q means net heat leaves the system. A negative value of W_{by} means net work enters it.

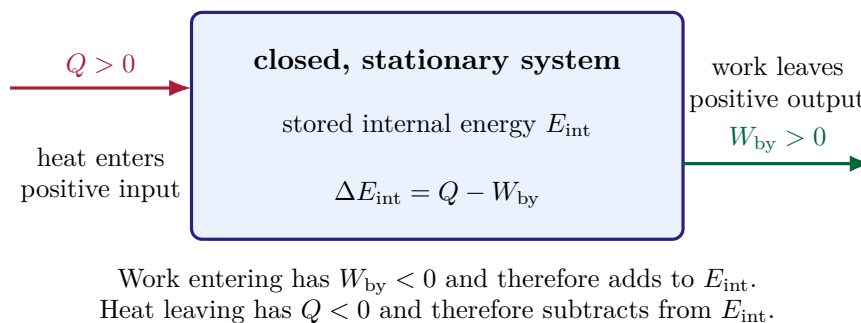


Figure 18.18: First-law energy directions. Signs refer to the chosen system boundary. The arrows show the positive reference directions rather than a claim that all real transfers follow those directions. Macroscopic kinetic and potential changes are neglected in this reduced ledger.

The supplied Halliday–Resnick–Walker book adopts positive work by the system, while Serway–Jewett uses positive work on the system. Their first-law equations are equivalent after $W_{\text{on}} = -W_{\text{by}}$:

$$\Delta E_{\text{int}} = Q - W_{\text{by}} = Q + W_{\text{on}}.$$

This translation is essential when comparing a solution written in the other convention [1, 2]. A change of notation cannot change a measured energy transfer.

18.6.2 Process labels and what they actually restrict

An *isochoric* process has fixed volume. For a rigid boundary this removes boundary work, but it does not remove electrical or shaft work. Only if these other work interactions are absent may we conclude $\Delta E_{\text{int}} = Q$. Heating a sealed rigid gas vessel without another work channel is a useful example. In a single-phase ideal-gas approximation its pressure increases with absolute temperature because $p/T = nR/V$ is constant.

An *isobaric* process has constant equilibrium pressure. Under mechanical quasi-equilibrium, $W_{\text{by}} = p(V_f - V_i)$. Constant pressure does not imply constant temperature: T changes with V for fixed n . Heating a gas beneath a freely moving frictionless loaded piston typically sends some input energy into internal energy and some into work.

An *adiabatic* process has $Q = 0$. Effective insulation or a process much shorter than the thermal exchange time may justify this approximation. With negligible macroscopic energy changes, $\Delta E_{\text{int}} = -W_{\text{by}}$. An adiabatic compression therefore increases internal energy when the net work-by is negative. Adiabatic is not a statement about constant temperature, fixed volume, or reversibility. The particular equilibrium relation for a reversible adiabatic ideal-gas path requires further material information and is developed in Chapter 19.

An *isothermal* process has constant temperature. For an ideal gas, using the additional property $E_{\text{int}} = E_{\text{int}}(T)$, its internal-energy change is zero and therefore $Q = W_{\text{by}}$. Heat may be positive

and substantial: during a quasistatic expansion it replaces energy leaving as work. The assertion “constant temperature means no heat transfer” fails because it confuses temperature with an energy-transfer quantity. For a nonideal substance, a temperature constraint alone need not imply constant internal energy.

These process labels do not specify a universal hierarchy. A process can be both adiabatic and constant-volume; it can be both quasistatic and isothermal; and it can be adiabatic and irreversible. Every calculation needs both the interaction restrictions and the material model.

18.6.3 Cycles and signed loop area

In a complete cycle every state variable returns to its starting value. If the system’s macroscopic motion and height also return, Eq. (18.24) gives

$$\Delta E_{\text{total}} = 0, \quad Q_{\text{net}} = W_{\text{by,net}}. \quad (18.26)$$

Net heat is the algebraic sum of all heat inputs and outputs. Net work is the algebraic sum over all work interactions; neither is generally equal to a single positive input measured on one portion of the cycle.

For a quasistatic pressure-volume cycle involving only boundary work, the integral $\oint p dV$ equals its signed loop area. Expansion at higher pressure than compression gives a clockwise cycle and positive net work-by. Reversing traversal reverses the sign. If other work interactions are present, the loop area gives only the boundary-work contribution, and the first-law ledger must include the others separately.

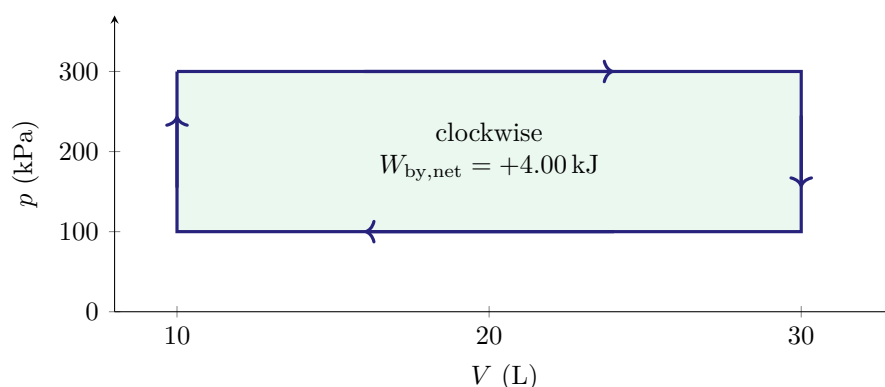


Figure 18.19: An analytical rectangular boundary-work cycle. Expansion contributes +6.00 kJ and compression -2.00 kJ, leaving the +4.00 kJ shaded area. Vertical legs contribute no boundary work. This energy bookkeeping does not establish an engine efficiency or a second-law limit; those require additional thermal information.

Example 18.15: A cycle ledger with a missing transfer

A closed device is reported to execute a complete cycle, returning its macroscopic motion and height as well as its internal state. The measured net heat entering is $Q_{\text{net}} = +7.2$ kJ. Reported net work done by the device is +5.9 kJ. Test consistency and identify the direction of an omitted work transfer if the heat and cycle claims are correct.

State condition. A complete cycle has $\Delta E_{\text{total}} = 0$. The listed transfers instead predict

$$\Delta E_{\text{total}} = 7.2 - 5.9 = +1.3 \text{ kJ},$$

so the record is inconsistent with a completed cycle.

Missing-channel inference. If heat and the cycle identification are correct, total net work-by must be +7.2 kJ. A work output of +1.3 kJ is missing from the reported 5.9 kJ. An unreported

electrical output is one possible channel. Alternatively, if the work record is complete, an omitted heat loss of 1.3 kJ would make actual net heat 5.9 kJ. A stored-energy increase of 1.3 kJ would mean the claimed cycle was incomplete.

Interpretation and check. These are distinct diagnoses, not evidence for which one occurred. In each repaired ledger $Q - W_{\text{by}} = 0$. The first law detects the missing energy but the measurements must identify its channel.

18.6.4 Free expansion: endpoints without an equilibrium path

In free expansion a gas fills an initially evacuated space after a partition is removed. Take a rigid insulated outer vessel and assume the partition's removal transfers negligible energy. There is no exterior moving-wall work, no external heat transfer, and no other work interaction. Equivalently, the gas front expands against zero resisting pressure in the idealized vacuum description. Total gas energy is conserved.

During the expansion the gas can have transient flow and spatial gradients. It is not a sequence of equilibrium states with a single pressure. At initial and final rest, with negligible change of external potential energy, Eq. (18.24) yields

$$Q = 0, \quad W_{\text{by}} = 0, \quad \Delta E_{\text{int}} = 0. \quad (18.27)$$

These restrictions are consequences of the full specified setup. Expansion into vacuum alone does not establish $Q = 0$ if the walls exchange heat. Insulation alone does not establish zero work if a driven partition or other interaction supplies energy.

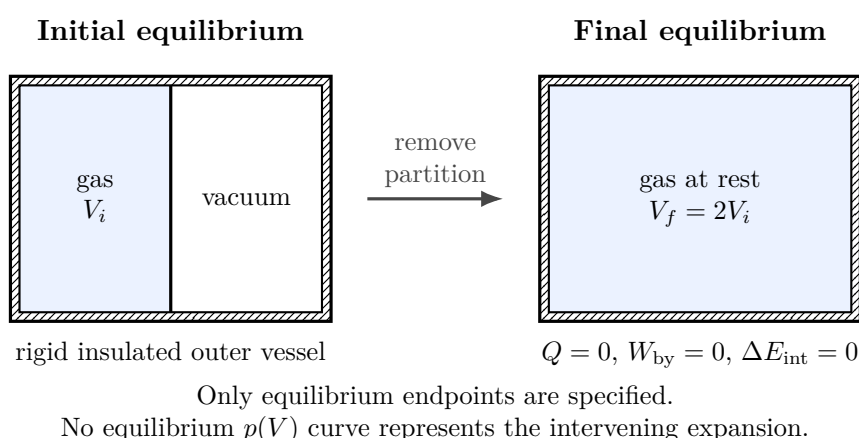


Figure 18.20: Free-expansion setup. The vessel geometry is schematic. Partition removal is assumed to contribute negligible energy. Initial and final gas states are at rest; the intermediate nonuniform evolution is deliberately not drawn as an equilibrium path.

For an ideal gas the independently supplied material property $E_{\text{int}} = E_{\text{int}}(T)$, with positive temperature derivative in the regime under study, then gives $T_f = T_i$. Since n is fixed, $p_f = p_i V_i / V_f$; doubling the occupied volume halves the final equilibrium pressure. A real gas can have internal energy depending on volume as well as temperature. Constant internal energy then need not give equal temperatures.

Equal initial and final ideal-gas temperatures do not turn free expansion into an equilibrium isothermal path. The isotherm joining the same endpoints in Fig. 18.15 describes another, quasistatic process which does positive work and receives heat. Substituting that curve into the free-expansion work integral would compute the work of the other process. The first law supports both ledgers because the endpoints have the same internal-energy change, while the transfers differ.

18.6.5 Electrical work in a rigid insulated vessel

Not all thermodynamic work is pressure-volume work. A heater inside a sealed rigid vessel can receive energy through electrical leads even though its boundary does not move. If the heater is part of the chosen system, electrical energy enters as work. Its subsequent conversion to internal energy occurs inside the boundary. If instead only the fluid is the system and the heater is excluded, energy crossing from the heater to the fluid through a temperature difference is heat. Both descriptions give compatible physical results when the boundary is kept consistent.

For a device receiving a steady electrical voltage $\mathcal{V}$ and current I , input power is $P_{\text{el}} = \mathcal{V}I$. The electrical work-on in a time τ is $W_{\text{on,el}} = \mathcal{V}I\tau$; equivalently, $W_{\text{by,el}} = -\mathcal{V}I\tau$. The electrical voltage is written $\mathcal{V}$ to distinguish it from thermodynamic volume V .

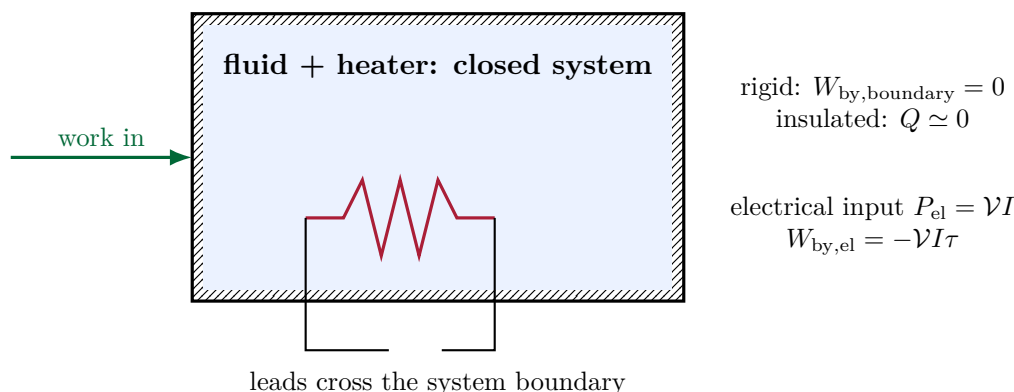


Figure 18.21: A schematic rigid vessel whose system includes fluid and heater. Electrical input is a work interaction through leads. Insulation suppresses heat transfer through the outer wall; it does not make the system isolated. With another boundary excluding the heater, transfer from heater to fluid is instead counted as heat.

Example 18.16: A fixed volume can still receive work

A rigid sealed vessel contains liquid and an electrical heater. Take liquid plus heater as the system. The voltage is 12.0 V, current 2.00 A, and operating time 180 s. During that time 0.360 kJ escapes as heat through imperfect insulation. Find the total work-by and internal-energy increase, assuming negligible macroscopic energy changes and no other transfer. Compare an ideal adiabatic run with the same electrical input.

Identify the interactions. The rigid wall gives zero boundary work. The heater receives electrical work; it lies inside the chosen system, so electrical conversion is an internal process. The wall heat loss gives $Q = -0.360$ kJ.

$$W_{\text{on,el}} = (12.0 \text{ V})(2.00 \text{ A})(180 \text{ s}) = 4320 \text{ J} = 4.320 \text{ kJ}, \quad W_{\text{by}} = -4.320 \text{ kJ}.$$

Consequently,

$$\Delta E_{\text{int}} = -0.360 - (-4.320) = +3.960 \text{ kJ}.$$

Energy check. Input 4.320 kJ equals stored increase 3.960 kJ plus heat loss 0.360 kJ. The current-voltage product has units of watts, so multiplying by seconds correctly yields joules.

Adiabatic comparison. If $Q = 0$, the same electrical input gives $\Delta E_{\text{int}} = +4.320$ kJ. It remains a closed system receiving work and is therefore not isolated. Fixed volume never justified discarding the electrical-work term. Predicting a temperature rise would additionally require the appropriate heat capacities and any phase-change information; the first law alone does not supply them.

Guided checks

Adiabatic compression. Compression gives $W_{\text{by}} < 0$; with $Q = 0$ and negligible bulk-energy changes, $\Delta E_{\text{int}} = -W_{\text{by}} > 0$. Thus an adiabatic process can increase internal energy.

Zero internal-energy change. This condition does not force the two transfers individually to vanish: they can be equal and nonzero. It does not establish a complete cycle, because other state variables may have changed.

A pressure-volume curve. Its area is boundary work only when the pressure plotted is the appropriate boundary traction, or the equilibrium pressure under mechanical quasi-equilibrium. An imagined line between irreversible endpoints has no such authority.

The first law tells us how energy transfers change stored energy. It supplies neither their rates nor the direction in which every conceivable process can occur. The following heat-transfer models quantify rates under stated geometrical and material assumptions. Chapter 20 introduces the additional restrictions associated with entropy and the second law.

18.7 Conduction, convection, and radiation

Conservation of energy determines how a transfer changes a system's stored energy. It does not determine how quickly the transfer occurs. A thin insulation layer and a thick insulation layer can separate the same two temperatures while transmitting different powers. A hot object in still air and the same object in a moving airstream also exchange energy at different rates. To predict those rates, the first law must be supplemented by geometry and a constitutive or empirical model.

Three mechanisms are distinguished here. *Conduction* transfers energy through a material without requiring a bulk flow of that material. *Convection* describes heat transfer between a surface and a moving fluid; fluid motion then redistributes energy away from the surface. *Thermal radiation* transfers energy by electromagnetic waves and can operate through vacuum. Several mechanisms commonly act on the same object. Choosing one mechanism for a calculation is therefore an approximation to be justified, rather than an identity of the object.

We use $\dot{Q}_{\text{out}}$ for a rate defined positive outward from a selected body. This is a directional bookkeeping choice for a *rate*; it does not change the chapter convention that Q is positive when heat enters the system. With no work or mass transfer, $dE_{\text{int}}/dt = -\dot{Q}_{\text{out}}$. A negative outward rate means inward transfer. This convention is especially useful when a cooling object eventually approaches its environment.

18.7.1 Fourier's law and the one-dimensional plane wall

Let x increase through a wall from the hot face toward the cold face. Let A be the area perpendicular to x and let $\kappa > 0$ be the thermal conductivity. Fourier's law in this one-dimensional model is

$$\dot{Q}_x = -\kappa A \frac{dT}{dx}. \quad (18.28)$$

Conductivity has units $\text{W m}^{-1}\text{K}^{-1}$. A temperature gradient has units K m^{-1} , so the product with area has units watts. If temperature decreases with increasing x , then $dT/dx < 0$ and $\dot{Q}_x > 0$: energy moves in the positive coordinate direction. Reversing the coordinate reverses the signed component, while the physical hot-to-cold transfer stays the same.

For a steady wall with no internal heat production, energy cannot accumulate in any thin slice. The heat rate entering the slice must equal the rate leaving it. Thus $\dot{Q}_x$ is independent of x . If A and κ are constant, Eq. (18.28) then requires a constant gradient. Integrating with $T(0) = T_1$ and

$T(L) = T_2$ gives

$$T(x) = T_1 + \frac{T_2 - T_1}{L}x, \quad (18.29)$$

$$\dot{Q}_x = \frac{\kappa A}{L}(T_1 - T_2). \quad (18.30)$$

The familiar wall formula therefore contains several assumptions: steady conditions, constant conductivity, one-dimensional conduction, no internal generation, and known *face* temperatures. Air temperatures on the two sides are generally not face temperatures. Surface films must be included if those are the specified boundary data.

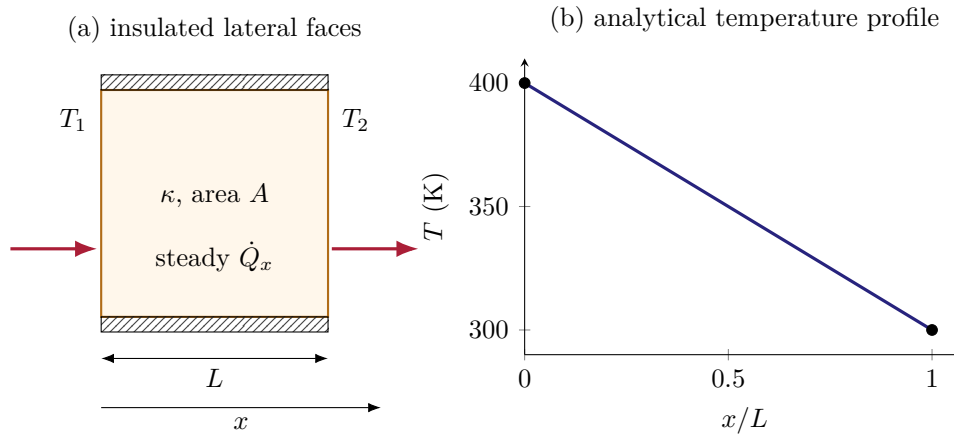


Figure 18.22: A plane wall with insulated lateral faces and fixed face temperatures. The physical geometry is schematic. The quantitative profile uses $T_1 = 400$ K and $T_2 = 300$ K, constant conductivity, and no internal generation. Equal heat rate through every cross section produces the straight temperature profile.

Figure 18.22 connects geometry, slope, and power. Doubling A doubles the number of parallel conduction paths and the rate. Doubling L halves the gradient for the same endpoint temperatures and halves the rate. Doubling κ doubles the rate without changing the linear profile when both face temperatures are prescribed. These are useful checks on any numerical wall calculation.

