

# Static Equilibrium and Elasticity

## Physics I - Chapter 12

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# Why balance and deformation belong together



An instrument mount can resist translation while a force couple still rotates it. A support can satisfy every algebraic balance equation while requiring a contact to pull. A balanced rod can also stretch or shorten.

## The physical route

Force and moment balance identify possible supported states. Contact constraints test whether those states can exist. Stress, strain and constitutive response then predict deformation and help determine redundant reactions.

For Space Technologies, the same reasoning supports laboratory instrument platforms, payload fixtures and elastic supports. Each calculation here uses the stated laboratory load model.

**Chapter support:** Revised Chapter 12: why balance and deformation belong together

# Learning route and prerequisites



**Balance:** isolate one body, resolve forces, and take signed moments. Then replace distributed loads and check the permitted contact reactions. **Response:** use stress and strain to separate material behaviour from specimen geometry. Extend axial response to shear, bulk compression and stored energy. **Inference:** fit force–extension data and assess uncertainty and apparatus compliance. Advanced work adds rigid-bar compatibility and stability.

## Tools to refresh

Vector components and cross products; Newton's laws; static friction; weighted means; elementary integrals, derivatives and simultaneous equations. Small strain and small rotation are explicit approximations.

**Chapter support:** Revised Chapter 12: learning route and prerequisites

# Learning goals



- 1 Isolate a body and solve force and moment balance, including a second-origin check.
- 2 Locate load resultants and assess sliding, tipping and unilateral contact.
- 3 Calculate axial, shear and bulk response with consistent signs and units.
- 4 Distinguish elastic recovery, proportionality and residual deformation; obtain energy from the loading path.
- 5 Infer stiffness and modulus from data, with uncertainty and apparatus effects.
- 6 In the advanced block, combine equilibrium, compatibility and stiffness, and distinguish equilibrium from stability.

**Chapter support:** Revised Chapter 12: learning goals

# Static equilibrium requires two balances



A body in static equilibrium is at rest in the chosen inertial frame. Its external forces and external torques, including applied couples, balance:

$$\sum_i \mathbf{F}_i = \mathbf{0}, \quad \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_j \mathbf{C}_j = \mathbf{0}.$$

Here  $\mathbf{r}_i$  runs from the chosen moment origin to the force application point. A couple  $\mathbf{C}_j$  has units of N m. **Planar convention:** +x right, +y up, and positive moments counterclockwise. The independent scalar balances are  $\sum F_x = 0$ ,  $\sum F_y = 0$ , and  $\sum \tau_z = 0$ . Zero resultant force alone allows rotation. Zero force and torque also allow uniform motion if the body was already moving. Rest is part of the static state.

**Chapter support:** Revised Chapter 12: static equilibrium requires two balances

# The chosen body determines its force diagram

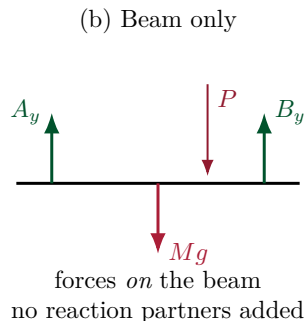
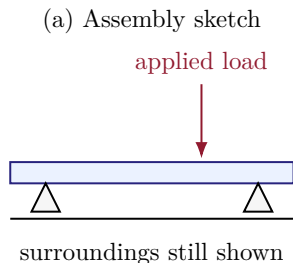


**Figure 12.1:** An assembly sketch and the free-body diagram of the isolated beam. The weight and support reactions are external to the beam system. Force lengths are schematic; the pin's horizontal component is omitted here only because this introductory loading has no horizontal action.

Action–reaction partners belong to different bodies. Internal interactions cancel only when both parts lie inside the system boundary.

**Chapter support:** Figure 12.1

# Moment arms and force couples



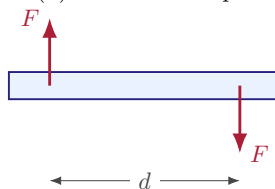
For a force applied at  $(x, y)$  relative to  $O$ ,  $\tau_{z,O} = xF_y - yF_x$ . Its lever arm is perpendicular to its line of action.

(a) Collinear



$$\sum \mathbf{F} = 0, \quad \tau_z = 0$$

(b) Clockwise couple



$$\sum \mathbf{F} = 0, \quad \tau_z = -Fd$$

**Figure 12.2:** Equal and opposite forces need not balance moments. In panel (b), each force gives a clockwise moment about the centre. The distance  $d$  measures the separation of their vertical lines of action.

$$\mathbf{C} = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}, \quad |\mathbf{C}| = Fd.$$

Chapter support: Figure 12.2

# Changing the moment origin



Let  $O'$  be displaced from  $O$  by  $\mathbf{a}$ . The same application point has position  $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{a}$ .

$$\begin{aligned}\tau_{O'} &= \sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i + \sum_j \mathbf{C}_j \\ &= \tau_O - \mathbf{a} \times \sum_i \mathbf{F}_i.\end{aligned}$$

When force balance holds, the total moment is independent of origin. Choose a support as origin to remove its force from the moment equation, then check another origin.

## An answered diagnostic

Choosing an origin on a force line can make that force's moment zero. It does not remove the force or establish equilibrium. The complete force sum must still vanish.

**Chapter support:** Revised Chapter 12: changing the moment origin

# Weight, centre of mass and centre of gravity



For point masses with total mass  $M$ , the centre of mass is a mass-weighted position:

$$\mathbf{r}_{\text{cm}} = \frac{\sum_i m_i \mathbf{r}_i}{M}.$$

If the same gravitational vector  $\mathbf{g}$  acts throughout the body,

$$\sum_i \mathbf{r}_i \times m_i \mathbf{g} = M \mathbf{r}_{\text{cm}} \times \mathbf{g}.$$

One weight  $M\mathbf{g}$  at the mass centre therefore preserves external force and torque. For a continuous body use  $M^{-1} \int \mathbf{r} dm$ . The mass distribution determines the centre of mass. Coincidence with the centre of gravity additionally requires the uniform-field approximation. A gravity gradient across an extended spacecraft requires distributed gravitational loading.

**Chapter support:** Revised Chapter 12: weight, centre of mass and centre of gravity

## Example 12.3: composite instrument platform



A rigid platform of negligible mass carries point-like components: 12.0 kg at  $(x, y) = (-0.250, 0.250)$  m, 20.0 kg at  $(0.180, 0.350)$  m and 8.00 kg at  $(0.300, 0.200)$  m. Its horizontal support interval is  $-0.300 \leq x \leq 0.300$  m. Find the initial centre of mass and the greatest added payload at  $x = 0.600$  m consistent with the gravity-line criterion. Assume the platform and attachment remain rigid.

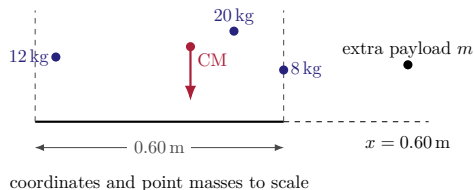


**Figure 12.4:** The three stipulated component masses determine the mass centre of a rigid platform. The support interval extends from  $-0.30$  to  $+0.30$  m. The overhanging payload changes the centre of mass; it is assumed attached rigidly, with its height irrelevant to the horizontal-platform threshold.

**Chapter support:** Example 12.3

## Example 12.3: mass centre and payload limit



The initial mass is  $M = 40.0$  kg. Use the weighted mean for each coordinate:

$$x_{\text{cm}} = \frac{12(-.250) + 20(.180) + 8(.300)}{40.0} = .0750 \text{ m},$$

$$y_{\text{cm}} = \frac{12(.250) + 20(.350) + 8(.200)}{40.0} = .290 \text{ m}.$$

Under gravity alone, the vertical weight line lies inside the horizontal support interval. An added payload  $m$  reaches the right-edge threshold when

$$\frac{3.00 + .600m}{40.0 + m} = .300 \implies m = 30.0 \text{ kg}.$$

**Independent moment check:** about the right edge, the initial restoring mass moment is  $12(.550) + 20(.120) + 8(0) = 9.00$  kg m. The payload gives  $30(.300) = 9.00$  kg m in the opposite sense. The common  $g$  cancels. This is an ideal gravity-only threshold, with no perturbation margin. Payload height would matter under tilt or horizontal acceleration.

**Chapter support:** Example 12.3

# A contact must balance force and moment



A nonadhesive base supplies nonnegative normal pressure. Its normal resultant must lie within the support region. On an incline use  $s$  uphill and  $n$  outward normal.

$$N = Mg \cos \theta, \quad f = Mg \sin \theta.$$

For a centred mass at normal height  $h$ , let the normal resultant act at base coordinate  $q$ . Moments about the base centre give

$$qN + hMg \sin \theta = 0 \implies q = -h \tan \theta.$$

For support width  $b$ , contact and friction require

$$|q| \leq b/2, \quad |f| \leq \mu_s N.$$

Thus  $\tan \theta_{\text{tip}} = b/(2h)$  and  $\tan \theta_{\text{slide}} = \mu_s$ . These are separate limits of the rigid, slowly tilted, unrestrained model.

