

Potential Energy and Conservation of Energy

Physics I - Chapter 8

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Why introduce potential energy?

Two routes can connect the same heights while having different lengths. A potential difference captures the work of an included conservative interaction using the endpoints.

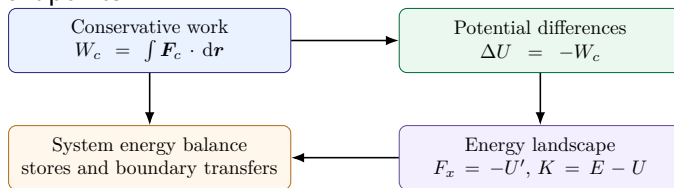


Figure 8.1: Four complementary representations. The conservative-force calculation supplies a potential difference; the potential supplies force and reachability information. Every representation is used within an explicitly chosen system energy balance. This is a conceptual diagram, not a physical energy-flow map.

Chapter support: Figure 8.1

Learning route and prerequisites



- ① **Work and potential differences.** Test path independence, then integrate a force or differentiate a potential.
- ② **State comparisons.** Choose reference values and derive the conditions for conserving $K + U$.
- ③ **Energy landscapes.** Infer forces, speeds, allowed regions, equilibrium, and contact limits.
- ④ **Complete accounting.** Track external work, internal conversion, and model validity.

Prerequisites. Chapter 7's work–kinetic-energy theorem, scalar products, single-variable calculus, Hooke's law, and Newton's second law. We refresh each tool before use. The advanced route adds Taylor expansion and partial derivatives to study local stiffness and coupled stability.

Chapter support: Revised Chapter 8 exposition

Learning goals



- Derive $\Delta U = -W_c$ and test a force model's path independence on its stated domain.
- Change an energy reference while preserving forces, speeds, and turning positions.
- Build and check energy balances for gravity, springs, external work, and sliding conversion.
- Read slopes, allowed regions, equilibria, and ordinary turns from $U(x)$.
- Test reachability along the path, including a track's contact constraint.
- Explain why internal warming and mechanical-energy loss can accompany total-energy conservation.

Advanced outcomes. Quantify a local approximation, calculate a gradient force, and test stability in every displacement direction.

Chapter support: Section 8.9

System, stores, and sign conventions



System. Name the objects and included interactions. Potential energy belongs to an interaction inside this boundary.

$$K = \frac{1}{2}mv^2, \quad E_{\text{mech}} = K + U, \quad E_{\text{int}} = \text{internal energy.}$$

Kinetic energy refers to a named inertial frame. Internal energy includes microscopic and deformation stores beyond the translational model. **Transfers.** $W_{\text{ext,on}} > 0$ adds energy through external work. $Q > 0$ adds energy by heating. All these quantities have SI unit joule, $\text{J} = \text{N m}$. For local numerical gravity models use $g = 9.81 \text{ m s}^{-2}$. Stipulated inputs describe teaching models rather than measured apparatus.

Chapter support: Revised Chapter 8 exposition

Conservative work and closed paths



For a particle, the work–kinetic-energy theorem gives

$$\Delta K = W_{\text{net}} = \int \mathbf{F}_{\text{net}} \cdot d\mathbf{r}.$$

A force is conservative on a stated domain when every admissible path between the same endpoints gives the same work:

$$W_{a \rightarrow b}[C_1] = W_{a \rightarrow b}[C_2].$$

Following C_1 outward and C_2 backward gives

$$\oint_C \mathbf{F}_c \cdot d\mathbf{r} = 0 \quad \text{for every admissible closed path.}$$

One nonzero loop disproves conservativity. One zero loop, including an exactly retraced route, proves only cancellation on that route.

Chapter support: Section 8.1, Equation 8.2

Potential differences and the minus sign



For an included conservative interaction, define

$$U_b - U_a = - \int_a^b \mathbf{F}_c \cdot d\mathbf{r} = -W_{c,a \rightarrow b}.$$

When that force supplies positive work to the moving object, the associated potential decreases. Energy changes form between interaction storage and kinetic energy. **Sign check.** An object falls through a positive downward distance. Gravity and displacement align, so $W_g > 0$ and $\Delta U_g < 0$. **Reverse route.** $W_{c,b \rightarrow a} = -W_{c,a \rightarrow b}$, hence the potential difference also reverses. For gravity the interaction is object–Earth. Its potential can be included in an object-plus-Earth system; the isolated object alone does not own that interaction store.

Chapter support: Equation 8.3, Problem 6

Gravity: the same endpoints on two routes

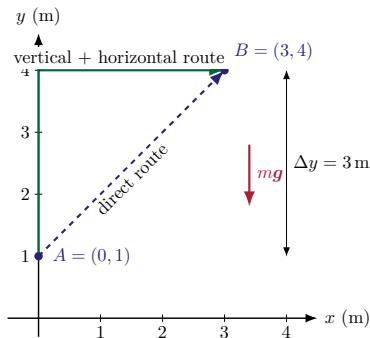


Figure 8.2: Two routes with the exact endpoints of Example 8.1. Gravity points downward and its work is $-mg\Delta y$ for either route. Coordinates and route geometry share one metre scale in both directions; the force-arrow length is schematic.

Gravity has no horizontal component. A longer path need not require more gravitational work.

Chapter support: Figure 8.2

Example 8.1: two gravitational paths



Complete manuscript statement

A 2.00 kg carriage moves from $A = (0, 1.0 \text{ m})$ to $B = (3.0 \text{ m}, 4.0 \text{ m})$. Path 1 consists of a vertical segment followed by a horizontal segment; path 2 is a direct ramp. Treat $\mathbf{g} = -g\hat{\mathbf{j}}$ with $g = 9.81 \text{ m s}^{-2}$. Find the gravitational work on each path and the corresponding change in gravitational potential energy.

Chapter support: Example 8.1

Example 8.1: work on both paths



Use the carriage–Earth interaction with uniform g . This calculation concerns gravity's work; no speed history is required.

$$\mathbf{F}_g \cdot d\mathbf{r} = (-mg\hat{\mathbf{j}}) \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}) = -mgdy.$$

$$W_g = -mg(y_B - y_A) = -(2.00)(9.81)(3.00) = -58.86 \text{ J}.$$

Path 1. The vertical leg gives -58.86 J and the horizontal leg gives zero.

Path 2. Integrating $-mgdy$ along the ramp gives the same -58.86 J .

$$\Delta U_g = +58.86 \text{ J} \quad \text{on either route.}$$

Reversing either route gives $W_g = +58.86 \text{ J}$ and $\Delta U_g = -58.86 \text{ J}$. Dimensions are $mg\Delta y$: $\text{N m} = \text{J}$.

Chapter support: Example 8.1, Equation 8.4

Integration and differentiation of a potential



On a connected one-dimensional interval, a time-independent, single-valued conservative position force satisfies

$$dU = -F_x(x)dx, \quad \boxed{F_x = -U'(x).}$$

$$\boxed{U(x) = U(x_0) - \int_{x_0}^x F_x(\xi)d\xi.}$$

The reference value fixes a constant. The dummy variable ξ runs inside the integral while x identifies its endpoint. A force–position area gives work in joules. A potential slope gives force in newtons. Positive force corresponds to a potential falling as x increases. A loading/unloading hysteresis loop needs branch or history information. Integrating one branch does not establish one universal position-only potential.

