

Motion Along a Straight Line

Physics I - Chapter 2

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Lecturer information and how to use this deck



Central question

How can position and time be turned into a quantitative description of one-dimensional motion without confusing coordinates, path length, rates, or model assumptions?

How to study from these slides

Read definitions together with their sign conventions and domains. Follow each worked example through its checks. Use the final problem block to practise choosing a model rather than merely inserting numbers into formulas.

Representations must agree

Words, a physical-track diagram, a position–time graph, a velocity graph and an equation are different representations of the same motion. Their event labels, signs and time domains must remain consistent.

Learning goals and route through Chapter 2



After the core, you should be able to

- define a one-dimensional observation frame;
- distinguish position, displacement and distance;
- distinguish average velocity, instantaneous velocity and speed;
- obtain and interpret local rates from graph slopes;
- diagnose speeding up or slowing down from signed v and a ;
- derive and apply constant-acceleration equations with valid domains;
- select physical roots and reconstruct general motion by signed integration.

Route

frame

$\rightarrow x(t) \rightarrow \bar{v} \rightarrow v(t) \rightarrow a(t)$
 $\rightarrow \text{constant } a \rightarrow \text{free fall}$
 $\rightarrow \int a \, dt, \int v \, dt$
 $\rightarrow \text{sampled data.}$

Supporting data strand

The final core pages add numerical differentiation and integration. Advanced Topics then quantify bias, noise and estimator choice.

Guideline, notation and pedagogical boundary



The guideline

Choose a reference frame, positive direction, origin and clock. Identify the interval and model that apply. Translate among words, diagrams, graphs and equations. Check signs, units, initial conditions and domain before interpreting a numerical result.

Signed quantities

x and y are coordinates. v and a are signed components, not magnitudes. Speed is $|v| \geq 0$. For vertical motion here, $+y$ is upward, $g > 0$, and ideal free-fall acceleration is $a_y = -g$.

Chapter support: Chapter 2 preface.

Kinematics before dynamics

This chapter describes motion before explaining its causes. Newton's laws, force diagrams, drag-force models and formal vector algebra are not prerequisites.

Calculus in context

Derivatives enter as limiting slopes; integrals enter as accumulated signed change. Taylor/variance/least-squares tools are enrichment, not hidden core prerequisites.

Position needs an observation frame

A position measurement compares an object with a reference. For a carriage on a straight rail, specify:

Spatial choices

- laboratory reference objects;
- one marked point as $x = 0$;
- a positive direction;
- the same physical marker on every observation.

Time choices

- a reproducible event as $t = 0$;
- positions and times referring to the same events;
- actual elapsed intervals, not assumed delays.

Do not read motion from one coordinate

A negative coordinate means only that the object lies on the negative side of the chosen origin. It says nothing by itself about distance travelled or direction of motion.

Displacement and distance answer different questions



For initial and final events on the same axis,

$$\Delta x = x_f - x_i.$$

Displacement is signed and depends only on endpoints. Distance travelled ℓ is the non-negative route length; for monotone straight-line legs,

$$\ell = \sum_j |x_{j+1} - x_j|.$$

Every actual reversal must be included among the leg endpoints.

Origin shift

With $x' = x - b$,

$$\Delta x' = (x_f - b) - (x_i - b) = \Delta x.$$

Distance is unchanged.

Axis reversal

With $x' = -x$,

$$\Delta x' = -\Delta x,$$

while distance is unchanged.

Chapter support: Chapter 2 §2.1.

Example 2.1 — Translation-stage return motion

Complete statement

A stage marker is at -0.20 m at $t = 0$, at $+0.60\text{ m}$ at 2.0 s , and at $+0.10\text{ m}$ at 4.0 s . It moves monotonically between successive observations, reversing at the middle observation. Find the displacement and distance over the full interval. Then move the origin to the middle position.

Plan before arithmetic

Use a fixed laboratory axis, positive to the right. There are two monotone legs. Add signed changes for displacement and magnitudes for distance. Then test invariance under the requested origin shift.

Example 2.1 — Solution and track representation



$$\Delta x_1 = 0.60 - (-0.20) = +0.80 \text{ m},$$

$$\Delta x_2 = 0.10 - 0.60 = -0.50 \text{ m}.$$

Therefore

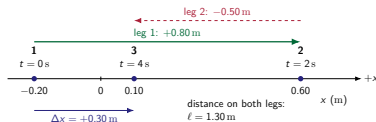
$$\Delta x = +0.30 \text{ m},$$

$$\ell = 1.30 \text{ m}.$$

Endpoint check:

$$0.10 - (-0.20) = +0.30 \text{ m}.$$

Shift the origin by $b = 0.60 \text{ m}$: the coordinates become $-0.80, 0, -0.50 \text{ m}$, while Δx and ℓ remain unchanged.



Interpretation

The return leg undoes part of the first displacement, so route length exceeds the magnitude of the endpoint displacement.

What sparse observations cannot establish



Three recorded positions do not reveal everything between them.

Why the monotonicity assumption matters

Without information excluding extra excursions, summing observed leg lengths gives only a lower bound on distance. A smooth curve through the samples is one possible interpolation, not an observation of the entire route.

$$\ell \geq |\Delta x|$$

for any motion on a line. Equality holds for monotone motion, including intervals of rest.

Return to the start

If a marker goes from $x = 2\text{ m}$ to 5 m and back to 2 m , then $\Delta x = 0$ but $\ell = 6\text{ m}$. Shifting the origin changes neither result.

Average velocity and average speed



Interval-average velocity

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(t_f) - x(t_i)}{t_f - t_i}, \quad t_f > t_i.$$

This definition assumes neither constant acceleration nor monotone motion.

Average speed

$$\text{average speed} = \frac{\ell}{\Delta t} \geq |\bar{v}|.$$

For Example 2.1, the leg averages are $+0.40$ and -0.25 m s^{-1} ; the whole-interval average is $+0.075 \text{ m s}^{-1}$, while average speed is 0.325 m s^{-1} .

Time weighting and the secant slope



If a motion is divided into intervals,

$$\bar{v} = \frac{\sum_j \bar{v}_j \Delta t_j}{\sum_j \Delta t_j}.$$

The whole-interval average is therefore a **time-weighted** average of the interval averages, not generally their arithmetic mean.

Graph interpretation

On a position–time graph, $\bar{v}$ is the slope of the secant through the two endpoint events:

$$\bar{v} = \frac{\Delta x}{\Delta t}.$$

A positive secant slope means positive net displacement over the interval; it does not require $v(t) > 0$ at every interior time.

Instantaneous velocity from a limiting slope



Compare positions over progressively shorter nonzero intervals:

$$\frac{x(t+h) - x(t)}{h}, \quad h \neq 0.$$

If these slopes approach a unique value,

$$v(t) = \lim_{h \rightarrow 0} \frac{x(t+h) - x(t)}{h} = \frac{dx}{dt}.$$

Why not substitute $h = 0$ immediately?

Before simplification, the quotient becomes $0/0$, which has no numerical value. The derivative is the *limit of valid nonzero-interval slopes*.