18.7.2 When conductivity varies with temperature

A straight steady temperature profile is not a general consequence of steady transfer. If $\kappa = \kappa(T)$, the rate can remain constant while the gradient changes. Fourier's law now separates as $\dot{Q}_x dx = -A\kappa(T) dT$. Integrating between the faces gives

$$\dot{Q}_x = \frac{A}{L} \int_{T_2}^{T_1} \kappa(T) dT. \quad (18.31)$$

An effective conductivity for this specific temperature interval is the average

$$\bar{\kappa} = \frac{1}{T_1 - T_2} \int_{T_2}^{T_1} \kappa(T) dT. \quad (18.32)$$

This is not a new universal property of the material. It depends on the temperature range. The formula reduces continuously to Eq. (18.30) when κ is constant.

The profile follows from the partial integral

$$\frac{x}{L} = \frac{\int_{T(x)}^{T_1} \kappa(T) dT}{\int_{T_2}^{T_1} \kappa(T) dT}. \quad (18.33)$$

A high local conductivity permits a smaller local gradient for the same heat rate. The slope is therefore gentler near a hotter face when conductivity increases with temperature. No extra source of energy is implied by the curvature.

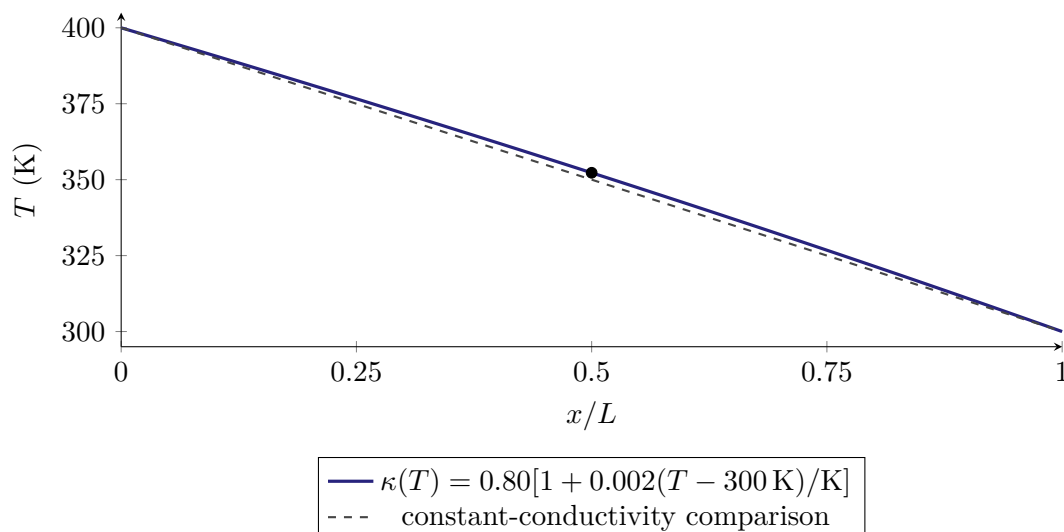


Figure 18.23: Steady temperature profiles with the same prescribed face temperatures. The stipulated variable-conductivity model gives $\bar{\kappa} = 0.88 \text{ W m}^{-1}\text{K}^{-1}$. Its midpoint is 352.27 K, rather than 350 K. Both curves are analytical; the small curvature records property variation, not transient energy accumulation.

Example 18.17: A wall with temperature-dependent conductivity

Statement. A slab has $A = 0.020 \text{ m}^2$, $L = 0.040 \text{ m}$, and fixed face temperatures $T_1 = 400 \text{ K}$, $T_2 = 300 \text{ K}$. Assume steady one-dimensional conduction, no heat production, and the stipulated law

$$\kappa(T) = \kappa_0[1 + b(T - 300 \text{ K})], \quad \kappa_0 = 0.80 \text{ W m}^{-1}\text{K}^{-1}, \quad b = 0.0020 \text{ K}^{-1}.$$

Find (a) the rate, (b) the midpoint temperature, and (c) the error from replacing conductivity by its cold-face value.

Strategy. Constant power does not require constant gradient. Integrate the constitutive law before evaluating the rate. Write $\theta = T - 300 \text{ K}$; the endpoint interval is $0 \leq \theta \leq 100 \text{ K}$.

$$\int_{300}^{400} \kappa(T) dT = \kappa_0 \left[\theta + \frac{b}{2} \theta^2 \right]_0^{100 \text{ K}} = (0.80)(110) \text{ W m}^{-1}.$$

Thus (a)

$$\dot{Q}_x = \frac{0.020}{0.040} (88) = 44.0 \text{ W}.$$

For (b), half the integral occurs before the midpoint:

$$\theta_m + \frac{b}{2} \theta_m^2 = 55 \text{ K}, \quad \theta_m = \frac{-1 + \sqrt{1 + 110b \text{ K}}}{b} = 52.27 \text{ K}.$$

The positive root gives $T_m = 352.27 \text{ K}$; the other root lies outside the allowed temperature interval. Figure 18.23 shows this displacement above the straight comparison line.

For (c), using only κ_0 would give 40.0 W, an underestimate of 4.0 W or 9.09% of the variable-property result. Using the conductivity at 350 K happens to give the exact average here because the stipulated law is linear in T . That midpoint rule is not exact for an arbitrary conductivity function.

Checks. The rate has units watts and remains positive because $T_1 > T_2$. Letting $b \rightarrow 0$ recovers both the constant-conductivity rate and the linear temperature profile. The positive conductivity throughout the interval ensures a monotonic profile.

18.7.3 Convection: a surface law and a moving fluid

Near a solid surface, heat crosses the thin fluid region through microscopic transport. Fluid motion continually replaces or redistributes that warmed or cooled fluid. A practical surface model collects this complicated process into

$$\dot{Q}_{\text{conv,out}} = hA_s(T_s - T_\infty). \quad (18.34)$$

Here A_s is the exposed surface area, T_s the surface temperature, and T_∞ a representative fluid temperature away from the immediate surface region. The coefficient h has units $\text{W m}^{-2}\text{K}^{-1}$. It is not an intrinsic solid property analogous to conductivity. It depends on fluid properties, flow speed, orientation, geometry, and flow regime. Treating h as constant means that these effects are sufficiently stable over the stated experiment or temperature interval.

In natural convection, buoyancy generates much of the motion. When warming lowers the fluid density, warm fluid rises and cooler fluid replaces it. This density argument needs qualification near water's density maximum: warming does not always lower liquid-water density over every interval. In forced convection, a fan or pump imposes a flow. Both cases require a surrounding fluid. An ideal vacuum has no convective heat transfer, although solid supports can still conduct and the exposed object can still radiate.

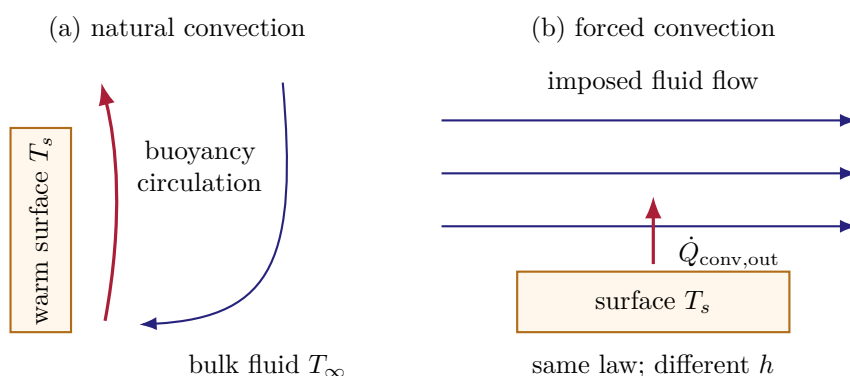


Figure 18.24: Natural and forced convection, shown schematically for a surface hotter than its fluid. Arrows identify fluid motion and outward thermal transfer; their lengths are not quantitative. The empirical coefficient h summarizes the boundary-layer transport for the specified regime.

A common confusion is to interpret a large h as a large heat capacity. They describe different properties. The product hA_s is a conductance in watts per kelvin: it connects a temperature difference to a power. A heat capacity C in joules per kelvin connects a temperature change to stored energy. Their ratio $C/(hA_s)$ has units seconds and can become a transient time scale, but only after an energy balance has been established.

18.7.4 Emission, absorption, and net thermal radiation

A surface radiates even while it absorbs radiation. For a grey surface, the emitted power is modelled as $\varepsilon_{\text{rad}}\sigma_{\text{SB}}A_sT_s^4$, where $0 \leq \varepsilon_{\text{rad}} \leq 1$ and $\sigma_{\text{SB}} = 5.670374419 \times 10^{-8} \text{ W m}^{-2}\text{K}^{-4}$. The word grey means that emissivity is treated as independent of wavelength over the range relevant to this model. A diffuse surface has direction-independent radiance. These approximations simplify the geometry and spectral accounting; they do not describe every real coating.

For a small opaque diffuse grey surface, a transparent nonparticipating intervening medium, and a much larger isothermal enclosure with view factor approximately one, the net outward rate becomes

$$\dot{Q}_{\text{rad,out}} = \varepsilon_{\text{rad}}\sigma_{\text{SB}}A_s(T_s^4 - T_{\text{env}}^4). \quad (18.35)$$

The enclosure acts as a nearly black radiation environment: it supplies irradiation approximately $\sigma_{\text{SB}}T_{\text{env}}^4$. This assumption explains why the expression is valid without separately specifying an

enclosure emissivity. Two finite grey plates generally require a different exchange calculation, including their areas, view factors, and repeated reflections. Multiplying the small-surface formula by an arbitrary view factor is not a complete solution for that case.

Every temperature entering a fourth power is absolute. With $T_s = 350$ K and $T_{\text{env}} = 300$ K, the correct fourth-power difference is $350^4 - 300^4$. Replacing it by $77^4 - 27^4$ changes the physics; Celsius zero is an arbitrary origin. Temperature *differences* in Fourier's law or the convection law have the same numerical size in kelvins and Celsius degrees.

The radiation law has a useful exact factorization:

$$\dot{Q}_{\text{rad,out}} = h_{\text{rad}} A_s (T_s - T_{\text{env}}), \quad (18.36)$$

$$h_{\text{rad}} = \varepsilon_{\text{rad}} \sigma_{\text{SB}} (T_s + T_{\text{env}}) (T_s^2 + T_{\text{env}}^2). \quad (18.37)$$

The representation is linear in the displayed temperature difference, but its coefficient still depends on the temperatures. Near a common temperature T_* , $h_{\text{rad}} \simeq 4\varepsilon_{\text{rad}} \sigma_{\text{SB}} T_*^3$. This approximation explains why radiation can resemble a constant added conductance over a narrow interval.

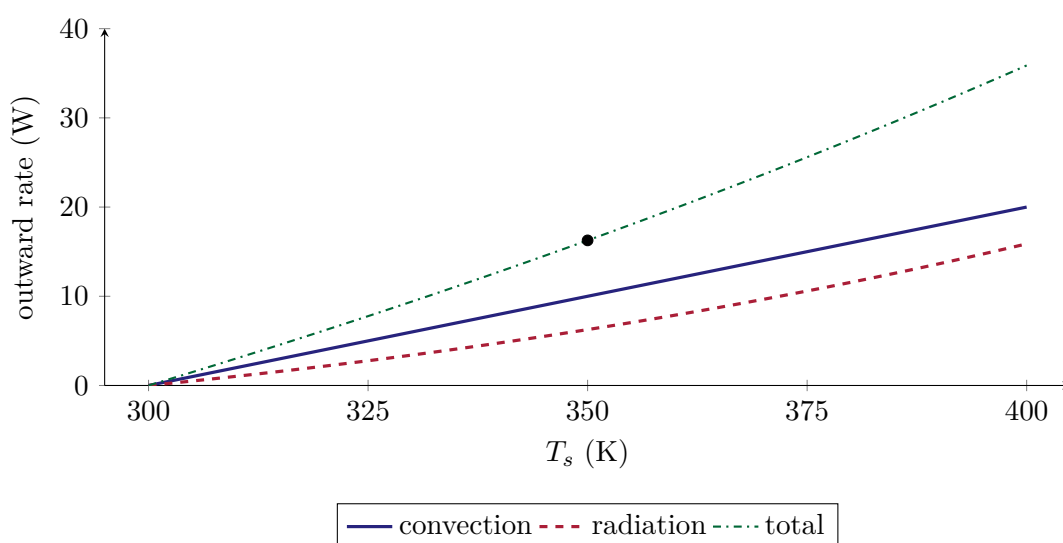


Figure 18.25: Analytical surface-transfer rates for $A_s = 0.020 \text{ m}^2$, $h = 10 \text{ W m}^{-2} \text{ K}^{-1}$, $\varepsilon_{\text{rad}} = 0.80$, and $T_\infty = T_{\text{env}} = 300$ K. Radiation and convection act in parallel on the same surface. Their sum at 350 K is 16.27 W. All curves vanish at environmental temperature.

Example 18.18: A component supplied by conduction

Statement. A component is idealized as a plane layer with $L = 0.040$ m, $\kappa = 0.80 \text{ W m}^{-1} \text{ K}^{-1}$, and $A = A_s = 0.020 \text{ m}^2$. Its rear face is maintained at 390.7 K and its exposed surface is quoted as 350 K. Air and a large enclosure are both at 300 K; use stipulated values $h = 10 \text{ W m}^{-2} \text{ K}^{-1}$ and $\varepsilon_{\text{rad}} = 0.80$. (a) Compare the three rates. (b) Find the steady surface temperature implied by those boundary data. (c) Judge whether radiation is negligible.

System and strategy. Choose a thin control region surrounding the exposed surface. At steady state it neither stores energy nor does work. The incoming conduction must equal the sum of outgoing convection and radiation.

For (a), evaluated at the quoted 350 K,

$$\begin{aligned} \dot{Q}_{\text{cond}} &= \frac{(0.80)(0.020)}{0.040} (390.7 - 350) = 16.280 \text{ W}, \\ \dot{Q}_{\text{conv,out}} &= (10)(0.020)(350 - 300) = 10.000 \text{ W}, \\ \dot{Q}_{\text{rad,out}} &= (0.80) \sigma_{\text{SB}} (0.020) (350^4 - 300^4) = 6.2658 \text{ W}. \end{aligned}$$

The outward sum is 16.2658 W. The residual 0.0142 W is a rounding-level mismatch in the constructed boundary data, not evidence of an additional mechanism.

For (b), retaining the stipulated rear temperature exactly requires solving

$$0.400(390.7 - T_s) = 0.200(T_s - 300) + (0.80)\sigma_{\text{SB}}(0.020)(T_s^4 - 300^4).$$

The incoming rate decreases as T_s rises, whereas the outgoing rate increases. Their unique intersection between 300 and 390.7 K is $T_s = 350.019$ K. Thus quoting 350 K is consistent with the accuracy of the original boundary values.

For (c), the radiation/convection ratio at 350 K is $6.2658/10.000 = 0.627$. Radiation supplies 38.5% of the total outward transfer and cannot reasonably be omitted. Its effective coefficient over this interval is $h_{\text{rad}} = 6.266 \text{ W m}^{-2}\text{K}^{-1}$, so $h + h_{\text{rad}} = 16.266 \text{ W m}^{-2}\text{K}^{-1}$.

Interpretation and checks. The three constitutive laws are linked by the first law at the surface; none independently guarantees balance. If all boundary temperatures become equal, every net rate vanishes. The small numerical residual at the rounded surface temperature has now been explained rather than silently set to zero. This is an idealized teaching case, not a measured design.

18.7.5 Radiation geometry and more than one environment

The *view factor* F_{sj} is the fraction of diffuse radiation leaving surface s that reaches surface j directly. For an unobstructed closed set of surroundings, $\sum_j F_{sj} = 1$. Geometry determines these fractions; they are not tunable fit coefficients. For black isothermal environmental surfaces and a small diffuse grey body, the incoming irradiation is $\sigma_{\text{SB}} \sum_j F_{sj} T_j^4$, giving

$$\dot{Q}_{\text{rad,out}} = \varepsilon_{\text{rad}} \sigma_{\text{SB}} A_s \left(T_s^4 - \sum_j F_{sj} T_j^4 \right). \quad (18.38)$$

This restricted formula is useful for a warm enclosure with a cold aperture. It still assumes a negligible contribution of the small body's radiation to the state of the surrounding surfaces. Grey finite surroundings, specular reflections, or a radiating/absorbing intervening medium require more information.

Example 18.19: A radiation shield changes equilibrium temperature

Statement. A small diffuse grey sensor is thermally isolated from solid supports and surrounded by vacuum. Black enclosure walls at 300 K fill a view factor 0.80; a black cold target at 200 K fills the remaining 0.20. Assume the sensor has no internal power and emissivity is nonzero and constant. Find (a) its radiation-equilibrium temperature and (b) whether reducing emissivity changes that equilibrium.

Solution. In equilibrium, the sensor's energy storage rate vanishes. With neither convection nor conduction, the net radiation rate must vanish:

$$T_s^4 = 0.80(300 \text{ K})^4 + 0.20(200 \text{ K})^4.$$

The physically positive root is (a)

$$T_s = 287.16 \text{ K}.$$

The arithmetic mean $0.80(300) + 0.20(200) = 280$ K is incorrect because the environment couples through T^4 . The sensor still emits radiation to both targets; zero *net* exchange does not mean zero emission.

For (b), the common factor $\varepsilon_{\text{rad}} \sigma_{\text{SB}} A_s$ cancels in the balance. Reducing a nonzero grey emissivity changes the rate of approach to equilibrium, but not this equilibrium temperature.

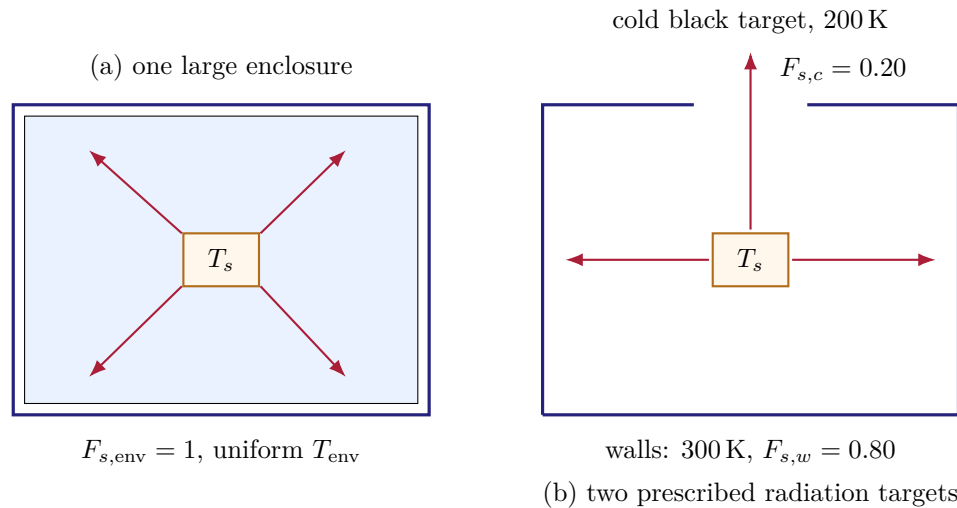


Figure 18.26: Radiation targets for a small body. Both drawings are schematic; their drawn angles do not establish the stipulated view factors. In (b), a cold black target fills the aperture, and its supplied view factor is 0.20. Radiation equilibrium depends on a weighted average of fourth powers, rather than an arithmetic mean temperature.