**Chapter support:** Revised Chapter 12: a contact must balance force and moment

# The gravity line reaches the downhill edge

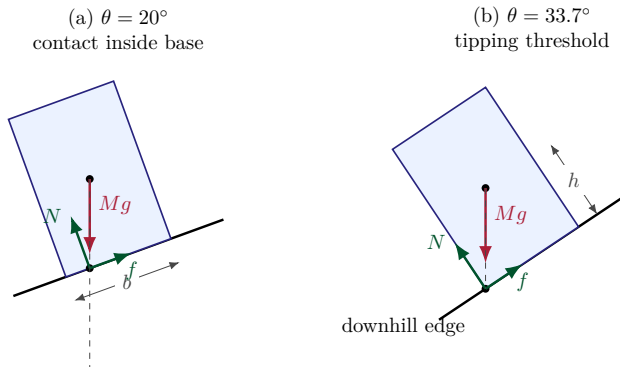


**Figure 12.5:** A housing tilts *with* its inclined platform. The weight arrows remain vertical in world coordinates. Geometry uses  $b : h = 0.60 : 0.45$ ; the rectangle height is drawn as  $2h$ . At the threshold the vertical mass-centre line meets the downhill edge. Reaction arrows are schematic, with tangential friction acting uphill.

Chapter support: Figure 12.5

## Example 12.4: tipping or sliding first?



A rectangular instrument housing has support width  $b = 0.60\text{ m}$  and a centred mass at height  $h = 0.45\text{ m}$  above its base. The whole platform is slowly tilted through angle  $\theta$ . The stipulated coefficient of static friction is  $\mu_s = 0.80$ . Which ideal limit occurs first, tipping or sliding? Check the friction requirement at the first limit.

Isolate the housing. The platform provides only contact forces. Compare the two independent thresholds:

$$\theta_{\text{tip}} = \arctan \frac{.60}{2(.45)} = 33.7^\circ, \quad \theta_{\text{slide}} = \arctan(.80) = 38.7^\circ.$$

Tipping occurs first. At  $33.7^\circ$ , the required ratio is  $f/N = \tan \theta = 2/3 = .667 < .80$ , so friction remains feasible. **Independent geometry check:** the mass centre's horizontal separation from the downhill edge is  $(b/2) \cos \theta - h \sin \theta$ . It vanishes at the same tipping angle. Both moments contain  $Mg$ , so this ideal threshold is independent of total mass.

Chapter support: Example 12.4

# A horizontal push changes the load model



On a horizontal surface, let weight  $W = Mg$  act through the base centre and a rightward push  $P$  act at height  $z_P$ . The contact supplies  $N = W$  and leftward friction  $P$ .

$$qW - Pz_P = 0 \quad \Rightarrow \quad q = \frac{Pz_P}{W}.$$

The right-edge and friction limits are therefore

$$P_{\text{tip}} = \frac{Wb}{2z_P}, \quad P_{\text{slide}} = \mu_s W.$$

The push height is the relevant lever arm. It need not equal the mass-centre height. The crossover is  $z_P = b/(2\mu_s)$ . As the push approaches the base, its overturning moment tends to zero and the tipping threshold increases. The sliding threshold remains finite. Re-derive the balance whenever the loading path changes.

**Chapter support:** Revised Chapter 12: a horizontal push changes the load model

# Support constraints and distributed-load models

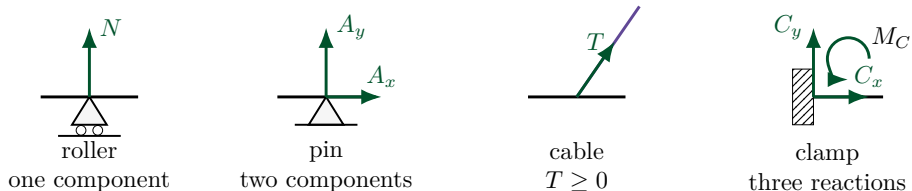


**Figure 12.7:** Ideal planar support models. The arrows show positive unknown directions, not guarantees of the final signs. A normal contact cannot pull, a cable cannot push, and a pin or clamp may exert either sign of force if its anchorage permits it. A clamp also supplies a reaction couple.

For downward load intensity  $w(x)$  in N/m, preserve its total force and first moment:

$$W = \int_a^b w(x) dx, \quad W\bar{x} = \int_a^b xw(x) dx \quad (W > 0).$$

**Chapter support:** Figure 12.7

# A distributed load preserves force and moment

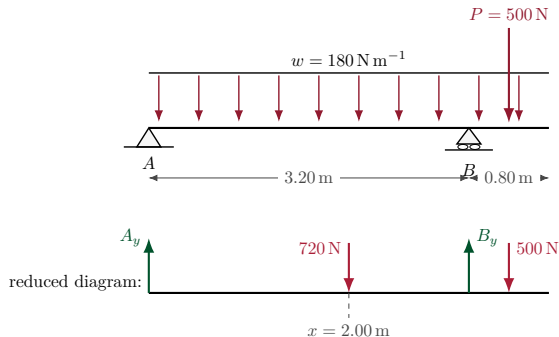


**Figure 12.8:** The sensor beam and its load-resultant diagram. Distances share the coordinate origin at A. The upper load arrows and the lower 720 N resultant belong to alternative representations, not simultaneous forces on one body. The vertical scale and force lengths are schematic.

**Chapter support:** Figure 12.8

## Example 12.6: the sensor beam



A 4.00 m horizontal beam is pinned at  $A$ ,  $x = 0$ , and rests on a horizontal roller at  $B$ ,  $x = 3.20$  m. A uniform downward load  $180 \text{ N m}^{-1}$  acts over the entire beam, including the overhang. A 500 N downward instrument load acts at  $x = 3.60$  m. Find all support reactions and verify the result about both supports.

The body is the entire beam. Use the geometry shown on slide 18. The roller supplies only  $B_y$ ; the pin supplies  $A_x$  and  $A_y$ . Both the allowed reactions and their restrictions must agree with that diagram.

$$W = wL = (180)(4.00) = 720 \text{ N}, \quad \bar{x} = L/2 = 2.00 \text{ m}.$$

The replacement includes the loaded overhang. Use the resultant or the original distributed load once. It preserves global force and moment, without supplying a beam-deflection or local bending model.

**Chapter support:** Example 12.6

## Example 12.6: reactions and a second origin



There is no horizontal applied load, so  $A_x = 0$ . Moments about the pin eliminate its two components:

$$3.20B_y - 720(2.00) - 500(3.60) = 0 \implies B_y = 1012.5 \text{ N}.$$

Vertical balance then gives  $A_y = 720 + 500 - 1012.5 = 207.5 \text{ N}$ . The upward roller reaction is admissible. **Independent check about  $B$ :**

$$-3.20(207.5) + 720(1.20) - 500(.40) = -664 + 864 - 200 = 0 \text{ N m}.$$

The uniform-load resultant lies left of  $B$  and gives a positive moment. The overhang load lies right of  $B$  and gives a negative moment. Force balance also gives  $207.5 + 1012.5 - 720 - 500 = 0 \text{ N}$ . Retaining guard digits makes the checks exact. The two reactions describe one supported configuration with unchanged external loading.

**Chapter support:** Example 12.6

# A triangular load has a different centroid

For  $w(x) = w_0x/L$  on  $0 \leq x \leq L$ ,

$$W = \frac{w_0L}{2}, \quad \int_0^L xw(x) dx = \frac{w_0L^2}{3}, \quad \bar{x} = \frac{2L}{3}.$$

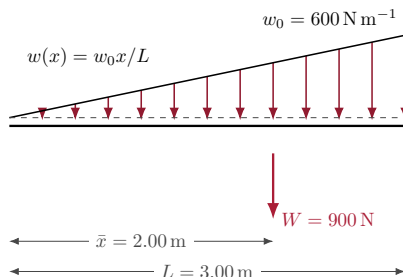


**Figure 12.10:** A linearly increasing load is represented in global balance by its triangular-area resultant at  $2L/3$  from the zero-intensity end. Load-intensity height and force-arrow length use separate schematic vertical scales; the horizontal coordinates are quantitative.

**Chapter support:** Figure 12.10

# Practice 1 / Example 12.7: nonuniform loading



A 3.00 m horizontal member is pinned at its left end and rests on a roller at its right end. A downward distributed load increases linearly from zero to  $600 \text{ N m}^{-1}$  across the full length. An additional 300 N point load acts 0.500 m from the left. Determine the reactions and quantify the error from placing the triangular-load resultant at the midpoint.

