

Rolling, Torque, and Angular Momentum

Physics I - Chapter 11

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What makes rolling a useful bridge?



A wheel's centre translates while the wheel rotates. The contact condition links these motions, but the forces decide whether that condition can be maintained. Earlier force and energy methods predict motion. This chapter adds **reference-aware torque and angular momentum**, so we can predict spin, capture, and momentum exchange as well.

The conceptual roadmap

Compose point velocities, solve rolling forces and energy, then use vector torque to track angular momentum. Test the contact regime and the chosen origin at every step.

A reaction wheel connects these ideas to Space Technologies: internal torques change spacecraft attitude while conserving the combined angular momentum in the ideal model.

Source: Revised Chapter 11, what makes rolling a useful bridge?

Learning route and prerequisites



- ① **Describe the motion.** Separate CM translation from rotation and specify the velocity of the supporting surface.
- ② **Choose a contact model.** Solve force and torque balances, then test the friction needed for no slip.
- ③ **Track angular momentum.** Select a system and origin before applying a torque balance or conservation law.
- ④ **Extend the vector picture.** Separate orbital and spin motion, introduce the inertia tensor, and test slow precession.

Prerequisites: Newton's laws, work and kinetic energy, rigid-body inertia, cross products, differentiation, and elementary integration. We refresh the sign and origin conventions before using them.

Source: Preface

Learning goals



- Calculate velocities and accelerations of material points, including a contact point with zero instantaneous velocity.
- Derive rolling acceleration and required friction, and identify when slipping replaces no slip.
- Evaluate torque and angular momentum about a stated origin, retaining vector signs and components.
- Apply angular impulse and conservation to capture, changing inertia, and a reaction wheel, with a separate energy account.
- In the advanced block, derive the orbital–spin split, use a principal-axis tensor, and assess the assumptions behind a precession estimate.

Each calculation ends with a physical check: constraint, units, sign, conservation, or a relevant limit.

Source: [ch:11 learninggoalsbox](#)

The model, frame, and sign convention



The laboratory frame is inertial. For horizontal rolling, $+x$ points right, $+y$ up, and $+z$ out of the page. Counterclockwise ω_z and α_z are positive; clockwise values are negative.

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}.$$

Reversing the order of a cross product reverses its sign. The CM-to-point vector is $\rho_P = r_{P/CM}$; it does not point from the contact to the CM.

A model declaration

State the body or system, the origin or axis, external forces, and the contact regime. Here bodies are rigid, contact is maintained, and $g = 9.81 \text{ m/s}^2$ unless specified otherwise.

The numerical inputs and later datasets are teaching models. They are not measurements of spacecraft hardware or material limits.

Source: Conventions and notation

Rigid-body velocity composition



For a material point P , separate the body's CM position from the point's position relative to the CM:

$$\mathbf{r}_P = \mathbf{R}_{\text{CM}} + \boldsymbol{\rho}_P, \quad \dot{\mathbf{r}}_P = \boldsymbol{\omega} \times \boldsymbol{\rho}_P.$$

Differentiation gives the instantaneous velocity field,

$$\mathbf{v}_P = \mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_P.$$

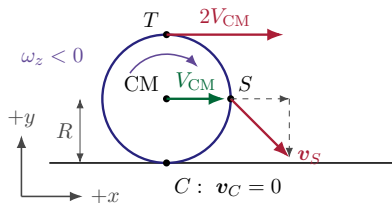
Every point shares the translational term. The rotational term is tangent to its CM-centred circle and perpendicular to both factors. At the bottom of a circular wheel, $\boldsymbol{\rho}_C = -R\hat{\mathbf{j}}$, so

$$\mathbf{v}_C = (V_{\text{CM}} + R\omega_z)\hat{\mathbf{i}}.$$

On stationary ground, no slip requires $V_{\text{CM}} + R\omega_z = 0$. A wheel moving right therefore spins clockwise. This is a kinematic constraint; it does not determine friction.

Source: Revised Chapter 11, rigid-body velocity composition

A wheel has several ground-frame speeds



$$\mathbf{v}_P = \mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_P$$

At the contact the terms cancel.

At the top the terms add.

At the side they are perpendicular.

Figure 11.1: Velocity field of a wheel rolling right without slipping on stationary ground. Red arrows represent ground-frame velocities and dashed lines resolve the side-point vector. The CM and top arrow lengths have the exact ratio 1 : 2; lengths use an arbitrary common velocity display scale.

The rim speed relative to the CM is always $R|\omega_z|$. Ground-frame velocities differ because the rotational vector changes direction around the rim.

Source: Figure 11.1

Example 11.1: three points on a rolling wheel



E11-01. A rigid wheel of radius $R = 0.30$ m rolls right on stationary ground with centre speed $V_{\text{CM}} = 1.80$ m/s and no slip. Find the angular velocity and the ground-frame velocities of its top, bottom and rightmost rim points. State their velocities relative to the CM.

Fixed ground gives $\omega_z = -1.80/0.30 = -6.00$ rad/s (clockwise).

Point	Velocity relative to CM (m/s)	Ground velocity (m/s)
Top	$+1.80\hat{i}$	$+3.60\hat{i}$
Bottom	$-1.80\hat{i}$	$\mathbf{0}$
Rightmost	$-1.80\hat{j}$	$1.80\hat{i} - 1.80\hat{j}$

The rightmost speed is $1.80\sqrt{2} = 2.55$ m/s, 45° below $+x$. All CM-relative rim speeds are $R|\omega_z| = 1.80$ m/s. As a check, distances $2R$, 0 , and $\sqrt{2}R$ from the instantaneous contact centre reproduce all three ground speeds. Contact is not a permanent pivot.

Source: Example 11.1

Acceleration follows one material point



Differentiate the velocity of the *same* material point. Both the angular rate and the CM-to-point direction can change:

$$\mathbf{a}_P = \mathbf{a}_{\text{CM}} + \boldsymbol{\alpha} \times \boldsymbol{\rho}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_P).$$

The last term is inward normal acceleration; for planar motion it is $-\omega_z^2 \boldsymbol{\rho}_P$. At the current bottom point on straight horizontal ground,

$$\mathbf{a}_C = (a + R\alpha_z)\hat{\mathbf{i}} + R\omega_z^2\hat{\mathbf{j}}.$$

No slip maintained through time gives $a + R\alpha_z = 0$, leaving

$$\mathbf{a}_C = R\omega_z^2\hat{\mathbf{j}}.$$

The wheel material leaves contact upward. Road material stays at its location, and the geometrical contact location advances. Those three histories must be distinguished.

Source: Revised Chapter 11, acceleration follows one material point

Bottom and top accelerations

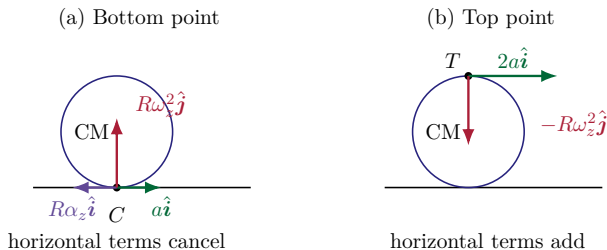


Figure 11.2: Acceleration of the material points presently at the bottom and top during accelerating rightward rolling without slip. For $a > 0$, $\alpha_z < 0$. Vector lengths are schematic; equal and opposite bottom horizontal components and twice the centre acceleration at the top are exact relations.

The bottom horizontal terms cancel; at the top they add. The inward term remains nonzero even during uniform rolling. A zero instantaneous velocity does not imply zero acceleration.

Source: Figure 11.2

Example 11.2: acceleration at contact



At the instant a wheel has $R = 0.25$ m, $V_{\text{CM}} = 2.00$ m/s and $a = 1.20$ m/s² rightward, it rolls without slipping on fixed horizontal ground. Find ω_z , α_z and the accelerations of the bottom, top and rightmost material points.

No slip gives $\omega_z = -8.00$ rad/s and $\alpha_z = -4.80$ rad/s². The inward rim acceleration is $R\omega_z^2 = 16.0$ m/s².

$$\boxed{\mathbf{a}_C = 16.0\hat{\mathbf{j}}, \quad \mathbf{a}_T = 2.40\hat{\mathbf{i}} - 16.0\hat{\mathbf{j}}} \quad \text{m/s}^2.$$

For the rightmost point, $\rho_S = R\hat{\mathbf{i}}$:

$$\boldsymbol{\alpha} \times \boldsymbol{\rho}_S = -1.20\hat{\mathbf{j}}, \quad -\omega_z^2 \boldsymbol{\rho}_S = -16.0\hat{\mathbf{i}} \quad \text{m/s}^2,$$

$$\boxed{\mathbf{a}_S = (-14.8\hat{\mathbf{i}} - 1.20\hat{\mathbf{j}}) \text{ m/s}^2.}$$

At contact, the horizontal terms $+1.20$ and -1.20 cancel exactly. At the side, inward acceleration dominates the rightward CM acceleration. Every component follows the same material-point equation.

