

Newton's Laws of Motion

Physics I - Chapter 5

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Why does motion change?



The earlier chapters described position, velocity and acceleration. Dynamics connects a change in motion to the interactions acting on a chosen system.

A coasting instrument

An instrument can keep moving along an ideal level guide without a horizontal drive. Its unchanged velocity requires zero horizontal resultant.

This chapter develops a route from interaction inventories to Newton's laws, then to force models and geometrical constraints. We will predict accelerations, infer applied forces and test whether a support or taut connector can sustain the assumed motion.

Source: Manuscript Chapter 5

Learning route and prerequisites



- 1 Select the system and observation frame, then identify external interactions.
- 2 Apply Newton's laws with signed vector components and explicit force models.
- 3 Derive geometrical constraints before combining the equations of different bodies.
- 4 Check individual equations, signs, limiting cases and contact domains.

Tools we will refresh

Vector components and scalar products; velocity and acceleration as time derivatives; constant-acceleration kinematics; elementary simultaneous equations. The advanced block adds a definite integral and dimensionless residuals.

Source: Manuscript Chapter 5

Learning goals



After the core lecture, you should be able to:

- choose a system and frame, and distinguish forces from motion information;
- use all three Newton laws without confusing force balance with an interaction pair;
- resolve weight, normal reaction, tension and spring forces in appropriate axes;
- derive rope constraints and solve connected-body equations;
- interpret apparent weight and reject inadmissible contact or tension solutions.

Advanced goals: derive a distributed tension profile, quantify a massless-link approximation, and test numerical candidates with physically scaled residuals.

Source: Manuscript Chapter 5

Systems and external interactions



A **system** is the body or collection of bodies selected for analysis. Everything outside it is the environment.

A force is external when its source is outside the chosen boundary. An interaction between two included bodies is internal to their composite system.

A free-body diagram

Draw only forces acting on the selected system. A velocity arrow, an acceleration arrow and the resultant of the drawn forces are different representations. They are not extra interactions.

Changing the boundary changes which forces are external. It does not remove the physical interaction.

Source: Manuscript Chapter 5

Inertial frames and the first law



An inertial frame allows an isolated particle to have constant velocity. A frame translating at constant velocity relative to it is also inertial in Newtonian mechanics.

$$\sum \mathbf{F}_{\text{ext}} = \mathbf{0} \implies \mathbf{a} = \mathbf{0} \implies \mathbf{v} = \text{constant.}$$

Rest is one constant-velocity state. A ground laboratory is an approximate inertial frame for the short, modest-precision models used here.

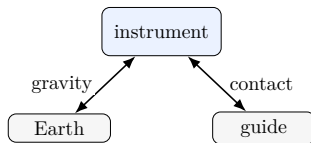
An answered turning-point check

A ball can have zero velocity at its highest point while gravity still gives downward acceleration. Zero instantaneous velocity alone does not establish equilibrium.

Source: Manuscript Chapter 5

Interaction map and force diagram

(a) Interaction map



Links denote physical interactions, not extra force arrows.

Only Earth and the guide interact with the idealised coasting instrument.

(b) Free-body diagram

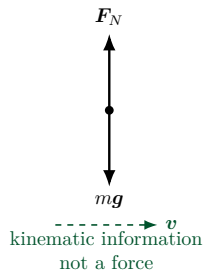


Figure 5.1: An instrument coasting on an ideal horizontal guide in an approximately inertial laboratory frame. The interaction map lists the physical partners; the free-body diagram contains only the two forces on the selected instrument. With negligible tangential resistance, the vertical forces balance and no horizontal force is required for constant horizontal velocity.

Two observers, one acceleration



Let S' translate at constant velocity $\mathbf{V}$ relative to inertial frame S . The axes are parallel and the origins coincide at $t = 0$.

$$\mathbf{r}' = \mathbf{r} - \mathbf{V}t, \quad \mathbf{v}' = \mathbf{v} - \mathbf{V}, \quad \mathbf{a}' = \mathbf{a}.$$

The final equality follows because $d\mathbf{V}/dt = \mathbf{0}$. Both observers can use the same interaction forces and acceleration in Newton's law.

A carriage can move east for one observer and west for the other while having zero acceleration for both. A changing speed or a turning observation platform requires a different frame analysis.

Source: Manuscript Chapter 5

Example 5.2: two inertial observers



A carriage moves on an ideal horizontal guide at $v_x = +0.60 \text{ m s}^{-1}$ relative to the laboratory. An observation platform moves parallel to the guide at $V = +1.00 \text{ m s}^{-1}$, with its origin passing the laboratory origin at $t = 0$. The carriage also passes that origin at $t = 0$. Find its velocity and position after 4.00 s for both observers. Decide whether the observation platform must introduce a new force to explain the carriage's westward motion relative to it.

Both frames translate without rotation. Vertical support and gravity balance; tangential resistance is neglected.

$$x = vt = (0.60)(4.00) = 2.40 \text{ m}, \quad v_x = +0.60 \text{ m s}^{-1}.$$

$$v'_x = 0.60 - 1.00 = -0.40 \text{ m s}^{-1}, \quad x' = 2.40 - 1.00(4.00) = -1.60 \text{ m}.$$

Both accelerations and horizontal resultants are zero. No new westward force is needed. Setting $V = v_x$ makes the carriage stationary in S' without changing that force balance.

Source: Manuscript Example 5.2

The same motion in two inertial frames

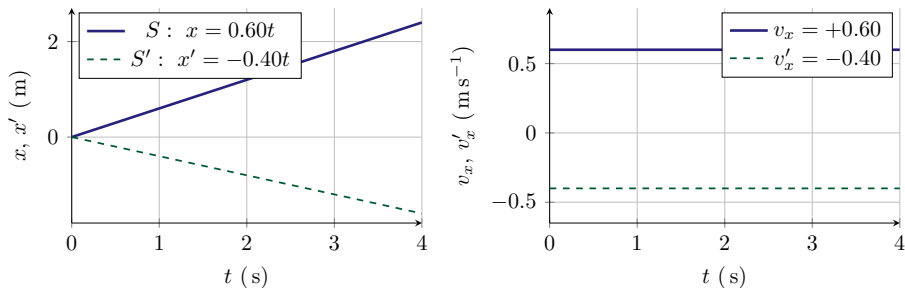


Figure 5.2: One coasting object described by two inertial observers. Frame S' moves along $+x$ at $V = 1.00 \text{ m s}^{-1}$ relative to S . The positions and velocities differ, while both accelerations are zero. Curves are exact analytical model predictions, not experimental records.

Mass and Newton's second law



For a constant-mass particle in an inertial frame,

$$\boxed{\sum \mathbf{F}_{\text{ext}} = m\mathbf{a}}, \quad \sum F_x = ma_x, \quad \sum F_y = ma_y, \quad \sum F_z = ma_z.$$

Inertial mass $m > 0$ measures the response to an external resultant. Force has SI unit $1\text{ N} = 1\text{ kg m s}^{-2}$. The law is empirical; its units do not prove it. Add or subtract force *components*. A magnitude is nonnegative, while a signed component may be negative.

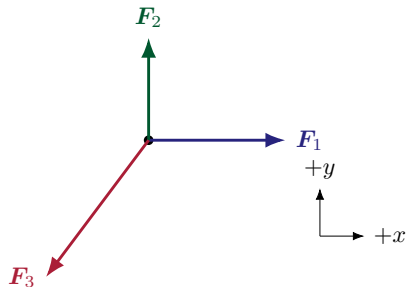
$$\mathbf{F}_3 = m\mathbf{a} - \mathbf{F}_1 - \mathbf{F}_2$$

infers the total missing interaction in a stated three-force model. It does not identify separate unmodelled sources.

Source: Manuscript Chapter 5

Force inventory and vector arithmetic

(a) Forces on the particle



(b) Vector sum construction

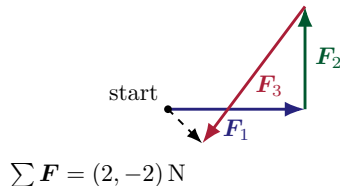


Figure 5.3: Recovering a missing interaction from a known acceleration. The left panel is a two-dimensional force diagram; support and gravity perpendicular to this plane cancel in the ideal horizontal-table model. The right panel performs vector addition and is not another free-body diagram. Arrow vectors share the stated scale 0.25 cm per newton.