Chapter support: Equation 8.5

Example 8.2: a variable force changes sign



Complete manuscript statement

An ideal actuator produces the position force

$$F_x(x) = a - bx, \quad a = 12.0 \text{ N}, \quad b = 6.00 \text{ N m}^{-1},$$

on a guide over $0 \leq x \leq 3.00 \text{ m}$. Treat this prescribed force as time independent and conservative on that interval. Given $U(0) = 4.00 \text{ J}$, find the potential, the work from $x = 0.500 \text{ m}$ to $x = 3.00 \text{ m}$, and the location where the force changes sign. Explain why a single average-force estimate can be misleading.

Chapter support: Example 8.2

Example 8.2: a force reversal and signed work



Integrate the stated force without assuming a constant speed:

$$U(x) = 4.00 \text{ J} - ax + \frac{1}{2}bx^2, \quad -U'(x) = a - bx.$$

The force reverses at $x = a/b = 2.00 \text{ m}$. It points right for $x < 2.00 \text{ m}$ and left for $x > 2.00 \text{ m}$.

$$U(0.500 \text{ m}) = -1.25 \text{ J}, \quad U(3.00 \text{ m}) = -5.00 \text{ J}.$$

$$\Delta U = -3.75 \text{ J}, \quad \boxed{W_c = +3.75 \text{ J}}.$$

Splitting at the force zero gives $+6.75 \text{ J}$ and -3.00 J , whose sum checks the endpoint result. The exact mean force here is $(9.00 - 6.00)/2 = 1.50 \text{ N}$ because F_x is linear. Using only the initial force gives $9.00(2.50) = 22.5 \text{ J}$ and misses the reversal. Endpoint averaging fails in general for curved force laws.

Chapter support: Example 8.2

Example 8.2: matched areas and slopes

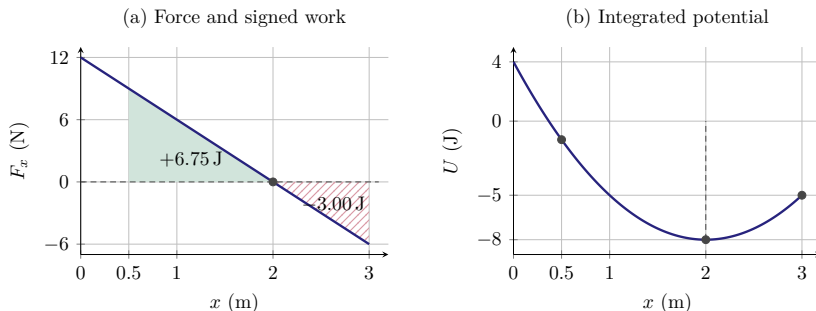


Figure 8.3: Matched force and potential for the affine model in Example 8.2. The shaded areas apply only to the motion from 0.500 to 3.00 m; their signed sum is 3.75 J. The potential decreases where the force is positive and increases where it is negative. Curve heights have different units in the two panels.

Chapter support: Figure 8.3

A position-dependent force can fail the path test



In two dimensions different routes sample different force components. For a prescribed field $\mathbf{F} = \alpha y \hat{\mathbf{i}}$,

$$\mathbf{F} \cdot d\mathbf{r} = \alpha y dx.$$

Vertical legs do no work. Horizontal work depends on the height at which the leg is traversed. **Complete manuscript statement: Example 8.4**

On the whole xy plane, let $\mathbf{F} = \alpha y \hat{\mathbf{i}}$ with $\alpha = 3.00 \text{ N m}^{-1}$. Compare the work from $A = (0, 0)$ to $B = (2.00 \text{ m}, 1.50 \text{ m})$ by the two routes in Fig. 8.5. Route I moves right along the bottom and then up. Route II moves up along the left and then right along the top. Find the work around the loop that follows I outward and II backward, and decide whether a single potential exists.

Chapter support: Example 8.4

Example 8.4: rectangle routes

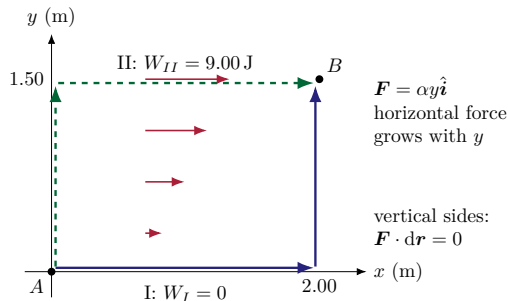


Figure 8.5: Two routes in the prescribed force field of Example 8.4. Solid and dashed lines distinguish routes even in grayscale. The field arrows indicate direction and relative magnitude; they are not motion arrows. The top horizontal segment does positive work on the outward route II, and negative work when traversed backward in the closed loop.

Chapter support: Figure 8.5

Example 8.4: unequal work forbids one potential



Route I. On the bottom $y = 0$; on the vertical leg $dx = 0$. Both contributions vanish, so $W_I = 0$. **Route II.** The vertical leg contributes zero. Along the top, $F_x = \alpha H = (3.00)(1.50) = 4.50 \text{ N}$:

$$W_{II} = \alpha HL = (4.50)(2.00) = 9.00 \text{ J.}$$

Following I outward and II backward gives

$$W_{\text{loop}} = W_I - W_{II} = -9.00 \text{ J.}$$

No single $U_B - U_A$ can equal both $-W_I = 0$ and $-W_{II} = -9.00 \text{ J}$. The force is nonconservative on the whole plane. The general result is $-\alpha LH$, with joule units. Reversing the loop reverses the sign. Exactly retracing one route gives zero and cannot repair the failed global test.

Chapter support: Example 8.4

Reference choices leave motion invariant



If a conservative interaction has potential $U(x)$, then

$$\tilde{U}(x) = U(x) + C$$

has the same force, because $dC/dx = 0$. The same physical state requires $\tilde{E} = E + C$. Therefore

$$\tilde{K} = \tilde{E} - \tilde{U} = (E + C) - (U + C) = E - U = K.$$

Forces, speeds, allowed regions, and turning positions are invariant. Shifting only U while leaving E fixed changes the state. A potential's numerical sign locates its reference. A lower graph does not imply a weaker force: compare slopes.

Chapter support: Section 8.2

Two zeros for one spring trajectory

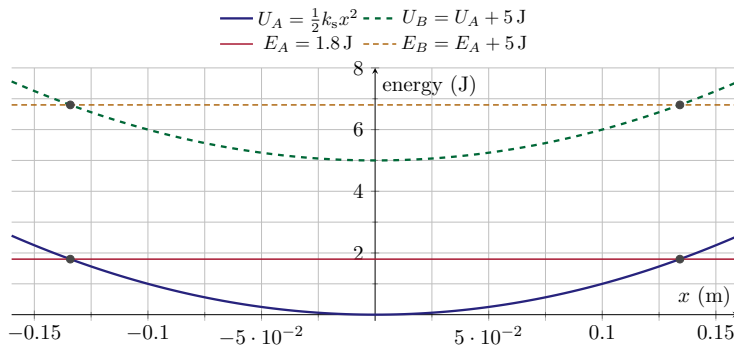


Figure 8.7: The same ideal spring under two potential-energy reference choices, with $k_s = 200 \text{ N m}^{-1}$. Both the potential curve and the total-energy line shift upward by 5 J, so their vertical separation $K = E - U$ and the turning positions remain unchanged.

$$|x_t| = \sqrt{2(1.80)/200} = 0.134164 \text{ m in both conventions.}$$

Chapter support: Figure 8.7, Example 8.6

Spring energy with signed deformation



Let x be displacement from the relaxed length, positive in extension. For a reversible ideal spring,

$$F_s = -k_s x, \quad k_s > 0,$$

$$U_s(x) - U_s(0) = - \int_0^x (-k_s \xi) d\xi = \frac{1}{2} k_s x^2.$$

Compression ($x < 0$) and extension ($x > 0$) both raise deformation energy relative to the relaxed configuration. Their forces have opposite signs. During a quasistatic displacement with negligible ΔK and no other work,

$$W_{\text{agent}} = -W_s = \Delta U_s.$$

A rapid input may also increase kinetic energy. The quadratic store assumes linear, reversible behavior over the entire stated range.