Source: Example 11.2

The cycloid of a marked rim point



A snapshot gives a velocity field. A trajectory follows one marked point. Let it start at the bottom with the CM at $(0, R)$, and let $\phi \geq 0$ be the clockwise rotation angle.

$$x_{\text{CM}} = R\phi, \quad \rho_P = -R \sin \phi \hat{\mathbf{i}} - R \cos \phi \hat{\mathbf{j}},$$

$$x_P = R(\phi - \sin \phi), \quad y_P = R(1 - \cos \phi).$$

For constant $\Omega = \dot{\phi}$, differentiation gives

$$v_{Px} = R\Omega(1 - \cos \phi), \quad v_{Py} = R\Omega \sin \phi.$$

Both vanish at a ground cusp. At $\phi = \pi$, the point is at height $2R$ with horizontal speed $2R\Omega$. During the second half-turn $v_{Py} < 0$: the point descends while the CM keeps moving right. The cusp reflects zero instantaneous speed, with no abrupt impact in ideal smooth rolling.

Source: Revised Chapter 11, the cycloid of a marked rim point

Example 11.3: position, velocity, and time



A wheel of radius $R = 0.20$ m rolls uniformly at $V_{\text{CM}} = 1.00$ m/s. A marked rim point starts at the bottom. Find its position and velocity after a quarter-turn, the time for one full turn, and the distance that the marked point travels along its cycloid during that turn.

Uniform rolling gives $\Omega = V_{\text{CM}}/R = 5.00$ rad/s. At $\phi = \pi/2$,

$$x_P = 0.20(\pi/2 - 1) = 0.114 \text{ m}, \quad y_P = 0.200 \text{ m},$$

$$\mathbf{v}_P = (1.00\hat{\mathbf{i}} + 1.00\hat{\mathbf{j}}) \text{ m/s.}$$

A full turn takes

$$T = \frac{2\pi}{\Omega} = 1.257 \text{ s}, \quad \Delta x_{\text{CM}} = 2\pi R = 1.257 \text{ m}.$$

These are the wheel's travel time and CM displacement. The marked point's path length requires integrating its speed, which we do on the next slide.

Source: Example 11.3

Example 11.3: the marked point travels farther



The marked point rises and falls. Its path length is therefore greater than its horizontal displacement. From the two velocity components,

$$\begin{aligned} |\mathbf{v}_P| &= R\Omega\sqrt{(1 - \cos\phi)^2 + \sin^2\phi} \\ &= 2R\Omega\sin(\phi/2), \quad 0 \leq \phi \leq 2\pi. \end{aligned}$$

The sine is nonnegative on this interval. With $dt = d\phi/\Omega$,

$$\begin{aligned} s_P &= \int_0^{2\pi} 2R\sin(\phi/2) d\phi \\ &= [-4R\cos(\phi/2)]_0^{2\pi} = 8R = \boxed{1.60 \text{ m}}. \end{aligned}$$

The speed formula independently gives zero at both contacts and $2V_{\text{CM}}$ at the top. The CM displacement and wheel circumference are $2\pi R$; the cycloid arc length is $8R$. They answer different geometrical questions.

Source: Example 11.3

Rolling on a moving belt



Let a straight belt move at laboratory velocity $U\hat{i}$. No slip equates the two contacting surface velocities:

$$V_{\text{CM}} + R\omega_z = U.$$

Spin is tied to the centre's speed *relative to the belt*. If the belt moves right faster than the CM, the wheel rotates counterclockwise even though its centre moves right. For constant U , the belt frame is inertial:

$$\mathbf{v}'_P = \mathbf{v}_P - U\hat{i}, \quad \omega'_z = \omega_z.$$

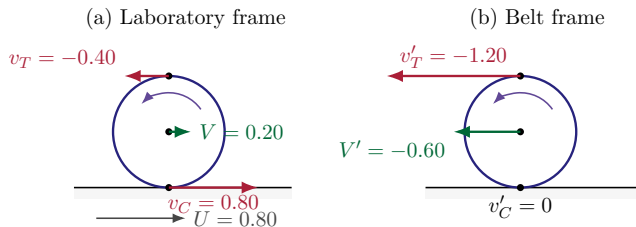
If U varies, the instantaneous velocity subtraction still applies, but the belt frame accelerates. In laboratory axes,

$$\mathbf{a} + R\alpha_z = \dot{U}, \quad \mathbf{a}_C = \dot{U}\hat{i} + R\omega_z^2\hat{j}.$$

Contact velocities agree instantaneously. The wheel and belt material points need not share the same full acceleration.

Source: Revised Chapter 11, rolling on a moving belt

One no-slip state in two inertial frames



All indicated velocity components are in m/s; $\omega_z = +3.00$ rad/s in both frames.

Figure 11.4: The same no-slip state viewed from the laboratory and the belt. The belt moves faster rightward than the centre, so the wheel rotates counterclockwise. Velocity arrows use a common scale; the wheel's drawn radius is schematic. Subtracting $U\hat{i}$ makes the bottom velocity zero in the belt frame.

Source: Figure 11.4

Example 11.4: a wheel turning against its travel



A wheel of radius $R = 0.20\text{ m}$ moves right at $V_{\text{CM}} = 0.20\text{ m/s}$ on a belt moving right at constant $U = 0.80\text{ m/s}$. It does not slip. Find its angular velocity, the laboratory top/contact velocities, and the corresponding belt-frame velocities.

The contact constraint determines the sign of spin:

$$\omega_z = \frac{U - V_{\text{CM}}}{R} = \frac{0.80 - 0.20}{0.20} = \boxed{3.00\text{ rad/s}} \quad (\text{counterclockwise}).$$

Horizontal component	Laboratory (m/s)	Belt frame (m/s)
CM	+0.20	-0.60
Contact	+0.80	0
Top	-0.40	-1.20

The belt-frame check is $V_{\text{CM}} - U + R\omega_z = 0$. The top velocity is twice the CM velocity in that frame. Spin is unchanged; at constant rates, $\mathbf{a}_C = 1.80\hat{\mathbf{j}}\text{ m/s}^2$.

Source: Example 11.4

Why kinetic energy separates at the CM



For a planar rigid body, expand the particle kinetic-energy sum:

$$K = \frac{1}{2} \sum_i m_i |\mathbf{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{\rho}_i|^2.$$

The CM identity $\sum_i m_i \boldsymbol{\rho}_i = 0$ makes the mixed term vanish:

$$K_{\text{mixed}} = \mathbf{V}_{\text{CM}} \cdot (\boldsymbol{\omega} \times \sum_i m_i \boldsymbol{\rho}_i) = 0.$$

The rotational squared terms sum to $I_{\text{CM}} \omega_z^2$. Hence

$$K = \frac{1}{2} M V_{\text{CM}}^2 + \frac{1}{2} I_{\text{CM}} \omega_z^2.$$

This split holds during slipping. Fixed-ground no slip further gives $\omega_z^2 = V_{\text{CM}}^2 / R^2$, so

$$\kappa = \frac{I_{\text{CM}}}{MR^2}, \quad K = \frac{1}{2} M (1 + \kappa) V_{\text{CM}}^2.$$

An arbitrary origin generally retains a mixed term.

Source: Revised Chapter 11, why kinetic energy separates at the cm

Mass distribution sets the energy partition

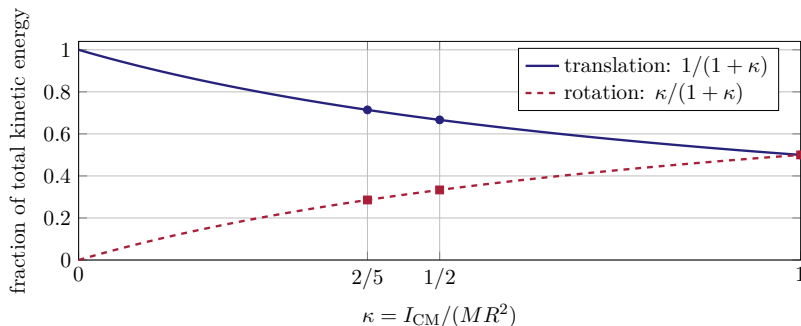


Figure 11.5: Energy partition for no-slip rolling on stationary ground. The two fractions sum to one at every inertia ratio. The marked solid-sphere, solid-disk and hoop values are quantitative; $\kappa = 0$ is an algebraic limiting case rather than a finite-mass rim-supported wheel.

A solid sphere has $\kappa = 2/5$, a solid disk $1/2$, and a thin hoop 1 . At a prescribed CM speed, larger κ requires more rotational energy.

Source: Figure 11.5

Contact power and gravitational energy



A force supplies power $\mathbf{F} \cdot \mathbf{v}_{\text{application}}$. Ideal stationary-ground contact has $\mathbf{v}_C = 0$, so static friction has zero total power on the wheel.

$$P_f = f_x V_{\text{CM}} + R f_x \omega_z = f_x (V_{\text{CM}} + R \omega_z) = 0.$$

The translational and spin contributions can be nonzero and opposite. CM force times CM displacement alone is not the actual contact work. For release from rest through vertical drop h , include body and Earth in the gravitational-energy system. With no slip and negligible losses,

$$Mgh = \frac{1}{2} M(1 + \kappa) V_{\text{CM}}^2,$$

$$V_{\text{CM}} = \sqrt{\frac{2gh}{1 + \kappa}}.$$

This predicts the final speed only after the required static friction has been shown available. A moving belt, deforming contact, or slipping contact needs a different work account.

