

Center of Mass and Linear Momentum

Physics I - Chapter 9

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Why describe a whole system?



Following every particle can be unnecessary: an instrument plate, a walking crew member, or a collision involves complicated internal motion.

Two collective quantities

The center of mass locates the mass-weighted system. Total momentum connects its motion to external force and impulse.

- Begin with mass distributions and the motion of their weighted center.
- Integrate forces, then choose the system and test conservation.
- Separate momentum from energy in collisions and fragmentation.
- Audit data; extend the same bookkeeping to rocket exhaust.

Chapter support: Physics I manuscript, Chapter 9

Learning route and prerequisites



- ① **Mass geometry:** turn an inventory or density into a weighted position.
- ② **Dynamics:** differentiate the weighted position; identify external force.
- ③ **Interactions:** integrate force, conserve justified components, and audit energy separately.
- ④ **Validation:** inspect frames, sampling, uncertainty, and momentum flux.

Tools to bring

Signed Cartesian vectors; Newton's laws; definite integrals; $K = \frac{1}{2}mv^2$; and work–energy balance. We refresh mass elements, frame shifts, and uncertainty propagation where needed.

Chapter support: Physics I manuscript, Chapter 9

What you should be able to do



- Calculate discrete and continuous centers of mass and check their locations.
- Derive $M\mathbf{A}_{\text{CM}} = \sum \mathbf{F}_{\text{ext}}$ for a fixed system.
- Integrate signed force components and distinguish contact from net impulse.
- Solve one- and two-dimensional interactions using justified balances.
- Test a calculation or dataset using momentum, energy, and uncertainty.

Advanced extension, slides 61–70

Separate CM and relative kinetic energy, then derive and apply open-system rocket momentum bookkeeping.

Chapter support: Physics I manuscript, Chapter 9

A mass-weighted position



For positive particle masses m_i , define $M = \sum_i m_i > 0$ in one Cartesian frame:

$$\mathbf{R}_{\text{CM}} = \frac{\sum_i m_i \mathbf{r}_i}{M}, \quad x_{\text{CM}} = \frac{\sum_i m_i x_i}{M}.$$

Each component is a weighted mean: heavier parts contribute larger first moments $m_i \mathbf{r}_i$. The CM need not be occupied by material.

An origin shift is a change of coordinates

If $\mathbf{r}_i^* = \mathbf{r}_i - \mathbf{a}$, then $\mathbf{R}_{\text{CM}}^* = \mathbf{R}_{\text{CM}} - \mathbf{a}$. The physical balance point stays in the same place.

For two masses separated by L , the distances satisfy $d_1 = m_2 L / M$, $d_2 = m_1 L / M$, and $m_1 d_1 = m_2 d_2$.

Chapter support: Physics I manuscript, Chapter 9

Choose the correct mass element



$$\mathbf{R}_{\text{CM}} = \frac{1}{M} \int \mathbf{r} \, dm, \quad M = \int dm.$$

$$dm = \lambda \, d\ell \text{ (wire)}, \quad dm = \sigma \, dA \text{ (lamina)}, \quad dm = \rho \, dV \text{ (volume)}.$$

Use symmetry of the *mass distribution* first. A symmetric outline with unequal densities need not have a symmetric CM.

Composite or cutout model

Add the masses and first moments of nonoverlapping pieces. Subtract a removed piece in both numerator and denominator. Negative bookkeeping mass represents removed positive material.

The CM coincides with the center of gravity in a uniform gravitational field; a nonuniform field can make the distinction relevant.

Chapter support: Physics I manuscript, Chapter 9

An off-center cutout: statement



Example 9.1 — complete statement

A uniform rectangular plate has width 0.60 m and height 0.40 m . The origin is at its lower-left corner. A rectangular hole of width 0.20 m and height 0.10 m is cut out. The hole center is at $(0.45, 0.30)\text{ m}$. Find the center of mass of the remaining plate.

Model before arithmetic

Use one lower-left origin for both rectangles. Uniform areal density cancels, so the signed areas can replace masses. Predict a shift down and left before calculating it.

Chapter support: Physics I manuscript, Chapter 9; Example 9.1

Cutout: calculate and check both coordinates



Example 9.1 — solution. The full plate and the hole have

$$A_0 = 0.240 \text{ m}^2, \quad (x_0, y_0) = (0.300, 0.200) \text{ m},$$

$$A_h = 0.020 \text{ m}^2, \quad (x_h, y_h) = (0.450, 0.300) \text{ m}.$$

The remaining area is 0.220 m^2 . Subtract each first moment:

$$X = \frac{0.240(0.300) - 0.020(0.450)}{0.220} = 0.2864 \text{ m},$$

$$Y = \frac{0.240(0.200) - 0.020(0.300)}{0.220} = 0.1909 \text{ m}.$$

Answer: $R_{\text{CM}} = (0.286\hat{x} + 0.191\hat{y}) \text{ m}$. Both shifts are negative, away from the removed material. A centered hole would give no shift.

Chapter support: Physics I manuscript, Chapter 9; Example 9.1

Cutout geometry and the small CM shift

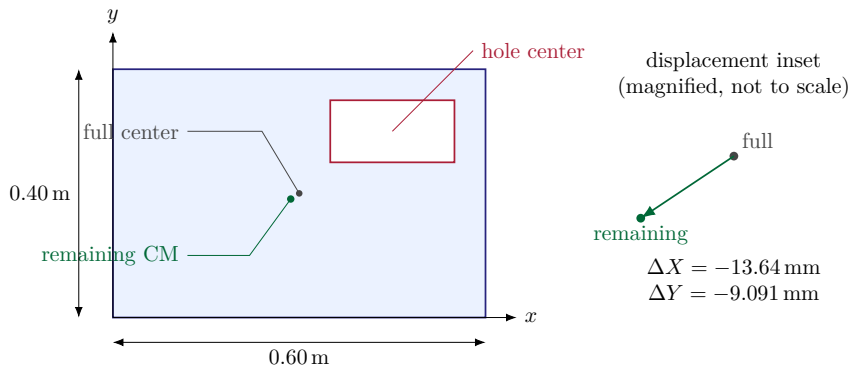


Figure 9.1: The plate and hole are drawn with equal physical axis scales; the hole measures $0.20 \text{ m} \times 0.10 \text{ m}$. Leader lines separate the nearly coincident full and remaining centers. The inset magnifies only the shift, which is directed away from the removed mass.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.1

Practice 1: three particles in a plane



Problem 2 — student practice

Three point masses are located in a plane: 2.0 kg at (0,0), 3.0 kg at (4.0, 0) m, and 5.0 kg at (0, 2.0) m. Find $\mathbf{R}_{\text{CM}}$.

Solution. Use all three masses, $M = 10.0$ kg:

$$X = \frac{2(0) + 3(4) + 5(0)}{10} = 1.20 \text{ m}, \quad Y = \frac{2(0) + 3(0) + 5(2)}{10} = 1.00 \text{ m}.$$

Thus $\mathbf{R}_{\text{CM}} = (1.20, 1.00) \text{ m}$. About this point, the first moments vanish: $2(-1.2) + 3(2.8) + 5(-1.2) = 0$ and $2(-1) + 3(-1) + 5(1) = 0$. An unweighted coordinate mean would describe a different mass distribution.

Chapter support: Physics I manuscript, Chapter 9; Problem 2

A CM belongs to the convex hull

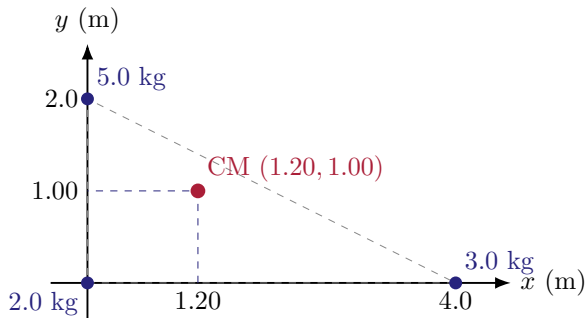


Figure 9.32: Solution geometry for Problem 2. Position coordinates are to scale; equal marker sizes identify ideal point particles and do not represent their masses. The dashed triangle is the convex hull of the three positive masses.