Guided practice: Example 5.3



A 2.00 kg puck translates on an ideal horizontal air table. The x and y axes lie in the table plane. Two calibrated actuators apply $\mathbf{F}_1 = 8.00\hat{x}\text{ N}$ and $\mathbf{F}_2 = 6.00\hat{y}\text{ N}$. A third actuator supplies an unknown vector force. Motion tracking finds $\mathbf{a} = (1.00\hat{x} - 1.00\hat{y})\text{ m s}^{-2}$. Find both components and the magnitude of the third force. Verify the measured acceleration by reconstructing the force sum.

Model before calculation

Select the puck in the laboratory frame. The table normal and gravity balance outside the x - y plane. The three actuators are the complete in-plane force inventory.

First form $m\mathbf{a}$, then subtract both known vectors by components. The full answer and an independent reconstruction follow on slide 15.

Source: Manuscript Example 5.3

Example 5.3: solution and reconstruction



$$m\mathbf{a} = (2.00, -2.00) \text{ N}.$$

$$F_{3x} = 2.00 - 8.00 = -6.00 \text{ N}, \quad F_{3y} = -2.00 - 6.00 = -8.00 \text{ N}.$$

Thus $\mathbf{F}_3 = (-6.00\hat{x} - 8.00\hat{y}) \text{ N}$.

$$|\mathbf{F}_3| = \sqrt{6.00^2 + 8.00^2} = 10.0 \text{ N}.$$

It points 53.1° below the negative x axis, in the southwest quadrant.

Independent check: $(8, 0) + (0, 6) + (-6, -8) = (2, -2) \text{ N}$. Dividing each component by 2.00 kg recovers the measured acceleration. The resultant magnitude is 2.83 N ; cancellation makes it smaller than the individual forces.

Source: Manuscript Example 5.3

Example 5.4: drive force and net force



With $+x$ along motion and a prescribed opposing resistance R ,

$$F_{\text{drive}} - R = ma_x, \quad F_{\text{drive}} = ma_x + R.$$

The drive-force line has slope m and intercept R .

A test carriage of mass $m = 18.0 \text{ kg}$ moves on a straight horizontal guide. A separately characterised resistance of magnitude $R = 12.0 \text{ N}$ acts opposite its motion. At one instant the measured acceleration is $a_x = 1.25 \text{ m s}^{-2}$. Determine the required drive force in the stated model.

In the constant-mass laboratory model, the vertical guide constraint gives zero vertical acceleration. Horizontally,

$$F_{\text{drive}} = (18.0)(1.25) + 12.0 = 34.5 \text{ N}.$$

The net force is 22.5 N . At $a_x = 0$, drive and resistance both equal 12.0 N , permitting constant velocity. Force and motion data must be obtained independently when testing this inference.

Source: Manuscript Example 5.4

Force direction and finite-time response



The resultant fixes acceleration. Initial velocity fixes the motion already present. For a constant resultant,

$$\mathbf{a} = \frac{\sum \mathbf{F}}{m}, \quad \mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}t, \quad \mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0t + \frac{1}{2}\mathbf{a}t^2.$$

Acceleration acts component by component. Using $v^2 = \mathbf{v} \cdot \mathbf{v}$,

$$\frac{d(v^2)}{dt} = 2\mathbf{v} \cdot \mathbf{a}.$$

An initially perpendicular force gives zero instantaneous speed derivative. It need not remain perpendicular as velocity changes, so speed can increase over a finite interval.

Source: Manuscript Chapter 5

A trajectory with an initially transverse force

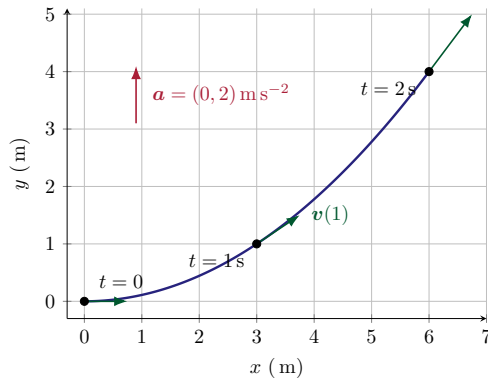


Figure 5.5: Motion under a constant northward in-plane force with an initially eastward velocity. The path is $x = 3t$, $y = t^2$ for $0 \leq t \leq 2$ s. Green tangent arrows are velocities, drawn at a common display multiplier of 0.25 s; the separate red arrow indicates acceleration direction and is not an added force. This is a motion diagram, not a free-body diagram.

Example 5.5: a transverse force



A 2.00 kg puck moves on an ideal horizontal air table with initial velocity $\mathbf{v}_0 = 3.00\hat{x} \text{ m s}^{-1}$. At $t = 0$ an actuator begins applying the constant in-plane force $\mathbf{F} = 4.00\hat{y} \text{ N}$. Set the initial in-plane position to zero. Find the position, velocity, speed and direction at $t = 2.00 \text{ s}$. Does the initially perpendicular force keep the speed fixed throughout the interval?

Physical model

Select the puck in the approximately inertial laboratory. Gravity and normal support cancel perpendicular to the table. The actuator is the only in-plane interaction.

Use $+x$ east and $+y$ north. The constant force gives $a_x = 0$ and $a_y = 4.00/2.00 = 2.00 \text{ m s}^{-2}$. Apply the integrated equations from slide 17.

Source: Manuscript Example 5.5

Example 5.5: trajectory and speed



Using t in seconds, $v_x = 3.00$, $v_y = 2.00t$, $x = 3.00t$ and $y = t^2$, with the corresponding SI units.

$$\mathbf{r}(2.00) = (6.00\hat{\mathbf{x}} + 4.00\hat{\mathbf{y}}) \text{ m}, \quad \mathbf{v}(2.00) = (3.00\hat{\mathbf{x}} + 4.00\hat{\mathbf{y}}) \text{ m s}^{-1}.$$

$$v = 5.00 \text{ m s}^{-1}, \quad \theta_v = \tan^{-1}(4/3) = 53.1^\circ \text{ north of east.}$$

The force is perpendicular to velocity only initially. For $t > 0$, the northward velocity component grows, giving

$$v(t) = \sqrt{9.00 + 4.00(t/\text{s})^2} \text{ m s}^{-1}.$$

Twice differentiating the position gives $(0, 2.00) \text{ m s}^{-2}$. At $t = 1 \text{ s}$, the path slope $2/3$ agrees with v_y/v_x . These checks recover both dynamics and tangent direction.

Source: Manuscript Example 5.5

Newton's third law: target and source



The notation $\mathbf{F}_{A \leftarrow B}$ means the force *on A due to B*. For one Newtonian interaction,

$$\mathbf{F}_{A \leftarrow B} = -\mathbf{F}_{B \leftarrow A}.$$

The partners occur simultaneously, have equal magnitudes and opposite directions, and act on different bodies.

An answered book-on-table check

The book's weight and normal force both act on the book, so they are not a third-law pair. The weight's partner acts on Earth; the normal force's partner acts on the table. Equality of weight and normal reaction comes from the book's force balance.

Source: Manuscript Chapter 5

A connector is a third body



For carts A, B and connector C , the direct pairs are

$$\mathbf{F}_{A \leftarrow C} = -\mathbf{F}_{C \leftarrow A}, \quad \mathbf{F}_{B \leftarrow C} = -\mathbf{F}_{C \leftarrow B}.$$

The forces on the two carts are not each other's direct partners. Isolating the straight connector gives

$$T_R - T_L = m_C a_C.$$

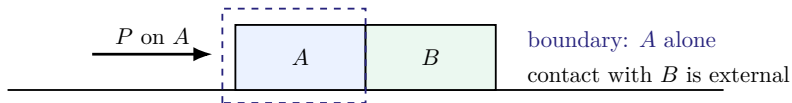
Equal endpoint tension follows when $m_C \rightarrow 0$ at finite acceleration. A finite-mass accelerating link generally needs unequal tensions.