Source: Revised Chapter 11, contact power and gravitational energy

The forces on an unpowered descending body

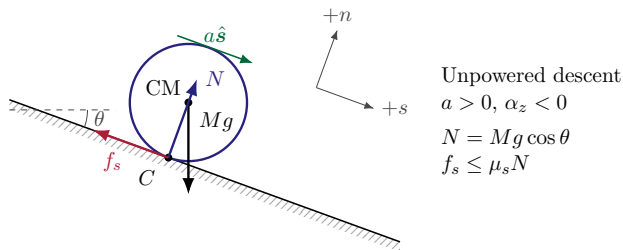


Figure 11.6: A round body descending a fixed incline to the right. Weight is vertical at the CM; the normal and uphill friction act at contact. The circle is tangent to the slope by construction. Force arrow lengths are schematic, and the green acceleration is a kinematic annotation rather than an additional force. Construction method adapted from Izaak Neutelings, TikZ.net.

Take $+s$ downhill and $+n$ away from the surface. Weight and normal have zero CM moment. Uphill friction supplies the clockwise spin torque.

Source: Figure 11.6

Incline acceleration and the friction threshold



Let f_s be the positive uphill friction magnitude for unpowered descent. Force, CM torque, and contact constraint give

$$Mg \sin \theta - f_s = Ma, \quad N = Mg \cos \theta, \quad -Rf_s = I_{\text{CM}}\alpha_z, \quad \alpha_z = -a/R.$$

Substituting the constraint gives $f_s = \kappa Ma$, hence

$$a = \frac{g \sin \theta}{1 + \kappa}, \quad f_s = \frac{\kappa}{1 + \kappa} Mg \sin \theta.$$

The force is determined by the coupled dynamics. Its available range is $0 \leq f_s \leq \mu_s N$, so no slip requires

$$\mu_s \geq \frac{\kappa}{1 + \kappa} \tan \theta.$$

Above threshold, larger μ_s does not change ideal rolling acceleration. Below it, no slip is unavailable. For $\theta \rightarrow 0$, both acceleration and required friction vanish.

Source: Revised Chapter 11, incline acceleration and the friction threshold

A friction coefficient tests a regime

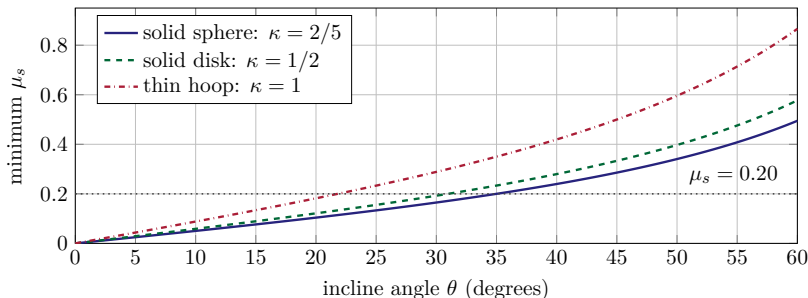


Figure 11.7: Required static-friction coefficient for unpowered no-slip descent. A specified material coefficient must lie on or above the relevant curve at the chosen angle. The horizontal line shows that $\mu_s = 0.20$ supports a larger angle for a solid sphere than for a hoop. The plot describes feasibility, not the actual friction force. Read the required coefficient at the specified angle, then compare it with the available material coefficient. The plot does not set $f_s = \mu_s N$ in ordinary rolling.

Source: Figure 11.7

Example 11.5: disk and hoop on one incline



E11-02. A uniform solid disk and a thin hoop have the same mass $M = 2.50$ kg and radius. Each is released from rest on the same 20° incline through vertical drop $h = 1.20$ m. The static coefficient is $\mu_s = 0.20$. Compare accelerations, friction, travel times and bottom speeds, and check no slip.

Assume rigid bodies, maintained contact, and negligible losses on the fixed incline. For both,

$$s = \frac{1.20}{\sin 20^\circ} = 3.509 \text{ m}, \quad N = 2.50(9.81) \cos 20^\circ = 23.05 \text{ N}.$$

For the disk, $\kappa = 1/2$:

$$a_d = \frac{9.81 \sin 20^\circ}{1.5} = 2.237 \text{ m/s}^2, \quad f_{s,d} = \frac{1}{3} Mg \sin 20^\circ = 2.80 \text{ N uphill}.$$

$$\mu_{\text{req},d} = \tan 20^\circ / 3 = 0.121 < 0.20,$$

$$t_d = \sqrt{2s/a_d} = 1.77 \text{ s}, \quad v_d = \sqrt{2a_d s} = 3.96 \text{ m/s}.$$

The disk satisfies its contact bound. The hoop and energy checks follow.

Source: Example 11.5

Example 11.5: comparison and energy check



For the hoop, $\kappa = 1$:

$$a_h = \frac{9.81 \sin 20^\circ}{2} = 1.678 \text{ m/s}^2, \quad f_{s,h} = \frac{1}{2} Mg \sin 20^\circ = 4.19 \text{ N uphill},$$

$$\mu_{\text{req},h} = \tan 20^\circ / 2 = 0.182 < 0.20, \quad t_h = 2.05 \text{ s}, \quad v_h = 3.43 \text{ m/s}.$$

Both required forces are below $\mu_s N = 4.61 \text{ N}$. The hoop lies closer to the contact limit and reaches the bottom later. Independently, the available gravitational energy is $Mgh = 29.43 \text{ J}$. The final speeds obey

$$v_d = \sqrt{\frac{2gh}{1.5}}, \quad v_h = \sqrt{\frac{2gh}{2}},$$

which reproduce the force-based results. Translation receives $2/3$ of the disk's total energy but only $1/2$ of the hoop's. If the coefficient were 0.15 , disk rolling would remain feasible while hoop rolling would fail. The energy equation alone cannot decide which contact branch applies.

Source: Example 11.5

Practice 1: an assumed rolling sphere



A uniform solid sphere rolls from rest down a 25° incline. Find its no-slip acceleration and the minimum μ_s . Is $\mu_s = 0.12$ sufficient?

Solution. For hypothetical no-slip rolling, $\kappa = 2/5$. CM torque gives $f_s = \kappa Ma$; force balance then gives $Mg \sin \theta = (1 + \kappa)Ma$:

$$a_{\text{roll}} = \frac{9.81 \sin 25^\circ}{1.4} = \boxed{2.961 \text{ m/s}^2},$$

$$\mu_{s,\text{min}} = \frac{2}{7} \tan 25^\circ = \boxed{0.1332} > 0.12.$$

The available static friction is insufficient. This acceleration is a hypothetical rolling value, not the actual acceleration on the stated surface. Translation and spin must evolve independently during slipping. A kinetic-friction law is needed; the specified μ_s alone does not determine the slipping acceleration. This feasibility failure is part of the answer.

Source: P11-11

When the rolling threshold fails



For release from rest below the static-friction threshold, the contact initially tends to slide downhill. Kinetic friction then acts uphill, with magnitude $f_k = \mu_k Mg \cos \theta$.

$$a = g(\sin \theta - \mu_k \cos \theta), \quad \alpha_z = -\frac{\mu_k g \cos \theta}{\kappa R}.$$

The two rates are independent. Define downhill contact slip $u = V_{\text{CM}} + R\omega_z$. Then

$$\dot{u} = g \left[\sin \theta - \mu_k \cos \theta \left(1 + \frac{1}{\kappa} \right) \right].$$

If $\mu_k \leq \mu_s < \kappa \tan \theta / (1 + \kappa)$, this derivative is positive, confirming continuing downhill slip from rest. Mechanical energy dissipates at rate $f_k |u|$. Keeping

$a = -R\alpha_z$ or conserving body–Earth mechanical energy after abandoning no slip would mix incompatible assumptions. Other initial spins can require the opposite friction direction and a piecewise solution.

Source: Revised Chapter 11, when the rolling threshold fails

Driving and braking require signed friction



Choose one wheel as the system. Let F_a be a signed horizontal axle force, T_z an external couple torque, and f_x signed ground friction. On fixed horizontal ground,

$$Ma = F_a + f_x, \quad I_{\text{CM}}\alpha_z = T_z + Rf_x, \quad a = -R\alpha_z.$$

Using $I_{\text{CM}} = \kappa MR^2$ and eliminating the constraint,

$$a = \frac{F_a - T_z/R}{M(1 + \kappa)}, \quad f_x = -\frac{\kappa F_a + T_z/R}{1 + \kappa}.$$

For pure clockwise motor drive, $F_a = 0$ and $T_z < 0$: both a and f_x are rightward. A counterclockwise brake gives both leftward. An axle pull with $T_z = 0$ gives rightward acceleration and leftward friction. In this isolated-wheel model, $N = Mg$ and $|f_x| \leq \mu_s Mg$. A vehicle wheel needs the actual axle forces and normal load; the isolated-wheel formulas do not automatically describe the whole vehicle.

Source: Revised Chapter 11, driving and braking require signed friction

The cause determines the friction direction

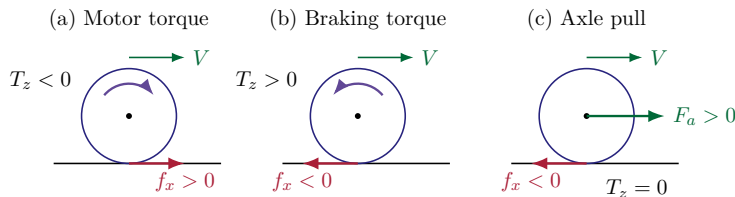


Figure 11.9: Horizontal force and torque causes can require different static-friction directions for wheels moving right without slip. Weight and the balancing normal force are omitted from this sign-comparison schematic. Panels (a) and (b) have $F_a = 0$; panel (c) has $T_z = 0$. Arrow lengths do not specify force magnitudes. Friction opposes relative slipping tendency at contact. The direction of the wheel's CM velocity alone does not determine its sign.

