

Vectors

Physics I — Chapter 3

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Why does direction belong in a physical description?

From motion along a line to geometry in space

One signed number distinguished the two directions on a line. A planar positioning stage can travel the correct distance and still miss its target. We need a quantity that retains direction and obeys a consistent addition rule.

This chapter starts with displacement arrows, constructs components in a stated basis, and develops projections and products. These tools predict an endpoint, the shortest geometric correction, a face area and its orientation.

A useful distinction

Changing the coordinate description can change every component while preserving the same physical vector.

Chapter support: Revised Chapter 3: Vectors

Learning route and prerequisites



- 1 **Geometry:** compare arrows, join endpoints, and identify signs before arithmetic.
- 2 **Coordinates:** resolve and reconstruct in one orthonormal basis, retaining quadrant information.
- 3 **Products:** use dot products for signed projections and cross products for oriented areas.
- 4 **Validation:** change axes, compare invariants, and test bounded survey records.

Tools to bring

Compatible units, signed displacement versus travelled distance, Pythagoras, sine and cosine, and elementary algebra. We introduce matrix multiplication and fixed-basis differentiation in slides 61–70 before applying them.

Chapter support: Revised Chapter 3: Vectors

Learning goals



- Distinguish a geometric vector, its signed components, and its nonnegative magnitude.
- Add and subtract displacements, recover a bearing, and reject impossible exact data.
- Derive projections and evaluate dot and cross products with correct units and orientation.
- Find a nearest line point, an oriented face area, and a signed volume, then check the results.
- Transform descriptions between rotated axes while preserving geometric invariants.

Advanced outcomes

Compose and invert planar rotations, compare a tangent rate with a finite rate, and quantify the sensitivity of a normal inferred from nearly parallel edges.

Chapter support: Revised Chapter 3: Section 3.9

Scalars, vectors and the zero vector



A scalar has a value and, where appropriate, a unit. A signed temperature difference remains a scalar. A list of three temperatures does not acquire a displacement transformation rule just because it contains three numbers.

$\mathbf{a}$: vector, $a = \|\mathbf{a}\| \geq 0$: magnitude.

Equality of free vectors

Equal magnitude and direction define equal free vectors. Translating their arrows preserves that equality. A position vector also refers to a specified origin.

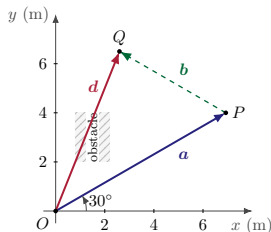
The zero case

The zero vector is an additive identity with magnitude zero and no unique direction.

Chapter support: Revised Chapter 3: Section 3.1

Two constructions of the same sum

(a) A route in physical space



(b) A common-tail construction

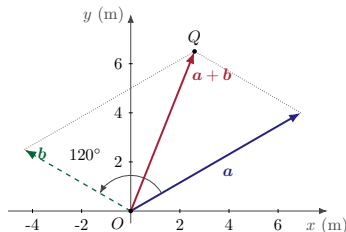


Figure 3.1: Two representations of the same constructed displacement. Both panels use the same horizontal and vertical scale: $\mathbf{a}$ has magnitude 8 m and bearing 30° ; $\mathbf{b}$ has magnitude 5 m and bearing 150° . In (a), $O \rightarrow P \rightarrow Q$ is the indirect route, and the hatched rectangle is a stipulated obstacle. In (b), translated copies complete the parallelogram. The solid resultant and the dashed second leg remain distinguishable without colour. The diagonal is a displacement, not a claim that the direct route is accessible.

Chapter support: Revised Chapter 3: Figure 3.1

Addition, reversal and scaling



Head-to-tail addition joins the first tail to the final head. Translation of a free arrow gives commutativity and associativity of the resultant.

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}, \quad \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

With common tails, $\mathbf{a} - \mathbf{b}$ points from the head of $\mathbf{b}$ to the head of $\mathbf{a}$. A negative scalar multiplier reverses direction.

$$\|\lambda \mathbf{a}\| = |\lambda| \|\mathbf{a}\|, \quad |a - b| \leq \|\mathbf{a} + \mathbf{b}\| \leq a + b.$$

Geometric and unit checks

The bounds follow from straight versus broken paths. Compatible quantities in a common unit can be added. Commutativity of a sum does not establish feasibility of both real routes.

Chapter support: Revised Chapter 3: Vectors

Subtraction has an endpoint order

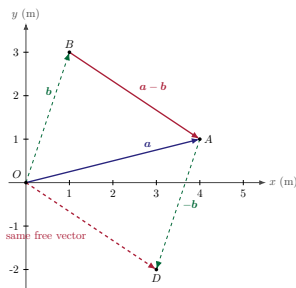


Figure 3.2: Two geometrically equivalent constructions of subtraction. Prescribed displacements are $\mathbf{a} = (4, 1) \text{ m}$ and $\mathbf{b} = (1, 3) \text{ m}$. The solid red arrow is $B \rightarrow A$; the dashed red arrow is its translated copy $O \rightarrow D$, where $D = (3, -2) \text{ m}$. The additional dashed green arrow implements $\mathbf{a} + (-\mathbf{b})$. Horizontal and vertical scales are equal. The subtraction order determines the orientation.

Chapter support: Revised Chapter 3: Figure 3.2

Example 3.1: comparing displacement records

Example 3.1 — Comparing two displacement records

Statement and model. Two inspection records, referred to one fixed origin and the same east–north axes, give $\mathbf{a} = (4, 1) \text{ m}$ and $\mathbf{b} = (1, 3) \text{ m}$. Find the displacement from the endpoint of $\mathbf{b}$ to that of $\mathbf{a}$, its length, and a command that returns from the endpoint of $\mathbf{b}$ to the origin. Treat both records as exact planar values.

Subtract the starting endpoint from the destination:

$$\mathbf{d} = \mathbf{a} - \mathbf{b} = (3, -2) \text{ m}, \quad d = \sqrt{13} \text{ m} \simeq 3.606 \text{ m}.$$

The return from B to the origin is $-\mathbf{b} = (-1, -3) \text{ m}$. The comparison points east and south. Adding back gives $(1, 3) + (3, -2) = (4, 1)$, so the ordered construction and the components agree.

Chapter support: Revised Chapter 3: Example 3.1

Resultant length depends on the included angle



Hold two positive lengths a, b fixed. Their common-tail angle is ϕ . The head-to-tail triangle has supplementary interior angle $\pi - \phi$.

$$R^2 = a^2 + b^2 - 2ab \cos(\pi - \phi) = a^2 + b^2 + 2ab \cos \phi.$$

Limiting geometries

Same direction: $R = a + b$. Opposite directions: $R = |a - b|$. Perpendicular directions: $R = \sqrt{a^2 + b^2}$. The last expression requires the right angle.

For an inverse problem, compute $\cos \phi = (R^2 - a^2 - b^2)/(2ab)$ and test its admissibility in $[-1, 1]$ before taking an inverse cosine.

Chapter support: Revised Chapter 3: Vectors

A quantitative angle–length relation

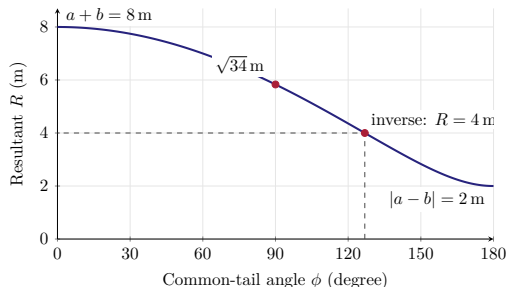


Figure 3.3: An analytical graph for two prescribed displacement lengths $a = 5 \text{ m}$ and $b = 3 \text{ m}$. Only their included common-tail angle varies. The solid curve is $R(\phi) = \sqrt{34 + 30 \cos \phi} \text{ m}$, with the trigonometric input expressed in degrees. The marked inverse construction selects $\phi = 126.87^\circ$ from $R = 4 \text{ m}$; the other point shows the perpendicular case. The axes represent unlike quantities and do not depict a spatial trajectory.