For $A + B + C$, all cart-link pairs are internal. A common acceleration additionally requires a straight taut inextensible geometry; force-pair equality alone does not supply it.

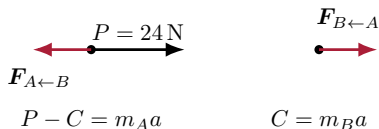
Source: Manuscript Chapter 5

Direct contact and useful system boundaries

(a) The interaction crosses a one-cart boundary



(b) Separate horizontal force diagrams



(c) Both carts

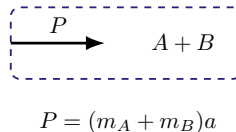


Figure 5.7: A direct cart–cart contact has a genuine third-law pair on different carts. For one cart the contact is external; for both carts it is internal. The lower panels show horizontal force content only: the omitted vertical normal forces and weights balance separately. Horizontal force lengths in panel (b) use 0.06 cm per newton; the apparatus and composite-system panel are schematic.

Example 5.7: two touching carts



Two touching carts of masses $m_A = 3.00\text{ kg}$ and $m_B = 5.00\text{ kg}$ stand on an ideal horizontal track. A 24.0 N external horizontal force pushes the rear cart A toward the front cart B . The carts maintain contact and translate without relevant rotation. Find the acceleration and the force each cart exerts on the other. Would applying the same push to the 5.00 kg cart, with that cart now at the rear, change these two results?

Model and boundary choice

Use the laboratory frame and a common $+x$ axis along the push. The maintained contact supplies a common acceleration. First select both carts to find a , then the front cart to find the contact magnitude $C \geq 0$.

$$P = (m_A + m_B)a, \quad C = m_{\text{front}}a.$$

Source: Manuscript Example 5.7

Example 5.7: contact and reversed order



$$a = \frac{24.0}{3.00 + 5.00} = 3.00 \text{ m s}^{-2}.$$

Initially the 5.00 kg front cart requires

$$C = 5.00(3.00) = 15.0 \text{ N}.$$

The force on B due to A is $+15.0 \hat{x} \text{ N}$; its partner on A is $-15.0 \hat{x} \text{ N}$. The rear equation checks $24.0 - 15.0 = 3.00(3.00) = 9.00 \text{ N}$.

Reversing the mass order preserves total mass and acceleration, but makes the 3.00 kg cart the front body:

$$C' = 3.00(3.00) = 9.00 \text{ N}, \quad 24.0 - 9.00 = 5.00(3.00).$$

Both contact magnitudes are positive. For a vanishing front mass, $C \rightarrow 0$. The third law fixes pair equality; the full force inventory and contact geometry fix motion.

Source: Manuscript Example 5.7

Weight, support, tension and spring models



- **Weight:** $\mathbf{W} = m\mathbf{g}$, with $g = 9.81 \text{ m s}^{-2} > 0$ in our numerical models. Mass measures inertia; weight depends on the gravitational environment.
- **Normal reaction:** perpendicular to a contact surface. Its value follows from the contact constraint, with $F_N \geq 0$ for a nonadhesive support.
- **Ideal tension:** a taut flexible connector pulls along its direction. A common magnitude requires negligible mass and ideal routing.
- **Linear spring:** $F_{s,x} = -k_s(x - x_0)$, with $k_s > 0$, only within its linear range.

Tangential contact may be required by a force balance. A friction-availability law will enter in Chapter 6.

Source: Manuscript Chapter 5

Inclined support: projections and constraints



Take $+s$ uphill along a fixed plane inclined by θ ; take $+n$ perpendicular outward.

$$\hat{\mathbf{t}} = (\cos \theta, \sin \theta), \quad \hat{\mathbf{n}} = (-\sin \theta, \cos \theta).$$

They are perpendicular unit vectors. Dotting $\mathbf{W} = (0, -mg)$ with each gives

$$W_s = -mg \sin \theta, \quad W_n = -mg \cos \theta.$$

While a body stays on a fixed straight smooth plane, $a_n = 0$. Therefore

$$F_N - mg \cos \theta = 0, \quad -mg \sin \theta = ma_s,$$

so $F_N = mg \cos \theta$ and $a_s = -g \sin \theta$. Normal acceleration is zero even though vertical acceleration need not be.

Source: Manuscript Chapter 5

Actual forces and weight projections

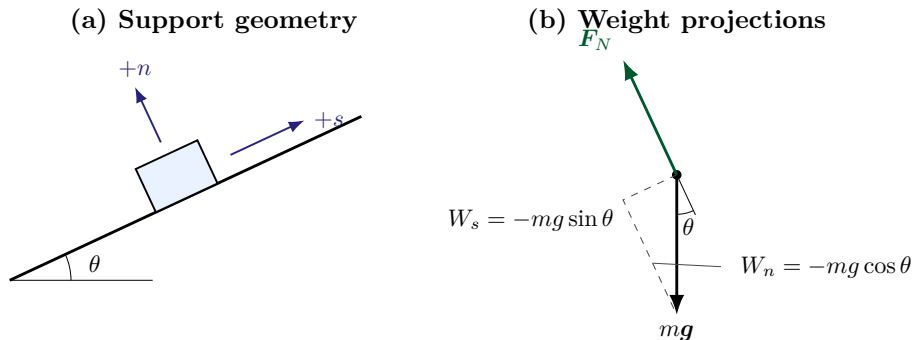


Figure 5.8: Weight components on a plane tilted by $\theta = 25.0^\circ$. The force diagram contains only gravity and the outward normal reaction. Dashed segments are projection guides for gravity, not extra interactions; they explain the signed components in the rotated basis. The shown force lengths use the 6.00 kg example and 0.035 cm per newton. Apparatus geometry is schematic.

Example 5.8: uphill motion and reversal



A 6.00 kg carriage is released on a fixed frictionless straight guide inclined at 25.0° above horizontal. At release it is moving uphill at 2.00 m s^{-1} . Find its normal force and tangential acceleration, the time until it is instantaneously at rest, and its maximum uphill displacement from release. Explain why the acceleration does not reverse when the velocity reverses.

Model and method

Select the carriage in the laboratory frame. The fixed frictionless straight guide supplies only a normal reaction. Use the uphill and outward axes of Figure 5.8; the constraint is $a_n = 0$.

Find forces and constant a_s first, then use $v_s = v_{s0} + a_s t$ and $\Delta s = v_{s0} t + \frac{1}{2} a_s t^2$ to locate the turning point.

Source: Manuscript Example 5.8

Example 5.8: forces and turning point



$$F_N = (6.00)(9.81) \cos 25.0^\circ = 53.3 \text{ N},$$
$$a_s = -9.81 \sin 25.0^\circ = -4.15 \text{ m s}^{-2}.$$

Using the unrounded magnitude $4.145885 \text{ m s}^{-2}$,

$$t_{\text{turn}} = \frac{2.00}{4.145885} = 0.482 \text{ s}, \quad \Delta s_{\text{max}} = \frac{2.00^2}{2(4.145885)} = 0.482 \text{ m}.$$

At the turning point $v_s = 0$, but neither gravity nor support changes. Acceleration remains downhill as the velocity reverses.

Independent Cartesian check:

$$a_x = -g \sin \theta \cos \theta, \quad a_y = -g \sin^2 \theta.$$

These are the components of $a_s \hat{\mathbf{t}}$; the vertical acceleration is nonzero while the normal acceleration is zero. As $\theta \rightarrow 0$, $a_s \rightarrow 0$ and a finite turning time disappears. The positive F_N supports the contact assumption.

Source: Manuscript Example 5.8

An angled applied force changes support load



On an incline, a tether directed outward by angle ϕ from uphill has components $T \cos \phi$ along $+s$ and $T \sin \phi$ along $+n$.

$$T \cos \phi - mg \sin \theta = ma_s, \quad F_N + T \sin \phi - mg \cos \theta = 0.$$

The outward component unloads the support. For equilibrium,

$$T_{\text{eq}} = \frac{mg \sin \theta}{\cos \phi}, \quad F_{N,\text{eq}} = mg(\cos \theta - \sin \theta \tan \phi).$$

For $0 < \theta < 90^\circ$, $0 \leq \phi < 90^\circ$, contact requires $\phi \leq 90^\circ - \theta$. Equality is a marginal state with $F_N = 0$.