Source: Figure 11.9

Example 11.7: motor torque and traction



A wheel has $M = 4.0 \text{ kg}$, $R = 0.20 \text{ m}$, $\kappa = 0.50$ and $\mu_s = 0.30$. There is no horizontal axle force. First apply a motor torque $T_z = -3.0 \text{ N m}$. Then consider the same wheel rolling right at 2.0 m/s while a brake applies $T_z = +2.0 \text{ N m}$. Find acceleration, friction and angular acceleration in each case. For braking, find stopping time and distance, and check energy accounting.

Motor case. $I_{\text{CM}} = 0.50(4.0)(0.20)^2 = 0.080 \text{ kg m}^2$. With $F_a = 0$ and $T_z = -3.0 \text{ N m}$,

$$a = 2.50 \text{ m/s}^2, \quad f_x = +10.0 \text{ N}, \quad \alpha_z = -12.5 \text{ rad/s}^2.$$

The CM torque checks directly:

$$-3.0 + 0.20(10.0) = -1.0 \text{ N m} = I_{\text{CM}}\alpha_z.$$

Available friction is $\mu_s Mg = 11.77 \text{ N} > 10.0 \text{ N}$, so the no-slip solution is feasible. The motor supplies energy; the ground supplies the net horizontal force. The next slide completes the brake case in the same statement.

Source: Example 11.7

Example 11.7: stopping and brake work



Brake case. With $T_z = +2.0 \text{ N m}$, the same equations give

$$a = -1.667 \text{ m/s}^2, \quad f_x = -6.67 \text{ N}, \quad \alpha_z = +8.33 \text{ rad/s}^2.$$

The initial spin is $\omega_{z0} = -2.0/0.20 = -10.0 \text{ rad/s}$. Both required friction values lie below 11.77 N .

$$t_{\text{stop}} = \frac{2.0}{1.666\dots} = 1.20 \text{ s}, \quad s_{\text{stop}} = \frac{2.0^2}{2(1.666\dots)} = 1.20 \text{ m}.$$

The independent energy account is

$$K_0 = \frac{1}{2}M(1 + \kappa)V_0^2 = 12.0 \text{ J}, \quad \Delta\theta_z = -s_{\text{stop}}/R = -6.00 \text{ rad},$$

$$W_{\text{brake}} = T_z\Delta\theta_z = -12.0 \text{ J} = K_f - K_0.$$

The brake removes the wheel's energy. Ideal ground friction has zero total work. The specified braking process ends at rest; it does not continue into an imposed backward acceleration.

Source: Example 11.7

Sliding on level ground can approach rolling



For a freely moving wheel on fixed horizontal ground, contact slip is $u = V + R\omega_z$. With constant $\mu_k > 0$ and $u_0 \neq 0$, friction opposes its initial sign until capture:

$$f_x = -\mu_k Mg \operatorname{sgn}(u_0), \quad \dot{V} = f_x/M, \quad \dot{\omega}_z = Rf_x/I_{\text{CM}}.$$

Therefore

$$\dot{u} = -\mu_k g \left(1 + \frac{1}{\kappa}\right) \operatorname{sgn}(u_0), \quad t_r = \frac{|u_0|}{\mu_k g (1 + 1/\kappa)}.$$

If rightward travel has too little clockwise spin, $u_0 > 0$ and friction acts left. If clockwise spin is excessive, $u_0 < 0$ and friction acts right. Both branches reduce $|u|$. Stop the kinetic-friction evolution at $u = 0$. In the ideal free-wheel model, zero static friction then maintains uniform rolling. Continuing the old friction sign past capture contradicts the sliding law.

Source: Revised Chapter 11, sliding on level ground can approach rolling

The captured state and dissipated energy



Integrate the linear and angular balances to t_r :

$$V_r = V_0 - \frac{\kappa}{1 + \kappa} u_0 = \frac{V_0 - \kappa R \omega_{z0}}{1 + \kappa}, \quad \omega_{zr} = -V_r/R.$$

At the sliding contact, $P_f = f_x u = -\mu_k Mg|u|$. Its magnitude decreases linearly to zero. The triangular time integral gives

$$\begin{aligned} K_0 - K_r &= \frac{1}{2} \mu_k Mg |u_0| t_r \\ &= \boxed{\frac{1}{2} M \frac{\kappa}{1 + \kappa} u_0^2}. \end{aligned}$$

Substitution of V_r and ω_{zr} into the CM energy split gives the same loss independently. For nonzero μ_k , the ideal final state and total loss do not depend on its value; the capture time does. As $\mu_k \rightarrow 0$, $t_r \rightarrow \infty$. At exactly zero friction an initially slipping free wheel never reaches rolling in finite time. At $u_0 = 0$, it is already rolling.

Source: Revised Chapter 11, the captured state and dissipated energy

A continuous change of contact regime

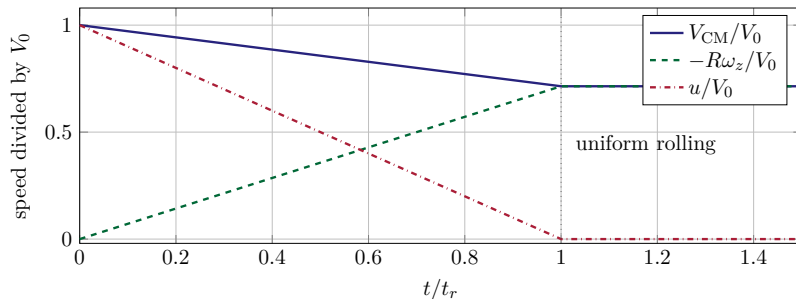


Figure 11.10: Capture into rolling for an initially nonspinning solid sphere with $\kappa = 2/5$. Kinetic friction decreases centre speed and increases clockwise spin until their associated speeds agree. The change of regime at $t = t_r$ keeps all velocities continuous; friction becomes zero after capture. The common final speed is $5V_0/7$.

Source: Figure 11.10

Example 11.8: a sliding sphere begins to roll



A uniform solid sphere with $M = 2.0 \text{ kg}$ and $R = 0.15 \text{ m}$ is launched along horizontal ground at $V_0 = 3.0 \text{ m/s}$ with no initial spin. Take $\mu_k = 0.20$ and $g = 9.81 \text{ m/s}^2$, neglect other resistance, and assume maintained contact. Find the time and centre displacement before rolling, the final speed and spin, and the kinetic-energy loss.

For a solid sphere, $\kappa = 2/5$ and $u_0 = 3.0 \text{ m/s} > 0$. Friction acts left:

$$f_x = -0.20(2.0)(9.81) = -3.924 \text{ N},$$

$$a = -1.962 \text{ m/s}^2, \quad R\alpha_z = -\frac{\mu_k g}{\kappa} = -4.905 \text{ m/s}^2.$$

The sum gives $\dot{u} = -6.867 \text{ m/s}^2$. Thus

$$t_r = \frac{3.0}{6.867} = 0.437 \text{ s}, \quad x_r = V_0 t_r + \frac{1}{2} a t_r^2 = \boxed{1.12 \text{ m}}.$$

The CM slows while clockwise spin grows. CM displacement differs from the relative slip distance used for the energy check next.

Source: Example 11.8

Example 11.8: speed, spin, and energy loss



The captured no-slip state is

$$V_r = \frac{3.0}{1.4} = 2.143 \text{ m/s}, \quad \omega_{zr} = -\frac{V_r}{0.15} = -14.29 \text{ rad/s}.$$

The constraint $V_r + R\omega_{zr} = 0$ checks the state before rounding. The endpoint kinetic energies are

$$K_0 = \frac{1}{2}(2.0)(3.0)^2 = 9.00 \text{ J}, \quad K_r = \frac{1}{2}(2.0)(1.4)V_r^2 = 6.43 \text{ J},$$

$$K_0 - K_r = 2.57 \text{ J}.$$

An independent contact-work check uses the relative slip distance,

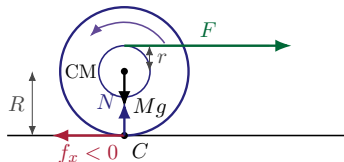
$$\int_0^{t_r} u \, dt = \frac{1}{2}u_0 t_r = 0.655 \text{ m}, \quad (3.924 \text{ N})(0.655 \text{ m}) \simeq 2.57 \text{ J}.$$

This agrees with $\frac{1}{2}M\kappa u_0^2/(1 + \kappa)$ and with the endpoint energy difference using guard digits. After capture, the ideal free sphere needs no contact friction; spin and translation continue at their matched values.

Source: Example 11.8

The string geometry fixes the spool torque

positive α_z : counterclockwise



String leaves the upper hub.

Friction shown for $q < \kappa$.

Actual rolling right has $\alpha_z < 0$.