Geometrically, $v(t)$ is the tangent slope of the position–time graph. Physically, it is a signed local rate in the chosen frame.

Derivative rules needed in the core



For

$$x(t) = x_0 + ut - ct^2,$$

the difference quotient simplifies, for $h \neq 0$, to

$$u - 2ct - ch,$$

so

$$v(t) = u - 2ct.$$

The core repeatedly uses:

$$\frac{d}{dt}(C) = 0, \quad \frac{d}{dt}(t^n) = nt^{n-1}, \quad \frac{d}{dt}(af + bg) = af' + bg'.$$

Checks

A constant shift in position disappears under differentiation. Units also change correctly: differentiating metres with respect to seconds gives m s^{-1} .

Example 2.2 — A polynomial path with a reversal

Complete statement

During $0 \leq t \leq 5 \text{ s}$, a marker obeys

$$x(t) = (-1 \text{ m}) + (4 \text{ m s}^{-1})t - (1 \text{ m s}^{-2})t^2.$$

Find the reversal time, final position, displacement, distance, whole-interval average velocity and average speed. Compare the average velocity with the instantaneous velocity at 1 s .

Strategy

Differentiate to locate sign changes of v . Split distance at an actual reversal rather than taking $|\Delta x|$.

Example 2.2 — Complete solution



$$v(t) = 4 - 2t \quad (\text{m s}^{-1}), \quad v = 0 \Rightarrow t_* = 2 \text{ s}.$$

Velocity is positive before and negative after 2 s, so this is a reversal.

$$x(0) = -1, \quad x(2) = 3, \quad x(5) = -6 \quad \text{m}.$$

Therefore

$$\Delta x = -5 \text{ m}, \quad \ell = 4 + 9 = 13 \text{ m},$$

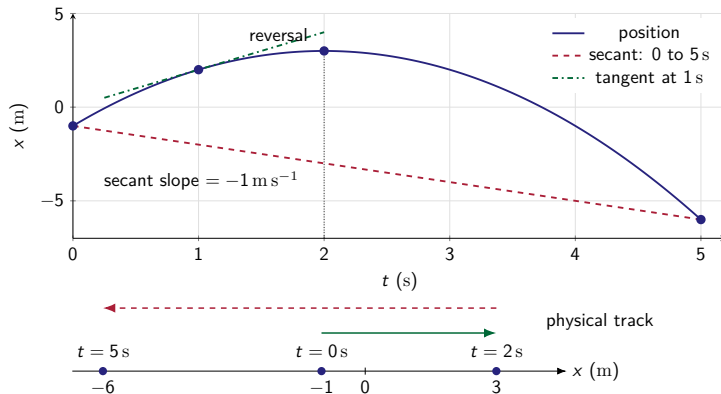
$$\bar{v} = -1.0 \text{ m s}^{-1}, \quad \text{average speed} = 2.6 \text{ m s}^{-1}.$$

At $t = 1 \text{ s}$, $x = 2 \text{ m}$ and $v = +2 \text{ m s}^{-1}$.

Interpretation

A positive instantaneous velocity can coexist with a negative whole-interval average because the later negative displacement is larger.

F02-02 — Secant, tangent, reversal and physical track



Read the representation carefully

The position–time graph is not a picture of the physical rail. The endpoint secant has slope -1 m s^{-1} , while the tangent at $t = 1 \text{ s}$ has slope $+2 \text{ m s}^{-1}$.

The same position does not imply the same motion



For the polynomial motion, $x = 2\text{ m}$ occurs at both $t = 1\text{ s}$ and $t = 3\text{ s}$:

$$v(1) = +2\text{ m s}^{-1}, \quad v(3) = -2\text{ m s}^{-1}.$$

Same coordinate, different event

A coordinate alone does not specify direction or local rate. The event requires both position and time.

Coordinate and time relabelling

Under $x' = x - b$, velocity is unchanged. Under $x' = -x$, velocity changes sign. Under a time-origin shift $t' = t - t_0$, physical rates do not change; only their labels do.

Acceleration is the rate of signed velocity



Average and instantaneous acceleration are

$$\bar{a} = \frac{v_f - v_i}{t_f - t_i}, \quad a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

Acceleration has units m s^{-2} .

Signed meaning

The sign of a states how the signed velocity changes along the chosen positive axis. It does *not* by itself tell whether the object is speeding up or slowing down.

For the polynomial example,

$$a = -2 \text{ m s}^{-2},$$

even though the object first slows, stops, reverses and then speeds up.

Speeding up and slowing down: use va



For $v \neq 0$, the sign of va gives the local change of speed:

$va > 0 \Rightarrow$ speeding up, $va < 0 \Rightarrow$ slowing down.

v	a	direction	speed change
+	+	toward $+x$	speeding up
+	-	toward $+x$	slowing down
-	+	toward $-x$	slowing down
-	-	toward $-x$	speeding up

Common mistake

“Negative acceleration” does not automatically mean “slowing down.” Direction matters.

Example 2.3 — Classify four motion intervals



Complete statement

Motion A is the polynomial of Example 2.2. Motion B is its reflection, $x_B = -x_A$, on the same 0–5 s interval. Classify direction and speed change for each motion before and after 2 s. What happens precisely at that time?

$$v_A = 4 - 2t, \quad a_A = -2; \quad v_B = -4 + 2t, \quad a_B = +2$$

in numerical SI units.

Plan

Reflection changes signed coordinate, velocity and acceleration together. Use the signs of v and a , not the acceleration sign alone.

Example 2.3 — Complete solution



For $0 < t < 2$:

$A: v_A > 0, a_A < 0 \Rightarrow \text{slowing}, \quad B: v_B < 0, a_B > 0 \Rightarrow \text{slowing}.$

For $2 < t < 5$:

$A: v_A < 0, a_A < 0 \Rightarrow \text{speeding up},$

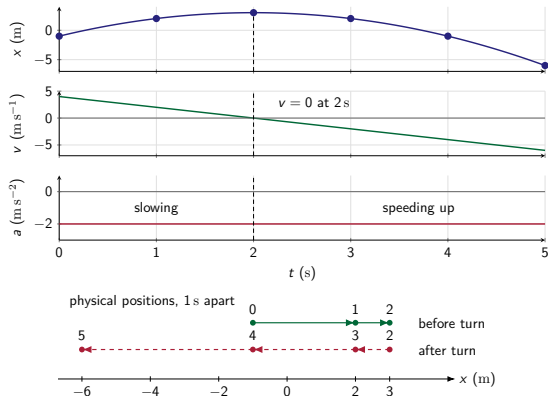
$B: v_B > 0, a_B > 0 \Rightarrow \text{speeding up}.$

At $t = 2\text{ s}$ both velocities vanish but neither acceleration does.

Check

Reflection changes v and a together, leaving va unchanged. Therefore the speed-change classification must remain unchanged.

F02-03 — Linked x , v , a graphs and motion diagrams



Reading across representations

At a shared time, the slope of $x(t)$ is $v(t)$ and the slope of $v(t)$ is $a(t)$. Equal-time physical markers spread out when speed is large and crowd together near the reversal.

