

Motion in Two and Three Dimensions

Physics I - Chapter 4

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science
AGH University of Krakow

Lecturer information and contact details



Lecturer

Hamzeh Khanpour

High Energy Particle Physicist, AGH University of Krakow || CERN-TH

How to contact me

E-mail: hamzeh.khanpour@fis.agh.edu.pl; hamzeh.khanpour@agh.edu.pl

CERN e-mail: hamzeh.khanpour@cern.ch

Where to find me

Office & Room: D10 & 218

You are welcome to contact me or stop by for discussion at any time.

Website and course materials

<https://view.fis.agh.edu.pl/staff/hamzeh.khanpour/>

<https://home.agh.edu.pl/~hamzeh.khanpour/>

<https://hamzeh-khanpour.web.cern.ch/>

A path needs a clock and an observer



A probe passes through a known point. That observation locates it, but a timed trajectory is needed to determine its direction, speed and acceleration.

What this chapter makes possible

We will differentiate a spatial path, coordinate arrival events, separate speeding up from turning, and translate motion between observers.

Sampled coordinates add another question: how accurately can the motion be reconstructed? The advanced section extends differentiation to polar directions that themselves turn.

Chapter 4 preface and opening; kinematics precedes Newton's laws.

Learning route and prerequisites



- 1 **Describe motion.** Fixed-basis vectors and derivatives turn coordinates into local velocity and acceleration.
- 2 **Predict events.** Component equations share one clock in projectile, circular and relative motion.
- 3 **Test a reconstruction.** Numerical refinement and coordinate uncertainty address different errors.
- 4 **Extend the representation.** Polar basis derivatives explain the advanced scan calculations.

Prerequisites: SI units, one-dimensional kinematics, vector components, dot products, elementary integration and the product and chain rules. Curvature and numerical update rules are introduced here.

Preface; conventions; Sections 4.1–4.7 and advanced section.

Learning goals



After the core lecture, you should be able to:

- Differentiate a three-dimensional trajectory and distinguish a vector from its magnitude.
- Solve a common-time arrival event and reject roots outside the physical model.
- Resolve acceleration into speed change and turning, including a local curvature radius.
- Compose velocities in named frames and minimize a time-limited separation.
- Check numerical event estimates and explain why differentiation amplifies coordinate errors.

Advanced goal: derive polar velocity and acceleration, test their changing-basis terms, and assess a complete scan interval.

Chapter learning goals and conventions.

Position in a fixed Cartesian frame



Choose the object, observer, origin, axes and time origin before calculating.
In a nonrotating Cartesian basis,

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}.$$

All three coordinates describe the *same event time*. Mixing coordinates from different times can create a point the object never occupied.

Conventions

Coordinates are in metres and time is in seconds. Bold symbols are vectors. Speed $v = |\mathbf{v}| \geq 0$, whereas a velocity component may be negative. A dot denotes differentiation with respect to the named frame's common time.

A fixed change of origin changes position but leaves its derivatives unchanged.

Conventions; Section 4.1; Eq. 4.1.

Velocity, acceleration and path slope



The shrinking displacement chord becomes the local tangent. For a differentiable path,

$$\mathbf{v} = \dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}), \quad \mathbf{a} = \dot{\mathbf{v}} = (\ddot{x}, \ddot{y}, \ddot{z}).$$

Velocity has units m s^{-1} and acceleration m s^{-2} . The norm gives speed:
 $v = \sqrt{v_x^2 + v_y^2 + v_z^2}.$

Two different kinds of slope

The slope of $y(t)$ is v_y . Where $v_x \neq 0$, the slope of the projected path is $dy/dx = v_y/v_x$, a dimensionless ratio of components.

A tangent direction of travel $\mathbf{v}/v$ requires $v > 0$. At a stop, Cartesian differentiation remains valid.

Section 4.1; Eqs. 4.3–4.5.

Example 4.1: the inspection probe



A point-like probe follows, for $0 \leq t \leq 4 \text{ s}$,

$$x = (0.60 \text{ m s}^{-1})t, \quad y = (0.20 \text{ m s}^{-2})t^2, \quad z = (0.050 \text{ m s}^{-3})t^3.$$

Find its position, velocity, speed and acceleration at $t = 2.0 \text{ s}$, and interpret the arrows in Fig. 4.1. **Model and first step.** The laboratory path is prescribed. Differentiate each function before inserting the event time:

$$\mathbf{v} = (0.60, 0.40t, 0.150t^2), \quad \mathbf{a} = (0, 0.40, 0.300t).$$

These are SI numerical formulas, with velocity and acceleration outputs in their respective units.

Example 4.1; results on slide 9, Figure 4.1 on slide 10.

Example 4.1: state and arrow interpretation



At $t = 2.0 \text{ s}$,

$$\mathbf{r} = (1.20, 0.80, 0.40) \text{ m},$$

$$\mathbf{v} = (0.60, 0.80, 0.60) \text{ m s}^{-1}, \quad v = \sqrt{1.36} = 1.17 \text{ m s}^{-1},$$

$$\mathbf{a} = (0, 0.40, 0.60) \text{ m s}^{-2}, \quad |\mathbf{a}| = \sqrt{0.52} = 0.721 \text{ m s}^{-2}.$$

Interpretation. The position arrow starts at the origin. Velocity and acceleration arrows attach to the local event. Their different units require different drawing scales. A positive y or z curvature in a coordinate-time graph produces the corresponding acceleration.

Checks. The path slopes are $v_y/v_x = 4/3$ and $v_z/v_x = 1$. The sum $0.60 + 0.80 + 0.60$ is not speed. Each derivative removes one time power from a coefficient's units.

Example 4.1; complete source drawing and caption follow on slide 10.

The probe: one event in two representations

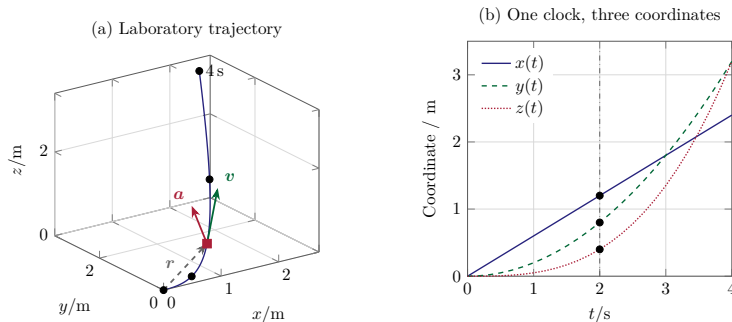


Figure 4.1: Calculated representations of the inspection path. The square marks the two-second event. Dots on the spatial curve are separated by one second. The local arrows at two seconds use the plotted displacement equivalents $(1.20\text{ s})\mathbf{v}$ and $(1.20\text{ s}^2)\mathbf{a}$; they are not additional position vectors. The three-dimensional panel is a projection, so apparent angles on the page are not physical three-dimensional angles. The component panel uses the same event time.

Displacement, distance and averages



For $\Delta t = t_2 - t_1 > 0$,

$$\mathbf{v}_{\text{avg}} = \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{\Delta t}, \quad s = \int_{t_1}^{t_2} |\mathbf{v}(t)| dt, \quad \bar{v} = \frac{s}{\Delta t}.$$

Refining a sequence of short path chords gives the speed integral. The triangle inequality gives $|\Delta \mathbf{r}| \leq s$, so $|\mathbf{v}_{\text{avg}}| \leq \bar{v}$. Equality requires a straight route without reversal, apart from rest intervals.

Answered check

Can one object return to its start with nonzero average speed? Yes. Its endpoint displacement is zero while its path length is positive. Average velocity can therefore be zero even when the object moved throughout the interval.

Section 4.1; Eq. 4.9.

The geometric path and its schedule



Write a fixed geometric curve as $\mathbf{R}(q)$, where q labels location. A schedule $q(t)$ supplies the clock. Two chain-rule applications give

$$\mathbf{v} = \mathbf{R}'(q)\dot{q}, \quad \mathbf{a} = \mathbf{R}''(q)\dot{q}^2 + \mathbf{R}'(q)\ddot{q}.$$

The first acceleration term comes from the changing geometric tangent. The second comes from changing progress along the path.

What a photograph leaves open

The same curve can be traversed at different speeds. If a schedule runs twice as fast, its velocity doubles and the geometric turning term is multiplied by four.

