

Chapter 10

Electromagnetic Induction and Inductance

First Edition

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Preface

This chapter introduces electromagnetic induction and inductance as the dynamical bridge between magnetic fields, electric circuits and energy transfer. In earlier chapters of Physics 2, magnetic fields were introduced through their forces on moving charges and currents, and magnetic fields produced by currents were calculated using symmetry, the Biot–Savart law and Ampere’s law. The present chapter asks a deeper question: what happens when the magnetic situation changes with time?

The answer is one of the central results of classical electromagnetism. A changing magnetic flux can produce an electromotive effect, and in field language it can produce a circulating, nonconservative electric field. This result is not only a beautiful piece of physics; it is the basis of generators, transformers, inductors, magnetic braking, pickup coils, contactless sensing, wireless charging, metal detectors, ignition systems, switched-mode power electronics and many measurement techniques. Modern electrical power infrastructure is essentially an organised use of Faraday’s law and energy conservation.

This chapter is written as a textbook chapter for Technical Physics, engineering, computer-science, applied-science and space-science students. It therefore does not present Faraday’s law as an isolated formula. Instead, it builds the idea of magnetic flux carefully, establishes a single orientation convention, explains the physical meaning of Lenz’s law, derives the standard motional-emf result, connects induced emf to nonconservative electric fields, introduces inductance as flux linkage per current, solves RL transients, derives magnetic energy storage, and closes with mutual induction and engineering applications.

The main conceptual difficulty is the sign. Students often remember that “induction opposes something” but then oppose the wrong thing. Faraday’s sign fixes the induced emf relative to the change in magnetic flux. In a resistance-dominated loop this also determines the current; with appreciable inductance, current additionally depends on its earlier state. This chapter repeatedly returns to that point because it is the key to professional reasoning with induction.

The presentation follows the scope of the Physics 2 master blueprint. The treatments in Halliday–Resnick–Walker, Serway–Jewett and OpenStax were used comparatively; the diagrams and problem statements here are redrawn and developed for this chapter. References at the end identify corresponding sections for further study.

The Guideline

Treat every induction problem first as an orientation problem and an energy problem. Choose the loop orientation, choose the associated area normal, write the magnetic flux with its algebraic sign, determine whether that flux is increasing or decreasing, and only then interpret the induced emf or current. The minus sign in Faraday’s law is not decoration; it is a compact statement of Lenz’s law for the orientation that you chose.

Pedagogical boundary

This chapter develops induction and inductance in the quasi-static circuit regime. It uses magnetic flux, line integrals, first-order differential equations and energy arguments. A fully local theory of time-dependent electromagnetic fields belongs to Maxwell’s equations and electromagnetic waves; those ideas are prepared here but are not used in their full generality.

Conventions and notation

Vectors are bold: $\mathbf{B}$ is magnetic field, $\mathbf{E}$ is electric field, and $d\mathbf{A} = \hat{\mathbf{n}} dA$ is oriented area. Choose a positive loop traversal, then use the right-hand rule to choose $\hat{\mathbf{n}}$. Flux and flux linkage are signed quantities:

$$\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{A}, \quad \lambda = \sum_{j=1}^N \Phi_{B,j}.$$

Only when all turns link the same signed flux does $\lambda = N\Phi_B$. Reversing the reference orientation reverses the signs of flux and emf, but does not change the physical current.

For a fixed loop and fixed spanning surface, Faraday's law is

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}.$$

For a continuously moving or deforming material loop, it becomes

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{A}.$$

Here $\mathbf{u}$ is the local wire velocity, not the charge drift velocity relative to the wire. Sliding contacts require the actual velocity of each conductor segment. The fixed-path and moving-loop forms are distinguished throughout the chapter.

Symbol	Meaning	SI unit
$\mathbf{B}$	magnetic field	T
$\mathbf{E}$	electric field, including induced fields	V m ⁻¹
$d\mathbf{A}$	oriented area element	m ²
$d\boldsymbol{\ell}$	oriented line element	m
Φ_B	magnetic flux through an oriented surface	Wb = T m ²
λ	flux linkage; $N\Phi_B$ for identical linked turns	Wb
$\mathcal{E}$	electromagnetic emf around a loop	V
I, i	electric current	A
R	resistance	Ω
L, M	self-inductance and mutual inductance	H
τ_L	inductive time constant L/R	s
U_B	stored magnetic-field energy	J
u_B	magnetic energy density	J m ⁻³
μ_0	vacuum permeability	H m ⁻¹

Note

Despite its historical name, electromotive force has units of volts, not newtons: $1 \text{ V} = 1 \text{ J/C}$. It measures electromagnetic work per unit charge around the stated path. Emf and terminal voltage need not be equal. In numerical estimates, $\mu_0 \simeq 4\pi \times 10^{-7} \text{ H/m}$ is adequate at the quoted precision.

Chapter 10

Electromagnetic Induction and Inductance

10.1 Motivation and roadmap

Electricity and magnetism become truly dynamic in this chapter. A steady current produces a magnetic field, but a changing magnetic flux produces an electromotive effect. The discovery that a changing magnetic environment can drive charge around a circuit was conceptually revolutionary because it turned magnetism into a source of electrical energy. It also showed that electric fields can exist in a form that is not produced simply by static charges and cannot be described globally as the negative gradient of a scalar potential.

The technological consequences are everywhere. In a power generator, mechanical work rotates a coil in a magnetic field and produces an alternating emf. In a transformer, a time-dependent current in one coil produces a time-dependent magnetic flux that induces an emf in another coil. In an inductor, a current stores energy in a magnetic field and resists changes in that current. In an eddy-current brake, induced currents dissipate mechanical energy without mechanical contact. In an induction cooktop, changing magnetic fields induce currents in a metal pan. In a pickup coil, a changing magnetic flux becomes a measurable voltage signal.

Figure 10.1 connects the two complementary descriptions used here: flux and fields, and circuits and energy. Chapters 11–13 continue with AC circuits, Maxwell’s equations and electromagnetic waves.

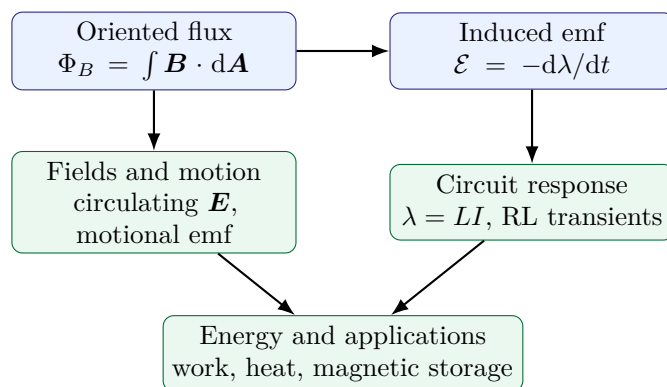


Figure 10.1: The chapter connects flux geometry to field and circuit descriptions, then tests both through energy conservation.

Learning goals

After studying this chapter, the student should be able to:

- define magnetic flux with a clear orientation convention;
- apply Faraday's law and Lenz's law to determine the magnitude and direction of induced emf and current;
- distinguish an induced emf from a mechanical force and from an electrostatic potential difference;
- derive and use the motional-emf result $\mathcal{E} = B\ell v$ as a special case;
- explain how energy is transferred in induction processes;
- interpret induced electric fields as nonconservative fields with nonzero circulation;
- define self-inductance and mutual inductance through flux linkage;
- derive the inductance of a long solenoid and estimate orders of magnitude;
- solve current growth and decay in simple RL circuits;
- compute magnetic energy $U_B = LI^2/2$ and magnetic energy density $u_B = B^2/(2\mu_0)$;
- connect induction to generators, transformers, eddy-current braking, wireless charging and sensors;
- use numerical simulation to model RL transients and rotating-coil emf waveforms.

Key message

Electromagnetic induction is the physics of change. Static magnetic flux through a stationary loop does not by itself imply an induced emf. The induced effect is controlled by the time derivative of the flux linkage, not merely by the existence of a magnetic field.

10.2 Required prior knowledge

This chapter relies on several earlier ideas. From mechanics, we need work, power, energy conservation and first-order differential equations. The power delivered by a force is $P = \mathbf{F} \cdot \mathbf{v}$, and the power dissipated in a resistor is $P = I^2 R$. From electrostatics and circuits, we need current as charge flow, potential difference, resistance and Kirchhoff's loop rule. From magnetism, we need magnetic field, magnetic force on a moving charge, magnetic force on a current-carrying wire, and the magnetic field inside an ideal long solenoid.

The mathematical tools are modest but must be used carefully. Magnetic flux is a surface integral. Faraday's law contains a time derivative. RL circuits require solving a first-order linear differential equation. The field form of Faraday's law uses a closed line integral. None of these tools is difficult when the geometry and signs are explicit; most errors come from unclear orientation choices.

Formula summary

Useful earlier results:

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}, \quad \mathbf{F}_B = I\boldsymbol{\ell} \times \mathbf{B}, \quad (10.1)$$

$$P_R = I^2 R = IV = \frac{V^2}{R}, \quad B_{\text{solenoid}} \simeq \mu_0 n I. \quad (10.2)$$

Here $n = N/\ell$ is the number of turns per unit length of a long solenoid. The solenoid expression is an idealisation valid near the middle of a long, tightly wound solenoid with negligible end effects.

Guided checks

Check 1. Which earlier magnetic-force equation explains the magnetic force on a current-carrying rod in a field?

Answer. $\mathbf{F} = I\boldsymbol{\ell} \times \mathbf{B}$. This will be used in motional-emf energy arguments.

Check 2. Why is $B = \mu_0 nI$ not exact for every solenoid?

Answer. It assumes a long solenoid and neglects field spreading near the ends.

Check 3. What is the SI unit of magnetic flux?

Answer. The weber, $1 \text{ Wb} = 1 \text{ T m}^2$.

Method

Dimensional habit. Check induction equations by units. Since $1 \text{ Wb} = 1 \text{ T m}^2$ and $1 \text{ V} = 1 \text{ Wb/s}$, the expression $-\text{d}\Phi_B/\text{d}t$ has the correct unit of emf. Since $1 \text{ H} = 1 \text{ Wb/A}$, the expression $L \text{ d}I/\text{d}t$ also has unit $\text{Wb/s} = \text{V}$.

10.3 Magnetic flux and orientation

Magnetic flux measures how much magnetic field passes through an oriented surface. For a surface S bounded by a closed curve C , the magnetic flux is

$$\Phi_B = \int_S \mathbf{B} \cdot \text{d}\mathbf{A} = \int_S B \cos \theta \text{ d}A, \quad (10.3)$$

where θ is the angle between $\mathbf{B}$ and the chosen local normal direction. If the field is uniform and the surface is flat, this becomes

$$\Phi_B = BA \cos \theta. \quad (10.4)$$

Figure 10.2 shows the paired loop and surface conventions. This compact expression hides a sign convention. The area vector is not simply an area; it is an oriented area. Reversing the chosen normal changes the sign of the flux.

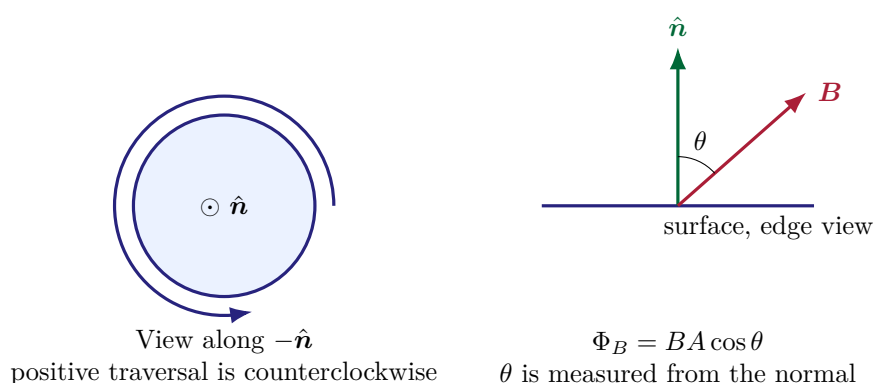


Figure 10.2: One orientation, shown in two views. The loop arrow fixes the normal; the angle in the dot product is between the field and that normal, not the plane.

Definition 10.1: Magnetic flux

The magnetic flux through an oriented surface S is $\Phi_B = \int_S \mathbf{B} \cdot \text{d}\mathbf{A}$. For a flat surface in a uniform magnetic field, $\Phi_B = BA \cos \theta$. Flux is an algebraic scalar whose sign depends on the chosen orientation of S .

Magnetic flux can change in several physically distinct ways. The magnetic field magnitude may change, the area may change, the orientation may change, or the circuit may move into or out of a

region where the magnetic field is present. Faraday's law does not care which mechanism produces the flux change; it responds to the total time derivative of the flux linkage.

$$\frac{d\Phi_B}{dt} = \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A}. \quad (10.5)$$

For a rigid flat coil of fixed area rotating in a uniform field, $\Phi_B = BA \cos \theta(t)$ and

$$\frac{d\Phi_B}{dt} = -BA \sin \theta \frac{d\theta}{dt}. \quad (10.6)$$

For a fixed loop in a spatially uniform but time-dependent magnetic field perpendicular to the loop, $\Phi_B = AB(t)$ and

$$\frac{d\Phi_B}{dt} = A \frac{dB}{dt}. \quad (10.7)$$

For a loop partly inside a uniform field region, the effective area inside the field may change with time, so $\Phi_B = BA_{\text{in}}(t)$.

Caution

Do not replace flux by BA automatically. The formula $\Phi_B = BA$ is valid only when the field is uniform, the surface is flat, and $\mathbf{B}$ is parallel to the chosen area normal. In most induction problems the sign and the changing part of the flux are the important features.

Guided checks

Check 1. A loop lies in the page and the chosen area normal points out of the page. A uniform magnetic field points into the page. Is the flux positive or negative?

Answer. Negative, because $\mathbf{B} \cdot d\mathbf{A} < 0$.

Check 2. A fixed loop is in a constant uniform field. The flux is nonzero. Is an emf induced?

Answer. No, not if the flux is constant in time. Induction depends on $d\Phi_B/dt$.

Check 3. Can the magnetic flux through a loop be zero even when B is nonzero everywhere on the loop area?

Answer. Yes. For example, the field may be tangent to the surface, or positive and negative contributions may cancel.

10.3.1 What surface should be used?

The surface in a flux integral is mathematical: no membrane has to fill a wire loop. A flat disk is usually convenient for a circular loop, but a curved surface with the same boundary gives the same flux if its orientation is consistent. To see why, join two such surfaces to make a closed surface, reversing the normal of the second one. Since magnetic field has zero net flux through a closed surface,

$$\int_{S_1} \mathbf{B} \cdot d\mathbf{A} - \int_{S_2} \mathbf{B} \cdot d\mathbf{A} = \oint_{S_1 \cup (-S_2)} \mathbf{B} \cdot d\mathbf{A} = 0.$$

This is the magnetic Gauss law, $\nabla \cdot \mathbf{B} = 0$, whose wider role is developed in Chapter 12. Here it ensures that a calculated emf cannot depend on an arbitrary choice of spanning surface. It does not make the flux through an *open* surface zero.

For a coil, add the fluxes through its individual turns with their winding orientations:

$$\lambda = \sum_{j=1}^N \Phi_{B,j}. \quad (10.8)$$

The turn number is dimensionless, so λ has SI unit weber. “Weber-turn” is sometimes used informally to emphasise that this is linkage rather than the flux through one turn. Multiplying by N twice is a common error: if a problem gives λ , Faraday’s law is already $\mathcal{E} = -\dot{\lambda}$.

10.3.2 Changing several things at once

For a uniform field and a flat loop, the product rule gives

$$\frac{d\Phi_B}{dt} = A \cos \theta \dot{B} + B \cos \theta \dot{A} - BA \sin \theta \dot{\theta}. \quad (10.9)$$

Each term describes a different way of changing linkage. The terms may reinforce or cancel. A field can increase while a loop turns toward an orientation with smaller projected area; the resulting emf depends on their combined rates. When the field is nonuniform, use the integral before differentiating. A central value of B need not equal its area average.

Guided checks

Question. Can the flux be zero at an instant while the emf is nonzero?

Answer. Yes. For $\Phi_B = BA \cos \omega t$, at $\omega t = \pi/2$ the flux is zero but its slope is $-BA\omega$. Conversely, at maximum flux its slope is zero. An instantaneous value and a time derivative answer different questions.

Example 10.1: Flux from a nonuniform magnetic field

Problem. A circular loop of radius a lies in a field $B(r) = B_0(1 - r^2/a^2)$ perpendicular to the loop for $0 \leq r \leq a$. Find the flux through the loop.

Solution. Use polar area element $dA = 2\pi r dr$ for an annulus. Then

$$\Phi_B = \int_0^a B_0 \left(1 - \frac{r^2}{a^2}\right) 2\pi r dr.$$

Thus

$$\Phi_B = 2\pi B_0 \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right]_0^a = 2\pi B_0 \left(\frac{a^2}{2} - \frac{a^2}{4} \right) = \frac{\pi B_0 a^2}{2}.$$

Interpretation and check. The result is half of what one would obtain from a uniform field B_0 over the full area, because the field decreases quadratically to zero at the edge.

10.4 Faraday–Lenz law

Faraday’s law states that a changing magnetic flux through a loop induces an emf around the loop. For one fixed loop,

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}. \quad (10.10)$$

For a coil of N turns, if the same flux links each turn, the flux linkage is $N\Phi_B$ and

$$\mathcal{E} = -N \frac{d\Phi_B}{dt}. \quad (10.11)$$

The minus sign is the mathematical encoding of Lenz’s law: the induced emf has the orientation that opposes the change in magnetic flux.