Zero derivatives require interpretation

Zero velocity need not mean reversal

For $x(t) = \beta t^3$ with $\beta > 0$,

$$v = 3\beta t^2 \geq 0.$$

At $t = 0$, $v = 0$ but the sign of velocity does not change. The object momentarily stops without reversing.

Zero acceleration at one instant

For the same motion,

$$a = 6\beta t,$$

so $a(0) = 0$ at one instant. This does not imply a constant-velocity interval.

Diagnosis

A zero derivative identifies a local condition. A reversal or a constant-rate interval requires information from a neighbourhood in time.

Constant acceleration: accumulate a into v



Suppose a phase begins at t_0 with position x_0 and velocity v_0 . Define local elapsed time

$$\tau = t - t_0.$$

For constant acceleration,

$$\frac{dv}{dt} = a \quad \Rightarrow \quad v - v_0 = \int_{t_0}^t a \, ds = a\tau,$$

so

$$v = v_0 + a\tau.$$

Why local time helps

Using $\tau = 0$ at the start of each phase makes the initial condition explicit and prevents mixing a global clock with phase-local formulas.

Accumulate v into position: signed area and integral



Adding short signed displacement contributions and taking the limit gives

$$x(t) - x_0 = \int_{t_0}^t v(s) ds.$$

For $v = v_0 + a(s - t_0)$,

$$x - x_0 = v_0\tau + \frac{1}{2}a\tau^2,$$

hence

$$x = x_0 + v_0\tau + \frac{1}{2}a\tau^2.$$

Graph meaning

The signed area under a velocity–time graph is displacement. Positive and negative areas can cancel even while the object travels a substantial distance.

Constant-acceleration equation set — and its domain



On one constant-acceleration phase:

$$v = v_0 + a\tau, \quad x = x_0 + v_0\tau + \frac{1}{2}a\tau^2,$$

$$\Delta x = \frac{1}{2}(v_0 + v)\tau, \quad v^2 = v_0^2 + 2a\Delta x.$$

Domain warning

These equations are exact only where the model assumes constant acceleration. The squared-velocity relation does not by itself determine the sign of v .

Checks

At $\tau = 0$, recover x_0 and v_0 . Differentiate the position formula to recover velocity and acceleration.

Piecewise motion: transfer the state and reset local time



For each phase:

- 1 record its start time, position x_0 , velocity v_0 , acceleration and allowed duration;
- 2 solve only inside that domain;
- 3 evaluate x and v at the phase end;
- 4 use those end values as the next phase's initial conditions;
- 5 reset local elapsed time for the new phase.

Continuity

When acceleration changes abruptly in an idealized program, acceleration can jump while position and velocity remain continuous.

Common error

Do not keep using a local-time quadratic beyond the phase for which its acceleration was prescribed.

Example 2.4 — Lift start–cruise–stop



Complete statement

An ideal lift starts from rest at $x = 0$ and $t = 0$, upward positive. It accelerates at $+1.20 \text{ m s}^{-2}$ for 2.00 s , cruises for 3.00 s , then slows at -0.80 m s^{-2} until it stops. The prescribed speed cap is 2.40 m s^{-1} . Find total duration, displacement and the continuous position function, and check the cap.

Model

Acceleration is constant within each phase; changes between phases are idealized as instantaneous. This is a kinematic program, not a drive-force or comfort model.

Example 2.4 — Start and cruise phases



Start, $0 \leq t \leq 2$ s:

$$v = 1.20t, \quad x = 0.60t^2.$$

At $t = 2$ s,

$$v_1 = 2.40 \text{ m s}^{-1}, \quad x_1 = 2.40 \text{ m}.$$

Cruise, $2 \leq t \leq 5$ s:

$$v = 2.40, \quad x = 2.40 + 2.40(t - 2).$$

Cruise displacement:

$$(2.40)(3.00) = 7.20 \text{ m},$$

so

$$x_2 = 9.60 \text{ m} \quad \text{at } t = 5 \text{ s}.$$

Cap check

The calculated peak speed is exactly 2.40 m s^{-1} , equal to the prescribed cap.

Example 2.4 — Stopping phase, result and checks



Let $\tau = t - 5$ s. Then

$$0 = 2.40 - 0.80\tau \Rightarrow \tau_{\text{stop}} = 3.00 \text{ s.}$$

For $5 \leq t \leq 8$ s,

$$v = 2.40 - 0.80(t - 5),$$

$$x = 9.60 + 2.40(t - 5) - 0.40(t - 5)^2.$$

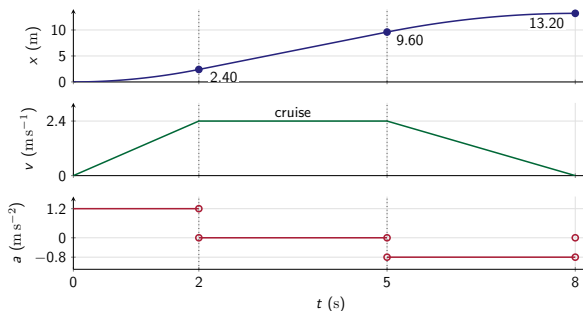
At $t = 8$ s,

$$v = 0, \quad x = 13.20 \text{ m.}$$

Independent checks

The velocity graph areas are $2.40 + 7.20 + 3.60 = 13.20$ m. Position and velocity agree from both sides at the phase joins. Continuing the last quadratic beyond 8 s would be outside the intended model.

F02-04 — Piecewise start–cruise–stop motion



What must remain continuous?

Position and velocity are continuous at $t = 2$ and 5 s; the idealized acceleration jumps. The velocity area reproduces the full displacement.

A kinematic constraint can change the motion program



For symmetric acceleration and braking magnitude A , a stage attempting to reach speed cap V needs ramp distance

$$D_{\text{ramps}} = \frac{V^2}{A}.$$

Two regimes

If $D \geq V^2/A$, the profile can be trapezoidal: accelerate, cruise, brake.

If $D < V^2/A$, there is no room to reach the cap. The profile is triangular with

$$v_{\text{peak}} = \sqrt{AD}, \quad T = 2\sqrt{\frac{D}{A}}.$$

Boundary check

At $D = V^2/A$, the triangular and trapezoidal descriptions meet continuously.

Example 2.5 — A short traverse cannot reach the speed cap

Complete statement

Take $D = 0.120 \text{ m}$, $A = 0.80 \text{ m s}^{-2}$ and $V = 0.40 \text{ m s}^{-1}$. Decide whether the stage reaches the cap and find the ideal duration.

Threshold:

$$\frac{V^2}{A} = \frac{0.40^2}{0.80} = 0.200 \text{ m} > D,$$

so the cap cannot be reached.

$$v_{\text{peak}} = \sqrt{AD} = 0.310 \text{ m s}^{-1},$$

$$T = 2\sqrt{D/A} = 0.775 \text{ s}.$$

Each ramp lasts 0.387 s and covers 0.0600 m .