Figure 11.11: The upper-hub pulled spool. The positive rotation arrow denotes the counterclockwise sign convention, while the actual angular acceleration for rightward rolling is clockwise. Leftward friction is shown for $q = r/R < \kappa$; it reverses across the threshold. Force arrow lengths are schematic.

The upper-hub force has CM torque $-rF\hat{\mathbf{k}}$; signed ground friction has torque $+Rf_x\hat{\mathbf{k}}$. The contact radius is R , not the hub radius r .

Source: Figure 11.11

A pulled spool can reverse its friction



Let $q = r/R$, $0 < q < 1$, and $I_{\text{CM}} = \kappa MR^2$. For an upper-hub horizontal pull $F > 0$ on stationary ground,

$$F + f_x = Ma, \quad -rF + Rf_x = I_{\text{CM}}\alpha_z, \quad \alpha_z = -a/R.$$

Divide the torque equation by R : $-qF + f_x = -\kappa Ma$. Eliminating $f_x = Ma - F$ gives

$$a = \frac{F(1+q)}{M(1+\kappa)}, \quad f_x = \frac{F(q-\kappa)}{1+\kappa}.$$

The CM accelerates right, but friction is leftward for $q < \kappa$, zero at equality, and rightward for $q > \kappa$. Without friction the contact would have horizontal acceleration $(F/M)(1 - q/\kappa)$. Its sign independently explains the friction direction. Check

$$\frac{F|q-\kappa|}{1+\kappa} \leq \mu_s Mg, \quad N = Mg > 0.$$

Source: Revised Chapter 11, a pulled spool can reverse its friction

Example 11.9: acceleration and feasible contact



E11-09. A spool has $M = 3.0 \text{ kg}$, $R = 0.20 \text{ m}$, $r = 0.080 \text{ m}$ and $I_{\text{CM}} = 0.50MR^2$. A horizontal force $F = 12 \text{ N}$ pulls the upper tangent of the hub; $\mu_s = 0.25$. Determine centre acceleration, angular acceleration and friction. Verify the rolling assumption. As an energy check, suppose it starts at rest and its centre moves 0.50 m .

Here $q = 0.40$, $\kappa = 0.50$, and $I_{\text{CM}} = 0.060 \text{ kg m}^2$. The coupled solution gives

$$a = \frac{12(1.40)}{3.0(1.50)} = 3.733 \text{ m/s}^2, \quad f_x = \frac{12(-0.10)}{1.50} = -0.800 \text{ N},$$

$$\alpha_z = -a/R = -18.67 \text{ rad/s}^2.$$

Checks: $F + f_x = 11.2 \text{ N} = Ma$ and

$$-rF + Rf_x = -0.080(12) + 0.20(-0.8) = -1.12 \text{ N m} = I_{\text{CM}}\alpha_z.$$

Rolling is feasible: $\mu_s Mg = 7.36 \text{ N} > |f_x| = 0.800 \text{ N}$.

Source: Example 11.9

Example 11.9: work through the string



As the CM travels $s = 0.50$ m, the clockwise turn has magnitude $s/R = 2.50$ rad. The string unwinds an additional length $r(s/R) = 0.20$ m.

$$\Delta x_{\text{string}} = s(1 + q) = 0.70 \text{ m}, \quad W_F = F \Delta x_{\text{string}} = \boxed{8.40 \text{ J}}.$$

For an inextensible massless string this is the force-application displacement. Constant-acceleration kinematics gives

$$V = \sqrt{2as} = 1.932 \text{ m/s}, \quad K = \frac{1}{2} M(1 + \kappa) V^2 = 8.40 \text{ J}.$$

The energy equality independently checks the $1 + q$ factor in the acceleration. Using $Fs = 6.00$ J would omit the unwinding displacement. The instantaneous form is equally useful: the upper-hub point has horizontal velocity $V - r\omega_z = (1 + q)V$ during no slip. Thus $P_F = F(1 + q)V$, while ideal fixed-ground friction supplies zero total power.

Source: Example 11.9

Vector torque about a specified origin



If force $\mathbf{F}$ acts at P , use the vector from the chosen origin O to that point:

$$\boldsymbol{\tau}_O = \mathbf{r}_{P/O} \times \mathbf{F}, \quad |\boldsymbol{\tau}_O| = rF \sin \phi = F\ell_{\perp}.$$

The lever arm $\ell_{\perp}$ is the perpendicular distance to the force's line of action. The distance OP alone is not generally the lever arm. Right-hand orientation fixes the torque vector's direction. For $\mathbf{r} = (x, y, z)$ and $\mathbf{F} = (F_x, F_y, F_z)$,

$$\tau_x = yF_z - zF_y,$$

$$\tau_y = zF_x - xF_z,$$

$$\tau_z = xF_y - yF_x.$$

Check $\boldsymbol{\tau} \cdot \mathbf{r} = \boldsymbol{\tau} \cdot \mathbf{F} = 0$. Torque is an axial vector in N m; work is scalar energy in J. Equal dimensions do not make them the same physical quantity.

Source: Revised Chapter 11, vector torque about a specified origin

Origin shifts and force couples



If $\mathbf{a}$ points from O to O' , then $\mathbf{r}_{P/O'} = \mathbf{r}_{P/O} - \mathbf{a}$. Consequently

$$\tau_{O'} = \tau_O - \mathbf{a} \times \mathbf{F}.$$

For several forces, replace $\mathbf{F}$ by their net force in the correction. Moving the origin parallel to a force leaves its moment unchanged; moving it onto that force's line of action makes its moment zero. A couple has $+\mathbf{F}$ at A and $-\mathbf{F}$ at B :

$$\mathbf{F}_{\text{net}} = 0, \quad \tau_{\text{couple}} = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}.$$

The couple moment is independent of origin because its net-force correction vanishes. It can change spin with no net translational force. A single off-centre force generally changes both. Selecting a convenient origin can eliminate a support force's moment from one equation. That force still belongs in the full force balance.

Source: Revised Chapter 11, origin shifts and force couples

Example 11.10: a three-dimensional bracket force



A force $\mathbf{F} = (30\hat{\mathbf{i}} + 20\hat{\mathbf{j}} - 10\hat{\mathbf{k}})$ N acts at $\mathbf{r}_{P/O} = (0.40\hat{\mathbf{i}} - 0.20\hat{\mathbf{j}} + 0.10\hat{\mathbf{k}})$ m. A second origin is displaced by $\mathbf{a} = (0.10\hat{\mathbf{i}} + 0.05\hat{\mathbf{j}})$ m. Find the torque about both origins and verify the shift identity.

For the specified force alone, the component calculation about O gives

$$\tau_x = 2 - 2 = 0, \quad \tau_y = 3 + 4 = 7.0, \quad \tau_z = 8 + 6 = 14.0,$$

$$\tau_O = (0\hat{\mathbf{i}} + 7.0\hat{\mathbf{j}} + 14.0\hat{\mathbf{k}}) \text{ N m.}$$

For O' , $\mathbf{r}' = (0.30\hat{\mathbf{i}} - 0.25\hat{\mathbf{j}} + 0.10\hat{\mathbf{k}})$ m gives

$$\tau_{O'} = (0.50\hat{\mathbf{i}} + 6.0\hat{\mathbf{j}} + 13.5\hat{\mathbf{k}}) \text{ N m.}$$

Independently, $\mathbf{a} \times \mathbf{F} = (-0.50\hat{\mathbf{i}} + 1.0\hat{\mathbf{j}} + 0.50\hat{\mathbf{k}})$ N m; subtracting it reproduces the shifted torque. For the original torque, $\tau_O \cdot \mathbf{r} = \tau_O \cdot \mathbf{F} = 0$, as required.

Source: Example 11.10

Angular momentum can accompany a straight line



For a constant-mass particle in a stated inertial frame,

$$\mathbf{L}_O = \mathbf{r}_{P/O} \times \mathbf{p}, \quad \mathbf{p} = m\mathbf{v}.$$

The magnitude is $rp \sin \phi = pb$, where b is the perpendicular offset to the momentum's instantaneous line. For a free particle moving along $y = b$ with $\mathbf{v} = v\hat{\mathbf{i}}$,

$$\mathbf{L}_O = (x\hat{\mathbf{i}} + b\hat{\mathbf{j}}) \times mv\hat{\mathbf{i}} = -bmv\hat{\mathbf{k}}.$$

The changing x coordinate is parallel to momentum and contributes zero. The angular momentum is constant although the distance from O changes. For a system, sum vectors about one origin using velocities in one inertial frame:

$$\mathbf{L}_O = \sum_i \mathbf{r}_{i/O} \times m_i \mathbf{v}_i, \quad \mathbf{L}_{O'} = \mathbf{L}_O - \mathbf{a} \times \mathbf{P}.$$

Here $\mathbf{P} = \sum_i m_i \mathbf{v}_i$. If $\mathbf{P} = 0$, angular momentum is independent of origin at that instant, even when it is nonzero. Its unit is $\text{kg m}^2/\text{s}$.

Source: Revised Chapter 11, angular momentum can accompany a straight line

A changing position need not change its moment

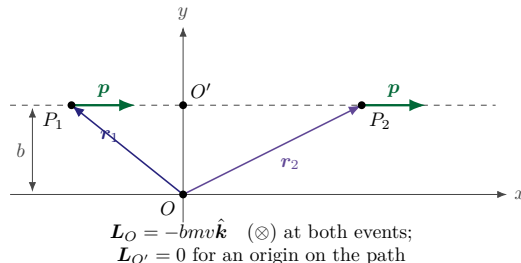


Figure 11.16: The exact two event positions of Example 11.12, drawn with a common position scale. Both have the same perpendicular offset from O and therefore the same angular momentum, despite different position vectors.