The coefficients m_i/M are nonnegative and sum to one. Therefore the CM lies in the convex hull of the positive mass distribution, even when that location contains no material.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.32

A graded rod: normalize the density



For a rod on $0 \leq x \leq L$, compute both $M = \int_0^L \lambda(x) dx$ and the first moment $\int_0^L x \lambda(x) dx$. Their units are kilograms and kilogram metres, respectively.

Example 9.4 — complete statement

A slender rod of length $L = 1.20$ m has linear density

$$\lambda(x) = \lambda_0 \left(1 + \alpha \frac{x}{L} \right), \quad \lambda_0 = 0.800 \text{ kg m}^{-1}, \quad \alpha = 2.00.$$

Find its mass and center of mass. Derive the dependence on α before substituting the numbers.

Chapter support: Physics I manuscript, Chapter 9; Example 9.4

Graded rod: derive before substituting



Example 9.4 — solution. Integrating the same density on $[0, L]$ gives

$$M = \lambda_0 L(1 + \alpha/2), \quad Q_x = \lambda_0 L^2(1/2 + \alpha/3).$$

$$X_{\text{CM}} = L \frac{1/2 + \alpha/3}{1 + \alpha/2}.$$

At $L = 1.20 \text{ m}$, $\lambda_0 = 0.800 \text{ kg m}^{-1}$, and $\alpha = 2$,

$$M = 1.92 \text{ kg}, \quad Q_x = 1.344 \text{ kg m}, \quad X_{\text{CM}} = 0.700 \text{ m}.$$

It is right of $L/2 = 0.600 \text{ m}$. Nonnegative density requires $\alpha \geq -1$: the limits are $L/3$ at -1 , $L/2$ at 0 , and $2L/3$ as $\alpha \rightarrow \infty$. These check the normalization and direction.

Chapter support: Physics I manuscript, Chapter 9; Example 9.4

Density and center are different functions

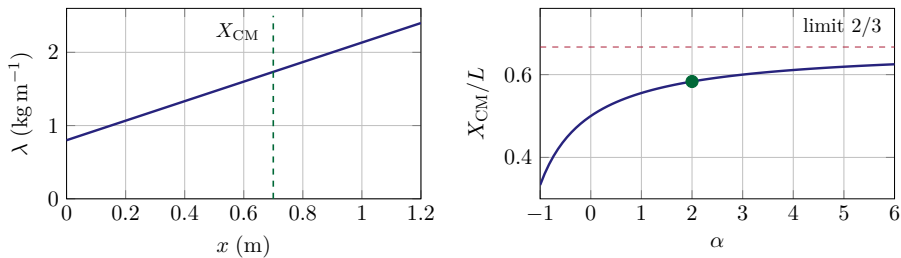


Figure 9.4: Left: the stipulated density of the graded rod. Right: its normalized center as the gradient varies through physically allowed values. The dashed limiting level is approached as $\alpha \rightarrow \infty$; it is not reached at any finite α .

Chapter support: Physics I manuscript, Chapter 9; Figure 9.4

Same outline, different mass measure



Symmetry gives $X_{\text{CM}} = 0$ for both upper semicircles. It does not determine the vertical mean. The wire has $dm = \lambda R d\theta$; the plate has $dm = \sigma r dr d\theta$.

Example 9.5 — complete statement

Compare the centers of mass of a uniform thin upper semicircular wire and a uniform thin upper semicircular lamina, each of radius $R = 0.600 \text{ m}$. Put the origin at the midpoint of the diameter, with y positive upward.

Chapter support: Physics I manuscript, Chapter 9; Example 9.5

Wire and lamina: complete vertical moments



Example 9.5 — solution. For $0 \leq \theta \leq \pi$:

$$M_w = \lambda \pi R, \quad Q_{y,w} = \lambda R^2 \int_0^\pi \sin \theta \, d\theta = 2\lambda R^2.$$

$$\boxed{Y_w = 2R/\pi = 0.382 \text{ m}}, \quad X_w = 0.$$

For the lamina, also integrate $0 \leq r \leq R$:

$$M_l = \sigma \pi R^2/2, \quad Q_{y,l} = \sigma \int_0^\pi \sin \theta \, d\theta \int_0^R r^2 \, dr = 2\sigma R^3/3.$$

$$\boxed{Y_l = 4R/(3\pi) = 0.255 \text{ m}}, \quad X_l = 0.$$

Both heights lie in $(0, R)$. The lamina includes mass nearer the diameter; its mean height is lower. The radius-independent ratio $Y_w/Y_l = 3/2$ is an additional check.

Chapter support: Physics I manuscript, Chapter 9; Example 9.5

Two separately derived centers

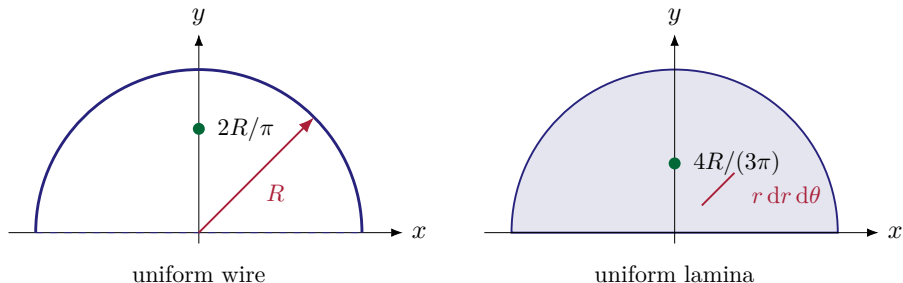


Figure 9.5: Semicircular wire and lamina use the same geometric scale. Green points show their separately derived centers. The small red segment on the right indicates the radial direction of an area element; it is schematic rather than an area drawn to differential scale.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.5

External force governs CM motion



For a **fixed set of constant-mass particles**, differentiate the weighted position:

$$M\mathbf{V}_{\text{CM}} = \sum_i m_i \mathbf{v}_i \equiv \mathbf{P}, \quad M\mathbf{A}_{\text{CM}} = \frac{d\mathbf{P}}{dt} = \sum_i \mathbf{F}_i.$$

Split forces by the chosen boundary:

$$\sum_i \mathbf{F}_i = \sum_i \mathbf{F}_{i,\text{ext}} + \sum_{i < j} (\mathbf{F}_{ij} + \mathbf{F}_{ji}).$$

Equal-and-opposite classical internal pairs cancel, giving

$$M\mathbf{A}_{\text{CM}} = \sum \mathbf{F}_{\text{ext}}.$$

Individual parts can accelerate while the CM moves uniformly. The particle-only proof assumes the selected system includes all relevant momentum carriers.

Chapter support: Physics I manuscript, Chapter 9

System boundary and relative displacement



For a person and a freely translating platform, their contact forces are internal to the combined system. Gravity and support cancel vertically; horizontal external impulse is neglected.

$$m\Delta x_a + M\Delta x_p = 0, \quad s = \Delta x_a - \Delta x_p.$$

The second equation is the *relative* walking distance. Solving both constraints yields

$$\Delta x_p = -\frac{m}{M+m}s, \quad \Delta x_a = \frac{M}{M+m}s.$$

The same mass-weighted balance applies to velocities. A tether to a wall introduces external impulse and changes the problem; zero net force does not require each part to stay still.