Area check

$\frac{1}{2} T v_{\text{peak}} = 0.120 \text{ m}$ and $v_{\text{peak}} < V$, as required.

Example 2.6 — Two carriages: statement and encounter equation

Complete statement

Two markers on parallel straight guides have

$$x_A = 0.30 + 0.20t, \quad x_B = 0.10t^2$$

in SI units for $0 \leq t \leq 4$ s. When are their longitudinal positions equal, and are their velocities then equal?

Set $x_A = x_B$:

$$0.10t^2 - 0.20t - 0.30 = 0 \Rightarrow (t - 3)(t + 1) = 0.$$

The algebra gives $t = -1$ s and $t = 3$ s. The model domain decides which event is physically admissible.

Example 2.6 — Physical root, velocities and check



The root $t = -1\text{ s}$ is outside the stated domain. The admissible encounter is

$$\boxed{t = 3.00\text{ s}}, \quad x_A = x_B = 0.90\text{ m}.$$

Differentiate:

$$v_A = 0.20\text{ m s}^{-1}, \quad v_B = 0.20t.$$

At the encounter,

$$v_B = 0.60\text{ m s}^{-1} \neq v_A.$$

Interpretation and check

The markers share a longitudinal position, not a velocity. $x_B - x_A$ changes from negative to positive at 3 s, so B passes A's longitudinal position. Parallel guides avoid assuming an actual collision.

Free fall: model, signs and domain



Near Earth's surface, when air resistance is negligible and the height range is small, vertical motion may be approximated by constant downward acceleration:

$$+y \text{ upward}, \quad g > 0, \quad a_y = -g.$$

For initial height y_0 and upward velocity w ,

$$v_y(t) = w - gt,$$

$$y(t) = y_0 + wt - \frac{1}{2}gt^2,$$

$$v_y^2 = w^2 - 2g(y - y_0).$$

Model boundary

Mass does not enter these kinematic equations. Air resistance, buoyancy, propulsion, supports or collisions can make the model inappropriate.

Landing at a lower level is an event condition



Let the arrival level be $H > 0$ below release:

$$y_L = y_0 - H.$$

Then

$$\frac{1}{2}gt^2 - wt - H = 0,$$

with roots

$$t_{\pm} = \frac{w \pm \sqrt{w^2 + 2gH}}{g}.$$

For $w > 0$ and $H > 0$, $t_- < 0$ and $t_+ > 0$; only t_+ lies in the post-launch domain.
At arrival,

$$v_{y,L} = -\sqrt{w^2 + 2gH}.$$

Important distinction

The time-free equation supplies the speed magnitude, but the event/domain analysis is what selects the negative descending sign.

Example 2.7 — Vertical launch to a lower platform

Complete statement

An ideal point-like object is launched vertically upward at 3.00 m s^{-1} from $y_0 = 1.80 \text{ m}$. It arrives at a lower level $y_L = 0.60 \text{ m}$. Use $g = 9.81 \text{ m s}^{-2}$ and neglect air resistance. Find the apex, arrival time and velocity, displacement and total distance before arrival.

Model boundary

Calculations end when the object first reaches the lower level. They do not describe bounce, penetration or contact forces.

Example 2.7 — Complete solution



Apex:

$$t_{\text{apex}} = \frac{3.00}{9.81} = 0.3058 \text{ s}, \quad y_{\text{apex}} = 1.80 + \frac{3.00^2}{2(9.81)} = 2.2587 \text{ m}.$$

With $H = 1.20 \text{ m}$,

$$t_L = \frac{3.00 + \sqrt{3.00^2 + 2(9.81)(1.20)}}{9.81} = 0.8873 \text{ s},$$

$$v_{y,L} = 3.00 - 9.81(0.8873) = -5.705 \text{ m s}^{-1}.$$

Displacement:

$$\Delta y = -1.20 \text{ m}.$$

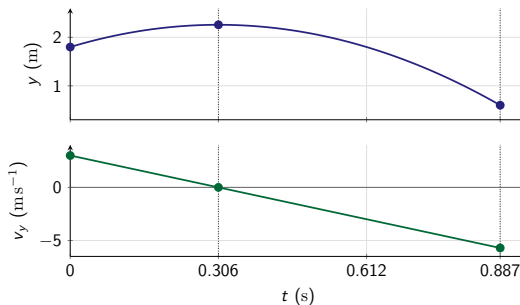
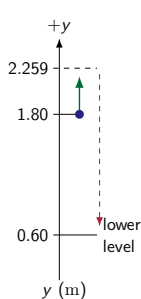
Distance:

$$\ell = H + \frac{w^2}{g} = 2.117 \text{ m}.$$

Checks

The other time root is negative. The time-free equation gives $v_{y,L}^2 = 32.544 \text{ m}^2 \text{ s}^{-2}$; substituting t_L in $y(t)$ returns 0.60 m .

F02-05 — Vertical geometry and matched motion graphs



Interpretation

At the apex $v_y = 0$ but $a_y = -g$ remains nonzero. The trajectory ends at first contact with the lower level; extending the parabola further would exceed the model domain.

Free-fall limits and alternative event questions

Same-height return

When $H \rightarrow 0$ with $w > 0$, the post-launch roots become

$$t = 0, \quad t = \frac{2w}{g}.$$

The nonzero root is the return to the launch height.

Released from rest

Set $w = 0$:

$$H = \frac{1}{2}gt^2, \quad |v| = \sqrt{2gH}.$$

Do not over-extend the model

A mathematical parabola may continue after a collision, but the free-flight model does not.

Coordinate consistency

Using downward-positive $z = y_0 - y$ gives $a_z = +g$ and reverses velocity signs while leaving physical times and speeds unchanged.

General motion from integration



When acceleration is not constant,

$$v(t) = v(t_0) + \int_{t_0}^t a(s) \, ds,$$

$$x(t) = x(t_0) + \int_{t_0}^t v(s) \, ds.$$

The total distance is

$$\ell = \int_{t_i}^{t_f} |v(t)| \, dt,$$

or equivalently a sum of signed-integral magnitudes split at reversals.

Initial conditions matter

Knowing $a(t)$ determines changes in velocity, not the initial velocity. Knowing $v(t)$ determines changes in position, not the initial position.

Example 2.8 — A reversible translation program

Complete statement

A marker has $x(0) = -0.20$ m. Its velocity varies linearly between:

t (s)	0	2	4	8	10
v (m s ⁻¹)	0	+0.40	+0.40	-0.40	0

Find the position at every phase boundary, the reversal event, net displacement and total distance. Assume the piecewise-linear velocity itself is the model.

Strategy

Use signed velocity areas to update position. Split distance wherever velocity changes sign.