Chapter support: Revised Chapter 3: Figure 3.3

Example 3.2: an inverse geometric problem



Example 3.2 — Inferring an angle from the resultant length

Statement. Two planar translations have lengths 5 m and 3 m , and their sum has length 4 m . Determine their unsigned common-tail angle. Explain what remains unknown and assess a proposed resultant of 9 m using the same two translation lengths.

Chapter support: Revised Chapter 3: Example 3.2

Example 3.2: angle, ambiguity and admissibility



Use the lengths without inventing an absolute bearing:

$$\cos \phi = \frac{16 - 25 - 9}{2(5)(3)} = -\frac{3}{5}, \quad \phi = 126.87^\circ.$$

A check can temporarily set $\mathbf{a} = (5, 0)$ m and $\mathbf{b} = (-1.8, 2.4)$ m. Their sum $(3.2, 2.4)$ m has length 4 m. Reflection changes the second component's sign without changing that length; a common rotation also leaves it unchanged.

What the data establish

The unsigned included angle is fixed. Absolute bearing and clockwise versus counterclockwise order remain undetermined.

For $R = 9$ m, $\cos \phi = 47/30 > 1$. Equivalently, $2 \leq R/(1 \text{ m}) \leq 8$ fails. The proposed exact record is impossible.

Chapter support: Revised Chapter 3: Example 3.2

Cartesian components are signed projections



Choose perpendicular, dimensionless unit directions $\hat{x}$, $\hat{y}$, $\hat{z}$ with right-handed orientation. A vector decomposes uniquely in this basis:

$$\mathbf{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}, \quad a = \sqrt{a_x^2 + a_y^2 + a_z^2}.$$

a_x is a signed scalar. $a_x \hat{x}$ is a vector part. The norm formula follows from two applications of Pythagoras and requires orthonormal directions.

An arrow away from the origin

In a common frame, $\overrightarrow{PQ} = \mathbf{r}_Q - \mathbf{r}_P$. Subtract the tail coordinates from the head coordinates. A component list describes a vector only after its basis and units are stated.

Add vectors component by component in that same basis. A unit vector carries no metre or millimetre unit.

Chapter support: Revised Chapter 3: Vectors

A bearing needs a convention and a quadrant



For $a > 0$ and a counterclockwise angle from positive x ,

$$a_x = a \cos \theta, \quad a_y = a \sin \theta, \quad \theta = \text{atan2}(a_y, a_x).$$

The ratio a_y/a_x loses quadrant information. Predict signs first. A negative returned bearing can be increased by 360° . The argument order is y, x .

Conversion and boundary cases

20° west of north is 110° under this convention. A vertical nonzero vector points at 90° or 270° ; it does not require division by zero. The zero vector has no bearing.

$180^\circ = \pi \text{ rad}$. Use the correct trigonometric input unit. Reconstructing both components from magnitude and bearing checks convention, signs, and angle units together.

Chapter support: Revised Chapter 3: Vectors

Four quadrants, one magnitude

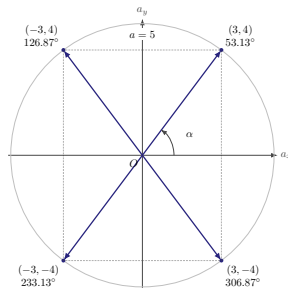


Figure 3.5: Four distinct directions with $|a_x| = 3$, $|a_y| = 4$ and $a = 5$ in a common arbitrary component unit. Equal horizontal and vertical scales make the endpoint circle quantitative. Bearings are measured counterclockwise from positive x and reported in $[0, 360^\circ)$. The acute reference angle is $\alpha = 53.13^\circ$; the component signs select the quadrant. No bearing is assigned to the origin.

Chapter support: Revised Chapter 3: Figure 3.5

Example 3.3: the survey around an obstacle

Restored Figure 3.1 data

The two legs are 8 m at 30° and 5 m at 150° , with bearings counterclockwise from east ($+x$) toward north ($+y$). The figure is on slide 7.

Example 3.3 — A survey around an obstacle

Use the two legs in Figure 3.1 to find the resultant displacement, its bearing and its relation to the route length. Treat the stipulated lengths and bearings as exact inputs of this geometric model; no surveying precision is being asserted.

Predict the signs: both legs point partly north, and only the second has a westward component. Add components before taking a magnitude. Distinguish the broken path length from the endpoint displacement.

Chapter support: Revised Chapter 3: Example 3.3, Figure 3.1

Example 3.3: components and an independent check



$$\mathbf{a} = (4\sqrt{3}, 4) \text{ m},$$

$$\mathbf{b} = (-5\sqrt{3}/2, 5/2) \text{ m},$$

$$\mathbf{d} = \mathbf{a} + \mathbf{b} = (3\sqrt{3}/2, 13/2) \text{ m}.$$

$$d_x \simeq 2.598 \text{ m}, \quad d_y = 6.500 \text{ m}, \quad d = \sqrt{27/4 + 169/4} \text{ m} = 7.000 \text{ m}.$$

The resultant is in quadrant I, with bearing $\text{atan2}(6.5, 2.598 \dots) = 68.21^\circ$.

The route length is 13 m, while the vector bound is $3 \leq d/(1 \text{ m}) \leq 13$.

Cosine-rule check

The common-tail angle is 120° . Therefore $d^2 = [8^2 + 5^2 + 2(8)(5)\cos 120^\circ] \text{ m}^2 = 49 \text{ m}^2$. The obstacle constrains the accessible path, not this endpoint identity.

Chapter support: Revised Chapter 3: Example 3.3

Reconstruction can have two branches or none



In a plane, a magnitude and one signed component give

$$a_y = \pm \sqrt{a^2 - a_x^2}.$$

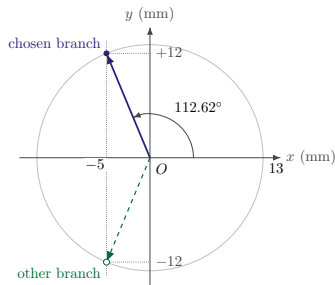
Two branches exist for $|a_x| < a$. A direction observation must select one. At $|a_x| = a$ the other component is zero. If $|a_x| > a$, no real vector satisfies the exact data.

Example 3.4 — Reconstructing a measured displacement

A hypothetical inspection record specifies a displacement magnitude $L = 13 \text{ mm}$ and signed horizontal component $d_x = -5 \text{ mm}$. An independent observation says that the endpoint lies above the horizontal through the start. Find the displacement and its bearing. Then assess a second record with $L = 10 \text{ mm}$ and $d_x = -11 \text{ mm}$.

Chapter support: Revised Chapter 3: Example 3.4

The source reconstruction and its alternative



What each constraint supplies

The circle specifies $\|d\| = 13$ mm. The vertical dotted line specifies $d_x = -5$ mm. Their two intersections give $d_y = \pm 12$ mm. The extra condition $d_y > 0$ selects the filled point.

Quadrant check

$d_x < 0$, $d_y > 0$ places the chosen vector in quadrant II. A negative answer from $\arctan(12/(-5))$ alone would select the wrong bearing.

Figure 3.4: A scaled geometric reconstruction with an explicit two-branch ambiguity. Both axes use millimetres and equal scales; the circle is calculated from $x^2 + y^2 = 169 \text{ mm}^2$. The angle is measured counterclockwise from positive x . The dashed arrow is an admissible alternative before the upward-direction observation is imposed, not an uncertainty interval.