At the singular ideal limit $\varepsilon_{rad} = 0$, this radiation model supplies no coupling and no mechanism forcing the sensor toward that temperature. Adding support conduction or internal power also prevents the cancellation from determining the temperature by itself.

Checks. The result lies between 200 and 300 K. Setting the cold-target view factor to zero gives 300 K; setting it to one gives 200 K. The calculation assumes the supplied view factors, rather than estimating them from the schematic drawing.

18.7.6 An idealized spacecraft thermal balance

In space, removing ordinary atmospheric convection does not remove the need for energy balance. A spacecraft surface absorbs external radiation, emits thermal radiation, conducts through connections, and may receive power dissipated inside the spacecraft. A steady model must identify the absorbing area as well as the emitting area. For a directional beam of irradiance S , the absorbed solar power is $\alpha_s S A_{proj}$, where A_{proj} is the area projected perpendicular to the beam. The radiating area A_r is generally different.

Solar absorptivity α_s and thermal-infrared emissivity ε_{rad} need not be equal. Equality between absorptivity and emissivity applies at the same wavelength and direction under the relevant equilibrium conditions; solar and thermal spectra weight different wavelengths. A surface's visible color therefore does not determine its entire thermal performance. The following model uses stipulated coefficients rather than a real mission specification.

Example 18.20: Radiator temperature in sunlight and eclipse

Statement. An idealized panel radiates from $A_r = 0.20 \text{ m}^2$ with $\varepsilon_{rad} = 0.80$ to a black sink at $T_e = 3 \text{ K}$, view factor one. It receives $P_{int} = 80 \text{ W}$ of internal dissipation. In sunlight, an imposed beam has $S = 1000 \text{ W m}^{-2}$, $\alpha_s = 0.20$, and $A_{proj} = 0.050 \text{ m}^2$. Neglect other radiation sources, support conduction, and temperature variation across the panel. Find (a) the sunlit equilibrium, (b) the eclipse equilibrium, and (c) the physical meaning of their difference.

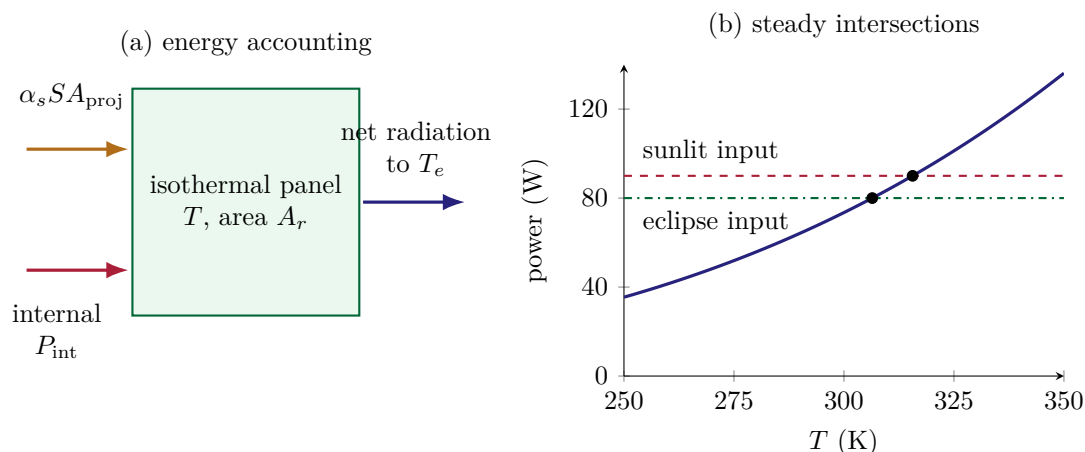


Figure 18.27: An idealized isothermal radiator and its energy balance. The schematic identifies input and output channels; the analytical plot uses $A_r = 0.20 \text{ m}^2$, $\varepsilon_{\text{rad}} = 0.80$, and a 3 K black sink with view factor one. Horizontal lines show stipulated sunlit and eclipse inputs. Their intersections are steady states, not instantaneous temperatures during an eclipse transition.

Balance. Internal dissipation and absorbed external power are supplied inputs. At steady state,

$$P_{\text{int}} + \alpha_s S A_{\text{proj}} = \varepsilon_{\text{rad}} \sigma_{\text{SB}} A_r (T^4 - T_e^4).$$

Solar absorption is $0.20(1000)(0.050) = 10.0 \text{ W}$, so (a)

$$T_{\text{sun}} = \left[T_e^4 + \frac{90.0 \text{ W}}{\varepsilon_{\text{rad}} \sigma_{\text{SB}} A_r} \right]^{1/4} = 315.59 \text{ K}.$$

In eclipse the beam input is zero; (b)

$$T_{\text{eclipse}} = \left[T_e^4 + \frac{80.0 \text{ W}}{\varepsilon_{\text{rad}} \sigma_{\text{SB}} A_r} \right]^{1/4} = 306.44 \text{ K}.$$

For (c), the difference is 9.16 K between two equilibria. The actual panel temperature is continuous and changes over a finite time governed by its heat capacity and transient balance. The calculation supplies no eclipse duration or heat capacity, so it cannot predict an eclipse time history.

Interpretation. Increasing radiating area lowers equilibrium temperature; increasing absorption raises it. In this model a doubling of total input changes T approximately by the factor $2^{1/4}$ when sink radiation is negligible. The assumptions intentionally omit Earth infrared radiation, reflected sunlight, obstructions, and gradients. The imposed irradiance is a teaching input, not a current solar constant or mission value.

18.8 Cooling models and thermal measurement

A steady transfer rate answers a different question from the time required for an object to cool. For a transient problem, define the stored energy and the rate laws together. We first assume that a body can be represented by one nearly uniform temperature $T(t)$. Its total heat capacity $C = dE_{\text{int}}/dT$ includes every part thermally equilibrated with the sensor: a water sample, its container, and any immersed probe when their contributions matter.

Choose the object as a closed system, neglect boundary work, and hold the ambient fluid temperature fixed. If convection alone supplies the outward transfer and C is approximately constant, the first

law becomes

$$C \frac{dT}{dt} = -hA_s(T - T_\infty). \quad (18.39)$$

A hotter object has a negative temperature derivative. An object colder than the ambient fluid has a positive derivative. Both cases approach the same environmental temperature. This sign test often catches an incorrect cooling equation before any solution is attempted.

18.8.1 Deriving the exponential and interpreting its time constant

Introduce the temperature excess $\theta = T - T_\infty$. For constant h , A_s , C , and T_∞ , separation gives $d\theta/\theta = -(hA_s/C)dt$. Integrating from $\theta_0 = T_0 - T_\infty$ to $\theta(t)$ yields

$$\ln \left| \frac{\theta(t)}{\theta_0} \right| = -\frac{t}{\tau}, \quad \tau = \frac{C}{hA_s}, \quad (18.40)$$

so

$$T(t) = T_\infty + (T_0 - T_\infty) \exp(-t/\tau). \quad (18.41)$$

After one time constant, the excess is θ_0/e ; after three time constants, only 4.98% remains. The temperature never reaches its asymptote at a finite time in this ideal differential equation. In an experiment, however, the remaining difference eventually falls below sensor resolution.

The derivation makes the role of each constant visible. A larger heat capacity slows cooling; a larger surface conductance speeds it up. Changing the initial temperature changes the amplitude but leaves τ unchanged only within the constant-coefficient model. If h changes with temperature or significant radiation is present, a fitted time constant can vary with the temperature interval.

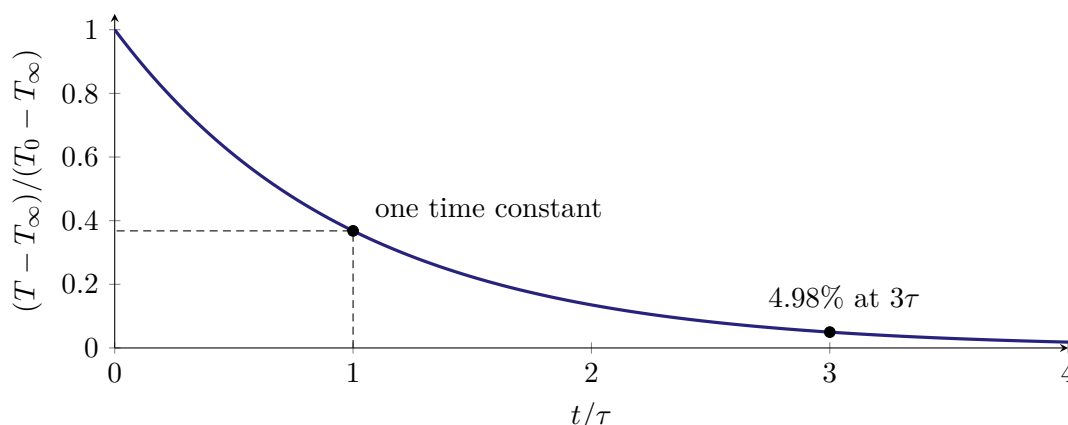


Figure 18.28: The dimensionless exponential prediction for a uniform-temperature body with constant surface conductance and heat capacity. Normalizing by the initial excess collapses all initial amplitudes onto one curve. This collapse is a model prediction; it does not by itself establish spatial uniformity inside a real object.

Example 18.21: A cooling time and an energy-loss check

Statement. A uniform body has $C = 120 \text{ J K}^{-1}$ and convective conductance $H = hA_s = 0.300 \text{ W K}^{-1}$. Ambient air remains at 22.0°C and the initial temperature is 45.0°C . Ignore radiation and other transfer paths. Find (a) the time to reach 30.0°C , (b) the temperature after 1200 s, and (c) the total heat leaving up to the time in (a), checked by two methods.

Solution. The model time constant is $\tau = C/H = 400 \text{ s}$. For (a), the final excess is 8.0 K and the initial excess is 23.0 K :

$$t = \tau \ln \frac{T_0 - T_\infty}{T - T_\infty} = (400) \ln(23/8) = 422.42 \text{ s}.$$

For (b), $1200\text{ s} = 3\tau$ gives

$$T = 22.0 + 23.0e^{-3} = 23.145^\circ\text{C}.$$

For (c), the first law with zero work gives $Q = C(T - T_0) = -1800\text{ J}$, hence 1800 J has left the body. Independently, integrating its outward power gives

$$\int_0^t H\theta_0 e^{-t'/\tau} dt' = H\theta_0\tau(1 - e^{-t/\tau}) = C(23.0 - 8.0) = 1800\text{ J}.$$

The agreement verifies conservation for the time-dependent rate, rather than merely reproducing the final temperature formula.

Interpretation. The initial outward power is $H(23.0) = 6.90\text{ W}$ and drops as the body cools. Dividing the total energy by this initial power would predict 261 s , too short because it incorrectly holds the power at its maximum. A constant-rate estimate is therefore distinguishable from a constant-coefficient cooling model.

18.8.2 Combining convection and radiation in a transient balance

When the same body exchanges energy with fluid and a large enclosure, both mechanisms enter its energy balance. For equal fixed fluid and radiation-environment temperatures, the model is

$$C \frac{dT}{dt} = -H(T - T_\infty) - \varepsilon_{\text{rad}}\sigma_{\text{SB}}A_s(T_K^4 - T_{\infty,K}^4), \quad H = hA_s. \quad (18.42)$$

The subscript K emphasizes that absolute temperatures enter the fourth powers. One may evolve the Celsius reading numerically because $dT_K/dt = dt_C/dt$, provided $T_K = t_C + 273.15\text{ K}$ is used inside the radiation term. Writing the ODE in Celsius does not permit taking fourth powers of Celsius readings.

With $T_{\text{env}} = T_\infty$, the exact factorization of radiation shows that the instantaneous conductance is $H + h_{\text{rad}}(T_K, T_{\infty,K})A_s$. Near ambient this tends to

$$G_\infty = H + 4\varepsilon_{\text{rad}}\sigma_{\text{SB}}A_sT_{\infty,K}^3, \quad \tau_\infty \simeq \frac{C}{G_\infty}. \quad (18.43)$$

Thus an apparently exponential tail can include substantial radiation. If the enclosure and fluid temperatures differ, the equilibrium is obtained by solving the zero-power balance and need not equal either environment temperature. An offset equilibrium must be found before an appropriate local linearization is made.

18.8.3 Synthetic cooling data: fit, residuals, and inference

A fit is most useful when it exposes which model assumptions are doing the work. The following dataset preserves the baseline teaching case and labels its provenance precisely: the values are *synthetic*, not measurements made at AGH. The construction uses Eq. (18.42) with $C = 120\text{ J K}^{-1}$, $H = 0.300\text{ W K}^{-1}$, $\varepsilon_{\text{rad}} = 0.80$, $A_s = 0.020\text{ m}^2$, $T_0 = 45.0^\circ\text{C}$, and ambient/enclosure 22.0°C . Small deterministic offsets were added and the displayed observations rounded to 0.001°C .

Time (s)	Synthetic reading (°C)	Offset from construction model (°C)
0	45.000	0.00000
120	37.436	+0.05988
240	32.262	−0.05013
360	28.970	+0.03973
480	26.634	−0.02985
600	25.161	+0.01953
720	24.077	−0.04030
900	23.202	+0.02961
1200	22.418	−0.02018

These offsets are part of a reproducible teaching construction, not independent random draws with an experimentally established uncertainty. A statistical uncertainty calculation below therefore introduces a separate, explicitly hypothetical observation model.

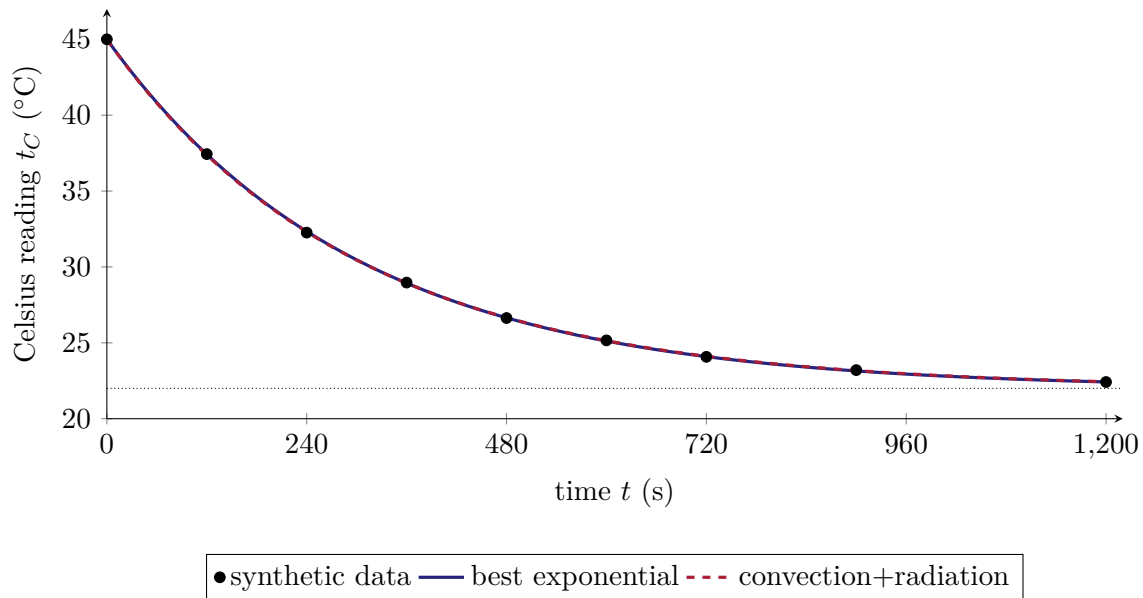


Figure 18.29: The declared synthetic cooling dataset and two fitted model predictions. The exponential fits only τ with T_0 and ambient fixed. The combined model fixes C , ε_{rad} , and area and fits only H . Its embedded numerical trace is sampled every 5 s; line segments join model predictions. Visual closeness of the two curves motivates examining their residuals.

For a reading y_i at time t_i and a model prediction m_i , define the residual $r_i = y_i - m_i$. Least squares chooses the fitted parameter to minimize $S = \sum_i r_i^2$. The root-mean-square residual used here is $\sqrt{S/N}$ for $N = 9$ displayed data points, including the fixed initial point. It is a description of model-data disagreement, not automatically an estimate of sensor uncertainty.

Example 18.22: A cooling fit and its conditional uncertainty

Statement. Use the nine synthetic readings above with fixed initial and ambient readings 45.0°C and 22.0°C . (a) Fit the exponential time constant. (b) Fit the convective conductance in the combined model with its other parameters fixed. (c) Check numerical integration error. (d) Under an assumed independent standard uncertainty $s_T = 0.050^\circ\text{C}$ for the noninitial observations, estimate the local parameter uncertainties and explain their limitations.

Exponential fit. Evaluating and minimizing

$$S(\tau) = \sum_i [y_i - 22.0 - 23.0 \exp(-t_i/\tau)]^2$$

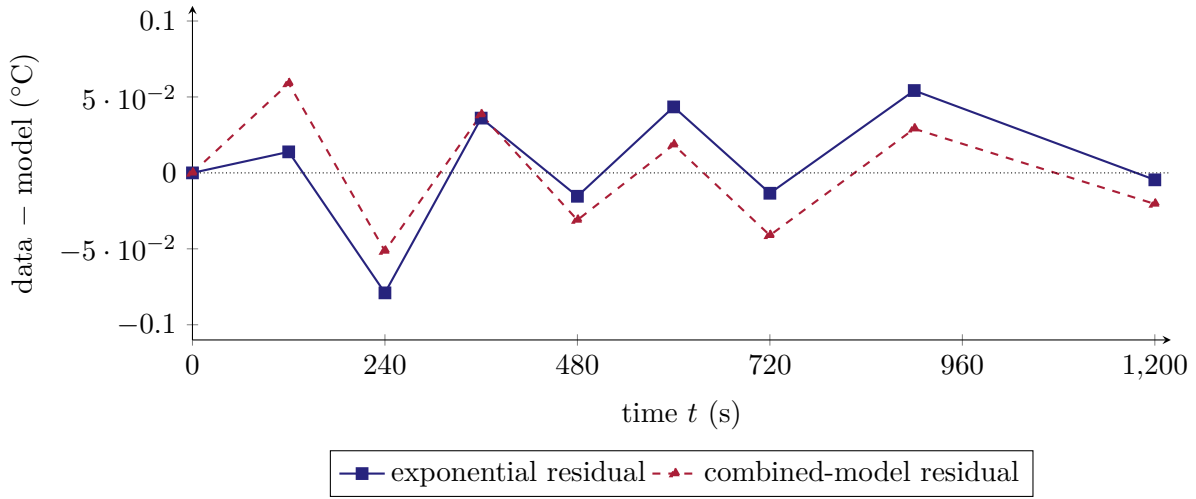


Figure 18.30: Residuals recalculated from the exact nine synthetic readings and the fitted models. Markers distinguish the curves in grayscale. Connecting lines guide the eye between discrete residuals; they are not an additional measured signal. The combined model has a slightly smaller RMS residual, but this modest difference alone does not establish unique physical parameters.

gives (a) $\tau_{\text{fit}} = 300.233 \text{ s}$ and RMS residual 0.038003°C . If interpreted as convection alone, this would imply

$$H_{\text{apparent}} = \frac{C}{\tau_{\text{fit}}} = 0.39969 \text{ W K}^{-1}.$$

This exceeds the construction value 0.300 W K^{-1} because the fitted exponential collects radiation into an effective cooling rate.