Example 2.8 — Signed areas, boundary positions and distance



Signed velocity areas over 0–2, 2–4, 4–8, 8–10 s are

$$+0.40, \quad +0.80, \quad 0, \quad -0.40 \quad \text{m}.$$

Hence

$$x(0) = -0.20, \quad x(2) = 0.20, \quad x(4) = 1.00, \quad x(8) = 1.00, \quad x(10) = 0.60 \quad \text{m}.$$

The zero signed area on 4–8 s does not mean no motion. Velocity crosses zero at 6 s, where $x = 1.40$ m.

$$\Delta x = +0.80 \text{ m}, \quad \ell = 2.40 \text{ m}.$$

Average velocity is 0.080 m s^{-1} ; average speed is 0.240 m s^{-1} .

Example 2.8 — Reconstructed position function



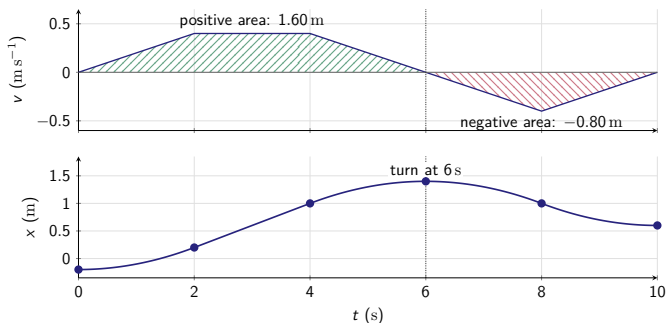
Integrating each phase and carrying the boundary position forward gives

$$x(t) = \begin{cases} -0.20 + 0.10t^2, & 0 \leq t \leq 2, \\ 0.20 + 0.40(t - 2), & 2 \leq t \leq 4, \\ 1.00 + 0.40(t - 4) - 0.10(t - 4)^2, & 4 \leq t \leq 8, \\ 1.00 - 0.40(t - 8) + 0.10(t - 8)^2, & 8 \leq t \leq 10. \end{cases}$$

Checks

Differentiating each branch recovers the programmed velocity. Position and velocity agree at every join. The largest position occurs at $t = 6$ s, inside a phase rather than at a listed phase boundary.

F02-06 — Signed velocity area and reconstructed position



Read signed area before endpoint formulas

The positive and negative velocity areas produce the boundary positions and reveal the interior maximum at the reversal.

Example 2.9 — Integrating a linearly increasing acceleration



Complete statement

A stage has

$$a(t) = \kappa t, \quad \kappa = 0.60 \text{ m s}^{-3},$$

with $v_0 = 0.40 \text{ m s}^{-1}$ and $x_0 = 0.20 \text{ m}$. Determine v and x during $0 \leq t \leq 2 \text{ s}$.
Test the endpoint-average-velocity shortcut.

Plan

Integrate acceleration with the velocity initial condition, then integrate velocity with the position initial condition. Only afterward test any shortcut.

Example 2.9 — Complete solution and shortcut test



$$v(t) = v_0 + \int_0^t \kappa s \, ds = v_0 + \frac{1}{2} \kappa t^2,$$

$$x(t) = x_0 + \int_0^t \left(v_0 + \frac{1}{2} \kappa s^2 \right) ds = x_0 + v_0 t + \frac{1}{6} \kappa t^3.$$

With $\alpha = \kappa/6 = 0.10 \text{ m s}^{-3}$,

$$x = x_0 + v_0 t + \alpha t^3, \quad v = v_0 + 3\alpha t^2.$$

At 2 s:

$$v = 1.60 \text{ m s}^{-1}, \quad x = 1.80 \text{ m},$$

so $\Delta x = 1.60 \text{ m}$ and $\bar{v} = 0.80 \text{ m s}^{-1}$.

Shortcut test

$(0.40 + 1.60)/2 = 1.00 \text{ m s}^{-1}$ overestimates displacement by 0.40 m. Endpoint averaging is exact for velocity only under the appropriate linear-in-time condition.

Finite motion data: forward and centred differences



For uniformly spaced samples $t_i = t_0 + ih$:

$$D_f x_i = \frac{x_{i+1} - x_i}{h}, \quad D_c x_i = \frac{x_{i+1} - x_{i-1}}{2h}.$$

Forward difference

For exact positions it is the interval-average velocity. Assigning it to t_i as an instantaneous estimate is an additional approximation.

Centred difference

It is naturally assigned to the interior time t_i . Symmetry removes some finite-step error, but endpoints require another rule or more data.

Calibration effects

A common additive position offset cancels. A scale error does not. Timing error also matters because time is in the denominator.

Example 2.10 — Recover velocity from constructed readings

Complete statement

Use

$$x = x_0 + v_0 t + \alpha t^3, \quad x_0 = 0.20 \text{ m}, \quad v_0 = 0.40 \text{ m s}^{-1}, \quad \alpha = 0.10 \text{ m s}^{-3},$$

with constructed readings spaced by $h = 0.25 \text{ s}$. Estimate v at 1 s and compare with the exact model.

$t \text{ (s)}$	$x_{\text{obs}} \text{ (m)}$	$v_{\text{exact}} \text{ (m/s)}$	$D_c x \text{ (m/s)}$
0.00	0.197651	0.400000	—
0.25	0.299079	0.418750	0.435448
0.50	0.415375	0.475000	0.484422
0.75	0.541290	0.568750	0.572026
1.00	0.701388	0.700000	0.713734
1.25	0.898157	0.868750	0.874312
1.50	1.138544	1.075000	1.075540
1.75	1.435927	1.318750	1.319754
2.00	1.798421	1.600000	—

Example 2.10 — Estimate, exact benchmark and error decomposition



At $t = 1$ s:

$$D_c x(1) = \frac{0.898157 - 0.541290}{0.50} = 0.713734 \text{ m s}^{-1}.$$

Exact model:

$$v(1) = 0.40 + 0.30(1)^2 = 0.700000 \text{ m s}^{-1}.$$

Observed signed error:

$$+0.013734 \text{ m s}^{-1}.$$

With exact unperturbed positions, the same finite difference is

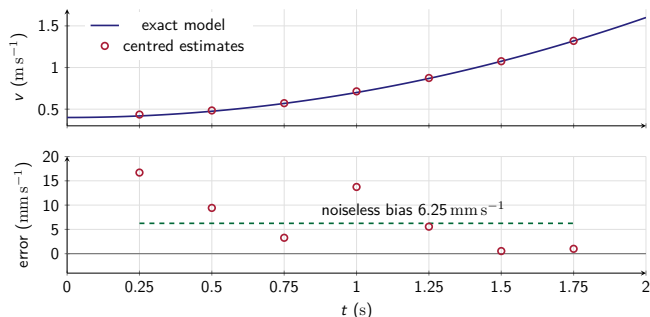
$$0.706250 \text{ m s}^{-1},$$

so finite-step bias is $+0.006250 \text{ m s}^{-1}$ and this particular record's noise contribution is $+0.007484 \text{ m s}^{-1}$.

Interpretation

The values are synthetic teaching data. Extra printed digits support reproducibility; they do not imply micrometre measurement accuracy.

F02-07 — Exact derivative, centred estimates and signed errors



Two different error sources

The dashed level is the noiseless finite-step bias for this cubic at $h = 0.25 \text{ s}$. Scatter about it comes from the particular added position errors.