Chapter support: Revised Chapter 3: Figure 3.4

Example 3.4: signs, bearing and impossible data



Start with the norm constraint, then use the observed direction:

$$d_y^2 = (13^2 - (-5)^2) \text{ mm}^2 = 144 \text{ mm}^2, \quad d_y = +12 \text{ mm}.$$

$$\mathbf{d} = (-5\hat{\mathbf{x}} + 12\hat{\mathbf{y}}) \text{ mm}, \quad \theta = \text{atan2}(12, -5) = 112.62^\circ.$$

The negative x and positive y components place the vector in quadrant II. Reconstructing its magnitude returns $\sqrt{25 + 144} = 13$ in millimetres.

The second record

$10^2 - (-11)^2 = -21$ in square millimetres. No real displacement meets these exact inputs. Taking an absolute value of the radicand changes the problem. Uncertainty could affect admissibility only if bounds were supplied.

Chapter support: Revised Chapter 3: Example 3.4

Three-dimensional direction and endpoint differences



For a nonzero vector, the positive-axis angles satisfy

$$\cos \alpha_i = \frac{a_i}{a}, \quad \cos^2 \alpha_x + \cos^2 \alpha_y + \cos^2 \alpha_z = 1.$$

This follows by dividing the norm square by a^2 . A negative component requires an obtuse positive-axis angle. The three angles are constrained; they need not sum to 180° .

Example 3.5 — A three-dimensional endpoint survey

Statement and assumptions. In one fixed orthonormal frame, two surveyed points are $P = (1, 1, 1)\text{ m}$ and $Q = (5, -2, 5)\text{ m}$. Find the displacement $P \rightarrow Q$, its magnitude, its unit direction and all three positive-axis angles. Determine which results survive a change of origin. The coordinates are exact values in a Euclidean geometric model.

Chapter support: Revised Chapter 3: Example 3.5

Two genuine right triangles in space

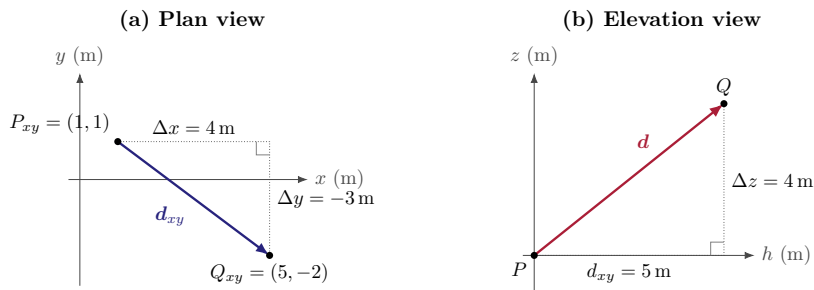


Figure 3.6: Orthogonal views of the displacement from $P = (1, 1, 1)$ m to $Q = (5, -2, 5)$ m. In the plan view, $d_{xy} = \sqrt{4^2 + (-3)^2} = 5$ m. The elevation is taken in the vertical plane containing that horizontal displacement; its horizontal coordinate h is distance along d_{xy} from P , not the original x coordinate. It combines 5 m horizontally with 4 m vertically. Each panel uses equal physical scales on its axes.

Chapter support: Revised Chapter 3: Figure 3.6

Example 3.5: length, direction and origin invariance



$$\mathbf{d} = \mathbf{Q} - \mathbf{P} = (4, -3, 4) \text{ m}, \quad d_{xy} = 5 \text{ m}, \quad d = \sqrt{41} \text{ m} \simeq 6.403 \text{ m}.$$

$$\hat{\mathbf{u}} = \frac{(4, -3, 4)}{\sqrt{41}}, \quad \alpha_x = \alpha_z \simeq 51.34^\circ, \quad \alpha_y \simeq 117.94^\circ.$$

The angles use $\arccos(4/\sqrt{41})$ and $\arccos(-3/\sqrt{41})$. The direction-cosine check is $(16 + 9 + 16)/41 = 1$.

Change of origin, parallel axes

For any fixed origin shift $\mathbf{c}$, $(\mathbf{r}_Q - \mathbf{c}) - (\mathbf{r}_P - \mathbf{c}) = \mathbf{d}$. The displacement, its length, unit direction and positive-axis angles survive. Rotating the axes preserves the length while changing the components and their axis angles.

Chapter support: Revised Chapter 3: Example 3.5

Practice 1: a complete return route



Problem 3.11 — The closing leg of a route

Three planar route legs are $(0.8, 0.6)$, $(-0.2, 0.9)$ and $(0.5, -0.4)$, all in kilometres and one fixed basis. Find the straight return displacement to the start and its bearing. Compare its magnitude with the length already travelled along the three legs.

Approach

First add the three signed displacements. The return command is the negative of that sum. Compute the travelled distance by adding three individual norms. The solution is on slide 27.

Chapter support: Revised Chapter 3: Problem 3.11

Practice 1: endpoint and path are different quantities



The route sum is $(1.1, 1.1)$ km. Therefore

$$\mathbf{d}_{\text{ret}} = (-1.1, -1.1) \text{ km}, \quad d_{\text{ret}} = 1.1\sqrt{2} \text{ km} \simeq 1.556 \text{ km}.$$

Both components are negative and equal. The return bearing is 225° . The three leg lengths are

$$1 \text{ km}, \quad \sqrt{0.85} \text{ km}, \quad \sqrt{0.41} \text{ km},$$

$$\ell_{\text{travelled}} = (1 + \sqrt{0.85} + \sqrt{0.41}) \text{ km} \simeq 2.562 \text{ km}.$$

The route distance exceeds the straight return distance, as the triangle inequality requires. Adding the return displacement to the three legs gives zero in both components. The multiplier used in a norm does not recover the path length.

Chapter support: Revised Chapter 3: Problem 3.11

The scalar product measures a signed comparison



Project $\mathbf{a}$ onto the direction of a nonzero $\mathbf{b}$, then multiply by b . The geometric definition is

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \theta, \quad 0 \leq \theta \leq \pi.$$

The result is a scalar with the product of the input units. Symmetric projection gives $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

Sign and domain

For two nonzero vectors, an acute angle gives a positive product, a right angle gives zero, and an obtuse angle gives a negative product. If either input is zero, the product is zero but an angle involving that vector is undefined.

The dot product is neither a vector nor an angle. To recover an angle, divide it by both nonzero magnitudes; the resulting cosine is dimensionless.

Chapter support: Revised Chapter 3: Vectors

The component formula follows from orthonormality



Projection on a fixed axis is additive. Distribute the product after expanding both vectors in the same basis. The unit directions obey

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

Only the three equal-axis terms survive:

$$\mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z, \quad \theta = \arccos \frac{\mathbf{a} \cdot \mathbf{b}}{ab} \quad (a, b > 0).$$

Interpretation and units

Two displacements give length squared. A force dotted with a unit direction gives force. A force dotted with a length gives N m, but a physical work interpretation requires a specified displacement and force process.

Chapter support: Revised Chapter 3: Vectors

Signed force components and length-valued references

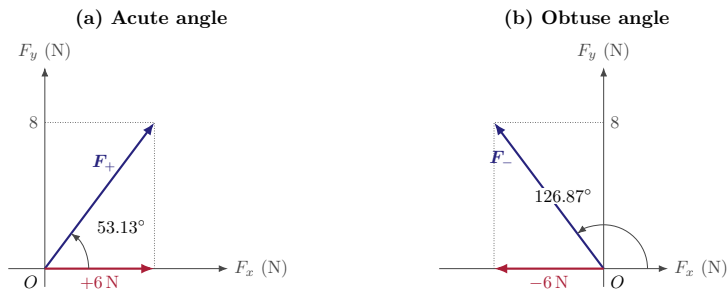


Figure 3.11: Two prescribed force vectors, each of magnitude 10 N, resolved along the same $+x$ reference direction. Both force-component plots have equal axis scales. The red horizontal arrows represent vector projections $\pm 6\hat{x}$ N; their signed scalar components are ± 6 N. The separate reference edge $\mathbf{b} = (2, 0)$ m is not drawn on these newton-valued axes. Its dot products with the forces are ± 12 N m, not ± 6 N.