Chapter support: Physics I manuscript, Chapter 9

Walking on a free platform: statement



Example 9.8 — complete statement

A 60 kg person walks 3.20 m to the right relative to a 180 kg platform. Initially both are at rest. Idealize the platform as translating horizontally without rotation, and neglect horizontal external force. (a) Find the laboratory displacements. (b) During a portion of the walk, the person moves at 0.400 m s^{-1} relative to the platform. Find both laboratory velocities and the combined kinetic energy. (c) Explain the final motion when the person stops relative to the platform.

Chapter support: Physics I manuscript, Chapter 9; Example 9.8

Platform: displacement, motion, and energy



Example 9.8 — solution. Total mass is 240 kg. **(a)** From the two displacement constraints,

$$\Delta x_p = -0.800 \text{ m}, \quad \Delta x_a = +2.40 \text{ m}.$$

Their difference is 3.20 m and $60(2.40) + 180(-0.800) = 0$. **(b)** Solve $v_a - v_p = 0.400$ and $60v_a + 180v_p = 0$:

$$v_a = +0.300 \text{ m s}^{-1}, \quad v_p = -0.100 \text{ m s}^{-1},$$

$$K = \frac{1}{2}(60)(0.300)^2 + \frac{1}{2}(180)(0.100)^2 = 3.60 \text{ J}.$$

(c) On stopping relative motion, both share V . Zero momentum gives $240V = 0$: both stop in the laboratory. Internal stores supplied the motion energy; stopping removes it. The combined CM stays fixed throughout.

Chapter support: Physics I manuscript, Chapter 9; Example 9.8

Snapshots preserve the same weighted center

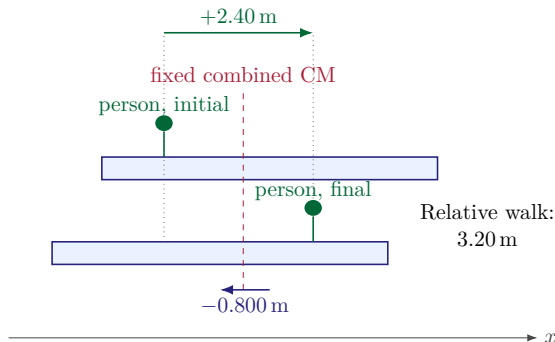


Figure 9.9: Horizontal displacement bookkeeping for Example 9.8. Initial platform and person coordinates are 2.70 and 1.00 m; final coordinates are 1.90 and 3.40 m. Both snapshots therefore give $X_{\text{CM}} = 2.275$ m. Vertical separation distinguishes snapshots and represents no vertical motion.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.9

Impulse is a signed vector integral



For a constant-mass particle, $\mathbf{p} = m\mathbf{v}$ and $\mathbf{F}_{\text{net}} = d\mathbf{p}/dt$. Integrating gives

$$\mathbf{J} = \int_{t_i}^{t_f} \mathbf{F}_{\text{net}}(t) dt = \mathbf{p}_f - \mathbf{p}_i.$$

Each component is a signed area: $J_x = \int F_x dt$ and $J_y = \int F_y dt$. Use seconds when force is in newtons.

State versus accumulated action

Momentum describes a state; impulse describes a force's accumulated action. Both use $\text{Ns} = \text{kg m s}^{-1}$. At intermediate t , $\mathbf{p}(t) = \mathbf{p}_i + \int_{t_i}^t \mathbf{F}(\tau) d\tau$.

A finite force jump changes the momentum slope, not its continuity.

Chapter support: Physics I manuscript, Chapter 9

Reversing force components: statement



Example 9.10 — complete statement

A particle of mass 0.500 kg starts with velocity $(-1.00, 2.00) \text{ m s}^{-1}$. During a 0.100 s interaction its net force components are

$$F_x(t) = \begin{cases} +60.0 \text{ N}, & 0 \leq t < 0.0400 \text{ s}, \\ -20.0 \text{ N}, & 0.0400 \leq t \leq 0.100 \text{ s}, \end{cases} \quad F_y(t) = +30.0 \text{ N}.$$

Find the net impulse, final velocity, and instant when the horizontal velocity first becomes zero.

Chapter support: Physics I manuscript, Chapter 9; Example 9.10

Signed pulse: integrate first, then add momentum



Example 9.10 — solution. The signed rectangular areas give

$$J_x = 60(0.040) - 20(0.060) = 1.20 \text{ N s}, \quad J_y = 30(0.100) = 3.00 \text{ N s}.$$

Thus $\mathbf{J} = (1.20, 3.00) \text{ N s}$, with magnitude 3.23 N s and direction 68.2° from $+x$.

$$\mathbf{p}_i = (-0.500, 1.00), \quad \mathbf{p}_f = \mathbf{p}_i + \mathbf{J} = (0.700, 4.00)$$

in kg m s^{-1} , so $\mathbf{v}_f = (1.40, 8.00) \text{ m s}^{-1}$. Initially $p_x(t) = -0.500 + 60t$ in SI units. Hence

$$t_{v_x=0} = 0.500/60 = 0.00833 \text{ s}.$$

At 40 ms , $p_x = 1.90$; it then falls to 0.700 , without a second reversal. Adding the final impulse to the initial momentum checks both components.

Chapter support: Physics I manuscript, Chapter 9; Example 9.10

Momentum remains continuous at the force switch

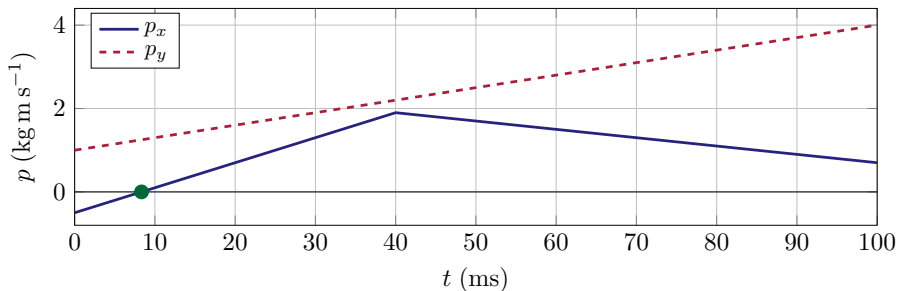


Figure 9.12: Integrating the component curves gives continuous momenta even though the force model has a discontinuity. The green point marks zero horizontal velocity; the force reversal occurs later, at 40 ms.

Integrating scalar magnitudes removes cancellation: here $\int |F_x| dt = 3.60 \text{ N s}$, whereas $J_x = 1.20 \text{ N s}$.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.12

One impulse, two force histories



The average force is $F_{\text{avg}} = J/T$. Duration and a specified pulse shape are needed to infer a peak; impulse alone does not determine it.

Example 9.9 — complete statement

A one-dimensional interaction delivers 14.4 N s of positive impulse. Compare (A) a triangular force pulse of duration 12 ms and peak 2.40 kN with (B) a half-sine pulse of duration 24 ms ,

$$F_B(t) = F_0 \sin\left(\frac{\pi t}{T}\right), \quad 0 \leq t \leq T.$$

Find F_0 so that both impulses are equal, and compare their average and peak forces.

Chapter support: Physics I manuscript, Chapter 9; Example 9.9

Equal areas: compare mean and peak



Example 9.9 — solution. Convert milliseconds and kilonewtons to SI units.

$$J_A = \frac{1}{2}(0.012)(2400) = 14.4 \text{ N s}, \quad \bar{F}_A = 14.4/0.012 = 1200 \text{ N}.$$

Integrate the stipulated half-sine rather than treating its peak as constant:

$$J_B = F_0 \left[-\frac{T}{\pi} \cos \frac{\pi t}{T} \right]_0^T = \frac{2F_0 T}{\pi}.$$

$$F_0 = \frac{14.4\pi}{2(0.024)} = 942 \text{ N}, \quad \bar{F}_B = 600 \text{ N}.$$

Pulse A peaks at 2400 N; pulse B peaks at 942 N. The longer model pulse has lower mean and peak here. Both have the same endpoint momentum change, while intermediate velocities and stresses can differ.