Reproducible centred differentiation: algorithm and tests

Essential algorithm

```
check same array length
check finite values
check strictly increasing time
check uniform spacing  $h$ 
for  $i = 1, \dots, n - 2$ :
     $v_i = (x_{i+1} - x_{i-1}) / (2h)$ 
return interior times and estimates
```

Validation sequence

- constant position $\rightarrow 0$;
- linear motion $\rightarrow$ exact constant v ;
- quadratic $\rightarrow$ exact centred derivative;
- cubic $\rightarrow$ finite-step bias;
- common offset $\rightarrow$ invariant estimate;
- $x \rightarrow -x \rightarrow v \rightarrow -v$.

Software validation is not calibration

Input checks do not prove that the physical position scale, clock or motion model is adequate.

Laboratory connection: build an auditable position record



A low-energy investigation can use a marker on a constrained track with a fixed camera or suitable position sensor. This is a proposed method, not a claim about available apparatus.

Record enough information to reproduce the coordinates

- reference object, origin and positive direction;
- the physical marker being tracked;
- actual timestamps and units;
- calibration/scale information;
- raw positions before smoothing;
- any rejected samples and the reason.

Analysis discipline

Differentiate the measured record with a declared estimator. Keep measured positions, inferred velocities and model curves visually and conceptually distinct.

Computational connection: integrate sampled acceleration



On a sampled grid, approximate acceleration and velocity linearly over each interval:

$$v_{i+1} = v_i + \frac{1}{2}(a_i + a_{i+1})h_i,$$

$$x_{i+1} = x_i + \frac{1}{2}(v_i + v_{i+1})h_i, \quad h_i = t_{i+1} - t_i > 0.$$

Refinement check

For Example 2.9 over 0–2 s, compare coarse and refined grids against the known analytic endpoint values. A converging numerical method should reduce integration error as the grid is refined while preserving initial conditions.

Do not confuse the operations

Differentiation tends to amplify high-frequency position error; integration tends to accumulate systematic offsets and model error.

Core formula sheet I — definitions and accumulation



Definitions on a fixed axis

$$\Delta x = x_f - x_i, \quad \bar{v} = \frac{\Delta x}{\Delta t},$$

$$v = \frac{dx}{dt}, \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2},$$

$$\ell = \int |v| dt.$$

Accumulation from initial conditions

$$v(t) = v(t_0) + \int_{t_0}^t a(s) ds,$$

$$x(t) = x(t_0) + \int_{t_0}^t v(s) ds.$$

For constant a :

$$v = v_0 + a\tau, \quad x = x_0 + v_0\tau + \frac{1}{2}a\tau^2.$$

Quick test

Every formula must carry its frame, sign convention, initial condition and valid interval.

Core formula sheet II — free fall, samples and final checks

Free fall: +y upward

$$v_y = w - gt, \quad y = y_0 + wt - \frac{1}{2}gt^2.$$

For a lower level H below release:

$$t_L = \frac{w + \sqrt{w^2 + 2gH}}{g},$$

$$v_{y,L} = -\sqrt{w^2 + 2gH}.$$

Sampled position

$$D_f x_i = \frac{x_{i+1} - x_i}{h},$$

$$D_c x_i = \frac{x_{i+1} - x_{i-1}}{2h}.$$

Four final questions

- 1 What model and time domain apply?
- 2 Which quantities are signed?
- 3 What independent check is available?
- 4 Am I using a model function, measured data, or an estimate derived from data?

Advanced Topics — Taylor expansion as a local model

Advanced Topics for Technical and Engineering Students

For a sufficiently smooth position function near time t ,

$$x(t+h) = x(t) + v(t)h + \frac{a(t)}{2}h^2 + \frac{x^{(3)}(t)}{6}h^3 + \frac{x^{(4)}(t)}{24}h^4 + \dots$$

The expansion is a *local model*. It requires the relevant derivatives to exist and remain controlled on the stencil neighbourhood.

Why it matters

Finite differences do not exactly equal derivatives at finite h . Taylor expansion exposes the deterministic truncation terms and their dependence on step size.

Advanced Topics — Forward-difference truncation error



Insert the Taylor expansion into

$$D_f x(t) = \frac{x(t+h) - x(t)}{h}.$$

Then

$$D_f x(t) = v(t) + \frac{1}{2}a(t)h + \frac{1}{6}x^{(3)}(t)h^2 + \dots.$$

Therefore the leading truncation error is

$$D_f x - v = \frac{1}{2}a h + \mathcal{O}(h^2).$$

Interpretation

At fixed smooth motion, the leading forward-difference bias decreases linearly with h . But this says nothing yet about measurement noise, which is divided by h .

Advanced Topics — Centred-difference cancellation



Expand $x(t + h)$ and $x(t - h)$. Odd powers change sign; even powers do not. Subtracting gives

$$D_c x(t) = \frac{x(t + h) - x(t - h)}{2h} = v(t) + \frac{x^{(3)}(t)}{6} h^2 + \mathcal{O}(h^4).$$

Thus the leading truncation error is second order:

$$D_c x - v = \frac{x^{(3)}}{6} h^2 + \mathcal{O}(h^4).$$

For the cubic $x = x_0 + v_0 t + \alpha t^3$, $x^{(3)} = 6\alpha$, so

$$b_c = \alpha h^2$$

exactly.

Advanced Topics — How measurement errors enter a difference



Write a measured position as

$$X_i = x(t_i) + \epsilon_i.$$

Assume initially independent, zero-mean errors with

$$\text{sd}(\epsilon_i) = u_x.$$

For a forward difference,

$$u_{v,f} = \frac{\sqrt{2} u_x}{h}.$$

For a centred difference,

$$u_{v,c} = \frac{u_x}{\sqrt{2} h}.$$

Tradeoff already visible

Shrinking h can reduce truncation bias but increases the velocity noise scale like $1/h$ when position uncertainty stays fixed.

Advanced Topics — Correlation, common offset and scale error



If the two readings in a centred difference have equal variance u_x^2 and correlation coefficient ρ ,

$$u_{v,c}^2 = \frac{u_x^2(1 - \rho)}{2h^2}.$$

Common additive offset

If every position has the same added constant, it cancels exactly from a difference. Strong positive correlation can therefore reduce difference noise.

Multiplicative scale error

If every position is multiplied by $1 + \delta$, every estimated velocity is also multiplied by $1 + \delta$. Differencing does not remove calibration-scale error.

Advanced Topics — Bias, variance, MSE and RMSE



For the cubic trajectory, the centred-estimate error combines deterministic bias αh^2 with zero-mean random noise:

$$\text{MSE}(h) = \alpha^2 h^4 + \frac{u_x^2}{2h^2},$$

$$\text{RMSE}(h) = \sqrt{\text{MSE}(h)}.$$

Minimizing the MSE gives

$$h_{\text{opt}} = \left(\frac{u_x^2}{4\alpha^2} \right)^{1/6}.$$

At the optimum

The noise variance equals twice the squared bias. An “optimal” spacing is therefore model- and uncertainty-dependent, not universal.