Chapter support: Revised Chapter 3: Figure 3.11

Example 3.9: a prescribed load direction



Example 3.9 — A prescribed load direction

Consider two possible prescribed force vectors $\mathbf{F}_+ = (6, 8) \text{ N}$ and $\mathbf{F}_- = (-6, 8) \text{ N}$ in the same plane. A reference edge is described by $\mathbf{b} = (2, 0) \text{ m}$. Find each force's angle to that edge, its signed component along the edge, and its dot product with the edge vector. No displacement under the force is specified.

Both forces have magnitude 10 N and the edge direction is $\hat{\mathbf{x}}$.

$$F_{\pm, \parallel} = \pm 6 \text{ N}, \quad \mathbf{F}_{\pm} \cdot \mathbf{b} = \pm 12 \text{ N m}.$$

The angle cosines are ± 0.6 , giving 53.13° and 126.87° . The negative sign agrees with the obtuse case. Reversing the edge reverses both signed products and replaces each angle by its supplement. These products are geometric comparisons; no work process is specified.

Chapter support: Revised Chapter 3: Example 3.9

Projection bounds and the information a product loses



For $\mathbf{b} \neq \mathbf{0}$, let $\hat{\mathbf{u}} = \mathbf{b}/b$, $p = \mathbf{a} \cdot \hat{\mathbf{u}}$, and $\mathbf{e} = \mathbf{a} - p\hat{\mathbf{u}}$. The residual has zero projection along $\hat{\mathbf{u}}$.

$$\|\mathbf{e}\|^2 = a^2 - 2p^2 + p^2 = a^2 - p^2 \geq 0 \implies |\mathbf{a} \cdot \mathbf{b}| \leq ab.$$

Equality for nonzero inputs requires parallel or antiparallel directions. A zero input also satisfies the bound without acquiring a direction.

Why cancellation fails

$\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$ implies only $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = 0$. For $\mathbf{a} = \hat{\mathbf{x}}$, $\mathbf{b} = \hat{\mathbf{y}}$, $\mathbf{c} = 2\hat{\mathbf{y}}$, both products vanish although the inputs differ. Transverse information was lost.

Chapter support: Revised Chapter 3: Problem 3.7

Scalar projection and vector projection



Choose a dimensionless unit direction $\hat{\mathbf{u}}$. Its signed scalar component and associated vector part are distinct:

$$a_{\parallel} = \mathbf{a} \cdot \hat{\mathbf{u}}, \quad \mathbf{a}_{\parallel} = a_{\parallel} \hat{\mathbf{u}}, \quad \mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel}.$$

The parallel and perpendicular vectors have the unit of $\mathbf{a}$. They reconstruct it, and their orthogonality gives

$$\mathbf{a}_{\perp} \cdot \hat{\mathbf{u}} = 0, \quad a^2 = a_{\parallel}^2 + \|\mathbf{a}_{\perp}\|^2.$$

A nonunit input direction

For $\mathbf{b} \neq \mathbf{0}$, use $\hat{\mathbf{u}} = \mathbf{b}/b$, or $\mathbf{a}_{\parallel} = [(\mathbf{a} \cdot \mathbf{b})/(\mathbf{b} \cdot \mathbf{b})]\mathbf{b}$. Reversing $\hat{\mathbf{u}}$ reverses the scalar component but preserves the vector projection onto the line.

Chapter support: Revised Chapter 3: Vectors

The parallel part and its perpendicular residual

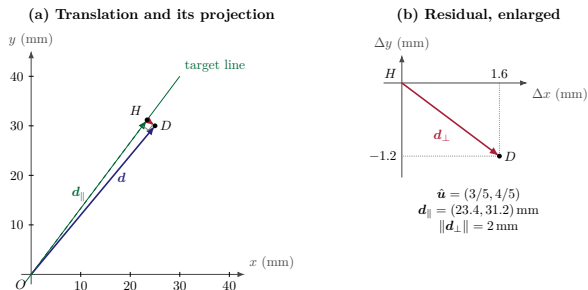


Figure 3.8: The net positioning displacement decomposed along an oblique line. The foot H is at (23.4, 31.2) mm and the endpoint D at (25, 30) mm. Panel (a) uses equal physical scales on its two axes; panel (b) translates the origin to H and enlarges the residual with its own equal millimetre scales. It is not a second displacement from the original origin. The displayed extent of the target line does not define the unit vector's magnitude.

Chapter support: Revised Chapter 3: Figure 3.8

Example 3.7: a stage and an oblique target

Example 3.7 — A positioning stage and an oblique target

Two commanded planar translations in one fixed laboratory basis are $\mathbf{p} = (40, 10)$ mm and $\mathbf{q} = (-15, 20)$ mm. A target line has direction $(3, 4)$. Find the net translation, its along-line component and its perpendicular error relative to a line through the starting point.

Model and order of operations

The target is a line through the starting point. First add commands in the common laboratory basis. Normalize the supplied direction $(3, 4)$. Then project the net displacement. The geometry is shown on slide 34.

Chapter support: Revised Chapter 3: Example 3.7

Example 3.7: projection, error and correction



$$\mathbf{d} = (40, 10) + (-15, 20) = (25, 30) \text{ mm}, \quad \hat{\mathbf{u}} = (3, 4)/5.$$

$$d_{\parallel} = 25(3/5) + 30(4/5) = 39 \text{ mm}, \quad \mathbf{d}_{\parallel} = (23.4, 31.2) \text{ mm}.$$

The residual is $\mathbf{d}_{\perp} = (1.6, -1.2) \text{ mm}$, of length 2 mm. The shortest correction is its negative, $(-1.6, 1.2) \text{ mm}$. With $\hat{\mathbf{n}} = (-4, 3)/5$, its signed transverse component is -2 mm .

Three independent checks

The two parts sum to $(25, 30) \text{ mm}$. The residual projection is $1.6(0.6) - 1.2(0.8) = 0$. The squared norms obey $25^2 + 30^2 = 39^2 + 2^2 = 1525$ in square millimetres.

Chapter support: Revised Chapter 3: Example 3.7

The closest point on a shifted line



Describe an infinite guide by a reference point $\mathbf{r}_0$ and a unit direction $\hat{\mathbf{u}}$: $\mathbf{r}(s) = \mathbf{r}_0 + s\hat{\mathbf{u}}$, with s a signed length. For point P , first subtract the guide reference:

$$\mathbf{a} = \mathbf{r}_P - \mathbf{r}_0, \quad s_* = \mathbf{a} \cdot \hat{\mathbf{u}}, \quad \mathbf{r}_H = \mathbf{r}_0 + s_*\hat{\mathbf{u}}, \quad \mathbf{e} = \mathbf{r}_P - \mathbf{r}_H.$$

The residual is perpendicular to the line. Any other candidate differs by a parallel segment, so

$$\|\mathbf{r}_P - \mathbf{r}(s)\|^2 = \|\mathbf{e}\|^2 + (s - s_*)^2.$$

Minimum and constraints

The extra square is nonnegative, proving the infinite-line minimum at s_* . For a finite allowed interval, use the nearest admissible s ; a constrained correction need not be perpendicular.

Chapter support: Revised Chapter 3: Vectors

Example 3.8: a guide with and without end stops

Example 3.8 — Recovering a closest command from point data

Statement and model. A planar guide is the infinite straight line through $\mathbf{r}_0 = (2, -1) \text{ mm}$ in unit direction $\hat{\mathbf{u}} = (3/5, 4/5)$. A positioning record gives $\mathbf{r}_P = (6, 3) \text{ mm}$. Find the closest point on the guide, the shortest corrective displacement from P , and the minimum distance. Then find the nearest allowed point if the guide instead permits only $0 \leq s \leq 4 \text{ mm}$. All coordinates refer to one orthonormal frame; guide thickness and measurement uncertainty are neglected.