Combined-model fit. For (b), integrate Eq. (18.42) for a trial H , evaluate its predictions at the nine times, and minimize their squared residuals. The result is

$$H_{\text{fit}} = 0.2999355 \text{ W K}^{-1}, \quad \text{RMS} = 0.036288^\circ\text{C}.$$

The value recovers the stipulated construction conductance to its relevant precision. The modest change in RMS should be read alongside the residual plot, rather than promoted into a general proof of the mechanism.

Integration check. For (c), a forward-Euler method updates $T_{n+1} = T_n + \Delta t f(T_n; H)$, where f is the right side of the temperature ODE. Refitting after each resolution change gives

Δt (s)	20	10	5	2.5
H_{fit} (W K^{-1})	0.286690	0.293241	0.296570	0.298248

Halving the step approximately halves the remaining bias, as expected for Euler's first-order global error. At 1.25 s the value is $0.299091 \text{ W K}^{-1}$, continuing toward the higher-order result. The published combined curve was calculated with an adaptive higher-order integrator. Tightening its relative/absolute tolerances from $10^{-8}/10^{-10} \text{ K}$ to $10^{-11}/10^{-12} \text{ K}$ changed the nine predicted temperatures by less than $5.3 \times 10^{-8} \text{ K}$. This numerical error is negligible compared with the displayed offsets.

Conditional uncertainty. For (d), linearize a one-parameter model around the optimum, $m_i(p + \delta p) \simeq m_i(p) + J_i \delta p$, with $J_i = \partial m_i / \partial p$. Independent equal observation uncertainties imply

$$s_p \simeq \frac{s_T}{\sqrt{\sum_i J_i^2}}.$$

For the exponential,

$$J_i = 23.0 \exp(-t_i/\tau) \frac{t_i}{\tau^2}, \quad s_\tau \simeq 0.854 \text{ s}.$$

A numerical sensitivity calculation for the combined model gives $s_H \simeq 0.00114 \text{ W K}^{-1}$. These are conditional standard uncertainties under the assumed observation model, with T_0 , ambient, C , ϵ_{rad} , and area treated as exact. They are not experimentally measured uncertainties for the synthetic offsets.

Interpretation. Unknown heat capacity rescales the cooling rate. Sensor offset, drift in ambient temperature, evaporation, and support conduction can bias the fitted parameter. If both H and $\epsilon_{\text{rad}}A_s$ are fitted over this narrow temperature range, their effects are strongly correlated because radiation is nearly linear in the excess. A fit alone identifies neither every physical coefficient nor uniformity of the body's internal temperature.

18.8.4 A proposed measurement workflow and a reasoned assessment

A classroom investigation can use warm water in a container at ordinary handling temperatures. The following proposal does not assert that specific apparatus is available. Its purpose is to separate directly observed quantities from inferred ones. Direct observations include sensor readings, sampling times, sample mass, and ambient temperature. Heat capacity and surface conductance are inferred through a model unless independently calibrated.

Before a cooling run, measure sensor offset in a stable reference bath and record resolution and response time. Estimate or calibrate the heat capacity of the container and probe. A sensor that responds slowly can initially report its own transient rather than the sample temperature. Start with warm water near 40–45°C, stir gently to reduce gradients, and record a baseline ambient reading. Avoid changes in airflow during a run unless airflow is the controlled variable.

Record the trace for several expected time constants or until the excess becomes comparable with sensor uncertainty. A stable ambient reading matters especially near the tail, because a small ambient error can dominate a small temperature excess. Compare a simple exponential first, inspect residuals, and repeat a condition to test reproducibility. Introduce radiation only with temperatures in kelvins and a specified environment. Consider support conduction, evaporation, container heat capacity, and changing fluid motion before attributing every departure to radiation.

Method

An assessment with its reference response. Propose one change to test whether the inferred conductance describes convection alone, one change to test spatial uniformity, and one repeatability test. Explain the expected result and an important ambiguity for each.

A defensible reference response varies airflow while holding surface finish and geometry fixed. Increased airflow should often increase the fitted conductance, but the resulting change can also alter internal mixing or evaporation. For uniformity, place calibrated probes at two separated locations and compare their time traces during the largest temperature excess; matching readings within combined uncertainty support, but do not prove, a uniform field throughout the body. For repeatability, repeat the same initial excess and geometry with ambient monitored; agreement should be judged against declared uncertainties, and drift across repeats can reveal an uncontrolled boundary condition.

Assessment criteria are a controlled variable, a measurable prediction, a model linking the change to the prediction, and a stated ambiguity. Alternative designs are acceptable when they meet those criteria. A single well-fitted trace is insufficient for all three tests.

Temperature differences also matter in calorimetry. If two independent temperature readings each have standard uncertainty s_T , their difference has $s_{\Delta T} = \sqrt{2} s_T$. For $s_T = 0.10 \text{ K}$, this is 0.141 K. A rise of 2.0 K carries a 7.1% relative temperature-difference uncertainty, whereas a rise of 10.0 K carries 1.4%. These percentages describe only that uncertainty contribution. Heat leakage, calibration,

mass, and supplied-energy uncertainty can be equally important.

Pedagogical boundary

A fitted time constant measures the rate of a particular observed temperature signal. It does not establish that all parts of a body share that signal. The advanced section compares internal conduction with surface transfer to decide when a one-temperature model is plausible. This validity test is separate from parameter fitting and from conservation of total energy.

18.9 Advanced Topics for Technical and Engineering Students

Two complementary engineering tools extend the preceding models. Thermal-resistance networks organize *steady* transfer paths and expose which temperature drops dominate. The Biot number tests whether internal conduction is sufficiently effective for a *transient* body to be represented by one temperature. The prerequisites are algebra, the steady conduction and convection laws, and a characteristic length. Neither tool replaces the need to specify geometry and boundary conditions.

18.9.1 Resistance, conductance, and area-normalized resistance

For a constant-conductivity plane layer, rearrange the steady law as

$$\dot{Q} = \frac{T_1 - T_2}{R_{\text{cond}}}, \quad R_{\text{cond}} = \frac{L}{\kappa A}. \quad (18.44)$$

For a surface convection film,

$$\dot{Q} = \frac{T_s - T_\infty}{R_{\text{conv}}}, \quad R_{\text{conv}} = \frac{1}{hA_s}. \quad (18.45)$$

Both resistances have units K W^{-1} and their reciprocal is a thermal conductance. A temperature drop divided by resistance yields watts. The formal similarity with an electrical network helps with algebra, but the elements encode different physical models: a convection coefficient and a solid conductivity cannot be interchanged.

Building-insulation discussions often use the area-normalized resistance $R'' = L/\kappa$, with units $\text{m}^2 \text{K W}^{-1}$. In that notation, $\dot{Q}/A = (T_1 - T_2)/R''$. The two quantities are related by $R = R''/A$. Halliday and Serway use an area-normalized R-value in their wall discussions; this chapter uses R for the *whole element* in network calculations and writes R'' when area-normalized resistance is intended. Recording the units prevents a factor-of-area error.

18.9.2 Series layers and contact resistance

A steady series path carries the same heat rate through every element. If the nodes are $T_0, T_1, \dots, T_n$, then $T_{i-1} - T_i = \dot{Q}R_i$. Summing drops telescopes to $T_0 - T_n$:

$$R_{\text{series}} = \sum_i R_i, \quad \dot{Q} = \frac{T_0 - T_n}{\sum_i R_i}. \quad (18.46)$$

The interface temperatures then follow sequentially from $T_i = T_{i-1} - \dot{Q}R_i$. A large resistance creates a large temperature drop for the same rate. This is a direct consequence of steady energy conservation, rather than an independent rule about insulation.

At a real contact, microscopic gaps and limited contact area can create a temperature discontinuity in an idealized one-dimensional model. A supplied contact resistance R_c gives $T_{c,1} - T_{c,2} = \dot{Q}R_c$. A perfect contact is the limiting case $R_c = 0$. A contact resistance must come from a justified estimate, specification, or measurement; it should not be invented after the calculation to force agreement with data.

Example 18.23: An insulated panel and its interface temperatures

Statement. A 1.00 m^2 panel separates indoor air at 293 K from outdoor air at 273 K . Use inside and outside film coefficients $h_i = 8.0$ and $h_o = 20 \text{ W m}^{-2} \text{K}^{-1}$. The insulation has $L_1 = 0.080 \text{ m}$, $\kappa_1 = 0.040 \text{ W m}^{-1} \text{K}^{-1}$; the board has $L_2 = 0.012 \text{ m}$, $\kappa_2 = 0.12 \text{ W m}^{-1} \text{K}^{-1}$. Assume steady one-dimensional transfer and neglect radiation, bypass paths, and contact resistance. (a) Find the rate and all node temperatures. (b) Determine the effect of adding a

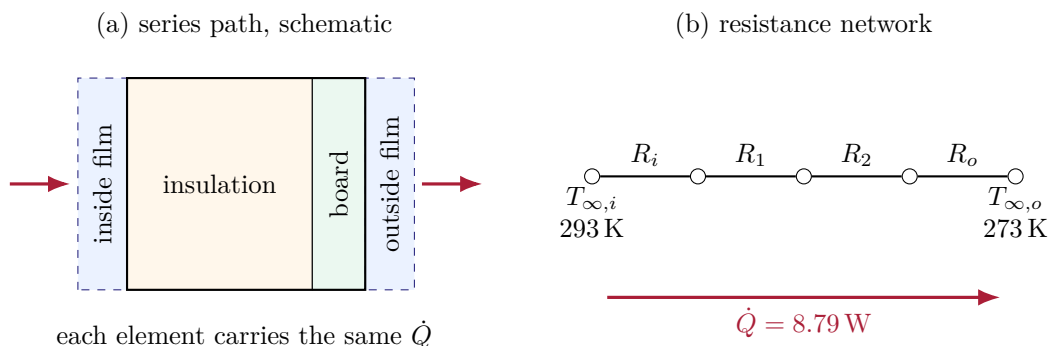


Figure 18.31: A composite panel and its whole-element resistance network. Dashed film regions symbolize fluid-side boundary transport; only the insulation and board are solid layers. Resistances are in K W^{-1} . The four elements describe the baseline panel example; a contact resistance, if specified, would be an additional series element.

specified $R_c = 0.050 \text{ K W}^{-1}$ contact resistance.

Reduction. The elements shown in Fig. 18.31 have

$$R_i = 0.125, \quad R_1 = 2.000, \quad R_2 = 0.100, \quad R_o = 0.050 \quad \text{K W}^{-1}.$$

Their sum is 2.275 K W^{-1} . For (a),

$$\dot{Q} = \frac{293 - 273}{2.275} = 8.7912 \text{ W}.$$

The individual temperature drops are

$$\Delta T_i = 1.0989, \quad \Delta T_1 = 17.5824, \quad \Delta T_2 = 0.8791, \quad \Delta T_o = 0.4396 \quad \text{K}.$$

Starting from the indoor air, the successive temperatures are 293.000, 291.901, 274.319, 273.440, and 273.000 K. The first and last solid face temperatures differ from the air temperatures precisely because the films have nonzero resistance.

For (b), the total becomes 2.325 K W^{-1} , giving $\dot{Q} = 8.6022 \text{ W}$. The contact jump is $\dot{Q}R_c = 0.4301 \text{ K}$. Adding resistance decreases the steady heat leak by 2.15%; it does not change the total imposed temperature difference.

Checks and interpretation. The four original drops sum to 20.000 K. Substituting each drop into its element's rate law gives the same 8.7912 W. Most of the temperature difference occurs across the insulation because it dominates the resistance. Adding layer heat capacities to this resistance sum would be dimensionally and physically wrong: capacities are needed for a transient model, not this steady reduction.

18.9.3 Parallel paths and thermal bridges

Two paths are in parallel only when they connect the same temperature nodes. Each sees the same difference ΔT , and their heat rates add. Hence

$$G_{\text{parallel}} = \sum_j G_j = \sum_j \frac{1}{R_j}, \quad \frac{1}{R_{\text{parallel}}} = \sum_j \frac{1}{R_j}. \quad (18.47)$$

This distinguishes a composite wall made of layers *through its thickness* from a wall with different materials occupying separate *areas*. Layers are in series; side-by-side bridges are in parallel under the stated common-boundary approximation. A narrow highly conducting fastener can transmit appreciable energy even when most of a panel is insulating.

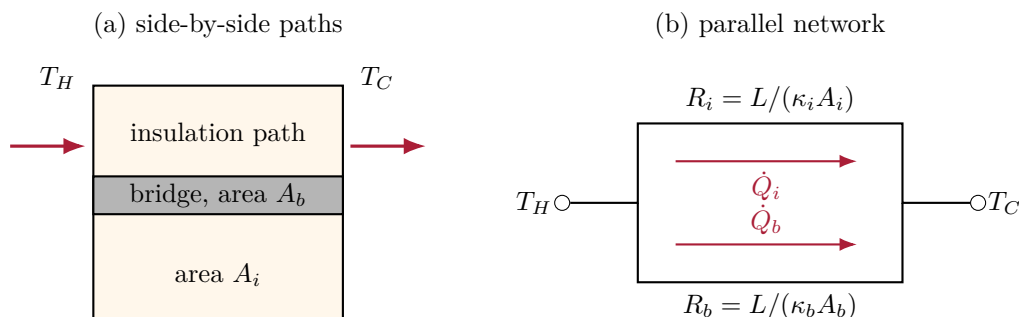


Figure 18.32: A thermal bridge and its parallel representation. The illustrated bridge is enlarged for clarity and is not drawn to its true area fraction. Each path connects the same prescribed face temperatures. The model neglects lateral spreading and interface resistance; a real fastener can require a multidimensional treatment.

Example 18.24: A small conducting bridge through insulation

Statement. A panel has fixed face temperatures differing by 20.0 K, thickness $L = 0.080$ m, and total face area 1.0000 m^2 . A stipulated bridge of conductivity $150 \text{ W m}^{-1}\text{K}^{-1}$ occupies $A_b = 0.00020 \text{ m}^2$. The remaining $A_i = 0.99980 \text{ m}^2$ has conductivity $0.040 \text{ W m}^{-1}\text{K}^{-1}$. Under independent one-dimensional parallel paths, find (a) each rate, (b) the increase over an entirely insulating panel, and (c) why averaging conductivities by thickness would fail.

Solution. The common temperature difference drives both paths:

$$\dot{Q}_i = \frac{(0.040)(0.99980)}{0.080}(20.0) = 9.998 \text{ W},$$

$$\dot{Q}_b = \frac{(150)(0.00020)}{0.080}(20.0) = 7.500 \text{ W}.$$

For (a), the total is 17.498 W. An entirely insulating panel would transmit 10.000 W, so (b) the bridge increases the rate by 74.98%. It carries 42.9% of the total despite occupying only 0.020% of the area.

For (c), thickness averaging would represent serial layers and enforce a common heat rate, contrary to this geometry. Here an area-weighted effective conductivity is possible only under the assumed common thickness and face-temperature constraints:

$$\kappa_{\text{eff}} = \frac{\kappa_i A_i + \kappa_b A_b}{A_i + A_b} = 0.069992 \text{ W m}^{-1}\text{K}^{-1}.$$

Using it in the homogeneous plane-wall formula reproduces 17.498 W. This is a geometry-specific effective value, not the intrinsic conductivity of either material.

Model boundary. A real bridge can have contact resistance, spreading resistance, and altered surface temperatures near it. The calculation isolates the mechanism by prescribing the same face temperatures for both paths. It should not be combined with independently prescribed air temperatures without adding their film model.

18.9.4 The Biot number and the one-temperature approximation

A body can cool slowly and still have substantial internal temperature gradients. The rate of surface loss and the effectiveness of internal conduction are separate issues. The surface transfer creates a temperature difference between the body surface and its surroundings. Internal conduction must transport energy from the interior to that surface fast enough to keep the interior-to-surface difference small.

Compare the characteristic internal conduction resistance $L_c/(\kappa A_s)$ with the surface resistance $1/(hA_s)$. Their ratio is

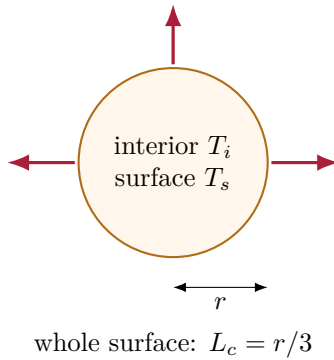
$$\text{Bi} = \frac{hL_c}{\kappa}, \quad L_c = \frac{V}{A_s}. \quad (18.48)$$

The characteristic length is defined using the volume V storing energy and the surface area A_s actually exchanging it. For a sphere of radius r cooled on its whole surface, $L_c = r/3$. For a slab of full thickness $2a$ exchanging through its two large faces with edge effects neglected, $L_c = a$. If only one face participates, the same slab has $L_c = 2a$. Choosing an arbitrary diameter instead changes the numerical screening value and must not be left unstated.

When $\text{Bi} \ll 1$, internal conductive resistance is small relative to surface resistance. This supports nearly uniform temperature after any very rapid internal adjustment. A commonly used engineering screening criterion is $\text{Bi} \lesssim 0.1$ for a lumped calculation, but its acceptable error depends on geometry, boundary conditions, and the observable. It is a practical screen, not an exact theorem that every point differs by less than a fixed percentage.

If convection and radiation operate against a common environmental temperature, use the relevant total surface coefficient $h_{\text{eff}} = h + h_{\text{rad}}$ for screening. For a narrow interval near T_* , $h_{\text{eff}} \simeq h + 4\epsilon_{\text{rad}}\sigma_{\text{SB}}T_*^3$. If radiation is strongly nonlinear, screen with a representative or conservative value over the interval and state that choice. Ignoring an appreciable radiation term underestimates surface transfer and can make the uniformity criterion too optimistic.

(a) competition of transfer scales



(b) qualitative spatial comparison

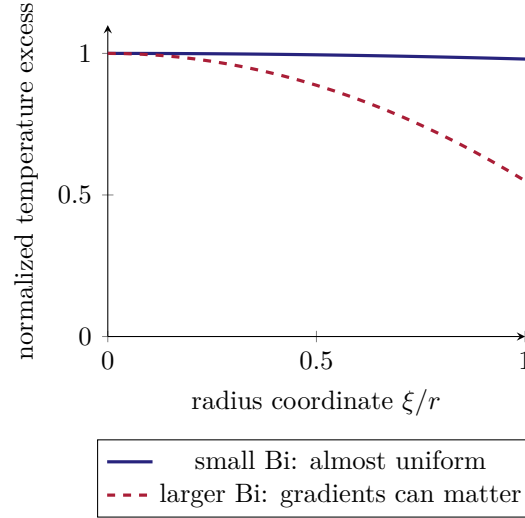


Figure 18.33: Internal conduction must supply surface losses. The spherical geometry establishes $L_c = r/3$ for whole-surface transfer. The comparison curves are explicitly qualitative sketches with a flat central slope; they are not solutions of a diffusion equation or predictions for a specified Biot number. Their role is to distinguish temporal cooling rate from spatial uniformity.

