

Fixed-Axis Rotation

Physics I - Chapter 10

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What determines a rotor's motion?



An encoder reports orientation, but a repeating display hides completed turns. Angular speed gives a rim speed only when the radius is known. A motor torque predicts angular acceleration only when the mass distribution about the axis is known.

This chapter connects those questions. We first describe signed angular motion, then calculate the inertia of an extended body, and finally predict its response to torque and energy transfer.

For a rotary stage, a flywheel, or a spacecraft mechanism, the chosen axis connects the geometry, the measured motion, and the dynamical model. Our core model is a rigid body with a fixed inertial axis and constant inertia.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Learning route and prerequisites



Description. Axis, reference direction, radians, and unwrapped angle establish a physical motion history.

Geometry and mass. Differentiation connects that history to point velocities and accelerations. Mass integrals then connect geometry to inertia and energy.

Dynamics and checks. Torque predicts acceleration. Work and power provide independent energy checks. Numerical refinement tests a discretized mass model.

Prerequisites. Resolve vectors in Cartesian components, differentiate and integrate a function, use Newton's second law, and apply kinetic and gravitational energy. We refresh the rotating basis and mass elements where needed.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Learning goals



- Recover signed angular changes and distinguish net displacement from total travel.
- Match constant-acceleration motion segments and connect angular variables to linear velocity and acceleration.
- Derive axis inertia from a mass distribution and use parallel axes correctly.
- Solve coupled translation and fixed-axis rotation with explicit signs and assumptions.
- Calculate work and power, check energy balances, and diagnose a braking or inference model.
- In the advanced block, verify a quadrature model and relate scalar inertia to a tensor projection.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Axis, angular state, and sign convention



Choose the fixed z axis and a laboratory reference direction. Positive rotation is counterclockwise when viewed from $+z$ toward the origin.

$$\Delta\theta = \theta_f - \theta_i, \quad \bar{\omega} = \frac{\Delta\theta}{\Delta t}, \quad \omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt}.$$

The signed variables have units rad , rad/s , and rad/s^2 ; $\bar{\omega}$ is the interval average using unwrapped angles. A rigid body shares one θ , ω , and α about this axis. Angular speed is $|\omega|$.

Sign check. If $\omega < 0$ and $\alpha > 0$, the body rotates in the negative sense while slowing toward rest. For $\omega \neq 0$, increasing speed requires $\omega\alpha > 0$.

A change of reference adds a constant to θ and leaves its derivatives unchanged. Reversing the positive sense changes the signs of all three angular variables.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Radians connect angle to arc length



$$1 \text{ rev} = 2\pi \text{ rad} = 360^\circ, \quad s = R\theta.$$

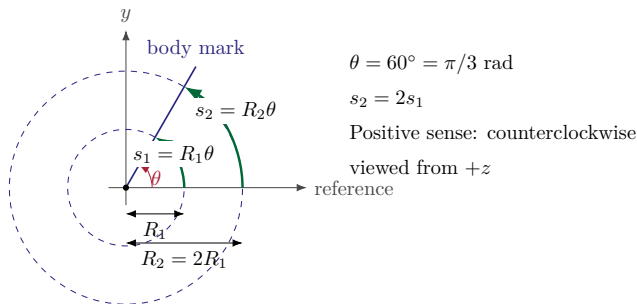


Figure 10.2: Radian geometry for two concentric paths. The angular displacement is shared; the traveled arc length is proportional to radius. The construction uses $\theta = \pi/3$ and $R_2 = 2R_1$, with arbitrary common length units.

Calculus convention. If ϕ is in degrees, use $\theta = (\pi/180)\phi$ before differentiating. The derivative of $\cos \theta$ uses radian measure.

Wrapped displays and stored turn counts

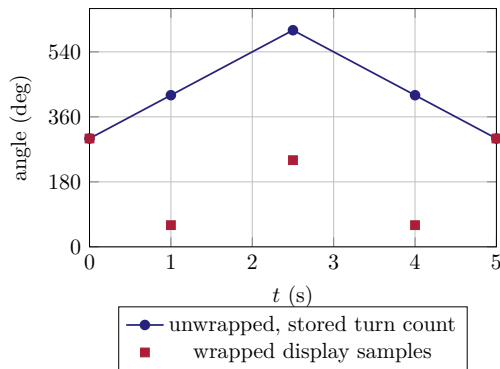
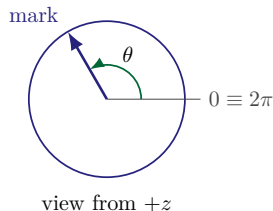


Figure 10.1: Encoder reference and recorded angular history. The blue record uses known turn counts; its straight segments describe the prescribed teaching motion. Red squares are discrete modulo- 360° readings, with no connecting lines that could falsely imply reverse motion across a wrap.

Unwrapped angle preserves accumulated turns. A wrapped display alone permits angular changes differing by integer multiples of 2π .

Example 10.1: encoder averages



Example 10.1. A controller stores the unwrapped angular positions 300° , 420° , 600° , 420° , 300° at times 0, 1.0, 2.5, 4.0, 5.0 s. Its front panel displays only the values modulo 360° . The stage moves counterclockwise until $t = 2.5$ s and then reverses. Find the average angular velocity over 0–2.5 s and 2.5–5.0 s.

The trusted turn count fixes the branch. Over the first interval,

$$\Delta\theta_1 = 300^\circ = \frac{5\pi}{3} \text{ rad}, \quad \bar{\omega}_1 = \frac{5\pi/3}{2.5} = +2.09 \text{ rad/s}.$$

The second displacement is -300° , giving $\bar{\omega}_2 = -2.09 \text{ rad/s}$. Equal opposite displacements in equal times check the signs and magnitudes. Over the full record, the net displacement is zero.

Chapter support: Physics I, Chapter 10: Example 10.1.

Net displacement and total angular travel



Angular displacement uses signed area. Angular travel adds the magnitudes of the areas on either side of every reversal.

$$\Delta\theta = \int_{t_1}^{t_2} \omega \, dt, \quad \mathcal{A} = \int_{t_1}^{t_2} |\omega| \, dt.$$

$$\bar{\omega} = \frac{\Delta\theta}{t_2 - t_1}, \quad \overline{|\omega|} = \frac{\mathcal{A}}{t_2 - t_1}.$$

For a point at fixed radius R , traveled distance is $R\mathcal{A}$. Signed arc displacement is $R\Delta\theta$. Neither generally equals the straight-line displacement between the endpoints.

Check. Equal forward and backward angular travel gives zero average velocity and a positive average speed.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Example 10.2: a return motion



Example 10.2. A prescribed fixed-axis motion has $\omega(t) = 3.0 - 1.5t$ in rad/s for $0 \leq t \leq 4.0$ s, with t in seconds. Find the reversal time, net angular displacement, total angular travel, and both angular averages. For a point 0.20 m from the axis, find the traveled distance.

Model. The radius is fixed and the angular velocity history is prescribed. A reversal requires a sign change of ω , not merely a zero of α .

Solve $3.0 - 1.5t = 0$: $t_* = 2.0$ s. The motion is positive before this instant and negative afterward. Integrate separately across t_* for total travel, and integrate the signed velocity once over the full interval for displacement.

Chapter support: Physics I, Chapter 10: Example 10.2.

Example 10.2: signed and absolute areas

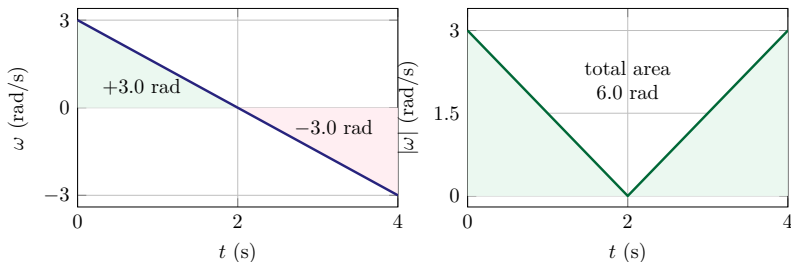


Figure 10.3: Signed and absolute areas for the return motion of Example 10.2. Identical horizontal scales make the reversal visible in both panels. Net displacement is zero; total angular travel is 6.0 rad.

$$\Delta\theta = [3t - 0.75t^2]_0^4 = 0, \quad \mathcal{A} = 3.0 + 3.0 = 6.0 \text{ rad.}$$

Thus $\bar{\omega} = 0$ and $\overline{|\omega|} = 1.5 \text{ rad/s}$. Traveled distance is $(0.20)(6.0) = 1.2 \text{ m}$. The point returns to its initial position while its traveled distance stays positive.