Chapter support: Revised Chapter 3: Example 3.8

The source geometry and squared-distance minimum

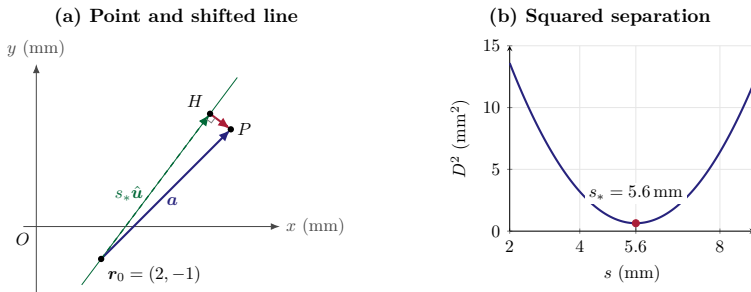


Figure 3.10: A target line through $(2, -1)$ mm with unit direction $\hat{u} = (0.6, 0.8)$ and a measured point $P = (6, 3)$ mm. Panel (a) has equal spatial scales and shows the orthogonal foot $H = (5.36, 3.48)$ mm; the small red residual is $H \rightarrow P$. Panel (b) is the independently derived analytical curve $D^2(s) = 0.64 \text{ mm}^2 + (s - 5.6 \text{ mm})^2$. Its minimum verifies the foot calculation. The plot's domain is a displayed interval, not a restriction of the infinite target line.

Chapter support: Revised Chapter 3: Figure 3.10

Example 3.8: the minimum changes under a constraint



For the infinite guide, $\mathbf{a} = (4, 4)$ mm and $s_* = 4(0.6) + 4(0.8) = 5.6$ mm. Hence

$$\begin{aligned} H &= (5.36, 3.48) \text{ mm}, & H - P &= (-0.64, 0.48) \text{ mm}, \\ D_{\min} &= 0.80 \text{ mm}. \end{aligned}$$

The residual $(0.64, -0.48)$ mm has zero dot product with $(0.6, 0.8)$. In numerical millimetres, $D^2(s) = 0.64 + (s - 5.6)^2$.

Finite guide: $0 \leq s \leq 4$ mm

The unconstrained foot is outside. The nearest allowed parameter is $s = 4$ mm, giving $E = (4.4, 2.2)$ mm and correction $(-1.6, -0.8)$ mm. Its length is $\sqrt{3.2}$ mm $\simeq 1.789$ mm. Substitution gives $D^2(4) = 3.2 \text{ mm}^2$, confirming the endpoint minimum.

Chapter support: Revised Chapter 3: Example 3.8

Practice 2: a projection in three dimensions



Problem 3.12 — A three-dimensional projection

Resolve $\mathbf{d} = (3, 4, 5)$ mm into parts parallel and perpendicular to the direction $(2, -1, 2)$. Find the signed scalar projection and check both perpendicularity and the squared-norm decomposition.

Approach

Normalize $(2, -1, 2)$, compute one signed scalar projection, and subtract the parallel vector. In three dimensions the residual is a vector in a plane, rather than one uniquely signed transverse scalar. The complete solution follows on slide 42.

Chapter support: Revised Chapter 3: Problem 3.12

Practice 2: reconstruction and orthogonality checks



$$\hat{\mathbf{u}} = \frac{(2, -1, 2)}{3}, \quad d_{\parallel} = \frac{6 - 4 + 10}{3} \text{ mm} = 4 \text{ mm}.$$

$$\mathbf{d}_{\parallel} = \frac{(8, -4, 8)}{3} \text{ mm}, \quad \mathbf{d}_{\perp} = \frac{(1, 16, 7)}{3} \text{ mm}.$$

Adding the vector parts returns $(3, 4, 5) \text{ mm}$. Perpendicularity is checked by

$$\mathbf{d}_{\perp} \cdot \hat{\mathbf{u}} = \frac{2 - 16 + 14}{9} \text{ mm} = 0.$$

The squared norm is 50 mm^2 . The parts have squared norms 16 mm^2 and $(1 + 256 + 49)/9 = 34 \text{ mm}^2$. Thus $50 = 16 + 34$, independently confirming the decomposition.

Chapter support: Revised Chapter 3: Problem 3.12

The vector product supplies an oriented area



Two nonparallel length vectors define a plane. Their parallelogram has base a and perpendicular height $b \sin \theta$. An ordered pair selects one normal by the right-hand rule:

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}, \quad \mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}.$$

The result is a vector with the product of the input units. For length inputs its magnitude is area, so a displayed normal arrow needs a separate scale or direction-only convention.

Orientation and degeneracy

With $+x$ right and $+y$ up in the page, $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ points toward the reader ($\odot$). Its negative points away ($\otimes$). Parallel edges or a zero edge give zero area and do not select a unique unit normal.

Chapter support: Revised Chapter 3: Vectors

Cross-product components and independent tests



Expand using $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$, $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$, $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$. Reversed cyclic orders are negative and equal-axis products vanish.

$$C_x = a_y b_z - a_z b_y,$$

$$C_y = a_z b_x - a_x b_z,$$

$$C_z = a_x b_y - a_y b_x, \quad \mathbf{C} = \mathbf{a} \times \mathbf{b}.$$

Checks that test different properties

$\mathbf{a} \cdot \mathbf{C} = \mathbf{b} \cdot \mathbf{C} = 0$ establishes perpendicularity. The right-hand rule or a signed coordinate-plane area establishes orientation. The identity $C^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2$ independently checks the area magnitude.

Perpendicularity alone accepts both $\mathbf{C}$ and $-\mathbf{C}$. It cannot detect an overall sign error. Cross products also require parentheses when repeated.

Chapter support: Revised Chapter 3: Vectors

A tilted mounting face and an orientation key

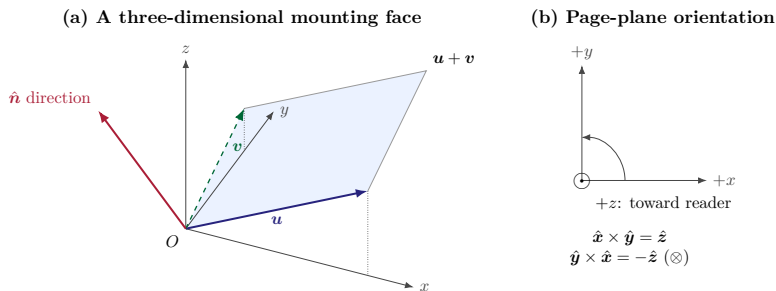


Figure 3.13: The bracket geometry and an independent orientation key. Panel (a) is an oblique linear projection of the exact Cartesian edge coordinates in Example 3.10; projected lengths and angles are not unforeshortened spatial measurements. The face area is 1400mm^2 and the normal direction is $(-3, -2, 6)/7$. Its displayed arrow length is arbitrary, not an area scale. Panel (b) is a schematic right-handed triad for axes lying in the page; its $\odot/\otimes$ convention is not an assertion that the tilted face lies in the page.

Chapter support: Revised Chapter 3: Figure 3.13

Example 3.10: area and normal of a mounting face

Example 3.10 — A mounting bracket's plane and normal

A hypothetical planar mounting face has adjacent edge vectors $\mathbf{u} = (40, 0, 20)$ mm and $\mathbf{v} = (0, 30, 10)$ mm. Find the oriented area vector, area and unit normal for the ordered pair $(\mathbf{u}, \mathbf{v})$. The face is a parallelogram, not an assumed rectangle in space.

Model

Use the ordered edges in a right-handed orthonormal frame. Calculate the cross product before normalizing it. The parallelogram is not assumed to be a rectangle, and the geometry supplies no material-strength conclusion.