The two event positions are different, but each has the same offset b from O . About an origin on the path, the position and momentum are parallel.

Source: Figure 11.16

Example 11.12: passing a detector offset



A particle with $m = 0.150 \text{ kg}$ moves freely with $\mathbf{v} = 12.0\hat{\mathbf{i}} \text{ m/s}$ along $y = 0.40 \text{ m}$. Find its angular momentum about the laboratory origin at $x = -0.50 \text{ m}$ and $x = +0.80 \text{ m}$. Compare an origin on its path.

At both events, $\mathbf{p} = 1.80\hat{\mathbf{i}} \text{ kg m/s}$. The parallel x coordinate contributes zero; the cross product reduces to

$$(x\hat{\mathbf{i}} + 0.40\hat{\mathbf{j}}) \times 1.80\hat{\mathbf{i}} = -0.720\hat{\mathbf{k}} \text{ kg m}^2/\text{s}.$$

Thus

$$\boxed{\mathbf{L}_{O,1} = \mathbf{L}_{O,2} = -0.720\hat{\mathbf{k}} \text{ kg m}^2/\text{s} .}$$

The direction is into the page. Independently, $r_1 = 0.640 \text{ m}$ and $r_2 = 0.894 \text{ m}$, but both have $r \sin \phi = 0.40 \text{ m}$; $bp = 0.720 \text{ kg m}^2/\text{s}$ agrees. For an origin on the path, the position is parallel to momentum and $\mathbf{L}_{O'} = 0$. Both fixed-origin descriptions have zero torque and constant angular momentum. Changing the origin changes the moment, not the physical motion.

Source: Example 11.12

Torque is the rate of angular momentum



For a fixed inertial origin, differentiate a particle's moment:

$$\frac{d\mathbf{L}_O}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{r} \times \mathbf{F} = \boldsymbol{\tau}_O.$$

The first term vanishes. For a closed particle system, an internal action–reaction pair has combined moment

$$(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}.$$

It vanishes for central forces. Equal opposite noncentral forces can instead form a couple; include compensating internal spin or other degrees of freedom when needed.

$$\dot{\mathbf{L}}_O = \boldsymbol{\tau}_{O,\text{ext}} \quad \text{under those system and origin conditions.}$$

A parallel torque changes the magnitude of nonzero $\mathbf{L}$; a perpendicular torque turns its direction instantaneously. A radial force has zero torque about its centre, so circular motion can have constant angular momentum despite changing linear momentum.

Source: Revised Chapter 11, torque is the rate of angular momentum

Moving origins and the exact CM exception



Positions relative to a moving point A can be combined with inertial particle momenta. This defines a specific quantity:

$$\mathbf{L}_A^{\text{abs}} = \sum_i (\mathbf{r}_i - \mathbf{a}) \times m_i \mathbf{v}_i.$$

With $\mathbf{V}_A = \dot{\mathbf{a}}$, product-rule differentiation gives

$$\dot{\mathbf{L}}_A^{\text{abs}} = \boldsymbol{\tau}_{A,\text{ext}} - \mathbf{V}_A \times \mathbf{P}.$$

The correction is present even for constant origin velocity if it is not parallel to total momentum. At the CM, $\mathbf{P} = M\mathbf{V}_{\text{CM}}$ makes that correction zero.

Also $\sum_i m_i \boldsymbol{\rho}_i = 0$, so using CM-relative velocities gives the same spin sum:

$$\dot{\mathbf{L}}_{\text{CM}} = \boldsymbol{\tau}_{\text{CM},\text{ext}}.$$

The CM may accelerate in these inertial, nonrotating axes. The balance using relative velocities about a general moving origin is assigned reading.

Source: Revised Chapter 11, moving origins and the exact cm exception

Angular impulse and a short off-centre impulse



For a fixed inertial origin, integrate the external-torque balance:

$$\Delta \mathbf{L}_O = \int_{t_i}^{t_f} \boldsymbol{\tau}_{O,\text{ext}} dt = \mathbf{J}_{\tau,O}.$$

The signed area of a torque-component graph gives the corresponding angular-momentum change. Its units are N m s, not J. A negative torque means a negative *rate of change*, not necessarily negative spin. For a linear impulse $\mathbf{J} = \int \mathbf{F} dt$ at nearly fixed lever arm during a short event,

$$\Delta \mathbf{L}_O \simeq \mathbf{r}_{P/O} \times \mathbf{J}.$$

The exact integral retains $\mathbf{r}(t)$ inside it. For a free platform, linear impulse determines CM translation and angular impulse about the CM determines spin. For fixed principal-axis inertia, use $L_z = I_z \omega_z$. Its energy $K_{\text{rot}} = L_z^2 / (2I_z)$ allows a separate work check.

Source: Revised Chapter 11, angular impulse and a short off-centre impulse

Practice 2: a torque history with a negative tail

A rigid rotor about a fixed principal axis has $I = 2.0 \text{ kg m}^2$ and initial $L_z = 2.0 \text{ kg m}^2/\text{s}$. Its net external torque is the piecewise-linear history in Figure 11.32. Find (a) the net angular impulse from 0 to 4 s, (b) L_z at $t = 3$ and 4 s, and (c) the final kinetic energy. (d) Explain whether the negative torque reverses the spin during the plotted interval.

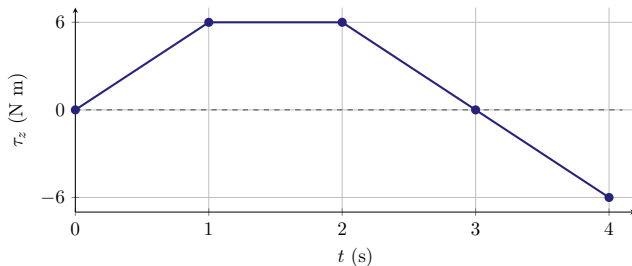


Figure 11.32: C11-P01. Stipulated net-torque history for Problem 37. Straight segments connect the displayed vertices; the angular impulse is its signed area. The graph gives torque, rather than angular momentum or power.

Source: P11-37; Figure 11.32

Practice 2: angular impulse, spin, and energy



(a) The successive signed areas over four one-second intervals are +3, +6, +3, and -3 N m s. The net angular impulse is +9 N m s.

(b) $L_z(3) = 2 + 3 + 6 + 3 = 14 \text{ kg m}^2/\text{s}$, $L_z(4) = 2 + 9 = 11 \text{ kg m}^2/\text{s}$.

$$(c) \quad \omega_f = \frac{11}{2} = 5.50 \text{ rad/s}, \quad K_f = \frac{11^2}{2(2)} = \boxed{30.25 \text{ J}}.$$

(d) The negative tail reduces L_z from 14 to 11, leaving it positive throughout. Spin does not reverse; torque specifies the slope of the angular-momentum history. The independent work check is

$$\Delta K = \int \tau_z \omega_z dt = \int_2^{11} \frac{L_z}{I_z} dL_z = \frac{11^2 - 2^2}{2(2)} = 29.25 \text{ J}.$$

Adding the initial $K_i = 1.00 \text{ J}$ recovers the final energy. Signed torque area and work involve different integrals.

Source: P11-37

Example 11.16: impulse on a free platform



A free rigid platform has $M = 20 \text{ kg}$ and $I_{\text{CM}} = 8.0 \text{ kg m}^2$ about its vertical principal axis. It starts at rest. A short impulse $\mathbf{J} = 30\hat{\mathbf{i}} \text{ N s}$ acts at $\mathbf{r}_{\text{A/CM}} = 0.25\hat{\mathbf{j}} \text{ m}$. Neglect all other impulses and geometric change during impact. Find the CM velocity and spin angular velocity.

The platform is free to translate. Linear impulse gives

$$M\mathbf{V}_{\text{CM},f} = \mathbf{J}, \quad \boxed{\mathbf{V}_{\text{CM},f} = 1.50\hat{\mathbf{i}} \text{ m/s.}}$$

The CM theorem remains valid during translation. The short-event spin impulse is

$$\Delta\mathbf{L}_{\text{CM}} = (0.25\hat{\mathbf{j}}) \times (30\hat{\mathbf{i}}) = -7.50\hat{\mathbf{k}} \text{ N m s},$$

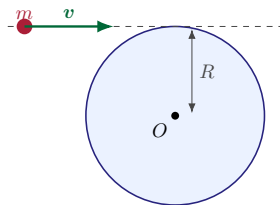
$$\boxed{\omega_{z,f} = -7.50/8.0 = -0.938 \text{ rad/s}} \quad (\text{clockwise}).$$

Endpoint energies are $K_{\text{trans}} = 22.5 \text{ J}$ and $K_{\text{spin}} = 3.52 \text{ J}$, totalling 26.0 J . A central impulse gives the same CM velocity and zero spin, checking the lever arm. The force history and detailed work transfer were not supplied.