It is crucial to state what is being opposed. For a resistance-dominated loop without another source, the following current interpretation applies: the induced current does not necessarily oppose $\mathbf{B}$

itself. If the external flux through the loop is increasing in the positive direction, the induced current produces a magnetic field whose own flux is negative. If the external flux is decreasing in the positive direction, the induced current produces a magnetic field whose own flux is positive. The induced effect opposes the change, not the mere presence of the field.

Definition 10.2: Faraday–Lenz law

For a fixed conducting loop or a fixed mathematical closed path C bounding a fixed oriented surface S , the electromagnetic emf is

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A}.$$

For N turns linked by the same flux, replace Φ_B by $N\Phi_B$. The sign is tied to the chosen orientation of C and S .

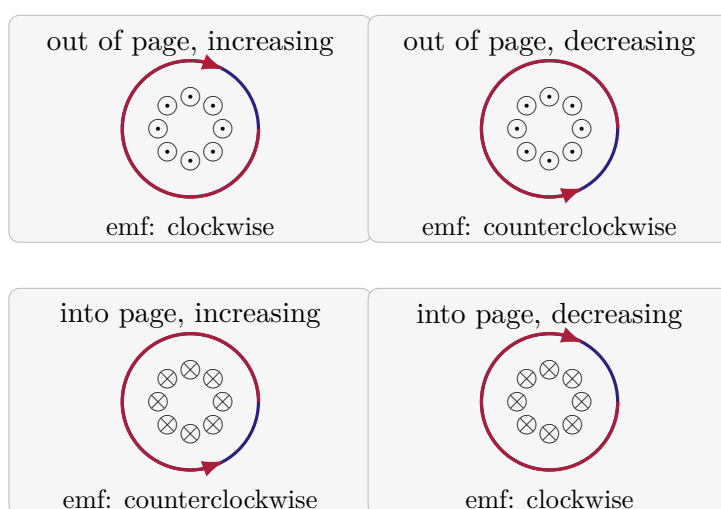


Figure 10.3: Four common Lenz-law cases for a circular loop. The arrows show the emf direction and the current in a resistance-dominated loop initially without current. The external field magnitude changes as stated; the induced response opposes that change.

Method

A reliable Faraday–Lenz algorithm.

1. Choose a positive traversal direction around the loop.
2. Use the right-hand rule to define the associated positive area normal.
3. Write the algebraic flux $\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$.
4. Determine the sign of $d\Phi_B/dt$.
5. Compute $\mathcal{E} = -d\Phi_B/dt$.
6. For a resistance-dominated closed loop, $I = \mathcal{E}/R$ gives the current direction. If self-inductance or another source matters, first solve the circuit equation.
7. Check the result with energy conservation, including any initial magnetic energy and all sources.

Key message

For an initially unenergised passive resistive loop, the induced current produces the reaction required for energy conservation. A pre-existing current or stored field energy can change the instantaneous force or current direction. Faraday's sign fixes the induced emf; the complete circuit determines the current.

Example 10.2: Flux change through a rectangular coil

A rectangular coil with $N = 120$ turns has area $A = 3.0 \times 10^{-3} \text{ m}^2$. Its area normal is chosen to be in the positive z direction. A uniform magnetic field is also in the z direction and changes from 0.10 T to 0.46 T in 0.080 s . Find the average induced emf using the chosen positive orientation.

Solution. The flux through one turn is $\Phi_B = BA$, so the change in flux linkage is

$$\Delta(N\Phi_B) = NA\Delta B = (120)(3.0 \times 10^{-3})(0.46 - 0.10) \text{ Wb} = 0.1296 \text{ Wb}.$$

The average induced emf is

$$\bar{\mathcal{E}} = -\frac{\Delta(N\Phi_B)}{\Delta t} = -\frac{0.1296 \text{ Wb}}{0.080 \text{ s}} = -1.62 \text{ V}.$$

The negative sign means that the emf is opposite to the positive loop traversal associated with the $+z$ area normal. The magnitude is 1.62 V (about 1.6 V to two significant figures).

Reasonableness check. A hundred-turn coil multiplies a millivolt-level single-turn effect into a volt-level signal. This is why pickup coils often use many turns.

Example 10.3: Lenz-law direction for a decreasing flux

A circular conducting loop lies in the horizontal plane. The positive area normal is upward. A uniform magnetic field points upward but decreases in magnitude. Determine the induced current direction as viewed from above.

Solution. With the upward normal, the external flux is positive. Because the field magnitude decreases, $d\Phi_B/dt < 0$. Faraday's law gives $\mathcal{E} = -d\Phi_B/dt > 0$, so the induced emf is in the positive traversal direction. The positive traversal direction corresponding to an upward normal is counterclockwise as viewed from above. Therefore the induced current is counterclockwise.

Physical interpretation. The upward magnetic flux is decreasing. The induced current produces an upward magnetic field through the loop to oppose that decrease. A counterclockwise current viewed from above produces an upward field by the right-hand rule.

Guided checks

Check 1. If a positive flux is increasing, what is the sign of the induced emf in the chosen orientation?

Answer. Negative. Faraday's law gives $\mathcal{E} = -d\Phi_B/dt < 0$.

Check 2. Does a negative emf mean that there is no induced current?

Answer. No. It means that the emf is opposite to the chosen positive traversal direction; the current follows it in a purely resistive loop.

Check 3. A loop has N turns. Why does the emf scale with N ?

Answer. Each turn contributes a similar induced emf if the same changing flux links each turn, so the total flux linkage is $N\Phi_B$.

10.4.1 From emf to current: external and self flux

An emf describes the electromagnetic drive around a path. It does not by itself specify a current. An open pickup coil can have a voltage signal with almost no current. A closed resistive loop has approximately $I = \mathcal{E}/R$ when its inductive response is negligible. A loop with appreciable inductance remembers its earlier current and requires a differential equation.

It is useful to distinguish externally imposed flux linkage λ_{ext} from self linkage LI . For a fixed linear loop with resistance R , no battery and no appreciable capacitance, applying Faraday's law to the total flux gives

$$RI = -\frac{d}{dt}(\lambda_{\text{ext}} + LI),$$

or, for constant L ,

$$L\dot{I} + RI = \mathcal{E}_{\text{ext}}, \quad \mathcal{E}_{\text{ext}} = -\dot{\lambda}_{\text{ext}}. \quad (10.12)$$

We will derive and solve this circuit response in detail. Its role here is to make the sign rule precise. Faraday's law fixes the emf sign. An already flowing current need not reverse immediately when the external emf reverses. Including LI in the flux and then adding $-L\dot{I}$ again would count self-induction twice.

For example, a constant positive external emf applied to an initially current-free loop produces an increasing positive current. If that emf is switched to zero, the current persists temporarily, even though the external flux is now constant. The decreasing self flux supplies the resistive loss. This explains both the usefulness and the limit of interpreting every Lenz-law arrow as an instantaneous current arrow. The four cases in Figure 10.3 apply directly to the resistance-dominated response.

10.4.2 Fixed paths and moving conductors

The equation $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -d\Phi_B/dt$ above applies to a fixed path. A conductor moving at local velocity $\mathbf{u}$ has an additional magnetic force per unit charge. For a continuously moving material circuit,

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B[C(t)]}{dt}. \quad (10.13)$$

The derivative follows the moving surface. This form includes both a time-dependent magnetic field and a changing loop geometry. The $\mathbf{u} \times \mathbf{B}$ term is essential even if $\partial\mathbf{B}/\partial t = 0$. Sliding-contact examples must use the velocity of each actual conductor segment; ordinary rail-and-rod circuits obey this flux rule, while a rotating disk needs more care. Section 10.16.1 gives the geometrical derivation and a counterexample to careless use of a fixed contour.

Method

Choose one complete calculation. Either evaluate the Lorentz-force integral on the actual circuit or calculate the derivative of flux through its properly moving boundary. For a simple sliding bar in a static field these are two evaluations of the same emf. Adding both answers would double-count induction.

10.5 Motional emf

Start with an isolated straight conducting rod. Let it be vertical, with its bottom end a and top end b , and let it move right with $\mathbf{u} = v\hat{\mathbf{x}}$ in $\mathbf{B} = -B\hat{\mathbf{z}}$. Then $\mathbf{u} \times \mathbf{B} = vB\hat{\mathbf{y}}$. Positive charge is driven upward; in a metal the mobile electrons initially move downward. Surface charge builds up until the electric field inside the open rod is

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} = -vB\hat{\mathbf{y}}.$$

This is an open-circuit equilibrium, with no sustained longitudinal current. At one laboratory time, in this static-field idealisation, the electric potential difference is

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\boldsymbol{\ell} = B\ell v. \quad (10.14)$$

Thus the upper end is positive. The motional contribution along the rod is also $\int_a^b (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = B\ell v$, but the *total* Lorentz force along an open rod vanishes after charge separation. Distinguishing these statements prevents the mistaken claim that a current flows indefinitely in an isolated rod.

Now connect the rod to stationary rails and a resistor as in Figure 10.4. Charges can circulate, and the charge distribution adjusts to drive that current. The open-circuit balance $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$ no longer holds everywhere in the resistive path. Only the moving segment contributes motional emf in the static field; the stationary rails have $\mathbf{u} = 0$. Choosing counterclockwise traversal and $+z$ normal gives $\Phi_{B,\text{ext}} = -B\ell x$, hence $\mathcal{E}_{\text{ext}} = +B\ell v$.

For a resistance-dominated circuit the current is counterclockwise, upward in the rod. A small rod resistance r would make its loaded terminal difference $V_b - V_a = B\ell v - Ir$ in the lumped approximation. As with a battery, source emf and loaded terminal voltage need not be equal.

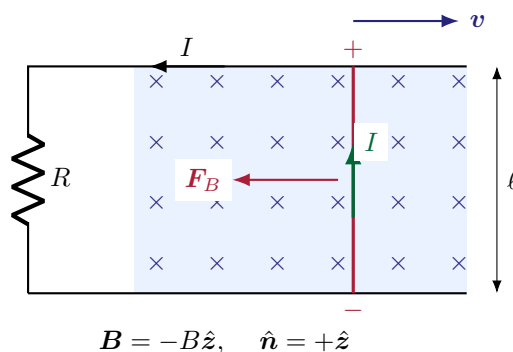


Figure 10.4: Closed sliding-rod circuit. Motion to the right increases the into-page external flux magnitude. The emf is counterclockwise, the rod current is upward, and the magnetic force is leftward. End signs indicate the rod's motional source polarity.

The same result follows from Faraday's law. If the rod slides on rails and forms a rectangular conducting loop, the area of the loop changes at rate $dA/dt = \ell v$. If $\mathbf{B}$ is perpendicular to the plane of the loop, the flux magnitude changes at rate $B\ell v$. Thus the magnitude of the induced emf is again $B\ell v$.

Caution

The expression $\mathcal{E} = B\ell v$ assumes a very specific geometry: a straight conducting rod of length ℓ moving perpendicular to a uniform magnetic field, with the rod perpendicular to the velocity and the field. For other geometries use $\mathcal{E} = \int (\mathbf{v} \times \mathbf{B}) \cdot d\boldsymbol{\ell}$ or Faraday's law.

Example 10.4: A rod sliding on rails

Neglect self-inductance and rail resistance except for the stated total R . A rod of length $\ell = 0.35$ m slides to the right at $v = 2.4$ m/s on conducting rails in a uniform magnetic field of magnitude $B = 0.80$ T into the page. The total circuit resistance is $R = 0.60$ Ω . Find the induced emf magnitude, the induced current magnitude and the magnetic force on the rod.

Solution. The motional emf is

$$\mathcal{E} = B\ell v = (0.80)(0.35)(2.4) = 0.672 \text{ V}.$$

The current magnitude is

$$I = \frac{\mathcal{E}}{R} = \frac{0.672}{0.60} = 1.12 \text{ A.}$$

The magnetic force on the rod has magnitude

$$F_B = I\ell B = (1.12)(0.35)(0.80) = 0.314 \text{ N.}$$

The direction is opposite the motion. As the rod moves right, the loop area and into-page flux increase. The induced current must create an out-of-page field, so it is counterclockwise. The current in the rod is upward, and $I\ell \times \mathbf{B}$ points left.

Energy check. The mechanical power needed to pull the rod at constant speed is $P_{\text{mech}} = F_B v = (0.314)(2.4) = 0.754 \text{ W}$. The resistor dissipates $P_R = I^2 R = (1.12)^2(0.60) = 0.753 \text{ W}$, equal within rounding.

Key message

Motional emf is not a different law of nature. It is Faraday's law plus the Lorentz force, applied to a moving conductor. The mechanical work required to keep the conductor moving becomes electrical energy and, in a resistive circuit, heat.

10.5.1 A loop crossing a finite field region

A rigid loop can move without any net motional emf. If the entire loop translates in a static uniform field, opposite sides give cancelling line integrals and the flux remains constant. Induction occurs at a boundary because one side enters or leaves the field before the other. The changing quantity is the overlap area, not the total wire-loop area.

Figure 10.5 uses a loop of horizontal width w and vertical height h , moving at speed v through a strip $0 < x < d$ with $d > w$. Let x be the leading-edge coordinate. For an out-of-page field $B > 0$ and a counterclockwise reference direction,

$$\Phi_B(x) = Bh \begin{cases} 0, & x \leq 0, \\ x, & 0 < x < w, \\ w, & w \leq x \leq d, \\ d + w - x, & d < x < d + w, \\ 0, & x \geq d + w. \end{cases} \quad (10.15)$$

Since $\dot{x} = v$, the entry emf is $-Bhv$, the interior emf is zero, and the exit emf is $+Bhv$. Thus entry current is clockwise and exit current counterclockwise in the resistive approximation. Although the currents reverse, the magnetic force opposes motion in both boundary regions. While fully inside, the ideal net force and emf vanish.

Example 10.5: Entry and exit pulses with an energy check

A square loop of side 0.10 m and resistance 0.20Ω crosses a 0.30 m strip containing $B = 0.80 \text{ T}$ out of the page. It moves right at 0.50 m/s. Neglect self-inductance. Determine the waveform and total heat generated in one passage.

Strategy. Use the leading-edge coordinate in Figure 10.5; the first boundary is reached at $t = 0$. Calculate emf from the slope of overlap area, then use $I^2 R$ only during entry and exit.

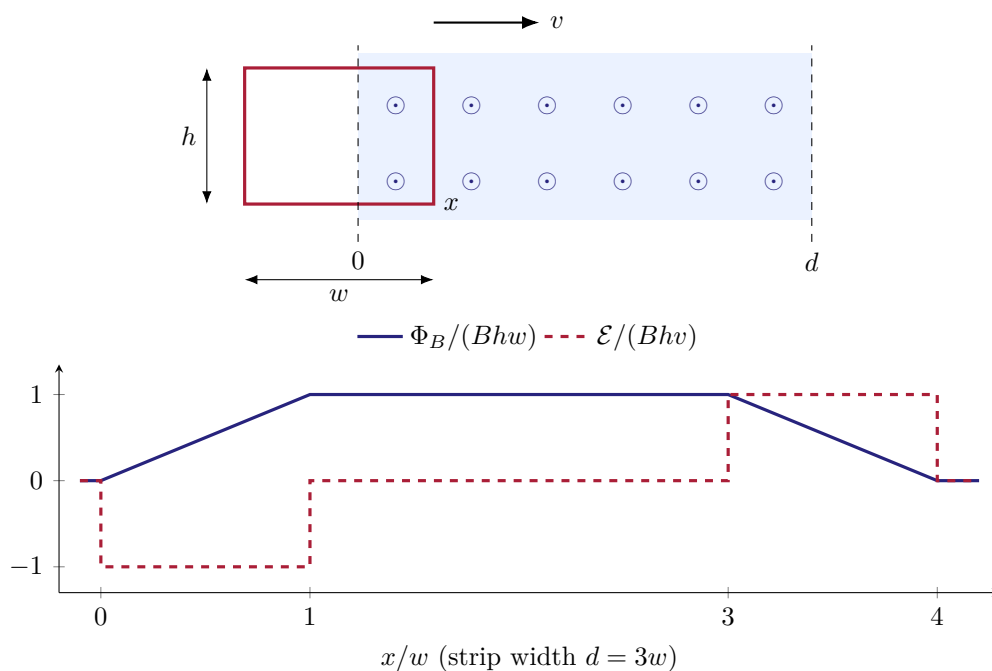


Figure 10.5: Geometry and calculated piecewise flux/emf for a loop crossing a sharp field boundary. The jumps are idealisations; fringing fields smooth the voltage transitions, and nonzero self-inductance smooths the current response.

Calculation. Entry lasts $w/v = 0.20$ s. The leading edge reaches the exit at $d/v = 0.60$ s, and the loop is wholly outside at 0.80 s. Hence

$$\mathcal{E}(t) = \begin{cases} -0.040 \text{ V}, & 0 < t < 0.20 \text{ s}, \\ 0, & 0.20 < t < 0.60 \text{ s}, \\ +0.040 \text{ V}, & 0.60 < t < 0.80 \text{ s}. \end{cases}$$

The current magnitude during either pulse is 0.20 A, and $P_R = (0.20)^2(0.20) = 0.0080$ W. The heat is $Q = P_R(0.40 \text{ s}) = 3.2 \times 10^{-3}$ J.

Check. The drag is $IBh = 0.016$ N and acts over distance $2w = 0.20$ m. External work is 0.0032 J, equal to the heat. Zero net signed pulse area means that final and initial flux are equal; it does not imply zero energy transfer, because power involves $\mathcal{E}^2/R$.

10.6 Induction and energy transfer

Induction problems are often easiest to understand through energy. In the sliding-rod example, the induced current produces a magnetic force opposing the rod's motion. An external force must therefore do positive work to keep the rod moving at constant speed. That work becomes thermal energy in the resistance. If the induced current instead helped the motion, the rod would accelerate while also heating the resistor, and energy conservation would be violated.