Chapter support: Equation 8.7

Example 8.7: signed spring motion



Complete manuscript statement

An ideal spring with $k_s = 160 \text{ N m}^{-1}$ is moved slowly from $x_i = -0.0750 \text{ m}$ to $x_f = +0.0450 \text{ m}$. The coordinate is measured from the relaxed length. Find the initial and final spring forces, the spring work, and the controlling agent's work. Repeat the potential accounting with the reference $\tilde{U}(0.0500 \text{ m}) = 0$.

Chapter support: Example 8.7

Example 8.7: release, storage, and shifted labels



The reversible quasistatic model gives

$$F_i = -k_s x_i = +12.0 \text{ N}, \quad F_f = -k_s x_f = -7.20 \text{ N}.$$

$$U_i = \frac{1}{2}(160)(0.0750)^2 = 0.450 \text{ J}, \quad U_f = 0.162 \text{ J}.$$

Thus $\Delta U_s = -0.288 \text{ J}$, $W_s = +0.288 \text{ J}$, and $W_{\text{agent}} = -0.288 \text{ J}$. The spring releases 0.450 J before reaching $x = 0$; the agent then supplies 0.162 J . The net controller work is negative because it receives energy overall.

$$\tilde{U}_s(x) = \frac{1}{2}k_s x^2 - 0.200 \text{ J},$$

since $U_s(0.0500 \text{ m}) = 0.200 \text{ J}$. The shifted endpoints are 0.250 J and -0.0380 J ; their difference is still -0.288 J . Differentiation restores the same forces. Relative to the relaxed state, the final deformation energy remains 0.162 J under either zero.

Chapter support: Example 8.7

Gravity signs and combined potentials



With y upward, uniform gravity gives

$$F_y = -mg, \quad U_g = mgy + C_g, \quad \Delta U_g = mg\Delta y.$$

With downward coordinate $s = -y$, the same interaction is

$$U_g = -mgs + C_g, \quad F_s = +mg.$$

The chain rule verifies the component change. A constant reference shift cannot repair an incorrectly signed slope. Independent conservative interactions add: $U_{\text{total}} = \sum_j U_j$. They must share a configuration coordinate or a stated geometric relation. For a vertical spring with downward extension s from relaxed length,

$$U(s) = \frac{1}{2}k_s s^2 - mgs + C.$$

Use constant g for local heights; orbital-scale gravity requires the later radial model.

Chapter support: Section 8.2

When mechanical energy is conserved



Split particle work into included conservative work and remaining work:

$$\Delta K = W_c + W_{\text{other}} = -\Delta U + W_{\text{other}}.$$

Therefore

$$\Delta(K + U) = W_{\text{other}}.$$

Only when the modeled mechanical stores receive no net change from other work or internal conversion throughout the process may we use

$$K_i + U_i = K_f + U_f.$$

State the system and frame, include the relevant interactions, fix references, and write the symbolic balance before numbers. Check $K \geq 0$ at every required configuration. A candidate endpoint can be energetically accessible while an intermediate barrier or contact constraint invalidates the proposed trajectory.

Chapter support: Section 8.3, Equation 8.10

Example 8.9: spring-assisted lift



Complete manuscript statement

A 0.500 kg carriage is launched upward by an ideal spring with $k_s = 300\text{ N m}^{-1}$ compressed by 0.150 m . A stationary guide lets the carriage separate from the spring at its relaxed length and subsequently rise. Take $U_g = 0$ at the initial carriage level. The carriage starts and reaches its highest point at rest, and the spring is relaxed after launch. First neglect all losses. Then repeat with a stipulated 0.600 J conversion from mechanical to internal energy.

Chapter support: Example 8.9

Example 8.9: an ideal rise and a declared conversion



Choose carriage+spring+guide+Earth. The initial and final carriage speeds vanish, and the spring finishes relaxed.

$$S = \frac{1}{2}k_sc^2 = \frac{1}{2}(300)(0.150)^2 = 3.375 \text{ J.}$$

Ideal rise. With no other transfer or conversion,

$$mgh_0 = S, \quad h_0 = \frac{3.375}{(0.500)(9.81)} = 0.688 \text{ m.}$$

Declared internal conversion. For the enlarged isolated system,

$$mgh + D = S, \quad D = 0.600 \text{ J,} \quad h = \frac{3.375 - 0.600}{4.905} = 0.566 \text{ m.}$$

Internal energy rises by D as mechanical energy falls. The limit $D \rightarrow 0$ restores the ideal height; $D > S$ makes the assumed positive-height endpoint inaccessible.

Chapter support: Example 8.9

Example 8.9: geometry and calibrated stores

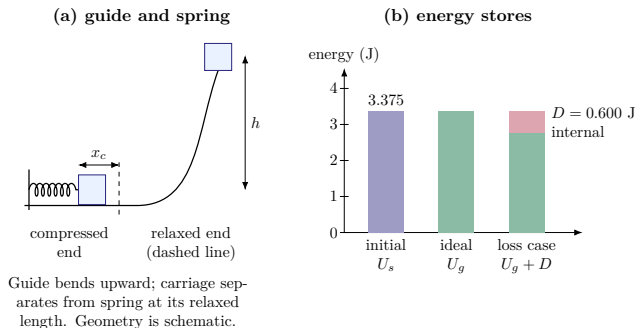


Figure 8.10: A compressed spring pushes a carriage along a stationary guide, after which the carriage rises freely along the guide. The system includes carriage, spring, guide and Earth; h is the rise from the initial carriage level and x_c is compression relative to the relaxed endpoint, not the carriage width. In Example 8.9, calibrated bars show 3.375 J initially, and either 3.375 J gravitational energy or 2.775 J gravitational plus 0.600 J internal energy at the final rest point.

Chapter support: Figure 8.10

Vertical spring: relaxed length and equilibrium



For a mass attached to a light ideal spring, take y downward from its relaxed position. The stationary support does no work.

$$F_y = mg - k_s y, \quad U(y) = \frac{1}{2} k_s y^2 - mgy.$$

The combined interaction reaches equilibrium at $y_{\text{eq}} = mg/k_s$. Completing the square gives

$$U(y) = \frac{1}{2} k_s (y - y_{\text{eq}})^2 - \frac{m^2 g^2}{2k_s}.$$

With $q = y - y_{\text{eq}}$ and a shifted total zero,

$$\tilde{U}(q) = \frac{1}{2} k_s q^2, \quad F_q = -k_s q.$$

This quadratic is the *combined* spring–gravity potential. The spring's elastic store at equilibrium is still $k_s y_{\text{eq}}^2/2$; its relaxed length and its loaded equilibrium are different positions.