Chapter support: Physics I, Chapter 10: Example 10.2.

When can encoder samples be unwrapped?



Successive wrapped samples permit

$$\Delta\theta_k = \theta_{w,k+1} - \theta_{w,k} + 2\pi n_k, \quad n_k \in \mathbb{Z}.$$

An independent continuous bound $|\omega(t)| \leq \Omega_{\max}$ gives

$$|\Delta\theta_k| \leq \Omega_{\max} \Delta t < \pi.$$

Exactly one candidate then lies in $(-\pi, \pi)$. The strict inequality excludes the indistinguishable half-turns $+\pi$ and $-\pi$.

With each angular error bounded by ε , a conservative margin is $\Omega_{\max} \Delta t + 2\varepsilon < \pi$. Lost samples increase the interval. Unique net displacement still does not determine reversals between samples or the exact traveled distance.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Example 10.3: sampling and branch selection



Example 10.3. An ideal encoder samples every 0.100 s. Its readings at $t = 0, 0.100, 0.200, 0.300, 0.400$ s are 5.800, 0.516815, 1.516815, 2.516815, and 3.516815 rad in $[0, 2\pi)$. An independently established limit is $|\omega| \leq 12.0$ rad/s. Recover the net displacement and average angular velocity. Explain whether the same readings determine the answer without the limit.

Model. These are rounded synthetic ideal samples. The speed limit applies throughout each interval and comes from independent information. Here $\Omega_{\max} \Delta t = 1.20$ rad, below π . For each difference, select the unique 2π branch with magnitude at most 1.20 rad. The resulting unwrapped history must reproduce all wrapped readings after reduction modulo 2π .

Chapter support: Physics I, Chapter 10: Example 10.3.

Example 10.3: reconstruction and ambiguity



The first raw difference is $0.516815 - 5.800 = -5.283185$ rad. Adding 2π gives $+1.000000$ rad to the supplied precision. The next three differences are already $+1.000000$ rad.

$$\theta_k : \quad 5.800, 6.800, 7.800, 8.800, 9.800 \text{ rad.}$$

$$\Delta\theta = 4.000 \text{ rad}, \quad \bar{\omega} = \frac{4.000}{0.400} = 10.00 \text{ rad/s.}$$

Without the bound, constant motion at

$$\omega' = 10.00 - \frac{2\pi}{0.100} = -52.8319 \text{ rad/s}$$

gives the same ideal wrapped samples. Its sampled angles differ by $-2\pi k$. The independent limit excludes it. Modulo reduction checks the permitted record within rounding. The sufficient safe interval is strictly $\Delta t < \pi/12.0 = 0.262$ s.

Chapter support: Physics I, Chapter 10: Example 10.3.

Constant angular acceleration



For constant α on one interval, let u be elapsed time from its initial state. Integrating the definitions gives

$$\omega(u) - \omega_i = \int_0^u \alpha \, du' = \alpha u,$$

$$\theta(u) - \theta_i = \int_0^u (\omega_i + \alpha u') \, du' = \omega_i u + \frac{1}{2} \alpha u^2.$$

Eliminating u gives $\omega^2 - \omega_i^2 = 2\alpha(\theta - \theta_i)$. Squaring removes the sign of ω , so the history must select the correct root.

For piecewise motion, each segment has its own initial state and elapsed time. Match θ and ω at a switch. Do not apply one constant- α relation across different accelerations.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Example 10.4: start, coast, and brake



Example 10.4. A rotary table begins at $\theta_0 = 0$ with $\omega_0 = 2.0$ rad/s. It accelerates at $+3.0$ rad/s² for 2.0 s, coasts for 1.5 s, and then brakes at -4.0 rad/s² until it stops. Find the stopping time measured from the start and the total angular displacement.

Model. Three intervals have different prescribed angular accelerations. The final angle and velocity of each interval become the next interval's initial values.

The first interval lasts 2.0 s. The second lasts 1.5 s with zero acceleration. The third ends at the first zero of ω under negative acceleration. Add their signed angle changes and their elapsed times.

Chapter support: Physics I, Chapter 10: Example 10.4.

Example 10.4: complete segment solution



Segment	Duration (s)	Final ω (rad/s)	$\Delta\theta$ (rad)
Acceleration	2.0	$2 + 3(2) = 8.0$	$2(2) + \frac{1}{2}(3)(2)^2 = 10.0$
Coast	1.5	8.0	$8(1.5) = 12.0$
Brake	$8/4 = 2.0$	0	$8(2) - \frac{1}{2}(4)(2)^2 = 8.0$

Thus the stopping time from the start is $2.0 + 1.5 + 2.0 = 5.5$ s.

$$\Delta\theta = 10.0 + 12.0 + 8.0 = 30.0 \text{ rad} = \frac{30.0}{2\pi} = 4.77 \text{ rev.}$$

Independent area check. The velocity graph contains a trapezoid, rectangle, and triangle with areas 10, 12, and 8 rad. All velocities before stopping are positive, so total travel equals net displacement here.

Chapter support: Physics I, Chapter 10: Example 10.4.

Practice 1: positive acceleration, negative motion



Problem 2. A shaft starts at $\omega_0 = -6.0$ rad/s and has constant $\alpha = +2.0$ rad/s². Find (a) when it stops momentarily and (b) the signed angular displacement by then. Is positive acceleration evidence of positive rotation?

Solution. The fixed-axis, constant-inertia problem-set model applies. For part (a), $0 = -6.0 + 2.0t$ gives $t_s = 3.00$ s. For part (b),

$$\Delta\theta = -6.0(3.0) + \frac{1}{2}(2.0)(3.0)^2 = -9.00 \text{ rad.}$$

The signed graph area is $\frac{1}{2}(-6.0)(3.0) = -9.00$ rad. Positive acceleration increases the signed velocity toward zero while the shaft rotates negatively. If the acceleration continues, positive rotation begins after this instant.

Chapter support: Physics I, Chapter 10: Problem 2.

Linear motion of a point on a rigid rotor



For fixed perpendicular radius R and radian angle, differentiate $s = R\theta$:

$$v_t = R\omega, \quad a_t = R\alpha, \quad a_n = \frac{v_t^2}{R} = R\omega^2 \quad (\text{inward}).$$

Here v_t and a_t are signed components along the positive tangent. Speed is $R|\omega|$. The normal magnitude is nonnegative and does not reveal the rotational sense.

$$|\mathbf{a}| = \sqrt{a_t^2 + a_n^2},$$

because radial and tangential directions are perpendicular. Uniform rotation has $\alpha = 0$ and nonzero inward acceleration. At an instantaneous reversal $\omega = 0$, the normal term vanishes while tangential acceleration can remain.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Two radii share angular motion

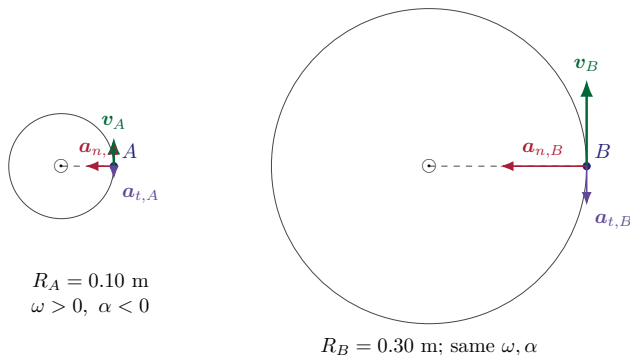


Figure 10.7: Two points of the same rotor, shown in separate views to keep their vectors distinct. The radii and the arrows within each vector family have the exact ratio 1 : 3. Velocity is upward, braking tangential acceleration downward, and normal acceleration inward. Different vector families use separate display scales. Axes point out of the page.