For a sliding rod, the energy relation can be derived explicitly. The induced current is

$$I = \frac{B\ell v}{R}. \quad (10.16)$$

The magnetic drag force has magnitude

$$F = I\ell B = \frac{B^2\ell^2 v}{R}. \quad (10.17)$$

Thus the mechanical power required for constant velocity is

$$P_{\text{mech}} = Fv = \frac{B^2 \ell^2 v^2}{R}. \quad (10.18)$$

The electrical power dissipated is

$$P_R = I^2 R = \left(\frac{B\ell v}{R} \right)^2 R = \frac{B^2 \ell^2 v^2}{R}. \quad (10.19)$$

The equality is not an accident. It is the quantitative expression of Lenz's law.

10.6.1 How can a magnetic force transfer mechanical energy?

For any charge with laboratory velocity $\mathbf{v}_q$, the magnetic part of the Lorentz force satisfies $q(\mathbf{v}_q \times \mathbf{B}) \cdot \mathbf{v}_q = 0$. A magnetic field does no work on that point charge. This remains true in the moving rod; induction does not invalidate the result from Chapter 8.

The charge velocity contains both the rod motion and drift relative to the rod. Magnetic forces redirect the charge motion and transmit a reaction to the lattice through the interactions that constrain the charges. The external agent does work against that macroscopic reaction. Electric fields and collisions deliver energy to the electrical degrees of freedom and ultimately to heat. The separation into rod motion and drift permits mechanical and electrical powers to be exchanged without a magnetic force doing net work on an individual charge.

In the resistance-dominated model, there is no accumulating magnetic energy, so $F_{\text{ext}}v = I\mathcal{E} = I^2 R$ at each instant. With an appreciable constant circuit inductance, the more general budget is

$$F_{\text{ext}}v = I^2 R + \frac{d}{dt} \left(\frac{1}{2} LI^2 \right) \quad (10.20)$$

at prescribed constant speed. A pre-energised inductor can return energy, so an instantaneous force need not always be a drag. For the initially unenergised examples here, the current is driven by the motion and the mechanical input builds field energy or heat. The simple formula $I = B\ell v/R$ is justified only if the current relaxes much faster than the mechanical speed changes.

Theoretical point 10.1: Lenz's law as stability of energy accounting

In the initially unenergised, resistance-dominated sliding-rod model, the external drive or the rod's initial kinetic energy supplies the dissipated heat. The magnetic reaction therefore opposes the motion. With stored magnetic energy or another source, determine the force from the actual current and check the complete energy budget.

Example 10.6: Magnetic damping of a sliding bar

A conducting bar of mass $m = 0.18 \text{ kg}$ slides without mechanical friction on horizontal rails separated by $\ell = 0.25 \text{ m}$. The circuit resistance is $R = 0.50 \Omega$, and a uniform magnetic field $B = 0.70 \text{ T}$ points perpendicular to the rails. The bar is given an initial speed $v_0 = 3.0 \text{ m/s}$ and then released. Find $v(t)$ in the resistance-dominated approximation $L/R \ll mR/(B^2 \ell^2)$.

Solution. At speed v , the induced current is $I = B\ell v/R$. The magnetic force opposing the motion has magnitude

$$F = I\ell B = \frac{B^2 \ell^2}{R} v.$$

Newton's second law gives

$$m \frac{dv}{dt} = -\frac{B^2 \ell^2}{R} v.$$

This first-order differential equation has solution

$$v(t) = v_0 \exp\left(-\frac{t}{\tau_m}\right), \quad \tau_m = \frac{mR}{B^2\ell^2}.$$

Numerically,

$$\tau_m = \frac{(0.18)(0.50)}{(0.70)^2(0.25)^2} = 2.94 \text{ s}.$$

Thus

$$v(t) = 3.0 \exp(-t/2.94 \text{ s}) \text{ m/s}.$$

Interpretation. Magnetic damping is proportional to velocity, so the decay is exponential, analogous to linear drag in mechanics.

Guided checks

Check 1. In a passive sliding-rod circuit, does the magnetic force assist or oppose the imposed motion?

Answer. It opposes the motion that changes the flux.

Check 2. Where does the energy dissipated in the resistor come from?

Answer. From mechanical work done by an external agent or from the kinetic energy of the moving conductor.

Check 3. Why is the drag force proportional to speed in the simple sliding-bar model?

Answer. Because $\mathcal{E} \propto v$, hence $I \propto v$, and the magnetic force on the current-carrying bar is proportional to I .

Example 10.7: Vertical sliding bar with gravity

Problem. A conducting bar of mass m slides vertically downward on frictionless rails separated by ℓ in a uniform magnetic field perpendicular to the circuit. The circuit resistance is R ; neglect self-inductance. Derive the terminal speed.

Solution. At speed v , the induced current magnitude is $I = B\ell v/R$. The magnetic force magnitude on the bar is $F_B = I\ell B = B^2\ell^2 v/R$, directed upward by Lenz's law. Terminal speed occurs when $mg = F_B$, so

$$mg = \frac{B^2\ell^2}{R}v_t.$$

Therefore

$$v_t = \frac{mgR}{B^2\ell^2}.$$

Interpretation and check. A larger resistance gives a larger terminal speed because weaker induced current gives weaker magnetic braking. A stronger field reduces terminal speed quadratically.

Example 10.8: Magnetic damping energy check

Problem. For a sliding bar with speed $v(t) = v_0 e^{-t/\tau_m}$, show that the total energy dissipated equals the initial kinetic energy.

Solution. The magnetic drag force is $F = bv$ with $b = B^2 \ell^2 / R$ and $\tau_m = m/b$. The dissipated power is $P = Fv = bv^2 = bv_0^2 e^{-2t/\tau_m}$. The total dissipated energy is

$$E_{\text{diss}} = \int_0^\infty bv_0^2 e^{-2t/\tau_m} dt = bv_0^2 \frac{\tau_m}{2}.$$

Since $\tau_m = m/b$,

$$E_{\text{diss}} = \frac{1}{2} m v_0^2.$$

Interpretation and check. The initial kinetic energy is fully converted into thermal energy in the circuit, assuming no other losses and no external work.

10.7 Induced electric fields

Faraday's law is sometimes introduced through conducting loops, but it is more general than wires. A changing magnetic field can induce an electric field in empty space. For a fixed path and fixed spanning surface, the field form is

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A}. \quad (10.21)$$

This equation says that the circulation of $\mathbf{E}$ around a closed path is determined by the rate of change of magnetic flux through a surface bounded by that path.

An electrostatic field produced by stationary charges satisfies $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = 0$ around any closed loop, so it can be written as $\mathbf{E} = -\nabla V$ in a simply connected region. An induced electric field produced by changing magnetic flux generally has nonzero circulation and cannot be represented globally by a single-valued electrostatic potential. This is why induction is not merely a new way of making an ordinary battery-like potential difference.

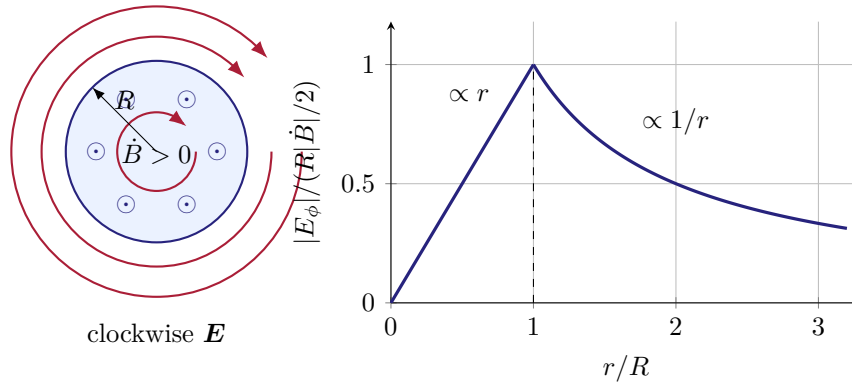


Figure 10.6: Induced field of an ideal long solenoid. Left: circulation for increasing out-of-page B . Right: the calculated magnitude, linear inside and inversely proportional to radius outside, continuous at $r = R$.

Consider the central region of an ideal long solenoid of radius R , with $\mathbf{B} = B(t)\hat{z}$ inside and negligible field outside. In the quasistatic approximation and absent an additional electrostatic field, cylindrical symmetry determines the azimuthal induced field. Write E_ϕ as a signed component, positive counterclockwise viewed from $+z$. For a circular path of radius $r < R$, E_ϕ is constant along the path. Faraday's law gives

$$E_\phi(2\pi r) = -\frac{d}{dt} (B\pi r^2), \quad (10.22)$$

so $E_\phi = -(r/2)\dot{B}$. In magnitude,

$$E(r) = \frac{r}{2} \left| \frac{dB}{dt} \right|, \quad r < R. \quad (10.23)$$

For a circular path outside the field region, $r > R$, the flux involves only the area πR^2 where B exists, and

$$E(r) = \frac{R^2}{2r} \left| \frac{dB}{dt} \right|, \quad r > R. \quad (10.24)$$

Example 10.9: Induced electric field around a changing solenoid field

A long solenoid produces a uniform magnetic field inside radius $R = 4.0$ cm and negligible field outside. The field increases at rate $dB/dt = 0.50$ T/s. Estimate the induced electric-field magnitude at $r = 2.0$ cm and at $r = 8.0$ cm.

Solution. For $r < R$,

$$E = \frac{r}{2} \frac{dB}{dt} = \frac{0.020}{2} (0.50) = 5.0 \times 10^{-3} \text{ V/m}.$$

For $r > R$,

$$E = \frac{R^2}{2r} \frac{dB}{dt} = \frac{(0.040)^2}{2(0.080)} (0.50) = 5.0 \times 10^{-3} \text{ V/m}.$$

The two numbers happen to be equal for these chosen radii. The direction is set by Lenz's law; if the magnetic field out of the page increases, the induced electric field circulates clockwise as viewed from the positive side.

Caution

An induced electric field is not confined to a wire. A wire only provides mobile charges that respond to the field. Faraday's law is a field statement, and the loop C in $\oint \mathbf{E} \cdot d\boldsymbol{\ell}$ may be an imaginary mathematical path.

10.7.1 What symmetry supplies, and what Faraday's law supplies

For the ideal solenoid, the full signed result is

$$E_\phi(r, t) = \begin{cases} -\frac{r}{2} \dot{B}(t), & r \leq R, \\ -\frac{R^2}{2r} \dot{B}(t), & r \geq R. \end{cases} \quad (10.25)$$

Both expressions give $-R\dot{B}/2$ at the boundary. For increasing $+z$ field, $E_\phi < 0$ and the circulation is clockwise. The units are $\text{m T/s} = \text{V/m}$. The field vanishes at the axis, grows linearly inside, and decreases outside as shown in Figure 10.6. Its nonzero value outside does not require a nonzero local magnetic field there: a circular path still encloses changing flux.

Faraday's law alone determines a circulation, not the entire electric field at every point. We also used the rotational symmetry of an ideal, very long source and excluded additional electrostatic fields. A nearby charged object could add a conservative field without changing the circulation. End effects and propagation delay invalidate the simple two-region field at sufficiently large distances or sufficiently fast changes. The result is a quasistatic local approximation, not a claim that a finite source changes fields instantaneously throughout space.

10.7.2 Voltage measurements and lead placement

In electrostatics, $V_b - V_a = -\int_a^b \mathbf{E} \cdot d\boldsymbol{\ell}$ is independent of path. In a changing magnetic environment, two paths P and Q from a to b need not agree. Define the path-based quantities $\mathcal{V}_P = -\int_P \mathbf{E} \cdot d\boldsymbol{\ell}$ and $\mathcal{V}_Q = -\int_Q \mathbf{E} \cdot d\boldsymbol{\ell}$. If C follows P from a to b and returns along Q , then

$$\mathcal{V}_P - \mathcal{V}_Q = -\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \frac{d\Phi_B[C]}{dt}. \quad (10.26)$$

A meter and its leads form part of the measurement circuit. Changing their routing can change the linked flux and therefore the reading. In ordinary lumped-circuit measurements, leads are kept close together and away from strongly changing flux so that this difference is negligible. Twisting a pair of leads also alternates the orientation of successive small loop areas, reducing pickup from fields varying slowly over a twist length. This is a geometrical consequence of flux cancellation, not a guarantee against all interference.

Example 10.10: An unintended measurement-loop voltage

Two routes for a pair of stationary voltage leads enclose a difference in effective area of 12 cm^2 perpendicular to a uniform field ramp of 30 T/s . Estimate the difference caused by induction.

Solution. Treat the two routes as the boundary of one comparison loop. Its flux rate has magnitude $A|\dot{B}| = (12 \times 10^{-4})(30) = 0.036 \text{ V}$. A 36 mV discrepancy can overwhelm a millivolt sensor signal. Reversing the loop normal reverses the algebraic sign. Reducing the enclosed area by a factor of ten reduces this contribution by the same factor; changing the meter's numerical precision does not remove it.

10.8 Inductors and inductance

An inductor is a device designed to produce magnetic flux linkage when current flows through it. For a fixed linear inductor, with no independent externally imposed flux, the flux linkage is proportional to current:

$$N\Phi_B = LI. \quad (10.27)$$

The proportionality constant L is the self-inductance:

$$L = \frac{N\Phi_B}{I}. \quad (10.28)$$

The SI unit is the henry,

$$1 \text{ H} = 1 \frac{\text{Wb}}{\text{A}} = 1 \frac{\text{Vs}}{\text{A}}. \quad (10.29)$$

Inductance depends on geometry and magnetic material properties, not on the current itself in an ideal linear device. In real magnetic cores, the relation between flux and current may be nonlinear, and then a single constant L is only an approximation around an operating point.

Definition 10.3: Self-inductance

For a linear inductor with N turns carrying current I , the self-inductance is $L = N\Phi_B/I$, where Φ_B is the magnetic flux through one turn due to the current in the inductor itself. The definition assumes that the same flux links all turns or that $N\Phi_B$ is interpreted as total flux linkage.

10.8.1 Inductance of a long solenoid

For a long solenoid of length ℓ_s , cross-sectional area A and N turns (Figure 10.7), the turn density is $n = N/\ell_s$. The magnetic field near the centre is approximately

$$B \simeq \mu_0 n I. \quad (10.30)$$

The flux through one turn is $\Phi_B = BA = \mu_0 n I A$. Therefore the flux linkage is

$$N\Phi_B = (n\ell_s)(\mu_0 n I A) = \mu_0 n^2 A \ell_s I. \quad (10.31)$$

Using $L = N\Phi_B/I$ gives

$$L \simeq \mu_0 n^2 A \ell_s, \quad \frac{L}{\ell_s} \simeq \mu_0 n^2 A. \quad (10.32)$$

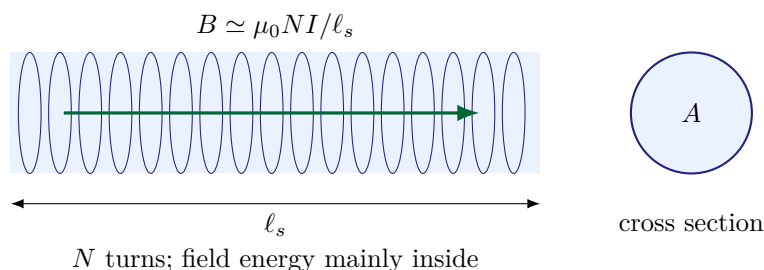


Figure 10.7: Long-solenoid model. The cross-sectional area is shown as an area, not a length. End and external fields are neglected in $L \simeq \mu_0 N^2 A / \ell_s$.

Example 10.11: Air-core solenoid inductance and flux linkage

An air-core solenoid has $N = 800$ turns, length $\ell_s = 0.40$ m and cross-sectional area $A = 5.0 \times 10^{-4} \text{ m}^2$. Estimate its inductance. If it carries $I = 1.5$ A, estimate the flux through one turn.

Solution. The turn density is

$$n = \frac{N}{\ell_s} = \frac{800}{0.40} = 2.0 \times 10^3 \text{ m}^{-1}.$$

The inductance is

$$L = \mu_0 n^2 A \ell_s = (4\pi \times 10^{-7})(2.0 \times 10^3)^2(5.0 \times 10^{-4})(0.40) = 1.0 \times 10^{-3} \text{ H}.$$

The field inside is $B = \mu_0 n I = (4\pi \times 10^{-7})(2.0 \times 10^3)(1.5) = 3.77 \times 10^{-3} \text{ T}$. The flux through one turn is

$$\Phi_B = BA = (3.77 \times 10^{-3})(5.0 \times 10^{-4}) = 1.89 \times 10^{-6} \text{ Wb}.$$

The flux linkage is $N\Phi_B = 1.51 \times 10^{-3} \text{ Wb}$, and $N\Phi_B/I = 1.0 \times 10^{-3} \text{ H}$, consistent with the inductance estimate.

Guided checks

Check 1. In the long-solenoid formula $L = \mu_0 n^2 A \ell_s$, does L depend on current?

Answer. Not for an ideal linear air-core solenoid.

Check 2. If the number of turns per unit length is doubled while A and ℓ_s remain fixed, how does L change?

Answer. It increases by a factor of four because $L \propto n^2$.

Check 3. Why is an iron core not simply included by replacing μ_0 with a universal constant?
Answer. Magnetic materials can be nonlinear and history-dependent; their effective permeability depends on material, field strength and operating conditions.

10.8.2 Inductance as a property of a complete circuit

The steps used for a solenoid are general: assume a current, compute its field, integrate the linked flux, and divide by the current. The relation is linear only if geometry and magnetic response remain fixed. For a solenoid at fixed N and A , increasing length lowers the turn density and gives $L \simeq \mu_0 N^2 A / \ell_s$. At fixed turn density n , increasing length adds turns and gives $L \propto \ell_s$. These are different design changes, not contradictory scalings.