On a horizontal platform, an applied downward component instead increases the normal reaction. The force direction enters both component equations.

Source: Manuscript Chapter 5

Example 5.10: a package against a wall



A package of mass $m = 3.20 \text{ kg}$ is pressed horizontally against a vertical wall by a constant force $P = 70.0 \text{ N}$. It is observed to remain at rest. Determine the normal force and the tangential contact force *required* by equilibrium. Do not assume a coefficient-of-friction law.

Select the package in the laboratory. Take $+x$ away from the wall and $+y$ upward. Rest gives $a_x = a_y = 0$.

$$F_N - P = 0 \quad \Rightarrow \quad F_N = 70.0 \text{ N}.$$

$$F_{\parallel, \text{req}} - mg = 0 \quad \Rightarrow \quad F_{\parallel, \text{req}} = 3.20(9.81) = 31.4 \text{ N upward}.$$

These are required contact components. Material-dependent availability needs the later friction model. The normal and gravitational forces both act on the package; their third-law partners act on the wall and Earth respectively.

Source: Manuscript Example 5.10

Upward pull: the supported and separated states



For a prescribed pull P at angle α above a smooth horizontal platform,

$$P \cos \alpha = ma_x, \quad F_N + P \sin \alpha - mg = 0$$

while contact gives $a_y = 0$. Thus $F_N = mg - P \sin \alpha$.

For a nonadhesive support, $F_N \geq 0$. If $P > mg$, the marginal contact angle is

$$\alpha_c = \sin^{-1}(mg/P).$$

Beyond that angle, a negative required reaction rejects the supported model. Immediately after separation, set the *actual* $F_N = 0$ and use

$$a_x = \frac{P \cos \alpha}{m}, \quad a_y = \frac{P \sin \alpha - mg}{m}.$$

These separated equations assume the force and its direction remain prescribed.

Source: Manuscript Chapter 5

The contact branch has a physical endpoint

(a) Force diagram at $\alpha = 30^\circ$

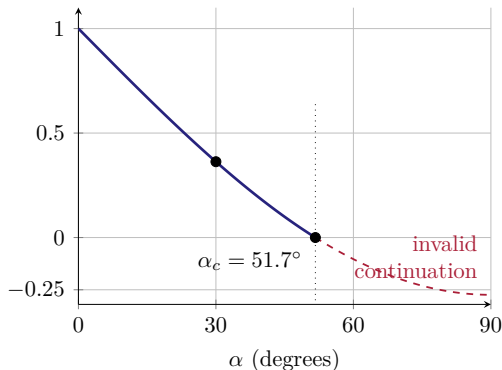
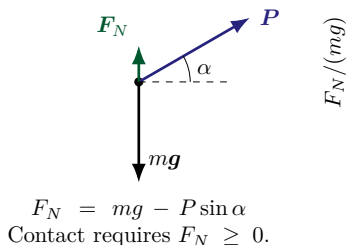


Figure 5.11: An upward angled pull unloads a horizontal support. The analytical curve uses $m = 4.00 \text{ kg}$, $P = 50.0 \text{ N}$, and $g = 9.81 \text{ m s}^{-2}$. The solid branch is the contact solution. The dashed negative continuation is an inadmissible required reaction, rather than the actual normal force after separation. Force arrows at $\alpha = 30.0^\circ$ use 0.04 cm per newton.

Example 5.12: leaving a horizontal support



A 4.00 kg case is pulled by a 50.0 N force on a frictionless horizontal platform. The force direction is prescribed. (a) Determine the normal force and horizontal acceleration at $\alpha = 30.0^\circ$ above horizontal. (b) Find the contact-limit angle. (c) Determine the accelerations immediately after separation if the same force is maintained at $\alpha = 60.0^\circ$.

Model and sign convention

Use the laboratory frame, $+x$ horizontal along the pull and $+y$ upward. The three possible real forces are the prescribed pull, gravity and a nonadhesive normal reaction. Uniform gravity and constant mass are assumed.

Test the supported solution separately at each angle. If its reaction would be negative, replace $a_y = 0$ by the separated force model from slide 33.

Source: Manuscript Example 5.12

Example 5.12: contact and separation



(a) At 30.0° : $mg = 39.24 \text{ N}$, so

$$F_N = 39.24 - 50.0 \sin 30.0^\circ = 14.2 \text{ N} > 0,$$

$$a_x = \frac{50.0 \cos 30.0^\circ}{4.00} = 10.8 \text{ m s}^{-2}, \quad a_y = 0.$$

(b) Marginal contact:

$$\sin \alpha_c = \frac{39.24}{50.0} = 0.7848, \quad \alpha_c = 51.7^\circ.$$

(c) At 60.0° : the supported expression demands $F_N = -4.06127 \text{ N}$, which is inadmissible. With actual $F_N = 0$,

$$a_x = 6.25 \text{ m s}^{-2}, \quad a_y = \frac{50.0 \sin 60.0^\circ - 39.24}{4.00} = 1.02 \text{ m s}^{-2}.$$

Upward acceleration confirms separation. Checks: at $\alpha = 0$, $F_N = mg$; at α_c , both branches give $a_y = 0$. A negative support magnitude rejects contact.

Source: Manuscript Example 5.12

Spring reference and loaded equilibrium



Define signed extension $e = x - x_0$, where x_0 is the unstretched coordinate.
A linear attached spring gives

$$F_{s,x} = -k_s e.$$

An extended spring pulls toward decreasing x ; a compressed spring can push toward increasing x if designed and attached to transmit that force.

With a constant actuator force P on a smooth horizontal guide,

$$ma_x = P - k_s(x - x_0), \quad x_{\text{eq}} = x_0 + \frac{P}{k_s}.$$

At the loaded equilibrium, spring force is $-P$, although net force is zero. Passing that coordinate fixes instantaneous acceleration, but gives no condition on velocity.

Source: Manuscript Chapter 5

Spring force and net force have different zeros

(a) Attached horizontal spring

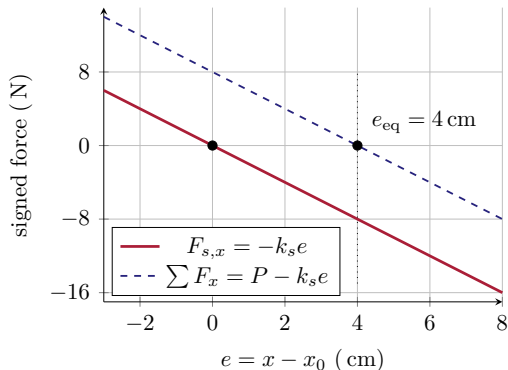
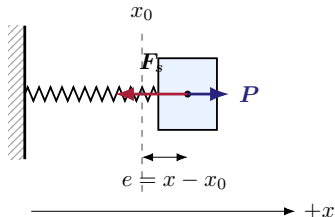


Figure 5.12: The zero of spring force differs from the zero of net force when another interaction is present. The analytical lines use $k_s = 200 \text{ N m}^{-1}$ and $P = 8.00 \text{ N}$. The horizontal axis is in centimetres, so each line's displayed slope is -2.00 N cm^{-1} . The left schematic shows $x = 0.270 \text{ m}$, $x_0 = 0.200 \text{ m}$; its force arrows use 0.08 cm per newton and show horizontal force content only.

Example 5.13: a loaded spring



A 1.50 kg instrument carriage is attached to a horizontal linear spring of stiffness $k_s = 200 \text{ N m}^{-1}$. The spring is unstretched when the carriage coordinate is $x_0 = 0.200 \text{ m}$. A constant actuator force $P = 8.00 \text{ N}$ acts along $+x$; all other horizontal resistance is neglected. The spring remains linear and can transmit both extension and compression over the range considered. (a) Find the loaded equilibrium coordinate. (b) Find the spring force, net force and instantaneous acceleration at $x = 0.270 \text{ m}$ and $x = 0.180 \text{ m}$. (c) Does a carriage passing the equilibrium coordinate necessarily come to rest there?