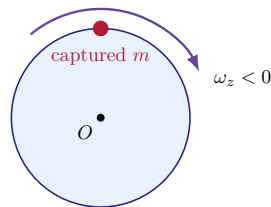
Source: Example 11.16

Capture: the axle is a useful torque origin

Choose particle plus disk as the system and the fixed axle as the origin. The axle impulse has zero axial moment. Other torque impulses must be negligible over the brief capture.



(a) incident particle; disk at rest



(b) common final clockwise rotation

Figure 11.21: An inelastic capture with a fixed frictionless vertical axle at O . The tangential incoming motion carries negative z angular momentum; the final motion has the same sign. The axle impulse can change total linear momentum while giving zero torque impulse about its own axis.

Conserve the axial angular momentum during capture. Sticking is inelastic, so evaluate the kinetic-energy loss separately.

Source: Figure 11.21

Example 11.17: tangential inelastic capture



A particle with $m = 0.20 \text{ kg}$ and speed $v = 6.0 \text{ m/s}$ strikes tangentially and sticks at the top rim of a stationary uniform disk with $M = 3.0 \text{ kg}$ and $R = 0.40 \text{ m}$. The disk has a fixed frictionless vertical axle. Use Fig. 11.21 to find the final angular speed and the kinetic-energy loss.

Take $+z$ out of the top view. Initially $L_{z,i} = -mvR = -0.480 \text{ kg m}^2/\text{s}$. The axle impulse has zero axial moment, so $L_{z,f} = L_{z,i}$:

$$I_f = MR^2/2 + mR^2 = 0.272 \text{ kg m}^2,$$
$$\omega_{z,f} = -0.480/0.272 = \boxed{-1.76 \text{ rad/s}}.$$

Initially $K_i = mv^2/2 = 3.60 \text{ J}$; finally $K_f = L_{z,i}^2/(2I_f) = 0.424 \text{ J}$. The loss is 3.18 J , or 88.2% . An independent ratio check gives $K_f/K_i = m/(M/2+m) = 0.1176$, between zero and one and tending to zero for a very massive disk. Inelastic capture transfers kinetic energy to internal energy. The axle can change linear momentum.

Source: Example 11.17

Changing inertia changes the endpoint energy



For a symmetric system on a fixed principal axis, $L_z = I(t)\omega_z$. Differentiation must retain the changing inertia:

$$\tau_z = I\alpha_z + \dot{I}\omega_z.$$

With negligible external axial torque,

$$I_i\omega_i = I_f\omega_f, \quad K_{\text{rot}} = \frac{L_z^2}{2I}.$$

Reducing inertia at fixed nonzero L_z increases spin rate and endpoint rotational energy. Internal work supplies the increase. Include radial kinetic energy during rearrangement when radial motion is present.

Conserved angular momentum does not supply energy. Constant angular momentum and constant angular speed impose different constraints.

Source: Revised Chapter 11, changing inertia changes the endpoint energy

A reaction wheel exchanges angular momentum



Use spacecraft body plus internal wheel as one ideal coaxial system. Let I_b exclude the wheel and let I_w describe it. Both rates below are measured in one inertial frame.

$$I_b\omega_b + I_w\omega_w = 0$$

for zero initial total angular momentum and negligible external torque impulse. The encoder reads the relative rate,

$$\Omega_{\text{rel}} = \omega_w - \omega_b.$$

Substitute $\omega_w = \omega_b + \Omega_{\text{rel}}$ to obtain

$$\omega_b = -\frac{I_w}{I_b + I_w}\Omega_{\text{rel}}, \quad \omega_w = \frac{I_b}{I_b + I_w}\Omega_{\text{rel}}.$$

The body counter-rotates. The sum $I_b + I_w$ appears because the encoder already contains the body's motion. An internal wheel cannot erase total angular momentum accumulated from external torque; it can redistribute it within the system.

Source: Revised Chapter 11, a reaction wheel exchanges angular momentum

Example 11.21: reaction-wheel rates and work



An ideal body has $I_b = 18.0 \text{ kg m}^2$ and its wheel has $I_w = 0.120 \text{ kg m}^2$. Initially both are at rest. The motor brings the relative wheel speed to $\Omega_{\text{rel}} = +60.0 \text{ rad/s}$. Find both inertial speeds, verify angular-momentum conservation, and calculate the minimum mechanical work for this ideal final state.

Part	Inertial rate (rad/s)	Momentum (kg m ² /s)
Body	-0.397351	-7.152318
Wheel	+59.602649	+7.152318

The rate difference is 60.0 rad/s and the momenta sum to zero. The minimum ideal mechanical work is

$$W_{\text{min}} = K_f = \frac{1}{2} \frac{I_b I_w}{I_b + I_w} \Omega_{\text{rel}}^2 = \boxed{214.570 \text{ J}}.$$

Sum $I_b \omega_b^2/2 + I_w \omega_w^2/2$ to check the final work.

Source: Example 11.21

What measurement and computation can establish



In a ramp experiment, path length s , elapsed time t , angle θ , and calibrated marker positions are direct observations. For a rest release with constant acceleration,

$$a_{\text{inferred}} = \frac{2s}{t^2}, \quad \kappa_{\text{inferred}} = \frac{g \sin \theta}{a_{\text{inferred}}} - 1.$$

A release push, biased timing, rolling resistance, or slipping can change this inference. Track both CM position and rotation: $V_{\text{CM}} = R|\omega_z|$ tests no slip more directly than arrival time alone. For computation, refine the integration step against a known answer and check the physical regime separately. Convergence cannot validate an infeasible contact constraint. **Further study in the revised chapter.** Example 11.19 propagates uncertainty in synthetic times; Example 11.20 tests Euler convergence. Position residuals assess a rest-release model. These are additional worked reading.

Source: Revised Chapter 11, what measurement and computation can establish

Core synthesis and the bridge to Chapter 12



Rolling joins translation and rotation through a contact constraint. Forces and torques determine the required friction; a bound check decides whether no slip is feasible. The force's application-point motion determines its work. Torque and angular momentum refer to specified origins. Conservation follows from external torque impulse. It can hold during inelastic capture or internally driven motion while kinetic energy changes.

Bridge to Static Equilibrium and Elasticity

Chapter 12 uses force and torque balances for extended bodies at rest, then relates force to small deformation. Choosing a moment origin can eliminate an unknown support moment, but all forces must still satisfy the full balance.

The advanced block now derives the CM orbital–spin split, explains why $\mathbf{L}$ need not be parallel to $\boldsymbol{\omega}$, and uses torque to turn a rapidly spinning rotor's momentum. These extensions retain the same system, origin, and model discipline.

Source: Revised Chapter 11, core synthesis and the bridge to chapter 12

Advanced: orbital and spin angular momentum



For a fixed particle system about inertial origin O , write $\mathbf{r}_i = \mathbf{R}_{\text{CM}} + \boldsymbol{\rho}_i$ and $\mathbf{v}_i = \mathbf{V}_{\text{CM}} + \dot{\boldsymbol{\rho}}_i$. The CM identities are

$$\sum_i m_i \boldsymbol{\rho}_i = 0, \quad \sum_i m_i \dot{\boldsymbol{\rho}}_i = 0.$$

Expand the definition into four terms:

$$\begin{aligned} \mathbf{L}_O &= \mathbf{R}_{\text{CM}} \times M \mathbf{V}_{\text{CM}} + \mathbf{R}_{\text{CM}} \times \sum_i m_i \dot{\boldsymbol{\rho}}_i \\ &\quad + \left(\sum_i m_i \boldsymbol{\rho}_i \right) \times \mathbf{V}_{\text{CM}} + \sum_i \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i. \end{aligned}$$

Both mixed terms vanish, leaving

$$\boxed{\mathbf{L}_O = \mathbf{R}_{\text{CM}/O} \times \mathbf{P} + \mathbf{L}_{\text{CM}}.}$$

This split is exact without rigidity. For a rigid body, $\dot{\boldsymbol{\rho}}_i = \boldsymbol{\omega} \times \boldsymbol{\rho}_i$; the last term is its spin angular momentum.

Source: Revised Chapter 11, orbital and spin angular momentum

Example 11.22: a translating and spinning disk

A symmetric disk has mass $M = 4.0 \text{ kg}$, CM position $\mathbf{R}_{\text{CM}} = (1.5\hat{i} + 0.50\hat{j}) \text{ m}$, and CM velocity $\mathbf{V}_{\text{CM}} = (2.0\hat{i} - 1.0\hat{j}) \text{ m/s}$. Its spin-axis inertia is $I_{\text{CM}} = 0.32 \text{ kg m}^2$ and $\omega = 5.0\hat{k} \text{ rad/s}$. Find its angular momentum about O and compare it with the CM-relative value.

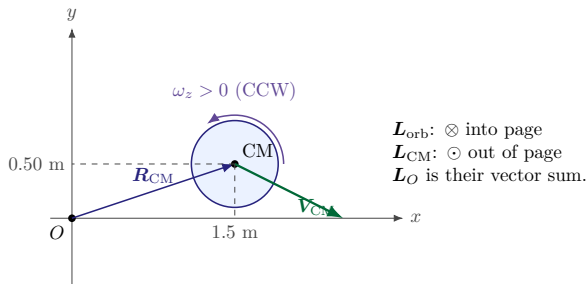


Figure 11.27: The translating disk of the next example. CM coordinates are drawn on a common scale; disk size and velocity-arrow length are schematic. Positive counterclockwise spin gives $\mathbf{L}_{\text{CM}}$ out of the page, whereas the CM position and momentum produce orbital angular momentum into it.