Inductance also exists when there is no visibly wound coil. A pair of separated conductors and its return path enclose field energy. Moving the return wire changes the field distribution and the loop inductance. Thus “the inductance of a wire” needs a specified return geometry or a carefully defined partial-inductance convention. We use complete circuits here. Section 10.16.2 derives an external inductance per unit length for a coaxial return path.

The definition $L = \lambda / I$ uses self linkage, not an independent external field. A permanent magnet can contribute linkage even at $I = 0$; that term must be written separately. The self-inductance remains well defined at zero current by the linear coefficient or limiting derivative, rather than by numerically evaluating $0/0$.

Example 10.12: Scaling a solenoid without hiding the constraints

An air-core solenoid is to have four times its original inductance. Compare doubling its turns at fixed length and area with doubling its length at fixed turn density and area.

Reasoning. At fixed A and ℓ_s , $L \propto N^2$, so doubling N gives $4L$. The same wire gauge roughly doubles the wire length and resistance if the winding radius stays comparable. At fixed n and A , doubling ℓ_s gives only $2L$, because it doubles the field volume while preserving the field at a given current.

Interpretation. A target inductance alone does not specify a viable design. Winding space, wire resistance, operating current and end effects must also be checked. The scaling laws describe the magnetic idealisation; they do not promise that arbitrarily many turns will fit in the original volume.

10.9 Self-induction

If the current in an inductor changes, the flux linkage produced by that current also changes. Faraday’s law then implies an induced emf in the same circuit. This is self-induction. Starting from

$$N\Phi_B = LI \quad (10.33)$$

for a constant linear inductance, Faraday’s law gives

$$\mathcal{E}_L = -\frac{d(N\Phi_B)}{dt} = -L\frac{dI}{dt}. \quad (10.34)$$

The self-induced emf opposes the change in current. If the current is increasing, the inductor’s emf acts against the increase. If the current is decreasing, the inductor’s emf acts to maintain the current.

This statement should not be exaggerated. An inductor does not make current changes impossible. It makes rapid changes require large voltage. In a real circuit, a very abrupt attempt to interrupt current

through an inductor can produce a large voltage spike, sometimes large enough to cause a spark or damage a switch. This is the reason for flyback diodes across relay coils and transient-suppression design in electronics.

10.9.1 Self-induced emf and terminal voltage

Choose a positive reference current I from terminal a to terminal b . Define the terminal voltage $v_L = v_a - v_b$, so the reference current enters the terminal marked positive. This is the passive sign convention. In the ideal fixed-inductor model,

$$v_L = L\dot{I}, \quad \mathcal{E}_L = -L\dot{I}. \quad (10.35)$$

The induced emf is an internal electromotive contribution along the chosen current path; the terminal voltage is the externally applied voltage drop that balances it. There is no sign contradiction. A coil with winding resistance R_w has $v_{ab} = R_w I + L \dot{I}$, as long as parasitic capacitance and other losses are negligible.

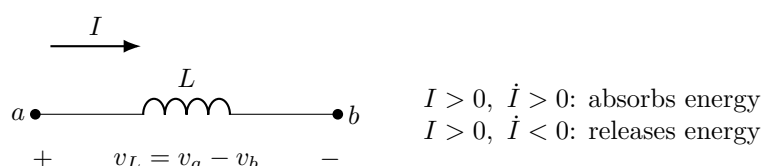


Figure 10.8: Reference polarity and current for an ideal inductor. The marked + is a voltage reference, not a claim that terminal a is always at higher potential.

Figure 10.8 fixes the references used in the circuit equations. The absorbed power is $p_L = v_L I = L I \dot{I}$. Its sign depends on *both* I and $\dot{I}$. If $I < 0$ and $\dot{I} > 0$, the magnitude of current is decreasing and the inductor releases energy. A positive derivative by itself does not prove energy absorption. In all cases $p_L > 0$ when I^2 increases.

Example 10.13: Voltage needed to ramp current in an inductor

A 25 mH inductor is driven so that its current increases linearly from 0 to 4.0 A in 2.0 ms. What is the magnitude of the self-induced emf?

Solution. The current slope is

$$\frac{dI}{dt} = \frac{4.0 \text{ A}}{2.0 \times 10^{-3} \text{ s}} = 2.0 \times 10^3 \text{ A/s.}$$

The emf magnitude is

$$|\mathcal{E}_L| = L \left| \frac{dI}{dt} \right| = (25 \times 10^{-3})(2.0 \times 10^3) = 50 \text{ V}.$$

The sign is opposite the assumed increasing-current direction in Faraday's-law orientation. In circuit design terms, a source must provide about 50 V across the ideal inductor to produce this linear ramp.

Caution

A common phrase is that an inductor “opposes current.” More precisely, an ideal inductor opposes changes in current. A steady current through an ideal inductor requires no inductor voltage, although real coils also have resistance.

10.9.2 Continuity and the voltage–time area

Integrating the ideal terminal relation between two times gives

$$I(t_2) - I(t_1) = \frac{1}{L} \int_{t_1}^{t_2} v_L(t) dt. \quad (10.36)$$

A bounded voltage over a vanishing interval gives a vanishing current change. Thus current is continuous at a finite-voltage switching instant. Voltage and the slope of current can change abruptly. A finite current jump would require a voltage impulse with nonzero integrated area; real switching instead involves capacitance, a discharge path, or breakdown.

The integral also explains current ripple in switched circuits. If $v_L \simeq 5.0 \text{ V}$ for $4.0 \mu\text{s}$ across $100 \mu\text{H}$, then $\Delta I = v_L \Delta t / L = 0.20 \text{ A}$. To return to the same current after a periodic cycle, the net voltage–time area across an ideal constant- L inductor must be zero. That statement concerns the *inductor* voltage, not the full supply voltage if resistive drops are present.

10.10 RL circuits

A resistor-inductor circuit is the magnetic analogue of the resistor-capacitor transient circuit. In an RL circuit, with finite terminal voltage the current cannot jump instantaneously because that would require an unbounded dI/dt and hence an infinite inductor voltage. Instead, the current approaches its final value exponentially.

Consider a series circuit with a constant source emf $\mathcal{E}_0$, resistance R and inductance L . With the switch closed at $t = 0$ and current initially zero, Kirchhoff's loop rule gives

$$\mathcal{E}_0 - IR - L \frac{dI}{dt} = 0. \quad (10.37)$$

Rearranging,

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{\mathcal{E}_0}{L}. \quad (10.38)$$

The solution is

$$I(t) = \frac{\mathcal{E}_0}{R} \left(1 - e^{-t/\tau_L} \right), \quad \tau_L = \frac{L}{R}. \quad (10.39)$$

When the source is removed and the inductor discharges through the resistor, the equation becomes

$$L \frac{dI}{dt} + RI = 0, \quad (10.40)$$

with solution

$$I(t) = I_0 e^{-t/\tau_L}. \quad (10.41)$$

10.10.1 Deriving the exponential response

The exponential follows from a feedback relation: as current grows, the resistor consumes more of the applied voltage, leaving less voltage to change the current. Define $I_\infty = \mathcal{E}_0/R$ and the remaining gap $J = I_\infty - I$. The differential equation gives

$$\dot{J} = -\frac{R}{L} J, \quad \frac{dJ}{J} = -\frac{R}{L} dt.$$

Integrating from 0 to t ,

$$\ln \frac{J(t)}{J(0)} = -\frac{Rt}{L}, \quad J(t) = J(0) e^{-tR/L}.$$

For an arbitrary initial current $I(0) = I_0$, this yields

$$I(t) = I_\infty + (I_0 - I_\infty) e^{-t/\tau_L}. \quad (10.42)$$

The familiar growth expression is its $I_0 = 0$ special case. Setting the source voltage to zero instead gives decay. These derivations assume $R > 0$, $L > 0$, constant component values and a closed circuit. For $R = 0$ and constant applied voltage, the current grows linearly, $I = I_0 + \mathcal{E}_0 t/L$, rather than approaching a finite limit.

The time constant is the time for the remaining gap to decrease to $1/e$ of its initial value. After one τ_L , growth from zero reaches 63.2% of I_∞ ; decay leaves 36.8% of I_0 . At early times $1 - e^{-t/\tau_L} \simeq t/\tau_L$, so $I \simeq \mathcal{E}_0 t/L$ and the initial slope is independent of R . The resistance controls the final value as well as the time constant, so a shorter time constant does not necessarily mean a larger current at a specified time.

10.10.2 Circuit topology at turn-off

Figure 10.9 shows the two *equivalent* circuits used for growth and decay. The transition must preserve a path for current, for example with a bypass switch or diode. Removing a battery and leaving the loop open is a different physical problem: the ideal RL equation alone then supplies no valid turn-off model.

The resistance in $\tau_L = L/R$ is the resistance actually seen by the inductor in the chosen switch state. Growth and decay can therefore have different time constants. Include winding resistance, any current-sense resistor, and the relevant source resistance. For a more complicated linear resistive network connected to one inductor, replace the network by its Thevenin voltage V_{th} and resistance R_{th} ; then $\tau = L/R_{th}$ and $I_\infty = V_{th}/R_{th}$. This is a circuit reduction, not a new induction law.

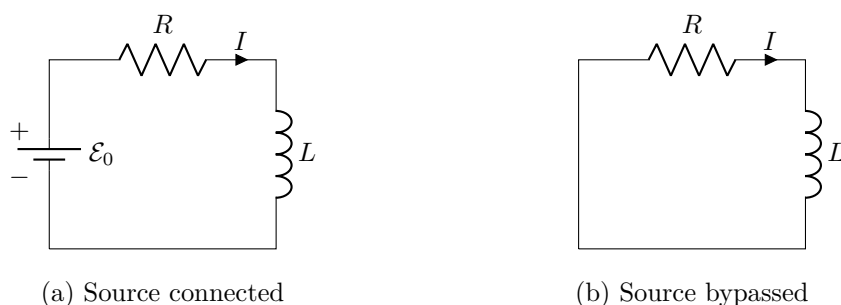


Figure 10.9: Equivalent RL circuits before and after a switching operation. In (b) the battery is removed while a closed path through R remains. Simply opening the loop does not produce the ideal exponential discharge shown here. Current retains its direction at switching.

Example 10.14: Current growth in an RL circuit

A 12 V battery is connected in series with $R = 6.0 \Omega$ and $L = 0.30 \text{ H}$. The switch is closed at $t = 0$, with zero initial current. Find the time constant, the final current, the current at $t = 0.10 \text{ s}$ and the inductor voltage at that time.

Solution. The time constant is

$$\tau_L = \frac{L}{R} = \frac{0.30}{6.0} = 0.050 \text{ s}.$$

The final current is

$$I_\infty = \frac{\mathcal{E}_0}{R} = \frac{12}{6.0} = 2.0 \text{ A}.$$

At $t = 0.10 \text{ s} = 2\tau_L$,

$$I(t) = 2.0(1 - e^{-2}) = 1.73 \text{ A}.$$

The resistor voltage is approximately $IR = 10.4 \text{ V}$. Keeping guard digits, Kirchhoff's rule and the inductor law agree:

$$V_L = \mathcal{E}_0 - IR = L \frac{dI}{dt} = \mathcal{E}_0 e^{-t/\tau_L} = 12e^{-2} = 1.62 \text{ V}.$$

Example 10.15: Current decay and stored energy

An inductor of $L = 0.80 \text{ H}$ initially carries $I_0 = 3.0 \text{ A}$ and is then connected across a resistor $R = 20 \Omega$. Find the current after 0.050 s and the initial magnetic energy.

Solution. The time constant is $\tau_L = L/R = 0.80/20 = 0.040 \text{ s}$. The current is

$$I(0.050) = 3.0e^{-0.050/0.040} = 0.860 \text{ A}.$$

The initial stored energy is

$$U_B(0) = \frac{1}{2}LI_0^2 = \frac{1}{2}(0.80)(3.0)^2 = 3.6 \text{ J}.$$

This energy is eventually dissipated as thermal energy in the resistor, assuming no other losses.

Guided checks

Check 1. What is the current through an ideal inductor immediately after a switch is closed if the current was initially zero?

Answer. It is still zero at the instant just after closing; it cannot jump discontinuously.

Check 2. What is the long-time voltage across an ideal inductor in a DC RL circuit?

Answer. Zero, because $dI/dt = 0$.

Check 3. Does a larger resistance make τ_L larger or smaller?

Answer. Smaller, since $\tau_L = L/R$.

10.10.3 Voltage and energy traces

For growth from zero, the two passive voltage drops are

$$v_R = \mathcal{E}_0(1 - e^{-t/\tau_L}), \quad v_L = \mathcal{E}_0 e^{-t/\tau_L}. \quad (10.43)$$

Their sum stays equal to the source voltage. At $t = 0^+$ the entire source voltage is across the inductor, even though its current is zero. At long times the ideal inductor drop is zero and the resistor takes the full voltage. In source-free decay, retaining the reference current direction gives $v_R = RI > 0$ and $v_L = -RI < 0$: the inductor supplies energy.

Figure 10.10 makes these comparisons on a common dimensionless time axis. It also anticipates $U_B \propto I^2$, derived next. Current and energy have different decay rates: $I/I_0 = e^{-t/\tau_L}$, but $U_B/U_B(0) = e^{-2t/\tau_L}$. The time to lose half the stored energy is $\tau_L \ln 2/2$, not $\tau_L \ln 2$.

10.10.4 Switching between stages

For a sequence of constant source values, solve each stage with its own I_∞ and τ_L , and carry the final current into the next stage. Reset the elapsed-time variable at every switch. The current stays continuous, but neither the final value nor its slope generally stays the same. A useful check is to evaluate the solution at both sides of every switching time before inserting numerical values.

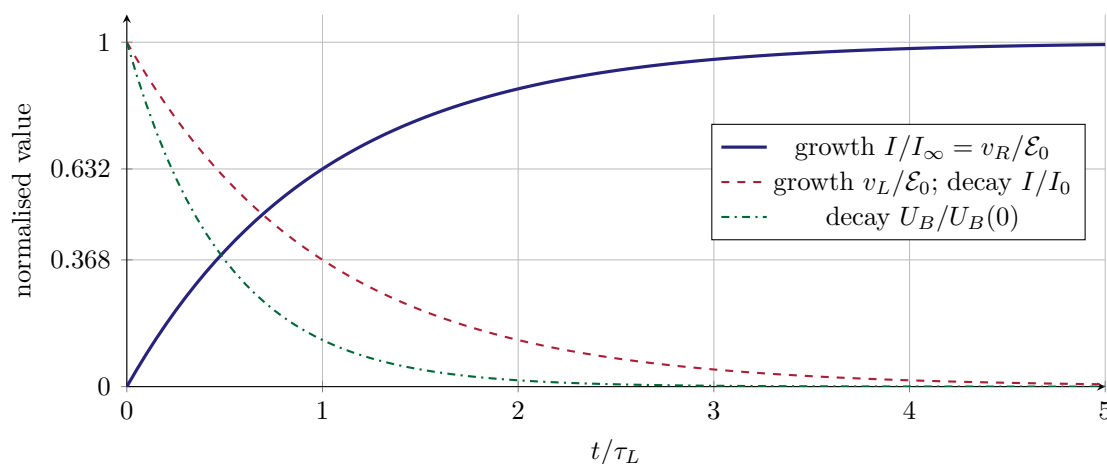


Figure 10.10: Exact normalised RL traces for the stated ideal circuits. The current scale is the final growth current or the initial decay current as indicated. The energy scale is the initial stored energy in decay.

Example 10.16: Piecewise switching of an RL current

A series RL circuit has $R = 5.0 \Omega$ and $L = 0.20 \text{ H}$. It is connected to a 10 V source for 0.080 s and then the source is replaced by a short circuit through the same resistor. Find the current immediately before switching off and 0.040 s after switching off.

Solution. The time constant is $\tau_L = L/R = 0.040 \text{ s}$ and the source-on final current would be $I_\infty = 10/5.0 = 2.0 \text{ A}$. At $t = 0.080 \text{ s} = 2\tau_L$,

$$I = 2.0(1 - e^{-2}) = 1.73 \text{ A}.$$

Current through an inductor is continuous, so this is the initial current for the decay stage. After another $0.040 \text{ s} = \tau_L$,

$$I = 1.73e^{-1} = 0.637 \text{ A}.$$

The current does not jump at the switching instant, but its derivative changes because the circuit equation changes.

10.11 Magnetic energy and energy density

An inductor stores energy in its magnetic field. The result follows from the power needed to increase the current. For an ideal inductor, the voltage associated with changing current is

$$V_L = L \frac{dI}{dt} \quad (10.44)$$

when using the passive sign convention. The instantaneous power delivered to the inductor is

$$P = IV_L = LI \frac{dI}{dt}. \quad (10.45)$$

Since $P = dU_B/dt$,

$$dU_B = LI dI. \quad (10.46)$$

Integrating from zero current to I gives

$$U_B = \frac{1}{2} LI^2. \quad (10.47)$$

This is analogous to the energy stored in a capacitor, $U_E = CV^2/2$, but the storage variable is current rather than voltage.

For a long solenoid, substituting $L = \mu_0 n^2 A \ell_s$ and $B = \mu_0 n I$ into $U_B = LI^2/2$ gives

$$U_B = \frac{1}{2}(\mu_0 n^2 A \ell_s) I^2 = \frac{B^2}{2\mu_0} (A \ell_s). \quad (10.48)$$

Since $A \ell_s$ is the volume of the field region, the magnetic energy density in vacuum is

$$u_B = \frac{B^2}{2\mu_0}. \quad (10.49)$$

Example 10.17: Energy stored in a solenoid field

A long air-core solenoid has radius 2.0 cm, length 0.50 m and 1000 turns. It carries 2.0 A. Estimate its inductance and stored magnetic energy.