Chapter support: Section 8.3.1, Equation 8.16

Example 8.10: release from relaxed length



Complete manuscript statement

A 0.800 kg mass is attached to an ideal vertical spring with $k_s = 160\text{ N m}^{-1}$. It is released from rest when the spring is relaxed. Neglect resistance and take $g = 9.81\text{ m s}^{-2}$. Find the equilibrium extension, the largest extension, and the speed at equilibrium.

Chapter support: Example 8.10

Example 8.10: equilibrium is passed at maximum speed



Use mass+spring+Earth with $U(0) = 0$. Release from rest gives $E = 0$:

$$\frac{1}{2}mv^2 = mgy - \frac{1}{2}k_sy^2.$$

$$y_{\text{eq}} = mg/k_s = 0.04905 \text{ m.}$$

At a turn, $0 = y(mg - k_sy/2)$. The roots are the initial $y = 0$ and

$$y_{\text{max}} = 2mg/k_s = 0.09810 \text{ m.}$$

At equilibrium,

$$K_{\text{eq}} = \frac{m^2g^2}{2k_s} = 0.192472 \text{ J,}$$

$$v_{\text{eq}} = g\sqrt{m/k_s} = 0.6937 \text{ m s}^{-1}.$$

Force is zero at equilibrium, so acceleration is zero there while speed is greatest. At the lower turn, $F_y = mg - k_s(2mg/k_s) = -mg$: it accelerates upward and reverses. The shifted description has the same turns $q = \pm 0.04905 \text{ m}$.

Chapter support: Example 8.10

Example 8.10: the combined energy landscape

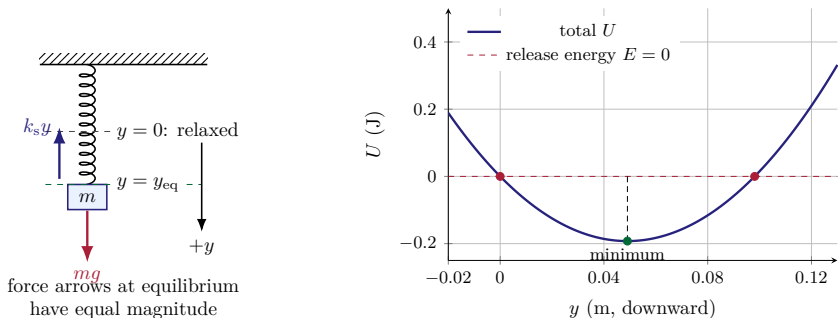


Figure 8.11: Attached vertical spring: the coordinate origin follows the relaxed spring, while the minimum of the combined potential follows the loaded equilibrium. The plot uses $m = 0.800 \text{ kg}$ and $k_s = 160 \text{ N m}^{-1}$. A release from $y = 0$ has $E = 0$ in this reference and two ordinary turning positions. The schematic and graph use the same downward-positive convention.

Chapter support: Figure 8.11

A launcher that pushes and a spring that remains attached



Let z increase upward from a compressed release position; initial compression is $c > 0$.

$$U_{\text{push}}(z) = mgz + \begin{cases} \frac{1}{2}k_s(c - z)^2, & 0 \leq z \leq c, \\ 0, & z \geq c. \end{cases}$$

Beyond $z = c$ a push-only spring is relaxed and detached. An attached spring continues to pull:

$$U_{\text{attached}}(z) = mgz + \frac{1}{2}k_s(z - c)^2.$$

Both potentials and their slopes agree at detachment, but their subsequent forces differ. An endpoint energy equation must use the actual contact model. Positive-speed detachment from rest requires $K(c) > 0$, which is stronger than merely having an initially upward acceleration.

Chapter support: Section 8.3.2, Equation 8.17

Identical release energy, different later forces

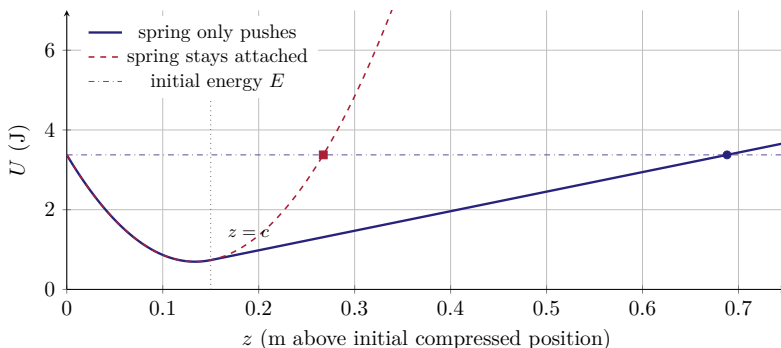


Figure 8.12: Two conservative models with identical initial energy but different later interactions. The solid potential follows a spring that can push but cannot pull; the dashed potential follows an attached spring. With $m = 0.500$ kg, $k_s = 300$ N m⁻¹ and $c = 0.150$ m, their noninitial turning heights are 0.6881 m and 0.2673 m respectively. The diagram is an energy comparison, not a time history.

Chapter support: Figure 8.12

Example 8.11: launch height and contact



Complete manuscript statement

Use the parameters of Fig. 8.12. Determine the speed at the relaxed spring position, the apex of a push-only launch, and the upper turning position if the carriage is instead attached permanently.

Restored figure context. $m = 0.500 \text{ kg}$, $k_s = 300 \text{ N m}^{-1}$, $c = 0.150 \text{ m}$. The carriage starts from rest at $z = 0$ with $U_g = 0$. Use $g = 9.81 \text{ m s}^{-2}$ and neglect losses.

Chapter support: Example 8.11

Example 8.11: detachment and push-only apex



Initially $E = k_s c^2 / 2 = 3.375 \text{ J}$. At $z = c$, both spring models have zero deformation energy:

$$K_c = E - mgc = 3.375 - 0.73575 = 2.63925 \text{ J},$$

$$v_c = \sqrt{2K_c/m} = 3.249 \text{ m s}^{-1}.$$

The positive upward speed confirms that the push-only carriage detaches. Above that position only gravitational potential increases:

$$mgz_{\max} = E, \quad z_{\max} = 0.6881 \text{ m}.$$

The rise after detachment is $z_{\max} - c = 0.5381 \text{ m}$. An independent free-rise check is

$$mg(z_{\max} - c) = K_c.$$

The apex is a zero-speed point with a downward gravitational force, so it is an ordinary turn rather than equilibrium.

Chapter support: Example 8.11

Example 8.11: attached turn and regimes



The attached spring remains in the balance:

$$mgz + \frac{1}{2}k_s(z - c)^2 = \frac{1}{2}k_sc^2.$$

Factoring out the initial root $z = 0$ gives the other turn,

$$z_{\text{turn}} = 2(c - mg/k_s) = 0.2673 \text{ m.}$$

The final extension is 0.1173 m. Its spring and gravity stores are 2.064 J and 1.311 J, summing to 3.375 J without dissipation. **Regime checks for the push-only model.**

- $k_sc \leq mg$: no initial upward launch from rest.
- $mg < k_sc < 2mg$: the carriage rises but turns before relaxation.
- $k_sc = 2mg$: it reaches relaxation with zero speed.
- $k_sc > 2mg$: it detaches with positive speed and has a free-rise stage.