Source: Manuscript Example 5.13

Example 5.13: signed spring states



(a) Set $P - k_s(x_{\text{eq}} - x_0) = 0$:

$$x_{\text{eq}} = 0.200 + \frac{8.00}{200} = 0.240 \text{ m.}$$

The spring force there is -8.00 N .

(b) Keep the extension signed and add the $+8.00 \text{ N}$ actuator force:

$x \text{ (m)}$	$e \text{ (m)}$	$F_{s,x} \text{ (N)}$	$\sum F_x \text{ (N)}$	$a_x \text{ (m s}^{-2}\text{)}$
0.270	+0.0700	-14.0	-6.00	-4.00
0.180	-0.0200	+4.00	+12.0	+8.00

(c) Passing x_{eq} gives $a_x = 0$, not necessarily $v_x = 0$. A carriage placed there at rest remains there.

Checks: $\sum F_x = ma_x$ holds in each row. Removing P gives $x_{\text{eq}} = x_0$; doubling k_s halves P/k_s . Acceleration points toward the loaded equilibrium on both sides.

Source: Manuscript Example 5.13

Free-body diagrams and independent constraints



- ① Select one body or a clearly marked composite system.
- ② Replace each external interaction by one force arrow acting on that system.
- ③ Choose axes and resolve the actual forces into signed components.
- ④ Write a Newton equation for each body that needs one.
- ⑤ Derive motion constraints from geometry and write them separately.

A component construction can replace a force in a calculation; adding both the force and its components would double count it.

A common acceleration is a geometrical result for a particular connection. Equal tension is a force-balance limit for a particular connector.

Source: Manuscript Chapter 5

A straight link connects two cart motions

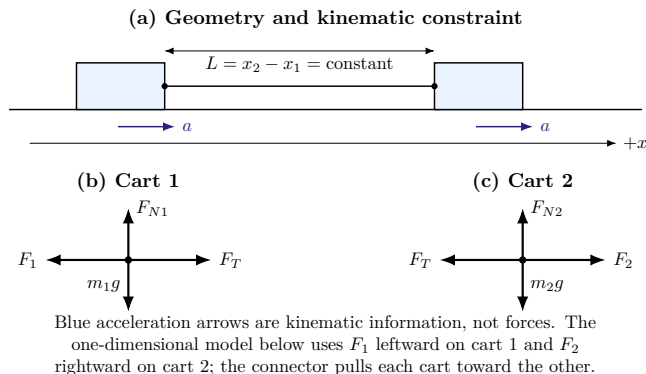


Figure 5.13: Two carts joined by a taut inextensible straight link. The fixed link length gives $a_{1x} = a_{2x}$; this constraint is written separately from each cart's Newton equation. The vertical reactions balance the weights in the level-track model.

Straight-link constraint and cart equations



For collinear attachment points connected by a taut inextensible straight link of fixed length L ,

$$x_2 - x_1 = L \quad \Rightarrow \quad v_{2x} = v_{1x} \quad \Rightarrow \quad a_{2x} = a_{1x} = a.$$

Negligible link mass additionally gives a common tension T . With $+x$ rightward,

$$T - F_1 = m_1 a, \quad F_2 - T = m_2 a.$$

Adding the equations eliminates T :

$$a = \frac{F_2 - F_1}{m_1 + m_2}, \quad T = F_1 + m_1 a.$$

Include the link in the composite boundary when interpreting cart-link forces as internal. Check each individual equation and $T \geq 0$, in addition to the combined equation.

Source: Manuscript Chapter 5

Example 5.14: connected carts



Two carts move on a horizontal track with negligible resistance. Cart 1 has mass $m_1 = 6.00 \text{ kg}$ and is pushed leftward by $F_1 = 14.0 \text{ N}$. Cart 2 has mass $m_2 = 4.00 \text{ kg}$ and is pushed rightward by $F_2 = 44.0 \text{ N}$. They are connected by a taut massless inextensible link. Find the common acceleration and the tension magnitude.

Use the level laboratory model and $+x$ rightward. The link gives common a , and negligible link mass gives common T .

$$T - 14.0 = 6.00a, \quad 44.0 - T = 4.00a.$$

$$a = \frac{44.0 - 14.0}{6.00 + 4.00} = 3.00 \text{ m s}^{-2}, \quad T = 14.0 + 6.00(3.00) = 32.0 \text{ N}.$$

Independent substitutions give $32.0 - 14.0 = 18.0 = 6.00a$ and $44.0 - 32.0 = 12.0 = 4.00a$, in newtons. Their sum is $30.0 \text{ N} = (10.00 \text{ kg})a$. Positive tension supports the assumed taut state.

Source: Manuscript Example 5.14

Student practice: Problem 20



Two 2.00 kg carts are separated by a flexible string that is initially straight. External forces of 50.0 N and 10.0 N both act to the right on the left and right carts, respectively. Assume, temporarily, that the string remains taut and that its tension direction is toward the other cart. Solve the common-acceleration equations and interpret the sign of the calculated tension. What physical state should replace the failed assumption?

Temporary model

Use a common rightward axis. Under the taut assumption the string pulls the left cart rightward and the right cart leftward. Its tension is one nonnegative magnitude in a physical taut state.

$$50.0 + T = 2.00a, \quad 10.0 - T = 2.00a.$$

Solve these equations first, then test whether the candidate lies in the string's physical domain. The replacement state is shown on slide 46.

Source: Manuscript Problem 20

Problem 20: rejecting negative tension



Adding the temporary equations gives

$$a = \frac{50.0 + 10.0}{2.00 + 2.00} = 15.0 \text{ m s}^{-2}, \quad T = 2.00(15.0) - 50.0 = -20.0 \text{ N}.$$

This would require the string to push the carts apart. A flexible string cannot carry compression, so the common-acceleration constraint fails.

With the same external forces maintained, the immediate slack state has

$$T = 0, \quad a_1 = \frac{50.0}{2.00} = 25.0 \text{ m s}^{-2}, \quad a_2 = \frac{10.0}{2.00} = 5.00 \text{ m s}^{-2}.$$

The relative acceleration $a_2 - a_1 = -20.0 \text{ m s}^{-2}$ closes the separation, consistent with slackening. Each independent cart equation is satisfied. These results apply before collision or a later change in the string state; those events require a new model.

Source: Manuscript Problem 20

A fixed pulley redirects the constraint



For a fixed ideal pulley, let q_1, q_2 both increase downward. The arc length and fixed offsets are constant:

$$q_1 + q_2 + C = L \quad \Rightarrow \quad \ddot{q}_1 + \ddot{q}_2 = 0.$$

If $m_2 > m_1$, define $a = \ddot{q}_2 = -\ddot{q}_1 > 0$. Choose upward positive for body 1 and downward positive for body 2:

$$T - m_1g = m_1a, \quad m_2g - T = m_2a.$$

Adding these *scalar* equations gives

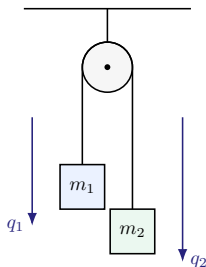
$$a = \frac{m_2 - m_1}{m_1 + m_2}g, \quad T = \frac{2m_1m_2}{m_1 + m_2}g.$$

Equal acceleration magnitudes do not mean equal vectors. Common tension requires negligible rope mass, pulley inertia and axle friction.

Source: Manuscript Chapter 5

Atwood apparatus and separate force diagrams

(a) Apparatus and coordinates (b) Body 1 (c) Body 2



Both position coordinates point downward.
 $q_1 + q_2 = \text{constant}.$

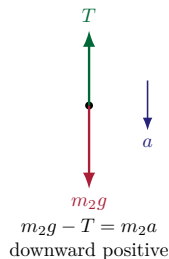
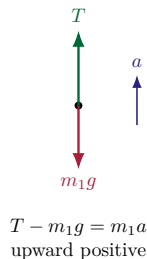


Figure 5.14: An ideal fixed-pulley system, often called an Atwood machine. Panels (b) and (c) are separate free-body diagrams; the blue acceleration arrows are outside them. For $m_2 > m_1$, the masses accelerate in opposite vertical directions with equal magnitudes. Force-arrow lengths indicate direction and qualitative ordering, not a calibrated force scale.