Scope: the later curvature calculation quantifies this local turning. A full description of a space curve's torsion is outside this chapter.

Section 4.1; Eq. 4.7.

Practice 1: Problem 33, same curve



Two points follow the same parabola with SI coordinate functions $\mathbf{r}_A(t) = (t, t^2)$ and $\mathbf{r}_B(t) = (2t, 4t^2)$ for $t \geq 0$. (a) Find when each passes through $(1, 1)$ m. (b) Compare their velocity and acceleration vectors at that spatial point. (c) Explain why path geometry alone does not determine acceleration. **Attempt.** Differentiate before evaluating. Use the x coordinate to find each arrival time, then test the y coordinate at that same time.

Complete Problem 33; fixed laboratory axes and SI coordinate convention; solution on slide 14.

Problem 33: the same point at different times



(a) Arrival. The x equations give $t_A = 1.00\text{ s}$ and $t_B = 0.500\text{ s}$. Both corresponding y values are 1 m .

$$\mathbf{v}_A = (1, 2t), \quad \mathbf{a}_A = (0, 2), \quad \mathbf{v}_B = (2, 8t), \quad \mathbf{a}_B = (0, 8).$$

(b) At that spatial point, the SI vectors are

$$\mathbf{v}_A = (1, 2), \quad \mathbf{v}_B = (2, 4)\text{ m s}^{-1}, \quad \mathbf{a}_A = (0, 2), \quad \mathbf{a}_B = (0, 8)\text{ m s}^{-2}.$$

Thus $\mathbf{v}_B = 2\mathbf{v}_A$ and $\mathbf{a}_B = 4\mathbf{a}_A$.

(c) Interpretation and check. Eliminating time gives $y = x^2$ for both SI paths. The curve fixes tangent and curvature, while the clock fixes their rates. Here $\mathbf{r}_B(t) = \mathbf{r}_A(2t)$, so the chain rule independently gives both factors.

Problem 33; all three subparts answered.

Integrating a constant acceleration vector



For a vector $\mathbf{a}$ constant in the chosen fixed frame, set $\tau = t - t_0$. Integration gives

$$\mathbf{v} - \mathbf{v}_0 = \int_{t_0}^t \mathbf{a} dt' = \mathbf{a}\tau,$$

$$\mathbf{r} - \mathbf{r}_0 = \int_{t_0}^t [\mathbf{v}_0 + \mathbf{a}(t' - t_0)] dt' = \mathbf{v}_0\tau + \frac{1}{2}\mathbf{a}\tau^2.$$

Each Cartesian component uses the same elapsed time and its own initial conditions. A constant acceleration *magnitude* does not justify removing a rotating vector from the integral.

Geometry. Every displacement lies in the plane spanned by $\mathbf{v}_0$ and $\mathbf{a}$. Three changing coordinates can still describe a planar path.

Section 4.2; Eq. 4.11.

Example 4.5: release above a moving belt



At $t = 0$, a small bead leaves a horizontal guide at $(x, y) = (0, 1.80) \text{ m}$ with laboratory velocity $(2.40, 0) \text{ m s}^{-1}$. A horizontal belt at $y = 0$ moves toward $+x$ at 0.60 m s^{-1} . Where should a marker on the belt be initially so that the bead first meets it at belt height? **Model.** After release, $\mathbf{a} = -g\hat{\mathbf{j}}$, with $g = 9.81 \text{ m s}^{-2}$ and negligible drag. The belt is long enough to contain the landing point. The model ends at first contact.

$$x_B = 2.40t, \quad y_B = 1.80 - 4.905t^2 \quad (\text{SI numerical values}).$$

Vertical event: $y_B(t_f) = 0$ gives the positive root $t_f = \sqrt{2(1.80)/9.81} = 0.605783 \text{ s}$.

Complete Example 4.5; horizontal matching on slide 17.

Example 4.5: matching the horizontal event



At the fall time, the bead's horizontal coordinate is $x_B(t_f) = 2.40t_f = 1.45388 \text{ m}$. The marker has

$$x_M = x_{M0} + 0.60t.$$

Coincidence requires $x_M(t_f) = x_B(t_f)$, hence

$$x_{M0} = (2.40 - 0.60)t_f = 1.09041 \text{ m}.$$

Place the marker about 1.09 m to the right of the release vertical.

Independent checks. The marker moves 0.36347 m , so its starting coordinate plus that distance equals 1.45388 m . The bead reaches the belt with $v_y = -gt_f = -5.94 \text{ m s}^{-1}$.

Transferable method. The vertical condition fixes the clock. Horizontal matching then uses that clock for both objects. Belt speed changes the required starting point, while the ideal fall time stays fixed.

Example 4.5; geometry and shared event lines on slide 18.

The landing event in space and time

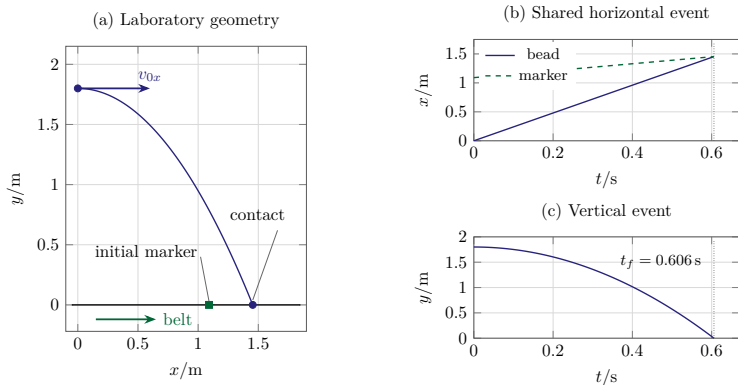


Figure 4.5: One landing, one time. The geometry has equal length scales; direction arrows are schematic in length. Both time panels use identical time-axis limits and the same dotted event line. The belt marker and bead coincide horizontally when the bead reaches $y = 0$, not when their speeds become equal.

Projectile motion: model and trajectory



Choose x horizontal and y upward. A point projectile in uniform downward gravity, without drag or prior contact, has

$$\begin{aligned}x &= x_0 + v_{0x}t, & v_x &= v_{0x}, \\y &= y_0 + v_{0y}t - \frac{1}{2}gt^2, & v_y &= v_{0y} - gt.\end{aligned}$$

The launch components are $v_{0x} = v_0 \cos \theta$ and $v_{0y} = v_0 \sin \theta$. Here $g = 9.81 \text{ m s}^{-2}$ is a stipulated model value.

For $v_{0x} \neq 0$, eliminate time:

$$y = y_0 + \frac{v_{0y}}{v_{0x}}(x - x_0) - \frac{g(x - x_0)^2}{2v_{0x}^2}.$$

A vertical launch still obeys the time equations. Its path cannot be written as this function $y(x)$. Free flight ends at the first relevant contact.

Section 4.3; Eqs. 4.13–4.14.

A height root is a candidate event



Solving the actual condition $y(t) = y_f$ gives

$$t_{\pm} = \frac{v_{0y} \pm \sqrt{v_{0y}^2 + 2g(y_0 - y_f)}}{g}.$$

A negative discriminant makes the height unreachable. A double root is the apex. Two positive roots can represent ascent and descent. Negative times lie before launch.

For $v_{0y} > 0$, $t_A = v_{0y}/g$ and $y_A = y_0 + v_{0y}^2/(2g)$. At the apex, $v_y = 0$ while v_x and downward acceleration can remain nonzero.

An arrival needs every condition

An interception also requires the target's horizontal position at the same time. A root after an earlier collision is outside the free-flight model. Check coordinates, direction and time domain before accepting it.

Section 4.3; Eqs. 4.16 and 4.15.

Example 4.6: a descending interception



A bead is launched from $(0, 0.50) \text{ m}$ with $\mathbf{v}_0 = (4.00, 5.00) \text{ m s}^{-1}$. A small target moves to the right at 1.00 m s^{-1} , at fixed height 1.50 m . Find its required initial x coordinate for interception while the bead is descending. Then test a target initially at $x_{T0} = 2.00 \text{ m}$. **Model.** Uniform $g = 9.81 \text{ m s}^{-2}$, no drag, and no obstacle before the requested event. The target holds its height.

$$1.50 = 0.50 + 5.00t - 4.905t^2, \quad t_{\pm} = \frac{5.00 \pm \sqrt{5.38}}{9.81} \text{ s}.$$

The roots are $t_- = 0.273243 \text{ s}$ and $t_+ = 0.746125 \text{ s}$. Since $t_A = 0.509684 \text{ s}$, select t_+ for descent.