## Model before calculation

Use upward reactions and counterclockwise moments as positive. The pin supplies  $A_x, A_y$  and the right roller supplies  $B_y$ . The triangle's centroid was established on slide 21.

First calculate the force and first moment of the distributed load. Then include the separate point load. The next slide answers every requested part and tests the midpoint approximation.

**Chapter support:** Example 12.7

## Practice 1 / Example 12.7: full solution



The triangle gives  $W = 900 \text{ N}$  at  $\bar{x} = 2.00 \text{ m}$ . There is no horizontal load, so  $A_x = 0$ .

$$3.00B_y - 900(2.00) - 300(.500) = 0 \implies B_y = 650 \text{ N}, \quad A_y = 550 \text{ N}.$$

Both reactions are positive. About the right support,

$$-550(3.00) + 900(1.00) + 300(2.50) = 0 \text{ N m}.$$

A midpoint replacement instead gives  $B_y = [900(1.50) + 300(.500)]/3 = 500 \text{ N}$  and  $A_y = 700 \text{ N}$ . The actual load then has moment residual

$$3(500) - 900(2) - 300(.5) = -450 \text{ N m}.$$

It underestimates  $B_y$  by  $150 \text{ N}$  and overestimates  $A_y$  by the same amount. Total force still balances. Preserving force without its first moment is insufficient.

**Chapter support:** Example 12.7

## Example 12.8: an upward-only pad



Use the 4.00 m beam and  $180 \text{ N m}^{-1}$  full-length load from Example 12.6, but change the supports:  $A$  is now a nonadhesive upward-only pad at  $x = 0$ , while  $B$  is an anchored pin at  $x = 3.20 \text{ m}$ . A variable downward load  $P \geq 0$  acts at  $x = 3.60 \text{ m}$ . Find the admissible load interval and assess  $P = 2500 \text{ N}$ .

### Restored context and the changed restraint

The beam runs from  $x = 0$  to  $4.00 \text{ m}$ . Its uniform downward load totals  $720 \text{ N}$  at  $x = 2.00 \text{ m}$ . The variable load acts on the overhang. In this example,  $A$  is an upward-only pad and  $B$  is the anchored pin.

Solve balance for the trial two-support state, then impose  $A_y \geq 0$ . A negative component at an anchored pin can be permitted. The same component at a nonadhesive pad cannot be supplied.

**Chapter support:** Example 12.8

## Example 12.8: the admissible load interval



Moment balance about  $A$  and vertical balance give, with forces in newtons,

$$B_y = \frac{720(2.00) + 3.60P}{3.20} = 450 + 1.125P,$$
$$A_y = 720 + P - B_y = 270 - .125P, \quad B_x = 0.$$

The pad requires  $A_y \geq 0$ , so  $0 \leq P \leq 2160$  N. At the upper endpoint,  $A_y = 0$  and  $B_y = 2880$  N. At  $P = 2500$  N, the formal reactions are  $A_y = -42.5$  N and  $B_y = 3262.5$  N. The pad would have to pull, so this two-support static state is inadmissible. **Independent threshold check about  $B$ :**

$$720(1.20) - P(.40) = 0 \implies P = 2160 \text{ N}.$$

Reversing the pad arrow would change its physical model. Motion after lift-off or a new contact configuration requires a new analysis.

**Chapter support:** Example 12.8

# Cable geometry fixes the force direction

A cable pulls along itself. At angle  $\alpha$  above the beam, its components are  $T_x = T \cos \alpha$  and  $T_y = T \sin \alpha$ . Include the force or its components once.

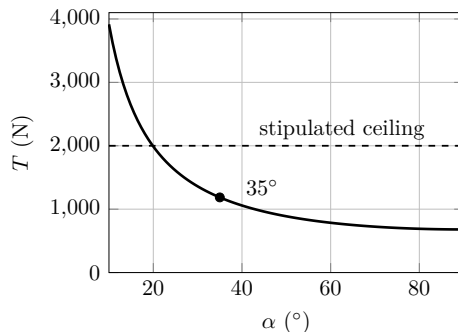
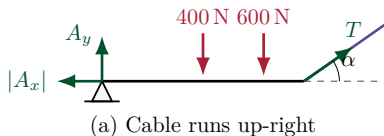


**Figure 12.12:** Cable direction and the quantitative angle sensitivity for the stipulated beam. The curve is  $T = 680 / \sin \alpha$ , with angle evaluated in degrees. The dashed ceiling intersects at  $19.9^\circ$ ; the filled marker gives the  $35^\circ$  example. Lengths in the force sketch are schematic except for the stated cable angle.

**Chapter support:** Figure 12.12

## Example 12.9: cable-supported beam



A horizontal 4.00 m beam is pinned at its left end. Its weight is 400 N at  $x = 2.00$  m, and an instrument of weight 600 N is attached at  $x = 3.20$  m. A cable attached at the right end runs *upward and to the right* toward an external anchor, making  $35.0^\circ$  with the beam. Find its tension and the pin components. Then determine the smallest cable angle allowed by a stipulated tension ceiling of 2000 N, retaining the same beam geometry.

The rigid beam is the chosen body. The cable pulls upward and rightward. In the zero-thickness horizontal geometry, its horizontal component acts along a line through the pin and makes zero moment there.

$$4.00 T \sin \alpha - 400(2.00) - 600(3.20) = 0 \quad \implies \quad T_y = 680 \text{ N}.$$

This required vertical component is fixed by the load moments. Changing the cable angle changes the total tension needed to provide it.

**Chapter support:** Example 12.9

## Example 12.9: tension, pin and angle limit



At  $\alpha = 35.0^\circ$ , use the unrounded tension for the components:

$$T = 680 / \sin 35.0^\circ = 1185.5 \text{ N},$$

$$T_x = 680 \cot 35.0^\circ = 971.1 \text{ N},$$

$$A_x = -971.1 \text{ N}, \quad A_y = 400 + 600 - 680 = 320 \text{ N}.$$

The pin force is leftward horizontally, balancing the specified rightward cable component. The tension is positive and permitted. For the stipulated ceiling,  $680 / \sin \alpha \leq 2000$ . Within  $0 < \alpha \leq 90^\circ$  this gives

$$\alpha_{\min} = \arcsin(.340) = 19.9^\circ.$$

As  $\alpha \rightarrow 90^\circ$ ,  $T \rightarrow 680 \text{ N}$  and  $A_x \rightarrow 0$ . As  $\alpha \rightarrow 0^+$ , both  $T$  and  $|A_x|$  diverge. These limits check the component model; the ceiling is a stated teaching input.

**Chapter support:** Example 12.9

# A ladder has three contact-force components

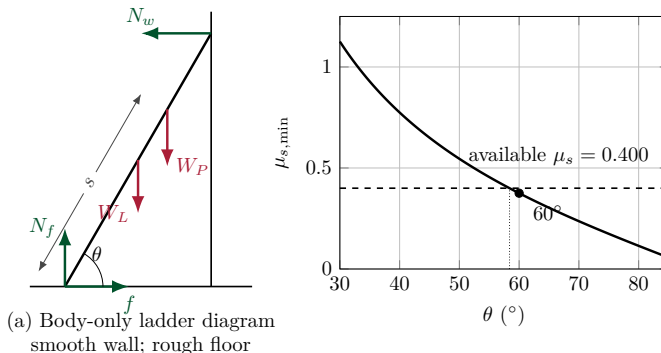


**Figure 12.13:** The specified loaded ladder and its friction requirement as a function of angle. The forces are all exerted *on* the ladder. The plotted demand is  $\mu_{s,\min} = 0.65 \cot \theta$  for the fixed 3.50 m load position. Its intersection with the dashed available coefficient sets the contact threshold. The drawing uses the quantitative  $60^\circ$  angle; force lengths are schematic.

**Chapter support:** Figure 12.13

# Ladder balance before the numerical example



The smooth wall supplies leftward  $N_w$ ; the floor supplies upward  $N_f$  and rightward  $f$ . Let  $L$  be ladder length,  $\theta$  its angle above the floor,  $W_L$  its weight and  $W_P$  a load at distance  $s$  along it. A uniform ladder's weight acts at  $L/2$ .

$$N_f = W_L + W_P, \quad f = N_w.$$

Moments about the foot remove both floor forces:

$$N_w L \sin \theta - W_L (L/2) \cos \theta - W_P s \cos \theta = 0.$$

Consequently,

$$N_w = \left( \frac{W_L}{2} + \frac{W_P s}{L} \right) \cot \theta, \quad \mu_{s,\min} = \frac{N_w}{W_L + W_P}.$$

The contacts also require nonnegative normals. The actual friction equals the equilibrium demand, with  $|f| \leq \mu_s N_f$ ; equality occurs at impending sliding. A shallower ladder needs more friction in this model.