Chapter support: Revised Chapter 3: Example 3.10

Example 3.10: area, orientation and Gram check



$$\mathbf{C} = \mathbf{u} \times \mathbf{v} = (-600, -400, 1200) \text{ mm}^2 = 200(-3, -2, 6) \text{ mm}^2.$$

$$A = 200\sqrt{9 + 4 + 36} \text{ mm}^2 = 1400 \text{ mm}^2, \quad \hat{\mathbf{n}} = (-3, -2, 6)/7.$$

The positive z component agrees with the positive ordered xy area. Reversing the edges reverses the normal and area vector, preserving area.

Independent checks

$40(-600) + 20(1200) = 0$ and $30(-400) + 10(1200) = 0$ establish perpendicularity. The edge norm squares are 2000 and 1000 in mm^2 , and their dot product is 200 mm^2 . Thus $2000(1000) - 200^2 = 1\,960\,000 = 1400^2$ in mm^4 .

Chapter support: Revised Chapter 3: Example 3.10

Triangle vertices must first become edge vectors



For ordered vertices P, Q, R , form two edge differences from the common vertex:

$$\mathbf{u} = Q - P, \quad \mathbf{v} = R - P, \quad \mathbf{A}_{\triangle} = \frac{1}{2} \mathbf{u} \times \mathbf{v}.$$

A diagonal halves the parallelogram. A common origin shift cancels from both edges. Exchanging Q, R reverses orientation while preserving area.

Problem 3.18 — A triangle from three points

In one right-handed frame, three vertices are $P = (1, 0, 0)$, $Q = (3, 1, 0)$ and $R = (1, 2, 2)$ in centimetres. Find the triangle area and the unit normal selected by the ordering P, Q, R . Explain the effect of exchanging Q and R .

Chapter support: Revised Chapter 3: Problem 3.18

Problem 3.18: the ordered triangular face



$$\mathbf{u} = (2, 1, 0) \text{ cm}, \quad \mathbf{v} = (0, 2, 2) \text{ cm}, \quad \mathbf{u} \times \mathbf{v} = (2, -4, 4) \text{ cm}^2.$$

$$\|\mathbf{u} \times \mathbf{v}\| = 6 \text{ cm}^2, \quad A_{\triangle} = 3 \text{ cm}^2, \quad \hat{\mathbf{n}} = (1, -2, 2)/3.$$

The unit-normal check is $(1 + 4 + 4)/9 = 1$. Its two edge products vanish: $2(1) + 1(-2) = 0$ and $2(-2) + 2(2) = 0$.

Geometric consistency

The independent Gram check is $(5)(8) - 2^2 = 36$ in cm^4 . Reversing Q, R gives $-\hat{\mathbf{n}}$ with the same 3 cm^2 area. Translating all three vertices leaves both edge differences unchanged.

Chapter support: Revised Chapter 3: Problem 3.18

A scalar triple product is a signed volume



Take $\mathbf{a}, \mathbf{b}$ as an ordered base, with area $A = \|\mathbf{a} \times \mathbf{b}\| > 0$ and normal $\hat{\mathbf{n}} = (\mathbf{a} \times \mathbf{b})/A$. The third edge has signed height $h_s = \mathbf{c} \cdot \hat{\mathbf{n}}$.

$$V_s = Ah_s = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}), \quad V = |V_s|.$$

Expanding components and collecting equal terms proves the cyclic equality. Exchanging any two edges reverses the sign. Length inputs give cubic length units.

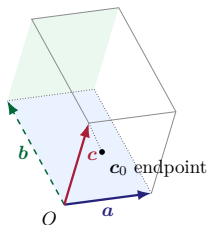
Meaning of zero and sign

Negative signed height means the opposite side of the chosen base, not a negative amount of space. If the base is nondegenerate, $V_s = 0$ makes $\mathbf{c}$ coplanar. Degenerate edges need not select a unique plane.

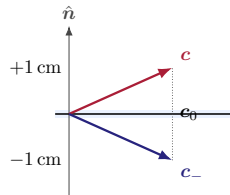
Chapter support: Revised Chapter 3: Vectors

A skew cell and its true perpendicular height

(a) The skew cell



(b) A normal section



$$V_s = Ah_s$$
$$A = 7 \text{ cm}^2$$

Figure 3.16: Signed volume for Example 3.12. Panel (a) uses an oblique linear projection of the exact cell vertices. The dotted segment joins the endpoint of $\mathbf{c}$ to its perpendicular projection in the base. Panel (b) is the physical section spanned by $\mathbf{c}_0$ and $\hat{\mathbf{n}}$, with equal length scales: $\|\mathbf{c}_0\| = \sqrt{5} \text{ cm}$ and the normal heights are $\pm 1 \text{ cm}$. The base-area value is an annotation, not a length on this section. Both third-edge choices give the same nonnegative volume.

Chapter support: Revised Chapter 3: Figure 3.16

Example 3.12: volume and projection into a base plane



Example 3.12 — A skew calibration cell: volume, height and coplanarity

A calibration cell is described by three edge vectors $\mathbf{a} = (2, 0, 1) \text{ cm}$, $\mathbf{b} = (0, 3, 1) \text{ cm}$ and $\mathbf{c} = (1, 1, 2) \text{ cm}$. Find its volume and signed height above the ordered $(\mathbf{a}, \mathbf{b})$ base. Then find a vector $\mathbf{c}_0$ in that base plane whose endpoint is the orthogonal projection of the endpoint of $\mathbf{c}$.

Method

Calculate the ordered base area vector. Its unit normal provides the signed height; subtract that normal part from the third edge to find its in-plane projection. Use the cell geometry on slide 51.

Chapter support: Revised Chapter 3: Example 3.12

Example 3.12: volume, height and coplanarity



$$\mathbf{a} \times \mathbf{b} = (-3, -2, 6) \text{ cm}^2, \quad A = 7 \text{ cm}^2, \quad \hat{\mathbf{n}} = (-3, -2, 6)/7.$$

$$V_s = (-3 - 2 + 12) \text{ cm}^3 = 7 \text{ cm}^3, \quad V = 7 \text{ cm}^3, \quad h_s = V_s/A = 1 \text{ cm}.$$

$$\mathbf{c}_0 = \mathbf{c} - h_s \hat{\mathbf{n}} = \frac{(10, 9, 8)}{7} \text{ cm} = \frac{5}{7} \mathbf{a} + \frac{3}{7} \mathbf{b}.$$

Its normal product is $(-30 - 18 + 48)/7 = 0$ in cubic centimetres, confirming the base-plane condition. Direct volume expansion gives $2(3 \cdot 2 - 1 \cdot 1) - 3 = 7$ in cubic centimetres. The third-edge length $\sqrt{6} \text{ cm}$ is not its perpendicular height.

Chapter support: Revised Chapter 3: Example 3.12

Passive rotation changes axes, keeping the vector fixed



Turn the axes counterclockwise by ϕ about their common origin. The new unit directions in the old basis are

$$\hat{\mathbf{x}}' = \cos \phi \hat{\mathbf{x}} + \sin \phi \hat{\mathbf{y}}, \quad \hat{\mathbf{y}}' = -\sin \phi \hat{\mathbf{x}} + \cos \phi \hat{\mathbf{y}}.$$

Project the unchanged vector onto them:

$$a'_x = a_x \cos \phi + a_y \sin \phi, \quad a'_y = -a_x \sin \phi + a_y \cos \phi.$$

Checks

At $\phi = 90^\circ$, $(a'_x, a'_y) = (a_y, -a_x)$. Squaring and adding cancels the mixed terms and preserves the norm. Transforming both inputs preserves their dot product and mutual angle. The bearing changes to $\theta - \phi$.

Chapter support: Revised Chapter 3: Vectors

The same displacement on two coordinate grids

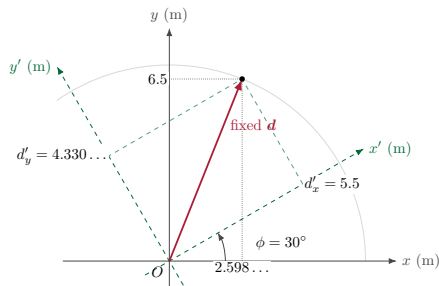


Figure 3.18: A passive axis rotation with one unchanged displacement arrow. The dashed primed axes are turned 30° counterclockwise from the solid unprimed axes. Both coordinate systems use metres and equal Cartesian scales. Dotted and dashed guides give the old and new projections respectively. The arc is part of the calculated locus of length 7 m, not an uncertainty band. Only the coordinate description changes.