Complete Example 4.6; horizontal condition and test on slide 22.

Example 4.6: one interception and one miss



Constructed interception. At t_+ , $x_B = 4t_+ = 2.98450$ m. Thus

$$x_{T0} = (4.00 - 1.00)t_+ = 2.23837 \text{ m.}$$

The contact velocity is $(4.00, -2.31948) \text{ m s}^{-1}$, with speed 4.62 m s^{-1} and direction -30.1° from $+x$. The negative vertical component independently confirms descent.

Test $x_{T0} = 2.00$ m. Horizontal coincidence needs $4t = 2 + t$, or $t = 2/3$ s. The bead is then at

$$y_B = 0.50 + 5(2/3) - 4.905(2/3)^2 = 1.65333 \text{ m.}$$

It is 0.15333 m above the target, so there is no interception. A shared height and a shared horizontal coordinate at different times do not form one contact event.

Example 4.6; both requested cases answered; Figure 4.6 on slide 23.

The constructed descending contact

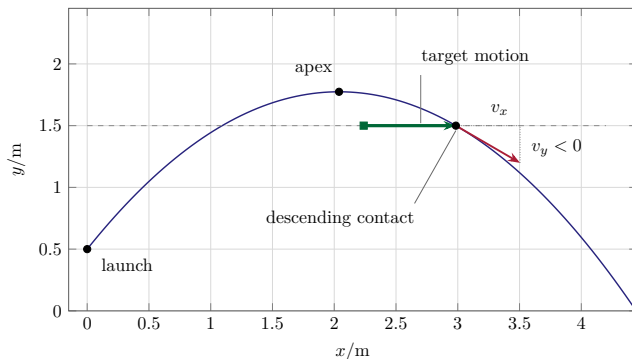


Figure 4.6: Analytical projectile path and target displacement during the constructed descending interception. Both axes have the same length scale. The green arrow joins the target's initial and final positions, not the bead's launch and arrival. The bead's red velocity arrow uses scale 0.13 s ; the dotted component triangle belongs to that velocity, not to a force. The continued curve after interception shows only the hypothetical no-contact path.

Range depends on the landing condition



For unobstructed return to $y_f = y_0$, with an upward launch,

$$T_f = \frac{2v_0 \sin \theta}{g}, \quad R = v_{0x} T_f = \frac{v_0^2 \sin(2\theta)}{g}.$$

At fixed speed and equal heights, $\sin 2\theta \leq 1$ gives the familiar optimum 45° . Complementary angles have equal range but different times and apex heights.

For a landing level $h \geq 0$ below launch, select its positive height root instead:

$$R(\theta) = \frac{v_0 \cos \theta}{g} \left[v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh} \right].$$

The extra falling time changes the optimum. Read the manuscript's envelope method and moving-target launch design for further developments.

Section 4.3; Eqs. 4.17 and 4.19; Examples 4.9–4.10 for reading.

Clearance is an event at a horizontal plane



A fence at $x = L$ fixes its own event time, provided $v_{0x} > 0$:

$$t_L = \frac{L - x_0}{v_{0x}}, \quad y_L = y_0 + v_{0y}t_L - \frac{1}{2}gt_L^2.$$

The point trajectory clears a fence of height H if $y_L > H$ and no earlier contact ended the flight. Solving $y = H$ alone finds crossings of a horizontal level.

For $x_0 = 0$, express the height at fixed launch angle as

$$y_L = y_0 + L \tan \theta - \frac{gL^2}{2v_0^2 \cos^2 \theta}.$$

Increasing speed reduces the downward-curvature term. The straight-ray limit $y_0 + L \tan \theta$ sets a feasibility bound before a finite speed is calculated.

Section 4.3; Eq. 4.18.

Example 4.8: a fence and a speed threshold



A projectile is released at $(0, 1.00) \text{ m}$ with speed 20.0 m s^{-1} and angle 35.0° . A narrow vertical fence at $L = 18.0 \text{ m}$ extends from ground level to $H = 7.00 \text{ m}$. There are no other obstacles. In the uniform-gravity, no-drag model, find the clearance and velocity when the projectile crosses the fence plane. At this fixed launch angle, find the threshold speed for the point trajectory to touch the fence top. **Event calculation.** With $g = 9.81 \text{ m s}^{-2}$,

$$(v_{0x}, v_{0y}) = (16.3830, 11.4715) \text{ m s}^{-1}, \quad t_L = \frac{18.0}{16.3830} = 1.09870 \text{ s}.$$

Substitution gives $y_L = 7.68274 \text{ m}$, hence clearance $y_L - H = 0.68274 \text{ m}$.

Complete Example 4.8; velocity and inverse threshold on slide 27.

Example 4.8: velocity and inverse design



At the fence, $\mathbf{v}_L = (16.3830, 0.69331) \text{ m s}^{-1}$. Its speed is 16.3977 m s^{-1} , at 2.42° above horizontal. It is still ascending: $t_L < t_A = 1.169 \text{ s}$.

Set $y_L = H$ at fixed angle and solve for the touching speed:

$$v_{\text{touch}} = \sqrt{\frac{gL^2}{2 \cos^2 \theta [y_0 + L \tan \theta - H]}} = 18.9379 \text{ m s}^{-1}.$$

Here the bracket is $6.60374 \text{ m} > 0$. Strict clearance requires $v_0 > v_{\text{touch}}$. If the bracket is nonpositive, no finite speed at that angle can pass above the top.

Checks. Inserting v_{touch} restores $y_L = 7.00 \text{ m}$. The given trajectory is above ground up to the fence. Expanding the velocity components also gives $v_L^2 = v_0^2 - 2g(y_L - y_0)$.

Example 4.8; all requested quantities; Figure 4.8 follows.

The same event in position and velocity space

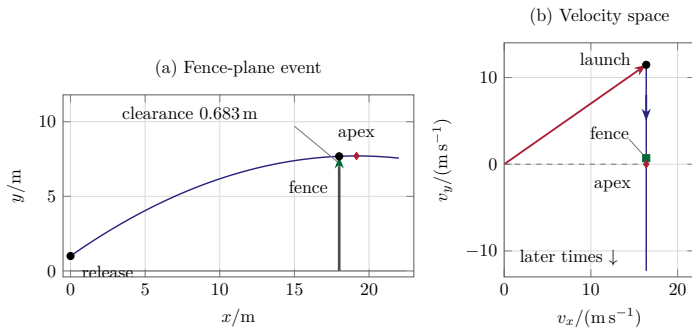


Figure 4.8: Two representations of the same clearance test. Panel (a) is physical geometry with equal length scales: the path passes the 7.00 m fence at $x = 18.0$ m before reaching its apex. Panel (b) is an analytical plot in *velocity* space, with equal component scales. Its vertical line expresses constant v_x and decreasing v_y ; an arrow shows the direction of advancing time. Distance from its origin equals speed, so the apex velocity is the closest point on that line to the origin. The launch vector in (b) is a velocity, not a displacement or force.

Why constant speed can require acceleration



On a circle of radius R , two equal-magnitude tangent velocities turn through the same angle as the radii. The isosceles triangles give

$$\frac{|\Delta \mathbf{v}|}{v} = \frac{|\Delta \mathbf{r}|}{R} = 2 \left| \sin \frac{\Delta \theta}{2} \right|.$$

Divide by the common elapsed time and take the short-interval limit:

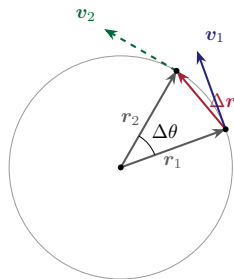
$$|\mathbf{a}| = \frac{v}{R} \lim_{\Delta t \rightarrow 0} \frac{|\Delta \mathbf{r}|}{\Delta t} = \frac{v^2}{R}.$$

The velocity difference approaches the inward direction. Acceleration therefore turns the velocity while remaining perpendicular to it.

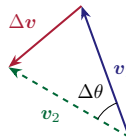
Meaning: centripetal describes an inward direction. A kinematic acceleration calculation does not identify an additional force.

Section 4.4; Eqs. 4.23–4.24.