At fixed ω and α , every shown linear magnitude scales in proportion to R . Normal acceleration scales as $R\omega^2$, rather than $R^2\omega^2$.

Example 10.6: two rotor markers

Example 10.6. At an instant a rotor has $\omega = +4.0$ rad/s and $\alpha = -2.0$ rad/s². Compare markers at $R_A = 0.10$ m and $R_B = 0.30$ m.

Quantity	A	B
Speed $R \omega $ (m/s)	0.400	1.20
Signed $a_t = R\alpha$ (m/s ²)	-0.200	-0.600
Inward $a_n = R\omega^2$ (m/s ²)	1.60	4.80
$ \mathbf{a} = \sqrt{a_t^2 + a_n^2}$ (m/s ²)	1.61	4.84

The radius ratio is 3, checking each family. Both accelerations tilt by

$$\psi = \tan^{-1}\left(\frac{|\alpha|}{\omega^2}\right) = \tan^{-1}(0.125) = 7.13^\circ$$

from inward toward the direction opposite motion. Shared angular velocity does not imply shared linear speed.

Chapter support: Physics I, Chapter 10: Example 10.6.

Acceleration in a rotating basis



The laboratory components of the instantaneous unit vectors are

$$\hat{\mathbf{e}}_r = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}, \quad \hat{\mathbf{e}}_\theta = -\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}.$$

Differentiate their directions as well as scalar coefficients:

$$\dot{\hat{\mathbf{e}}}_r = \omega \hat{\mathbf{e}}_\theta, \quad \dot{\hat{\mathbf{e}}}_\theta = -\omega \hat{\mathbf{e}}_r.$$

For $\mathbf{r}_\perp = R\hat{\mathbf{e}}_r$ at constant R , the product rule gives

$$\mathbf{v} = R\omega \hat{\mathbf{e}}_\theta, \quad \mathbf{a} = R\alpha \hat{\mathbf{e}}_\theta - R\omega^2 \hat{\mathbf{e}}_r.$$

The second term is always inward. The result assumes a fixed inertial axis and fixed radius. Radial sliding would require additional terms.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Example 10.7: laboratory components



Example 10.7. At a particular instant, a point on a rigid rotor is at $R = 0.180$ m and $\theta = 30.0^\circ$. The rotor has $\omega = -5.00$ rad/s and $\alpha = +4.00$ rad/s². Find the point's Cartesian velocity and acceleration, the acceleration magnitude, and whether its speed is increasing.

Geometry. The positive tangent is fixed by increasing θ , even when $\omega < 0$. At 30.0° ,

$$\hat{e}_r = (0.866025, 0.500000), \quad \hat{e}_\theta = (-0.500000, 0.866025).$$

The signed tangential velocity is $R\omega = -0.900$ m/s. The acceleration coefficients are $R\alpha = +0.720$ m/s² and $-R\omega^2 = -4.50$ m/s².

Chapter support: Physics I, Chapter 10: Example 10.7.

Example 10.7: velocity, acceleration, and speed



$$\mathbf{v} = -0.900\hat{\mathbf{e}}_\theta = (0.450\hat{\mathbf{i}} - 0.779\hat{\mathbf{j}}) \text{ m/s.}$$

$$a_x = -4.50 \cos 30^\circ - 0.720 \sin 30^\circ = -4.26 \text{ m/s}^2,$$

$$a_y = -4.50 \sin 30^\circ + 0.720 \cos 30^\circ = -1.63 \text{ m/s}^2.$$

The magnitude is $\sqrt{4.50^2 + 0.720^2} = 4.56 \text{ m/s}^2$, also recovered from the unrounded Cartesian components.

Because $\omega < 0$, $d(R|\omega|)/dt = -R\alpha = -0.720 \text{ m/s}^2$: speed decreases. Independently,

$$\mathbf{v} \cdot \mathbf{a} = R^2\omega\alpha = -0.648 \text{ m}^2/\text{s}^3 < 0,$$

so the rate of change of $v^2/2$ is negative. The vector norm and dot product check the component transformation and speed interpretation.

Chapter support: Physics I, Chapter 10: Example 10.7.

Example 10.7: vector construction

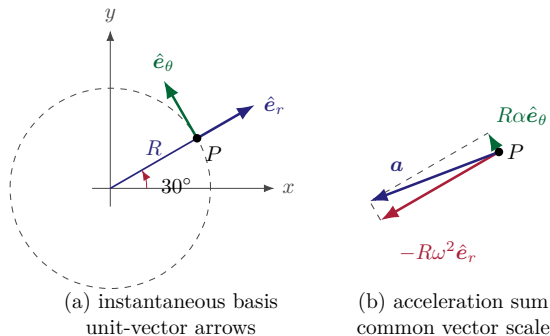


Figure 10.8: A rotating basis and the acceleration components for Example 10.7. The right panel uses 0.50 cm per m/s^2 for all acceleration arrows; the left panel has schematic length units. The inward component and positive tangential component combine to produce the laboratory acceleration. Polar point placement and angle-relative vector directions were studied in Izaak Neutelings's *Uniform circular motion*, TikZ.net; this basis/acceleration construction uses independently derived geometry.

Rotational energy defines axis inertia



Each particle in a fixed-axis rigid body has speed $v_i = r_{\perp i}|\omega|$. Sum its translational kinetic energy:

$$K_{\text{rot}} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \left(\sum_i m_i r_{\perp i}^2 \right) \omega^2.$$

$$I_z = \sum_i m_i r_{\perp i}^2, \quad K_{\text{rot}} = \frac{1}{2} I_z \omega^2.$$

Inertia has units kg m^2 and depends on the chosen axis. It is a weighted mass distribution, with large radii amplified by a square. Kinetic energy is nonnegative and unchanged when the rotation sense reverses.

For a continuous body, replace the sum by $I_z = \int r_{\perp}^2 dm$. The distance and mass element must describe the same body and axis.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Mass elements and a uniform slender rod



Use $dm = \lambda dx$, σdA , or ρdV for line, surface, or volume density. State the domain and normalize density to the specified total mass.

For a uniform slender rod along $0 \leq x \leq L$, $\lambda = M/L$. A perpendicular axis at coordinate a has distance $|x - a|$:

$$I_{z,a} = \frac{M}{L} \int_0^L (x - a)^2 dx = M \left(\frac{L^2}{3} - La + a^2 \right).$$

Completing the square gives

$$I_{z,a} = \frac{ML^2}{12} + M(a - L/2)^2.$$

Thus $I_{CM,z} = ML^2/12$ and $I_{end,z} = ML^2/3$. The center is the minimum for these parallel axes. Finite thickness contributes to transverse inertia and is neglected in this line model.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

A uniform annulus: normalization and limits



For a thin annulus about its central normal axis, all points of a narrow ring share the radius r .

$$\sigma = \frac{M}{\pi(R_o^2 - R_i^2)}, \quad dm = 2\pi r \sigma dr.$$

$$\begin{aligned} I_z &= 2\pi\sigma \int_{R_i}^{R_o} r^3 dr = \frac{\pi\sigma}{2}(R_o^4 - R_i^4) \\ &= \frac{1}{2}M(R_i^2 + R_o^2). \end{aligned}$$

Setting $R_i = 0$ gives $I_z = MR_o^2/2$. Concentrating a fixed total mass into a thin ring near R gives $I_z \rightarrow MR^2$.

At equal mass and outer radius, moving mass outward raises I_z . Comparing a solid disk and an annulus at equal mass requires different surface densities.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Solid and annular flywheel geometry

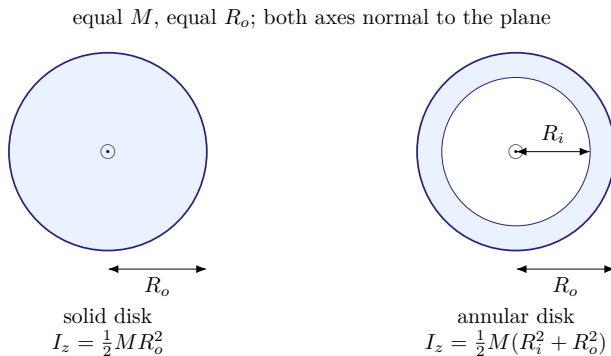


Figure 10.9: Solid and annular flywheels with equal mass and outer radius. Here $R_i/R_o = 0.75$, matching Example 10.8. Maintaining equal mass requires different surface densities. The annulus places a larger fraction of its mass farther from the normal symmetry axis.