The same conclusion follows from comparing time scales. With density ρ and specific heat c , the thermal diffusivity is $\alpha = \kappa/(\rho c)$. A conduction-adjustment scale is $t_{\text{diff}} \sim L_c^2/\alpha$, whereas the lumped surface-cooling scale is $\tau = \rho c V/(h_{\text{eff}} A_s) = \rho c L_c/h_{\text{eff}}$. Their ratio is of order $h_{\text{eff}} L_c/\kappa$. Geometry-dependent factors remain in a precise diffusion solution, but this comparison explains the physical content of Bi: internal adjustment should be faster than overall surface-driven cooling.

Example 18.25: Identical cooling time constants, different validity

Statement. Consider two idealized spheres of radius $r = 0.010$ m, each having the stipulated volumetric heat capacity $\rho c = 2.4 \times 10^6$ J m⁻³K⁻¹. One has $\kappa_m = 200$ W m⁻¹K⁻¹, the other $\kappa_p = 0.20$ W m⁻¹K⁻¹. They exchange through their whole surfaces by convection with $h = 10$ W m⁻²K⁻¹ and radiation with $\varepsilon_{\text{rad}} = 0.80$ near a common 300 K environment. (a) Screen the lumped approximation for each. (b) Calculate the nominal lumped time constant. (c) Explain why matching time constants would not prove matching internal fields.

Surface model. Over a small temperature excess near 300 K,

$$h_{\text{rad}} \simeq 4\varepsilon_{\text{rad}}\sigma_{\text{SB}}(300\text{ K})^3 = 4.8992\text{ W m}^{-2}\text{K}^{-1},$$

so $h_{\text{eff}} = 14.8992\text{ W m}^{-2}\text{K}^{-1}$. Whole-surface geometry gives $L_c = r/3 = 0.003333$ m.

For (a),

$$\text{Bi}_m = \frac{14.8992(0.003333)}{200} = 2.4832 \times 10^{-4}, \quad \text{Bi}_p = \frac{14.8992(0.003333)}{0.20} = 0.24832.$$

The high-conductivity sphere strongly supports a lumped model; the low-conductivity sphere fails the usual 0.1 screen and needs a spatial model or independent evidence to justify the desired accuracy.

For (b), the nominal lumped time scale would be identical for both:

$$\tau = \frac{\rho c V}{h_{\text{eff}} A_s} = \frac{\rho c r}{3 h_{\text{eff}}} = 536.94\text{ s}.$$

The word nominal matters for the low-conductivity sphere because its assumed uniform temperature has not been justified. The inferred time scale cannot validate the assumption used to derive it.

For (c), heat capacity and surface conductance control this nominal time scale, while conductivity controls the internal gradients. The two conductivities differ by a factor 1000, despite identical external geometry and storage capacity. Multiple internal probes, a spatial conduction calculation, or an appropriately small Biot number provide information that a single cooling fit cannot supply.

Check. Omitting radiation would reduce the screening Biot numbers by the factor 10/14.8992 and enlarge the nominal time scale. Including it consistently improves both the energy model and the validity test. At a much larger temperature excess, the local 300 K radiation coefficient must be reconsidered.

Guided checks

Three model checks with their answers. A small Biot number does not guarantee an accurately known h : it compares internal and surface resistance for the assumed coefficient, rather than calibrating that coefficient. A large Biot number does not prohibit a uniform field at the initial instant: a body can be prepared uniformly, but surface exchange may subsequently create gradients. Finally, $R = L/(\kappa A)$ is not a heat capacity: resistance has units K W⁻¹ and governs steady rate, whereas heat capacity has units J K⁻¹ and governs stored-energy change.

For comparison with the insulated panel, an isolated slab of insulation of full thickness 0.080 m exchanging through both large faces would have $L_c = 0.040$ m. Using $h = 20$ W m⁻²K⁻¹ and $\kappa = 0.040$ W m⁻¹K⁻¹ gives $\text{Bi} = 20$, far from the lumped regime. This is only a scale illustration for an isolated slab. The actual composite panel has different adjoining layers and boundary conditions; a transient wall analysis must use that full geometry. Insulation is designed to resist internal conduction, so assigning it one spatially uniform temperature is often a particularly poor transient

approximation.

Take-home message

A useful model states both its prediction and its range of validity. Series and parallel networks organize steady paths and expose temperature drops or bypasses. A transient lumped balance adds storage, and the Biot number tests the spatial assumption behind that balance. Conservation, constitutive laws, geometry, parameter inference, and validity checks each answer a distinct question.

18.10 Chapter summary and a working formula sheet

Summary

Temperature labels thermal equilibrium; heat and work are transfers. Celsius and kelvin intervals have the same numerical size, while absolute temperature ratios and radiation laws require kelvins. Thermal expansion is a constitutive material response; perfect constraint can convert free thermal strain into stress. Calorimetry is an energy balance whose boundary must include relevant heat capacities and phase branches. For a closed system with the adopted work-by convention, $\Delta E_{\text{int}} = Q - W_{\text{by}}$. Boundary work depends on process and, for the stated piston model, uses p_{ext} . Conduction, convection, and radiation are separate rate models with different coefficients and validity conditions. Lumped cooling is an approximation, not an automatic consequence of a good exponential fit.

Formula summary

Temperature scales

$$\frac{t_C}{^\circ\text{C}} = \frac{T}{\text{K}} - 273.15, \quad \Delta T[\text{K}] = \Delta t_C[^\circ\text{C}].$$

Thermal expansion, small uniform change

$$\begin{aligned} \Delta L &\simeq \alpha_L L_0 \Delta T, & \Delta V/V_0 &\simeq 3\alpha_L \Delta T \quad (\text{isotropic}), \\ \sigma &= -E\alpha_L \Delta T \quad (\text{perfect axial restraint}). \end{aligned}$$

Calorimetry and phase change

$$Q \simeq mc\Delta T, \quad Q = \pm mL_{\text{phase}}, \quad \sum_j Q_j = 0 \quad \text{for the stated insulated mixing boundary.}$$

Here $L_{\text{phase}} > 0$: melting/vaporization takes the positive sign; freezing/condensation takes the negative sign. **Equilibrium ideal-gas bridge**

$$pV = nRT.$$

Boundary work and first law

$$W_{\text{by}} = \int p_{\text{ext}} dV, \quad \Delta E_{\text{int}} = Q - W_{\text{by}}.$$

Elementary transfer rates

$$\dot{Q}_x = -\kappa A \frac{dT}{dx}, \quad \dot{Q}_{\text{conv}} = hA(T_s - T_\infty), \quad \dot{Q}_{\text{rad}} = \varepsilon_{\text{rad}} \sigma_{\text{SB}} A (T_s^4 - T_{\text{env}}^4).$$

Lumped cooling

$$C \frac{dT}{dt} = -hA(T - T_\infty), \quad T - T_\infty = (T_0 - T_\infty)e^{-t/\tau}, \quad \tau = C/(hA).$$

Advanced transport tools

$$R_{\text{cond}} = \frac{L}{\kappa A}, \quad R_{\text{conv}} = \frac{1}{hA}, \quad \dot{Q} = \frac{\Delta T}{\sum R_i}, \quad \text{Bi} = \frac{hL_c}{\kappa}.$$

Every expression above carries conditions. The formula sheet is a map back to those conditions, not a substitute for them.

18.11 Exercises and problems

The following progression begins with temperature and model selection, then develops energy accounting, transport, graph interpretation, measurement and computation. Problems 1–30 retain the original practice set; Problems 31–40 combine the ideas in new settings. Every item has a complete solution below. Use $\sigma_{\text{SB}} = 5.670374419 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ and, where needed, $R = 8.314462618 \text{ J mol}^{-1} \text{ K}^{-1}$. All material properties and transfer coefficients are *stipulated teaching values*, rather than claims about measured specimens. Use $\Delta E_{\text{int}} = Q - W_{\text{by}}$ for closed-system balances and state any further assumptions.

A. Temperature, calibration and expansion

1. A thermometer has a constant -1.5°C offset. It reads 36.8°C . Find the corrected Celsius and kelvin temperatures. It reads 10.0°C at one state and 30.0°C at another: find the true interval. A student calls the second state “three times hotter” because $30/10 = 3$. Diagnose this claim using absolute temperature.
2. A metal bench and a wooden bench have reached the same room temperature. Explain why the metal can feel colder. Identify the physical response of the skin, and whether touch provides a reliable equilibrium temperature comparison.
3. Two systems separately equilibrate with the same thermometer and produce the same stable reading. Explain what the zeroth law implies about their temperatures, and what it does not imply about their masses or internal energies.
4. A 1.50 m member with $\alpha_L = 17 \times 10^{-6} \text{ K}^{-1}$ warms by 80 K. Find its free expansion. If $E = 70 \text{ GPa}$ and perfect axial restraint prevents extension, find the ideal stress using a tension-positive convention. State one mechanical limitation of this ideal prediction.
5. Starting from an isotropic cube, derive the leading volume-expansion coefficient $3\alpha_L$. When $x = \alpha_L \Delta T = 2.0 \times 10^{-3}$, estimate the quadratic correction relative to the linear volume change. Distinguish this relative correction from the total fractional volume change.

B. Calorimetry and phase change

6. A 0.150 kg metal sample, $c_m = 900 \text{ J kg}^{-1} \text{ K}^{-1}$, initially at 80°C is immersed in 0.250 kg water at 20°C . A calorimeter of heat capacity $C_{\text{cal}} = 60 \text{ J K}^{-1}$ starts at the water temperature. Neglect losses and use $c_w = 4180 \text{ J kg}^{-1} \text{ K}^{-1}$. Find the common equilibrium temperature.
7. Repeat Problem 6 while incorrectly neglecting C_{cal} . Find the predicted temperature and quantify its bias relative to the stated full model. Explain the direction of the bias.

8. Add 0.020 kg ice at 0°C to 0.100 kg water at 15°C in an ideal negligible-capacity container. Use $L_f = 334 \text{ kJ kg}^{-1}$ and $c_w = 4180 \text{ J kg}^{-1} \text{ K}^{-1}$. First test whether all the ice melts; then determine the final temperature and the masses in each phase.
9. A calorimeter temperature rise is $\Delta T = T_2 - T_1 = 2.0 \text{ K}$. Each independent reading has standard uncertainty 0.10 K. Find $\sigma_{\Delta T}$ and the fractional standard uncertainty. Repeat for a 10.0 K rise and interpret the comparison. State how a common constant offset differs from independent reading errors.
10. A solution for heating ice from -15°C to liquid water at $+10^\circ\text{C}$ uses one $Q = mc\Delta T$. Diagnose the error and write the correct piecewise energy path. Explain what must be checked if the available heat is insufficient to complete that path.

C. Systems, work and the first law

11. Classify the stated systems as closed, open or isolated: (a) a rigid sealed steel tank exchanging heat with a room; (b) the same tank inside ideal insulation, with no electrical, shaft or other work interaction; (c) a steady turbine control volume with inlet and outlet streams; (d) a sealed insulated piston-cylinder compressed by an external load. Explain each classification.
12. A gas expands quasistatically against constant external pressure 250 kPa from 0.010 to 0.028 m³. Find W_{by} and W_{on} for this interaction. Explain the sign physically.
13. A gas is compressed from 0.050 to 0.020 m³ at constant external pressure 300 kPa. Find W_{by} . If $Q = +2.0 \text{ kJ}$, find ΔE_{int} and explain how heat and compression work combine.
14. Between the same equilibrium endpoints, Path A expands at 500 kPa from 0.010 to 0.030 m³ and then changes pressure at fixed volume. Path B changes pressure first and then expands at 200 kPa. Treat both as quasistatic boundary-work paths. Find their works. If $\Delta E_{\text{int}} = +1.0 \text{ kJ}$ between the endpoints, find the two heat transfers. Identify what is path dependent.
15. A closed rigid vessel receives 12 kJ of heat and 3 kJ of electrical work from its surroundings. Express the electrical interaction as W_{by} and find ΔE_{int} . Explain why rigid walls do not imply zero *total* work.
16. An adiabatic compression has $W_{\text{by}} = -4.5 \text{ kJ}$. Find ΔE_{int} . Does this determine the temperature change for an arbitrary material? State what further material information is required.
17. A device is reported to return to its initial state, while measurements give $Q_{\text{net}} = -3.0 \text{ kJ}$ and $W_{\text{by,net}} = +1.0 \text{ kJ}$. Quantify the inconsistency and give two plausible experimental or model explanations. Do not invent a violation of energy conservation.
18. A fixed amount of ideal gas has equilibrium endpoints satisfying $p_i V_i = p_f V_f$. Does this prove that its process was isothermal? Explain the distinction between an endpoint relation and a condition along the entire path.

D. Transfer and cooling

19. A plane layer has $A = 0.50 \text{ m}^2$, $L = 0.020 \text{ m}$, $\kappa = 0.80 \text{ W m}^{-1} \text{ K}^{-1}$, and face temperatures 330 and 300 K. Find the steady one-dimensional conduction rate and direction. Assume uniform properties and no internal generation.
20. A surface at 340 K with $A = 0.10 \text{ m}^2$ exchanges heat by convection with air at 295 K. For $h = 12 \text{ W m}^{-2} \text{ K}^{-1}$, find the outward rate.

21. For Problem 20, take $\epsilon_{\text{rad}} = 0.70$ and a large isothermal enclosure at 295 K. Find net radiation and compare it with convection. State the radiation geometry and surface model. What is the net radiative rate at equal temperatures, and how does its sign change if the surface is colder?
22. A body with $C = 250 \text{ J K}^{-1}$ and $hA = 2.5 \text{ W K}^{-1}$ cools from 70°C in a 20°C environment under the lumped convection model. Find τ and its temperature at 120 s.
23. A cooling trace follows $T - 20 = (60)e^{-t/180}$, with temperatures in degrees Celsius and t in seconds. Find the fractions of initial excess after one, two and three time constants. Explain why a Celsius *difference* is allowed here although Celsius readings cannot be raised to the fourth power in radiation.
24. Design a validation test for a numerical solver of Eq. (18.39). Give at least three checks that test physical predictions. Supply specific inputs, an expected result or limit, and a criterion for deciding whether each check passes.
25. A cooling-data fit gives $\tau = 200 \text{ s}$ with small residuals. Explain why this alone does not establish that the independently measured convection coefficient satisfies $hA = C/(200 \text{ s})$. Give two omitted mechanisms that could contribute to the fitted decay rate, and identify a further measurement that helps distinguish them.

E. Thermal networks and scaling

26. A wall has two series layers, each with $A = 1.0 \text{ m}^2$: $L_1 = 0.040 \text{ m}$, $\kappa_1 = 0.040 \text{ W m}^{-1}\text{K}^{-1}$, and $L_2 = 0.020 \text{ m}$, $\kappa_2 = 0.20 \text{ W m}^{-1}\text{K}^{-1}$. The outside solid faces are held at 300 and 280 K. Find the rate and interface temperature. Assume steady one-dimensional conduction and perfect contact.
27. Add convection films $h_i = 10$ and $h_o = 25 \text{ W m}^{-2}\text{K}^{-1}$ to the wall of Problem 26, now with bulk-fluid temperatures 305 and 275 K. Find the total resistance, rate and both solid surface temperatures. Check that all temperature drops sum to the bulk-fluid difference.
28. A compact body has $V = 1.0 \times 10^{-4} \text{ m}^3$, exposed area $A_s = 0.020 \text{ m}^2$, $h = 15 \text{ W m}^{-2}\text{K}^{-1}$, and $\kappa = 150 \text{ W m}^{-1}\text{K}^{-1}$. Calculate $L_c = V/A_s$ and Bi. Assess support for lumping. Repeat for $\kappa = 0.15 \text{ W m}^{-1}\text{K}^{-1}$, and distinguish a screening criterion from experimental proof.
29. Every linear dimension of a geometrically similar body is doubled, with material properties and h unchanged. Find the factors by which L_c , $C \propto V$, A_s , $\tau = C/(hA_s)$ and Bi change. Discuss thermal response time and internal gradients.
30. Derive the addition of series resistances for a one-dimensional steady composite panel. Give a physical situation in which that network fails even when every listed L , κ , A and h is known exactly. Identify the missing physics rather than treating an accurate parameter as an adequate model.

F. Graphs, measurement and integrated challenges

31. **Calibration from a graph.** Figure 18.34 gives two exact calibration points for a linear sensor: true temperatures 0 and 100°C produce indicated readings 2 and 104°C . (a) Find the calibration $r = a + bt$ and the corrected temperature, in both scales, when $r = 53^\circ\text{C}$. (b) With calibration coefficients treated as exact and $\sigma_r = 0.20 \text{ K}$, find the standard uncertainty of the corrected temperature. (c) Later the ice-point reading becomes 3°C while the slope stays fixed. For another 53°C reading, find the bias caused by using the old calibration. Explain why one calibration point cannot generally establish both offset and slope.

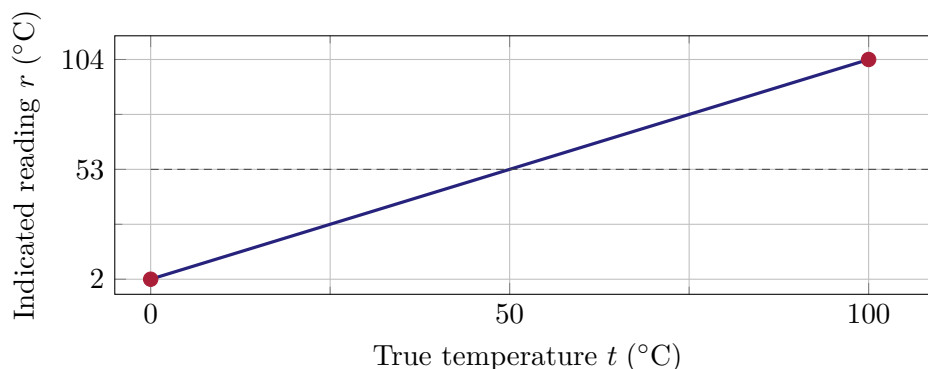


Figure 18.34: Exact calibration points and the indicated reading used in Problem 31. The straight line is the stipulated sensor model, not measured intermediate data.

- 32. Heating-curve inference.** A 0.300 kg sample follows the synthetic equilibrium heating curve in Figure 18.35. The horizontal axis is cumulative heat delivered to the sample; the container has negligible capacity. The phase transition occurs at fixed pressure. (a) Infer c_i , L_f and c_w from the three segments. (b) At $Q = 90.0$ kJ, find the temperature and masses of ice and water. (c) Explain why the plateau is consistent with energy conservation, and why interpolating temperature along one straight line from the first to last point would fail.

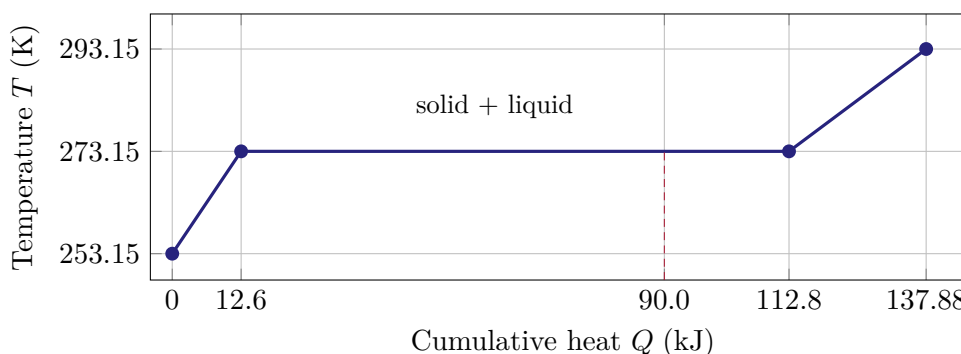


Figure 18.35: Synthetic heating curve for Problem 32. Vertex coordinates define the exact piecewise model; there are no losses, supercooling or temperature gradients.