Equal-angle geometry and a velocity difference



(a) Position chord and local tangents



(b) Velocity-space difference

$$\mathbf{v}_2 = \mathbf{v}_1 + \Delta \mathbf{v}$$

Figure 4.11: An original equal-angle construction with $\Delta\theta = 40^\circ$, enlarged to make the geometry readable. The circle radii and the two velocity magnitudes are each equal within their own scale. Position and velocity panels have different units and scales. The velocity-difference arrow joins the tip of $\mathbf{v}_1$ to the tip of $\mathbf{v}_2$; it is not their vector sum. The limiting construction, rather than the finite drawing alone, establishes acceleration at one event.

Circular motion in Cartesian components



For a fixed circle centre at the origin and constant signed ω ,

$$\begin{aligned}\theta &= \theta_0 + \omega t, & \mathbf{r} &= R(\cos \theta, \sin \theta), \\ \mathbf{v} &= R\omega(-\sin \theta, \cos \theta), & \mathbf{a} &= -R\omega^2(\cos \theta, \sin \theta).\end{aligned}$$

Positive angular rate is counterclockwise viewed from $+z$. Its sign reverses velocity at a fixed angle. Acceleration depends on ω^2 and remains inward.

$$v = R|\omega|, \quad a_n = R\omega^2 = \frac{v^2}{R}, \quad T = \frac{2\pi}{|\omega|} \quad (\omega \neq 0).$$

Derivative formulas require radians. For a centre at $\mathbf{r}_C$, use $\mathbf{a} = -\omega^2(\mathbf{r} - \mathbf{r}_C)$. The radius begins at the circle centre.

Section 4.4; Eqs. 4.25–4.28.

Example 4.11: the rotating inspection disk



A marker is $R = 0.300\text{ m}$ from the centre of an ideal horizontal inspection disk. Its angular rate is $\omega = 4.00\text{ rad s}^{-1}$, counterclockwise, and it begins on the positive x axis. Find its velocity and acceleration at $t = \pi/12\text{ s}$, and compare with the events at $\theta = 0$ and $\theta = 120^\circ$. **Main event.** The laboratory origin is the centre and the axes do not rotate. At $t = \pi/12\text{ s}$, $\theta = 4t = \pi/3$, so

$$\mathbf{v} = (-1.03923, 0.600)\text{ m s}^{-1}, \quad \mathbf{a} = (-2.400, -4.15692)\text{ m s}^{-2}.$$

These follow by inserting $R = 0.300\text{ m}$ and $\omega = 4.00\text{ s}^{-1}$ into the Cartesian derivatives on slide 31.

Complete Example 4.11; comparison events and checks on slide 33.

Example 4.11: magnitudes and other events



At every event, $v = R|\omega| = 1.20 \text{ m s}^{-1}$, $|\mathbf{a}| = R\omega^2 = 4.80 \text{ m s}^{-2}$, and $T = \pi/2 = 1.57 \text{ s}$.

Angle	$\mathbf{v}/(\text{m s}^{-1})$	$\mathbf{a}/(\text{m s}^{-2})$
0°	$(0, 1.20)$	$(-4.80, 0)$
60°	$(-1.03923, 0.600)$	$(-2.400, -4.15692)$
120°	$(-1.03923, -0.600)$	$(2.400, -4.15692)$

Checks. Each event satisfies $\mathbf{r} \cdot \mathbf{v} = 0$, $\mathbf{v} \cdot \mathbf{a} = 0$ and $\mathbf{a} = -16 \text{ s}^{-2} \mathbf{r}$. The vectors change even though their magnitudes stay constant.

Comparing opposite uniform senses of rotation at one position reverses velocity and retains inward acceleration. A physical reversal would require an additional transition model.

Example 4.11; complete comparison requested in the statement.

Acceleration separates speed change and turning



Where $v > 0$, write $\mathbf{v} = v\hat{\mathbf{t}}$, with forward unit tangent $\hat{\mathbf{t}} = \mathbf{v}/v$. The product rule gives

$$\mathbf{a} = \dot{v}\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{dt}.$$

Because $\hat{\mathbf{t}} \cdot \hat{\mathbf{t}} = 1$, its derivative is perpendicular to $\hat{\mathbf{t}}$. Define local curvature by $|d\hat{\mathbf{t}}/ds| = 1/R_c$, with inward principal normal $\hat{\mathbf{n}}$. Since $\dot{s} = v$,

$$\mathbf{a} = a_t\hat{\mathbf{t}} + a_n\hat{\mathbf{n}}, \quad a_t = \dot{v}, \quad a_n = \frac{v^2}{R_c}, \quad |\mathbf{a}| = \sqrt{a_t^2 + a_n^2}.$$

The forward-tangential component is signed. The chosen principal-normal component is nonnegative. On a straight segment it vanishes. At a stop, use Cartesian derivatives because $\mathbf{v}/v$ is undefined.

Section 4.5; Eqs. 4.29–4.31.

Example 4.12: speeding up through a bend



A point-like carriage follows a horizontal circular bend of radius 5.00 m. Over $0 \leq t \leq 5.00$ s, its speed increases according to $v(t) = 1.00 \text{ m s}^{-1} + (0.400 \text{ m s}^{-2})t$. Find the acceleration at five seconds. Choose the fixed axes so the carriage is at the rightmost point of the circle at that event, moving upward on the page. **At the event**, $v = 3.00 \text{ m s}^{-1}$, $\hat{\mathbf{t}} = \hat{\mathbf{j}}$ and $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$. The components are

$$a_t = 0.400 \text{ m s}^{-2}, \quad a_n = \frac{3.00^2}{5.00} = 1.80 \text{ m s}^{-2}.$$

The signs follow the event geometry: speed grows in the upward tangent direction and turning is inward to the left.

Complete Example 4.12; vector, magnitude and Cartesian check on slide 36.

Example 4.12: vector sum and independent check



The complete acceleration at five seconds is

$$\mathbf{a} = (-1.80, 0.400) \text{ m s}^{-2}, \quad |\mathbf{a}| = \sqrt{1.80^2 + 0.400^2} = 1.84 \text{ m s}^{-2}.$$

It points 77.5° from the forward tangent toward the inward normal, or 12.5° forward of inward.

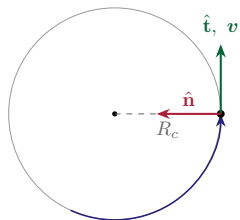
Cartesian check. Integrate speed to obtain $s = t + 0.200t^2$ in SI numerical values. Set $\theta = -2 + s/5$ so $\theta(5) = 0$, and differentiate $\mathbf{r} = 5(\cos \theta, \sin \theta)$.

$$\begin{aligned} \dot{\theta} &= 0.600 \text{ s}^{-1}, & \ddot{\theta} &= 0.0800 \text{ s}^{-2}, \\ a_x &= -R\dot{\theta}^2 = -1.80, & a_y &= R\ddot{\theta} = 0.400. \end{aligned}$$

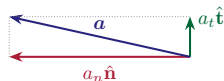
The last components are in m s^{-2} . Removing speed change leaves the inward part. Making the bend straight leaves the tangential part.

Example 4.12; Figure 4.12 on slide 37.

The bend and its acceleration triangle



(a) Geometry at $t = 5$ s
direction arrows not to a common scale



(b) Acceleration vectors
1.80 inward, 0.400 forward
in m s^{-2}

Figure 4.12: The bend and the acceleration triangle for the same event. The arc indicates the final 2.00 rad of the prescribed motion. In panel (b) all acceleration arrows use the same scale, so their vector sum is geometrically quantitative. No force arrows are shown.

Local curvature from velocity and acceleration



For $v > 0$, project onto the actual direction of travel:

$$\hat{\mathbf{t}} = \frac{\mathbf{v}}{v}, \quad a_t = \frac{\mathbf{v} \cdot \mathbf{a}}{v}, \quad \mathbf{a}_\perp = \mathbf{a} - a_t \hat{\mathbf{t}}.$$

Then $a_n = |\mathbf{a}_\perp|$, $\hat{\mathbf{n}} = \mathbf{a}_\perp / a_n$ and $R_c = v^2 / a_n$, where $a_n > 0$. This works in space as well as in a plane.

$$|\mathbf{v} \times \mathbf{a}| = va_n, \quad R_c = \frac{v^3}{|\mathbf{v} \times \mathbf{a}|}.$$

The cross product checks curvature independently. A local normal need not point toward the coordinate origin. For a stipulated total limit, orthogonality requires $a_t^2 + v^4 / R_c^2 \leq a_{\max}^2$. Holding two of these quantities fixed constrains the others.