Solution. The area is $A = \pi(0.020)^2 = 1.26 \times 10^{-3} \text{ m}^2$ and $n = 1000/0.50 = 2.0 \times 10^3 \text{ m}^{-1}$. Thus

$$L = \mu_0 n^2 A \ell_s = (4\pi \times 10^{-7})(2.0 \times 10^3)^2(1.26 \times 10^{-3})(0.50) = 3.16 \times 10^{-3} \text{ H}.$$

The energy is

$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}(3.16 \times 10^{-3})(2.0)^2 = 6.3 \times 10^{-3} \text{ J}.$$

The field is $B = \mu_0 n I = 5.03 \times 10^{-3} \text{ T}$. The energy-density method gives

$$u_B = \frac{B^2}{2\mu_0} = 10.1 \text{ J/m}^3,$$

multiplied by volume $A \ell_s = 6.28 \times 10^{-4} \text{ m}^3$, again giving $6.3 \times 10^{-3} \text{ J}$.

Key message

In the geometric-inductance model used here, the stored energy is magnetic-field energy. Carrier kinetic inductance is neglected; it can matter in other regimes, for example some superconducting devices. When the current changes, energy flows between the circuit and the field.

Caution

The formula $u_B = B^2/(2\mu_0)$ is the vacuum or air-core expression. In magnetic materials, energy density can be more subtle because $\mathbf{B}$ and $\mathbf{H}$ may be nonlinear and hysteretic.

10.11.1 A complete energy budget during current growth

Multiplying the loop equation by current gives an instantaneous conservation statement,

$$\mathcal{E}_0 I = RI^2 + \frac{dU_B}{dt}. \quad (10.50)$$

The source supplies energy; part heats the resistor and part accumulates in the field. This identity is valid at every time, not just after a transient. For zero initial current, let $x = t/\tau_L$ and $U_\infty = LI_\infty^2/2$.

Integrating each term from zero to the finite time t gives

$$W_s(t) = \frac{\mathcal{E}_0^2}{R} [t - \tau_L(1 - e^{-x})], \quad (10.51)$$

$$Q_R(t) = \frac{\mathcal{E}_0^2}{R} \left[t - 2\tau_L(1 - e^{-x}) + \frac{\tau_L}{2}(1 - e^{-2x}) \right], \quad (10.52)$$

$$U_B(t) = \frac{1}{2}LI_\infty^2(1 - e^{-x})^2. \quad (10.53)$$

These expressions follow by integrating $1 - e^{-x}$ and $(1 - e^{-x})^2$; expanding the latter before integrating avoids missing the cross term. Direct subtraction verifies $W_s - Q_R = U_B$.

At one time constant,

$$\frac{W_s}{U_\infty} = \frac{2}{e} = 0.7358, \quad \frac{U_B}{U_\infty} = (1 - e^{-1})^2 = 0.3996, \quad \frac{Q_R}{U_\infty} = 0.3362.$$

There is no general rule that growth heating equals the final magnetic energy. If the source remains connected indefinitely, the resistor continues heating at $\mathcal{E}_0^2/R$ and its accumulated heat diverges, while magnetic energy approaches a finite limit. A claim about “heat during establishment” is meaningless without a time interval or a precisely defined subtraction of steady heating.

Example 10.18: Checking the finite-time growth budget

Use the earlier 12 V, 6.0 Ω , 0.30 H circuit. How much energy has the source supplied after 0.050 s, and where is it?

Strategy. This time equals τ_L . The final current is 2.0 A, giving $U_\infty = \frac{1}{2}(0.30)(2.0)^2 = 0.60$ J. Use the integrated budget rather than multiplying final power by elapsed time.

Result. $W_s = 0.7358(0.60) = 0.441$ J, $U_B = 0.3996(0.60) = 0.240$ J, and $Q_R = 0.3362(0.60) = 0.202$ J. The sum of field energy and heat agrees with the source input within rounding.

Check. Both powers start at zero because $I(0) = 0$, even though the inductor voltage is initially 12 V. Treating the resistor as if it carried the final current throughout the interval would greatly overestimate the heat.

10.11.2 Discharge and field-energy integration

For source-free decay through R , $I = I_0 e^{-t/\tau_L}$ and

$$Q_R(0, t) = \int_0^t RI_0^2 e^{-2t'/\tau_L} dt' = \frac{1}{2}LI_0^2(1 - e^{-2t/\tau_L}).$$

Consequently, the total heat over a complete discharge is exactly the *initial* magnetic energy. This equality is correct for discharge and should not be imported into the continuously driven growth problem.

For an arbitrary vacuum field distribution the field description is

$$U_B = \int_{\text{all space}} \frac{B^2}{2\mu_0} dV. \quad (10.54)$$

The integration region must include fringing and leakage fields if they are appreciable. A spatially varying field cannot be replaced by its average before squaring: generally $\langle B^2 \rangle \neq \langle B \rangle^2$. In a fixed linear magnetic medium the reversible expression can be written $\mathbf{B} \cdot \mathbf{H}/2$ under the appropriate constitutive assumptions. Hysteresis requires an explicit account of loss; it is discussed in Chapter 12.

10.12 Mutual induction

Mutual induction occurs when a changing current in one circuit changes the magnetic flux through another circuit. If current I_1 in coil 1 produces flux linkage $N_2\Phi_{21}$ through coil 2, the mutual inductance is

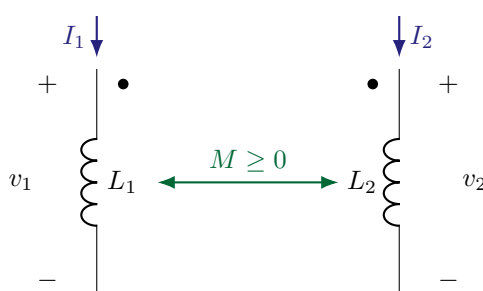
$$M_{21} = \frac{N_2\Phi_{21}}{I_1}. \quad (10.55)$$

For fixed geometry and linear reciprocal magnetic media, the mutual inductance is symmetric: $M_{21} = M_{12} = M$. The mutual contribution to the emf in coil 2 is

$$\mathcal{E}_2 = -M \frac{dI_1}{dt}, \quad (10.56)$$

and similarly

$$\mathcal{E}_1 = -M \frac{dI_2}{dt}. \quad (10.57)$$



Both reference currents enter dots.
Mutual terms in terminal voltages have positive signs.

Figure 10.11: Coupled-coil reference convention. Dots encode relative winding orientation; the central arrow denotes coupling, not a magnetic field line. With these references, $v_2 = L_2\dot{I}_2 + M\dot{I}_1$.

Mutual induction is the principle behind transformers. In an ideal transformer, the same time-dependent core flux links primary and secondary windings. If losses and leakage flux are neglected, the emf ratio is approximately the turns ratio:

$$\frac{|\mathcal{E}_2|}{|\mathcal{E}_1|} = \frac{N_2}{N_1} \quad (\text{common flux, corresponding windings}). \quad (10.58)$$

This transformer relation will be used more fully in the AC circuits chapter, where sinusoidal steady-state operation is studied.

Example 10.19: Induced emf from mutual inductance

Two coils have mutual inductance $M = 18 \text{ mH}$. The current in coil 1 changes according to $I_1(t) = (0.50 \text{ A/s}^2)t^2$, where t is in seconds. Find the magnitude of the emf induced in coil 2 at $t = 3.0 \text{ s}$.

Solution. The current derivative is

$$\frac{dI_1}{dt} = (1.0 \text{ A/s}^2)t.$$

At $t = 3.0 \text{ s}$, $dI_1/dt = 3.0 \text{ A/s}$. Therefore

$$|\mathcal{E}_2| = M \left| \frac{dI_1}{dt} \right| = (18 \times 10^{-3})(3.0) = 5.4 \times 10^{-2} \text{ V}.$$

The sign depends on the winding orientation convention; the magnitude is 54 mV.

Guided checks

Check 1. If the current in coil 1 is constant, is there an induced emf in coil 2 from mutual induction?

Answer. No, not from mutual induction, because $dI_1/dt = 0$.

Check 2. What does a larger mutual inductance mean physically?

Answer. More flux linkage in one coil per unit current in the other.

Check 3. Why are transformer cores made of magnetic material?

Answer. To guide and enhance shared magnetic flux, increasing coupling between windings.

10.12.1 Winding signs and the meaning of mutual emf

Figure 10.11 fixes a circuit convention: both currents enter dotted terminals, and both voltages are measured from dot to undotted terminal. Write $M \geq 0$ for the coupling magnitude. Then

$$\lambda_1 = L_1 I_1 + M I_2, \quad \lambda_2 = M I_1 + L_2 I_2,$$

and the passive terminal relations contain *positive* mutual terms, $v_2 = L_2 \dot{I}_2 + M \dot{I}_1$. The Faraday emf referred along the winding current path has the opposite sign. If the secondary is open and $\dot{I}_1 > 0$, then $I_2 \simeq 0$ and $v_2 = M \dot{I}_1 > 0$: its dotted terminal is positive. If a resistor closes the secondary, current leaves that positive terminal through the resistor; expressed in our reference it has $I_2 < 0$. This is Lenz's opposition to the increasing primary linkage.

Reversing one winding reference changes the algebraic sign of the mutual terms. One may instead use a signed M ; then the coupling magnitude is $k = |M|/\sqrt{L_1 L_2}$. The two conventions are equivalent, but mixing them produces incorrect signs. Section 10.16.3 derives the energy bound and the reaction of a loaded secondary on the primary.

10.12.2 A geometrical estimate of mutual inductance

Place a secondary winding of N_2 turns around the central part of a long primary solenoid with N_1 turns, length ℓ_s , and field area A_1 . If the secondary encloses that whole area, the primary field is approximately $\mu_0 N_1 I_1 / \ell_s$, so

$$M \simeq \frac{\mu_0 N_1 N_2 A_1}{\ell_s}. \quad (10.59)$$

Making the secondary loop larger does not increase the linked primary flux once it already encloses the entire ideal field region. A smaller coaxial secondary within the uniform interior instead links only its own area A_2 . This is the same overlap-area reasoning used for the induced electric field outside a solenoid.

Example 10.20: Choosing the flux area for a pickup winding

A long air-core primary has $N_1 = 1000$, $\ell_s = 0.50$ m and radius 2.0 cm. A coaxial $N_2 = 40$ turn pickup of radius 3.0 cm surrounds its central region. Find the approximate M and the open-circuit secondary voltage magnitude when $\dot{I}_1 = 20$ A/s.

Model and calculation. The field area is $A_1 = \pi(0.020)^2$, not the pickup area. Hence $M \simeq 1.26 \times 10^{-4}$ H and $|v_2| = M|\dot{I}_1| = 2.53$ mV. With corresponding dots and increasing positive primary current, the pickup dot is positive.

Check. The units of $M\dot{I}$ are henry ampere per second, or volt. Using the pickup area would overestimate the result by $(3/2)^2 = 2.25$. Field leakage and end effects determine the accuracy of the long-solenoid estimate.

10.13 Engineering and applied-science connections

Induction is unusually rich in engineering applications because it converts between mechanical, magnetic and electrical forms of energy. This section surveys several applications while keeping the assumptions visible.

10.13.1 Generators

A simple generator consists of a coil rotating in a magnetic field. If the coil has N turns, area A , angular speed ω and field B , then for an ideal orientation

$$\Phi_B(t) = BA \cos(\omega t), \quad (10.60)$$

so

$$\mathcal{E}(t) = NBA\omega \sin(\omega t). \quad (10.61)$$

The amplitude is $\mathcal{E}_0 = NBA\omega$. The mechanical torque required to rotate the coil under load is the mechanical counterpart of the electrical power delivered.

For a coil with its normal initially aligned with $\mathbf{B}$, the flux starts at its maximum while the emf starts at zero. After a quarter turn the normal is perpendicular to $\mathbf{B}$: the flux is zero and the emf reaches its positive maximum. Figure 10.12 shows these exactly calculated phase relations. The phase is fixed by differentiation, not by an arbitrary time shift added after the calculation.

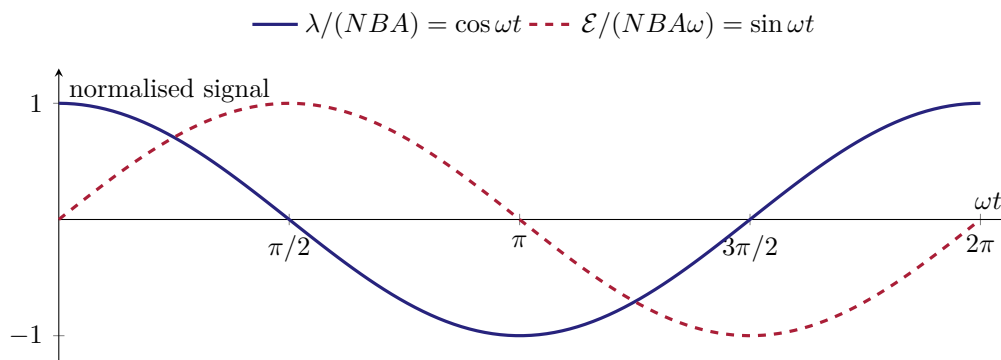


Figure 10.12: Flux linkage and emf for a coil rotating at constant angular speed. Curves are normalised separately; the emf amplitude grows with ω while the flux amplitude does not.

Under a purely resistive load R_{tot} and negligible coil inductance, $I = \mathcal{E}/R_{\text{tot}}$. The magnetic torque opposes the imposed rotation. Its magnitude is $N|I|AB|\sin \theta|$, and multiplication by angular speed gives

$$\tau_{\text{ext}}\omega = I\mathcal{E} = \frac{\mathcal{E}^2}{R_{\text{tot}}}.$$

The magnetic field determines the coupling, but the energy comes from the drive. A generator at open circuit can exhibit a substantial emf while delivering almost no electrical power. Connecting a load increases current and reaction torque. With appreciable inductance, current is no longer instantaneously proportional to emf; that phase-dependent case belongs to Chapter 11.

For a sinusoid, the root-mean-square value is defined by $\mathcal{E}_{\text{rms}}^2 = T^{-1} \int_0^T \mathcal{E}^2 dt = \mathcal{E}_0^2/2$. This short definition suffices for power estimates here; a full treatment of rms quantities and power factor follows in the AC chapter.

Example 10.21: Rotating coil design estimate

A small generator coil has $A = 2.5 \times 10^{-3} \text{ m}^2$ and rotates at 50 Hz in a field of 0.30 T. How many turns are needed for an emf amplitude of approximately 20 V?

Solution. The angular speed is $\omega = 2\pi f = 314 \text{ rad/s}$. The ideal amplitude is $\mathcal{E}_0 = NBA\omega$, so

$$N = \frac{\mathcal{E}_0}{BA\omega} = \frac{20}{(0.30)(2.5 \times 10^{-3})(314)} = 84.9.$$

A design estimate would choose about 85 turns, before accounting for winding resistance, load, mechanical losses and magnetic-circuit details.

10.13.2 Eddy-current braking

When a conducting sheet enters or leaves a finite magnetic-field region, changing flux can drive circulating currents within the sheet. Uniform translation wholly inside a static uniform field need not produce eddy currents. These currents dissipate energy through the sheet's resistance and produce magnetic forces opposing the motion. This is the principle of eddy-current braking in some trains, exercise machines and laboratory demonstrations. The effect is contactless, smooth and speed-dependent, but it produces heating and depends strongly on conductor geometry and field distribution.

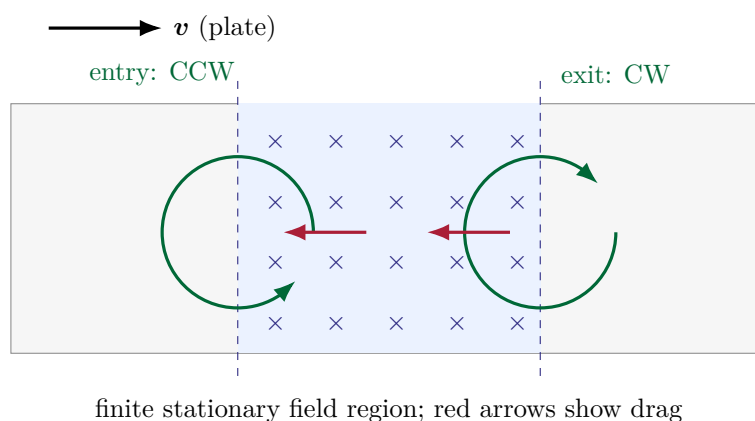


Figure 10.13: Qualitative eddy-current loops near the edges of a finite into-page field region as the plate moves right. The two circulations differ, but the force on the portions inside the field is leftward in both cases. These loops illustrate direction, not a solved current-density map.

Figure 10.13 makes the field boundaries explicit. As an element of plate enters an into-page field, its into-page flux increases, so a local counterclockwise circulation is induced. On exit the flux decreases and the circulation reverses. The forces on current-carrying parts within the field oppose translation. Actual currents form a continuous two-dimensional distribution, not independent round wires; the drawing is a sign model.

Cutting slots across the large current paths increases their effective resistance and greatly weakens damping. This explains the contrast between a solid conducting pendulum bob and a slotted one. Insulated laminations in a transformer core likewise restrict large eddy-current paths; their planes are chosen to carry the useful magnetic flux while making the principal induced current loops narrow. A changing flux is still present, but the unwanted heating is reduced.

A falling magnet in a copper pipe is another version of the same mechanism. The flux through circular paths changes differently above and below the magnet; the net force opposes descent. The pipe need not be ferromagnetic. At terminal speed gravitational power mgv equals the electrical losses. If the magnet is held still, the currents decay and the eddy-current drag disappears. A

quantitative force requires the nonuniform magnet field and the pipe's distributed current paths; the sliding-rod coefficient cannot simply be reused for this geometry.