Chapter support: Example 8.11

Energy landscapes: height, slope, and curvature



For one-dimensional conservative motion at fixed E ,

$$K(x) = E - U(x) \geq 0, \quad |v(x)| = \sqrt{2[E - U(x)]/m}.$$

The vertical gap to the energy line gives kinetic energy. Velocity still needs a sign or initial-direction information. **Slope.** $F_x = -U'(x)$ determines force and acceleration. Force arrows do not determine velocity direction. **Stationary points.** Equilibrium requires $U'(x_0) = 0$. For a smooth nondegenerate point, $U''(x_0) > 0$ gives a local minimum and restoring response; $U''(x_0) < 0$ gives an unstable maximum. **Zero speed.** $E = U$ alone does not mean equilibrium. An ordinary turn has force directed back into the neighboring allowed region.

Chapter support: Section 8.4, Section 8.4.1

The stipulated double-well landscape

$$U(x) = U_*[(x/\ell)^2 - 1]^2, \quad U_*, \ell > 0.$$

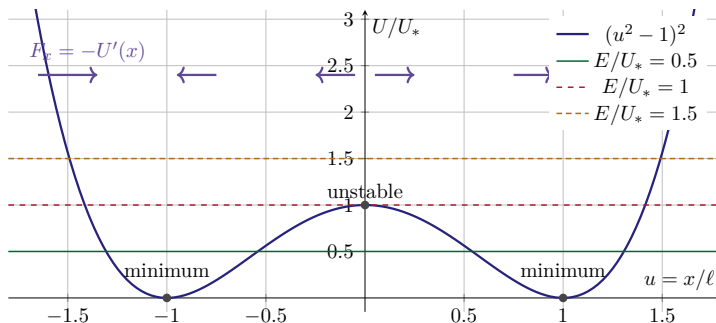


Figure 8.13: Calculated double-well potential from Eq. (8.20). Horizontal lines show energies below, at and above the barrier. Motion is allowed where an energy line lies at or above the potential. The purple arrows indicate the force direction from $F_x = -U'(x)$; x is position, not time.

Chapter support: Equation 8.20, Figure 8.13

Double-well force and equilibrium stability



Introduce $u = x/\ell$. The chain rule gives

$$U'(x) = \frac{4U_*}{\ell} u(u^2 - 1), \quad F_x = \frac{4U_*}{\ell} u(1 - u^2).$$

$$U''(x) = \frac{4U_*}{\ell^2} (3u^2 - 1).$$

Thus $x = \pm\ell$ are minima with $U = 0$ and positive curvature $8U_*/\ell^2$. The point $x = 0$ is a maximum with $U = U_*$ and negative curvature $-4U_*/\ell^2$. Between a minimum and the central maximum the force points toward that minimum. Outside each well it points inward. At a minimum, a moving particle can have maximum speed while instantaneous acceleration is zero. The force scale U_*/ℓ has units newton; curvature has units N m^{-1} .

Chapter support: Equation 8.21, Section 8.11

Example 8.12: carriage in a double-well landscape



Complete manuscript statement

Take $U_* = 4.00 \text{ J}$ and $\ell = 0.500 \text{ m}$. Determine allowed intervals and key points for $E/U_* = 0.50, 1.00$ and 1.50 .

Chapter support: Example 8.12

Example 8.12: roots and allowed intervals



Set $e = E/U_*$ and solve $e = (u^2 - 1)^2$:

$$u^2 = 1 \pm \sqrt{e}, \quad x = \ell u.$$

$e = 0.50$. Four ordinary turning roots are $u = \pm 0.5412, \pm 1.3066$. Allowed intervals are

$$[-1.3066, -0.5412] \cup [0.5412, 1.3066].$$

In metres these are approximately $[-0.6533, -0.2706] \cup [0.2706, 0.6533]$. Initial position selects one well; motion cannot jump the forbidden gap. $e = 1.00$. The position inequality allows $u \in [-\sqrt{2}, \sqrt{2}]$, with outer turns $x = \pm 0.7071$ m. The root $u = 0$ is unstable rest or an asymptotic approach, not an ordinary rebound or finite-time crossing. $e = 1.50$. The allowed interval is $u \in [-1.4916, 1.4916]$, with boundary roots $x = \pm 0.7458$ m. Barrier crossing has nonzero speed. Substitution checks every root. A common shift of U and E preserves $E - U$, hence every interval and classification.

Chapter support: Example 8.12

Example 8.13: force and velocity directions



Complete manuscript statement

A 0.500 kg carriage moves in the stipulated double well with $U_* = 4.00$ J and $\ell = 0.500$ m. At $x = +0.250$ m its mechanical energy is $E = 3.00$ J. Find its possible velocities and its acceleration. If it is moving left, is it speeding up or slowing down at this instant?

Chapter support: Example 8.13

Example 8.13: one acceleration, two velocities



Here $u = 0.250/0.500 = 0.500$. Energy gives

$$U = 4.00(0.500^2 - 1)^2 = 2.250 \text{ J}, \quad K = 3.00 - 2.250 = 0.750 \text{ J},$$

$$v = \pm \sqrt{2K/m} = \pm 1.732 \text{ m s}^{-1}.$$

Force instead follows from the position:

$$F_x = \frac{4(4.00)}{0.500} (0.500)(1 - 0.500^2) = +12.00 \text{ N},$$

$$a_x = F_x/m = +24.00 \text{ m s}^{-2}.$$

A left-moving carriage slows down because $F_x v < 0$. Its power is -20.78 W ; a right-moving one has $+20.78 \text{ W}$ and speeds up. Since $E < U_*$, both velocity choices remain in the right well. Reversing velocity preserves energy and force but changes the instantaneous dynamical state.

Chapter support: Example 8.13

Ordinary turns and the barrier threshold



For right-moving motion,

$$dt = \sqrt{m/2} \frac{dx}{\sqrt{E - U(x)}}.$$

Near a regular right-hand turn x_t with $U'(x_t) > 0$, $E - U(x) \simeq U'(x_t)(x_t - x)$. The time integral behaves as $\int dx/\sqrt{x_t - x}$ and is finite; nonzero leftward force reverses the motion. At the double-well barrier with $E = U_*$,

$$E - U = U_*(2u^2 - u^4), \quad |v| \simeq 2\sqrt{U_*/m} |u|.$$

Now time contains $\int dx/|x|$, which diverges logarithmically. A trajectory approaches the maximum without finite-time crossing. For $E > U_*$, $|v(0)| = \sqrt{2(E - U_*)/m} > 0$, so crossing is finite. The threshold is an infimum of crossing energies, not an attained minimum crossing energy.

Chapter support: Section 8.4.3, Equation 8.22

Flat stationary points and stable motion



The test $U''(x_0) = 0$ is inconclusive. Let $u = (x - x_0)/\ell$ and $U_* > 0$.