Section 4.5; Eqs. 4.32 and 4.36.

Example 4.15: infer a bend and test a limit



A planar inspection carriage passes a local point chosen as $(0,0)$, with laboratory vectors

$$\mathbf{v} = (3.00, 4.00) \text{ m s}^{-1}, \quad \mathbf{a} = (-1.00, 2.00) \text{ m s}^{-2}.$$

Infer its instantaneous curvature circle. Then assess an illustrative trajectory requirement $|\mathbf{a}| \leq 2.00 \text{ m s}^{-2}$, assuming the same tangential acceleration and bend radius must be retained. If it fails, calculate a permitted speed or the radius needed at the original speed. **Local inference.** $v = 5.00 \text{ m s}^{-1}$, $\hat{\mathbf{t}} = (0.600, 0.800)$, and

$$a_t = \frac{3(-1) + 4(2)}{5} = 1.00 \text{ m s}^{-2}, \quad \mathbf{a}_\perp = (-1.60, 1.20) \text{ m s}^{-2}.$$

Thus $a_n = 2.00 \text{ m s}^{-2}$, $\hat{\mathbf{n}} = (-0.800, 0.600)$, $R_c = 12.5 \text{ m}$, and the local centre is $(-10.0, 7.50) \text{ m}$.

Complete Example 4.15; revised requirements on slide 40.

Example 4.15: a permitted speed or radius



The request has $|\mathbf{a}| = \sqrt{1^2 + 2^2} = 2.23607 \text{ m s}^{-2}$, exceeding the stipulated 2.00 m s^{-2} limit.

Orthogonal components require $a_t^2 + v^4/R_c^2 \leq a_{\max}^2$. Retaining $a_t = 1.00 \text{ m s}^{-2}$ and $R_c = 12.5 \text{ m}$ gives

$$v_{\max} = \sqrt{R_c \sqrt{a_{\max}^2 - a_t^2}} = 4.65302 \text{ m s}^{-1}.$$

Alternatively, retain $v = 5.00 \text{ m s}^{-1}$:

$$R_c \geq \frac{v^2}{\sqrt{a_{\max}^2 - a_t^2}} = 14.4338 \text{ m}.$$

Checks. The determinant $3(2) - 4(-1) = 10$ gives $R_c = 125/10 = 12.5 \text{ m}$ independently. At either limiting revision, $a_n = \sqrt{3} \text{ m s}^{-2}$, restoring a total magnitude of 2.00 m s^{-2} . This tests a prescribed motion, with no actuator or material capacity supplied.

Example 4.15; all alternatives answered.

A local curvature circle at the prescribed event

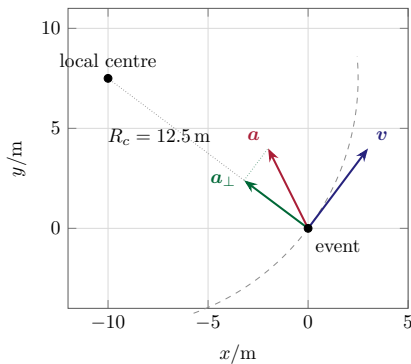


Figure 4.15: A prescribed event and its local curvature circle. Position axes use equal metre scales. The velocity arrow is drawn with scale 1.00 s ; the two acceleration arrows use scale 2.00 s^2 . Their displayed lengths therefore represent different physical units, and only the acceleration arrows may be compared directly. The dotted closing segment is the tangential acceleration contribution. The dashed circle arc shares the event tangent and curvature, without asserting a global circular path.

Relative motion starts with positions



Use $\mathbf{r}_{A/B}$ for A relative to B. With parallel nonrotating axes and a nonrelativistic common time,

$$\mathbf{r}_{A/C} = \mathbf{r}_{A/B} + \mathbf{r}_{B/C}.$$

Travel from C to B, then from B to A. Differentiating this head-to-tail identity gives

$$\mathbf{v}_{A/C} = \mathbf{v}_{A/B} + \mathbf{v}_{B/C}, \quad \mathbf{a}_{A/C} = \mathbf{a}_{A/B} + \mathbf{a}_{B/C}.$$

Constant relative translation makes the origin acceleration zero, so the two observers agree on acceleration while generally assigning different velocities.

Conditions matter

An accelerating translating origin adds its own acceleration. Rotating axes require basis derivatives. Adding speeds cannot replace these vector identities.

Section 4.6; Eqs. 4.38–4.40.

A heading and a ground track answer different questions



Let C , W and S denote craft, water and shore:

$$\mathbf{v}_{C/S} = \mathbf{v}_{C/W} + \mathbf{v}_{W/S}.$$

Choose $+x$ downstream and $+y$ across. For width W , downstream endpoint D , current $w\hat{\mathbf{i}}$, and positive arrival time T ,

$$\mathbf{v}_{C/S} = \left(\frac{D}{T}, \frac{W}{T} \right), \quad \mathbf{v}_{C/W} = \left(\frac{D}{T} - w, \frac{W}{T} \right).$$

If water-relative speed is prescribed as U , impose its norm:

$$(D - wT)^2 + W^2 = U^2 T^2.$$

Thus $(U^2 - w^2)T^2 + 2DwT - (D^2 + W^2) = 0$. Reject unavailable times and test the required speed and both endpoint coordinates.

Section 4.6; Eqs. 4.41–4.43.

Example 4.16: an upstream channel crossing



An idealized craft crosses a 60.0 m-wide channel to a point 30.0 m upstream of its start. The current is 1.00 m s^{-1} downstream. The craft maintains a speed 2.00 m s^{-1} relative to the water. Find the constant heading and arrival time. Treat the craft as a point, with spatially uniform, steady current and no course corrections. **Event model.** Set $D = -30.0 \text{ m}$, $W = 60.0 \text{ m}$, $w = 1.00 \text{ m s}^{-1}$ and $U = 2.00 \text{ m s}^{-1}$. The SI time equation is

$$3T^2 - 60T - 4500 = 3(T - 50)(T + 30) = 0.$$

Only $T = 50.0 \text{ s}$ describes a later crossing. Compute the two frame-specific velocity vectors from this same time.

Complete Example 4.16; heading and coordinate checks on slide 45.

Example 4.16: the heading and arrival check



$$\mathbf{v}_{C/S} = (-0.600, 1.200) \text{ m s}^{-1}, \quad \mathbf{v}_{C/W} = (-1.600, 1.200) \text{ m s}^{-1}.$$

The heading is $\tan^{-1}(1.6/1.2) = 53.1^\circ$ upstream of directly across. The ground track is $\tan^{-1}(0.6/1.2) = 26.6^\circ$ upstream of directly across.

Checks. The water-relative norm is $\sqrt{1.6^2 + 1.2^2} = 2.00 \text{ m s}^{-1}$. Adding $(1, 0)$ gives the shore velocity. Multiplying it by 50.0 s gives the requested endpoint $(-30.0, 60.0) \text{ m}$.

Feasibility. With current $w \geq U$, an upstream destination across the channel cannot be reached under this model. Producing the full upstream component at $w = U$ would leave no crossing component. The manuscript treats stronger-current downstream crossings as further reading.

Example 4.16; Figure 4.17 follows.

The shore path and the velocity triangle

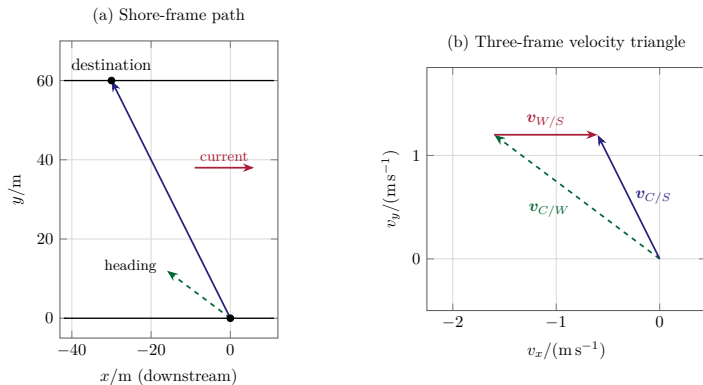


Figure 4.17: The upstream destination and its velocity solution. Each panel uses equal scales on its two axes, but one panel is in metres and the other in metres per second. The dashed arrow in (a) indicates the water-relative heading, not another shore-frame trajectory. Panel (b) closes exactly: $(-1.6, 1.2) + (1, 0) = (-0.6, 1.2) \text{ m s}^{-1}$.