Example 5.15: unequal hanging masses



Two bodies of masses $m_1 = 2.00 \text{ kg}$ and $m_2 = 3.50 \text{ kg}$ are joined as in Fig. 5.14. The rope is taut, massless and inextensible; the fixed pulley has negligible rotational inertia and axle friction. The bodies are released from rest without striking a stop. Find their accelerations, the tension, and their speeds after 0.600 s .

Use separate laboratory-frame systems with the ideal routing from slide 48. The heavier body accelerates downward:

$$a = \frac{3.50 - 2.00}{3.50 + 2.00}(9.81) = 2.68 \text{ m s}^{-2}, \quad T = 2.00(g + a) = 25.0 \text{ N}.$$

Body 1 accelerates upward at the same magnitude. From rest, both speeds after 0.600 s are $at = 1.61 \text{ m s}^{-1}$.

Using unrounded values, $T = 24.970909 \text{ N}$ and $3.50g - T = 9.364091 \text{ N} = 3.50a$. The displacement magnitude is $\frac{1}{2}at^2 = 0.482 \text{ m}$, so the relevant clearance is required. The tension lies between the two weights.

Source: Manuscript Example 5.15

Atwood limits and the mass-ratio graph

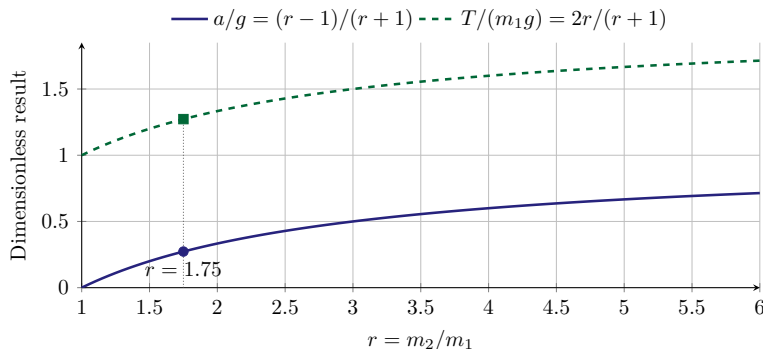
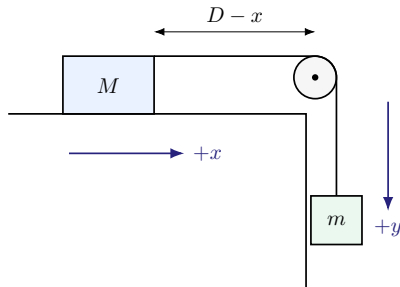


Figure 5.15: Analytical mass-ratio dependence for the ideal Atwood model. The markers use Example 5.15. The two curves are normalized differently, as indicated in the legend; their vertical separation is not a difference between dimensional forces and accelerations.

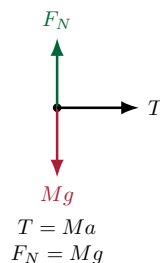
At $r = 1$, $a = 0$ and $T = m_1g$. At fixed m_1 , $r \rightarrow \infty$ gives $a \rightarrow g$ from below and $T \rightarrow 2m_1g$. The plotted example has $r = 1.75$.

Cart and hanger: a perpendicular route

(a) Level track and rope route



(b) Cart



(c) Hanger

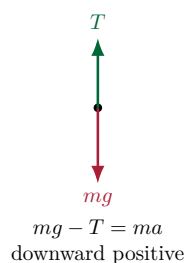


Figure 5.16: A frictionless cart and hanging body connected by an ideal rope over a fixed pulley. The route gives $a_y = a_x$ for the indicated axes. The cart's normal force and weight are included even though they cancel in the vertical equation. The common tension requires negligible rope mass, pulley inertia and axle friction.

The route gives $(D - x) + y + C = L$, so $a_y = a_x = a$ for rightward x and downward y . Combining $T = Ma$ with $mg - T = ma$ gives $a = mg/(M + m)$.

Example 5.16: a hanging drive



A test cart and all its attached fixtures have total mass $M = 1.80 \text{ kg}$. A hanging body has mass $m = 0.600 \text{ kg}$. Use the ideal arrangement in Fig. 5.16, with negligible track resistance, level motion and no initial velocity. Find the acceleration, rope tension, cart normal force, and time to move 0.500 m . Assume sufficient hanging-body clearance.

The separate cart and hanger models give

$$a = \frac{0.600}{1.80 + 0.600}(9.81) = 2.45 \text{ m s}^{-2}, \quad T = 1.80a = 4.41 \text{ N}.$$

Vertical balance gives $F_N = 1.80g = 17.7 \text{ N}$. Starting from rest,

$$t = \sqrt{\frac{2(0.500)}{2.45250}} = 0.639 \text{ s}.$$

The hanger travels 0.500 m downward. With unrounded values, $5.88600 - 4.41450 = 1.47150 \text{ N} = 0.600a$, and $0 < a < g$. Equal scalar acceleration magnitudes have perpendicular vector directions.

Source: Manuscript Example 5.16

A moving pulley creates a two-to-one constraint



Let z be the load's downward coordinate and y the free body's downward coordinate. Two rope legs change with the lower pulley:

$$2z + y + C = L \Rightarrow a_y = -2a_z.$$

The load assembly receives two upward rope tensions:

$$Mg - 2T = Ma_z, \quad mg - T = ma_y = -2ma_z.$$

From the second equation $T = mg + 2ma_z$. Substitute into the first:

$$(M - 2m)g = (M + 4m)a_z, \quad a_z = \frac{M - 2m}{M + 4m}g.$$

The factor $M + 4m$ comes from constraint elimination, not the total physical mass. Both pulleys and the rope are ideal; attachments are included in M .

Source: Manuscript Chapter 5

The moving-pulley geometry and force content

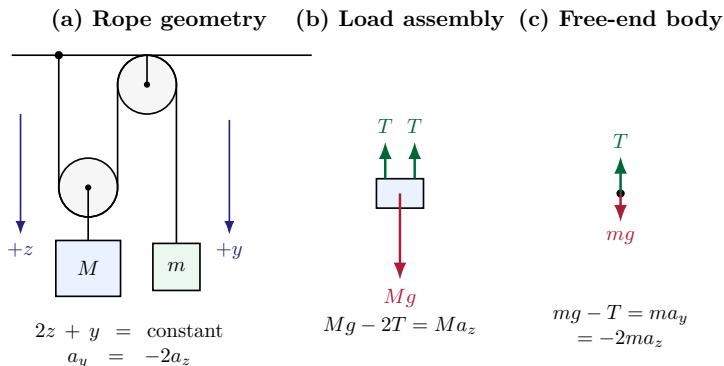


Figure 5.17: An ideal movable-pulley arrangement and two subsystem force diagrams. Each of the two vertical rope legs bordering the lower pulley lengthens by the load's downward displacement Δz . The free-end body must therefore move upward by $2\Delta z$. Both y and z are defined downward; a negative a_y denotes an upward acceleration. The force diagrams use the numerical example below with a common force scale of 0.025 cm N^{-1} .

Example 5.17: two-to-one motion



The apparatus of Fig. 5.17 has load-assembly mass $M = 6.00$ kg and free-end body mass $m = 2.00$ kg. It is released from rest. Find both signed accelerations, the tension, and the displacements after 0.400 s. Verify each body's Newton equation and the rope-length change.