10.13.3 Wireless charging and inductive coupling

Wireless charging pads use time-dependent currents in a transmitter coil to create time-dependent magnetic flux. A receiver coil in the device intercepts part of this flux and develops an induced emf. Practical systems use resonant circuits, ferrite flux guides, power electronics and communication between charger and device to control efficiency and safety. The basic physics, however, is mutual induction.

10.13.4 Pickup coils and sensors

A pickup coil converts changing magnetic flux into voltage. In a guitar pickup, ferromagnetic strings disturb the magnetic flux through a coil as they vibrate. In a speed sensor, a toothed wheel modulates flux through a coil. In magnetic diagnostics, search coils measure time-varying magnetic fields through calibrated coil area and turns. The measured voltage is proportional to $-Nd\Phi_B/dt$, so recovering $B(t)$ may require numerical integration.

10.13.5 Switching circuits and transient protection

Inductors in relays, motors and power converters store magnetic energy. When current is interrupted, the inductor attempts to keep current flowing, which can produce a large voltage across the switch. Protective components such as flyback diodes, snubber networks and transient-voltage suppressors provide controlled paths for the stored energy.

10.13.6 Turn-off time as an energy-management choice

Suppose a coil initially carries $I_0 > 0$ and its supply is disconnected. A dissipative clamp opposes the old current with approximately constant drop $V_c > 0$, while winding resistance is R_w . During conduction,

$$L\dot{I} + R_w I + V_c = 0, \quad I(t) = \left(I_0 + \frac{V_c}{R_w}\right) e^{-R_w t/L} - \frac{V_c}{R_w}. \quad (10.62)$$

This expression holds only until the current first reaches zero. A unidirectional diode or clamp then stops conducting; extending the expression to negative current would describe a different circuit. Setting $I(t_{\text{off}}) = 0$ gives

$$t_{\text{off}} = \frac{L}{R_w} \ln \left(1 + \frac{R_w I_0}{V_c}\right) \rightarrow \frac{L I_0}{V_c} \quad (R_w \rightarrow 0). \quad (10.63)$$

The constant-voltage approximation produces nearly linear decay when $V_c \gg R_w I$. A low-voltage flyback path reduces voltage stress but prolongs current and therefore can delay release of an actuator. A higher-voltage clamp reduces turn-off time at the cost of a larger required blocking voltage. The actual switch voltage depends on circuit topology and supply voltage; V_c here is the opposing clamp drop in the discharge loop.

Example 10.22: Two turn-off choices for the same coil

A coil has $L = 0.20$ H, $R_w = 10 \Omega$, and $I_0 = 0.50$ A. Compare idealised constant clamp drops 0.70 V and 24 V.

Calculation. The resistance time scale is $L/R_w = 20$ ms. The low-voltage path gives $t_{\text{off}} = 20 \ln(1 + 5/0.70)$ ms = 41.9 ms. The 24 V clamp gives $20 \ln(1 + 5/24)$ ms = 3.78 ms.

Energy and limits. The initial energy is $LI_0^2/2 = 0.025$ J in both cases. It is divided between winding heat and clamp absorption, not removed by choosing a smaller voltage. Real diode

voltage varies with current, so the constant-0.70 V result is a model estimate, especially near zero current. The ideal pure-inductor estimate LI_0/V_c gives 4.17 ms for the high-voltage clamp, close because V_c dominates the winding drop.

Take-home message

In engineering, induction is rarely only about finding a voltage. One must also ask where the energy comes from, where it goes, how rapidly the current can change, what voltages appear during switching, what losses occur, and how geometry controls flux linkage.

10.14 Laboratory and computational connection

The most useful experimental checks test a relationship, not just a single number. A moving magnet and pickup coil test the sign and integral of Faraday's law. A controlled RL step tests a differential equation and its time scale. A numerical model can then distinguish a genuine physical effect from a sampling or integration error.

10.14.1 Reconstructing flux change from a pickup signal

Connect a coil to a high-input-impedance voltage sensor with tightly paired leads. Mark a positive coil normal and define the measured voltage polarity to agree with its positive winding emf. Start with the magnet stationary, move it to a second stationary position, and record before, during and after motion. If loading is negligible,

$$\Delta\lambda = - \int_{t_1}^{t_2} \mathcal{E}(t) dt. \quad (10.64)$$

Changing the speed generally changes the peak and duration, but for the same endpoints it leaves the signed integral unchanged. If the magnet begins and ends far away, the total signed area is approximately zero even though each lobe is nonzero. The sum of absolute lobe areas measures something different and must not be substituted for the signed integral.

The measured signal can be modelled as $V_{\text{meas}} = \mathcal{E} + V_{\text{off}} + \eta$, with offset V_{off} and noise η . Estimate the offset from stationary intervals and subtract it before integration. A remaining offset error δV over duration T causes $\delta\lambda = -\delta V T$: integration removes rapid fluctuations effectively but accumulates a systematic baseline error.

For a uniform field aligned with the coil normal, $\Delta B = -\int \mathcal{E} dt / (NA)$. A pickup coil measures changes, so $B(t_0)$ requires independent information if an absolute field is wanted. With a nonuniform field, the integral measures the linked flux change or area-averaged normal field, not necessarily the field at the coil centre.

Example 10.23: A synthetic pickup trace and its calibration

An illustrative, noise-free voltage trace has a triangular positive pulse of height 0.80 V and full duration 20 ms. A 200-turn coil has area $2.0 \times 10^{-4} \text{ m}^2$. Determine the field change for a uniform aligned field and estimate the bias from a residual 2.0 mV offset over a 0.10 s integration window.

Strategy and result. Integrate the pulse geometrically: its area is $(1/2)(0.80)(0.020) = 0.0080 \text{ Vs}$. Thus $\Delta\lambda = -0.0080 \text{ Wb}$ and $\Delta B = -0.0080 / (200 \times 2.0 \times 10^{-4}) = -0.20 \text{ T}$. The field along the chosen normal decreases.

The offset creates a spurious area $(0.0020)(0.10) = 2.0 \times 10^{-4} \text{ V s}$ and a field bias of -0.0050 T , or 2.5% of the pulse result in magnitude. The bias scales with the entire integration window, not only the pulse duration. These numbers illustrate a measurement model; they are not experimental data.

10.14.2 Measuring an inductance from a transient

Use a low-voltage source, a known series sense resistor R_s , and the coil. Measure $v_s = IR_s$ to infer current. Include the coil's winding resistance R_w and source resistance R_g : the total is $R_{\text{tot}} = R_s + R_w + R_g$. For an initially current-free coil, fit

$$v_s(t) = R_s I_\infty \left[1 - \exp\left(-\frac{t - t_0}{\tau_L}\right) \right]$$

after the switching time t_0 , allowing a measured offset if needed. Then $L = R_{\text{tot}} \tau_L$. A decay measurement with a closed discharge path provides an independent check and may have a different resistance.

The instrument should resolve the initial curvature: a sample interval small compared with τ_L is needed, and the record should extend several time constants. The switch rise time must be short compared with τ_L if a step model is used. Systematic residuals may indicate an incorrect t_0 , finite source rise time, heating, saturation or parasitic capacitance. They should not automatically be interpreted as random noise.

Example 10.24: Inductance and an uncertainty budget

An illustrative transient fit gives $\tau_L = (12.0 \pm 0.3) \text{ ms}$. The independently measured total resistance is $(18.0 \pm 0.2) \Omega$. Treat these as independent standard uncertainties. Estimate L and its uncertainty.

Calculation. $L = R\tau_L = 0.216 \text{ H}$. First-order propagation gives

$$\left(\frac{\sigma_L}{L}\right)^2 = \left(\frac{\sigma_R}{R}\right)^2 + \left(\frac{\sigma_\tau}{\tau_L}\right)^2,$$

so $\sigma_L/L = \sqrt{(0.2/18)^2 + (0.3/12)^2} = 0.0274$ and $L = (0.216 \pm 0.006) \text{ H}$.

Interpretation. This uncertainty covers the specified fit and resistance uncertainties, not a failure of the constant- L model. If the fitted value changes systematically with pulse amplitude, test saturation or heating rather than merely collecting more points at the same amplitude.

For flux calibration $\Phi = BA \cos \theta$, independent small standard uncertainties give the more general absolute relation

$$\sigma_\Phi^2 = (A \cos \theta \sigma_B)^2 + (B \cos \theta \sigma_A)^2 + (BA \sin \theta \sigma_\theta)^2.$$

Angles in this expression are in radians. Dividing by Φ^2 produces the familiar relative term $(\tan \theta \sigma_\theta)^2$, but that form is unsuitable when Φ is near zero. Absolute uncertainty remains the meaningful quantity there. Correlated calibration errors require covariance terms.

10.14.3 Numerical simulation of an RL transient

The physical question is whether a numerical method reproduces monotonic relaxation and energy balance on the known time scale. For a constant step source,

$$\dot{I} = \frac{\mathcal{E}_0 - RI}{L}, \quad I_{k+1} = I_k + \Delta t \frac{\mathcal{E}_0 - RI_k}{L}$$

is the explicit Euler update. The example below shows why even a plausible time step can yield a visible bias.

Example 10.25: Numerical time-step check for an RL circuit

For $R = 10\ \Omega$, $L = 0.50\ \text{H}$ and $\mathcal{E}_0 = 5.0\ \text{V}$, use one Euler step with $\Delta t = 0.010\ \text{s}$ starting from $I_0 = 0$. Compare with the analytic current at the same time.

Solution. The initial derivative is

$$\left. \frac{dI}{dt} \right|_0 = \frac{5.0 - 10(0)}{0.50} = 10\ \text{A/s}.$$

One Euler step gives

$$I_1 = 0 + (0.010)(10) = 0.100\ \text{A}.$$

The time constant is $\tau_L = L/R = 0.050\ \text{s}$ and the final current is $0.50\ \text{A}$. The analytic result is

$$I(0.010) = 0.50(1 - e^{-0.010/0.050}) = 0.0906\ \text{A}.$$

The Euler step overestimates because it uses the initial slope throughout the interval. Smaller time steps reduce the error.

10.14.4 Numerical stability is not physical oscillation

Define the error from equilibrium $e_k = I_k - I_\infty$ and $h = \Delta t/\tau_L$. Euler's rule gives $e_{k+1} = (1 - h)e_k$. Hence numerical errors decay only when $|1 - h| < 1$, or $0 < h < 2$. For $1 < h < 2$ they alternate in sign: Euler predicts overshoot and oscillation even though this first-order passive RL system has neither. The response is monotonic for $0 < h \leq 1$; the endpoint $h = 1$ jumps to equilibrium in one numerical step and is very inaccurate. Accuracy usually requires $h \ll 1$, a stronger demand than stability.

For piecewise constant source voltage, the exact update is equally simple:

$$I_{k+1} = I_{\infty,k} + (I_k - I_{\infty,k})e^{-\Delta t/\tau_L}, \quad I_{\infty,k} = V_k/R. \quad (10.65)$$

Align step boundaries with source-switching times. Applying one constant-source update across an unresolved switch is not exact. For variable source values within a step, use an appropriate integrator or the convolution derived in the advanced section.

The following complete Python example uses only the standard library. It compares Euler with the analytic solution and checks the integrated energy budget using trapezoidal quadrature. Values used for current, power and energy are evaluated at the same sampling times.

```

from math import exp

def check_rl(steps):
    R, L, V, T = 6.0, 0.30, 12.0, 0.25
    dt = T / steps
    tau, Iinf = L / R, V / R
    current = 0.0
    source_work = heat = 0.0
    max_error = 0.0
    for j in range(steps):
        new = current + dt * (V - R * current) / L
        source_work += 0.5 * dt * V * (current + new)
        heat += 0.5 * dt * R * (current**2 + new**2)
        current = new
        exact = Iinf * (1.0 - exp(-(j + 1) * dt / tau))
        max_error = max(max_error, abs(current - exact))
    stored = 0.5 * L * current**2
    residual = source_work - heat - stored
    return dt / tau, max_error, residual

for steps in (50, 100, 200, 400):
    print(steps, check_rl(steps))

```

Halving the time step should approximately halve the leading global Euler error once the steps are small. The energy residual should also approach zero. A small residual alone is insufficient: it must be accompanied by the correct waveform and time-step convergence. For comparison, replace the Euler update by Equation (10.65); current then matches the analytic values to roundoff while numerical integration of the powers still has quadrature error.

10.14.5 A reproducible convergence comparison

For the $R = 10\ \Omega$, $L = 0.50\ \text{H}$, $V = 5.0\ \text{V}$ circuit, the exact current after one time constant is $0.316060\ \text{A}$. Repeating the Euler step gives the closed form $I_n = I_\infty[1 - (1 - h)^n]$. At $t = \tau_L$ the comparison is

$\Delta t / \tau_L$	Steps	$I_{\text{Euler}}\ (\text{A})$	Relative error
0.20	5	0.336160	6.36%
0.10	10	0.325661	3.04%
0.05	20	0.320757	1.49%
0.025	40	0.318384	0.735%

The systematic overestimate follows from using the larger, earlier slope throughout each step. These are calculated values, not measured data. Extending this exercise to a switched pulse tests both time-step convergence and continuity at the switching instant.

10.15 Common mistakes and diagnostic questions

Before accepting an induction solution, identify the changing quantity, the orientation, the current-response model and the energy source. These four checks catch errors that a correct unit conversion cannot.

Misleading shortcut	Correct diagnostic question
“The field is nonzero, so there is emf.”	Is the linked flux changing? A constant nonzero flux produces zero emf.
“Lenz’s law opposes the field.”	Which signed flux change is opposed? A decreasing positive flux can produce a positive induced field.
“The current has the emf sign.”	Is the loop resistance dominated, or does it contain stored energy, inductance or another source?
“Use $\oint \mathbf{E} \cdot d\boldsymbol{\ell}$ for every loop.”	Is the actual conductor moving? If so include $\mathbf{u} \times \mathbf{B}$.
“An inductor voltage is always negative.”	Which terminal is the reference positive terminal? Use $v_L = L\dot{I}$ with the passive convention.
“Opening the switch gives exponential decay.”	What closed discharge path remains? Otherwise capacitance or breakdown enters.
“Current and energy have the same decay time.”	Is the observable proportional to I or to I^2 ?
“More mutual coupling means the same efficiency.”	What are the winding losses and load? k describes coupling, not efficiency.

Guided checks

Diagnostic reversal. A loop enters an out-of-page field and then exits it at the same speed. Which signs reverse?

Answer. The flux derivative, emf and resistance-dominated current reverse. Magnetic drag continues to oppose the velocity, and heat production remains positive. A result that reverses the heat or predicts drag aiding the imposed motion in this initially unenergised resistive model is inconsistent.

10.16 Advanced Topics for Technical and Engineering Students

The core chapter can be studied independently of this section. The following modules extend its models in specific directions: moving boundaries, nonuniform field geometry, coupled circuits, nonlinear magnetic response, mechanical motion and induction acceleration. Each begins from the same flux and energy laws. Vector calculus and additional differential-equation reasoning are introduced where needed, rather than assumed throughout the core.

10.16.1 Moving boundaries and the limits of the flux rule

For smooth fields and a fixed surface, Stokes' theorem turns Faraday's circulation law into

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{A} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}.$$

Since the equality holds for arbitrary small surfaces,

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (10.66)$$

This is a local field law. It contains a partial derivative at a fixed position; it does not include the change in flux caused by moving an integration boundary.

To include that motion, compare a material loop at t with its nearby position at $t + dt$. The field changes on the old surface, and a narrow ribbon is swept out by each boundary element. With positive contour orientation paired to the surface normal, the ribbon contributes $-(\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} dt$. Summing the ribbons gives the transport identity

$$\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{A} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} - \oint_{C(t)} (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}. \quad (10.67)$$

The magnetic Gauss law removes dependence on how the interior surface is moved. Combining this identity with the local Faraday law produces Equation (10.13). As a sign check, take a right-moving rod in a $+z$ field and counterclockwise contour: the rod element points upward while $\mathbf{u} \times \mathbf{B}$ points downward. The motional integral is negative, and the swept-area flux derivative is positive, exactly as the minus sign requires.

The conductor velocity is not the full charge velocity. For a thin wire, the drift velocity is tangent to the wire, so its magnetic force has no component along the wire: $(\mathbf{v}_{\text{drift}} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = 0$. The tangential electromagnetic driving term is therefore $\mathbf{E} + \mathbf{u} \times \mathbf{B}$. This argument also makes clear why the rule describes the actual conductor, not any geometrical contour drawn on a moving object.

Example 10.26: A rotating rod and the missing factor of two

A conducting rod of length a rotates counterclockwise in the xy plane about one end O , with constant angular speed $\omega > 0$ in $\mathbf{B} = B\hat{\mathbf{z}}$. Find the open-circuit outer-end potential relative to the pivot.

Strategy. The local speed is $u = \omega r$, so the rod has no single translational speed. Integrate the local magnetic force per charge. Here $\mathbf{u} = \omega r \hat{\boldsymbol{\phi}}$ and $\mathbf{u} \times \mathbf{B} = B\omega r \hat{\mathbf{r}}$ points outward.

Calculation. In open-circuit equilibrium $E_r = -B\omega r$, giving

$$V(a) - V(0) = - \int_0^a E_r dr = \int_0^a B\omega r dr = \frac{1}{2} B\omega a^2.$$

The outer end is positive. For $a = 0.20$ m, $B = 0.50$ T, and $\omega = 30$ s⁻¹, the result is 0.30 V.