Example 10.8: flywheel energy comparison



Example 10.8. Two ideal flywheels have mass $M = 20$ kg and outer radius $R_o = 0.40$ m. Flywheel A is a uniform solid disk. Flywheel B is a uniform annulus with inner radius $R_i = 0.30$ m. At $\omega = 30$ rad/s, compare their inertias and rotational kinetic energies.

	Solid A	Annular B
I_z (kg m ²)	$\frac{1}{2}(20)(0.40)^2 = 1.60$	$\frac{1}{2}(20)(0.30^2 + 0.40^2) = 2.50$
$K = \frac{1}{2}I_z(30)^2$ (J)	720	1125

The energy ratio is $K_B/K_A = I_B/I_A = 1.5625$. To store the same 720 J, the annulus needs

$$\omega_B = \sqrt{2(720)/2.50} = 24.0 \text{ rad/s.}$$

The mass and outer-radius bounds bracket both inertias. This compares energetic scales in an ideal rigid-body model; structural stresses and bearing loads require additional models.

Chapter support: Physics I, Chapter 10: Example 10.8.

Parallel-axis theorem



Let $\mathbf{r}'$ be the perpendicular CM-plane position of a mass element and $\mathbf{d}$ the separation of parallel axes. Expand its squared distance:

$$|\mathbf{r}' + \mathbf{d}|^2 = r'^2 + d^2 + 2\mathbf{r}' \cdot \mathbf{d}.$$

Integrating over the body gives

$$I_z = I_{\text{CM},z} + Md^2 + 2\mathbf{d} \cdot \int \mathbf{r}' dm.$$

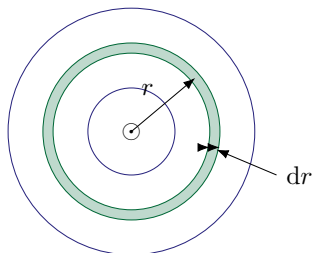
The last integral vanishes at the center of mass, so

$$I_z = I_{\text{CM},z} + Md^2.$$

The axes must be parallel, and one must pass through the CM. Shifting away from that axis increases inertia. To infer the CM inertia from an offset value, subtract Md^2 and check that the result is nonnegative.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Ring elements and parallel normal axes



complete ring element
 $dm = 2\pi r \sigma dr$

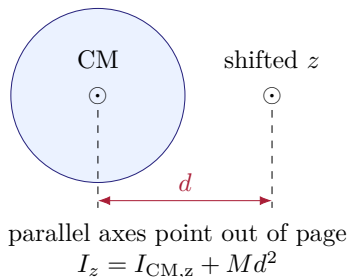


Figure 10.11: The inertia integral uses a full thin ring, not a rectangular mass patch. The right panel is a plan view of two parallel axes normal to the disk. The separation d lies in the plane perpendicular to both axes. Ring thickness is enlarged for clarity.

Example 10.9: an attached offset component



Example 10.9. A uniform disk of mass 4.0 kg and radius 0.20 m rotates about its central z axis. A smaller uniform disk of mass 1.0 kg and radius 0.050 m is rigidly attached with its center 0.25 m from that axis; its own symmetry axis is parallel to z . Find the composite I_z .

Calculate each component about the same main axis, then add:

$$\begin{aligned} I_1 &= \frac{1}{2}(4.0)(0.20)^2 = 0.0800 \text{ kg m}^2, \\ I_2 &= \frac{1}{2}(1.0)(0.050)^2 + (1.0)(0.25)^2 \\ &= 0.00125 + 0.06250 = 0.06375 \text{ kg m}^2. \end{aligned}$$

Thus $I_z = 0.14375 \approx 0.144 \text{ kg m}^2$. The shift term dominates the small disk's own central inertia. Direct expansion of the small disk's offset mass integral gives the same two terms because its CM first moment vanishes.

Chapter support: Physics I, Chapter 10: Example 10.9.

Other axes and useful inertia bounds



Axis names carry physical information. For a uniform cylinder of length H and radius R , its symmetry-axis inertia is $I_z = MR^2/2$, while a transverse CM axis has

$$I_x = M(3R^2 + H^2)/12.$$

For a thin rectangular plate with sides a along x and b along y ,

$$I_x = Mb^2/12, \quad I_y = Ma^2/12, \quad I_z = M(a^2 + b^2)/12.$$

The planar identity $I_z = I_x + I_y$ does not apply to a general thick body. Parallel-axis shifts change position, not axis direction.

The radius of gyration is $k_z = \sqrt{I_z/M}$. Nonnegative mass between $r_{\min}$ and $r_{\max}$ implies $Mr_{\min}^2 \leq I_z \leq Mr_{\max}^2$. These bounds test a model before detailed calculation.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Torque component about the chosen axis



For force $\mathbf{F}$ applied at $\mathbf{r}$ from an origin on the fixed axis,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}, \quad \tau_z = (\mathbf{r} \times \mathbf{F})_z = xF_y - yF_x.$$

Only the axial component drives the allowed rotational degree of freedom. A large axial force F_z produces no z torque, although it can produce loads and transverse torques.

For a planar force perpendicular to the axis,

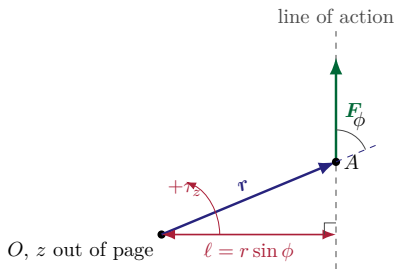
$$|\tau_z| = rF|\sin \phi| = F\ell = r|F_t|,$$

where ℓ is the perpendicular distance to the line of action. Determine the sign by its turning sense. A radial force has zero axial torque while supplying inward acceleration.

Torque uses N m. Work uses J. Equal base dimensions do not make the quantities interchangeable.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Lever arm and force line of action



planar geometry; force at A
lever arm ends on the line of action

Figure 10.17: The force lever arm is a perpendicular distance to its line of action. It need not end at the force application point. Here the upward force at the right-hand point produces a positive z torque, and its lever arm is the horizontal distance to the vertical force line. The force-arrow length is schematic. The distance r to the application point equals the lever arm only for a tangential force. Moving the origin along the same z axis leaves x and y , and hence τ_z , unchanged.

Fixed-axis dynamics from particle mechanics



For material particle i , project Newton's second law on its positive tangent. Rigidity gives $a_{t,i} = r_i\alpha$:

$$F_{t,i}^{\text{ext}} + F_{t,i}^{\text{int}} = m_i r_i \alpha.$$

Multiply by r_i and sum. Internal interaction torques cancel for the complete ordinary rigid-body model. Then

$$\sum \tau_z^{\text{ext}} = \left(\sum_i m_i r_i^2 \right) \alpha = I_z \alpha.$$

The axis is fixed in an inertial frame and the mass distribution is constant. Include bearing-resistance couples. An ideal axle force on the axis has zero axial moment; transverse constraint torques can maintain the axis direction. This is an axial scalar equation. A freely rotating asymmetric body requires angular-momentum reasoning rather than a universal identity $\tau = I_z \alpha$.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Example 10.14: applied and net axis torque



Example 10.14. Statement. A force is applied at $(x, y, z) = (0.300, 0.100, 0)$ m on a rigid body with $I_z = 0.0800 \text{ kg m}^2$. Its components are $(F_x, F_y, F_z) = (12.0, 20.0, 40.0)$ N. A separate bearing-resistance torque is -0.800 N m. Find the applied z torque, net z torque and angular acceleration, and explain why the 40.0 N component does not enter these results.

$$\tau_z = (0.300)(20.0) - (0.100)(12.0) = 4.80 \text{ N m},$$

$$\tau_{z,\text{net}} = 4.80 - 0.800 = 4.00 \text{ N m}, \quad \alpha = 4.00/0.0800 = +50.0 \text{ rad/s}^2.$$

The 40.0 N component is parallel to z and has zero axial torque. The full cross product is $(4.00, -12.0, 4.80)$ N m.