Closest approach within an allowed time window



With constant nonzero relative velocity $\mathbf{V}$, relative separation is $\mathbf{R} = \mathbf{R}_0 + \mathbf{V}t$. Minimize its squared norm:

$$d^2 = R_0^2 + 2(\mathbf{R}_0 \cdot \mathbf{V})t + V^2 t^2, \quad t_* = -\frac{\mathbf{R}_0 \cdot \mathbf{V}}{V^2}.$$

The positive second derivative $2V^2$ makes this a minimum. At an interior minimum, $\mathbf{R}(t_*) \cdot \mathbf{V} = 0$.

If the model lasts only on $[0, H]$, use $t_{\min} = 0$ for $t_* < 0$, t_* for $0 \leq t_* \leq H$, and H for $t_* > H$. An endpoint minimum need not satisfy perpendicularity.

Special case

If $\mathbf{V} = 0$, the separation is constant. Every allowed time minimizes it; dividing by V^2 is unnecessary and invalid.

Section 4.6; Eqs. 4.45 and 4.47.

Example 4.19: a limited three-dimensional approach



Two tracked points have relative position and relative velocity

$$\mathbf{R}_0 = (8.00, -6.00, 4.00) \text{ m}, \quad \mathbf{V} = (-2.00, 1.00, -1.00) \text{ m s}^{-1}.$$

Find their closest separation in the next $H = 3.00 \text{ s}$. Then find the unrestricted future minimum if this constant-velocity model remains valid. **Stationary event.** The dot product is $-26.0 \text{ m}^2 \text{ s}^{-1}$, and $V^2 = 6.00 \text{ m}^2 \text{ s}^{-2}$. Thus

$$d^2 = 116 - 52t + 6t^2 \quad (\text{SI values}), \quad t_* = \frac{13}{3} \text{ s} > H.$$

The closest point in the stated window is its final event:

$$\mathbf{R}(3) = (2, -3, 1) \text{ m}, \quad d_{\min} = \sqrt{14} = 3.74166 \text{ m}.$$

Complete Example 4.19; conditional unrestricted answer on slide 49.

Example 4.19: the conditional later minimum



If the same motion remains valid after the three-second window, the unrestricted future minimum is

$$t_* = 4.33333 \text{ s}, \quad \mathbf{R}_* = \frac{1}{3}(-2, -5, -1) \text{ m}, \quad d_* = \frac{\sqrt{30}}{3} = 1.82574 \text{ m}.$$

Checks. At the later event, $\mathbf{R}_* \cdot \mathbf{V} = 0$. At the window end, $\mathbf{R}(3) \cdot \mathbf{V} = -8.00 \text{ m}^2 \text{ s}^{-1}$, so separation is still decreasing.

An independent geometric decomposition uses the coordinate along $\hat{\mathbf{V}}$:

$$q_0 = -\frac{26}{\sqrt{6}} \text{ m}, \quad d^2(t) = (q_0 + \sqrt{6} t)^2 + d_*^2 \quad \text{in SI values.}$$

The perpendicular distance stays fixed. The longitudinal coordinate reaches zero only after the allowed window ends. The nonzero minimum separation also rules out a point collision.

Example 4.19; both requested domains answered; Figure 4.19 follows.

The window and the spatial separation

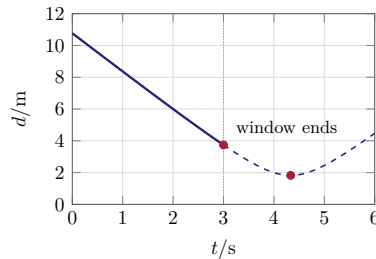
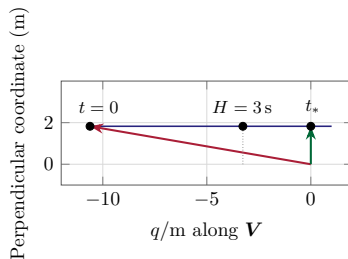


Figure 4.19: A closest-approach calculation in three dimensions. Left: an exact orthonormal plane containing the relative-motion line and the origin, with one axis parallel to $\mathbf{V}$. This is a geometrical reduction, rather than an x - y projection that discards the z separation. Right: actual separation versus time. The solid curve is inside the observation window; the dashed extension requires the additional assumption that the same motion continues.

Forward reconstruction: explicit midpoint



For a supplied time-only acceleration, solve $\dot{\mathbf{r}} = \mathbf{v}$, $\dot{\mathbf{v}} = \mathbf{a}(t)$ from a given initial state. On $t_n = nh$, $h > 0$, use

$$\begin{aligned}\mathbf{v}_{n+1/2}^{\text{pred}} &= \mathbf{v}_n + \frac{1}{2}h\mathbf{a}(t_n), \\ \mathbf{r}_{n+1} &= \mathbf{r}_n + h\mathbf{v}_{n+1/2}^{\text{pred}}, \\ \mathbf{v}_{n+1} &= \mathbf{v}_n + h\mathbf{a}(t_n + h/2).\end{aligned}$$

The position update needs a predicted midpoint *velocity*. Taylor expansion gives a one-step error of order h^3 , typically accumulating to h^2 over a fixed smooth interval.

For constant acceleration, this includes the $\frac{1}{2}\mathbf{a}h^2$ increment and gives exact nodes in exact arithmetic. Forward Euler omits that term and accumulates position error $-\frac{1}{2}\mathbf{a}Th$ at fixed T .

Section 4.7; midpoint equations; Problem 25's constant-acceleration check.

An event can lie between numerical samples



For a first upward crossing of $y = y_e$, find adjacent nodes with $y_n < y_e \leq y_{n+1}$. A linear event estimate is

$$\lambda = \frac{y_e - y_n}{y_{n+1} - y_n}, \quad t_e^{(h)} = t_n + \lambda h, \quad \mathbf{r}_e^{(h)} = \mathbf{r}_n + \lambda(\mathbf{r}_{n+1} - \mathbf{r}_n).$$

Require $0 \leq \lambda \leq 1$ and a nonzero denominator. Use the same fraction for time and every coordinate.

Two errors to test

State integration approximates the nodes. Event interpolation approximates the curve between them. Even exact quadratic nodes can give an inexact linearly interpolated time.

Refine the grid for a smooth crossing. Near a tangency, endpoint signs can miss two crossings; inspect an extremum and search each monotone side.

Section 4.7; Eq. 4.52; grazing discussion.

Practice 2: Example 4.21, a sampled path



Use the midpoint update to reconstruct a planar path from sampled values of the prescribed acceleration

$$\mathbf{a}(t) = [(0.600 \text{ m s}^{-3})t]\hat{\mathbf{i}} + (1.00 \text{ m s}^{-2})\hat{\mathbf{j}}, \quad \mathbf{r}_0 = 0, \quad \mathbf{v}_0 = (1.00, 0) \text{ m s}^{-1}.$$

Estimate the first upward crossing of $y = 0.500 \text{ m}$ and test grid refinement.

Exact benchmark. Integrating the supplied model gives the SI formulas

$$\mathbf{v} = (1 + 0.3t^2, t), \quad \mathbf{r} = (t + 0.1t^3, 0.5t^2).$$

The event $y = 0.500 \text{ m}$ is exactly $t_e = 1.000 \text{ s}$, $x_e = 1.100 \text{ m}$. Test midpoint grids with $h = 0.300 \text{ s}$ and successive halvings through a fixed final time of 1.2 s .

Complete Example 4.21; attempt the bracket calculation before slides 54–55.

Example 4.21: the coarse interpolated event



For $h = 0.300$ s, the midpoint nodes bracketing the upward event are

$$(t, x, y) = (0.9, 0.9648, 0.4050), \quad (1.2, 1.3620, 0.7200),$$

with times in seconds and coordinates in metres. Therefore

$$\lambda = \frac{0.500 - 0.405}{0.720 - 0.405} = 0.301587,$$

$$t_e^{(h)} = 0.9 + 0.300\lambda = 0.990476 \text{ s},$$

$$x_e^{(h)} = 0.9648 + \lambda(1.3620 - 0.9648) = 1.084590 \text{ m}.$$

The predicted event is early and to the left of the exact event. The first node above the threshold, $t = 1.2$ s, would be a poorer time estimate.