**Chapter support:** Revised Chapter 12: ladder balance before the numerical example

## Example 12.10: loaded ladder and thresholds



A uniform rigid ladder of length  $5.00\text{ m}$  and weight  $200\text{ N}$  rests at  $60.0^\circ$  on a rough horizontal floor against a smooth vertical wall. A  $600\text{ N}$  load is located  $3.50\text{ m}$  along the ladder from its foot. The floor has stipulated  $\mu_s = 0.400$ . Find all contact forces, test equilibrium, find the minimum permissible angle for this load position, and find the limiting load position at  $60.0^\circ$ .

### The complete model

Use the body-only forces on slide 29 and the laws on slide 30. The wall has no friction. Both normal contacts can push only. The load position is measured along the ladder, so its horizontal moment arm is  $s \cos \theta$ .

The next two slides determine all contact forces, test their admissibility, and solve the separate angle and load-position limits.

**Chapter support:** Example 12.10

## Example 12.10: forces and contact feasibility



Vertical balance gives  $N_f = 200 + 600 = 800$  N. The foot-origin moment equation gives

$$N_w = \left( 100 + \frac{600(3.50)}{5.00} \right) \cot 60.0^\circ = 300.2 \text{ N.}$$

Horizontal balance requires  $f = 300.2$  N toward the wall. Both normals are positive. The floor can supply up to  $\mu_s N_f = .400(800) = 320$  N, exceeding this demand.

$$\mu_{s,\min} = 300.2/800 \simeq .375.$$

**Independent moment check about the top:**

$$\begin{aligned} fL \sin \theta - N_f L \cos \theta \\ + W_L(L/2) \cos \theta + W_P(L - s) \cos \theta = 0. \end{aligned}$$

The floor friction gives the positive moment and the floor normal the negative moment. Substituting unrounded forces makes this residual vanish.

**Chapter support:** Example 12.10

## Example 12.10: angle and load-position limits



For the same  $s = 3.50$  m, limiting friction requires

$$520 \cot \theta_{\min} = 320 \quad \Rightarrow \quad \theta_{\min} = \arctan(520/320) = 58.4^\circ.$$

For the same  $60.0^\circ$  angle, set the wall-force demand to 320 N and solve instead for position:

$$\begin{aligned} s_{\max} &= \frac{L}{W_P} \left[ \mu_s (W_L + W_P) \tan \theta - \frac{W_L}{2} \right] \\ &= \frac{5.00}{600} [320 \tan 60.0^\circ - 100] = 3.79 \text{ m}. \end{aligned}$$

This is inside the ladder length and beyond the stated 3.50 m position. Substitution returns the 320 N limiting friction. Angle and position are different loading variables. The calculation tests the stated two-contact rigid model; it supplies no member-strength or footing-deformation calculation.

**Chapter support:** Example 12.10

# Internal loading survives external balance

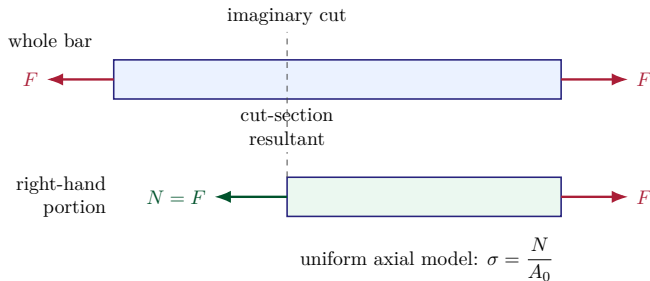


**Figure 12.14:** An internal force becomes an external force when a portion of the bar is isolated. In the stated uniform axial model, each cut transmits the same tensile resultant. Arrow lengths and specimen dimensions are schematic.

For a uniform central axial load, the signed average normal stress is  $\sigma = N/A_0$ , with tension positive. A leftward cut force on the right-hand portion can therefore represent positive tensile stress. For nonuniform stress,  $N = \int_A \sigma \, dA$  gives a sectional resultant, rather than a local stress field.

**Chapter support:** Figure 12.14

# Strain and Young modulus add material response



Reference marks define stress-free gauge length  $L_0$ . The signed length change and engineering strain are

$$\delta = L - L_0, \quad \varepsilon = \frac{\delta}{L_0}.$$

Strain is dimensionless. A strain of 0.001 is 0.1%, with its dimensional extension set by gauge length. Compression gives  $\delta < 0$  and  $\varepsilon < 0$ . For the stated small-strain linear elastic range,

$$\sigma = E_Y \varepsilon, \quad E_Y > 0 \quad [\text{Pa}].$$

This constitutive relation adds physical information beyond equilibrium. A larger modulus gives less strain at the same stress. Small geometric strain does not establish a material's proportional range. That range is a response assumption or must be tested by loading and unloading observations.

**Chapter support:** Revised Chapter 12: strain and young modulus add material response

## Practice 2 / Example 12.11: signed compression



A straight short specimen has reference length  $L_0 = 0.250 \text{ m}$ , area  $A_0 = 200 \text{ mm}^2$  and stipulated  $E_Y = 120 \text{ GPa}$ . Opposite central forces compress it with magnitude  $18.0 \text{ kN}$ . Neglect its weight and assume it remains straight in its small-strain linear elastic range. Find the signed stress, strain, length change and loaded length. Explain why equilibrium does not imply zero strain.

### A consistent sign model

Use  $N < 0$  for compression even though the two external end forces have opposite global directions. Keep  $A_0$  and  $L_0$  as reference geometry. Convert area to  $\text{m}^2$  and modulus to pascals before substitution.

The next slide gives the four requested results and explains why force and moment balance coexist with nonzero shortening.

**Chapter support:** Example 12.11

## Practice 2 / Example 12.11: full solution



Compression gives  $N = -18.0 \times 10^3 \text{ N}$ , and  $A_0 = 200 \times 10^{-6} \text{ m}^2$ .

$$\sigma = \frac{-18.0 \times 10^3}{200 \times 10^{-6}} = -90.0 \text{ MPa},$$

$$\varepsilon = \frac{-90.0 \times 10^6}{120 \times 10^9} = -7.50 \times 10^{-4},$$

$$\delta = .250\varepsilon = -.1875 \text{ mm},$$

$$L = 250.0000 - .1875 = 249.8125 \text{ mm} \simeq 249.81 \text{ mm}.$$

The shortening magnitude is 0.188 mm and fractional magnitude 0.0750%.

The geometry is consistent with the stated small-strain model. **Physical and**

**algebraic checks:** opposite external forces balance while cut portions carry internal compression. Substitution into  $N = E_Y A_0 \delta / L_0$  returns  $-18.0 \text{ kN}$ . Doubling area at the same load halves both stress and shortening. The assumed straight configuration is part of the model.

**Chapter support:** Example 12.11

# Material modulus and specimen stiffness



Combine the uniform axial definitions with the constitutive law:

$$\frac{N}{A_0} = E_Y \frac{\delta}{L_0} \implies \delta = \frac{NL_0}{E_Y A_0}, \quad k = \frac{E_Y A_0}{L_0}.$$

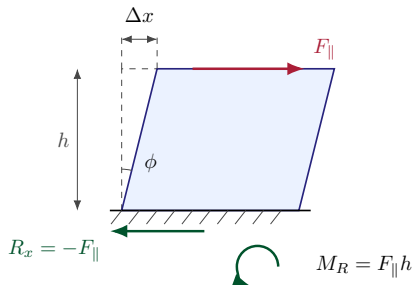
Thus  $N = k\delta$ , with stiffness in  $\text{N m}^{-1}$ . Compliance is  $C = 1/k = L_0/(E_Y A_0)$  in  $\text{m N}^{-1}$ . The slope of stress against strain is the material modulus. The slope of force against extension is a specimen stiffness, combining modulus, area and gauge length.

## An answered geometry check

At the same tensile force, doubling area halves stress, strain and extension while doubling stiffness. Young modulus remains unchanged. Doubling length instead doubles extension and compliance.

**Chapter support:** Revised Chapter 12: material modulus and specimen stiffness

# Shear changes shape and needs moment balance



reference top corner:  $(0, h)$

shifted same corner:  $(\Delta x, h)$

$$\gamma = \Delta x/h = \tan \phi$$

clamp supplies force  
and a counterclockwise moment

**Figure 12.19:** Correct simple-shear displacement measured from the same reference corner. The dashed outline is undeformed. The top-force/bottom-force pair alone has a clockwise moment; the clamp also supplies the counterclockwise reaction moment shown. The display distortion is exaggerated. The reference-overlay construction draws on the method in I. Neutelings, *Mechanical stress*, TikZ.net; geometry and equilibrium are constructed independently here.