Check. In the planar projection, $F_{xy} = 23.3238$ N and $\ell = |xF_y - yF_x|/F_{xy} = 0.205798$ m, giving $F_{xy}\ell = 4.80$ N m. Its full torque magnitude does not enter the axial equation.

Chapter support: Physics I, Chapter 10: Example 10.14.

A massive pulley couples translation and rotation

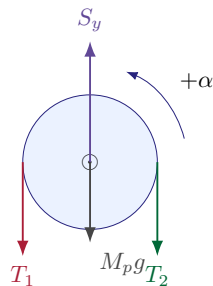
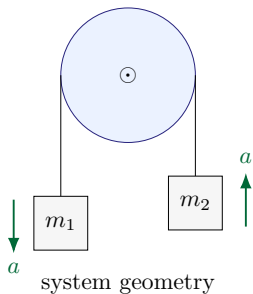


Figure 10.19: Atwood geometry and the isolated pulley. Left-side downward motion corresponds to counterclockwise rotation, so $\tau_z = (T_1 - T_2)R$. The axle support S_y and pulley weight pass through the axis and produce no z torque. Force lengths are schematic; the no-slip constraint is $a = R\alpha$ with the shown signs. No slip links the cord acceleration to the pulley: $a = R\alpha$. The two cord tensions can differ because their torque accelerates a pulley with nonzero inertia.

Example 10.15: massive-pulley Atwood machine



Example 10.15. Masses $m_1 = 4.0$ kg and $m_2 = 2.0$ kg are connected by a light inextensible cord over a uniform solid pulley of mass $M_p = 2.0$ kg and radius $R = 0.10$ m. The cord does not slip and axle friction is negligible. The system is released from rest. Find the linear acceleration, the two tensions and the pulley angular acceleration. Then independently check the speed after m_1 descends $h = 0.80$ m.

Constants and signs. Use the chapter's $g = 9.81$ m/s². Take m_1 downward, m_2 upward, and counterclockwise pulley rotation as positive. The solid pulley has $I_z = \frac{1}{2}M_p R^2 = 0.0100$ kg m². Separate free-body equations are essential: the support and pulley weight act on its axis and have zero axial moment.

Chapter support: Physics I, Chapter 10: Example 10.15.

Example 10.15: acceleration and tensions



For the two masses and the pulley,

$$m_1g - T_1 = m_1a, \quad T_2 - m_2g = m_2a,$$

$$(T_1 - T_2)R = I_z\alpha, \quad \alpha = a/R.$$

Eliminate T_1 and T_2 :

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I_z/R^2} = \frac{2(9.81)}{4 + 2 + 1} = 2.803 \text{ m/s}^2.$$

Then $T_1 = m_1(g - a) = 28.03 \text{ N}$, $T_2 = m_2(g + a) = 25.23 \text{ N}$, and $\alpha = a/R = 28.03 \text{ rad/s}^2$.

Using guard digits, $(T_1 - T_2)R = 0.280286 \text{ N m}$ equals $I_z\alpha$. Setting equal tensions would remove the torque required for pulley acceleration.

Chapter support: Physics I, Chapter 10: Example 10.15.

Example 10.15: independent energy check



After m_1 descends $h = 0.80$ m, constant-acceleration kinematics gives

$$v = \sqrt{2ah} = 2.118 \text{ m/s}, \quad \omega = v/R.$$

The gravitational energy decrease is

$$(m_1 - m_2)gh = 2(9.81)(0.80) = 15.696 \text{ J}.$$

Independently, no slip and negligible losses give

$$(m_1 - m_2)gh = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}I_z(v/R)^2.$$

With unrounded v , the translational contribution is 13.4537 J and the pulley contribution is 2.24229 J, totaling 15.696 J.

The independent energy equation yields the same speed. In the limit $I_z \rightarrow 0$, acceleration becomes $(m_1 - m_2)g/(m_1 + m_2)$ and the two tensions coincide.

Chapter support: Physics I, Chapter 10: Example 10.15.

Equilibrium and constraint forces



Under the fixed-axis model, $\sum \tau_z = 0$ gives $\alpha = 0$. The rotor can be at rest or rotating at a constant signed angular velocity. Rotational equilibrium therefore does not determine its speed.

Support forces can remain large because the material needs inward acceleration. A force at a fixed pivot produces no moment about that pivot and does no work there, yet it can change the body's momentum.

For a uniform rod released horizontally about an end pivot, choose the downward angle ϕ as positive. The gravitational moment arm changes with orientation:

$$\tau_\phi = Mg(L/2) \cos \phi, \quad \alpha_\phi = \frac{3g}{2L} \cos \phi.$$

At the vertical, the torque vanishes while the rod can still move. Constant initial torque cannot describe the entire fall.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Rotational work and instantaneous power



For a tangential force, $ds = r d\theta$ and $\tau_z = rF_t$. Thus

$$dW = F_t ds = \tau_z d\theta, \quad W = \int_{\theta_i}^{\theta_f} \tau_z(\theta) d\theta.$$

For net torque, constant I_z and a fixed axis,

$$W_{\text{net}} = \Delta K = \frac{1}{2} I_z (\omega_f^2 - \omega_i^2).$$

Differentiate with respect to time:

$$P = \tau_z \omega = I_z \omega \alpha = \dot{K}.$$

Net power is positive when the net torque adds kinetic energy and negative when it removes energy. The sign of torque alone is insufficient. Individual motor power can feed both the rotor and dissipative loads.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Angle-dependent torque and its work area

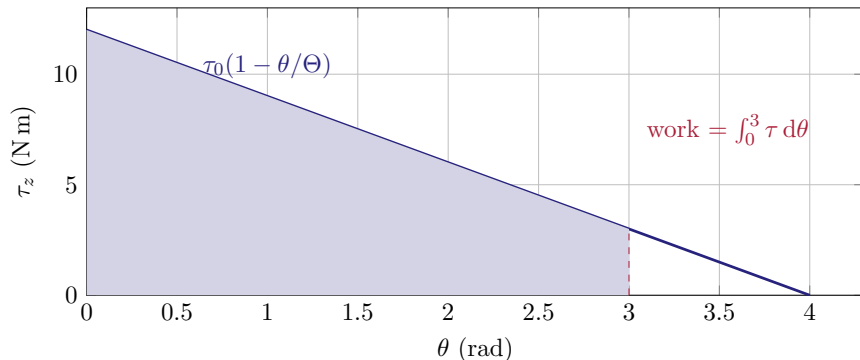


Figure 10.23: Prescribed variable torque for Example 10.18. The shaded area is rotational work, with units $\text{N m} = \text{J}$. The torque itself is still labelled in N m , not J . For motion along the specified positive-angle branch, integrate the actual torque law. An angle-dependent torque generally gives changing angular acceleration.

Example 10.18: a positioning drive



Example 10.18. A rotor with $I_z = 3.0 \text{ kg m}^2$ begins at $\theta = 0$ with $\omega_0 = 1.0 \text{ rad/s}$. Over $0 \leq \theta \leq 4.0 \text{ rad}$, a drive applies

$$\tau_z(\theta) = \tau_0 \left(1 - \frac{\theta}{\Theta}\right), \quad \tau_0 = 12 \text{ N m}, \quad \Theta = 4.0 \text{ rad}.$$

Neglect losses. Find the speed at $\theta = 3.0 \text{ rad}$ and the instantaneous power there.

Model. The quoted drive torque is the net torque because losses are neglected. Integrate from 0 to 3.0 rad, add the initial rotational energy, and then choose the positive velocity branch consistent with the motion.

Chapter support: Physics I, Chapter 10: Example 10.18.