Advanced Topics — Example 2.11: choosing a differentiation interval

Complete statement

At $t = 1 \text{ s}$ in the cubic model, take $\alpha = 0.10 \text{ m s}^{-3}$ and independent position errors with $u_x = 0.002 \text{ m}$. Compare $h = 0.50, 0.25$ and 0.125 s , and find the continuous optimum.

$h \text{ (s)}$	bias	noise s.d.	RMSE
0.500	0.025000	0.002828	0.025159
0.250	0.006250	0.005657	0.008430
0.125	0.001563	0.011314	0.011421

All error columns are in m s^{-1} .

Lesson

The smallest spacing has the smallest bias but not the smallest expected total error.

Advanced Topics — Example 2.11 solution and F02-08



$$h_{\text{opt}} = \left(\frac{0.002^2}{4(0.10)^2} \right)^{1/6} = 0.2154 \text{ s}.$$

At the optimum:

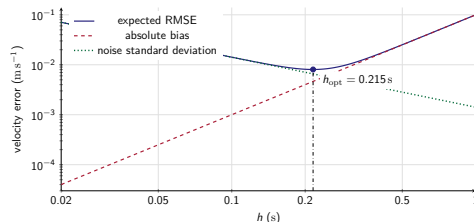
$$b = 0.004642 \text{ m s}^{-1},$$

$$\text{RMSE} = 0.008039 \text{ m s}^{-1}.$$

The centred stencil spans $2h$, not h .

Data limitation

The existing 0.25 s grid cannot create independent observations at h_{opt} merely by interpolation.



Advanced Topics — Five-point straight-line fit



For five readings X_k at offsets kh , $k = -2, \dots, 2$, fit $p + qkh$ by least squares. Symmetry gives

$$q = \frac{-2X_{-2} - X_{-1} + X_1 + 2X_2}{10h}.$$

For independent equal-variance position errors,

$$u_q = \frac{u_x}{\sqrt{10} h}.$$

For the cubic trajectory,

$$q - v(t) = 3.4 \alpha h^2.$$

Tradeoff

At fixed adjacent spacing, the five-point fit lowers noise variance but spans a longer time window, creating larger cubic-curvature bias than the three-point centred difference.

Advanced Topics — Example 2.12: quieter can still be worse

Complete statement

Compare the three-point centred estimate with the five-point straight-line fit at $t = 1$ s, $h = 0.25$ s, for the same cubic motion and independent-noise model.

estimator	bias	noise s.d.	RMSE
3-point centred	0.006250	0.005657	0.008430
5-point fit	0.021250	0.002530	0.021400

All error columns are in m s^{-1} .

Check

Exact cubic positions give $q = 0.721250 \text{ m s}^{-1}$, confirming the 0.021250 bias relative to the true 0.700000.

Conclusion

Lower scatter does not guarantee lower total error when estimator bias changes.

Advanced Topics — Second differences amplify position noise



A symmetric acceleration estimate is

$$D^2 x_i = \frac{X_{i+1} - 2X_i + X_{i-1}}{h^2}.$$

For independent equal-variance position errors,

$$\text{Var}(D^2 x_i) = \frac{1 + 4 + 1}{h^4} u_x^2 = \frac{6u_x^2}{h^4}.$$

Therefore

$$u_a = \frac{\sqrt{6} u_x}{h^2}.$$

Stronger noise amplification

Halving h multiplies this acceleration-noise standard deviation by four. Numerical second derivatives can therefore be much noisier than first derivatives.

Advanced Topics — A reproducible estimator investigation



Four-stage investigation

- 1 Use exact polynomial data to verify predicted powers of h without noise.
- 2 Add errors from a declared distribution with a recorded seed; inspect signed errors.
- 3 Repeat many realizations to compare empirical bias, variance and RMSE with formulas.
- 4 Change the estimator or window while holding the comparison criterion explicit.

What to report

Model, estimator formula, time assignment, grid spacing, uncertainty model, correlation assumptions, random seed, number of repetitions, and the exact benchmark used for verification.

Advanced Topics — Smoothness limits and practical choice

Smooth-error formulas have domains

A centred stencil that straddles an abrupt acceleration switch samples two phases. Taylor-order statements require a neighbourhood in which the assumed derivatives exist and remain controlled.

Practical estimator choice

Balance:

- model curvature and smoothness;
- position/timing uncertainty and correlation;
- required temporal resolution and endpoint availability;
- whether the target metric is bias, scatter or total expected error.

Engineering lesson. “More points” and “smaller spacing” are not universal rules; state the error model and comparison criterion.

Problem 2.4 — A negative average during initially positive motion



Complete problem statement

For $0 \leq t \leq 4 \text{ s}$ let

$$x = 1 + 0.60t - 0.20t^2$$

in numerical SI values. Find the reversal event, displacement, distance, average velocity, average speed and instantaneous velocity at 1 s. Draw the endpoint secant and indicate whether its slope is positive or negative. Diagnose a report of distance 0.80 m obtained by taking the absolute endpoint displacement.

Before solving

Distance requires locating the actual reversal; $|\Delta x|$ is only a lower bound in general.

Problem 2.4 — Solution



$$v = 0.60 - 0.40t, \quad v = 0 \Rightarrow t_* = 1.50 \text{ s.}$$
$$x(0) = 1.00, \quad x(1.5) = 1.45, \quad x(4) = 0.20 \text{ m.}$$

Thus

$$\Delta x = -0.80 \text{ m,}$$
$$\ell = (1.45 - 1.00) + (1.45 - 0.20) = 1.70 \text{ m.}$$
$$\bar{v} = -0.20 \text{ m s}^{-1}, \quad \text{average speed} = 0.425 \text{ m s}^{-1}.$$

At $t = 1 \text{ s}$:

$$v = +0.20 \text{ m s}^{-1}.$$

The endpoint secant is $x = 1 - 0.20t$ and has negative slope.

Independent area check

The velocity graph has $+0.45 \text{ m}$ positive area and -1.25 m negative area. Their signed sum is -0.80 m and magnitude sum is 1.70 m .

Problem 2.12 — Does the kinematic stopping model fit the clearance?



Complete problem statement

A hypothetical stage enters a braking phase with $v_0 = 0.75 \text{ m s}^{-1}$. The model assumes an immediate constant acceleration -1.50 m s^{-2} and permits 0.160 m further travel. Is the modelled stop inside that distance? Find the minimum constant braking magnitude that would suffice with zero delay.

Model disclaimer

This is an ideal kinematic calculation, not a machine-safety specification. Real design must account for delays, uncertainty, actuator limits and safety margins.