Chapter support: Revised Chapter 3: Figure 3.18

Example 3.14: one survey, two component descriptions



Restored context

The first survey leg is $\mathbf{a} = (4\sqrt{3}, 4)$ m, of length 8 m and bearing 30° . The resultant from Example 3.3 is $\mathbf{d} = (3\sqrt{3}/2, 13/2)$ m.

Example 3.14 — One survey, two coordinate grids

Return to the displacement in Figure 3.18, $\mathbf{d} = (3\sqrt{3}/2, 13/2)$ m from Example 3.3. Rotate the axes counterclockwise through $\phi = 30^\circ$, keeping the vector fixed. Then compare its dot product with the first survey leg before and after the change.

Use the passive projection rule and transform both vectors before comparing their dot products. The complete calculation is on slide 57.

Chapter support: Revised Chapter 3: Example 3.14, Example 3.3

Example 3.14: a changed list and unchanged invariants



$$d'_x = \left(\frac{3\sqrt{3}}{2} \frac{\sqrt{3}}{2} + \frac{13}{2} \frac{1}{2} \right) \text{ m} = 5.500 \text{ m},$$

$$d'_y = \left(-\frac{3\sqrt{3}}{2} \frac{1}{2} + \frac{13}{2} \frac{\sqrt{3}}{2} \right) \text{ m} = \frac{5\sqrt{3}}{2} \text{ m} \simeq 4.330 \text{ m}.$$

The squared norm is $(11/2)^2 + (5\sqrt{3}/2)^2 = 49$ in square metres. The new bearing is $68.21^\circ - 30^\circ = 38.21^\circ$.

Transform both inputs

The first leg has primed components $(8, 0)$ m. In the old basis, $\mathbf{d} \cdot \mathbf{a} = 18 + 26 = 44 \text{ m}^2$. In the new basis it is $(11/2)(8) + 0 = 44 \text{ m}^2$. A passive turn of -30° recovers the old components.

Chapter support: Revised Chapter 3: Example 3.14

Active turns and origin shifts have different meanings



An active counterclockwise turn moves the vector in fixed axes:

$$\tilde{a}_x = a_x \cos \phi - a_y \sin \phi, \quad \tilde{a}_y = a_x \sin \phi + a_y \cos \phi.$$

It equals the passive component formula with $\phi \mapsto -\phi$. Both preserve norms and dot products. A positive quarter turn maps an active $+x$ arrow to $+y$; a passive positive quarter turn gives that fixed arrow components $(0, -1)$.

A new origin, with parallel axes

Point positions change by $\mathbf{r}' = \mathbf{r} - \mathbf{c}$. Displacements do not acquire an origin offset, since it cancels between their endpoints. For a combined change, subtract the new origin before resolving the point on the new axes.

Chapter support: Revised Chapter 3: Section 3.6

Problem 3.25: a bounded survey record



A conservative absolute bound on a component sum is the sum of the component bounds. The resulting interval tests compatibility with closure.

Problem 3.25 — A closing error outside stipulated bounds

A constructed planar survey has legs $(0.50, 0.20)$, $(-0.10, 0.70)$ and $(-0.39, -0.92)$ in metres. Every reported component has a stipulated absolute error bound of 0.004 m. Find the closing vector and decide whether exact closure is compatible with all the bounds. Can you identify the incorrect reading uniquely?

The reported closure is $(0.01, -0.02)$ m, of length 0.02236 m. Each closure component has worst-case bound $3(0.004) = 0.012$ m:

$$c_x \in [-0.002, 0.022] \text{ m}, \quad c_y \in [-0.032, -0.008] \text{ m}.$$

The y interval excludes zero, so exact closure is incompatible with all stated bounds. This does not identify one incorrect reading: an open route, inconsistent axes, or inadequate bounds could also explain it. These are deterministic intervals, with no supplied confidence probability.

Chapter support: Revised Chapter 3: Problem 3.25

Core synthesis and the bridge to spatial motion



A dependable vector calculation

Declare the physical quantity, origin when needed, basis, units and orientation. Construct the geometry before arithmetic. Preserve signs in components and scalar projections, and test norms, reconstruction, bounds and ordered directions.

Perpendicularity misses a normal's overall sign. Norm preservation misses active/passive choices. Loop closure can miss a common scale or axis error. These checks complement one another.

From geometry to kinematics

Chapter 4, *Motion in Two and Three Dimensions*, turns position and displacement into velocity and acceleration. A change of direction is a vector change even when the magnitude is constant. The advanced block now previews derivatives and the reliability of inferred directions.

Chapter support: Revised Chapter 3: Section 3.8, Bridge to Chapter 4

Advanced: matrices organize the projection rules



Write a component column in a stated basis. Matrix multiplication takes the dot product of each row with that column:

$$\begin{pmatrix} a'_x \\ a'_y \end{pmatrix} = R(\phi) \begin{pmatrix} a_x \\ a_y \end{pmatrix}, \quad R(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.$$

Transpose swaps rows and columns. With $c = \cos \phi$, $s = \sin \phi$,

$$R^T R = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c^2 + s^2 & 0 \\ 0 & c^2 + s^2 \end{pmatrix} = I.$$

Thus $R^{-1}(\phi) = R(-\phi)$. Multiplying a second passive turn gives $R(\psi)R(\phi) = R(\phi + \psi)$ by the sine and cosine addition identities. This commutation applies to planar turns about the same origin.

Chapter support: Revised Chapter 3: Vectors

Example 3.18: composition and an inverse check



Example 3.18 — Two coordinate changes and an inverse check

A displacement has laboratory components $(3, -1)$ mm. Axes are first turned counterclockwise by 30° and then by another 60° about the same origin. Find the final components and verify the inverse operation.

$$R(30^\circ) \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} (3\sqrt{3} - 1)/2 \\ -(3 + \sqrt{3})/2 \end{pmatrix} \text{ mm.}$$

The next multiplication by $R(60^\circ)$ gives $(-1, -3)$ mm. Independently, a total 90° passive turn maps (x, y) to $(y, -x)$ and gives the same result.

$$R(-90^\circ)(-1, -3)^T = (3, -1)^T \text{ mm.}$$

The original and final squared norms are both 10 mm^2 . Inverse operations reverse the order of a general matrix chain; these particular same-axis planar rotations commute.

Chapter support: Revised Chapter 3: Example 3.18

A vector derivative requires a stated basis



For a differentiable curve in fixed Cartesian unit directions,

$$\mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = x'(t)\hat{\mathbf{x}} + y'(t)\hat{\mathbf{y}} + z'(t)\hat{\mathbf{z}}.$$

The chord rate approaches a tangent rate. A position derivative with respect to time has velocity units. Its direction exists where it is nonzero. A finite rate need not equal the derivative at the initial time.

Magnitude and basis changes

Differentiating $\|\mathbf{r}\|^2 = \mathbf{r} \cdot \mathbf{r}$ gives $d\|\mathbf{r}\|/dt = (\mathbf{r} \cdot \mathbf{r}')/\|\mathbf{r}\|$ for $\mathbf{r} \neq 0$. This is a radial projection, not generally $\|\mathbf{r}'\|$. For a moving basis, the product rule additionally gives $\sum_i r_i \hat{\mathbf{e}}'_i$.

Trigonometric differentiation uses radians. The path's prescribed interval is part of its model; evaluating a formula outside it does not justify extrapolation.