**Chapter support:** Figure 12.19

# Bulk compression fixes the volume-change sign



The reduced shear law is  $\tau_{\text{shear}} = F_{\parallel}/A_0 = G_s \gamma$ , with shear modulus  $G_s$ . In simple shear,  $\gamma = \Delta x/h = \tan \phi$  is exact geometry;  $\phi \simeq \gamma$  uses small angles in radians. The reduced approximately uniform shear model neglects end effects and bending. A positive pressure increase  $\Delta p$  acts inward on the boundary. Define the small volume strain  $\varepsilon_V = \Delta V/V_0$ . For positive bulk modulus  $B$ ,

$$\Delta p = -B \frac{\Delta V}{V_0} \implies \Delta V = -\frac{V_0 \Delta p}{B}.$$

The minus sign gives contraction under increasing pressure. Pressure is a positive compression magnitude, unlike tension-positive axial stress. Hydrostatic loading compresses in all directions. A force pair acting only at the ends of a bar is a different stress state. The finite-change law assumes nearly constant  $B$  over a small range. Its local definition,  $B = -V dp/dV$ , depends on thermodynamic conditions.

**Chapter support:** Revised Chapter 12: bulk compression fixes the volume-change sign

## Example 12.14: three deformation modes



Consider three independent stipulated specimens, each in its small-deformation linear range. (a) An axial bar has  $F = 30.0 \text{ kN}$ ,  $A_0 = 300 \text{ mm}^2$ ,  $L_0 = 0.500 \text{ m}$  and  $E_Y = 100 \text{ GPa}$ . (b) A clamped shear block has  $F_{\parallel} = 18.0 \text{ kN}$ ,  $A_0 = 600 \text{ mm}^2$ ,  $h = 0.200 \text{ m}$  and  $G_S = 25.0 \text{ GPa}$ . (c) A body of volume  $V_0 = 0.0150 \text{ m}^3$  has  $B = 80.0 \text{ GPa}$  and experiences a uniform pressure increase  $\Delta p = 12.0 \text{ MPa}$ . Find the associated stresses, strains and dimensional changes. For the shear block also determine the angle and clamp reactions.

The models are independent. Their three stipulated moduli describe three separate specimens. Each strain measure is dimensionless, but the associated change is axial length, transverse displacement, or volume.

**Chapter support:** Example 12.14

## Example 12.14: axial and shear solutions



### (a) Axial bar:

$$\sigma = 30000 / (300 \times 10^{-6}) = 100 \text{ MPa},$$

$$\varepsilon = \sigma / E_Y = .00100, \quad \delta = .500\varepsilon = .500 \text{ mm}.$$

### (b) Shear block:

$$\tau_{\text{shear}} = 18000 / (600 \times 10^{-6}) = 30.0 \text{ MPa},$$

$$\gamma = \tau_{\text{shear}} / G_S = .00120, \quad \Delta x = .200\gamma = .240 \text{ mm},$$

$$\phi = \arctan(.00120) = .0011999994 \text{ rad} = .06875^\circ.$$

Clamp force is  $R_x = -18.0 \text{ kN}$ . Its counterclockwise couple is  $M_R = F_{\parallel} h = 3.60 \text{ kN m}$ . Both global balances close. The fractional small-angle error is about  $\gamma^2/3 = 4.8 \times 10^{-7}$ . Modulus times strain recovers each stress.

**Chapter support:** Example 12.14

## Example 12.14: bulk result and mode comparison



### (c) Hydrostatic compression:

$$\varepsilon_V = -\frac{12.0 \times 10^6}{80.0 \times 10^9} = -1.50 \times 10^{-4},$$

$$\begin{aligned}\Delta V &= (.0150)(-1.50 \times 10^{-4}) = -2.25 \times 10^{-6} \text{ m}^3 \\ &= -2.25 \text{ cm}^3.\end{aligned}$$

Check:  $-B\Delta V/V_0 = 12.0 \text{ MPa}$ , with contraction.

| Mode  | Strain                     | Dimensional change              |
|-------|----------------------------|---------------------------------|
| Axial | $\varepsilon = .00100$     | $\delta = .500 \text{ mm}$      |
| Shear | $\gamma = .00120$          | $\Delta x = .240 \text{ mm}$    |
| Bulk  | $\varepsilon_V = -.000150$ | $\Delta V = -2.25 \text{ cm}^3$ |

The loading geometry selects the modulus. Equal units do not make the three moduli interchangeable.

**Chapter support:** Example 12.14

# Lateral response and an isotropic bridge



For a uniaxial test with free lateral faces, Poisson's ratio is

$$\nu = -\frac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{axial}}}.$$

Positive  $\nu$  gives lateral contraction in tension. For the same homogeneous, isotropic, infinitesimal linear elastic model,

$$G_s = \frac{E_Y}{2(1 + \nu)}, \quad B = \frac{E_Y}{3(1 - 2\nu)}.$$

**Bulk derivation:** superpose three normal stresses  $-p$ . In each direction,  $\varepsilon_x = -p/E_Y + 2\nu p/E_Y$ . Summing gives  $\Delta V/V_0 = -3(1 - 2\nu)p/E_Y$ . For shear, axes rotated by  $45^\circ$  have normal stresses  $+\tau_{\text{shear}}$  and  $-\tau_{\text{shear}}$ . Their strain difference is  $2(1 + \nu)\tau_{\text{shear}}/E_Y = \gamma$ , giving the shear relation. These connections require the same material state and isotropic model. They do not link the three independent specimens in Example 12.14.

**Chapter support:** Revised Chapter 12: lateral response and an isotropic bridge

# Elastic recovery and proportionality differ

Elastic recovery requires unloading; proportionality concerns linearity. Persistent unloaded deformation is plastic in this model.

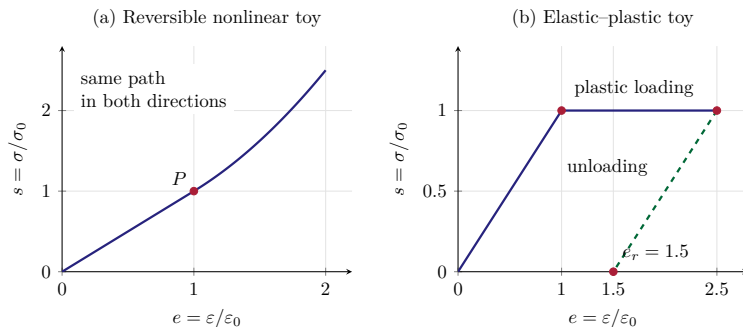


**Figure 12.22:** Two quantitative normalized teaching models, each with a specified loading and unloading law. In (a)  $P$  marks the end of exact proportionality, yet the whole path is reversible. In (b) loading beyond the plateau's start leaves a calculated residual strain. The models are not empirical material curves.

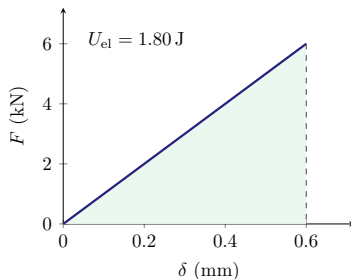
**Chapter support:** Figure 12.22

# Elastic energy comes from the loading path



Quasi-static reversible linear loading stores the triangular force–extension area:  $U_{el} = \int_0^\delta kq \, dq = \frac{1}{2} N\delta$ .

(a) Whole specimen



(b) Per reference volume

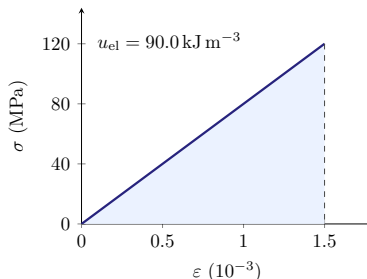


**Figure 12.23:** The same stipulated linear specimen represented by force–extension and stress–strain areas. Multiplying panel (b)'s energy density by  $V_0 = 2.00 \times 10^{-5} \text{ m}^3$  gives panel (a)'s energy. Axis scale factors must be included when converting graphical area to SI energy.

**Chapter support:** Figure 12.23

## Example 12.17: energy definitions and statement



The linear law  $N = k\delta$  gives  $U_{\text{el}} = \frac{1}{2}k\delta^2$ . Dividing by reference volume  $V_0 = A_0L_0$  gives

$$u_{\text{el}} = \frac{U_{\text{el}}}{V_0} = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E_Y\varepsilon^2 = \frac{\sigma^2}{2E_Y}.$$

Compression also stores positive energy because both  $N$  and  $\delta$  are negative. For nonlinear response integrate the actual curve; for plastic response distinguish loading work and recoverable unloading area. A stipulated uniform axial specimen has  $L_0 = 0.400\text{ m}$ ,  $A_0 = 50.0\text{ mm}^2$  and  $E_Y = 80.0\text{ GPa}$ . Its tensile stress reaches  $120\text{ MPa}$  within its small-strain linear elastic range. Find strain, extension, force and energy. Verify the energy from force–displacement, stress–strain and stiffness descriptions.