Example 10.18: speed and shaft power



$$W = 12 \left[\theta - \frac{\theta^2}{8} \right]_0^3 = 12(3 - 9/8) = 22.5 \text{ J}.$$

Initially $K_i = \frac{1}{2}(3.0)(1.0)^2 = 1.5 \text{ J}$, so $K_f = 24.0 \text{ J}$ and

$$\omega_f = +\sqrt{2K_f/I_z} = +4.00 \text{ rad/s}.$$

At $\theta = 3.0 \text{ rad}$, $\tau_z = 12(1 - 3/4) = 3.0 \text{ N m}$, hence

$$P = \tau_z \omega_f = 12.0 \text{ W}.$$

Independent check. The torque graph is a trapezoid with sides 12 and 3 N m and width 3 rad. Its area is $\frac{1}{2}(12 + 3)(3) = 22.5 \text{ J}$. Positive torque and positive motion increase energy throughout the selected interval.

Chapter support: Physics I, Chapter 10: Example 10.18.

Practice 2: power during negative rotation



Problem 41. At an instant a rotor has $I_z = 0.50 \text{ kg m}^2$, $\omega = -8.0 \text{ rad/s}$, and net torque $+3.0 \text{ N m}$. (a) Find α and net power. (b) Decide whether speed and rotational energy are increasing. (c) If this torque is held constant until the rotor stops, find stopping time and signed displacement and check the work. State what happens if the torque continues afterward.

Model. The problem-set rotor is rigid with a fixed inertial axis and constant inertia. A prescribed positive torque can either remove or add energy, depending on ω .

Use $\alpha = \tau_z / I_z$ for the motion and $P = \tau_z \omega$ for energy transfer. Locate the first zero of velocity, retain the signed displacement, and compare $\tau_z \Delta\theta$ with the initial kinetic energy.

Chapter support: Physics I, Chapter 10: Problem 41.

Practice 2: complete solution and torque continuation



- (a) $\alpha = 3.0/0.50 = +6.00 \text{ rad/s}^2$ and $P = 3.0(-8.0) = -24.0 \text{ W}$.
(b) Negative ω and positive α decrease speed. Negative net power decreases energy. Initially $K_i = \frac{1}{2}(0.50)(8.0)^2 = 16.0 \text{ J}$.
(c) $\omega = -8 + 6t$ gives $t_s = 4/3 \text{ s}$. Then

$$\Delta\theta = -8(4/3) + \frac{1}{2}(6)(4/3)^2 = -16/3 \text{ rad},$$

$$W = 3.0(-16/3) = -16.0 \text{ J} = 0 - K_i.$$

The signed velocity-area triangle independently gives $-16/3 \text{ rad}$. If the torque continues after rest, it drives positive rotation and adds energy. Its power changes sign because the velocity changes sign.

Chapter support: Physics I, Chapter 10: Problem 41.

Constant and viscous braking laws



For positive initial speed and a constant opposing torque,

$$I_z \dot{\omega} = -\tau_c, \quad \omega = \omega_0 - \frac{\tau_c}{I_z} t, \quad t_s = \frac{I_z \omega_0}{\tau_c}.$$

The stopping angle is the triangular velocity area, $\Delta\theta_s = I_z \omega_0^2 / (2\tau_c)$. This model ends at rest. Continuing the same torque would drive reversal instead of passive braking.

For viscous resistance, $I_z \dot{\omega} = -b\omega$. Separation gives $d\omega/\omega = -(b/I_z)dt$. Integrating yields

$$\omega = \omega_0 e^{-t/t_b}, \quad t_b = I_z/b, \quad \Delta\theta = \omega_0 t_b (1 - e^{-t/t_b}).$$

Here b has units N m s. Power $-b\omega^2 \leq 0$ is passive for either sense. Exact zero speed is approached asymptotically.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Equal initial braking torque, different histories

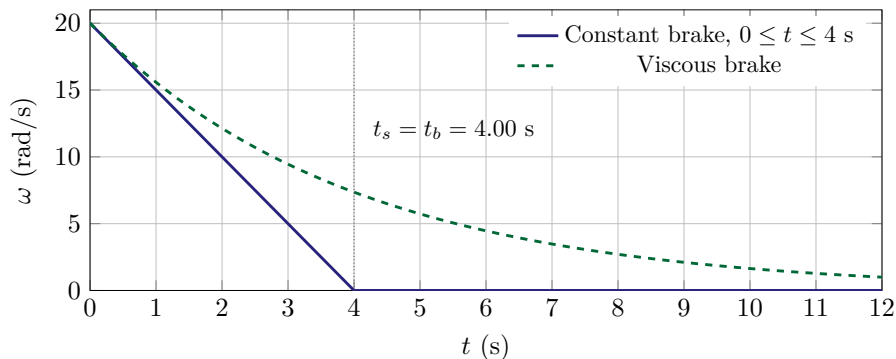


Figure 10.24: Brakes with equal initial torque can have different histories. Here $I_z = 0.0800 \text{ kg m}^2$, $\omega_0 = 20.0 \text{ rad/s}$, $\tau_c = 0.400 \text{ N m}$, and $b = 0.0200 \text{ N m s}$. The constant brake switches to a rest regime at 4 s; the dashed exponential still has positive speed.

Example 10.19: constant-brake solution



Example 10.19. Use the parameters in Figure 10.24. Find the constant-brake stopping angle and the viscous-brake speed and angle after 4.00 s. Check energy removal and find the viscous time to reach 1% of the initial speed.

Restored figure parameters. $I_z = 0.0800 \text{ kg m}^2$, $\omega_0 = 20.0 \text{ rad/s}$, $\tau_c = 0.400 \text{ N m}$, and $b = 0.0200 \text{ N m s}$.

The initial energy is $K_0 = \frac{1}{2}(0.0800)(20.0)^2 = 16.0 \text{ J}$. The constant-brake acceleration is $-0.400/0.0800 = -5.00 \text{ rad/s}^2$, giving $t_s = 4.00 \text{ s}$ and

$$\Delta\theta_s = \frac{I_z\omega_0^2}{2\tau_c} = 40.0 \text{ rad}.$$

The independent work check gives $-\tau_c\Delta\theta_s = -16.0 \text{ J}$. All initial kinetic energy is removed at the stated rest event.

Chapter support: Physics I, Chapter 10: Example 10.19.

Example 10.19: viscous solution and energy check



The viscous time scale is $t_b = I_z/b = 4.00$ s. At $t = 4.00$ s,

$$\omega = 20e^{-1} = 7.358 \text{ rad/s}, \quad \Delta\theta = 80(1 - e^{-1}) = 50.57 \text{ rad}.$$

Remaining energy is $K = 16e^{-2} = 2.165$ J. Independently integrate positive dissipated power:

$$E_{\text{loss}} = \int_0^4 b\omega^2 dt = b\omega_0^2 \frac{t_b}{2} (1 - e^{-2}) = 13.835 \text{ J}.$$

It equals $K_0 - K(4)$. For $\omega/\omega_0 = 0.01$, $e^{-t/4} = 0.01$ gives $t = 4 \ln 100 = 18.42$ s. The limiting total angle is $\omega_0 t_b = 80.0$ rad.

Equal initial torque does not fix the later history. Both models remove energy, but only the constant brake reaches its ideal rest regime at 4.00 s.

Chapter support: Physics I, Chapter 10: Example 10.19.

Engineering interpretation and drive limits



Flywheel energy scales as $I_z \omega^2 / 2$, while instantaneous shaft power is $\tau_z \omega$. A larger radius can improve an energetic scale and also change structural and bearing demands.

A constant-power operating law requires $\tau = P_0 / \omega$ for $\omega > 0$. It cannot extend to rest with bounded torque. A torque-limited startup joins it at $\omega_c = P_0 / \tau_{\max}$.

At steady speed, motor and load torques can balance while motor power remains positive. Inertia determines transient response; the torque-speed intersection determines the steady speed.

Space Technologies connection. A fixed spacecraft mechanism axis makes the same geometry and energy bookkeeping useful. A freely reorienting spacecraft requires vector angular momentum. Specify which constraint a proposed model actually has.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Inferring inertia from an accelerated hanging mass



An encoder's finite difference gives interval-average velocity. A quadratic unwrapped angle fit estimates constant acceleration. Sampling and turn counting must support that history.