Chapter support: Revised Chapter 3: Vectors

Example 3.21: a prescribed spatial path

Example 3.21 — A prescribed three-dimensional path

Consider

$$\mathbf{r}(t) = at\hat{\mathbf{x}} + bt^2\hat{\mathbf{y}} + ct^3\hat{\mathbf{z}}, \quad 0 \leq t \leq 2\text{ s},$$

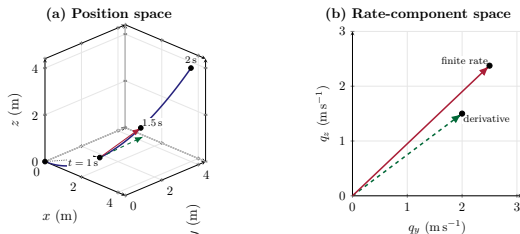
with $a = 2\text{ m s}^{-1}$, $b = 1\text{ m s}^{-2}$ and $c = 0.5\text{ m s}^{-3}$. Find the tangent rate at $t = 1\text{ s}$ and compare it with the finite rate from 1 to 1.5 s.

What is held fixed

The basis directions do not depend on time. The curve is prescribed geometrically, without a force law. Each coefficient's units make every position component a length. We compare the derivative with a chord rate using two times inside the stated interval.

Chapter support: Revised Chapter 3: Example 3.21

The source path and two distinct rate vectors



Both rates have $q_x = 2\text{ m s}^{-1}$, which is not displayed in this two-component inset.

Figure 3.25: An analytical vector curve and two different rate vectors. Panel (a) uses $\mathbf{r}(t) = (2t, t^2, 0.5t^3)$ in the declared numerical-SI convention for $0 \leq t/(1\text{ s}) \leq 2$. The solid chord is $\Delta\mathbf{r}$ from 1 to 1.5 s; the dashed tangent segment is $(0.6\text{ s})\mathbf{r}'(1\text{ s})$, so both displayed segments have metre units. The 3D axes use equal unit lengths before projection. Panel (b) is a separate rate-component space, not another position plot: it compares $\Delta\mathbf{r}/(0.5\text{ s})$ with $\mathbf{r}'(1\text{ s})$. All coordinates and 101 curve samples follow the exact polynomial, not a hand-drawn trajectory.

Chapter support: Revised Chapter 3: Figure 3.25

Example 3.21: tangent rate versus finite rate



Differentiate in the fixed basis:

$$\mathbf{r}'(t) = (a, 2bt, 3ct^2), \quad \mathbf{r}'(1\text{ s}) = (2, 2, 1.5) \text{ m s}^{-1}.$$

Its magnitude is $\sqrt{10.25} \text{ m s}^{-1}$ (about 3.2016 m s^{-1}). Its unit tangent is $(4, 4, 3)/\sqrt{41}$. The two positions are

$$\mathbf{r}(1\text{ s}) = (2, 1, 0.5) \text{ m}, \quad \mathbf{r}(1.5\text{ s}) = (3, 2.25, 1.6875) \text{ m}.$$

Thus $\Delta \mathbf{r} = (1, 1.25, 1.1875) \text{ m}$, and the finite rate is $(2, 2.5, 2.375) \text{ m s}^{-1}$. Its difference from the initial tangent rate is $(0, 0.5, 0.875) \text{ m s}^{-1}$. Independently, polynomial expansion gives the chord rate $(a, 2bt + bh, 3ct^2 + 3cth + ch^2)$. The extra terms vanish as $h \rightarrow 0$, explaining the limit.

Chapter support: Revised Chapter 3: Example 3.21

A computed normal can be sensitive to its input



Use ordered edges $\mathbf{p} = (L, 0, 0)$, $\mathbf{q} = (L, \varepsilon, \delta)$, where $L > 0$, $\varepsilon > 0$, and all three symbols have length units. The nominal plane has $\delta = 0$.

$$\mathbf{p} \times \mathbf{q} = (0, -L\delta, L\varepsilon), \quad \hat{\mathbf{n}} = \frac{(0, -\delta, \varepsilon)}{\sqrt{\varepsilon^2 + \delta^2}}.$$

The signed tilt from $+z$ toward $-y$ is

$$\alpha = \arctan \frac{\delta}{\varepsilon}, \quad \frac{\partial \alpha}{\partial \delta} = \frac{\varepsilon}{\varepsilon^2 + \delta^2} \quad (\text{radians}).$$

Here $\partial/\partial\delta$ changes the perturbation while holding L, ε fixed. At the nominal plane the sensitivity is $1/\varepsilon$. The approximation $\alpha \simeq \delta/\varepsilon$ requires $|\delta| \ll \varepsilon$. Increasing L alone scales area but leaves the inferred direction unchanged.

Chapter support: Revised Chapter 3: Vectors

Exact tilt and the local approximation

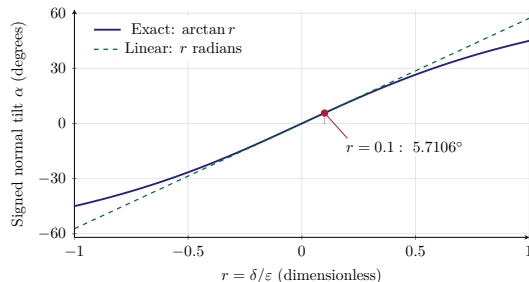


Figure 3.26: Calculated sensitivity of the normal for $\mathbf{p} = (L, 0, 0)$ and $\mathbf{q} = (L, \varepsilon, \delta)$ with $L, \varepsilon > 0$. The signed tilt is measured from $+z$ toward $-y$. The solid curve is the exact model; the dashed line is the local approximation $\alpha \simeq \delta/\varepsilon$ in radians, converted to degrees for the vertical axis. Their agreement near the origin does not justify the approximation at large $|\delta/\varepsilon|$. The marked point is the constructed perturbation of Example 3.22; no measured data or statistical band is shown.

Chapter support: Revised Chapter 3: Figure 3.26

Advanced challenge: a perturbed normal



Restored perturbation model

Use ordered edges $\mathbf{p} = (L, 0, 0)$, $\mathbf{q} = (L, \varepsilon, \delta)$ in a fixed right-handed frame. The signed tilt is measured from $+z$ toward $-y$. These are stipulated exact geometric inputs.

Problem 3.34 — A perturbed normal (advanced)

Use Equation (3.58) with $L = 1\text{ m}$, $\varepsilon = 0.020\text{ m}$ and $\delta = 0.002\text{ m}$. Find the area vector, its magnitude, the unit normal and its signed tilt. What changes if only δ changes sign?

Predict the y -component sign before calculating. Distinguish a sign change of the perturbation from reversing the entire ordered pair. The complete solution and geometric checks are on slide 70.

Chapter support: Revised Chapter 3: Problem 3.34

Advanced challenge: magnitude, tilt and sign checks



$$\mathbf{C} = (0, -0.002, 0.020) \text{ m}^2, \quad A = 0.002\sqrt{101} \text{ m}^2 \simeq 0.0200998 \text{ m}^2.$$

$$\hat{\mathbf{n}} = \frac{(0, -1, 10)}{\sqrt{101}}, \quad \alpha = \arctan(0.1) \simeq +5.7106^\circ.$$

The normal has unit norm. Both edge products vanish: $\mathbf{p} \cdot \hat{\mathbf{n}} = 0$ and $\mathbf{q} \cdot \hat{\mathbf{n}} = (-0.020 + 10(0.002))/\sqrt{101} = 0$ in metres.

Change only δ to -0.002 m

The area vector becomes $(0, +0.002, 0.020) \text{ m}^2$. Its magnitude is unchanged, the normal's y component reverses, and the signed tilt becomes -5.7106° . Its positive z component stays fixed.

The perturbation is only 0.2% of L , yet tilts the normal by about 5.7° . If $\varepsilon = \delta = 0$, the pair supplies no unit normal. Geometric checks and input sensitivity must both be considered.

Chapter support: Revised Chapter 3: Problem 3.34