**Chapter support:** Example 12.17

## Example 12.17: complete energy solution



The linear material law and reference geometry give

$$\varepsilon = \frac{120 \times 10^6}{80.0 \times 10^9} = .00150, \quad \delta = .400\varepsilon = .600 \text{ mm},$$

$$N = \sigma A_0 = 6000 \text{ N}, \quad k = \frac{E_Y A_0}{L_0} = 10.0 \text{ MN m}^{-1}.$$

**Force–extension route:**  $U = \frac{1}{2}(6000)(.000600) = 1.80 \text{ J}$ . **Stress–strain**

**route:**  $u = \frac{1}{2}(120 \times 10^6)(.00150) = 90.0 \text{ kJ m}^{-3}$ . Since  $V_0 = 50.0 \times 10^{-6}(.400) = 2.00 \times 10^{-5} \text{ m}^3$ ,  $uV_0 = 1.80 \text{ J}$ . **Stiffness route:**  $\frac{1}{2}k\delta^2 =$

$\frac{1}{2}(10.0 \times 10^6)(.000600)^2 = 1.80 \text{ J}$ . All three descriptions agree in the same model. Doubling stress in its declared linear range doubles extension and quadruples stored energy.

**Chapter support:** Example 12.17

# Series members add compliance



For collinear members joined end to end, negligible weight and no axial load at the joint imply the same axial force  $N$  in each. Compatibility adds extensions:

$$\delta_{\text{tot}} = \sum_i \delta_i = N \sum_i \frac{L_i}{E_{Y,i} A_i}, \quad \frac{1}{k_{\text{series}}} = \sum_i \frac{1}{k_i}.$$

The assembly is more compliant than either member alone. Its energy is  $\frac{1}{2} N \delta_{\text{tot}}$ , equal to the sum of member energies under proportional linear loading. A gradually varying member is a continuous series of short intervals.

First find  $N(x)$  from equilibrium, then integrate local strain:

$$d\delta = \frac{N(x)}{E_Y(x) A_0(x)} dx, \quad \delta = \int_0^{L_0} \frac{N(x)}{E_Y(x) A_0(x)} dx.$$

Force transmission, constitutive response and displacement compatibility supply distinct steps.

**Chapter support:** Revised Chapter 12: series members add compliance

# A taper weights the inverse area



For constant tensile  $F > 0$  and  $E_Y$ , let  $A_0(x) = A_a(1 + \alpha x/L_0)$ , with  $A_a > 0$  and  $\alpha > -1$ .

$$\begin{aligned}\delta &= \frac{F}{E_Y A_a} \int_0^{L_0} \frac{dx}{1 + \alpha x/L_0} \\ &= \frac{FL_0}{E_Y A_a} \frac{\ln(1 + \alpha)}{\alpha} \quad (\alpha \neq 0).\end{aligned}$$

The logarithm follows by substituting  $u = 1 + \alpha x/L_0$ . As  $\alpha \rightarrow 0$ ,  $\ln(1 + \alpha)/\alpha \rightarrow 1$ , recovering the uniform-bar result. For  $\alpha > 0$ , the uniform bars with the largest and smallest areas bound the extension:

$$\frac{FL_0}{E_Y A_a(1 + \alpha)} < \delta < \frac{FL_0}{E_Y A_a}.$$

The inverse of mean area is generally different from the mean of inverse area. The one-dimensional model neglects local multiaxial stress near grips, joints and taper transitions.

**Chapter support:** Revised Chapter 12: a taper weights the inverse area

# From force–extension data to a material modulus



Fit  $F_i = k\delta_i + b + r_i$ :  $k$  is stiffness,  $b$  is an intercept and  $r_i$  is observed minus predicted force. Ordinary least squares assumes known extensions and independent, equal-variance force scatter.

$$\hat{k} = \frac{\sum(\delta_i - \bar{\delta})(F_i - \bar{F})}{\sum(\delta_i - \bar{\delta})^2}, \quad \hat{b} = \bar{F} - \hat{k}\bar{\delta}.$$

|               |      |      |      |      |      |      |      |
|---------------|------|------|------|------|------|------|------|
| $\delta$ (mm) | .040 | .100 | .160 | .220 | .280 | .340 | .400 |
| $F$ (N)       | 410  | 1015 | 1592 | 2210 | 2798 | 3418 | 3990 |

These are the seven synthetic points of source Table 12.1. Convert the fitted stiffness to  $E_Y = kL_0/A$  only if the displacement is gauge extension and the axial model applies. For small independent input uncertainties, first-order propagation gives  $u_r(E_Y)^2 = u_r(k)^2 + u_r(L_0)^2 + u_r(A)^2$ , where  $u_r(q) = u(q)/|q|$ . Statistical scatter and physical bias require separate checks.

**Chapter support:** Revised Chapter 12: from force–extension data to a material modulus

## Example 12.19: fit, inference and uncertainty



**Statement.** Fit all seven points of Table 12.1 to  $F = k\delta + b$ . Take  $L_0 = 0.400\text{ m}$  and  $A = 50.0\text{ mm}^2$ . Find stiffness, intercept, residual scatter, and inferred modulus. Compare with a line forced through zero. Estimate combined standard uncertainty when independent geometry uncertainties are  $u(L_0)/L_0 = 0.25\%$  and  $u(A)/A = 1.0\%$ , and slope uncertainty is estimated from these residuals.

### Restored data and assumptions

Use all seven  $(\delta, F)$  pairs on slide 51. Extensions are treated as known; force scatter is independent with common variance. The intercept is fitted rather than suppressed. Geometry uncertainties are independent of the regression slope.

Slides 53–55 give the fit, scatter, modulus, through-zero comparison and complete combined-standard-uncertainty result. These are synthetic data and contain no unloading observations.

**Chapter support:** Example 12.19

## Example 12.19: fit and residual scatter



Using extensions in metres gives  $\hat{k} = 9.971428571 \text{ MN m}^{-1}$  and  $\hat{b} = 11.0 \text{ N}$ .  
Observed-minus-predicted residuals, in newtons, are

$$(.143, 6.857, -14.429, 5.286, -5.000, 16.714, -9.571).$$

Their sum of squares is  $\text{RSS} = 679.142857 \text{ N}^2$ . The descriptive RMS and estimated scatter have different denominators:

$$r_{\text{RMS}} = \sqrt{\text{RSS}/7} = 9.84989 \text{ N}, \quad s = \sqrt{\text{RSS}/(7-2)} = 11.65455 \text{ N}.$$

Two fitted parameters leave five residual degrees of freedom. With  $S_{xx} = \sum(\delta_i - \bar{\delta})^2$ ,

$$u(\hat{k}) = s/\sqrt{S_{xx}} = 36708.4 \text{ N m}^{-1}, \quad u_r(\hat{k}) = .368136\%.$$

The independent normal-equation checks are  $\sum r_i = 0$  and  $\sum \delta_i r_i = 0$  before rounding. A small scatter estimates precision under the adopted model.

**Chapter support:** Example 12.19

## Example 12.19: modulus and the intercept choice



The gauge model converts the fitted stiffness using independent geometry:

$$\widehat{E}_Y = \frac{\widehat{k}L_0}{A} = \frac{(9.971428571 \times 10^6)(.400)}{50.0 \times 10^{-6}} = 79.7714 \text{ GPa.}$$

A line forced through zero instead minimizes  $\sum(F_i - k_0\delta_i)^2$ . Differentiation gives

$$k_0 = \frac{\sum \delta_i F_i}{\sum \delta_i^2} = 10.0099636 \text{ MN m}^{-1}, \quad E_{Y0} = 80.0797 \text{ GPa.}$$

This modulus is 0.386% above the two-parameter value, relative to  $\widehat{E}_Y$ . Suppressing an intercept changes the assumed model. With the intercept fit,  $u(b) = s\sqrt{1/7 + \bar{\delta}^2/S_{xx}} = 9.20 \text{ N}$ . The fitted 11 N intercept is about 1.2 standard uncertainties, so the data do not establish a resolved nonzero preload.