- $U - U_0 = +U_*u^4$: the shape rises on both sides and $F_x = -(4U_*/\ell)u^3$ restores toward zero. The point is stable.
- $U - U_0 = -U_*u^4$: the shape falls on both sides and the force reinforces a displacement. The point is unstable.
- $U - U_0 = U_*u^3$: a horizontal tangent has no extremum; $F_x = -(3U_*/\ell)u^2$ points left on both sides. A leftward perturbation departs, so the point is unstable.

Stability means sufficiently small disturbances remain close. Settling at rest needs dissipation. A flat quartic minimum has no positive quadratic stiffness; the usual harmonic frequency formula does not describe its finite-amplitude motion.

Chapter support: Section 8.4.4

Loop contact requires energy and force constraints



For a particle on the inside of a smooth vertical circle, let θ start at the bottom. While contact holds,

$$h = R(1 - \cos \theta), \quad v^2 = 2g[H - R(1 - \cos \theta)].$$

Inward Newton's law gives

$$N - mg \cos \theta = mv^2/R,$$

$$N = mg(2H/R - 2 + 3 \cos \theta).$$

The surface can push inward, so $N \geq 0$. At the top $N_{\text{top}} = mg(2H/R - 5)$ is smallest:

$$H \geq \frac{5}{2}R \quad \text{for ideal continuous contact.}$$

The energy-only top condition $H \geq 2R$ is weaker. At departure the circular constraint ends and the subsequent motion is free flight.

Chapter support: Section 8.4.5, Equation 8.23

Loop contact and the rejected continuation

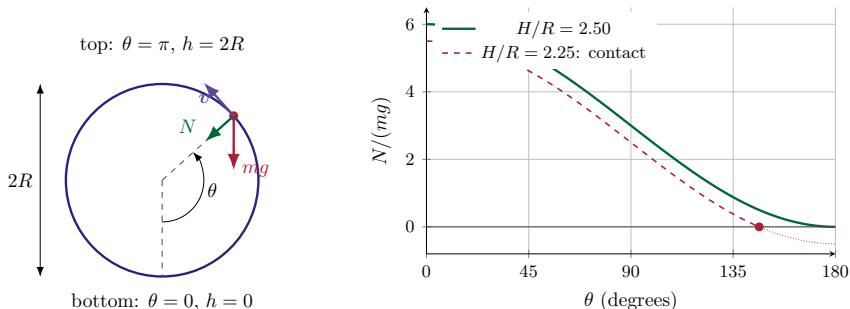


Figure 8.18: Contact must satisfy both energy conservation and an admissible inward normal force. The left schematic shows forces while contact holds. In the right plot, the dotted negative-normal continuation for $H/R = 2.25$ is a rejected mathematical continuation, not a physical contact force. The particle departs the circular constraint where the dashed curve first reaches zero.

Chapter support: Figure 8.18

Example 8.16: testing loop contact



Complete manuscript statement

A 0.200 kg particle is released from rest at $H = 0.900\text{ m}$ above the bottom of a smooth inside loop of radius $R = 0.400\text{ m}$. An energy-only calculation predicts a positive speed at the top. Test that conclusion and find where continuous contact actually ends.

Chapter support: Example 8.16

Example 8.16: an invalid contact state



For $m = 0.200$ kg, $H = 0.900$ m, and $R = 0.400$ m, the formal contact-path top speed is

$$v_{\text{top}} = \sqrt{2g(H - 2R)} = \sqrt{2(9.81)(0.100)} = 1.401 \text{ m s}^{-1}.$$

But the corresponding reaction would be

$$N_{\text{top}} = mg(2H/R - 5) = (0.200)(9.81)(4.50 - 5) = -0.9810 \text{ N}.$$

The inside surface cannot supply this outward pull. The assumed contact path fails before reaching the top. Set the reaction to zero at its first zero during ascent:

$$\cos \theta_d = (2 - 2H/R)/3 = -5/6,$$

$$\theta_d = 146.44^\circ.$$

The energy equation remains valid; the geometry constrained to the circle is what must stop at departure.

Chapter support: Example 8.16

Example 8.16: departure and radial check



At the departure angle,

$$h_d = R(1 - \cos \theta_d) = 0.73333 \text{ m},$$

$$v_d = \sqrt{2g(H - h_d)} = 1.808 \text{ m s}^{-1}.$$

The inward centripetal requirement is

$$mv_d^2/R = 1.635 \text{ N} = -mg \cos \theta_d,$$

so gravity alone supplies the radial requirement at the instant $N = 0$. This independently checks departure. Continuous ideal contact instead needs

$$H_{\min} = 2.5R = 1.000 \text{ m}, \quad v_{\text{top}} = \sqrt{gR} = 1.981 \text{ m s}^{-1}.$$

After separation the velocity is tangent and acceleration is gravity alone. Continue with a free-flight model rather than allowing the reaction to become negative.

Chapter support: Example 8.16

External work and changes of system boundary



For the closed mechanical-plus-internal model used here,

$$\Delta(K + U + E_{\text{int}}) = W_{\text{ext,on}} + Q.$$

The storage and transfer terms must exhaust the chosen model. Closed means no mass transfer; work and heating can still cross the boundary. For a cart alone in a particle approximation, gravity is external:

$$\Delta K = W_T + W_g.$$

For cart+Earth, gravity is represented internally by U_g :

$$\Delta K + \Delta U_g = W_T, \quad W_g = -\Delta U_g.$$

Count the interaction once. Equivalent work-by signs are $W_{\text{on}} = -W_{\text{by}}$ and $\Delta E = Q - W_{\text{by}}$.

Chapter support: Section 8.5, Equation 8.24

Example 8.18: cable work



Complete manuscript statement

A cable raises a 5.00 kg cart through 2.00 m . Its initial speed is zero and its final upward speed is 1.00 m s^{-1} . The cart is treated as rigid; drag and heating are negligible. Find the cable work and show two consistent system choices.

Chapter support: Example 8.18

Example 8.18: cable work under two boundaries



For cart+Earth, the cable supplies both kinetic and gravitational changes:

$$W_T = \Delta K + \Delta U_g = \frac{1}{2}m(v_f^2 - v_i^2) + mg\Delta y.$$

$$\Delta K = \frac{1}{2}(5.00)(1.00)^2 = 2.50 \text{ J}, \quad \Delta U_g = (5.00)(9.81)(2.00) = 98.10 \text{ J}.$$

$$W_T = 100.60 \text{ J}.$$

For cart alone, $W_g = -98.10 \text{ J}$, so

$$W_T + W_g = 100.60 - 98.10 = 2.50 \text{ J} = \Delta K.$$

No constant tension or acceleration was assumed. A quasistatic lift ending at rest would require 98.10 J, providing a limiting-state check. Loss-free motion can still receive external work. Cable work alone does not determine the motor's electrical consumption.

Chapter support: Example 8.18

Example 8.17: slowing and warming



Complete manuscript statement

A 2.00 kg object slows from 4.00 to 2.00 m s⁻¹ while sliding horizontally. Treat the same trajectory in two ways. **Translational change.**

$$\Delta K = \frac{1}{2}(2.00)(2.00^2 - 4.00^2) = -12.0 \text{ J.}$$

Object alone. The centre-of-mass force integral is -12.0 J. For a real warming object, its complete energy balance is instead

$$\Delta K + \Delta E_{\text{int,obj}} = W_{\text{contact,on}} + Q.$$

Actual contact transfer cannot be inferred from ΔK alone.