Chapter support: Physics I manuscript, Chapter 9; Example 9.9

Constructed force curves, equal impulse

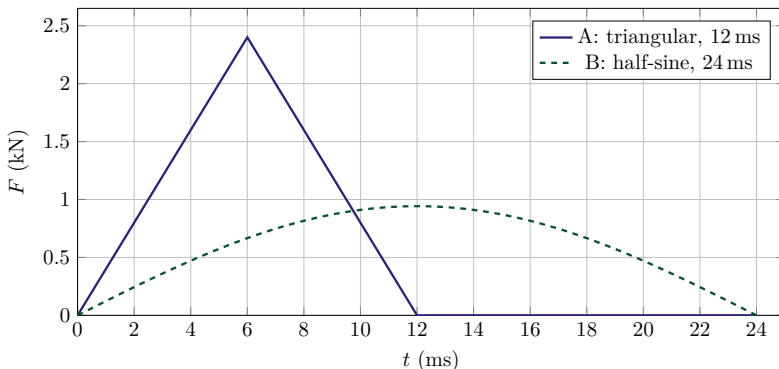


Figure 9.10: Analytical force–time models from Example 9.9. Both enclosed areas are 14.4 N s , although duration, average force, and peak force differ. These are constructed teaching curves, not measured impact data.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.10

Contact impulse and net impulse differ



Choose upward positive. A sensor's contact force $N(t)$ acts alongside weight, so

$$J_{\text{contact}} - mgT = m(v_f - v_i) = J_{\text{net}}.$$

Gravity can be neglected only after comparing mgT with the required accuracy.

Example 9.11 — complete statement

A 0.200 kg ball reaches a horizontal compliant surface at $v_i = -3.00\text{ m s}^{-1}$ and leaves at $v_f = +2.40\text{ m s}^{-1}$. Contact lasts $T = 0.0600\text{ s}$. Use $g = 9.81\text{ m s}^{-2}$. Find the net and contact impulses and the mean contact force. If a half-sine models the contact force, find its peak.

Chapter support: Physics I manuscript, Chapter 9; Example 9.11

Rebound: all requested impulses and forces



Example 9.11 — solution.

$$J_{\text{net}} = 0.200[2.40 - (-3.00)] = 1.08 \text{ N s},$$

$$J_g = -0.200(9.81)(0.0600) = -0.11772 \text{ N s},$$

$$J_{\text{contact}} = 1.19772 \text{ N s}, \quad \bar{N} = J_{\text{contact}}/T = 19.962 \text{ N} \simeq 20.0 \text{ N}.$$

For $N(t) = N_0 \sin(\pi t/T)$, its area is $2N_0 T/\pi$, giving

$$N_0 = \pi J_{\text{contact}}/(2T) = 31.4 \text{ N}.$$

The mean net force is 18.0 N; the difference is the weight 1.962 N. Ignoring gravity underestimates contact impulse by 9.83% of its true value. The contact peak is a property of this stipulated pulse shape.

Chapter support: Physics I manuscript, Chapter 9; Example 9.11

Net force can point down during an upward impulse

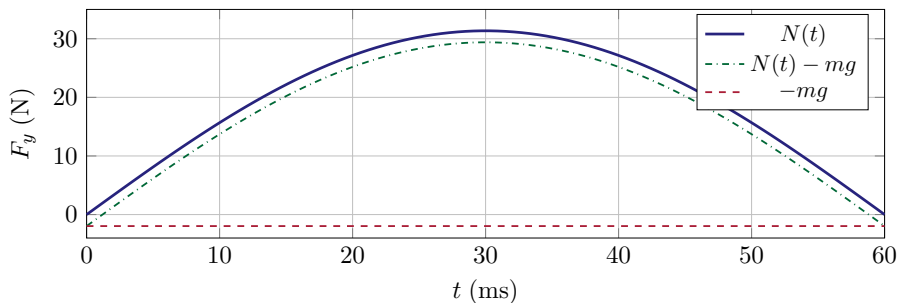


Figure 9.14: Contact and net-force models for Example 9.11. Subtracting the constant weight changes the signed area by 0.11772 N s and produces small negative net-force intervals near the endpoints.

The symmetric pulse implies $\Delta y = (v_i + v_f)T/2 = -0.0180 \text{ m}$. The model permits different contact and separation heights; it is not an equal-height rebound model for an unchanged surface.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.14

When is momentum conservation justified?



For the same fixed system before and after an interaction,

$$\mathbf{P}_f - \mathbf{P}_i = \mathbf{J}_{\text{ext}} = \int \sum \mathbf{F}_{\text{ext}} dt.$$

Conserve only components with negligible external impulse over the selected interval. Internal force can be large without changing the system total.

A quantitative tolerance test

A 0.12 N force bound over 8 ms gives $|J_x| \leq 9.6 \times 10^{-4} \text{ N s}$. Relative to a nonzero 0.5 N s scale this is 0.192%: acceptable for a 2% demonstration, insufficient to justify a 0.1% claim.

Use a characteristic interaction momentum when total initial momentum is zero; do not divide by zero to form a percentage.

Chapter support: Physics I manuscript, Chapter 9

Two carts latch: the system and statement



Collision models. Sticking imposes a common final velocity. Elasticity instead imposes conserved kinetic energy. Momentum conservation requires its own external-impulse test in either case.

Example 9.13 — complete statement

A cart of mass $m_1 = 0.80 \text{ kg}$ moves at $u_1 = +2.50 \text{ m s}^{-1}$. A second cart of mass $m_2 = 1.20 \text{ kg}$ moves at $u_2 = -0.50 \text{ m s}^{-1}$. They collide on a horizontal guide and latch together. The external horizontal impulse during the collision is negligible. Find their common final velocity.

Choose both carts and the laboratory $+x$ axis. Their initial momentum is $P_i = 0.80(2.50) + 1.20(-0.50) = 1.40 \text{ N s}$.

Chapter support: Physics I manuscript, Chapter 9; Example 9.13

Latching: momentum and a separate energy audit



Example 9.13 — solution. The latched mass is $M = 2.00$ kg, so

$$P_f = Mv_f = P_i, \quad \boxed{v_f = +0.700 \text{ m s}^{-1}}.$$

A positive weighted mean lies between the incoming signed velocities. Common final velocity does not mean rest in the laboratory. Compute energy from those actual velocities:

$$K_i = \frac{1}{2}(0.80)(2.50)^2 + \frac{1}{2}(1.20)(0.50)^2 = 2.65 \text{ J},$$

$$K_f = \frac{1}{2}(2.00)(0.700)^2 = 0.490 \text{ J}, \quad K_i - K_f = 2.16 \text{ J}.$$

The decrease goes to internal energy and other channels. Momentum has remained 1.40 N s ; it does not require kinetic energy to remain constant.

Chapter support: Physics I manuscript, Chapter 9; Example 9.13

The CM frame subtracts one common velocity



In a second inertial frame moving at constant $\mathbf{U}$,

$$\mathbf{v}'_i = \mathbf{v}_i - \mathbf{U}, \quad \mathbf{P}' = \mathbf{P} - M\mathbf{U}.$$

Choose $\mathbf{U} = \mathbf{V}_{\text{CM}} = \mathbf{P}/M$. Then $\mathbf{P}' = 0$, but individual momenta generally remain nonzero.

The retained latching example

Here $V_{\text{CM}} = 0.700 \text{ m s}^{-1}$. Incoming velocities become $u'_1 = +1.80$ and $u'_2 = -1.20 \text{ m s}^{-1}$. Their momenta are $+1.44$ and -1.44 N s . After latching both CM-frame velocities are zero.

The value of momentum changes with frame; conservation holds in each inertial frame for the same fixed isolated system.