**Chapter support:** Example 12.19

## Example 12.19: the combined uncertainty



Linearize the gauge conversion:  $dE_Y/E_Y = dk/k + dL_0/L_0 - dA/A$ . Independent contributions add as variances:

$$\begin{aligned}u_r(E_Y) &= \sqrt{(.00368136)^2 + (.00250)^2 + (.0100)^2} \\&= .0109454 = 1.09454\%, \\u(E_Y) &= (79.7714)(.0109454) = .873 \text{ GPa}.\end{aligned}$$

An appropriate report is  $E_Y = (79.8 \pm .9) \text{ GPa}$ , with  $\pm$  indicating combined standard uncertainty under these assumptions. The slope, length and area contribute 11.31%, 5.22% and 83.47% of the variance. Improving area measurement has the largest effect in this budget. For a circular section,  $A = \pi d^2/4$  gives  $u_r(A) \simeq 2u_r(d)$ . The result is not a confidence interval.

Grip motion, alignment or calibration bias absent from the input model is not quantified by this uncertainty. Correlated inputs require covariance terms.

**Chapter support:** Example 12.19

# Apparatus motion adds series compliance

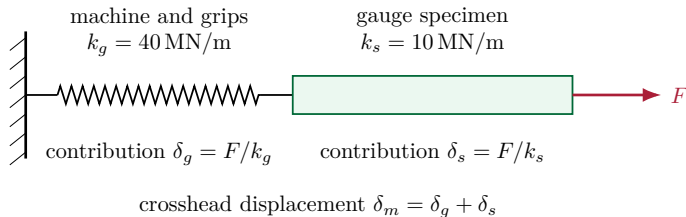


**Figure 12.28:** A series-compliance model for extension measured by a loading machine. The spring represents the combined grip/machine compliance; its drawn length is schematic. The same force acts through both components, whereas their displacements add. Measuring gauge extension separately changes the inference.

$$\delta_m = \delta_s + \delta_g = F \left( \frac{1}{k_s} + \frac{1}{k_g} \right), \quad \frac{1}{k_m} = \frac{1}{k_s} + \frac{1}{k_g}.$$

Crosshead travel includes apparatus motion, so  $k_m < k_s$ . Linear series elements give a straight curve: residuals alone cannot identify this bias.

Chapter support: Figure 12.28

## Example 12.20: correcting an apparent modulus



**Statement.** A specimen has  $L_0 = 0.400\text{ m}$ ,  $A = 50.0\text{ mm}^2$ , and true stipulated  $E_Y = 80.0\text{ GPa}$ . The series grip/machine stiffness is  $k_g = 40.0\text{ MN m}^{-1}$ . Find the stiffness measured using crosshead displacement, the apparent modulus from that measurement, and the gauge and grip extensions at  $F = 4.00\text{ kN}$ . Derive a correction when  $k_g$  is known independently.

The specimen has  $k_s = E_Y A / L_0 = 10.0\text{ MN/m}$ . The machine measures

$$k_m = \frac{k_s k_g}{k_s + k_g} = 8.00\text{ MN m}^{-1}, \quad E_{Y\text{app}} = k_m L_0 / A = 64.0\text{ GPa}.$$

At  $4.00\text{ kN}$ ,  $\delta_s = .400\text{ mm}$  and  $\delta_g = .100\text{ mm}$ , giving  $\delta_m = .500\text{ mm}$ . The apparent modulus is 20% low. Rearranging the reciprocal-stiffness law gives

$$k_s = \frac{k_m k_g}{k_g - k_m}, \quad E_Y = \frac{k_m k_g L_0}{(k_g - k_m) A} = 80.0\text{ GPa}.$$

Positive  $k_s$  requires  $k_g > k_m$ . Rigid grips recover  $k_m \rightarrow k_s$  as  $k_g \rightarrow \infty$ .

**Chapter support:** Example 12.20

# What observations validate the physical model?



A regression tests arithmetic and scatter within a chosen statistical model. Interpreting its slope as a material modulus additionally requires the correct geometry and displacement measurement. **Reversibility:** compare increasing and decreasing loads and revisit zero after unloading. The seven synthetic loading points alone cannot establish recovery. **Range:** compare systematic residual structure with independently estimated repeatability. A visually attractive short fit is insufficient evidence of a proportional range. **Apparatus:** compare local gauge extension with crosshead travel. Inspect alignment and dimensions along the gauge section. Repeated precise observations can still share a bias. For an instrument-support test, these observations separate material response from mount and fixture motion. The chapter proposes this investigation; it does not report an experiment already conducted.

**Chapter support:** Revised Chapter 12: what observations validate the physical model?

# Core synthesis: three layers of reasoning



**Equilibrium** asks whether the external actions on one chosen body balance. Use  $\sum \mathbf{F} = 0$  and  $\sum \boldsymbol{\tau} = 0$ , then verify a second origin. **Constraints** ask whether the surroundings can supply the required actions. Check  $N \geq 0$ ,  $T \geq 0$ ,  $|f| \leq \mu_s N$  and a support resultant within its contact region. **Constitutive response and compatibility** ask how loads deform the material and how connected displacements can coexist. For a uniform axial member,

$$\sigma = \frac{N}{A_0}, \quad \varepsilon = \frac{\delta}{L_0}, \quad \sigma = E_Y \varepsilon, \quad k = \frac{E_Y A_0}{L_0}.$$

These levels also organize inverse measurement. A slope estimates stiffness; geometry converts it to modulus; uncertainty and apparatus checks assess that inference. Advanced supports use all three levels together. Static balance at one configuration still leaves a separate question about stability after a perturbation.

**Chapter support:** Revised Chapter 12: core synthesis: three layers of reasoning

# Bridge to Chapter 13: gravitation



Here a nearly uniform gravitational field allowed one weight to act at the mass centre for global balance. Chapter 13 develops attraction, gravitational fields and potential energy. For an extended object in a varying field, integrate the gravitational loading. Supported laboratory equilibrium has zero resultant force, while orbital motion requires a nonzero resultant force. Potential energy connects equilibrium to the response under nearby displacements. For Space Technologies, carry forward the distinction between a laboratory payload under uniform gravity, an accelerating mount, and an extended body in a gravity gradient. Their external-load models differ.

## Reading and continued practice

Use the revised chapter's exercises and complete reasoned solutions. Its bibliography lists Halliday–Resnick–Walker, Serway–Jewett and OpenStax. The advanced block follows on slides 61–70.

**Chapter support:** Revised Chapter 12: bridge to chapter 13: gravitation

# Advanced: compatibility needs a motion model

A rigid bar can translate and rotate. In a small clockwise rotation model, its downward displacement is  $d(x) = d_0 + \theta x$ . Equal support displacement requires  $\theta = 0$ , imposed here by an antirotation guide.

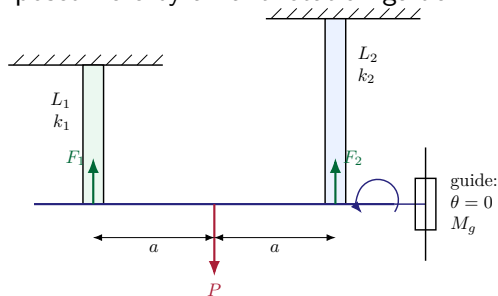


**Figure 12.29:** A guided crosshead can translate while remaining horizontal. Different upper-anchor levels allow different rod lengths; heights are schematic. The ideal guide supplies a balancing reaction couple and no vertical reaction, so both force and moment equilibrium are satisfied with unequal rod forces.

**Chapter support:** Figure 12.29

## Example 12.21: unequal rods with a guide



**Statement.** Rod 1 has  $E_1 = 70.0$  GPa,  $A_1 = 300$  mm<sup>2</sup>, and  $L_1 = 1.75$  m. Rod 2 has  $E_2 = 200$  GPa,  $A_2 = 160$  mm<sup>2</sup>, and  $L_2 = 4.00$  m. They attach at  $x = \mp 0.400$  m to the guided crosshead of Fig. 12.29. A centered 30.0 kN downward load acts on it. Find the rod forces, common displacement, and guide couple; check both balances.

The guide geometry is on slide 61. Equal end displacement gives

$$d_1 = d_2 = d_0, \quad F_i = k_i d_0, \quad d_0 = \frac{P}{k_1 + k_2}.$$

Thus  $F_i = Pk_i/(k_1 + k_2)$ . With counterclockwise-positive physical moments about the crosshead centre,

$$-aF_1 + aF_2 + M_g = 0 \quad \implies \quad M_g = a(F_1 - F_2).$$

Equal displacement need not mean equal force. The guide supplies the missing moment when stiffnesses differ.