Object+stationary track. If insulated with no external work, $\Delta K + \Delta E_{\text{int,total}} = 0$, giving $\Delta E_{\text{int,total}} = +12.0 \text{ J}$. This determines only the combined internal increase. Its allocation between object and track requires independent thermal information.

Chapter support: Example 8.17

Sliding conversion and a moving belt



For an enlarged isolated contact pair with only mechanical-to-internal conversion,

$$\Delta E_{\text{mech}} = -D, \quad \Delta E_{\text{int}} = +D, \quad D \geq 0.$$

The Coulomb model gives $D = \mu_k N s_{\text{rel}}$, using relative surface travel. On a belt moving at $V > v > 0$, friction $+f$ accelerates a slower package. Laboratory powers are

$$P_{\text{pkg}} = fv > 0, \quad P_{\text{belt}} = -fV,$$

$$P_{\text{pkg}} + P_{\text{belt}} = -f(V - v), \quad \dot{D} = f|V - v| \geq 0.$$

A motor holding belt speed constant supplies its lost power. Positive friction work on one body and positive contact conversion coexist. Under a common Galilean frame shift, body work can change sign while relative slip stays the same.

Chapter support: Section 8.6, Equation 8.31

Student practice: Problem 23



Complete manuscript statement

A 2.0 kg block slides 3.0 m relative to a stationary horizontal surface, with $\mu_k = 0.25$. (a) Compute Coulomb dissipation. (b) If its initial speed is 4.0 m s^{-1} , can it complete that distance without another input? Find its speed there if reachable. **Shared exercise convention.** $g = 9.81\text{ m s}^{-2}$.

Inputs are stipulated. State the system, reference, and loss assumptions before calculating.

Chapter support: Problem 23

Problem 23: a reachable slide and its stopping limit



Model. Include block+stationary surface. Height stays fixed; the constant contact conversion is internal and there is no other input. **(a)** Convert the Coulomb loss:

$$f_k = \mu_k mg = (0.25)(2.0)(9.81) = 4.905 \text{ N},$$

$$D = f_k(3.0) = 14.715 \text{ J}.$$

(b) $K_i = \frac{1}{2}(2.0)(4.0)^2 = 16.000 \text{ J}$, so

$$K_f = 16.000 - 14.715 = 1.285 \text{ J} > 0,$$

$$v_f = \sqrt{2K_f/m} = 1.134 \text{ m s}^{-1}.$$

Independent check: $s_{\text{stop}} = v_i^2 / (2\mu_k g) = 3.262 \text{ m} > 3.0 \text{ m}$. The loss grows monotonically, confirming full-path reachability. Stop the slip model at rest.

Chapter support: Section 8.11

Student practice: Problem 25



Complete manuscript statement

A spring store has $k_s = 8.0 \text{ kN m}^{-1}$ and usable compression 25 mm. (a) Find ideal stored energy. (b) If a stipulated 15% converts to internal energy before launch, find the remaining modeled mechanical output. **Solution (a).**

Convert $8.0 \text{ kN m}^{-1} = 8000 \text{ N m}^{-1}$ and $25 \text{ mm} = 0.025 \text{ m}$:

$$S = \frac{1}{2}(8000)(0.025)^2 = 2.500 \text{ J.}$$

Solution (b). The declared internal conversion is $0.15S = 0.375 \text{ J}$ and the remaining mechanical output is

$$E_{\text{out}} = 0.85S = 2.125 \text{ J.}$$

The two channels sum to 2.500 J. This is a stipulated energy audit. Recoverability of a real component also needs loading/unloading and range checks; a force curve from one branch cannot establish a complete conservative cycle.

Chapter support: Problem 25, Section 8.11

Core synthesis and the bridge to momentum



- **Interactions:** conservative work gives $\Delta U = -W_c$; force is a local slope, $F_x = -U'$.
- **Motion:** $E - U \geq 0$ supplies candidate speeds and allowed regions. Initial state, barrier time, and contact determine trajectory validity.
- **Accounting:** $\Delta(K + U + E_{\text{int}}) = W_{\text{ext,on}} + Q$ tracks storage and transfer within the chosen model.
- **Validation:** compare units, signs, limits, and independent force/work checks. Numerical drift and calibration bias can imitate physical changes.

Further manuscript study. Section 8.7 develops velocity-Verlet diagnostics, uncertainty budgets, and sensor-bias reconstruction. The exercise bank supplies further complete solutions. **Next chapter.** Chapter 9 develops centre of mass and linear momentum for interacting bodies. Continue to distinguish system boundaries and internal interactions.

Chapter support: Section 8.8, Section 8.7, Section 8.12

Advanced: local curvature and effective stiffness



Near a smooth stationary point x_0 , put $q = x - x_0$. Taylor expansion gives

$$U(x_0 + q) = U(x_0) + \frac{1}{2} U''(x_0) q^2 + O(q^3).$$

If $k_{\text{eff}} = U''(x_0) > 0$, then $F_x \simeq -k_{\text{eff}} q$. Newton's law becomes

$$m\ddot{q} \simeq -k_{\text{eff}} q, \quad \omega \simeq \sqrt{k_{\text{eff}}/m}, \quad f = \omega/(2\pi).$$

At the right double-well minimum, let $\delta = q/\ell$:

$$U/U_* = 4\delta^2 + 4\delta^3 + \delta^4, \quad k_{\text{eff}} = 8U_*/\ell^2.$$

The leading quadratic term is symmetric, while the cubic correction is asymmetric. These frequencies describe small oscillations about one minimum. A flat minimum or barrier-scale motion requires another approximation.

Chapter support: Section 8.9, Equation 8.38, Equation 8.39

Advanced problem 31: local frequencies



Complete manuscript statement

(a) Derive the double well's local effective stiffness directly from U'' . (b) Find angular and ordinary small-oscillation frequencies near $x = +\ell$ for $U_* = 4.0$ J, $\ell = 0.50$ m and $m = 0.80$ kg. **(a)** Differentiate directly, with $u = x/\ell$:

$$U''(x) = \frac{4U_*}{\ell^2}(3u^2 - 1), \quad \boxed{k_{\text{eff}} = U''(\ell) = 8U_*/\ell^2.}$$

(b) For the stipulated parameters,

$$k_{\text{eff}} = 8(4.0)/(0.50)^2 = 128 \text{ N m}^{-1},$$

$$\boxed{\omega = \sqrt{128/0.80} = 12.65 \text{ s}^{-1}, \quad f = 2.01 \text{ Hz.}}$$

The curvature agrees with the coefficient of $q^2/2$ in the exact local expansion. Units of k_{eff}/m are s^{-2} . Higher-order terms limit these values to sufficiently small amplitude.

Chapter support: Problem 31, Section 8.11

Advanced challenge 32: error of the local force



Differentiation of the exact local polynomial gives

$$\frac{F_{\text{exact}} \ell}{U_*} = -(8\delta + 12\delta^2 + 4\delta^3), \quad \frac{F_{\text{local}} \ell}{U_*} = -8\delta.$$

The relative error uses the exact force as denominator:

$$\varepsilon_F = \frac{|F_{\text{local}} - F_{\text{exact}}|}{|F_{\text{exact}}|}, \quad F_{\text{exact}} \neq 0.$$

Complete manuscript statement

(a) Derive the exact relative force-error expression for the local double-well approximation, using $\delta = (x - \ell)/\ell$. (b) Explain why relative error is inappropriate at the minimum and identify any other denominator zeros. **Attempt**

first. Factor the exact-force polynomial before deciding where this error measure is defined.