Check and extension. Using Ba times the tip speed gives twice the correct result because all interior elements move more slowly. If a stationary return conductor and circular sliding

contact close a resistive circuit, the swept sector has area $a^2\theta/2$, reproducing the same emf magnitude. For outward rod current I , the magnetic torque is opposing:

$$|\tau_B| = \int_0^a rIBdr = \frac{1}{2}IBa^2, \quad |\tau_B|\omega = I\mathcal{E}.$$

Thus the power balance agrees with the translational case.

A conducting disk rotating in an axial field provides a useful caution. With stationary brushes at its axle and rim, the actual material has velocity $\omega r\hat{\phi}$ even if a chosen radial path through the disk is fixed in laboratory coordinates. The motional contribution along that radial path is $B\omega a^2/2$. The external circuit and the drawn radial path may bound a fixed surface with constant flux; its flux derivative alone would incorrectly predict zero. That boundary is not a material circuit moving with the disk. The Lorentz-force integral, evaluated with the local conductor velocities, remains the reliable calculation. This is the essential distinction in a homopolar generator; a static magnetic field can couple mechanical drive to electrical output.

10.16.2 Nonuniform fields: inductance from geometry and energy

The long-solenoid result is convenient because its field is nearly uniform. Many useful devices require an integral. A toroid with rectangular cross section has inner radius a , outer radius b , axial height h and N tightly wound turns. In the ideal continuous-winding model,

$$B(r) = \frac{\mu_0 NI}{2\pi r}, \quad a < r < b.$$

The flux through one turn is calculated across a radial cross section, not across the central hole:

$$\Phi_B = \int_a^b B(r)hdr = \frac{\mu_0 NIh}{2\pi} \ln \frac{b}{a}.$$

Consequently,

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}. \quad (10.68)$$

The logarithm is dimensionless and $\mu_0 h$ has unit henry. For a thin cross section $b-a \ll r_m = (a+b)/2$, $\ln(b/a) \simeq (b-a)/r_m$, giving $L \simeq \mu_0 N^2 A/(2\pi r_m)$ with $A = h(b-a)$. This is the mean-radius estimate used in design. A magnetic core can be included with constant μ only if a linear, reversible material law and negligible leakage are adequate.

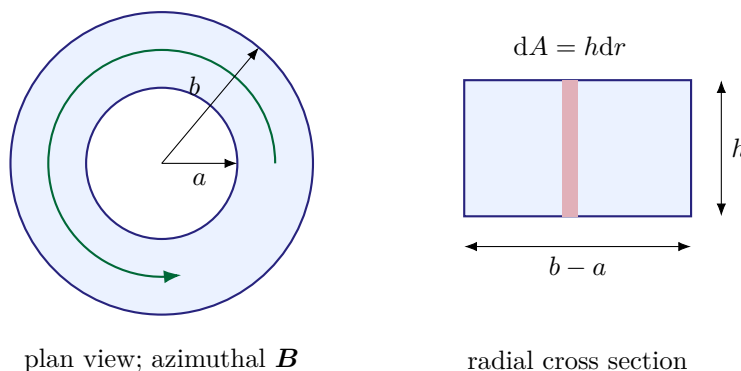


Figure 10.14: Toroid geometry. Flux through a winding turn crosses the radial rectangle. The field varies as $1/r$, so using a single mean radius is an additional approximation.

The radial strip in Figure 10.14 identifies the integration area explicitly. The same $1/r$ field appears in a coaxial cable: an inner conductor of radius a carries $+I$, and an outer cylindrical return with

inner radius b carries $-I$. In the space between them,

$$B(r) = \frac{\mu_0 I}{2\pi r}.$$

Neglect end effects and fields inside the conductors. The energy per unit length is

$$U'_B = \int_a^b \frac{B(r)^2}{2\mu_0} 2\pi r dr = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a}, \quad (10.69)$$

$$L'_{\text{ext}} = \frac{2U'_B}{I^2} = \frac{\mu_0}{2\pi} \ln \frac{b}{a}. \quad (10.70)$$

The prime means “per unit length”, not a derivative. Units are H/m. This calculation gives the *external* inductance associated with the field between the conductors. A uniform DC current density inside a solid inner conductor contributes an additional internal inductance; at sufficiently high frequency the current distribution changes. Neglecting internal fields must therefore be a declared model choice, not an unnoticed assumption.

Example 10.27: Comparing two field geometries

Estimate the inductance of an air-core toroid with $N = 500$, $a = 3.0$ cm, $b = 5.0$ cm and $h = 1.0$ cm. Also find the external inductance of 2.0 m of coaxial cable with $b/a = 3$.

Calculation. The toroid gives

$$L = (2.0 \times 10^{-7})(500)^2(0.010) \ln(5/3) = 2.55 \times 10^{-4} \text{ H}.$$

The mean-radius approximation gives 2.50×10^{-4} H, about 2.1% smaller. For the cable, $L'_{\text{ext}} = (2.0 \times 10^{-7}) \ln 3 = 2.20 \times 10^{-7}$ H/m, so the length has $L_{\text{ext}} = 0.439 \mu\text{H}$.

Interpretation. A cable can have inductance without turns, and a wound device gains a factor N^2 . The cable result is not a full high-frequency transmission-line analysis; capacitance and propagation belong to a distributed model.

10.16.3 Coupled circuits and positivity of magnetic energy

With the dot convention of Figure 10.11, the two-winding model is

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} L_1 & M \\ M & L_2 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}, \quad v_j = \dot{\lambda}_j. \quad (10.71)$$

Reciprocity makes the off-diagonal coefficients equal for the fixed linear reciprocal system considered here. It is not an unrestricted statement about all magnetic media or actively driven devices.

The electrical power entering the field is $p = I_1 \dot{\lambda}_1 + I_2 \dot{\lambda}_2$. With constant coefficients,

$$p = L_1 I_1 \dot{I}_1 + L_2 I_2 \dot{I}_2 + M(I_1 \dot{I}_2 + I_2 \dot{I}_1) = \frac{d}{dt} \left(\frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2 \right).$$

Taking zero energy at zero currents therefore gives

$$U_B = \frac{1}{2} L_1 I_1^2 + \frac{1}{2} L_2 I_2^2 + M I_1 I_2. \quad (10.72)$$

The mutual term can be negative for opposite-signed currents. Total energy must still be nonnegative for all currents. Completing the square,

$$U_B = \frac{1}{2} L_1 \left(I_1 + \frac{M}{L_1} I_2 \right)^2 + \frac{1}{2} \left(L_2 - \frac{M^2}{L_1} \right) I_2^2,$$

shows that

$$M^2 \leq L_1 L_2, \quad k = \frac{|M|}{\sqrt{L_1 L_2}} \leq 1. \quad (10.73)$$

At perfect ideal coupling one current combination cancels the common magnetic field and the matrix becomes singular. Real leakage generally makes it positive definite. Coupling k describes flux geometry, whereas efficiency also depends on resistances, frequency and load.

Example 10.28: Measuring mutual inductance by series connection

Two linear coils have $L_1 = 4.0 \text{ mH}$ and $L_2 = 9.0 \text{ mH}$. A series-aiding connection is measured as 23 mH . Find M , the coupling magnitude, and the predicted series-opposing inductance.

Derivation. Aiding currents enter both dots, so $I_1 = I_2 = I$ and the energy is $\frac{1}{2}(L_1 + L_2 + 2M)I^2$. Thus $L_+ = L_1 + L_2 + 2M$. Opposing currents give $L_- = L_1 + L_2 - 2M$.

Result. $M = (23 - 4 - 9)/2 = 5.0 \text{ mH}$; $k = 5/\sqrt{4 \times 9} = 0.833$; and $L_- = 3.0 \text{ mH}$. The positivity bound is $|M| \leq 6.0 \text{ mH}$, so the values are consistent. Measuring both configurations gives $M = (L_+ - L_-)/4$ and checks the assumed fixed linear behaviour.

How a passive secondary reacts back on the primary

Connect a resistance R_2 across the secondary and apply a voltage V to the primary through R_1 . Both winding currents retain their dot-entering references. The equations are

$$L_1 \dot{I}_1 + M \dot{I}_2 + R_1 I_1 = V, \quad (10.74)$$

$$M \dot{I}_1 + L_2 \dot{I}_2 + R_2 I_2 = 0. \quad (10.75)$$

At the initial moment with both currents zero, the second equation gives $\dot{I}_2 = -(M/L_2)\dot{I}_1$. Consequently,

$$\dot{I}_1(0^+) = \frac{V}{L_1 - M^2/L_2} = \frac{V}{L_1(1 - k^2)}.$$

The induced secondary current opposes the primary flux buildup and lowers the effective inductance seen during this initial response. In the ideal $k \rightarrow 1$ limit this calculation becomes singular; resistances, leakage, source rise time and parasitic capacitance cannot then be ignored. At long times under DC, $I_2 \rightarrow 0$ and $I_1 \rightarrow V/R_1$.

Multiplying the two equations by their currents and adding produces

$$V I_1 = R_1 I_1^2 + R_2 I_2^2 + \dot{U}_B.$$

The secondary's load energy ultimately comes from the primary source. Treating the primary current as an externally prescribed function is often convenient for a pickup calculation, but the source required to impose that function must supply this reaction and energy.

10.16.4 Nonlinear linkage, energy and magnetic force

For a fixed, reversible nonlinear magnetic system, λ can be a single-valued function of I without being proportional to it. The secant and differential inductances differ:

$$L_{\text{sec}} = \frac{\lambda}{I}, \quad L_{\text{diff}} = \frac{d\lambda}{dI}, \quad v_L = L_{\text{diff}} \dot{I}.$$

The slope, not the ratio, controls a small current change at a given operating point. If saturation reduces that slope, a given voltage causes a larger current slope. A model using a constant low-current inductance can therefore underestimate current ripple or turn-on current.

From $p = I\dot{\lambda}$, the reversible stored energy at fixed geometry is

$$U_B(I) = \int_0^{\lambda(I)} I(\lambda') d\lambda' = \int_0^I I' \frac{d\lambda}{dI'} dI'. \quad (10.76)$$

The related magnetic coenergy is $W'_B(I) = \int_0^I \lambda(I') dI'$. Integration by parts gives $U_B + W'_B = I\lambda$. For a linear inductor they are equal, each $LI^2/2$. In a nonlinear system they generally are not. Neither $\lambda I/2$ nor $L_{\text{diff}} I^2/2$ is a general energy formula.

Example 10.29: Energy for a specified nonlinear model

Consider the reversible model $\lambda(I) = L_0 I + \alpha I^3$ with $L_0 > 0$ and $\alpha > 0$. Find both inductances and stored energy. This model illustrates nonlinearity; it is not a saturation model.

Calculation. $L_{\text{sec}} = L_0 + \alpha I^2$ and $L_{\text{diff}} = L_0 + 3\alpha I^2$. Equation (10.76) gives

$$U_B = \frac{1}{2} L_0 I^2 + \frac{3}{4} \alpha I^4, \quad W'_B = \frac{1}{2} L_0 I^2 + \frac{1}{4} \alpha I^4.$$

For $L_0 = 0.10 \text{ H}$, $\alpha = 0.020 \text{ H/A}^2$ and $I = 2.0 \text{ A}$, the secant and differential values are 0.18 H and 0.34 H . The stored energy is 0.44 J , whereas $\lambda I/2 = 0.36 \text{ J}$. Checking $U_B + W'_B = I\lambda = 0.72 \text{ J}$ confirms the bookkeeping.

Now let geometry change through a mechanical coordinate x . Even a locally linear system has $\lambda = L(x)I$ and therefore

$$v_L = L\dot{I} + I \frac{dL}{dx} \dot{x}. \quad (10.77)$$

The second term is an electromechanical contribution; it is present even at constant current. Assuming a lossless magnetic system, terminal electrical work divides into stored-energy increase and mechanical work:

$$I d\lambda = dU_B + F_x dx.$$

For the linear model $U_B = LI^2/2$, substitution yields

$$F_x = \frac{1}{2} I^2 \frac{dL}{dx}. \quad (10.78)$$

At constant current, motion toward greater inductance increases the stored field energy while also delivering positive mechanical work. The source supplies both. It is incorrect to set $F_x = -\partial(LI^2/2)/\partial x$ at fixed current and forget the source. The energy derivative with a minus sign applies at fixed flux linkage, while coenergy gives $F_x = (\partial W'_B/\partial x)_I$.

For example, if $dL/dx = 0.80 \text{ H/m}$ and $I = 2.0 \text{ A}$, the force is 1.6 N toward increasing L . At speed 0.10 m/s , the motional voltage is 0.16 V and the source supplies 0.32 W : half increases field energy and half is mechanical power. Winding heat would be an additional term. Hysteresis makes the reversible single-valued model inadequate and requires explicit dissipative bookkeeping.

10.16.5 General forcing and electromechanical dynamics

For fixed L and R , a general source $v_s(t)$ satisfies

$$\dot{I} + \frac{R}{L} I = \frac{v_s(t)}{L}.$$

Multiplying by e^{t/τ_L} turns the left side into a derivative. Integrating and applying the initial current gives

$$I(t) = I(0)e^{-t/\tau_L} + \frac{1}{L} \int_0^t e^{-(t-t')/\tau_L} v_s(t') dt'. \quad (10.79)$$

The first term remembers the initial condition; the second adds the decaying contributions from the entire voltage history. The kernel has unit $1/H$, so voltage times the kernel times time has unit ampere. A constant source immediately reproduces the exponential growth law. A short voltage pulse with area $\mathcal{A}_V$ changes the current by approximately $\mathcal{A}_V/L$ if the pulse duration is much shorter than τ_L .

For an ideal inductor without resistance, integrating a sinusoidal voltage also leaves an arbitrary constant current. Thus there is no decay mechanism selecting a unique zero-mean sinusoidal state. When the phrase “current lags voltage by 90° ” is used for an ideal inductor, it refers to the sinusoidal component. The complete AC treatment in Chapter 11 includes the relevant steady-state assumptions.

Magnetic damping when circuit inductance matters

For a sliding bar of mass m , let $K = B\ell$. With a constant total circuit inductance as a model approximation, the coupled equations are

$$L\dot{I} + RI = Kv, \quad m\dot{v} = F_{\text{ext}} - KI. \quad (10.80)$$

The earlier drag model sets $L\dot{I} \simeq 0$, giving $I \simeq Kv/R$. This is justified when $\tau_e = L/R \ll \tau_m = mR/K^2$. When the time scales are comparable, current lags motion and cannot be eliminated instantaneously.

For a freely released bar, differentiating the mechanical equation and eliminating current gives

$$\ddot{v} + \frac{R}{L}\dot{v} + \frac{K^2}{mL}v = 0.$$

There can be damped electromechanical exchange between kinetic and magnetic energy, even though the circuit alone is first order. With $I(0) = 0$, $\dot{v}(0) = 0$, rather than the instantaneous-drag slope of the reduced model. This is a physical oscillatory possibility in a coupled system, distinct from spurious Euler oscillations in the single RL equation.

The decisive conservation check follows without solving the equations:

$$\frac{d}{dt} \left(\frac{1}{2}mv^2 + \frac{1}{2}LI^2 \right) = F_{\text{ext}}v - RI^2. \quad (10.81)$$

For a real rail geometry, L can depend on position; then both the $I\dot{L}$ voltage and the associated magnetic force must be included consistently. Equation (10.80) deliberately assumes that such changes are negligible.

Back emf in a motor

A simple fixed-field motor model has

$$L\dot{I} + RI + K\omega = V, \quad J\dot{\omega} = KI - \tau_{\text{load}} - b\omega,$$

where J is rotor inertia, $b\omega$ is viscous drag, $K\omega$ is back emf, and KI is electromagnetic torque in consistent SI units. The common constant K expresses equality of conversion powers $I(K\omega) = (KI)\omega$ in the ideal coupling model.

At the exact start of a finite-voltage pulse the current remains continuous. The often quoted “starting current V/R ” is the stationary-rotor electrical equilibrium after the inductive transient, not the instantaneous current at switch closure. As rotation develops, back emf reduces the voltage available for current growth. If an external drive raises the speed sufficiently and the electrical circuit accepts returned power, the current can reverse and the machine can operate as a generator. Multiplying and adding the equations gives the complete magnetic-plus-kinetic energy budget, with winding heat, drag and load work explicitly separated.

10.16.6 Induction acceleration and the betatron condition

A circulating induced electric field can accelerate a charge without a wire. Consider an electron moving on a circular orbit of fixed radius r_0 around a symmetry axis. A magnetic field at the orbit bends its momentum; the flux through the orbit determines the accelerating electric field. These involve different measures of the magnetic field and must not be confused.

Let the azimuthally symmetric field point along $+z$ and increase. Faraday's law gives clockwise $E_\phi = -\dot{\Phi}_B/(2\pi r_0)$. An electron moving counterclockwise experiences a counterclockwise electric force because its charge is $-e$, so its momentum magnitude grows at rate

$$\frac{dp}{dt} = \frac{e}{2\pi r_0} \frac{d\Phi_B}{dt}.$$

The local magnetic field B_{orb} supplies the radial force. Since $pv/r_0 = evB_{\text{orb}}$, the momentum condition is $p = er_0 B_{\text{orb}}$, valid also relativistically when p is relativistic momentum. Differentiating at fixed radius and equating the two expressions gives

$$\frac{d\Phi_B}{dt} = 2\pi r_0^2 \frac{dB_{\text{orb}}}{dt}.$$

Define the area-average field $\bar{B} = \Phi_B/(\pi r_0^2)$. Then

$$\frac{d\bar{B}}{dt} = 2 \frac{dB_{\text{orb}}}{dt}, \quad \bar{B} - 2B_{\text{orb}} = \text{constant}. \quad (10.82)$$

For a proportional field ramp with matching reference values, this is the usual betatron condition $\bar{B} = 2B_{\text{orb}}$. The derivative relation is the more general statement; injection conditions determine the constant. A field uniform over the orbit disk has $\bar{B} = B_{\text{orb}}$ and cannot satisfy the condition during a nonzero ramp. A tailored radial field profile is essential.