Independent node identity. Summing the omitted cubic increment yields $x_n = t_n + 0.1t_n^3 - 0.1t_nh^2$ in SI values. The velocity and vertical nodes are exact for this model.

Example 4.21; refinement and error definitions on slide 55.

Example 4.21: grid refinement is a separate test



Hold the model, initial state, final time 1.2 s, and event $y = 0.500$ m fixed. Errors are absolute differences from the exact endpoint or event:

h/s	$t_e^{(h)}/s$	E_x^{end}/m	E_t/s	E_x^{event}/m
0.30000	0.99047619	0.01080000	0.00952381	0.01540952
0.15000	0.99743590	0.00270000	0.00256410	0.00406603
0.07500	0.99938272	0.00067500	0.00061728	0.00099124
0.03750	0.99984277	0.00016875	0.00015723	0.00025106
0.01875	0.99996106	0.00004219	0.00003894	0.00006237

The node identity gives $E_x^{\text{end}} = 0.12h^2$ in metres, decreasing exactly by four at each halving. Event errors approach second order with a changing bracket fraction.

Interpretation. Event-time error comes from interpolation of exact y nodes. Event-position error combines integration and interpolation. These deterministic errors are distinct from measurement uncertainty.

Example 4.21; manuscript convergence table; all five rows independently recomputed.

Coordinates to derivatives: central differences



For equally spaced coordinate samples, Taylor expansion about the central time gives

$$\mathbf{v}_{x,i}^{\text{est}} = \frac{x_{i+1} - x_{i-1}}{2h}, \quad \mathbf{a}_{x,i}^{\text{est}} = \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}.$$

Odd Taylor terms cancel in the second difference and even terms in the first. Both estimates have smooth-function truncation error of order h^2 . Their time is the *central* sample time.

The next source example uses the benchmark

$$x = t + 0.1t^3, \quad y = 0.5t^2 \quad (\text{SI numerical values}).$$

Its exact derivatives are $\mathbf{v} = (1 + 0.3t^2, t)$ and $\mathbf{a} = (0.6t, 1)$. This restores the referenced path before examining the perturbed samples.

Section 4.7; Eqs. 4.53–4.54; context for Example 4.22.

Example 4.22: synthetic coordinate samples



Consider the following synthetic coordinates, sampled at $h = 0.200$ s. They are generated from the analytical path of Example 4.21, with a deterministic alternating x perturbation $0.001(-1)^i$ m, where $i = 0$ at the first row. The y coordinates are unperturbed.

t/s	x_i/m	y_i/m
0.0	0.0010	0.0000
0.2	0.1998	0.0200
0.4	0.4074	0.0800
0.6	0.6206	0.1800
0.8	0.8522	0.3200
1.0	1.0990	0.5000
1.2	1.3738	0.7200

Estimate the velocity and acceleration at $t = 0.600$ s, then compare with the known model.

Complete Example 4.22; referenced analytical path is restored on slide 56; solution on slide 58.

Example 4.22: estimates and their causes



At $t = 0.600$ s, with $h = 0.200$ s, the central stencils give

$$v_x^{\text{est}} = \frac{0.8522 - 0.4074}{0.400} = 1.112 \text{ m s}^{-1},$$
$$a_x^{\text{est}} = \frac{0.8522 - 2(0.6206) + 0.4074}{0.0400} = 0.460 \text{ m s}^{-2}.$$

The unperturbed y data give $v_y^{\text{est}} = 0.600 \text{ m s}^{-1}$ and $a_y^{\text{est}} = 1.00 \text{ m s}^{-2}$. Exact model values are $v_x = 1.108 \text{ m s}^{-1}$ and $a_x = 0.360 \text{ m s}^{-2}$.

Cause check. Equal outer perturbations cancel in the first difference. Its remaining error is the cubic bias $0.1h^2 = 0.004 \text{ m s}^{-1}$. The cubic has no second-difference acceleration bias, but the alternating perturbations give $0.004/h^2 = 0.100 \text{ m s}^{-2}$.

Small coordinate errors can therefore create a much larger apparent acceleration change. The values are constructed data, not laboratory observations.

Example 4.22; all velocity and acceleration components compared with the model.

Noise, sampling and what a camera can establish



For independent equal coordinate standard uncertainties σ_x , with negligible timing error, squared stencil weights give

$$\sigma_{v_x} = \frac{\sigma_x}{\sqrt{2}h}, \quad \sigma_{a_x} = \frac{\sqrt{6}\sigma_x}{h^2}.$$

For a separate random-error model with $\sigma_x = 0.001 \text{ m}$ and $h = 0.200 \text{ s}$, these are 0.00354 m s^{-1} and 0.0612 m s^{-2} . They do not describe the deterministic alternating errors as independent random noise.

Halving spacing reduces smooth truncation error but doubles velocity uncertainty and quadruples acceleration uncertainty at fixed σ_x . Fits need a justified model and window, followed by residual checks.

An ideal camera view along z records only (x, y) . Without an established motion plane or depth information, it cannot determine the missing velocity and acceleration components.

Section 4.7; Eq. 4.55; fitting and projection subsections for reading.

Core synthesis and the bridge to Newton's laws



- A timed trajectory in a named frame supplies velocity and acceleration. Geometry alone leaves the schedule open.
- An event must satisfy every relevant coordinate condition at one time, inside the physical model's domain.
- Acceleration changes speed and direction through perpendicular local contributions.
- Relative motion uses vectors in named frames. Numerical convergence and measurement uncertainty require separate checks.

Next chapter: Newton's laws relate the acceleration to the net force on the selected object. The present calculations describe the motion; identifying interactions is the next task.

Reading: the current Chapter 4, including helix motion, additional projectile designs, strong-current crossings, fitting and grazing events. Its bibliography points to HRW Vol. 1, Serway–Jewett and OpenStax Vol. 1.

Chapter summary; evidenced bridge to Newton's Laws of Motion.

Advanced: the polar basis changes with angle



Range and bearing coordinates are useful for a scanner, but their directions rotate as the point moves. For $r > 0$, define

$$\hat{\mathbf{e}}_r = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}, \quad \hat{\mathbf{e}}_\theta = -\sin \theta \hat{\mathbf{i}} + \cos \theta \hat{\mathbf{j}}.$$

The basis is orthonormal. $\hat{\mathbf{e}}_r$ points outward from the origin and $\hat{\mathbf{e}}_\theta$ toward increasing counterclockwise angle. These directions generally differ from the path's normal and tangent.

Differentiate their Cartesian components with the chain rule:

$$\frac{d\hat{\mathbf{e}}_r}{dt} = \dot{\theta} \hat{\mathbf{e}}_\theta, \quad \frac{d\hat{\mathbf{e}}_\theta}{dt} = -\dot{\theta} \hat{\mathbf{e}}_r.$$

At $\theta = 0$ with $\dot{\theta} > 0$, the radial direction turns toward $+y$ and the transverse direction toward $-x$, checking both signs.

Advanced section; Eqs. 4.60–4.62.

Both polar unit vectors turn through the same angle

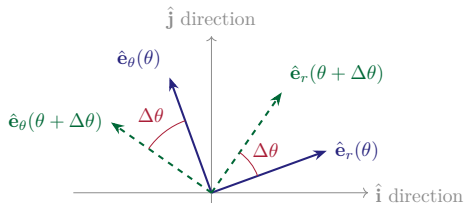


Figure 4.26: Both polar basis vectors rotate by the same angle. Equal arrow lengths represent dimensionless unit vectors; this is a basis-space construction, not a trajectory or a drawing of four velocities. The solid vectors are at 20° , the dashed vectors at 55° , with the same 35° increment.

Polar velocity from the product rule



Position is $\mathbf{r} = r\hat{\mathbf{e}}_r$. Differentiating both factors gives

$$\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + r\frac{d\hat{\mathbf{e}}_r}{dt} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta.$$

Therefore $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$. Angular rate alone is not a velocity component: multiplication by radius supplies units of m s^{-1} .

$$v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}, \quad \hat{\mathbf{t}} = \frac{\dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta}{v} \quad (v > 0).$$

An outward-moving scanner has a tilted tangent, even with a constant angular rate. At the coordinate origin, polar angle and basis are not uniquely defined; Cartesian coordinates avoid that singular representation.