Problem 2.12 — Solution



Write $a = -A$ with $A > 0$. At first stop,

$$0 = v_0^2 - 2A\Delta x.$$

Hence

$$\Delta x_{\text{stop}} = \frac{0.75^2}{2(1.50)} = 0.1875 \text{ m.}$$

Stopping time:

$$t_{\text{stop}} = \frac{0.75}{1.50} = 0.50 \text{ s.}$$

The modelled stop exceeds the allowed distance by

$$0.1875 - 0.160 = 0.0275 \text{ m.}$$

Require

$$\frac{v_0^2}{2A} \leq 0.160 \Rightarrow A \geq 1.7578 \text{ m s}^{-2}.$$

Independent check

The triangular velocity area is $\frac{1}{2}(0.50)(0.75) = 0.1875 \text{ m.}$

Problem 2.15 — Unequal launch and arrival heights



Complete problem statement

An ideal object is launched upward at 4.00 m s^{-1} from $y_0 = 2.00 \text{ m}$ and arrives at $y = 0$. Find the apex time and height, arrival time and signed velocity. A student uses $2v_0/g$ for the arrival time. Explain the error, recover the return-time formula when arrival height equals launch height, and write the same trajectory using a downward-positive coordinate measured from release.

Model

Use ideal free fall with $+y$ upward and $g = 9.81 \text{ m s}^{-2}$.

Problem 2.15 — Solution



$$t_{\text{apex}} = \frac{4.00}{9.81} = 0.40775 \text{ s},$$

$$y_{\text{apex}} = 2.00 + \frac{4.00^2}{2(9.81)} = 2.81549 \text{ m}.$$

Arrival:

$$0 = 2.00 + 4.00t - \frac{1}{2}(9.81)t^2,$$

$$t_L = \frac{4.00 + \sqrt{4.00^2 + 2(9.81)(2.00)}}{9.81} = 1.1654 \text{ s},$$

$$v_{y,L} = -\sqrt{55.24} = -7.432 \text{ m s}^{-1}.$$

$2v_0/g = 0.81549 \text{ s}$ is the return to $y = 2.00 \text{ m}$, not to $y = 0$. With downward-positive $z = 2.00 - y$:

$$z = -4.00t + \frac{1}{2}gt^2, \quad v_z = -4.00 + gt, \quad a_z = +g.$$

The time and speed are unchanged; the velocity sign reverses with the axis.

Problem 2.20 — A reversal under time-dependent acceleration



Complete problem statement

Let

$$a(t) = (0.20 \text{ m s}^{-3})t, \quad v(0) = -0.40 \text{ m s}^{-1}, \quad x(0) = 0.70 \text{ m}$$

for $0 \leq t \leq 3 \text{ s}$. Derive $v(t)$ and $x(t)$, find the reversal, and compute full-interval displacement and distance.

Plan

Integrate twice with initial conditions. Then split total distance at the actual velocity sign change.

Problem 2.20 — Solution I: integrate and locate the reversal



Integrate acceleration:

$$v(t) = -0.40 + 0.10t^2.$$

Integrate again:

$$x(t) = 0.70 - 0.40t + \frac{t^3}{30}.$$

Set $v = 0$:

$$-0.40 + 0.10t^2 = 0 \Rightarrow t = \pm 2 \text{ s}.$$

Only

$$t_* = 2.00 \text{ s}$$

lies in the model domain. Velocity is negative before and positive afterward, so position has a minimum there.

$$x(2) = 0.70 - 0.80 + \frac{8}{30} = \frac{1}{6} \text{ m}.$$

At $t = 3 \text{ s}$:

$$x(3) = 0.40 \text{ m}.$$

Chapter support: Chapter 2 detailed solution to Problem 2.20.

Problem 2.20 — Solution II: distance and independent checks



Full-interval displacement:

$$\Delta x = 0.40 - 0.70 = -0.30 \text{ m.}$$

Split distance at the reversal:

$$\ell = \left(0.70 - \frac{1}{6}\right) + \left(0.40 - \frac{1}{6}\right) = \frac{23}{30} \text{ m} = 0.7667 \text{ m.}$$

Independent signed-integral check

$$\int_0^2 v \, dt = -\frac{8}{15} \text{ m}, \quad \int_2^3 v \, dt = +\frac{7}{30} \text{ m.}$$

Signed sum: -0.30 m . Magnitude sum: $23/30 \text{ m}$.

Differentiating $x(t)$ twice returns the stipulated $a(t) = 0.20t$.

Problem 2.23 — Compute and diagnose two finite differences



Complete problem statement

Using Table 2.1, calculate a centred velocity at 0.50 s and a forward estimate assigned to $t = 0$. Compare each with the exact velocity at its assigned time. A student concludes that taking half the spacing must always improve these estimates. What is missing from that conclusion? Also evaluate the centred difference at 0.50 s using exact cubic positions at 0.25 and 0.75 s, and separate finite-step bias from the noise contribution.

Diagnosis target

Different stencils, different assigned times and different noise combinations must not be compared as though they were one universal error sequence.

Problem 2.23 — Solution I: compute the two estimates



Centred estimate at 0.50 s:

$$D_c x(0.50) = \frac{0.541290 - 0.299079}{0.50} = 0.484422 \text{ m s}^{-1}.$$

Exact:

$$v(0.50) = 0.40 + 0.30(0.50)^2 = 0.475000 \text{ m s}^{-1}.$$

Signed error: $+0.009422 \text{ m s}^{-1}$.

Forward estimate assigned to $t = 0$:

$$D_f x(0) = \frac{0.299079 - 0.197651}{0.25} = 0.405712 \text{ m s}^{-1}.$$

Exact $v(0) = 0.400000 \text{ m s}^{-1}$; error $+0.005712 \text{ m s}^{-1}$.

Time assignment

The forward quotient is exactly an interval average for exact positions; treating it as $v(0)$ is an approximation.

Problem 2.23 — Solution II: bias, noise and spacing



For exact cubic positions,

$$x(0.25) = 0.3015625 \text{ m}, \quad x(0.75) = 0.5421875 \text{ m}.$$

Thus

$$D_c x(0.50) = \frac{0.5421875 - 0.3015625}{0.50} = 0.481250 \text{ m s}^{-1}.$$

Exact:

$$v(0.50) = 0.475000 \text{ m s}^{-1}.$$

Finite-step bias:

$$0.006250 \text{ m s}^{-1}.$$

The rounded noisy estimate is 0.484422, so this record's noise contribution is

$$0.003172 \text{ m s}^{-1}.$$

Why “halve h ” is incomplete

Reducing spacing can reduce smooth-function truncation error while increasing sensitivity to position error. Existing samples do not create new independent half-spacing measurements by interpolation.

Chapter 2 synthesis — and the bridge to vectors

Habits to carry forward

- 1 define frame, axis, origin and clock;
- 2 keep signed quantities distinct from magnitudes;
- 3 distinguish route quantities from endpoint quantities;
- 4 connect graph slope to rate;
- 5 connect signed area to accumulated change;
- 6 preserve initial conditions;
- 7 use constant- a equations only on their valid phase;
- 8 check roots and domains;
- 9 distinguish model functions from measured data;
- 10 report estimates with approximation/noise assumptions.

Next: Vectors

One-dimensional signed components prepare the language of magnitude, direction, components and projections needed for multidimensional motion.

Final principle

A complete result states the frame, model, interval, sign convention and an independent check.