- 33. Phase calorimetry with uncertainty.** Mix $m_i = 0.0400$ kg ice at $t_i = -10.0^\circ\text{C}$ with $m_w = 0.200$ kg water at $t_w = 25.0^\circ\text{C}$. A calorimeter has $C_{\text{cal}} = 80 \text{ J K}^{-1}$ and initially shares the water temperature. Use $c_i = 2100$, $c_w = 4180 \text{ J kg}^{-1}\text{K}^{-1}$ and $L_f = 334000 \text{ J kg}^{-1}$, with no losses. (a) Test the phase branch and determine the final state. (b) Find the result when the calorimeter is omitted and the resulting signed bias. (c) Propagate independent standard uncertainties $\sigma_{t_w} = \sigma_{t_i} = 0.20 \text{ K}$, $\sigma_{m_i} = 0.0005 \text{ kg}$ and $\sigma_{C_{\text{cal}}} = 5 \text{ J K}^{-1}$; treat all other inputs as exact. Check whether this uncertainty could plausibly change the chosen phase branch near the nominal inputs.
- 34. Cycle from a p - V diagram.** A closed system makes the quasistatic cycle $A \rightarrow B \rightarrow C \rightarrow A$ in Figure 18.36; the sloping segment is straight. Pressure equals the external boundary pressure in the quasistatic limit. Measured internal-energy changes are $\Delta E_{\text{int},AB} = +6$, $\Delta E_{\text{int},BC} = +9$ and $\Delta E_{\text{int},CA} = -15 \text{ kJ}$. (a) Find work and heat on every segment. (b) Find the net work and heat and compare them with the signed enclosed area. (c) Explain whether the device supplies or receives net mechanical work, and why the first law alone does not establish that an imagined implementation is physically possible.

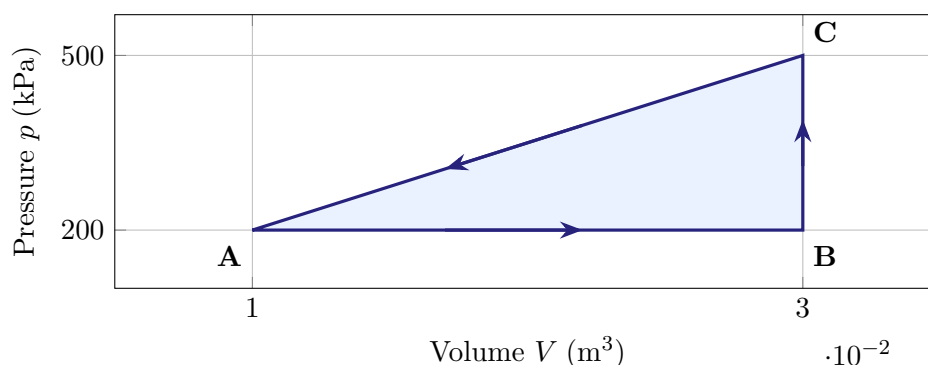


Figure 18.36: Specified equilibrium cycle for Problem 34. The traversal direction fixes the sign of the boundary work; the filled triangle is an area aid.

- 35. Actual free expansion and a comparison path.** An insulated rigid chamber initially contains 1.00 mol ideal gas at 300 K in $V_i = 0.0100 \text{ m}^3$; the other half is vacuum. A partition is removed without transferring significant energy to the gas, and final equilibrium occupies $V_f = 0.0200 \text{ m}^3$ (Figure 18.37). Stipulate $E_{\text{int}} = CT + \text{constant}$ for this sample, with $C > 0$. (a) Find actual Q , W_{by} , ΔE_{int} and T_f . (b) Calculate endpoint pressures and explain why the intermediate event cannot be represented by a quasistatic equilibrium curve. (c) For a different, quasistatic isothermal piston process between the same endpoints, find W_{by} and Q . Explain how the endpoints can agree while the transfers differ.

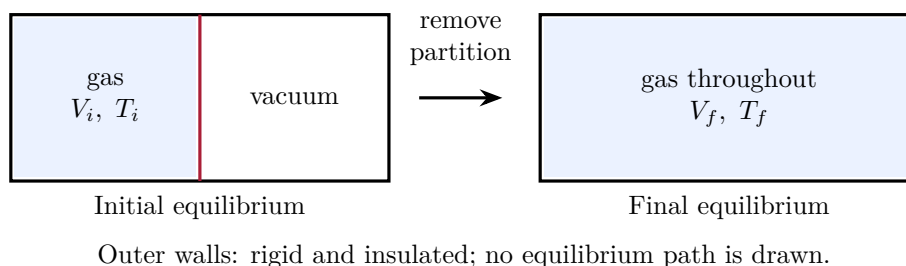


Figure 18.37: Endpoint geometry for the free-expansion event of Problem 35. The arrow represents partition removal, not a curve of intermediate equilibrium states.

- 36. A thermal bridge in a panel.** A panel of area 1.20 m^2 and thickness 0.080 m has parallel portions: 1.14 m^2 insulation with $\kappa_i = 0.040$, and a 0.060 m^2 bridge with $\kappa_b = 15.0 \text{ W m}^{-1} \text{ K}^{-1}$. Ideal thin face plates enforce uniform temperature on each solid face. The bulk-fluid temperatures are 305 and 275 K, with $h_i = 10$ and $h_o = 25 \text{ W m}^{-2} \text{ K}^{-1}$. Assume no other transfer or contact resistance. (a) Form the parallel-core/series-film resistance and find the total rate and face temperatures. (b) Find the heat through each core portion and the bridge's fraction of the total. (c) Replace the bridge with insulation and compare rates. Explain why an area-weighted conductivity is usable in this stipulated geometry but not in every composite structure.
- 37. Radiative balance and heater duty.** A surface has emitting/projected area $A = 0.50 \text{ m}^2$, shortwave absorptivity $\alpha_s = 0.25$, longwave emissivity $\varepsilon_{\text{rad}} = 0.80$, and incident shortwave irradiance $S = 900 \text{ W m}^{-2}$. It sees an ideal large black longwave enclosure at 250 K with view factor one. Other surfaces are insulated; shortwave input and enclosure longwave input are separate spectral terms. There is no convection or contact conduction. (a) Find the equilibrium temperature with a 20.0 W heater. (b) Find the heater power needed for 320 K and the duty fraction of a 50.0 W heater, assuming rapid switching and negligible temperature ripple. (c) Linearize the net outward radiation rate about 320 K. State what happens to the equilibrium temperature if emissivity increases while shortwave absorptivity stays fixed.
- 38. What a cooling fit establishes.** The following *synthetic teaching data* refer to a body in a fixed 293.15 K ambient. Its independently known total heat capacity is $C = 400 \text{ J K}^{-1}$. Define

$\theta = T - 293.15 \text{ K}$ and dimensionless $y = \ln[\theta/(1 \text{ K})]$. Figure 18.38 shows the transformed data.

$t \text{ (s)}$	0	100	200	300	400
$T \text{ (K)}$	343.15	323.28	311.69	304.21	299.97
$\theta \text{ (K)}$	50.00	30.13	18.54	11.06	6.82

(a) Fit $y = a + bt$ using unweighted least squares; derive τ and the fitted initial excess. (b) Give residuals in y and the effective hA if only convection is assumed. (c) If independent temperature readings have $\sigma_T = 0.20 \text{ K}$ and ambient temperature is exact, compare uncertainty in y at the first and last points. Explain why an unweighted log fit does not match equal-uncertainty temperature measurements. (d) Interpret the small residuals, the small number of data points and the claim of uniform internal temperature. Propose an additional measurement to test the latter.

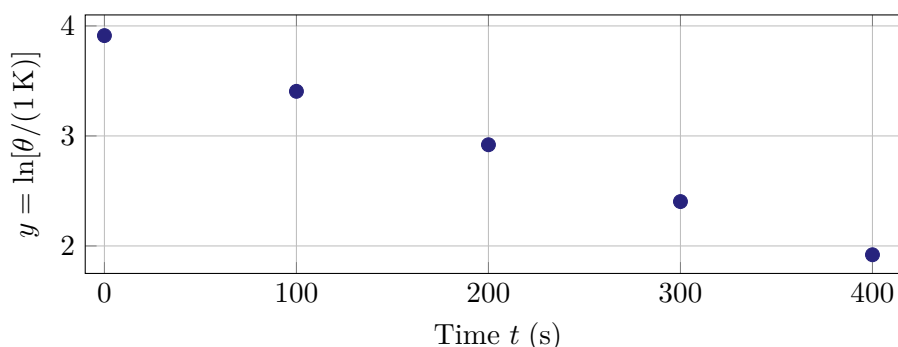


Figure 18.38: Logarithms of the synthetic excess-temperature data in Problem 38. No fitted line is supplied: the reader must infer and assess it.

- 39. A spherical sensor and the lumped criterion.** A sphere has $r = 0.020 \text{ m}$, $\rho = 2700 \text{ kg m}^{-3}$, $c = 900 \text{ J kg}^{-1} \text{ K}^{-1}$, $\kappa = 150 \text{ W m}^{-1} \text{ K}^{-1}$, and $h = 20 \text{ W m}^{-2} \text{ K}^{-1}$. It cools by convection alone from 353.15 K toward 293.15 K . (a) Derive L_c , compute Bi , C and τ , and predict $T(600 \text{ s})$ if lumping is supported. (b) Change only κ to $0.15 \text{ W m}^{-1} \text{ K}^{-1}$. Evaluate validity and find the maximum radius for the screening condition $\text{Bi} \leq 0.10$. (c) In the original high-conductivity case, double r and find the new τ and Bi .
- 40. A solver can conserve energy and still give a poor trajectory.** Consider convection-only lumped cooling with $C = 300 \text{ J K}^{-1}$, $hA = 3.00 \text{ W K}^{-1}$, and initial excess $\theta_0 = 50.0 \text{ K}$. (a) Derive the forward-Euler update for θ and the step conditions for monotonic nonnegative cooling and for linear stability. (b) Calculate $\theta(200 \text{ s})$ with steps 50, 25 and 12.5 s, compare with the analytic value, and assess convergence. (c) Use $\Delta t = 150 \text{ s}$ for one step and interpret the result. (d) Give pseudocode that checks accumulated convection against the energy change. Explain why this discrete energy check alone is insufficient for validation.

18.12 Complete reasoned solutions

Each solution uses the model and data stated in its problem. A temperature written in degrees Celsius below is a scale reading; every radiation power, thermodynamic temperature ratio and ideal-gas evaluation uses kelvin. Numerical uncertainties are standard uncertainties, not guaranteed bounds.

Solution 1: offset and absolute temperature

The instrument reads 1.5°C below the true reading, so subtracting its negative offset gives

$$t = 36.8 - (-1.5) = 38.3^\circ\text{C}, \quad T = 38.3 + 273.15 = 311.45 \text{ K}.$$

A constant offset cancels from a difference: $(30 + 1.5) - (10 + 1.5) = 20.0$ K. The corrected endpoint temperatures are 284.65 and 304.65 K, whose ratio is 1.0703. Thus the second absolute temperature is about 7.03% larger; the ratio of the uncorrected Celsius readings is dependent on the arbitrary scale zero and has no thermodynamic meaning. Even an absolute-temperature ratio does not by itself specify heat content, which also depends on material and amount.

Solution 2: tactile temperature

Before contact, the equilibrium temperature of both benches is the same. Skin is usually warmer than the benches, so touching either initiates a transient transfer of energy from the skin. Metal ordinarily transports that energy away from the contact region more quickly, maintaining a larger cooling rate at the skin. Wood develops a warmer contact region and reduces the transfer rate sooner.

The sensation responds to the skin's local temperature change and transient transfer rate, which depend on conductivity, volumetric heat capacity, contact resistance and contact duration. An ideal semi-infinite-contact model organizes these effects through thermal effusivity $\sqrt{\kappa\rho c}$; conductivity alone is not a complete quantitative contact model. Touch therefore cannot replace an equilibrium thermometer comparison. If the benches were hotter than the skin, the same rapid transfer could make metal feel hotter.

Solution 3: the zeroth-law conclusion

Provided the same calibrated thermometer does not substantially disturb either system and both readings represent equilibrium, each system is in thermal equilibrium with the thermometer at the same state. The zeroth law then implies that the two systems have equal temperature and would have no sustained thermal driving difference if brought into thermal contact under the same conditions.

Temperature equality does not determine their masses, compositions, volumes, heat capacities or internal energies. For example, a small and a large water sample can share a temperature while storing different total internal energies. The law is a statement about the transitivity of thermal equilibrium, not equality of extensive properties.

Solution 4: free expansion and restraint

Strategy. Apply small-strain linear expansion first, then impose zero total axial strain in a linear elastic model. Free warming gives

$$\Delta L = \alpha_L L \Delta T = (17 \times 10^{-6})(1.50)(80) = 2.04 \times 10^{-3} \text{ m} = 2.04 \text{ mm}.$$

Under perfect restraint, $0 = \epsilon_{\text{mech}} + \alpha_L \Delta T$, and $\sigma = E\epsilon_{\text{mech}}$, so

$$\sigma = -E\alpha_L \Delta T = -95.2 \text{ MPa}.$$

The negative sign means compression. The support suppresses extension and must exert a compressive reaction; a positive tensile result would contradict this constraint. The prediction presumes uniform temperature, linear elasticity and perfect restraint. It does not test buckling, yielding or support compliance, any of which could invalidate the actual stress prediction.

Solution 5: cubic expansion and approximation error

For initial edge L and constant linear coefficient, the warmed edge is $L(1 + x)$, where $x = \alpha_L \Delta T$. Consequently

$$\frac{\Delta V}{V} = (1 + x)^3 - 1 = 3x + 3x^2 + x^3.$$

The leading coefficient is therefore $\beta \simeq 3\alpha_L$ for an isotropic solid at small strain. At $x = 0.002$, the linear fractional change is 0.006 and the exact expression gives 0.006012008 within this constant-coefficient model. The quadratic term divided by the linear term is $x = 0.002$, or 0.20%. This is the relative correction to the predicted *change*; the total volume change is about 0.6012% of the original volume. The cubic term contributes another relative $x^2/3$, much smaller than the quadratic correction. An anisotropic solid need not have the same coefficient in three directions.

Solution 6: metal, water and calorimeter

Take the metal, water and calorimeter together as the closed, externally adiabatic system. Boundary work is neglected and all components reach one temperature. The heat lost by the metal equals the heat gained by water and calorimeter:

$$m_m c_m (80 - t_f) = (m_w c_w + C_{\text{cal}})(t_f - 20).$$

The heat capacities are $m_m c_m = 135 \text{ J K}^{-1}$ and $m_w c_w + C_{\text{cal}} = 1045 + 60 = 1105 \text{ J K}^{-1}$. Thus

$$t_f = \frac{135(80) + 1105(20)}{135 + 1105} = 26.53^\circ\text{C}.$$

The result lies between the two initial temperatures and much closer to the water temperature because the water-plus-container heat capacity is much larger. Substitution into either side of the balance gives the same energy transfer; the common rounding of t_f introduces only rounding-level disagreement.

Solution 7: omitting the container

With C_{cal} omitted, the balance becomes $135(80 - t'_f) = 1045(t'_f - 20)$, giving

$$t'_f = \frac{135(80) + 1045(20)}{1180} = 26.86^\circ\text{C}, \quad t'_f - t_f = +0.332 \text{ K}.$$

The sign is positive: the simplified calculation assigns all energy released by the metal to the water, although the real stipulated container also absorbs energy. Since that container starts cold, including its capacity lowers the equilibrium temperature. A small container correction in this example should not be assumed negligible in all experiments: its relevance depends on its capacity relative to the sample capacities and on the required measurement accuracy.

Solution 8: first test the phase branch

Cooling the warm water to the melting point releases

$$Q_{\text{avail}} = m_w c_w (15) = 6270 \text{ J},$$

whereas melting all the ice would require $m_i L_f = 6680 \text{ J}$. Since the available energy is insufficient, the system cannot end with all-liquid water above the melting point. It ends as an ice-water mixture at 0°C (273.15 K) in the ideal equilibrium model.

$$m_{\text{melted}} = \frac{6270}{334000} = 0.01877 \text{ kg}.$$

There are 1.23 g of ice and 118.77 g of liquid water. These masses sum to the initial 120 g. The water supplies all the latent transfer by cooling; no temperature rise is available while any ice remains. Solving an all-melt balance without this test would produce an unphysical branch.

Solution 9: uncertainty of a difference

For independent reading errors, propagation through $\Delta T = T_2 - T_1$ gives

$$\sigma_{\Delta T}^2 = \sigma_{T_1}^2 + \sigma_{T_2}^2, \quad \sigma_{\Delta T} = \sqrt{2}(0.10) = 0.141 \text{ K}.$$

The fractional standard uncertainty is 7.1% for a 2.0 K signal and 1.4% for a 10.0 K signal. The absolute uncertainty is unchanged; the larger signal makes it less important in relative terms. A common constant offset cancels from the difference instead of adding in quadrature. More generally, correlation contributes $-2 \text{cov}(T_1, T_2)$ to the variance of the difference, so the stated independent-error result must not be applied to correlated calibration errors without modification.

Solution 10: heat across a phase transition

The stipulated path first warms solid ice to its melting point, then melts it at that temperature, then warms the liquid. At fixed pressure, using constant properties within each phase,

$$Q_{\text{total}} = mc_i(15 \text{ K}) + mL_f + mc_w(10 \text{ K}).$$

The first and last terms use different heat capacities, and the latent term transfers energy without a temperature change. One $mc\Delta T$ misses both distinctions.

For a limited heat input, compare it with the first threshold $mc_i(15 \text{ K})$ and then with that threshold plus mL_f . Below the first threshold the sample remains solid; between thresholds it is a mixture at the melting point; above the second threshold, leftover energy raises the liquid temperature. A finite-capacity container or heat losses add terms to the corresponding energy balance. No numerical final state is determined here because neither mass nor total supplied heat is specified; the piecewise method is the complete requested answer.

Solution 11: boundaries and interactions

(a) The sealed tank is *closed*: no matter crosses its boundary. Heat crosses, so it is not isolated. (b) The sealed, ideally insulated tank is *isolated under the additional stated no-work model*: neither matter nor energy crosses. Insulation alone would establish only an adiabatic boundary, not isolation. (c) The turbine control volume is *open*: inlet and outlet streams carry matter and energy across its boundary. Steady flow does not change this classification. (d) The sealed piston-cylinder is *closed and adiabatic*, but it is not isolated because the external load transfers compression work into the gas. A moving boundary need not permit mass transfer.