Chapter support: Physics I manuscript, Chapter 9

Same collision in two inertial frames

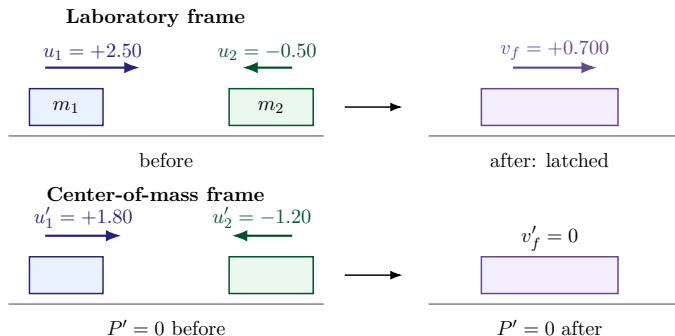


Figure 9.16: The same perfectly inelastic collision viewed in two inertial frames. The coupled carts move at $+0.700 \text{ m s}^{-1}$ in the laboratory but are at rest after collision in the center-of-mass frame. Velocity labels are in m s^{-1} ; these arrows indicate directions and have schematic lengths.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.16

Elastic contact: derive the separating branch



For an isolated elastic one-dimensional encounter, momentum and kinetic energy give

$$m_1(u_1 - v_1) = m_2(v_2 - u_2),$$

$$m_1(u_1 - v_1)(u_1 + v_1) = m_2(v_2 - u_2)(v_2 + u_2).$$

For the nonzero momentum-transfer branch, division yields

$$v_2 - v_1 = u_1 - u_2.$$

Together with momentum, this determines both velocities. The removed branch is $v_1 = u_1$, $v_2 = u_2$: no interaction. For body 1 initially left of body 2 with $u_1 > u_2$, a separating contact needs $v_2 \geq v_1$; unchanged velocities keep closing and do not describe that outcome. If $u_1 = u_2$, no ordinary approach exists.

Chapter support: Physics I manuscript, Chapter 9

Same incoming state: the elastic outcome



Example 9.14 — complete statement

Use the same initial state as Example 9.13: $m_1 = 0.80 \text{ kg}$, $m_2 = 1.20 \text{ kg}$, $u_1 = +2.50 \text{ m s}^{-1}$, $u_2 = -0.50 \text{ m s}^{-1}$.

Complete comparison. Sticking gave $v_f = +0.700 \text{ m s}^{-1}$ and $K_f = 0.490 \text{ J}$. For elastic separation solve

$$0.80v_1 + 1.20v_2 = 1.40, \quad v_2 - v_1 = 3.00.$$

Thus $v_1 = -1.10 \text{ m s}^{-1}$, $v_2 = +1.90 \text{ m s}^{-1}$. The checks are $P_f = -0.880 + 2.280 = 1.40 \text{ N s}$ and $K_f = \frac{1}{2}(0.80)(1.10)^2 + \frac{1}{2}(1.20)(1.90)^2 = 2.65 \text{ J}$. Subtracting 0.700 gives $v'_1 = -1.80$, $v'_2 = +1.20 \text{ m s}^{-1}$: each incoming CM velocity reverses.

Chapter support: Physics I manuscript, Chapter 9; Example 9.14

Restitution supplies an extra contact condition



For an initially closing one-dimensional contact, let $g = u_1 - u_2 > 0$ and $M = m_1 + m_2$. Define

$$e_r = \frac{v_2 - v_1}{g}, \quad V = \frac{m_1 u_1 + m_2 u_2}{M}.$$

Combine $v_2 - v_1 = e_r g$ with momentum to obtain

$$\boxed{v_1 = V - \frac{m_2}{M} e_r g}, \quad \boxed{v_2 = V + \frac{m_1}{M} e_r g}.$$

For the passive model $0 \leq e_r \leq 1$: $e_r = 1$ gives elastic separation; $e_r = 0$ gives zero normal separating speed. Permanent sticking also needs adhesion or a latch. Restitution is a model parameter, not a conservation law. In oblique impacts it usually concerns a contact-normal component, not total speed magnitudes.

Chapter support: Physics I manuscript, Chapter 9

Partial head-on impact: statement and model



Example 9.15 — complete statement

Two carts approach on a straight frictionless track. Their masses and incoming velocities are $m_1 = 1.50 \text{ kg}$, $m_2 = 2.50 \text{ kg}$, $u_1 = +4.00 \text{ m s}^{-1}$, and $u_2 = -1.00 \text{ m s}^{-1}$. The restitution coefficient is $e_r = 0.600$. Find both final velocities, verify separation and momentum conservation, and determine the translational kinetic-energy decrease.

Model. Put cart 1 initially on the left. The approach speed is $g = 4.00 - (-1.00) = 5.00 \text{ m s}^{-1}$. With both carts selected,

$$P = 1.50(4.00) + 2.50(-1.00) = 3.50 \text{ N s}, \quad V = P/4.00 = 0.875 \text{ m s}^{-1}.$$

Chapter support: Physics I manuscript, Chapter 9; Example 9.15

Partial impact: velocities, separation, and energy



Example 9.15 — solution. Use $e_r g = 0.600(5.00) = 3.00$:

$$v_1 = 0.875 - \frac{2.50}{4.00}(3.00) = \boxed{-1.00 \text{ m s}^{-1}},$$

$$v_2 = 0.875 + \frac{1.50}{4.00}(3.00) = \boxed{+2.00 \text{ m s}^{-1}}.$$

The positive separation speed is 3.00; $P_f = 1.50(-1) + 2.50(2) = 3.50 \text{ N s}$.

$$K_i = 13.25 \text{ J}, \quad K_f = 5.75 \text{ J}, \quad \boxed{K_i - K_f = 7.50 \text{ J}}.$$

As an independent algebraic check, substituting the two velocities gives $K_i - K_f = \frac{1}{2}\mu(1 - e_r^2)g^2$, with $\mu = 0.9375 \text{ kg}$: again 7.50 J. Separating bodies can still lose kinetic energy.

Chapter support: Physics I manuscript, Chapter 9; Example 9.15

Ballistic pendulum: two distinct intervals



The short capture uses horizontal momentum. The subsequent ideal swing uses mechanical energy; suspension and gravity exert external forces during that swing.

Example 9.17 — complete statement

A 12.0 g projectile embeds in a 1.988 kg catcher initially at rest at the bottom of a pendulum. The combined mass can be idealized as a point at distance $L = 1.20$ m from the pivot. It subsequently rises to 30.0° from the vertical. Use $g = 9.81 \text{ m s}^{-2}$. (a) Find the speed immediately after impact and the projectile's initial speed. (b) Calculate the kinetic energy removed during capture. (c) State which conservation law applies in each stage.

Chapter support: Physics I manuscript, Chapter 9; Example 9.17

Pendulum: infer the two speeds



Example 9.17 — solution (a). The point catcher rises by

$$h = L(1 - \cos \theta) = 1.20(1 - \cos 30.0^\circ) = 0.16077 \text{ m.}$$

On the lossless swing, $\frac{1}{2}(m + M)V^2 = (m + M)gh$, hence

$$V = \sqrt{2gh} = 1.7760 \text{ m s}^{-1}.$$

Convert 12.0 g to $m = 0.0120 \text{ kg}$; the combined mass is 2.000 kg. During capture the string is vertical and horizontal external impulse is neglected:

$$mu = (m + M)V, \quad u = \frac{2.000}{0.0120}(1.7760) = 296.0 \text{ m s}^{-1}.$$

Reinsert V into the swing energy and u into the capture momentum. Catcher rotation or dissipative swing loss would require a different inversion.

Chapter support: Physics I manuscript, Chapter 9; Example 9.17

Pendulum: where did the incoming energy go?