For a light cord with no slip on a fixed hub of effective radius R_h , take descent as positive. Use $g = 9.81 \text{ m/s}^2$; the next apparatus rotates clockwise. Model a constant opposing loss torque τ_{loss} :

$$mg - T = mR_h\alpha, \quad TR_h - \tau_{\text{loss}} = I_z\alpha.$$

Eliminate the tension:

$$m(g - R_h\alpha)R_h = I_z\alpha + \tau_{\text{loss}}.$$

The corrected torque plotted against α has slope I_z and intercept τ_{loss} . Using mgR_h would omit accelerated tension. Test constant radius, no slip, and loss dependence on speed.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Problem 39: separate inertia and bearing loss

Problem 39. A cord on a hub of radius $R_h = 0.050$ m drives a platform as in Fig. 10.38. Constructed runs with hanging masses 0.20 kg and 0.40 kg give constant positive angular accelerations 3.0 and 7.0 rad/s^2 , respectively. Model bearing resistance as a common constant torque $\tau_{\text{loss}} > 0$ opposing positive rotation. (a) Derive two equations for platform inertia and loss torque, including acceleration of the hanging mass. (b) Determine both quantities. (c) Explain which measurement checks would test the model.

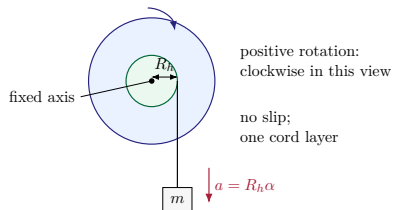


Figure 10.38: Idealized hanging-mass inertia experiment for Problem 39. The cord pulls tangentially on the hub; the hanging mass and platform must be modeled together.

Problem 39: complete inverse-model solution



(a) Combining the body equations gives $mgR_h - mR_h^2\alpha = I_z\alpha + \tau_{\text{loss}}$. For the two runs, in N m,

$$0.09660 = 3I_z + \tau_{\text{loss}}, \quad 0.18920 = 7I_z + \tau_{\text{loss}}.$$

(b) Subtraction and back-substitution give

$$I_z = \frac{0.18920 - 0.09660}{7 - 3} = 0.02315 \text{ kg m}^2,$$

$$\tau_{\text{loss}} = 0.09660 - 3(0.02315) = 0.02715 \text{ N m}.$$

Check tensions $T = m(g - R_h\alpha)$: 1.932 and 3.784 N reproduce the corrected torques after multiplication by R_h .

(c) Check quadratic unwrapped histories, stable axis and radius, no slip, repeated runs, and speed dependence of losses. Additional runs and residuals test the common-loss model. Two synthetic runs determine its parameters; they do not experimentally validate it.

Chapter support: Physics I, Chapter 10: Problem 39.

Core synthesis: one axis links the model



The angular state describes orientation. Differentiation gives velocity and acceleration. The same axis determines the mass distances in I_z and the force moments in τ_z .

$$\mathbf{v} = R\omega\hat{\mathbf{e}}_\theta, \quad \mathbf{a} = R\alpha\hat{\mathbf{e}}_\theta - R\omega^2\hat{\mathbf{e}}_r,$$

$$I_z = \int r_\perp^2 dm, \quad I_z = I_{\text{CM},z} + Md^2,$$

$$\sum \tau_z = I_z \alpha, \quad K = \frac{1}{2} I_z \omega^2, \quad W_{\text{net}} = \Delta K, \quad P_{\text{net}} = \tau_{z,\text{net}} \omega.$$

Choose the axis and signs, write the mass and force models, solve, then check dimensions, limits, and energy. A correct number cannot establish an unsupported physical assumption.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Reading route and bridge to Chapter 11



Chapter reading. Revisit the detailed graded-rod, cylinder, plate, and perforated-rotor integrations. Extend the coupled model with the driven pulley and falling rod. The motor law, time-step test, encoder uncertainty, and residual diagnosis provide further technical practice.

The advanced block now tests radial mass grading and numerical inertia, then projects the inertia tensor onto a chosen direction.

Bridge to Chapter 11. The manuscript's next step permits translation with rotation, including rolling, and develops angular momentum for more general motion. The reference point or axis and the physical constraints continue to matter.

For a second core explanation, the manuscript lists Halliday–Resnick–Walker, Serway–Jewett, and OpenStax Volume 1. Edition-specific sections were not checked during this production.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Advanced: a radially graded disk

For a thin axisymmetric disk, prescribe

$$\sigma(r) = \sigma_0[1 + \beta(r/R)^2], \quad 0 \leq r \leq R.$$

Use the same ring element for mass and inertia:

$$M = 2\pi\sigma_0 \int_0^R (r + \beta r^3/R^2) dr = \pi\sigma_0 R^2(1 + \beta/2),$$
$$I_z = 2\pi\sigma_0 \int_0^R (r^3 + \beta r^5/R^2) dr = \pi\sigma_0 R^4(1/2 + \beta/3).$$

At fixed M and R , normalization must change between profiles:

$$\sigma_0(\beta) = \frac{M}{\pi R^2(1 + \beta/2)}, \quad q(\beta) = \frac{I_z}{MR^2} = \frac{1/2 + \beta/3}{1 + \beta/2}.$$

For $\sigma_0 > 0$, nonnegative density throughout the disk requires $\beta \geq -1$.
Refinement of one numerical disk must retain its chosen density.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Advanced: density domain and normalized inertia

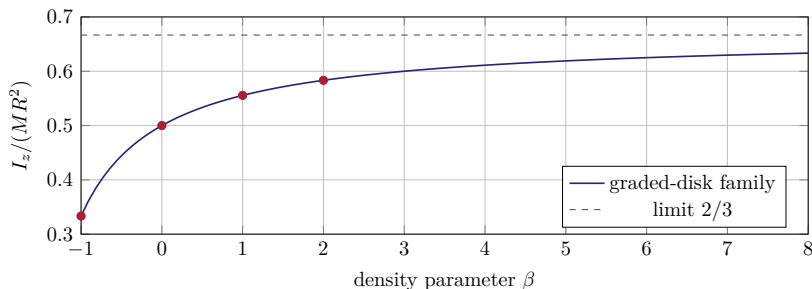


Figure 10.31: Normalized inertia for the nonnegative polynomial-density family. Total mass and radius are fixed between curves' parameter values, with the central density adjusted explicitly. The asymptote is $2/3$, because this family does not concentrate all mass at the rim.

$$q'(\beta) = \frac{1}{12(1 + \beta/2)^2} > 0 \quad (\beta \geq -1).$$

Here $q(-1) = 1/3$ and $q(0) = 1/2$. The limit $2/3$ retains interior mass.

Advanced Problem 26: two graded profiles



Problem 26. For the graded disk $\sigma(r) = \sigma_0[1 + \beta(r/R)^2]$ with the prescribed total mass kept fixed, evaluate $I_z/(MR^2)$ for (a) $\beta = -0.5$ and (b) $\beta = 2$. Explain the radial redistribution and check that both density profiles remain nonnegative.

Solution. Apply the normalized $q(\beta)$ derived on slide 61:

$$(a) \quad q(-0.5) = \frac{1/2 - 1/6}{1 - 1/4} = \frac{4}{9} = 0.444444,$$

$$(b) \quad q(2) = \frac{1/2 + 2/3}{1 + 1} = \frac{7}{12} = 0.583333.$$

Both bracket the uniform value $1/2$. Their minimum density factors are 0.5 and 1, respectively, so both are nonnegative. At fixed M , the first profile shifts relative mass inward and the second outward.

Independent check. Directly integrate each normalized density and its r^2 weight. Mass returns M and inertia returns qMR^2 . Holding σ_0 fixed would change M and represent a different comparison.

Chapter support: Physics I, Chapter 10: Problem 26.