Source: Example 11.22; Figure 11.27

Example 11.22: orbital and spin signs



The body's linear momentum is $\mathbf{P} = (8\hat{i} - 4\hat{j}) \text{ kg m/s}$. Its CM orbital contribution is

$$\mathbf{L}_{\text{orb}} = [1.5(-4) - 0.50(8)]\hat{k} = -10.0\hat{k} \text{ kg m}^2/\text{s}.$$

The disk spins about its principal symmetry axis, so

$$\mathbf{L}_{\text{CM}} = I_{\text{CM}}\boldsymbol{\omega} = +1.60\hat{k} \text{ kg m}^2/\text{s}.$$

Adding vectors, rather than magnitudes,

$$\mathbf{L}_O = -8.40\hat{k} \text{ kg m}^2/\text{s}.$$

The orbital contribution about O dominates and has the opposite sign to CM spin. As an independent check, shift the origin to the CM:

$$\mathbf{L}_{\text{CM}} = \mathbf{L}_O - \mathbf{R}_{\text{CM}} \times \mathbf{P} = [-8.40 - (-10.0)]\hat{k} = +1.60\hat{k} \text{ kg m}^2/\text{s}.$$

The reference change removes only the orbital term; it does not change the body's motion or spin.

Source: Example 11.22

The inertia tensor is a linear map



For a rigid body relative to its CM, insert $\dot{\boldsymbol{\rho}} = \boldsymbol{\omega} \times \boldsymbol{\rho}$ into the spin sum. The triple-product identity gives

$$\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) = \rho^2 \boldsymbol{\omega} - (\boldsymbol{\rho} \cdot \boldsymbol{\omega}) \boldsymbol{\rho}.$$

Let δ_{ab} be 1 for $a = b$, otherwise 0. Then

$$\mathbf{L}_{\text{CM}} = \mathbf{I}_{\text{CM}} \boldsymbol{\omega}, \quad I_{ab} = \int (\rho^2 \delta_{ab} - \rho_a \rho_b) dm.$$

$I_{xx} = \int (y^2 + z^2) dm$ is an ordinary axial inertia; $I_{xy} = - \int xy dm$ couples directions. The tensor is real and symmetric, allowing an orthonormal principal-axis basis.

$$\mathbf{L} = (I_1 \boldsymbol{\omega}_1, I_2 \boldsymbol{\omega}_2, I_3 \boldsymbol{\omega}_3), \quad K_{\text{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}.$$

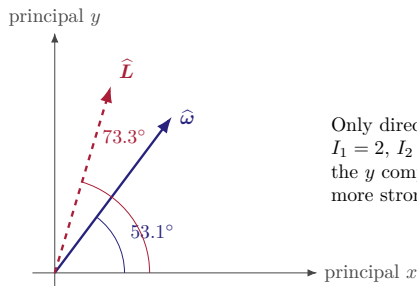
A single occupied principal direction gives parallel $\mathbf{L}$ and $\boldsymbol{\omega}$. With unequal principal inertias and several occupied directions, they generally differ in direction.

Source: Revised Chapter 11, the inertia tensor is a linear map

Example 11.24: an oblique spin



In principal-axis coordinates, take $(I_1, I_2, I_3) = (2.0, 5.0, 6.0) \text{ kg m}^2$ and $\boldsymbol{\omega} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \text{ rad/s}$. Find $\mathbf{L}$, its angular separation from $\boldsymbol{\omega}$, and the rotational energy.



Only directions are compared.
 $I_1 = 2, I_2 = 5$:
the y component is weighted
more strongly in $\mathbf{L}$.

Figure 11.29: Principal-axis inertia changes the direction as well as the magnitude of an angular-velocity vector. Both vectors have been normalized separately to show their directions on the same geometric scale; their different physical units prohibit a direct magnitude comparison.

Source: Example 11.24; Figure 11.29

Example 11.24: direction and energy checks



Principal-axis matrix multiplication gives

$$\mathbf{L} = (2 \cdot 3\hat{\mathbf{i}} + 5 \cdot 4\hat{\mathbf{j}}) = \boxed{(6\hat{\mathbf{i}} + 20\hat{\mathbf{j}}) \text{ kg m}^2/\text{s}}.$$

The first-quadrant vector directions are

$$\theta_{\omega} = \arctan(4/3) = 53.1301^{\circ}, \quad \theta_L = \arctan(20/6) = 73.3008^{\circ},$$

so their separation is $\phi = 20.1707^{\circ}$. Independently,

$$\cos \phi = \frac{\boldsymbol{\omega} \cdot \mathbf{L}}{|\boldsymbol{\omega}| |\mathbf{L}|} = \frac{98}{5\sqrt{436}}.$$

The energy is

$$K_{\text{rot}} = \frac{1}{2}[2(3^2) + 5(4^2)] = \boxed{49 \text{ J}} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}.$$

Both checks agree. These are instantaneous results; principal axes can rotate with the body, so the displayed components cannot simply be held constant in inertial axes to predict torque-free evolution.

Source: Example 11.24

Gyroscopic torque changes a vector direction

A symmetric rotor is supported at fixed O . Its CM is $d\hat{s}$ from the support and its axis is tilted by θ from upward vertical:

$$\tau_O = d\hat{s} \times (-Mg\hat{k}), \quad |\tau_O| = Mgd \sin \theta.$$

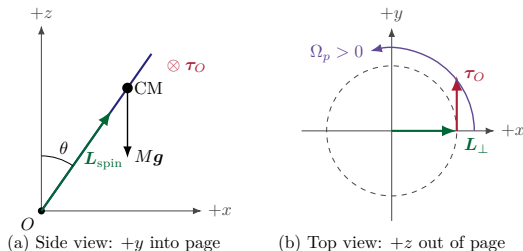


Figure 11.30: Consistent side and top views of slow precession at the displayed instant. In the side view, torque is perpendicular to the x – z plane, into the page. In the top view, it is tangent to the circle swept by the horizontal angular-momentum component. Positive precession is counterclockwise about $+z$. Arrow lengths are schematic.

Source: Figure 11.30

Slow precession and its approximation tests



For spin-dominated, approximately steady precession, $\mathbf{L}_O \simeq L_{\text{spin}} \hat{\mathbf{s}}$ with $L_{\text{spin}} = I_s \omega_s$. A vector turning about vertical obeys

$$\dot{\mathbf{L}}_O \simeq \Omega_p \hat{\mathbf{k}} \times \mathbf{L}_{\text{spin}}, \quad |\dot{\mathbf{L}}_O| \simeq \Omega_p L_{\text{spin}} \sin \theta.$$

Equate this to gravitational torque, for nonzero tilt:

$$Mgd \sin \theta = \Omega_p L_{\text{spin}} \sin \theta, \quad \boxed{\Omega_p \simeq \frac{Mgd}{I_s \omega_s}}.$$

The projection factors cancel. At vertical alignment, $0 = 0$ cannot determine an observable precession rate.

$$\eta = \left| \frac{\Omega_p}{\omega_s} \right|, \quad \epsilon = \frac{I_{\perp} |\Omega_p|}{I_s |\omega_s|} \ll 1.$$

$I_{\perp}$ denotes transverse inertia about the support. Small η tests rate separation; spin dominance needs small ϵ . Negligible nutation and steady release remain additional assumptions.

Source: Revised Chapter 11, slow precession and its approximation tests

Advanced challenge: the supplied rotor



A rotor has $M = 2.0$ kg, CM offset $d = 0.080$ m, spin-axis inertia $I_s = 0.015$ kg m², and $\omega_s = 200$ rad/s. Its axis makes $\theta = 35^\circ$ with the vertical during approximately steady precession. Estimate the rate and state exactly what can be checked from these inputs. Use $g = 9.81$ m/s².

Under the stated steady, spin-dominated model,

$$L_{\text{spin}} = I_s \omega_s = (0.015)(200) = 3.00 \text{ kg m}^2/\text{s},$$

$$\tau = Mgd \sin 35^\circ = 0.900286 \text{ N m}.$$

The vector turning relation then gives

$$\Omega_p = \frac{Mgd}{L_{\text{spin}}} = 0.5232 \text{ rad/s}.$$

The next slide checks this rate and distinguishes the approximation tests supported by the inputs from those that remain conditional.

Source: Example 11.25

Advanced challenge: verification and model limits



The estimated rate gives

$$f_p = \Omega_p / (2\pi) = 0.08327 \text{ Hz}, \quad T_p = 2\pi / \Omega_p = 12.01 \text{ s}.$$

The vector-rate check reproduces the torque:

$$\Omega_p L_{\text{spin}} \sin 35^\circ = 0.900286 \text{ N m}.$$

Using $\tau / L_{\text{spin}} = 0.300095 \text{ s}^{-1}$ would omit the projection factor and give the wrong rate.

$$\eta = \Omega_p / \omega_s = 0.002616, \quad \epsilon = (I_\perp / I_s)(0.002616).$$

The inputs establish a slow rate relative to spin. Without $I_\perp$, full spin-dominance verification is conditional on the inertia ratio. These inputs also cannot establish absence of nutation or startup behaviour. The estimate is consistent with its stated approximation. It cannot be extrapolated toward zero spin, where the dominant-spin model fails.

Source: Example 11.25