Example 9.17 — solutions (b) and (c). Keep guard digits from the speed calculation:

$$K_i = \frac{1}{2}(0.0120)(296.0057)^2 = 525.716 \text{ J},$$

$$K_c = \frac{1}{2}(2.000)(1.776034)^2 = 3.15430 \text{ J},$$

$$K_i - K_c = 522.562 \text{ J}.$$

Only $K_c/K_i = m/(M + m) = 0.00600 = 0.600\%$ remains as translational kinetic energy after embedding. That retained energy becomes gravitational potential energy on the swing.

Two balances, not one

Capture conserves horizontal momentum approximately and dissipates kinetic energy. The ideal swing conserves mechanical energy and does not conserve projectile–catcher momentum. Total energy is not destroyed by the capture loss.

Chapter support: Physics I manuscript, Chapter 9; Example 9.17

Planar momentum: conserve components



Momentum supplies $P_{x,f} = P_{x,i}$ and $P_{y,f} = P_{y,i}$. Two final velocity vectors contain four components, so an arbitrary two-body outcome needs additional information.

Example 9.18 — complete statement

A parent body of mass 5.0 kg moves with velocity

$$\mathbf{v}_0 = (4.0 \hat{\mathbf{x}} + 1.0 \hat{\mathbf{y}}) \text{ m s}^{-1}.$$

It separates rapidly into fragments of masses 2.0 kg and 3.0 kg. During the short separation interval the external impulse is negligible. The 2.0 kg fragment is measured to have

$$\mathbf{v}_A = (7.0 \hat{\mathbf{x}} + 3.0 \hat{\mathbf{y}}) \text{ m s}^{-1}.$$

Find the velocity of the other fragment and the change in total kinetic energy.

Chapter support: Physics I manuscript, Chapter 9; Example 9.18

Fragmentation: vector subtraction and energy



Example 9.18 — solution. In kg m s^{-1} ,

$$\mathbf{P}_i = 5(4, 1) = (20, 5), \quad \mathbf{p}_A = 2(7, 3) = (14, 6),$$

$$\mathbf{p}_B = (6, -1), \quad \boxed{\mathbf{v}_B = (2.00, -0.333) \text{ m s}^{-1}}.$$

Check both components: $(14, 6) + (6, -1) = (20, 5)$.

$$K_i = \frac{1}{2}(5)(4^2 + 1^2) = 42.5 \text{ J},$$

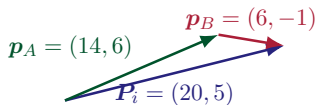
$$K_f = \frac{1}{2}(2)(7^2 + 3^2) + \frac{1}{2}(3)(2^2 + 1/9) = 64.17 \text{ J}.$$

Thus $\boxed{\Delta K = +21.67 \text{ J}}$ must be supplied by another energy store. A mechanism releasing only 20.0 J would be short by 1.67 J, even though the proposed momenta balance.

Chapter support: Physics I manuscript, Chapter 9; Example 9.18

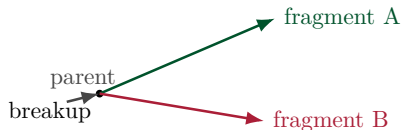
A momentum polygon is not a trajectory

Momentum polygon



vectors may be translated
head-to-tail

Possible spatial sketch



trajectory directions come from
velocities in physical space

Figure 9.23: Momentum-vector bookkeeping for Example 9.18 compared with a spatial trajectory sketch. The momentum polygon is an addition diagram; its arrow placement is not a record of where the fragments travelled. Momentum components are in kg m s^{-1} ; one drawn centimeter represents 5 N s before the common display scale of 0.9. Spatial arrow directions match the fragment velocities, but spatial lengths are schematic.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.23

Equal-mass elastic scattering: a special geometry



For an equal stationary target, momentum gives $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$ and elasticity gives $u^2 = v_1^2 + v_2^2$. Squaring the vector sum leaves $2\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. Both nonzero outgoing velocities are therefore perpendicular; a zero vector has no direction.

Example 9.19 — complete statement

A 0.400 kg projectile moves at 5.00 m s^{-1} along $+x$ toward an equal stationary target. The collision is elastic and planar. The projectile's outgoing direction is measured to be 35.0° above $+x$. Find the two outgoing speeds and the target's direction, and verify both conserved quantities.

Chapter support: Physics I manuscript, Chapter 9; Example 9.19

Scattering: solve speeds and verify the vectors



Example 9.19 — solution. The target goes below $+x$ at -55.0° . The component equations are

$$v_1 \sin 35^\circ - v_2 \cos 35^\circ = 0, \quad 5 = v_1 \cos 35^\circ + v_2 \sin 35^\circ.$$

Therefore $v_1 = 5 \cos 35^\circ = 4.096 \text{ m s}^{-1}$ and $v_2 = 5 \sin 35^\circ = 2.868 \text{ m s}^{-1}$.

$$\mathbf{v}_1 = (3.3551, 2.3492), \quad \mathbf{v}_2 = (1.6449, -2.3492)$$

in m s^{-1} . Their sum $(5.0000, 0)$ verifies both momentum components after multiplication by 0.400 kg . The energies are 3.3551 J and 1.6449 J , summing to 5.0000 J initially. Equal mass and elasticity alone do not fix a deflection; the measured angle supplies the remaining information.

Chapter support: Physics I manuscript, Chapter 9; Example 9.19

The circle is in velocity space

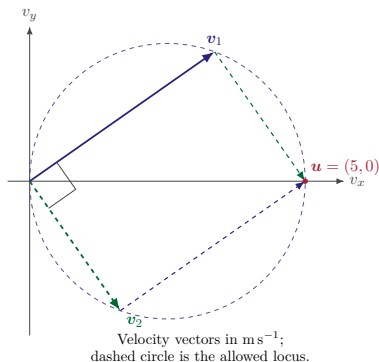


Figure 9.24: Velocity-space geometry for equal-mass elastic scattering. Either outgoing vector lies on the circle with incident velocity as a diameter. The vectors sum to the incident velocity and meet at a right angle when both are nonzero. This is a vector diagram, not a spatial trajectory map.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.24

A useful computational collision audit



For the same positive masses and common inertial frame, calculate before and after:

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i, \quad K = \frac{1}{2} \sum_i m_i |\mathbf{v}_i|^2, \quad \mathbf{V}_{\text{CM}} = \mathbf{P}/M.$$

Return $\Delta \mathbf{P} = \mathbf{P}_f - \mathbf{P}_i$ and $\Delta K = K_f - K_i$; the code does not decide whether the physical system is isolated.

- Reorder particle labels: every total must remain unchanged.
- Check $\Delta \mathbf{P} = M(\mathbf{V}_f - \mathbf{V}_i)$, even for a nonzero residual.
- The elastic carts on slide 40 must return $(\Delta P, \Delta K) = (0, 0)$; latching on slide 36 returns $(0, -2.16 \text{ J})$.
- A common frame shift leaves $\Delta \mathbf{P}$ unchanged. Energy residual obeys $\Delta K' = \Delta K - \mathbf{U} \cdot \Delta \mathbf{P}$ for fixed mass.

Chapter support: Physics I manuscript, Chapter 9

Practice 2: sampled force and trapezoids



Between adjacent samples, the trapezoidal model integrates the straight joining segment:

$$J_{\text{trap}} = \sum_k \frac{F_k + F_{k+1}}{2} (t_{k+1} - t_k).$$

Problem 14 — student practice

Sampled force data are recorded at $t = (0, 2, 4, 6, 8)$ ms with $F = (0, 400, 800, 400, 0)$ N. Use the trapezoidal rule to estimate the impulse. Convert the 2 ms spacing to 0.002 s. The approximation describes the sample interpolant; it does not establish the unmeasured force between readings.