Chapter support: Problem 32, Equation 8.39

Problem 32: force zeros and error range



(a) Subtract the exact and local forces:

$$\varepsilon_F = \frac{|12\delta^2 + 4\delta^3|}{|8\delta + 12\delta^2 + 4\delta^3|}.$$

(b) The denominator is $4\delta(\delta + 1)(\delta + 2)$, so its zeros are $\delta = 0, -1, -2$. At $\delta = 0$, both forces vanish: relative error is undefined, although its nearby limit is zero. Use absolute error or curvature there. The other zeros correspond to the central maximum and opposite minimum, outside the intended local neighborhood.

Finite-range checks. For nonzero δ near zero, $\varepsilon_F = |\delta(3 + \delta)|/|(1 + \delta)(2 + \delta)|$. Substitution gives 7.08% at $\delta = 0.050$ and 13.4% at $\delta = 0.100$. Solving $\varepsilon_F < 0.10$ gives approximately $-0.0619 < \delta < 0.0723$, excluding zero. Cubic corrections make this range asymmetric.

Chapter support: Section 8.11, Example 8.26

Advanced: gradients and two-coordinate force



A partial derivative changes one coordinate while holding the other fixed. For independent Cartesian coordinates,

$$\mathbf{F} = -\nabla U = -(\partial_x U, \partial_y U), \quad dU = \partial_x U dx + \partial_y U dy.$$

Thus $dU = -\mathbf{F} \cdot d\mathbf{r}$ and $W_{A \rightarrow B} = U(A) - U(B)$. The force is perpendicular to an equal-potential contour where the gradient is nonzero. For a coupled quadratic model,

$$U = \frac{1}{2}(ax^2 + 2bxy + cy^2), \quad F_x = -ax - by, \quad F_y = -bx - cy.$$

Here a, b, c have stiffness units N m^{-1} . The cross term couples the coordinates. For smooth forces, vanishing curl is necessary locally and sufficient on a simply connected domain. A hole in the domain needs a separate global loop test.

Chapter support: Section 8.9.3, Equation 8.44

Coupled energy contours and restoring forces

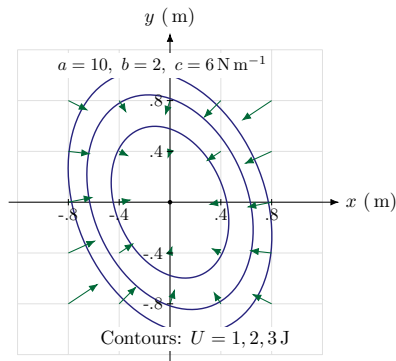


Figure 8.28: An original two-coordinate energy landscape. The nested ellipses are contours of the same potential used to compute every force arrow. Arrows point across contours toward lower U , rather than along the contours; arrow length represents force magnitude with scale 0.025 m N^{-1} . The origin is a stable minimum. The energy contours are representations of configuration energy, not outlines of a physical bowl.

Chapter support: Figure 8.28

Example 8.27: two coupled-field routes



Complete manuscript statement

A particle is guided quasistatically from $O = (0, 0)$ to $P = (0.20, 0.10)$ m in the field of Eq. (8.44), with $a = 10$, $b = 2$ and $c = 6 \text{ N m}^{-1}$. Find the field's work on the particle along the two axis-segment routes, find the final force, and explain the external agent's energy transfer.

Restored equation context.

$$U(x, y) = \frac{1}{2}(ax^2 + 2bxy + cy^2), \quad \mathbf{F} = (-ax - by, -bx - cy).$$

The two routes run horizontal then vertical, or vertical then horizontal. The potential is zero at O , and the quasistatic guided transfer has negligible ΔK .

Chapter support: Example 8.27

Example 8.27: direct work along both routes



Horizontal, then vertical. On $y = 0$, integrate $F_x = -10x$; on $x = 0.20$ m, integrate $F_y = -0.400 - 6y$:

$$W_1^{(x)} = \int_0^{0.20} -10x dx = -0.200 \text{ J},$$

$$W_1^{(y)} = \int_0^{0.10} (-0.400 - 6y) dy = -0.0700 \text{ J}.$$

Vertical, then horizontal. On $x = 0$, integrate $F_y = -6y$; on $y = 0.10$ m, integrate $F_x = -10x - 0.200$:

$$W_2^{(y)} = -0.0300 \text{ J}, \quad W_2^{(x)} = \int_0^{0.20} (-10x - 0.200) dx = -0.240 \text{ J}.$$

Both totals: $W_1 = W_2 = -0.270 \text{ J}$. Segment works differ; totals agree. The endpoint check follows on the next slide. A global smooth potential, rather than two path samples alone, establishes conservativity.

Chapter support: Example 8.27

Example 8.27: endpoint and stability checks



An independent endpoint calculation gives

$$U(P) = \frac{1}{2}[10(0.20)^2 + 4(0.20)(0.10) + 6(0.10)^2] = 0.270 \text{ J.}$$

Since $U(O) = 0$, $W = -\Delta U = -0.270 \text{ J}$. The final force is

$$\mathbf{F}(P) = (-2.20\hat{i} - 1.00\hat{j}) \text{ N.}$$

The quasistatic agent supplies $+0.270 \text{ J}$. Count that transfer once; it balances the field work while storing interaction energy. **Preparation for stability.** The Hessian H collects second partial derivatives, and the local restoring force is $\mathbf{F} \simeq -H\mathbf{q}$. For $a > 0$,

$$aq_x^2 + 2bq_xq_y + cq_y^2 = a(q_x + bq_y/a)^2 + \frac{ac - b^2}{a}q_y^2.$$

Thus $a > 0$ and $ac - b^2 > 0$ give a positive quadratic energy increase for every nonzero displacement. For this model $ac - b^2 = 56 \text{ N}^2 \text{ m}^{-2} > 0$.

Chapter support: Example 8.27, Section 8.9.4, Equation 8.46

Advanced challenge: a hidden saddle



Original teaching extension based on Section 8.9.4.

For $U = \frac{1}{2}(ax^2 + 2bxy + cy^2)$ with $a = 10$, $b = 8$, $c = 6 \text{ N m}^{-1}$, (a) classify the stationary origin despite its upward x and y axis cuts; (b) exhibit a downhill displacement and explain the instability. **(a)** Both axis cuts rise:

$U(x, 0) = 5x^2$ and $U(0, y) = 3y^2$ in SI units. But

$$ac - b^2 = 60 - 64 = -4 \text{ N}^2 \text{ m}^{-2} < 0.$$

The quadratic form has one positive and one negative principal stiffness, so the origin is a saddle. **(b)** Choose $x = -0.8s$, $y = s$, with s in metres:

$$U(-0.8s, s) = \frac{1}{2}(6.4 - 12.8 + 6)s^2 = -0.20 \text{ N m}^{-1}s^2 < 0 \quad (s \neq 0).$$

Any arbitrarily small displacement in this downhill direction lowers U . The force reinforces departure in a negative-stiffness direction. Axis checks alone miss it; stability needs *every* direction. A constant reference shift cannot change this result.

Chapter support: Section 8.9.4