Solution 12: expansion work

For the specified constant resisting pressure, boundary work is

$$W_{\text{by}} = \int_{V_i}^{V_f} p_{\text{ext}} dV = (250 \text{ kPa})(0.018 \text{ m}^3) = +4.50 \text{ kJ}.$$

The gas pushes its surroundings outward, so work by the gas is positive and $W_{\text{on}} = -4.50 \text{ kJ}$. The unit check is $\text{kPa m}^3 = \text{kJ}$. The relevant pressure for boundary work is the external resistance; quasistatic mechanical equilibrium allows it to coincide with the gas pressure to the needed accuracy.

Solution 13: compression and heat input

The volume decreases, so

$$W_{\text{by}} = (300 \text{ kPa})(0.020 - 0.050) \text{ m}^3 = -9.0 \text{ kJ}.$$

Then the first law gives $\Delta E_{\text{int}} = Q - W_{\text{by}} = 2.0 - (-9.0) = +11.0 \text{ kJ}$. Both interactions transfer energy into the gas: 2.0 kJ as heat and 9.0 kJ as work on it. The internal-energy increase exceeds the heat input because heat is not the only interaction. No temperature change can be calculated without an appropriate caloric relation for the gas.

Solution 14: common endpoints, different paths

Only the volume-changing segments contribute to boundary work; a fixed-volume pressure change has $dV = 0$. Consequently

$$W_{\text{by}}^{(A)} = (500)(0.020) = 10.0 \text{ kJ}, \quad W_{\text{by}}^{(B)} = (200)(0.020) = 4.0 \text{ kJ}.$$

The common internal-energy change fixes the heat through $Q = \Delta E_{\text{int}} + W_{\text{by}}$:

$$Q_A = 11.0 \text{ kJ}, \quad Q_B = 5.0 \text{ kJ}.$$

The higher-pressure expansion exports more mechanical energy, so it requires more heat to reach the same final stored energy. Internal energy is a state function; the areas under the two pressure-volume paths, and therefore their work and heat transfers, differ. This comparison depends on the stated equilibrium path and the specified work model, not on an assumption that all expansions follow one gas law.

Solution 15: electrical work at rigid walls

The electrical interaction brings 3 kJ into the system as work on it. In the work-by convention its contribution is $W_{\text{by}} = -3 \text{ kJ}$, hence

$$\Delta E_{\text{int}} = 12 - (-3) = 15 \text{ kJ}.$$

Rigid walls eliminate moving-boundary $p dV$ work. They do not prevent electrical energy from crossing terminals, shaft work through a sealed bearing or other specified work modes. The answer therefore requires accounting for *total* work, not replacing total work with boundary work without justification. The chosen boundary must include the element in which electrical energy is ultimately stored as internal energy.

Solution 16: adiabatic does not mean constant temperature

Adiabatic means $Q = 0$. The first law then gives

$$\Delta E_{\text{int}} = -W_{\text{by}} = +4.5 \text{ kJ}.$$

An internal-energy increase does not, for an arbitrary material, determine a unique temperature rise from the information given. Internal energy may depend on volume, phase or other state variables as well as temperature. A temperature calculation needs a constitutive relation such as $E_{\text{int}}(T, V)$ and sufficient final-state information. For an ideal gas with known constant total heat capacity C_V , one could write $\Delta T = 4.5 \text{ kJ}/C_V$, but neither that material model nor C_V is supplied here. Introducing the reversible adiabatic power law would also be unjustified from the word “adiabatic” alone.

Solution 17: an inconsistent cyclic ledger

A true cycle returns all state variables and stored energies to their initial values, so $\Delta E_{\text{int}} = 0$ under the chapter’s negligible kinetic/potential storage model. The reported transfers instead imply

$$Q_{\text{net}} - W_{\text{by,net}} = -3.0 - 1.0 = -4.0 \text{ kJ}.$$

Thus the measurements imply an energy decrease of 4.0 kJ despite the claimed return. One possible explanation is an omitted energy input of 4.0 kJ, such as unmeasured electrical work or heat through a support. Another is that the device did not complete a true cycle: an internal component may remain cooler or have different stored mechanical/electrical energy. A sign convention mistake is another plausible cause. A correct investigation checks the boundary, instruments and return of all storage variables before attributing the discrepancy to a new physical effect.

Solution 18: equal endpoint temperatures

For fixed n and equilibrium ideal-gas endpoints, $pV = nRT$ gives

$$T_i = \frac{p_i V_i}{nR} = \frac{p_f V_f}{nR} = T_f.$$

This proves only equality of endpoint temperatures. Intermediate temperatures could rise and then fall; the gas could even pass through a nonequilibrium transient where a single gas temperature is not well defined. An isothermal equilibrium process requires the same temperature at *every* intermediate equilibrium state. The distinction is analogous to returning to the same elevation without staying at that elevation during a trip. Fixed amount matters: if matter crossed the boundary, equal pV alone would not establish equal T .

Solution 19: conduction through a layer

Choose the positive direction from the 330 K face to the 300 K face. Fourier conduction for a uniform plane layer gives

$$\dot{Q} = \frac{\kappa A}{L}(T_h - T_c) = \frac{(0.80)(0.50)}{0.020}(30) = 600 \text{ W}.$$

Energy flows toward the colder face. In derivative form, $\dot{Q} = -\kappa A dT/dx$; the negative temperature gradient in the chosen direction produces a positive rate. The unit is $(\text{W m}^{-1}\text{K}^{-1})(\text{m}^2)(\text{K})/\text{m} = \text{W}$. Reversing the coordinate axis changes the signed component, not the hot-to-cold physical direction. Steady state and no internal generation make the rate equal across every cross section.

Solution 20: convection

The surface is warmer than the bulk air, so the stipulated Newton transfer law predicts outward convection:

$$\dot{Q}_{\text{conv}} = hA(T_s - T_{\infty}) = 12(0.10)(340 - 295) = 54.0 \text{ W}.$$

The coefficient h summarizes the flow and boundary-layer model; it is not a material conductivity. The result is a transfer rate at these temperatures, not a total heat unless a duration and temperature history are also supplied. Equal surface and bulk temperatures would produce zero net convection in this model.

Solution 21: net radiation and its direction

Assume an opaque diffuse grey surface viewing an effectively large black isothermal enclosure with view factor one, no participating medium and no additional illuminated surfaces. Emission and absorbed enclosure radiation then give

$$\dot{Q}_{\text{rad}} = \epsilon_{\text{rad}} \sigma_{\text{SB}} A (T_s^4 - T_{\text{env}}^4) = 0.70 \sigma_{\text{SB}} (0.10) (340^4 - 295^4) = 22.98 \text{ W}.$$

Relative to the 54.0 W convection rate, radiation is 42.6%. Under these teaching conditions its omission is substantial. The convection and radiation rates can be added because they are separate simultaneous outward interactions of the same surface.

At $T_s = T_{\text{env}}$, the fourth-power difference vanishes even though the surface still emits and absorbs radiation individually. If $T_s < T_{\text{env}}$, the outward net rate is negative, meaning net energy transfer into the surface. The difference must use kelvin: replacing both temperatures by Celsius produces a different fourth-power difference and breaks the physical model.

Solution 22: a lumped exponential

For uniform body temperature, constant C , constant hA and fixed ambient temperature, $C dT/dt = -hA(T - T_\infty)$. The time constant is

$$\tau = \frac{C}{hA} = \frac{250}{2.5} = 100 \text{ s.}$$

Therefore

$$t(120) = 20 + (70 - 20)e^{-120/100} = 35.06^\circ\text{C}.$$

This lies between the initial and ambient readings, and its excess has decreased by the factor $e^{-1.2}$. The Celsius differences have the same numerical magnitude as kelvin differences; the ambient and body readings share the same offset. This does not validate uniform internal temperature experimentally: lumping is an input assumption that must be assessed independently.

Solution 23: meaning of a time constant

The model identifies $\tau = 180 \text{ s}$. At $t = n\tau$, the fraction of initial excess is e^{-n} , giving

$$e^{-1} = 0.3679, \quad e^{-2} = 0.1353, \quad e^{-3} = 0.04979.$$

Thus about 36.8%, 13.5% and 4.98% of the original temperature difference remain. An exponential never reaches the ambient exactly at a finite time in this ideal model, although the remaining excess can become smaller than a measurement resolution.

The Celsius zero cancels in $T - T_{\text{ambient}}$, so the difference may be expressed numerically in either Celsius intervals or kelvin. By contrast, $T^4 - T_{\text{env}}^4$ contains absolute fourth powers: changing the zero changes all cross terms. That expression requires absolute temperatures.

Solution 24: a worked validation plan

An exemplar uses $C = 250 \text{ J K}^{-1}$, $hA = 2.5 \text{ W K}^{-1}$, fixed ambient 293.15 K and initial temperature 343.15 K . The following independent checks test different physical requirements.

Analytic benchmark: compare the full numerical trace with $293.15 + 50e^{-t/100} \text{ K}$, including the 308.21 K result at 120 s . Before running, choose an accuracy criterion appropriate to the teaching goal, for example a maximum 0.05 K error through 500 s . If a chosen integration step fails, refine it rather than weakening the criterion.

Equilibrium and direction: start exactly at ambient and require it to remain there to arithmetic precision. For an initially hot body require nonnegative excess and monotonic decay with a sufficiently resolved step; for an initially cold body require monotonic warming toward ambient. There should be no artificial crossing of the ambient in these first-order physical solutions.

Energy ledger: verify that $C[T(t) - T(0)] = -\int_0^t hA[T(s) - T_\infty] ds$ with a separately assessed integration error. **Refinement and limits:** halve the step and check that errors against the analytic result decrease consistently with the method's order. Setting $hA = 0$ must leave temperature constant. These checks cannot prove that a real specimen is lumped; they validate the solver for the stipulated model.

A defensible rubric rewards correct model inputs, a benchmark or exact limit, an explicit acceptance criterion, a conservation/direction check, and a refinement test. Merely obtaining a plot or a successful program exit earns no physical validation. Other numerical choices can pass if the same requirements are met.

Solution 25: fitted decay is not mechanism identification

Under *only* constant-coefficient convection with known uniform C , the model yields $hA = C/\tau$. A close exponential fit shows that the observed trace resembles an exponential over the sampled interval; it does not establish that all those physical assumptions hold.

For instance, near a limited temperature range radiation can be linearized as another conductance. Contact conduction through a mount can also give a conductance proportional to the body's excess. Their combination gives approximately $C\dot{\theta} = -(hA + G_{\text{rad}} + G_{\text{contact}})\theta$, so C/τ measures the sum rather than convection alone. A sensor lag or unresolved internal gradient can further distort the observed decay.

One useful follow-up is to repeat the experiment after thermally isolating the mount and compare decay rates; another is to change surface emissivity while keeping geometry and airflow controlled. Independent airflow/convection measurements and simultaneous centre/surface thermometry test different aspects. Small residuals alone cannot choose among mechanisms that predict nearly the same curve.

Solution 26: two layers in series

At steady state with perfect contact, both layers carry the same rate. Their resistances are

$$R_1 = \frac{0.040}{(0.040)(1.0)} = 1.00 \text{ K W}^{-1}, \quad R_2 = \frac{0.020}{(0.20)(1.0)} = 0.100 \text{ K W}^{-1}.$$

Hence $R_{\text{tot}} = 1.10 \text{ K W}^{-1}$ and

$$\dot{Q} = \frac{300 - 280}{1.10} = 18.18 \text{ W}, \quad T_{12} = 300 - \dot{Q}R_1 = 281.82 \text{ K}.$$

The low-conductivity first layer produces most of the drop. From the cold side, $280 + \dot{Q}R_2$ gives the same interface temperature. The interface must lie between the imposed face temperatures, which it does. These are solid-face temperatures, so convection films do not belong in this particular balance.

Solution 27: films and solids

The relevant endpoints are now bulk fluids. Add both film resistances to the unchanged solid resistances:

$$R_i = \frac{1}{h_i A} = 0.100, \quad R_1 = 1.000, \quad R_2 = 0.100, \quad R_o = \frac{1}{h_o A} = 0.040 \text{ K W}^{-1}.$$

Thus $R_{\text{total}} = 1.240 \text{ K W}^{-1}$ and $\dot{Q} = 30/1.240 = 24.19 \text{ W}$. Using the unrounded rate gives

$$T_{s,i} = 305 - \dot{Q}R_i = 302.58 \text{ K}, \quad T_{s,o} = 275 + \dot{Q}R_o = 275.97 \text{ K}.$$

The interface is $305 - \dot{Q}(R_i + R_1) = 278.39 \text{ K}$. The drops are $\dot{Q}$ times 0.100, 1.000, 0.100 and 0.040 K W^{-1} ; their sum is exactly $\dot{Q}(1.240) = 30 \text{ K}$ before rounding. The solid surface temperatures differ from the bulk-fluid temperatures because transfer through each film needs a driving difference. Using the bulk temperatures directly across the solids would overestimate the rate.

Solution 28: assessing the lumped assumption

The specified characteristic length is $L_c = V/A_s = (10^{-4})/(0.020) = 0.0050 \text{ m}$. Thus

$$\text{Bi}_{\text{high k}} = \frac{15(0.0050)}{150} = 5.0 \times 10^{-4}, \quad \text{Bi}_{\text{low k}} = \frac{15(0.0050)}{0.15} = 0.50.$$

The first value is well below the usual 0.1 screening scale, suggesting that internal conduction is fast compared with external convection and supporting a lumped approximation for a suitable

homogeneous compact body. The second value is not small, so spatial gradients cannot be dismissed; a distributed-temperature model may be needed.

A low Biot number is a model assessment, not evidence that sensors have measured equal internal temperatures. Geometry, contact heating, initial gradients and any unmodelled transfer still require attention. For example, a highly localized support interaction may not be characterized by the same external h used in this screening calculation.

Solution 29: doubling a similar body

For length scale factor s , volume scales as s^3 and exposed area as s^2 . The material volumetric heat capacity remains fixed, so $C \propto s^3$ and $L_c = V/A_s \propto s$. Therefore

quantity	L_c	C	A_s	$\tau = C/(hA_s)$	$\text{Bi} = hL_c/\kappa$
factor for $s = 2$	2	8	4	2	2

The lumped model predicts slower cooling because the energy storage grows faster than the transfer area. At the same time the larger internal conduction length raises Biot number, reducing support for uniform temperature. If the doubled Biot number is no longer small, even the simple doubled time constant ceases to be a reliable prediction. The conclusion depends on the stipulated unchanged h ; in a real flow, changing body size can also change convection.

Solution 30: why series resistances add

Let a steady one-dimensional chain have no generation, leakage or branching. Energy conservation then requires one common $\dot{Q}$ in every element. If each element obeys $\Delta T_j = \dot{Q}R_j$, adjacent interface temperatures cancel in the sum:

$$T_{\text{hot}} - T_{\text{cold}} = \sum_j \Delta T_j = \dot{Q} \sum_j R_j, \quad R_{\text{equivalent}} = \sum_j R_j.$$

This derivation states both the linear constitutive assumption and the shared-rate condition. It is not just an analogy to an electrical formula.

As a counterexample, a metal fastener can bridge an insulation layer and create a parallel path. Knowing the insulation's thickness, conductivity, area and film coefficients exactly still does not account for the fastener's path or lateral spreading. Likewise, transient storage makes the incoming and outgoing rates of an element differ. A valid answer identifies one such omitted mechanism and explains which step of the shared-rate/linear-drop derivation fails. Radiation, variable properties or imperfect contact require additional or nonlinear elements rather than being cured by greater precision in the original parameters.

Solution 31: slope, inverse calibration and drift

(a) **Read the model from two points.** The intercept is $a = 2^\circ\text{C}$ and the slope is

$$b = \frac{104 - 2}{100 - 0} = 1.02, \quad r = 2 + 1.02t.$$

Here r and t are numerical Celsius readings; the slope is dimensionless. Inverting the calibration at $r = 53$ gives $t = (53 - 2)/1.02 = 50.00^\circ\text{C}$ or $T = 323.15\text{ K}$. The indicated reading is not a true thermodynamic temperature until this calibration is applied.

(b) **Propagate the reading uncertainty.** Since the coefficients are stipulated exact, $\partial t/\partial r = 1/b$ and

$$\sigma_t = \frac{\sigma_r}{b} = \frac{0.20}{1.02} = 0.196\text{ K}.$$

Adding the exact Celsius–kelvin offset does not change that uncertainty. In an actual calibration, the uncertainties and covariance of a and b would also contribute; the stated answer intentionally excludes them.

(c) Test the later offset. With $a' = 3$ and the same slope, the new true reading for $r = 53$ is $(53 - 3)/1.02 = 49.02^\circ\text{C}$. Continuing to use the old calibration reports 50.00°C , a signed bias of $+0.980\text{ K}$. The drift is systematic and exceeds the stipulated single-reading standard uncertainty. One known calibration point supplies one equation for two unknowns, a and b : many straight lines pass through it. A second distinct temperature, or independent knowledge of the slope, is necessary to distinguish gain from offset.

Solution 32: interpreting the phase curve

(a) Use energy intervals within each segment. Before melting, the sample warms through 20.00 K with 12.60 kJ input. Its inferred solid heat capacity per mass is

$$c_i = \frac{12600}{(0.300)(20.00)} = 2100\text{ J kg}^{-1}\text{K}^{-1}.$$

The plateau extends from 12.60 to 112.80 kJ , so $mL_f = 100.20\text{ kJ}$ and $L_f = 334\text{ kJ kg}^{-1}$. The last segment transfers $137.88 - 112.80 = 25.08\text{ kJ}$ while liquid temperature rises by 20.00 K . Thus $c_w = 25080/[(0.300)(20.00)] = 4180\text{ J kg}^{-1}\text{K}^{-1}$. These are properties of the stipulated piecewise teaching model.

(b) Determine the phase before calculating temperature. At $Q = 90.0\text{ kJ}$ the state lies on the plateau. After the first 12.6 kJ has warmed the ice, the remaining 77.4 kJ melts

$$m_w = \frac{77400}{334000} = 0.23174\text{ kg}.$$

The temperature is 273.15 K and the remaining ice mass is 0.06826 kg . Total mass is still 0.300 kg .

(c) Explain the horizontal segment. During coexistence, heat changes the phase fraction at fixed transition temperature. At fixed pressure, the latent heat is the appropriate phase-change heat; the first law still includes any small boundary work accompanying the phase change. A single straight interpolation would assign a rising temperature while solid and liquid coexist and would predict the wrong phase fractions. The graph's different slopes and horizontal interval encode distinct physical relations; they must not be replaced by one averaged slope.

Solution 33: a branch-tested calorimetric measurement

(a) Test all melting before solving. Let $A = m_w c_w + C_{\text{cal}} = 916\text{ J K}^{-1}$. Cooling the initial water and calorimeter to 0°C supplies $A(25) = 22900\text{ J}$. Warming the ice to 0°C costs $m_i c_i(10) = 840\text{ J}$; melting it then costs $m_i L_f = 13360\text{ J}$. There is a surplus of 8700 J after complete melting. The final state therefore contains only liquid above the melting point, and the surplus warms the final water and calorimeter:

$$B = (m_w + m_i)c_w + C_{\text{cal}} = 1083.2\text{ J K}^{-1}, \quad t_f = \frac{8700}{1083.2} = 8.03^\circ\text{C} = 281.18\text{ K}.$$

An explicit all-melt energy balance is

$$A(t_w - t_f) = m_i c_i(0 - t_i) + m_i L_f + m_i c_w t_f.$$

The temperatures in this equation are Celsius *intervals from the melting point*; adding 273.15 to each physical temperature changes none of those differences. The computed positive t_f confirms the selected branch.