Chapter support: Physics I manuscript, Chapter 9; Problem 14

Sampled force: solution and acquisition limits



Problem 14 — solution. Sum the four trapezoids:

$$J = 0.002 \left[\frac{0 + 400}{2} + \frac{400 + 800}{2} + \frac{800 + 400}{2} + \frac{400 + 0}{2} \right] = \boxed{3.20 \text{ N s}}.$$

The contributions 0.40, 1.20, 1.20, 0.40 N s also form a triangular interpolant of height 800 N and base 0.008 s.

Two answered diagnostics

A constant +20 N baseline adds $bT = 0.160 \text{ N s}$ over this record, a 5% bias; finer quadrature does not remove it. Estimate a stable baseline from a known force-free pre-trigger interval.

A 3 ms pulse sampled every 5 ms can be missed entirely. Interpolation cannot recover unmeasured impulse without extra data or a validated model.

Chapter support: Physics I manuscript, Chapter 9; Problem 14

A controlled convergence study



For the *known* $T = 24 \text{ ms}$ half-sine on slides 28–30, hold the model fixed and sample both endpoints plus $N - 1$ interior points.

$$J_N = \frac{F_0 T}{N} \cot \frac{\pi}{2N}, \quad \frac{J_N}{14.4} = z \cot z, \quad z = \frac{\pi}{2N}.$$

N	spacing (ms)	J_N (N s)	error (%)
4	6.000	13.652	−5.19
8	3.000	14.214	−1.29
16	1.500	14.354	−0.321
32	0.750	14.388	−0.0803

Here error means $100(J_N - 14.4)/14.4$. Since $z \cot z = 1 - z^2/3 + \dots$, doubling N reduces the leading error by four. A known smooth-model benchmark validates the algorithm, not an under-resolved sensor signal.

Chapter support: Physics I manuscript, Chapter 9

Synthetic cart data: define an audit task



For independent inputs q_j , first-order propagation is

$$\sigma_f^2 = \sum_j \left(\frac{\partial f}{\partial q_j} \right)^2 \sigma_{q_j}^2.$$

Reused masses correlate the two totals. Propagate each residual directly.

Teaching extension M1 — cart audit.

Compute the momentum and kinetic-energy changes and their standard uncertainties. The independent inputs below reuse each mass before and after.

Quantity	Cart 1	Cart 2
m (kg)	0.500 ± 0.002	0.750 ± 0.002
u (m s ⁻¹)	$+0.600 \pm 0.006$	-0.100 ± 0.006
v (m s ⁻¹)	-0.238 ± 0.006	$+0.457 \pm 0.006$

These are synthetic values, not measurements.

Chapter support: [Physics I manuscript, Chapter 9; synthetic-data teaching extension](#)

Momentum residual: propagate the shared inputs



M1 — momentum solution.

$$P_i = 0.22500 \text{ N s}, \quad P_f = 0.22375 \text{ N s}, \quad \boxed{\Delta P = -0.00125 \text{ N s}}.$$

Use $\Delta P = \sum_i m_i(v_i - u_i)$, not independent before/after estimates:

$$\sigma_{\Delta P}^2 = \sum_i \left[(v_i - u_i)^2 \sigma_{m_i}^2 + m_i^2 (\sigma_{v_i}^2 + \sigma_{u_i}^2) \right].$$

$$\sigma_{\Delta P} = 0.007908854 \text{ N s}, \quad \boxed{|\Delta P|/\sigma_{\Delta P} = 0.158}.$$

The residual is consistent with zero at this stipulated precision. It does not establish a real momentum loss or exclude calibration/systematic effects. An external-impulse model remains part of the experiment.

Chapter support: Physics I manuscript, Chapter 9; synthetic-data teaching extension

Energy residual: a separate statistical question



M1 — energy solution.

$$K_i = 0.09375 \text{ J}, \quad K_f = 0.09248 \text{ J}, \quad \boxed{\Delta K = -0.001271 \text{ J}}.$$

For $\Delta K = \frac{1}{2} \sum_i m_i (v_i^2 - u_i^2)$, propagate the same independent inputs:

$$\sigma_{\Delta K}^2 = \sum_i \left[\frac{1}{4} (v_i^2 - u_i^2)^2 \sigma_{m_i}^2 + m_i^2 (v_i^2 \sigma_{v_i}^2 + u_i^2 \sigma_{u_i}^2) \right].$$

$$\sigma_{\Delta K} = 0.002883 \text{ J}, \quad |\Delta K|/\sigma_{\Delta K} = 0.44.$$

The central 1.36% decrease does not distinguish the event from elastic at this precision. Report both balances and the uncertainty assumptions; a common timing calibration would require additional covariance terms.

Chapter support: Physics I manuscript, Chapter 9; synthetic-data teaching extension

Core synthesis: a hierarchy of balances



- **Geometry:** $R_{\text{CM}} = M^{-1} \int \mathbf{r} \, dm$ needs a correct mass inventory.
- **Fixed-system motion:** $\mathbf{P} = M\mathbf{V}_{\text{CM}}$ and external force controls its rate of change.
- **Interaction:** external impulse controls $\Delta\mathbf{P}$; kinetic energy needs a separate balance.
- **Validation:** check components, units, limits, sampling, and uncertainty.

Before accepting a solution, name the system, frame, external forces, interval, and energy model.

Next: rotation about a fixed axis

Chapter 10 introduces how mass distribution relative to an axis controls rotational motion. The advanced pages now deepen CM energy and open-system momentum flux. Revisit the manuscript's unselected examples and Problems 1–41 for further practice.

Chapter support: Physics I manuscript, Chapter 9

Advanced: separate translation and relative motion



Define $\mathbf{v}'_i = \mathbf{v}_i - \mathbf{V}$ with $\mathbf{V} = \mathbf{P}/M$. Then

$$\sum_i m_i \mathbf{v}'_i = \mathbf{P} - M\mathbf{V} = \mathbf{0}.$$

Expand the kinetic energy using that *mass-weighted* zero:

$$\begin{aligned} K &= \sum_i \frac{1}{2} m_i |\mathbf{V} + \mathbf{v}'_i|^2 \\ &= \frac{1}{2} M V^2 + \mathbf{V} \cdot \sum_i m_i \mathbf{v}'_i + \sum_i \frac{1}{2} m_i |\mathbf{v}'_i|^2 \\ &= \boxed{\frac{p^2}{2M} + K_{\text{rel}}}. \end{aligned}$$

This is an instantaneous nonrelativistic identity, with no collision or elasticity assumption. In a fixed isolated collision M and $\mathbf{P}$ are unchanged: all change in translational kinetic energy occurs in relative motion.

Chapter support: Physics I manuscript, Chapter 9

Reduced mass, energy bounds, and frame shifts



For two masses let $\mathbf{w} = \mathbf{v}_1 - \mathbf{v}_2$ and $M = m_1 + m_2$:

$$\mathbf{v}'_1 = \frac{m_2}{M} \mathbf{w}, \quad \mathbf{v}'_2 = -\frac{m_1}{M} \mathbf{w}, \quad \mu = \frac{m_1 m_2}{M}.$$

Substitution into the relative-energy sum gives

$$\boxed{K_{\text{rel}} = \frac{1}{2} \mu w^2}, \quad \boxed{K \geq P^2 / (2M)}.$$

Common final velocity attains the bound. Sticking removes the initial relative energy; restitution retains e_r^2 of it in the one-dimensional model.

$$K^* = K - \mathbf{U} \cdot \mathbf{P} + \frac{1}{2} M U^2.$$

If before and after have the same M and $\mathbf{P}$, the boost terms cancel in their energy difference. Absolute kinetic energies are frame dependent; the isolated inelastic decrease is unchanged.

Chapter support: Physics I manuscript, Chapter 9

Advanced challenge: an independent CM-energy check



Problem 35 — student practice

For two particles of masses 2.0 and 1.0 kg with velocities $+3.0$ and -1.0 m s^{-1} , calculate the total kinetic energy, the CM translational kinetic energy, and K_{rel} . Verify $K_{\text{rel}} = \frac{1}{2}\mu(v_1 - v_2)^2$.