Using the ideal assembly model and downward-positive coordinates,

$$a_z = \frac{6.00 - 2(2.00)}{6.00 + 4(2.00)}g = 1.40143 \text{ m s}^{-2}, \quad a_y = -2.80286 \text{ m s}^{-2}.$$

$$T = 2.00(g + 2a_z) = 25.2257 \text{ N}, \quad \Delta z = 0.112114 \text{ m}, \quad \Delta y = -0.224229 \text{ m}.$$

Here $\Delta z = \frac{1}{2}a_z(0.400)^2$ and similarly for y . Independent checks using unrounded values give

$$58.86 - 2T = 6.00a_z, \quad 19.62 - T = 2.00a_y, \quad 2\Delta z + \Delta y = 0.$$

The load moves down and the free end up. Tension is positive; the equilibrium limit $M = 2m$ gives $a_z = a_y = 0$.

Source: Manuscript Example 5.17

Apparent weight is a contact force



A force-calibrated scale responds to the normal reaction, called apparent weight. Gravitational weight in this chapter is mg .

Choose upward positive in the ground-frame approximation. While the body shares the floor acceleration a_y ,

$$F_N - mg = ma_y, \quad \boxed{F_N = m(g + a_y)}.$$

- Upward acceleration gives $F_N > mg$.
- Moderate downward acceleration gives $0 < F_N < mg$.
- Zero acceleration gives $F_N = mg$, for upward or downward constant-speed travel.

Contact requires $F_N \geq 0$. At $a_y = -g$, the ideal reading is zero while gravity still acts. A demanded negative reaction rejects shared floor acceleration.

Source: Manuscript Chapter 5

Acceleration changes the scale reading

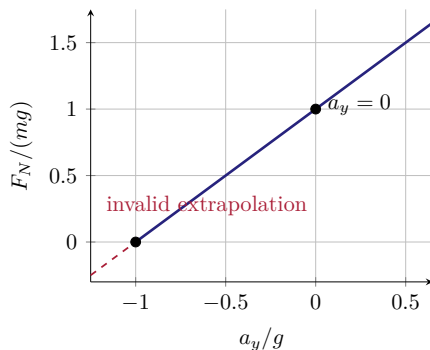
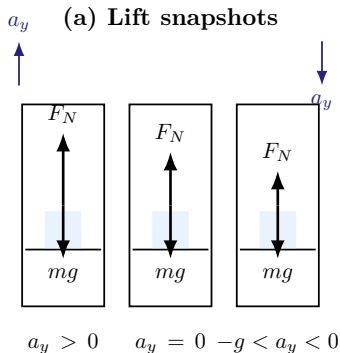


Figure 5.18: Apparent weight in a vertically accelerating lift. The exact contact-branch relation is $F_N/(mg) = 1 + a_y/g$. The branch is physically admissible only for $a_y \geq -g$ while the body remains in contact with the floor; the dashed extrapolation below $-g$ is invalid for floor contact. The actual force after separation is zero, not this negative continuation. Snapshot arrow lengths distinguish the qualitative load cases; velocity is unspecified.

Example 5.18: a scale in a lift



A 68.0 kg instrument stands on a force-calibrated platform scale in a lift. Use $g = 9.81\text{ m s}^{-2}$. Determine the scale reading for: (a) upward acceleration $a_y = +1.40\text{ m s}^{-2}$; (b) constant-speed motion; (c) downward acceleration $a_y = -1.40\text{ m s}^{-2}$; and (d) the free-fall limit $a_y = -g$. What would it mean if someone inserted $a_y = -12.0\text{ m s}^{-2}$ into the contact formula?

Model and strategy

Select the instrument and use the approximately inertial ground frame with upward positive. The upright body shares the horizontal floor's acceleration only while a nonnegative reaction can maintain contact.

Use $F_N = m(g + a_y)$ for each admissible subpart, then test the proposed -12.0 m s^{-2} floor acceleration. The answer table follows on slide 59.

Source: Manuscript Example 5.18

Example 5.18: readings and contact loss



Every admissible reading uses $F_N = 68.0(9.81 + a_y)$:

Case	a_y (m s^{-2})	F_N (N)
(a) Upward acceleration	+1.40	$762.28 \approx 762$
(b) Constant-speed motion	0	$667.08 \approx 667$
(c) Downward acceleration	-1.40	$571.88 \approx 572$
(d) Free-fall limit	-9.81	0

At a prescribed floor acceleration -12.0 m s^{-2} , the contact equation demands -148.92 N . This rejects contact.

After separation, actual $F_N = 0$ and the ideal instrument has $a_y = -g$. Its acceleration relative to the floor is $-9.81 - (-12.0) = +2.19 \text{ m s}^{-2}$, consistent with separation. Each table row gives $F_N - mg = ma_y$. Scale readings depend on acceleration, not travel direction.

Source: Manuscript Example 5.18

Core synthesis and the next modelling steps



Newton's law is short:

$$\sum \mathbf{F}_{\text{ext}} = m\mathbf{a}.$$

Its use depends on the selected boundary, frame, interaction models and separately derived constraints.

Test the *individual* force equations as well as a composite equation. Positive tension and nonnegative reaction are part of physical admissibility.

For a measurement comparison, acquire motion and force independently. Check moving mass, sensor zero, track level and common time interval. A fitted intercept can include resistance or a zero offset; a line alone does not distinguish them.

Chapter 6 develops friction regimes, circular-motion applications, drag and accelerating frames. The present method remains the foundation. Optional reading: the chapter's synchronized lift journey, force-fit investigation and unselected practice problems.

Source: Manuscript Chapter 5

Advanced Topics for Technical and Engineering Students

Model a straight inextensible link in uniform translation. Let s increase from its left end, let λ be linear mass density, and let every material point have the same rightward acceleration a . There is no unbalanced distributed axial force.

A segment of length Δs has mass $\lambda \Delta s$:

$$T(s + \Delta s) - T(s) = \lambda \Delta s a.$$

Dividing by Δs and taking its limit gives

$$\frac{dT}{ds} = \lambda a, \quad T(s) = T_L + \lambda a s, \quad T_R - T_L = m_\ell a.$$

The gradient has units N m^{-1} . This local Newton equation integrates to the whole-link equation introduced in the core.

Source: Manuscript Chapter 5

Example 5.22: a massive connecting link



A 4.00 kg left cart and a 6.00 kg right cart are joined by a straight uniform link of mass 2.00 kg. The track resistance is negligible. An external force of 20.0 N acts leftward on the left cart and an external force of 70.0 N acts rightward on the right cart. The link remains straight, taut and inextensible. Find the common acceleration and the tension at each end. Then compare with the massless-link approximation.

Select both carts *and the link*. The straight taut inextensible geometry gives a common rightward acceleration.

$$F_{\text{ext}} = 70.0 - 20.0 = 50.0 \text{ N}, \quad m_{\text{total}} = 4.00 + 6.00 + 2.00 = 12.00 \text{ kg}.$$

$$a = \frac{50.0}{12.00} = 4.166666 \dots \text{ m s}^{-2}.$$

The endpoint tensions are external to each one-cart boundary. Apply separate equations to find them and compare with the massless approximation on slide 63.

Source: Manuscript Example 5.22

Example 5.22: endpoint forces and model error



Left and right cart equations give

$$T_L = 20.0 + 4.00a = 36.666666 \dots \text{ N} \approx 36.7 \text{ N},$$

$$T_R = 70.0 - 6.00a = 45.0 \text{ N}.$$

The independent link check is $T_R - T_L = 8.333333 \dots \text{ N} = 2.00a$.

Neglecting the same 2.00 kg link gives

$$a_0 = \frac{50.0}{10.00} = 5.00 \text{ m s}^{-2}, \quad T_0 = 20.0 + 4.00(5.00) = 40.0 \text{ N}.$$

$$\frac{a_0 - a}{a} = \frac{5.00 - 4.166666 \dots}{4.166666 \dots} = 0.200 = 20.0\%.$$

The approximation overpredicts acceleration and hides the $8.333333 \dots \text{ N}$ endpoint difference. Both true tensions are positive; acceptable approximation error depends on the specified observable and tolerance.