**Chapter support:** Example 12.21

## Example 12.21: load sharing and guide couple



From the stated rod data,

$$k_1 = 70.0 \times 10^9 (300 \times 10^{-6}) / 1.75 = 12.0 \text{ MN/m},$$

$$k_2 = 200 \times 10^9 (160 \times 10^{-6}) / 4.00 = 8.00 \text{ MN/m}.$$

The common displacement is  $d_0 = 30000 / (20.0 \times 10^6) = 1.50 \text{ mm}$ . Hence  $F_1 = 18.0 \text{ kN}$  and  $F_2 = 12.0 \text{ kN}$ .

$$M_g = .400(18.0 - 12.0) = 2.40 \text{ kN m} \quad \text{counterclockwise.}$$

**Independent checks:**  $18 + 12 - 30 = 0 \text{ kN}$  and  $-.4(18) + .4(12) + 2.4 = 0 \text{ kN m}$ . Both  $F_i/k_i$  return  $1.50 \text{ mm}$ . Rod strains are  $8.57 \times 10^{-4}$  and  $3.75 \times 10^{-4}$ . As  $k_2 \rightarrow 0$ , rod 1 carries  $P$  and  $M_g \rightarrow aP$ . As  $k_2 \rightarrow \infty$ , rod 2 carries  $P$  and  $M_g \rightarrow -aP$ . Equal stiffness gives equal forces and zero couple.

**Chapter support:** Example 12.21

## Example 12.22: removing the guide



Without a guide couple, a centered load and symmetric locations require

$$F_1 + F_2 = P, \quad -aF_1 + aF_2 = 0 \quad \implies \quad F_1 = F_2 = P/2.$$

The rigid bar can rotate. Its two extensions determine translation and rotation through  $d(\mp a) = d_0 \mp a\theta = F_{1,2}/k_{1,2}$ . **Statement.** Remove the guide from Example 12.21. Keep the centered 30.0 kN load, rod stiffnesses 12.0 and 8.00 MN m<sup>-1</sup>, and locations  $x = \mp 0.400$  m. Find reaction forces, both extensions, center translation, and clockwise rotation. Diagnose the proposed answer  $F_1 = 18.0$  kN,  $F_2 = 12.0$  kN with a horizontal bar.

The rods and positions are those on slides 61–63. The removed guide changes the constraints. A straight bar preserves collinearity, without requiring a horizontal orientation.

**Chapter support:** Example 12.22

## Example 12.22: rotation and a rejected answer



Force and moment balance give  $F_1 = F_2 = 15.0 \text{ kN}$ .

$$d_1 = 15000 / (12.0 \times 10^6) = 1.250 \text{ mm},$$

$$d_2 = 15000 / (8.00 \times 10^6) = 1.875 \text{ mm},$$

$$d_0 = (d_1 + d_2) / 2 = 1.5625 \text{ mm},$$

$$\theta = \frac{d_2 - d_1}{.800} = 7.8125 \times 10^{-4} \text{ rad} = .0448^\circ.$$

The denominator in the last row is in metres, and the displacement difference is converted to metres. Positive clockwise rotation makes the right end descend farther. Substitution into  $d_0 \mp .400\theta$  returns both extensions. The rotation is small under the stipulated axial and first-order geometry model. The proposed 18/12 kN split passes force balance but leaves  $-.400(18) + .400(12) = -2.40 \text{ kN m}$ . No guide couple remains to balance it. Those forces belong to the guided model.

**Chapter support:** Example 12.22

# Three supports require compatibility



Three vertical reactions have only force and one planar moment balance. For all engaged posts,  $R_i = k_i(d_0 + \theta x_i)$ . A downward load  $P$  at  $x = e$  gives

$$\begin{pmatrix} \sum k_i & \sum k_i x_i \\ \sum k_i x_i & \sum k_i x_i^2 \end{pmatrix} \begin{pmatrix} d_0 \\ \theta \end{pmatrix} = \begin{pmatrix} P \\ Pe \end{pmatrix}.$$

The rows are  $\sum R_i = P$  and  $\sum x_i R_i = Pe$ . Constitutive response and the rigid displacement profile close equilibrium. For positive stiffnesses at separated positions, the determinant satisfies

$$(\sum k_i)(\sum k_i x_i^2) - (\sum k_i x_i)^2 = \sum_{i < j} k_i k_j (x_i - x_j)^2 > 0.$$

Solve for the motion, recover reactions, and test every contact sign. A compression-only post with negative predicted force must be removed from the trial active set before re-solving.

**Chapter support:** Revised Chapter 12: advanced modelling

## Advanced challenge / Example 12.23: three posts



**Statement.** A rigid bar rests on three linear compression-only posts at  $x = -0.500, 0, +0.500$  m with stiffnesses  $4.00, 6.00, 8.00$  MN m<sup>-1</sup>. Their unloaded tops lie on the same level, and there are no initial gaps. A downward load  $18.0$  kN acts at  $x = +0.100$  m. Neglect bar weight. Find center displacement, rotation, and all post reactions. Check contact admissibility and compare with assigning one-third of the load to every post.

The compatibility model is already established

Use the first-order rigid profile  $d_i = d_0 + \theta x_i$  and  $R_i = k_i d_i$  for the trial all-contact state. The matrix on slide 66 combines these with the two relevant balance equations.

The following slide solves the motion, checks every contact, tests the equal-load guess and verifies stored energy independently.

**Chapter support:** Example 12.23

## Example 12.23: full compatibility solution



The three sums are  $18.0 \times 10^6 \text{ N/m}$ ,  $2.00 \times 10^6 \text{ N}$  and  $3.00 \times 10^6 \text{ N m}$ . Thus

$$18 \times 10^6 d_0 + 2 \times 10^6 \theta = 18000, \quad 2 \times 10^6 d_0 + 3 \times 10^6 \theta = 1800.$$

Solving gives  $d_0 = 1.008 \text{ mm}$  and  $\theta = -7.20 \times 10^{-5} \text{ rad}$ . The negative clockwise rotation means less downward motion on the right.

$$(d_1, d_2, d_3) = (1.044, 1.008, .972) \text{ mm},$$
$$(R_1, R_2, R_3) = (4.176, 6.048, 7.776) \text{ kN}.$$

All contacts push. Checks give  $\sum R_i = 18.000 \text{ kN}$ ,  $-.500R_1 + .500R_3 = 1.800 \text{ kN m} = Pe$ , and  $(d_1 + d_3)/2 = d_2$ . Equal  $6.00 \text{ kN}$  forces give zero moment and extensions  $(1.500, 1.000, .750) \text{ mm}$ , violating midpoint compatibility. Independently,  $\sum R_i^2/(2k_i) = 9.0072 \text{ J} = \frac{1}{2}P(d_0 + \theta e)$  for proportional loading.

**Chapter support:** Example 12.23

## Advanced stability / Example 12.24: a hinged link



Consider a rigid link of length  $\ell$ , hinged at its base, with a restoring torsional spring  $-K_\theta\theta$ . A fixed downward dead load  $P$  acts at its tip. Relative to upright, the tip height changes by  $\ell(\cos\theta - 1)$ .

$$\Pi(\theta) = \frac{1}{2}K_\theta\theta^2 + P\ell(\cos\theta - 1), \quad \Pi'(\theta) = K_\theta\theta - P\ell\sin\theta.$$

The upright state has  $\Pi'(0) = 0$  for every  $P$ . Stability concerns nearby admissible tilts. Its quadratic curvature is  $\Pi''(0) = K_\theta - P\ell$ . **Statement.**

In the hinged-link model take  $K_\theta = 120 \text{ N m}$  and  $\ell = 0.600 \text{ m}$ . Determine the upright equilibrium's linearized stability for  $P = 100, 200, 300 \text{ N}$ , and the critical load. Explain whether that critical load is an Euler buckling result for a real slender rod.

Positive curvature is restoring and negative curvature is unstable. Zero curvature requires the next nonzero term in this actual potential.

**Chapter support:** Example 12.24

## Example 12.24: balance, stability and the critical case

The critical linear stiffness occurs at  $P_c = K_\theta/\ell = 120/.600 = 200 \text{ N}$ .

| $P \text{ (N)}$ | $K_\theta - P\ell \text{ (N m)}$ | <b>Upright response</b>             |
|-----------------|----------------------------------|-------------------------------------|
| 100             | 60                               | positive linear restoring stiffness |
| 200             | 0                                | higher-order test required          |
| 300             | -60                              | unstable despite exact balance      |

Using  $\cos \theta = 1 - \theta^2/2 + \theta^4/24 + \dots$  gives

$$\Pi = \frac{1}{2}(K_\theta - P\ell)\theta^2 + \frac{P\ell}{24}\theta^4 + \mathcal{O}(\theta^6).$$

At  $P = 200 \text{ N}$ , the leading term  $5\theta^4 \text{ J}$  is positive for small nonzero tilt: a local minimum remains with zero linear stiffness. Differentiating the exact potential twice checks the curvatures. This rigid-link threshold is not an Euler-column result. Buckling needs elastic bending, cross-section, length, end restraints and load conditions beyond axial stress and static balance.

**Chapter support:** Example 12.24