Advanced section; Eqs. 4.63–4.64.

Polar acceleration: every differentiated factor matters



Differentiate the two velocity contributions separately:

$$\begin{aligned}\frac{d}{dt}(r\hat{\mathbf{e}}_r) &= \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta, \\ \frac{d}{dt}(r\dot{\theta}\hat{\mathbf{e}}_\theta) &= (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{e}}_\theta - r\dot{\theta}^2\hat{\mathbf{e}}_r.\end{aligned}$$

Collecting them yields

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta.$$

The two transverse product-rule contributions produce the factor two. Coordinate second derivatives alone omit basis rotation.

Independent limits. A fixed ray gives $\mathbf{a} = \ddot{r}\hat{\mathbf{e}}_r$. A constant-radius path gives $-R\dot{\theta}^2\hat{\mathbf{e}}_r + R\ddot{\theta}\hat{\mathbf{e}}_\theta$. Constant angular rate then restores the core inward circular result.

Advanced section; Eq. 4.65; exact kinematics for differentiable $r > 0$.

Example 4.26: a prescribed expanding spiral



Restored source equation: $r = r_0 + ut$, $\theta = \Omega t$. A point-like scanner follows Eq. (4.66) for $0 \leq t \leq 4.00$ s, with $r_0 = 0.400$ m, $u = 0.100$ m s⁻¹ and $\Omega = 1.00$ rad s⁻¹. Find its velocity and acceleration at $t = \pi/2$ s. Is its actual direction of motion the polar transverse direction? **Model and event.** $\dot{r} = u$, $\ddot{r} = 0$, $\dot{\theta} = \Omega$, and $\ddot{\theta} = 0$. Consequently,

$$\mathbf{v} = u\hat{\mathbf{e}}_r + r\Omega\hat{\mathbf{e}}_\theta, \quad \mathbf{a} = -r\Omega^2\hat{\mathbf{e}}_r + 2u\Omega\hat{\mathbf{e}}_\theta.$$

At $t = \pi/2$ s, $r = 0.557080$ m, $\hat{\mathbf{e}}_r = \hat{\mathbf{j}}$, and $\hat{\mathbf{e}}_\theta = -\hat{\mathbf{i}}$. The complete Cartesian state and tangent check follow.

Complete Example 4.26; Eq. 4.66 restored above; solution on slide 66.

Example 4.26: the actual tangent and a Cartesian check



At the stated event,

$$\begin{aligned}\mathbf{v} &= (-0.557080, 0.100) \text{ m s}^{-1}, & v &= 0.565984 \text{ m s}^{-1}, \\ \mathbf{a} &= (-0.200, -0.557080) \text{ m s}^{-2}, & |\mathbf{a}| &= 0.591893 \text{ m s}^{-2}.\end{aligned}$$

The tangent tilts outward from $\hat{\mathbf{e}}_\theta$ by $\beta = \tan^{-1}[u/(r\Omega)] = 10.2^\circ$, so it is not the polar transverse direction.

Independent Cartesian check. For $x = r \cos \Omega t$, $y = r \sin \Omega t$, direct product-rule differentiation gives

$$\begin{aligned}\ddot{x} &= -2u\Omega \sin \Omega t - r\Omega^2 \cos \Omega t, \\ \ddot{y} &= 2u\Omega \cos \Omega t - r\Omega^2 \sin \Omega t.\end{aligned}$$

Inserting $\Omega t = \pi/2$ reproduces both acceleration components. Setting $u = 0$ recovers a circle; setting $\Omega = 0$ gives constant-velocity outward motion.

Example 4.26; Figure 4.27 follows.

The spiral and the actual velocity direction

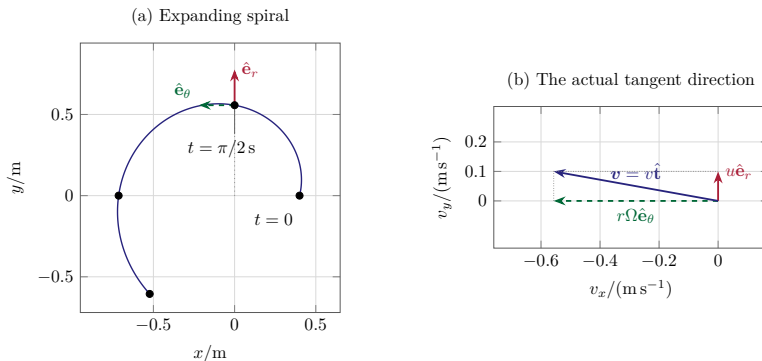


Figure 4.27: The calculated spiral and its velocity decomposition at the same event. Panel (a) has equal metre scales and direction-only unit-vector glyphs; these are not velocity arrows. Panel (b) uses a common velocity scale and shows the actual tangent $\hat{\mathbf{t}}$, distinct from $\hat{\mathbf{e}}_\theta$ because $\dot{r} \neq 0$. The changing-basis geometry underlying this calculation is shown separately in Fig. 4.26.

Advanced challenge: Problem 28



A different prescribed planar path has $r(t) = 0.300 + 0.0200t^2$ in metres and $\theta(t) = 2.00t + 0.100t^2$ in radians, with t entered in seconds and $0 \leq t \leq 2$ s. Find v_r , v_θ , a_r , a_θ and speed at one second. Keep radial coordinate acceleration distinct from the radial component of vector acceleration. **Attempt.**

First differentiate the schedules. Then use the full polar component formulas from slide 64. The coordinate radial acceleration and the radial vector component have the same units but represent different operations.

At one second, the intermediate values are $r = 0.320$ m, $\theta = 2.10$ rad, $\dot{r} = 0.0400$ m s⁻¹, $\dot{\theta} = 2.20$ s⁻¹, $\ddot{r} = 0.0400$ m s⁻², and $\ddot{\theta} = 0.200$ s⁻².

Complete Problem 28; fixed laboratory frame; solution on slide 69.

Problem 28: coordinate rates and vector components



Use the basis at $\theta = 2.10$ rad. The requested components are

$$v_r = \dot{r} = 0.0400 \text{ m s}^{-1},$$

$$v_\theta = r\dot{\theta} = (0.320)(2.20) = 0.704 \text{ m s}^{-1},$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = 0.0400 - 0.320(2.20)^2 = -1.5088 \text{ m s}^{-2},$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.320(0.200) + 2(0.0400)(2.20) = 0.240 \text{ m s}^{-2}.$$

Speed is $\sqrt{0.0400^2 + 0.704^2} = 0.705135 \text{ m s}^{-1}$.

Although $\ddot{r} > 0$, the vector's radial acceleration is inward because basis rotation contributes the larger negative term. Independently differentiating $(r \cos \theta, r \sin \theta)$ twice and projecting onto $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta)$ reproduces both acceleration components.

Problem 28; every requested component and speed answered.

Problem 30: a rate valid for the whole scan



Retain the full 0–4.00 s spiral interval, $r_0 = 0.400 \text{ m}$ and $u = 0.100 \text{ m s}^{-1}$, but allow the positive constant angular rate Ω to change. Find the largest Ω consistent throughout the interval with the illustrative acceleration bound $|\mathbf{a}| \leq 0.700 \text{ m s}^{-2}$. Explain why checking the final radius is sufficient for this model. **Restored model:** $r = r_0 + ut$, $\theta = \Omega t$, so $|\mathbf{a}|^2 = r^2\Omega^4 + 4u^2\Omega^2$.

Its derivative is $2ru\Omega^4 > 0$. The greatest radius, $r(4) = 0.800 \text{ m}$, sets the strongest bound.

Let $q = \Omega^2$. The limiting SI equation is $0.640q^2 + 0.0400q - 0.490 = 0$. Its positive root is

$$q = \frac{-0.0400 + \sqrt{0.0400^2 + 4(0.640)(0.490)}}{1.280} = 0.84430786 \text{ s}^{-2}.$$

Thus $\Omega_{\max} = 0.91886226 \text{ rad s}^{-1}$. The negative root cannot equal a real squared rate. Substitution gives 0.700 m s^{-2} at the end and smaller values earlier. The bound is therefore met throughout the interval.

Complete Problem 30; interval reasoning, root choice and answer on this page.