Advanced: radial cells and a continuum benchmark



Let $h = R/N$, $r_j = (j + \frac{1}{2})h$. Midpoint density approximates both cell mass and inertia:

$$\Delta m_j \approx 2\pi r_j \sigma(r_j) h, \quad I_N = \sum_{j=0}^{N-1} r_j^2 \Delta m_j.$$

For the source benchmark $M = 6.00$ kg, $R = 0.300$ m, and $\beta = 1$, keep $\sigma_0 = M/[\pi R^2(1 + \beta/2)] = 14.1471$ kg/m² fixed. Analytic integration gives $I = 0.300000$ kg m².

An alternative uses each cell's exact mass at r_j . Correct mass still leaves a representative-radius error in inertia.

For nonnegative density and cell edges a_j, b_j , exact cell masses also provide bounds:

$$\sum_j a_j^2 \Delta m_j \leq I \leq \sum_j b_j^2 \Delta m_j.$$

Refine the grid with the physical density held fixed and report mass and inertia separately.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Advanced challenge: area-weighted disk refinement



Problem 43. Model a uniform disk by concentric annular cells with edges $r_j = jR/N$. Assign each cell its exact mass and place that mass at radial midpoint $\bar{r}_j = (j + \frac{1}{2})R/N$. (a) Derive Δm_j . (b) Prove $I_N = MR^2[\frac{1}{2} - 1/(4N^2)]$. (c) Calculate relative inertia errors for $N = 10, 20$ and contrast this convergent construction with Problem 30. Exact mass in each cell does not imply exact inertia in each cell.

Restored Problem 30 context. Its incorrect construction assigns equal masses M/N at equally spaced midpoint radii. A uniform disk instead has annular area proportional to rdr .

Strategy. Use $\sigma = M/(\pi R^2)$ and the exact cell area. Write $I_N/(MR^2)$ as a finite sum, prove the sum identity, and compare its limit and error with the equal-mass construction.

Chapter support: Physics I, Chapter 10: Problem 43.

Advanced challenge: complete finite-sum solution



(a) Exact area weights are $\Delta m_j = M[(j+1)^2 - j^2]/N^2 = M(2j+1)/N^2$. Their sum telescopes to M .

(b) Put $q_j = j + 1/2$. Then

$$\frac{I_N}{MR^2} = \frac{2}{N^4} \sum_{j=0}^{N-1} q_j^3 = \frac{1}{2} - \frac{1}{4N^2}.$$

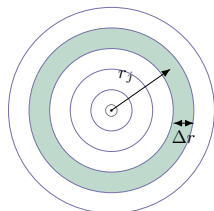
To prove the identity, expand $q_j^3 = j^3 + \frac{3}{2}j^2 + \frac{3}{4}j + \frac{1}{8}$ and substitute $\sum j = N(N-1)/2$, $\sum j^2 = N(N-1)(2N-1)/6$, and $\sum j^3 = N^2(N-1)^2/4$. This gives $\sum q_j^3 = N^4/4 - N^2/8$.

(c) Relative error is $e_N = 1/(2N^2)$: $e_{10} = 0.00500 = 0.500\%$, $e_{20} = 0.00125 = 0.125\%$. Doubling N reduces it by four.

Equal masses give $I_N/(MR^2) = 1/3 - 1/(12N^2) \rightarrow 1/3$, whereas the area-weighted model tends to $1/2$. Refinement cannot repair the wrong mass distribution.

Chapter support: Physics I, Chapter 10: Problem 43.

Advanced: graded-disk convergence in the source



complete annular cells
 $\Delta m_j = 2\pi r_j \sigma(r_j) \Delta r$

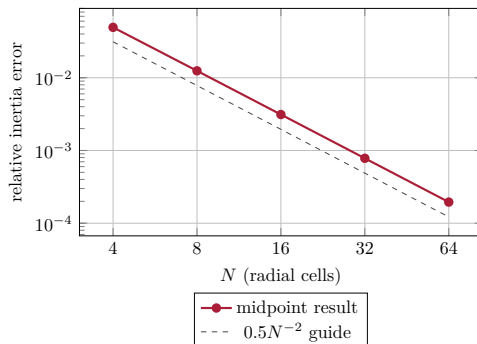


Figure 10.32: Radial midpoint approximation to the graded disk and its convergence. Each lump represents a complete annulus. Density is held fixed as the grid is refined; neither mass nor inertia is silently renormalized. Embedded errors come from the same midpoint sums as the table.

For this graded model, $e_{16} = 0.00312233$ and $e_{32} = 0.000781083$. Their unrounded ratio is 3.99744, giving $p = \log_2(e_{16}/e_{32}) = 1.99907$. This checks second-order behavior at fixed density.

Advanced: tensor geometry and scalar projection



For a rigid body about its center of mass, the inertia tensor is

$$I_{ij} = \int (r^2 \delta_{ij} - r_i r_j) dm.$$

The Kronecker delta δ_{ij} equals 1 for $i = j$ and 0 otherwise. In principal coordinates the matrix is diagonal, and

$$\mathbf{L} = I\boldsymbol{\omega} = (I_x\omega_x, I_y\omega_y, I_z\omega_z).$$

Unequal moments can make $\mathbf{L}$ and $\boldsymbol{\omega}$ nonparallel. For a unit direction $\mathbf{n}$ and constrained $\boldsymbol{\omega} = \omega\mathbf{n}$,

$$I_n = \mathbf{n}^T I \mathbf{n} = \int [r^2 - (\mathbf{r} \cdot \mathbf{n})^2] dm, \quad K = \frac{1}{2} I_n \omega^2.$$

The scalar is an axis projection of the tensor. Ideal supports can maintain a nonprincipal axis. This preview does not predict an unconstrained body's later trajectory.

Chapter support: Physics I, Chapter 10. Full source locations in the companion map.

Advanced Problem 45: constrained tensor direction



Problem 45. At the illustrated instant, a body's principal moments are $(I_x, I_y, I_z) = (1, 3, 4) \text{ kg m}^2$. Bearings constrain its rotation to the fixed unit direction $\mathbf{n} = (\hat{i} + \hat{j})/\sqrt{2}$, with signed speed $\omega = 2.0 \text{ rad/s}$. Figure 10.39 gives the principal-plane geometry. (a) Find $\mathbf{L} = I(\omega\mathbf{n})$ and $L_n = \mathbf{n} \cdot \mathbf{L}$. (b) Find $I_n = \mathbf{n}^T \mathbf{I} \mathbf{n}$ and rotational energy. (c) Explain how $\tau_n = I_n \alpha$ can be valid while $\mathbf{L}$ is not parallel to ω . Do not assume that this is free rotation without bearing reactions.

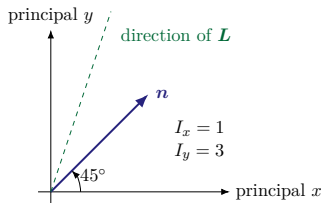


Figure 10.39: Principal-plane directions for Problem 45. The dashed direction indicates the effect of unequal principal moments; its length is not a scale for angular momentum units.

Chapter support: Physics I, Chapter 10: Problem 45.

Advanced Problem 45: momentum, energy, and supports



(a) $\boldsymbol{\omega} = 2\mathbf{n} = (\sqrt{2}, \sqrt{2}, 0)$ rad/s. Multiplication in principal coordinates gives

$$\mathbf{L} = (\sqrt{2}, 3\sqrt{2}, 0) \text{ kg m}^2/\text{s}, \quad L_n = \mathbf{n} \cdot \mathbf{L} = 4.00 \text{ kg m}^2/\text{s}.$$

(b) $I_n = (1 + 3)/2 = 2.00 \text{ kg m}^2$, so $K = \frac{1}{2}(2.00)(2.0)^2 = 4.00 \text{ J}$. Independently, $K = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L} = \frac{1}{2}(2 + 6) = 4.00 \text{ J}$.

(c) The fixed direction gives $L_n = I_n\omega$. Rigidity and the bearing constraint keep I_n constant. Projecting $\boldsymbol{\tau} = d\mathbf{L}/dt$ along fixed $\mathbf{n}$ therefore gives $\tau_n = I_n\alpha$.

The unequal component ratios show nonparallel vectors. Transverse bearing torques maintain the axis and change the transverse momentum direction. They do not replace the valid axial equation with a universal scalar vector law.

Chapter support: Physics I, Chapter 10: Problem 45.