Solution. $M = 3.0 \text{ kg}$ and $P = 2(3) + 1(-1) = 5 \text{ N s}$:

$$K = \frac{1}{2}(2)(3)^2 + \frac{1}{2}(1)(-1)^2 = \boxed{9.50 \text{ J}},$$

$$K_{\text{CM}} = \frac{5^2}{2(3)} = \boxed{25/6 \text{ J}}, \quad K_{\text{rel}} = 9.50 - 25/6 = \boxed{16/3 \text{ J}}.$$

Independently, $\mu = 2/3 \text{ kg}$ and $v_1 - v_2 = 4 \text{ m s}^{-1}$ give $\frac{1}{2}(2/3)(4)^2 = 16/3 \text{ J}$. The CM velocities $4/3$ and $-8/3 \text{ m s}^{-1}$ also sum to zero momentum and reproduce the same relative energy.

Chapter support: Physics I manuscript, Chapter 9; Problem 35

An open vehicle and a fixed material system

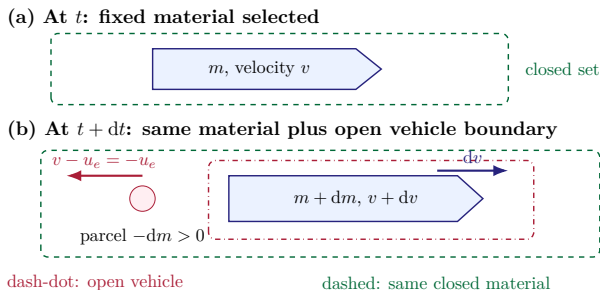


Figure 9.28: Schematic differential rocket bookkeeping in the instantaneously comoving inertial frame, where the initial vehicle velocity is $v = 0$. Therefore the expelled parcel points backward in this drawing. In an arbitrary laboratory frame $v - u_e$ can have either sign. The dashed boundary follows the fixed selected material; the dash-dot boundary excludes the expelled parcel and defines an open vehicle subsystem. Arrow lengths and distances are schematic.

Chapter support: Physics I manuscript, Chapter 9; Figure 9.28

Derive the rocket equation with momentum flux



In one inertial frame the vehicle has mass m and velocity v ; $dm < 0$ is its mass change. The expelled positive mass $-dm$ has velocity $v - u_e$, where $u_e > 0$ is the backward relative exhaust speed.

$$mv = (m + dm)(v + dv) + (-dm)(v - u_e).$$

Discard the second-order product $dm dv$ and cancel mv :

$$m dv = -u_e dm, \quad \boxed{\Delta v = u_e \ln(m_i/m_f)}.$$

The logarithm follows from $\int_{m_i}^{m_f} dm/m$; its mass ratio is dimensionless. This integral assumes constant u_e and zero external force. With positive outflow $\dot{m}_e = -\dot{m}$, the more general balance is $m\dot{v} = F_{\text{ext}} + u_e \dot{m}_e$. Differentiating mv alone for an open vehicle omits the exhaust momentum.

Chapter support: Physics I manuscript, Chapter 9

Rocket mass-flow audit: statement and solution



Example 9.22 — complete statement

A spacecraft in deep space is idealized as one-dimensional, with negligible external force. Its mass decreases from $m_i = 1200 \text{ kg}$ to $m_f = 900 \text{ kg}$ while the effective exhaust speed is constant at $u_e = 2200 \text{ m s}^{-1}$. Estimate the ideal velocity increment. If propellant is expelled at 3.0 kg s^{-1} , also find the ideal thrust magnitude and burn duration under these assumptions.

Solution. With zero external force and backward relative exhaust,

$$\Delta v = 2200 \ln(1200/900) = 632.9 \text{ m s}^{-1},$$

$$T = u_e \dot{m}_e = 6600 \text{ N},$$

$$t_b = (1200 - 900)/3.0 = 100 \text{ s}.$$

The mass ratio exceeds one and the increment is positive. If the masses were equal, the logarithm and increment would vanish. Burn time sets the time history, but not this ideal force-free endpoint increment.

Chapter support: Physics I manuscript, Chapter 9; Example 9.22

Inverse sizing needs a burn-by-burn mass ledger



Invert the force-free logarithmic relation:

$$m_i = m_f \exp(\Delta v / u_e), \quad m_{\text{prop}} = m_f [\exp(\Delta v / u_e) - 1].$$

Burnout mass includes every component still attached, including dry structure. Propellant is the mass expelled at the specified exhaust speed.

Staging changes the next burn

Discarding empty structure with no relative separation speed produces no instantaneous velocity jump. It reduces the mass accelerated in the subsequent burn. Apply a separate logarithm to each burn using its own attached initial and final masses.

A launch-to-payload mass ratio would incorrectly count jettisoned dry structure as expelled propellant. Changing outflow rate changes thrust and duration; under these force-free assumptions it does not change the increment for a fixed mass ratio.

Chapter support: Physics I manuscript, Chapter 9

Inverse sizing and staging: complete statement



Example 9.23 — complete statement

All motion is along one horizontal line in deep space, with external forces neglected and nonrelativistic exhaust. (a) A single-stage vehicle must gain 1800 m s^{-1} , has 700 kg attached at burnout, and uses constant effective exhaust speed 2400 m s^{-1} . Find initial mass, propellant mass, and burn time at 2.00 kg s^{-1} outflow. (b) Independently consider a two-stage stack with 400 kg payload, upper-stage dry mass 150 kg and propellant 600 kg , and lower-stage dry mass 250 kg and propellant 1000 kg . The lower and upper effective exhaust speeds are 2200 and 2800 m s^{-1} . Find its ideal velocity increment. (c) Compare retaining the empty lower structure throughout the upper burn.

Chapter support: Physics I manuscript, Chapter 9; Example 9.23

Inverse sizing and staged burns: parts (a) and (b)



Example 9.23 — solutions (a) and (b). For (a), $m_i/m_f = \exp(1800/2400) = 2.11700$:

$$m_i = 1481.90 \text{ kg}, \quad m_{\text{prop}} = 781.90 \text{ kg}, \quad t_b = 781.90/2.00 = 390.95 \text{ s}.$$

For (b), include both dry stages, both propellants, and the payload:

$$2400 \xrightarrow{\text{lower burn}} 1400 \xrightarrow{\text{jettison 250}} 1150 \xrightarrow{\text{upper burn}} 550 \text{ (kg)}.$$

$$\Delta v_1 = 2200 \ln(2400/1400) = 1185.79 \text{ m s}^{-1},$$

$$\Delta v_2 = 2800 \ln(1150/550) = 2065.28 \text{ m s}^{-1}.$$

Thus $\Delta v_{\text{staged}} = 3251.07 \text{ m s}^{-1}$. Jettison adds no increment; 550 kg includes payload plus upper dry mass.

Chapter support: Physics I manuscript, Chapter 9; Example 9.23

Retain or discard structure: part (c)



Example 9.23 — solution (c). Retaining lower dry structure leaves the first burn unchanged. The upper burn now runs from 1400 to 800 kg:

$$\Delta v_{\text{retained}} = 1185.79 + 2800 \ln(1400/800) = 2752.72 \text{ m s}^{-1}.$$

$$\Delta v_{\text{staged}} - \Delta v_{\text{retained}} = 498.35 \text{ m s}^{-1}.$$

The same propellant and exhaust speeds accelerate less spent structure during the staged upper burn. Check each logarithm against its mass ledger.

Advanced synthesis

Fixed momentum places a lower bound on kinetic energy. An open subsystem additionally requires boundary momentum flux. The common discipline is to define the material, frame, interval, and energy stores before calculating.

The ideal comparison excludes gravity, drag, and separation recoil; these would require their own force or impulse terms.

Chapter support: Physics I manuscript, Chapter 9; Example 9.23