(b) Quantify the omitted-container bias. Set $C_{\text{cal}} = 0$ in the same branch-tested balance. It gives $t'_f = 6.68^\circ\text{C}$, a bias $t'_f - t_f = -1.35\text{ K}$. Here the container initially belongs to the *warm* subsystem and releases heat as it cools; omitting it lowers the predicted equilibrium temperature. This direction is the opposite of Problem 7, where the omitted container initially belonged to the cold subsystem.

(c) Propagate the specified independent errors. Write the solution as

$$t_f = \frac{At_w + m_i c_i t_i - m_i L_f}{B}.$$

For the four uncertain inputs, the sensitivities at the nominal point are

$$\begin{aligned} \frac{\partial t_f}{\partial t_w} &= \frac{A}{B} = 0.845643, & \frac{\partial t_f}{\partial t_i} &= \frac{m_i c_i}{B} = 0.077548, \\ \frac{\partial t_f}{\partial m_i} &= \frac{c_i t_i - L_f - c_w t_f}{B} = -358.727\text{ K kg}^{-1}, & \frac{\partial t_f}{\partial C_{\text{cal}}} &= \frac{t_w - t_f}{B} = 0.015665\text{ K}^2\text{ J}^{-1}. \end{aligned}$$

The first two derivatives are dimensionless. Independent quadrature yields

$$\sigma_{t_f} = \left[(0.845643 \cdot 0.20)^2 + (0.077548 \cdot 0.20)^2 + (358.727 \cdot 0.0005)^2 + (0.015665 \cdot 5)^2 \right]^{1/2} = 0.26\text{ K}.$$

The nominal temperature may therefore be reported as 281.18 K with standard uncertainty 0.26 K under this model. This includes only the listed input uncertainties; it does not quantify systematic heat loss or uncertainty of the stipulated material properties.

The branch margin is $M = At_w + m_i c_i t_i - m_i L_f = 8700\text{ J}$. Applying the same propagation to M gives $\sigma_M \simeq 285\text{ J}$. The margin exceeds thirty standard uncertainties, so nearby input variations of the stated scale do not approach the melting threshold. This supports using the smooth all-melt sensitivities here. Near $M = 0$, the piecewise branch changes and a single linear uncertainty propagation would require much more care. Standard uncertainties are not strict bounds and do not justify a claim of mathematical impossibility for every error distribution.

Solution 34: the signed area of a cycle

(a) Integrate each segment. With pressure in kPa and volume in m^3 , work comes out in kJ. The horizontal segment gives $W_{\text{by}}^{AB} = 200(0.030 - 0.010) = +4.0\text{ kJ}$. The vertical segment has $W_{\text{by}}^{BC} = 0$. Along the straight compression segment the mean pressure is $(500 + 200)/2 = 350\text{ kPa}$, so $W_{\text{by}}^{CA} = 350(0.010 - 0.030) = -7.0\text{ kJ}$. Applying $Q = \Delta E_{\text{int}} + W_{\text{by}}$ segment by segment gives

segment	AB	BC	CA
$W_{\text{by}}\text{ (kJ)}$	+4	0	-7
$\Delta E_{\text{int}}\text{ (kJ)}$	+6	+9	-15
$Q\text{ (kJ)}$	+10	+9	-22

These heat signs cannot be inferred solely from whether a segment expands or compresses; they require the supplied internal-energy changes.

(b) Check both loop closure and area. The internal-energy changes sum to zero, while the work sums to -3.0 kJ and the heat to -3.0 kJ . The geometric triangular area is $\frac{1}{2}(0.020)(300) = 3.0\text{ kJ}$. The counterclockwise traversal assigns it negative work-by sign, agreeing with the explicit integrals.

(c) Interpret the net interaction. The system receives 3.0 kJ of net mechanical work and exports the same net energy as heat. These data satisfy the first-law cyclic ledger. That consistency does not alone establish the feasibility of any chosen reservoir arrangement or prove reversibility: those questions need additional process information and the second-law restrictions developed in Chapter 20.

Solution 35: free expansion is not isothermal piston work

(a) **Apply the actual boundary conditions.** Insulation gives $Q = 0$. The rigid outer wall does no boundary work, and expansion into vacuum has no resisting pressure on the advancing gas front. With the stipulated negligible partition-removal interaction and no other work, $W_{\text{by}} = 0$. Therefore $\Delta E_{\text{int}} = 0$. The stated $E_{\text{int}} = CT + \text{constant}$ with $C > 0$ gives $T_f = T_i = 300 \text{ K}$.

(b) **Calculate equilibrium endpoints only.** The ideal-gas equation gives

$$p_i = \frac{(1.00)R(300)}{0.0100} = 249.43 \text{ kPa}, \quad p_f = \frac{(1.00)R(300)}{0.0200} = 124.72 \text{ kPa}.$$

During partition removal and redistribution, the gas can have nonuniform density and velocity; there is no single equilibrium gas pressure assigned to every intermediate occupied volume. A hyperbola joining the endpoints is therefore *not* the actual free-expansion path. In particular, its area cannot be substituted for actual work. Equal endpoint temperatures do not mean that a well-defined common temperature describes the whole transient.

(c) **Analyze a separate comparison process.** In a quasistatic isothermal piston process, the resisting pressure follows $p_{\text{ext}} \simeq nRT/V$. Then

$$W_{\text{by}}^{\text{comp}} = \int_{V_i}^{V_f} \frac{nRT}{V} dV = nRT \ln \frac{V_f}{V_i} = 1.729 \text{ kJ}.$$

The comparison process has the same zero internal-energy change, so it requires $Q^{\text{comp}} = +1.729 \text{ kJ}$ to replace the exported work. The actual process has both transfers zero. The endpoint equality is consistent because stored energy is a state property while heat and work are path interactions. The quasistatic comparison requires a controlled piston and thermal contact, physically different boundary conditions from the insulated partition-removal event.

Solution 36: a small-area bridge dominates

(a) **Form the correct network.** The enforced uniform face temperatures give the two core branches the same ΔT_s . Their conductances, rather than resistances, add:

$$G_i = \frac{\kappa_i A_i}{L} = 0.570 \text{ W K}^{-1}, \quad G_b = \frac{\kappa_b A_b}{L} = 11.25 \text{ W K}^{-1}.$$

The core resistance is $R_{\text{core}} = 1/(G_i + G_b)$; both convection films use the full area 1.20 m^2 . Thus

$$R_{\text{tot}} = \frac{1}{12} + \frac{1}{11.82} + \frac{1}{30} = 0.20127 \text{ K W}^{-1}, \quad \dot{Q} = \frac{30}{R_{\text{tot}}} = 149.05 \text{ W}.$$

The solid-face temperatures are $T_{s,i} = 305 - \dot{Q}/12 = 292.58 \text{ K}$ and $T_{s,o} = 275 + \dot{Q}/30 = 279.97 \text{ K}$. Their difference is the core driving difference, while the films account for the rest of the 30 K bulk difference.

(b) **Split the core heat.** Multiplying the unrounded face difference by each conductance gives $\dot{Q}_i = 7.19 \text{ W}$ and $\dot{Q}_b = 141.87 \text{ W}$. Their unrounded sum equals the total rate; the sum of rounded values differs by 0.01 W only. The bridge share is $G_b/(G_i + G_b) = 95.18\%$. Although it occupies only 5% of the area, it carries nearly all the core heat because its conductivity is so much larger.

(c) **Remove the bridge.** With insulation across the full area, $G_{\text{core}} = 0.040(1.20)/0.080 = 0.600 \text{ W K}^{-1}$. The films stay unchanged and the rate becomes

$$\dot{Q}_{\text{no bridge}} = \frac{30}{1/12 + 1/0.600 + 1/30} = 16.82 \text{ W}.$$

An area-weighted conductivity works here because parallel portions share the same thickness and exactly the same face temperatures; it reproduces $\sum \kappa_j A_j / L$. If faces are not isothermal, if lateral spreading matters, or if portions are arranged in series, that averaging does not in general reproduce the network. The ideal face plates in the statement make the simple parallel model explicit rather than hiding that condition.

Solution 37: fourth-power equilibrium and control

(a) Balance separate spectral terms. Absorbed shortwave power is $P_s = \alpha_s SA$. Substitution gives $0.25(900)(0.50) = 112.5 \text{ W}$. Net longwave loss is $\varepsilon_{\text{rad}}\sigma_{\text{SB}}A(T^4 - 250^4)$. At equilibrium, the heater plus absorbed shortwave input must equal that loss:

$$T_{\text{eq}} = \left[250^4 + \frac{112.5 + 20.0}{0.80\sigma_{\text{SB}}(0.50)} \right]^{1/4} = 314.22 \text{ K}.$$

Using different shortwave absorptivity and longwave emissivity is consistent with the stipulated spectral model; applying a single bolometric equality to those two distinct bands would be an unwarranted material assumption.

(b) Determine controllable heater power. At 320 K, the net radiative loss is 149.23 W, so the required additional heater input is

$$P_h = 149.23 - 112.5 = 36.73 \text{ W}, \quad d = \frac{P_h}{50.0} = 0.735 \text{ approximately.}$$

The duty is 73.5%, within the physically available interval from zero to one. It represents average power only under the rapid-switching/negligible-ripple assumption. Slow switching would require a transient balance to determine the temperature ripple; averaging powers would not by itself give a constant temperature.

(c) Linearize a local change. For $T = 320 \text{ K} + \delta T$ with small $|\delta T|$,

$$\dot{Q}_{\text{rad}}(T) \simeq \dot{Q}_{\text{rad}}(320) + G_{\text{rad}}\delta T, \quad G_{\text{rad}} = 4\varepsilon_{\text{rad}}\sigma_{\text{SB}}A(320)^3 = 2.973 \text{ W K}^{-1}.$$

This is a tangent approximation to the fourth-power law, not a new globally constant radiative coefficient. For fixed positive total input, the equilibrium relation gives $T^4 - 250^4 = P_{\text{in}}/(\varepsilon_{\text{rad}}\sigma_{\text{SB}}A)$. Increasing emissivity while holding absorptivity and inputs fixed lowers the equilibrium temperature toward the enclosure temperature. At the same fixed target temperature it instead raises the required heater power.

Solution 38: fit arithmetic and inferential limits

(a) Compute the specified fit. A convection-only exponential would give $\theta = \theta_0 e^{-t/\tau}$ and hence $y = \ln[\theta_0/(1 \text{ K})] - t/\tau$. For the stated unweighted least-squares fit,

$$b = \frac{\sum_i (t_i - \bar{t})(y_i - \bar{y})}{\sum_i (t_i - \bar{t})^2}, \quad a = \bar{y} - b\bar{t}.$$

Using $\bar{t} = 200 \text{ s}$ and the logarithms of the tabulated excesses gives $b = -0.0049865 \text{ s}^{-1}$ and $a = 3.90944$. Consequently

$$\tau = -1/b = 200.54 \text{ s}, \quad \theta_{0,\text{fit}} = (1 \text{ K})e^a = 49.87 \text{ K}.$$

The fitted intercept is free; it need not pass exactly through the initial observation. Units in the logarithm are handled by dividing excess temperature by the specified reference 1 K.

(b) Assess the actual residuals. Define residual as observation minus fitted value, $r_i = y_i - a - bt_i$. In time order they are

$$(r_0, r_1, r_2, r_3, r_4) = (0.00259, -0.00526, 0.00780, -0.01015, 0.00503).$$

The convection-only interpretation would be $hA = C/\tau = 1.9946 \text{ W K}^{-1}$. Without excluding other interactions this is an *effective* conductance, as in Problem 25. The table, graph, fit and residuals all use the same five synthetic observations; there is no separate measured dataset.

(c) Inspect uncertainty after transformation. With exact ambient and small independent temperature errors, $\partial y/\partial T = 1/\theta$, so $\sigma_y \simeq \sigma_T/\theta$. It is 0.0040 initially but 0.0293 at the last point. Equal-uncertainty temperatures therefore produce increasing log uncertainty as the body approaches ambient. Unweighted least squares in y does not reflect that error model. Approximate weighted log fitting would use weights proportional to θ^2/σ_T^2 ; fitting the exponential in the original temperature domain is another way to respect equal temperature uncertainties. Uncertain ambient temperature would introduce a shared error and correlated excesses, requiring further treatment.

(d) Limit the conclusion. Only five points are fitted with two parameters, leaving three residual degrees of freedom. Small residuals show compatibility with a simple exponential over this short range; they provide weak evidence about detailed residual patterns, and they cannot identify a unique physical mechanism. In particular, surface data cannot prove internal temperature uniformity. A worked follow-up places calibrated sensors near the centre and surface and compares their difference and lag during the transient, accounting for sensor response and placement effects. A defensible experimental answer specifies that comparison, its uncertainty and the temperature range over which agreement is expected. No real experiment has been performed here, so neither fitted parameters nor proposed follow-ups are presented as measured material properties.

Solution 39: a sphere's geometry and validity

(a) Derive the characteristic length before using it. For a sphere, $V = 4\pi r^3/3$ and $A_s = 4\pi r^2$, so $L_c = r/3 = 0.006667$ m. Thus

$$\text{Bi} = \frac{20(r/3)}{150} = 0.000889, \quad C = \rho c V = 81.43 \text{ J K}^{-1}.$$

For convection alone, $\tau = C/(hA_s) = \rho c r/(3h) = 810$ s. The Biot number strongly supports lumping under the stated homogeneous-body model, giving

$$T(600) = 293.15 + 60e^{-600/810} = 321.76 \text{ K}.$$

The value remains between the initial and ambient temperatures. The characteristic length is $r/3$, not r , for the specified volume-to-area screening convention.

(b) Reduce conductivity without changing external transfer. With $\kappa = 0.15$, the Biot number becomes 0.889, too large to justify neglecting internal gradients. Although inserting the unchanged ρ , c , h and r into the lumped time constant still returns 810 s, that formal result is not a validated physical temperature prediction for the low-conductivity sphere. A spatial conduction model would now be required. The radius meeting $h(r/3)/\kappa \leq 0.10$ is

$$r \leq \frac{3(0.10)\kappa}{h} = 0.00225 \text{ m} = 2.25 \text{ mm}.$$

At the threshold, lumping remains an approximate screening decision whose accuracy depends on the desired observable and tolerance.

(c) Double the original radius. For the original high-conductivity material and unchanged h , both τ and Bi scale linearly with r . The doubled values are 1620 s and 0.001778. Lumping remains well supported for this stipulated larger sphere, while its thermal response becomes slower.

Solution 40: convergence, overshoot and conservation

(a) Derive the update and inspect its factor. The ODE is $\dot{\theta} = -\theta/\tau$ with $\tau = C/(hA) = 100$ s. Forward Euler gives

$$\theta_{n+1} = \theta_n - \frac{\Delta t}{\tau} \theta_n = \left(1 - \frac{\Delta t}{\tau}\right) \theta_n.$$

For positive initial excess, monotonic nonnegative decay requires $0 < \Delta t \leq \tau$; equality produces an artificial one-step arrival at ambient. Strictly positive decay needs $0 < \Delta t < \tau$. Linear asymptotic

stability requires $|1 - \Delta t/\tau| < 1$, or $0 < \Delta t < 2\tau$. Thus stability alone permits sign-alternating, physically incorrect trajectories when $\tau < \Delta t < 2\tau$. At $\Delta t = 2\tau$ there is no decay and above it the magnitude grows.

(b) Compare with the exact solution. At 200 s, the exact excess is $50e^{-2} = 6.7668$ K. All stated steps divide the final time exactly, so the Euler result is $50(1 - \Delta t/100)^{200/\Delta t}$:

Δt (s)	50	25	12.5
steps	4	8	16
θ_{Euler} (K)	3.1250	5.0056	5.90335
$\theta_{\text{exact}} - \theta_{\text{Euler}}$ (K)	3.6418	1.7611	0.8634

Halving the step reduces the error by factors 2.068 and 2.040, approaching the factor two expected for first-order global convergence. The coarse result cools too rapidly; smaller steps approach the analytic trace. These three checks demonstrate convergence for this model but do not imply that the coarsest step meets any particular accuracy requirement.

(c) Show stable but unphysical overshoot. With $\Delta t = 150$ s, the first step gives $\theta_1 = (1 - 1.5)(50) = -25$ K. The body numerically crosses below ambient even though this convection-only ODE approaches ambient monotonically from above. Subsequent step magnitudes decrease because the multiplier is -0.5 , so the scheme is linearly stable but physically unacceptable at this resolution.

(d) Track a discrete ledger and its limits. One reproducible algorithm, with an integer number of equal steps, is

```
from math import exp
dt, final_time = 25.0, 200.0      # s
number_of_steps = round(final_time/dt)
C, G, theta = 300.0, 3.0, 50.0    # J/K, W/K, K
initial = theta
removed = 0.0                     # J
for n in range(number_of_steps):
    outward = G * theta            # W at the start of this step
    removed += outward * dt        # J, Euler quadrature
    theta -= outward * dt / C      # K
ledger_error = C*(initial-theta) - removed
exact = initial * exp(-final_time*G/C)
trajectory_error = abs(theta-exact)
```

The Euler update makes $C(\theta_n - \theta_{n+1}) = G\theta_n\Delta t$. Summing telescopes, so this discrete ledger agrees to arithmetic roundoff even with a poor step. At 200 s, the exact removed energy is $C(50 - 50e^{-2}) \simeq 12970$ J; a coarse discrete method can conserve its own approximate ledger while removing the wrong amount in the wrong time. Validation therefore needs conservation *and* analytic/error, physical-direction and refinement checks. An assessment rubric should credit the correctly derived update, both step conditions, the three independently calculated results, convergence interpretation, overshoot diagnosis and this distinction between a ledger identity and trajectory accuracy.

18.13 Bridge to Chapter 19

The present chapter has treated $pV = nRT$ as an empirical relation among equilibrium variables and E_{int} as a macroscopic state function whose change is constrained by energy conservation. It has not yet explained why dilute gases follow the ideal-gas law, why temperature is connected with molecular kinetic energy, or why different gases and solids have characteristic heat capacities.

Chapter 19 develops those questions microscopically. It connects pressure to momentum transfer in molecular collisions, temperature to distributions of particle motion, and heat capacities to accessible energy modes. It also derives the ideal-gas results needed to justify reversible adiabatic power laws. The conceptual discipline developed here remains essential: a molecular model must still respect system boundaries, state variables, transfer signs, and stated approximation limits.

Bibliography

- [1] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 12th ed., Vol. 1, Wiley, Chapter 18 and Section 19.2 of the supplied edition.
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th ed., Brooks/Cole, Chapters 19–20 of the supplied edition.
- [3] Bureau International des Poids et Mesures, *SI base unit: kelvin (K)*, <https://www.bipm.org/en/si-base-units/kelvin>.
- [4] National Institute of Standards and Technology, *2022 CODATA Recommended Values of the Fundamental Constants of Physics and Chemistry*, NIST SP 959 (May 2024), https://physics.nist.gov/cuu/pdf/wallet_2022.pdf.
- [5] I. Neutelings, *PV diagrams*, TikZ.net, https://tikz.net/thermodynamics_pv_diagrams/.
- [6] I. Neutelings, *Heat baths*, TikZ.net, https://tikz.net/heat_baths/.