Source: Manuscript Example 5.22

Nonuniform links and cumulative mass



Let the local density be $\lambda(s)$. For the same uniformly translating straight-link model,

$$\frac{dT}{ds} = \lambda(s)a.$$

Integration from the left endpoint accumulates the mass to the left of the cut:

$$T(s) = T_L + a \int_0^s \lambda(u) du, \quad T_R - T_L = m_\ell a.$$

For $\lambda(s) = \lambda_0(1 + \beta s/L)$,

$$m_\ell = \lambda_0 L(1 + \beta/2), \quad T(s) = T_L + \lambda_0 a \left(s + \frac{\beta s^2}{2L} \right).$$

Require $\lambda_0 > 0$ and $1 + \beta s/L > 0$ on $0 \leq s \leq L$. Total mass fixes the endpoint difference; density distribution additionally fixes the interior profile.

Source: Manuscript Chapter 5

Equal endpoints can hide different profiles

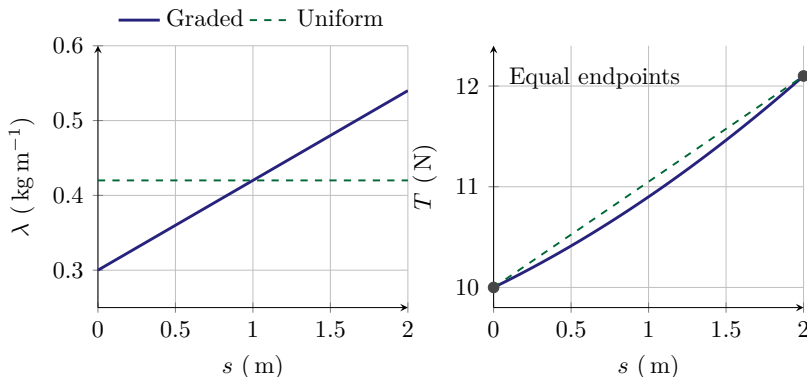


Figure 5.24: Graded (solid) and uniform (dashed) connectors with the same length, total mass, left tension and acceleration. The uniform comparison uses $\lambda = m_\ell/L = .420 \text{ kg m}^{-1}$. Equal endpoint tensions do not establish equal interior tension profiles; the cumulative mass distribution determines the difference.

Example 5.23: graded-link calculation



A straight translating connector has $L = 2.00 \text{ m}$, $\lambda_0 = .300 \text{ kg m}^{-1}$, and $\beta = .800$. Its prescribed common acceleration is $a = 2.50 \text{ m s}^{-2}$, rightward, and its left-end tension is $T_L = 10.0 \text{ N}$. Find its mass, tension profile, right-end tension and midpoint tension. Compare the midpoint with a uniform connector having the same total mass and length.

Integrate the density and use the left tension as the integration constant:

$$m_\ell = 0.300(2.00)(1 + 0.800/2) = 0.840 \text{ kg},$$

$$T(s) = 10.0 \text{ N} + (0.750 \text{ N m}^{-1})s + (0.150 \text{ N m}^{-2})s^2.$$

Thus $T_R = 12.100 \text{ N}$ and $T(L/2) = 10.900 \text{ N}$. The uniform comparison has $\lambda_U = 0.420 \text{ kg m}^{-1}$, $T_U(s) = 10.0 \text{ N} + (1.050 \text{ N m}^{-1})s$, and midpoint 11.050 N , higher by 0.150 N .

Checks: $T_R - T_L = 2.100 \text{ N} = m_\ell a$; differentiation gives $T'(s) = \lambda(s)a$. All tensions remain positive.

Source: Manuscript Example 5.23

Residuals need explicit physical scales



Acceleration and tension have different units. Choose positive scales a_* and F_* , and define

$$\alpha = a/a_*, \quad \tau = T/F_*, \quad \mu_i = m_i a_*/F_*, \quad p_i = F_i/F_*.$$

The straight-cart equations become a dimensionless system:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mu_1 & -1 \\ \mu_2 & 1 \end{bmatrix}, \quad \tilde{\mathbf{q}} = \begin{bmatrix} \alpha \\ \tau \end{bmatrix}, \quad \tilde{\mathbf{b}} = \begin{bmatrix} -p_1 \\ p_2 \end{bmatrix}.$$

For a candidate, compute $\tilde{\mathbf{r}} = \tilde{\mathbf{A}}\tilde{\mathbf{q}} - \tilde{\mathbf{b}}$. Its determinant is $\mu_1 + \mu_2 > 0$, so the equations have a unique candidate. Physical admissibility remains a separate check. Define a vector infinity norm as its largest absolute entry, and a matrix infinity norm as its largest absolute row sum.

Source: Manuscript Chapter 5

Example 5.24: testing a numerical candidate



For the baseline cart data $m_1 = 6.00 \text{ kg}$, $m_2 = 4.00 \text{ kg}$, $F_1 = 14.0 \text{ N}$ and $F_2 = 44.0 \text{ N}$, evaluate the candidate $a = 3.00 \text{ m s}^{-2}$, $F_T = 31.0 \text{ N}$. Use $a_* = 1.00 \text{ m s}^{-2}$ and $F_* = 10.0 \text{ N}$.

With $a_* = 1.00 \text{ m s}^{-2}$ and $F_* = 10.0 \text{ N}$,

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0.6 & -1 \\ 0.4 & 1 \end{bmatrix}, \quad \tilde{\mathbf{q}} = \begin{bmatrix} 3.00 \\ 3.10 \end{bmatrix}, \quad \tilde{\mathbf{b}} = \begin{bmatrix} -1.40 \\ 4.40 \end{bmatrix}.$$

$$\tilde{\mathbf{r}} = \begin{bmatrix} 1.80 - 3.10 + 1.40 \\ 1.20 + 3.10 - 4.40 \end{bmatrix} = \begin{bmatrix} +0.10 \\ -0.10 \end{bmatrix}.$$

In the earlier Newton-equation convention, the dimensional residuals are $\mathcal{R}_1 = T - F_1 - m_1 a = -1.00 \text{ N}$ and $\mathcal{R}_2 = F_2 - T - m_2 a = +1.00 \text{ N}$.

Source: Manuscript Example 5.24

Example 5.24: residual size and interpretation



The norms of the candidate are

$$\|\tilde{\mathbf{A}}\|_{\infty} = 1.60, \quad \|\tilde{\mathbf{q}}\|_{\infty} = 3.10, \quad \|\tilde{\mathbf{b}}\|_{\infty} = 4.40, \quad \|\tilde{\mathbf{r}}\|_{\infty} = 0.10.$$

Use the declared dimensionless measure

$$\epsilon_{\text{res}} = \frac{\|\tilde{\mathbf{r}}\|_{\infty}}{\|\tilde{\mathbf{A}}\|_{\infty}\|\tilde{\mathbf{q}}\|_{\infty} + \|\tilde{\mathbf{b}}\|_{\infty}} = \frac{0.10}{1.60(3.10) + 4.40} = 0.0106838.$$

The residual is about 1.07%, far above ordinary floating-point roundoff for these equations. Replacing 31.0 N by 32.0 N makes both residuals zero.

The incorrect tension still satisfies the combined equation because T cancels there. Individual equations detect it. Even a zero residual cannot reveal a force omitted when the equations were constructed.

Source: Manuscript Example 5.24

Advanced challenge: Problem 39



Context for chapter Eq. (5.74): its error denominator is the finite-link acceleration.

For two carts with combined mass M joined by a link of mass m_ℓ , prove Eq. (5.74). What maximum ratio m_ℓ/M keeps the massless-link acceleration error below 2.0% when the error is measured relative to the finite-link result?

Hold the same nonzero external resultant F and translating geometry:

$$a = \frac{F}{M + m_\ell}, \quad a_0 = \frac{F}{M}.$$

Taking their ratio eliminates the unknown nonzero resultant:

$$\frac{a_0 - a}{a} = \frac{a_0}{a} - 1 = \frac{M + m_\ell}{M} - 1 = \frac{m_\ell}{M}.$$

Strictly below 2.0% requires $m_\ell/M < 0.020$. Equality gives exactly 2.0%, so it is the boundary. For $F < 0$, the ratio measures magnitude overprediction. At $F = 0$, both accelerations vanish and the ratio is undefined. This criterion applies to acceleration, not individual tension errors.