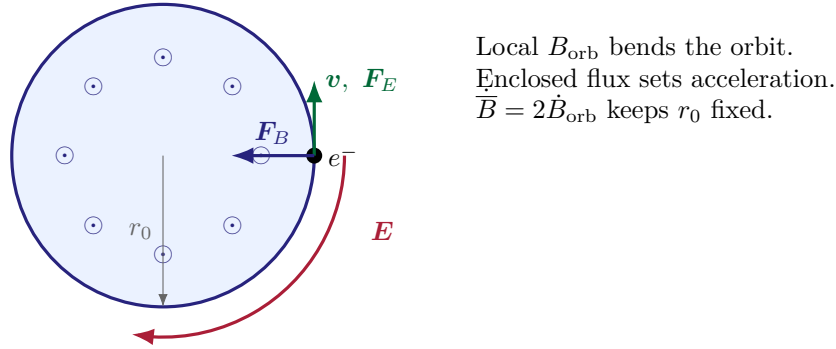


Figure 10.15: Idealised induction acceleration. Increasing out-of-page field induces clockwise electric field; the electron accelerates counterclockwise while its magnetic force is inward. Field dots show direction only, not a uniform radial profile.

Figure 10.15 displays the separate accelerating and bending roles. As a scale estimate, take $r_0 = 0.20$ m and $\dot{B}_{\text{orb}} = 0.50$ T/s. A fixed-radius ramp requires $\dot{\bar{B}} = 1.0$ T/s; hence $|E_\phi| = (r_0/2)\dot{\bar{B}} = 0.10$ V/m. The work per electron revolution is approximately $e 2\pi r_0 |E_\phi| = 0.126$ eV, assuming the field changes little over one revolution. This is an illustrative ramp, not a specification for an accelerator. Orbit focusing, injection, radiation losses and practical magnetic design lie beyond the model.

10.16.7 When the models need to be enlarged

The lumped description assumes that one current characterises each winding and that electromagnetic propagation across the apparatus is fast compared with the signal variation. A necessary scale check is $a/c \ll T$ for system size a and variation time T , though this alone does not guarantee negligible

capacitance, radiation or material losses. Real inductors also have inter-turn capacitance, winding resistance, frequency-dependent current distributions and sometimes hysteretic cores.

Use the simplest model that answers the question: a resistance-dominated loop for slow imposed flux changes; an RL equation when current memory matters; coupled circuit equations when receiver loading matters; and a distributed field model when propagation or geometry cannot be reduced to lumped elements. The formulas are most useful when accompanied by these criteria. Chapters 11–13 develop resonance, magnetic media and electromagnetic propagation in their proper setting.

10.17 Chapter summary and formula guide

Magnetic flux is an oriented geometrical quantity. Induction responds to its rate of change, whether the change comes from the field or the moving circuit. For a fixed path, the emf is the electric-field circulation; for a moving conductor, the Lorentz-force integral includes $\mathbf{u} \times \mathbf{B}$. The emf drives a circuit response whose direction and time dependence also depend on resistance, inductance, other sources and initial conditions.

Self-inductance packages the relation between current and its own flux linkage. A fixed linear inductor obeys $v_L = L\dot{I}$ in the passive terminal convention, stores $LI^2/2$, and keeps its current continuous for finite applied voltage. Mutual inductance describes cross linkage between windings; its signs require defined current and winding references. Energy conservation connects these circuit laws to magnetic braking, generator torque and controlled switching.

Relation	Assumptions and interpretation
$\Phi_B = \int_S \mathbf{B} \cdot d\mathbf{A}$	Oriented surface. Uniform flat case: $BA \cos \theta$.
$\lambda = \sum_j \Phi_{B,j}$	Linkage; equals $N\Phi_B$ only for identical linked fluxes.
$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}$	Fixed path and surface. No conducting wire is required.
$\oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\dot{\Phi}_B$	Continuously moving material boundary; use actual conductor velocities at contacts.
$ \mathcal{E}_{\text{mot}} = B\ell v$	Straight perpendicular rod, uniform field.
$L = \lambda_{\text{self}}/I$	Fixed linear system; independent external flux excluded.
$L \simeq \mu_0 N^2 A / \ell_s$	Long air-core solenoid, end effects neglected.
$\mathcal{E}_L = -L\dot{I}$, $v_L = +L\dot{I}$	Induced emf and passive terminal drop are different signed quantities.
$I = I_\infty + (I_0 - I_\infty)e^{-t/\tau_L}$	Constant source and series $R > 0$, $L > 0$; $I_\infty = V/R$, $\tau_L = L/R$.
$I = I_0 e^{-t/\tau_L}$	Closed source-free RL discharge path.
$U_B = LI^2/2$, $p_L = LI\dot{I}$	Fixed linear inductance; $p_L > 0$ means absorption.
$u_B = B^2/(2\mu_0)$	Vacuum field energy density; integrate over field volume.
$\mathcal{E}_{2,\text{mut}} = -M\dot{I}_1$	Fixed coupling and consistent winding orientation; not necessarily the total secondary emf.
$k = M /\sqrt{L_1 L_2} \leq 1$	Passive reciprocal linear coupled-inductor model.

Take-home message

For a complete induction solution, state the orientation, the changing linkage, the circuit response and the energy budget. Check the special-case assumptions before using a shortcut, and check both initial and limiting behaviour after solving a differential equation.

10.18 Practice problems

These problems progress from interpretation to modelling. Unless a problem states otherwise, use SI units, $\mu_0 \simeq 4\pi \times 10^{-7} \text{ H/m}$ and ideal fixed linear inductors. Treat a moving-loop current as resistance dominated only where stated. For every direction calculation, give the normal and positive traversal. Selected detailed solutions follow; problems without a solution are intended for independent practice or discussion.

Conceptual reasoning

- C1.** A loop in the page has positive normal out of the page. An into-page field first grows and then decreases. Give the signs of flux, flux derivative and emf in each stage. Under what assumption can the emf arrows also be used as current arrows?
- C2.** Give one example of zero instantaneous flux with nonzero emf and one example of nonzero flux with zero emf. Sketch flux versus time near the relevant instant in each case.
- C3.** An isolated rod moves right in an into-page field. Explain why its upper end becomes positive even though electrons carry the current in a metal. Why is there no sustained current in the open rod?
- C4.** Two students calculate $B\ell v$ for a sliding rod, one from the Lorentz force and the other from the flux derivative. A third adds their answers. Explain the error and state a situation in which electric and motional contributions really must both be included.
- C5.** A circuit has positive inductor reference current but negative v_L . Is it absorbing or releasing energy? Repeat when both I and v_L are negative. Explain why the sign of $\dot{I}$ alone is insufficient.
- C6.** A constant positive voltage is applied to an ideal inductor with zero resistance. Does the current approach a finite value? Contrast this with a series RL circuit and identify the singular step in applying $I_\infty = V/R$ at $R = 0$.
- C7.** A magnet falls through a copper tube, then is held motionless inside it. Explain the difference in eddy currents and force. Why is ferromagnetism unnecessary?
- C8.** A perfectly coupled pair of coils has $k = 1$. Does this alone imply unity power-transfer efficiency? Explain the roles of winding resistance, load and operating frequency.

Analytical and numerical applications

- A1.** A 75-turn coil of area $4.0 \times 10^{-3} \text{ m}^2$ has its positive normal parallel to a field decreasing from 0.20 to 0.05 T in 0.030 s. Find the signed average emf. Does this information determine the peak emf?
- A2.** A circular loop of radius 0.12 m has positive normal along $\mathbf{B}$, with $B(t) = (0.40 \text{ T/s}^2)t^2$. Find the signed emf at $t = 2.0 \text{ s}$ and the change in flux between 0 and 2.0 s.
- A3.** A 0.50 m rod slides right at 3.0 m/s in a 0.60 T into-page field, closing a left-hand rail circuit of resistance 0.90Ω . Neglect inductance. Find emf, current direction, force on the rod and external mechanical power. Verify the heat rate.
- A4.** A square loop of side 0.20 m and resistance 0.40Ω crosses an out-of-page 0.80 T strip of width 0.60 m at 1.5 m/s. Neglect self-inductance. Taking $t = 0$ at first entry, give the signed emf intervals, total heat and net signed voltage integral.

- A5.** An air-core solenoid has 600 turns, length 0.30 m and radius 1.5 cm. Estimate L and the stored energy at 2.0 A. Check the radius-to-length ratio and explain what the idealisation omits.
- A6.** A 0.15 H ideal inductor has current ramped from -2.0 to $+1.0$ A uniformly in 0.050 s. Find v_L , the net field-energy change and the time at which power changes sign. Distinguish net energy from total energy absorbed during only the latter part of the ramp.
- A7.** A 24 V step is applied to $R = 8.0\ \Omega$, $L = 0.40$ H with $I(0) = 0$. Find τ_L , $I(0.10\text{ s})$, v_L and $\dot{I}$ at that time. Then compute source work, resistor heat and stored energy up to $t = 0.10$ s.
- A8.** A 2.0 H inductor initially carrying 1.2 A discharges through $5.0\ \Omega$. Find current and stored energy after 0.50 s, the heat released by that time, and the time for the energy to fall to 10% of its initial value.
- A9.** An ideal long solenoid of radius 0.050 m has $\dot{B} = +1.2\text{ T/s}$ along $+z$. Find signed E_ϕ at radii 0.020, 0.050 and 0.10 m. Explain why the outer value is nonzero even if local B is negligible.
- A10.** A receiver is coupled to a transmitter with $M = 8.0\ \mu\text{H}$. The primary current is $I_1 = (2.0\text{ A})\sin(\omega t)$ at $f = 100\text{ kHz}$. Find the open-circuit secondary voltage amplitude and phase relative to I_1 with corresponding dotted terminals. Which information is missing for calculating delivered power?
- A11.** A coil with $N = 200$, $A = 1.0 \times 10^{-3}\text{ m}^2$ rotates at 60 Hz in $B = 0.40\text{ T}$. Find emf amplitude and rms value. If the total load plus winding resistance is $30\ \Omega$ and inductance is negligible, find mean drive power and mean reaction torque.
- A12.** A series circuit with $R = 5.0\ \Omega$ and $L = 0.20\text{ H}$ is driven from zero current by $+10\text{ V}$ for 0.080 s, then by -5.0 V . Find the current at switching, its derivative immediately after switching, and the elapsed time after reversal when current crosses zero.

Derivations and graphical interpretation

- D1.** Derive Equation (10.9). Construct a case with nonzero $\dot{B}$ and $\dot{A}$ but zero emf at fixed $\theta = 0$; specify which of the two quantities increases.
- D2.** A fixed circular loop of radius a has $\mathbf{B} = B_0(t)(1 - r^2/a^2)\hat{\mathbf{z}}$ over its disk. Derive flux and emf. If this cylindrically symmetric field is negligible outside the disk, derive signed $E_\phi(r, t)$ inside and outside, and show continuity at $r = a$.
- D3.** Derive the general-initial-current RL solution with an integrating factor. Show directly that the solution satisfies both the differential equation and the initial condition.
- D4.** For RL growth from zero, derive $W_s(t)$ and $Q_R(t)$ by integrating power. Evaluate their ratios to U_∞ at $t = \tau_L$. Explain why the claim “growth heating equals final stored energy” lacks a valid general basis.
- D5.** From the curves in Figure 10.10, find analytically the time when $v_R = v_L$ during growth and the time when half the initial energy remains during decay. Explain why these two times differ.
- D6.** Derive the open-circuit voltage of a rod rotating about one end in a uniform axial field. Then derive magnetic torque for a current-carrying rod with a stationary return path, verifying equality of mechanical and electrical conversion powers.

Laboratory, computation and uncertainty

- L1.** A synthetic baseline-subtracted coil signal is sampled at $t = (0, 10, 20, 30, 40)$ ms with voltages $(0, 0.20, 0.40, 0.20, 0)$ V. Use trapezoidal integration to find $\Delta\lambda$. For $N = 100$ and $A = 5.0 \times 10^{-4} \text{ m}^2$ find the uniform aligned field change. What bias results if an unremoved 5.0 mV offset is present over this window?
- L2.** A fitted time constant is (20.0 ± 0.4) ms. The external resistor is $(12.0 \pm 0.1) \Omega$ and winding resistance $(3.0 \pm 0.2) \Omega$. Treat independent standard uncertainties and negligible source resistance. Estimate $L \pm \sigma_L$ and the bias from ignoring the winding resistance.
- L3.** Implement the complete RL code in the chapter. Replace the Euler update by the exact update, compare maximum current error and energy residual, and explain why an exact current update does not by itself make trapezoidal power integration exact.
- L4.** For RL decay, compare Euler steps $\Delta t/\tau_L = 0.2, 1.2, 2.2$. Starting with $I_0 > 0$, calculate the first four values normalised to I_0 . Identify accurate, oscillatory-stable and unstable behaviour, and compare all three with the physical exponential.
- L5.** Design a measurement to test the signed pulse-area invariance when a magnet follows the same path at different speeds. State how you would set the voltage polarity, choose integration endpoints, remove baseline drift and assess whether sensor loading is negligible.
- L6.** Two voltage-lead routes enclose a difference area of 8.0 cm^2 in a uniform perpendicular 15 T/s field ramp. Estimate the induced difference in readings. Propose a geometrical change that reduces it and explain why a higher-resolution voltmeter alone is insufficient.

Engineering and advanced challenges

- E1.** A coil with $L = 40 \text{ mH}$ and $R_w = 2.0 \Omega$ must follow $I(t) = I_f t/T$ from 0 to $I_f = 5.0 \text{ A}$ in $T = 1.0 \text{ ms}$. Derive the required source waveform, its initial and final voltages, and the energy delivered over the ramp.
- E2.** A coil with $L = 0.10 \text{ H}$, $R_w = 4.0 \Omega$ and initial $I_0 = 2.0 \text{ A}$ discharges through a clamp with fixed drop $V_c = 20 \text{ V}$. Find turn-off time, initial v_L , total clamp energy and winding heat. Assume the clamp stops conduction at zero current.
- E3.** A long primary solenoid has radius a , turn density n_1 and current $I_1(t)$. A coaxial circular pickup of N_2 turns has radius b . Derive M for $b < a$ and $b > a$. Sketch $M(b)$ and state why the larger-loop branch saturates in this idealisation.
- E4.** Two coils have $L_1 = 10 \text{ mH}$ and $L_2 = 40 \text{ mH}$. The series-aiding and opposing inductances are 78 and 22 mH. Find M and k and verify both measurements. Would $M = 25 \text{ mH}$ be possible in the same passive model?
- E5.** A magnetic actuator has $L(x) = L_0 + \alpha x$ over a short displacement, with $\alpha = 0.80 \text{ H/m}$. It is held at $I = 2.0 \text{ A}$. Derive the force, field-energy increase, mechanical work and source work for displacement $\Delta x = 5.0 \text{ mm}$. Neglect resistance.
- E6.** Derive the external coaxial inductance per unit length using field energy. State why this may not equal the complete DC inductance for a solid inner conductor, and identify which region of the energy integral was omitted.
- E7.** In the coupled bar model, take $m = 0.10 \text{ kg}$, $L = 0.020 \text{ H}$, $R = 0.10 \Omega$, and $K = B\ell = 0.20 \text{ Tm}$. Determine whether velocity dynamics is underdamped, critically damped or overdamped. Compare L/R with mR/K^2 , and assess the instantaneous-drag approximation.

- E8.** Derive the fixed-radius betatron condition from electric acceleration and magnetic bending. If $r_0 = 0.15$ m and $\dot{B}_{\text{orb}} = 0.80$ T/s, find the required average-field ramp, tangential electric-field magnitude and electron energy gain per revolution in eV. State the initial-condition caveat in replacing the derivative condition by $\bar{B} = 2B_{\text{orb}}$.

10.19 Selected detailed solutions

These solutions emphasise decisions that transfer to other problems. Numerical results retain a few guard digits where they make an energy or continuity check clearer; the input precision limits their physical significance.

C1: Signs and the current-response assumption

Choose counterclockwise traversal, paired with the out-of-page normal. Write the into-page field as $\mathbf{B} = -B(t)\hat{\mathbf{z}}$ with $B(t) > 0$. Then $\Phi_B = -AB(t)$ is negative in both stages. During increasing magnitude, $\dot{\Phi}_B = -A\dot{B} < 0$ and $\mathcal{E} > 0$; the emf is counterclockwise. During decreasing magnitude, $\dot{\Phi}_B > 0$ and $\mathcal{E} < 0$; it is clockwise.

In a resistance-dominated loop without another source, $I = \mathcal{E}/R$, so these are also current directions. If L matters, solve $L\dot{I} + RI = \mathcal{E}_{\text{ext}}$. A current present when the field trend reverses cannot instantly reverse under a finite emf. The direction of the *induced drive* is fixed by the flux derivative, while the actual current also depends on its history. The original field direction never changes in this example, yet the emf does; this directly diagnoses the phrase “opposes the field” as inadequate.

A1: A signed average does not determine a peak

With the normal parallel to the field, $\lambda = NBA$. Its change is

$$\Delta\lambda = (75)(4.0 \times 10^{-3})(0.05 - 0.20) = -4.5 \times 10^{-2} \text{ Wb}.$$

Thus $\bar{\mathcal{E}} = -\Delta\lambda/\Delta t = +1.5$ V. The positive sign denotes the right-hand traversal associated with the selected normal. Since Wb/s = V, the dimensions check.

Only an average has been determined. If $B(t)$ falls linearly throughout the interval, the instantaneous emf is a constant +1.5 V. A faster change during a shorter part of the interval produces a larger peak with the same signed integral. Endpoints alone cannot distinguish those histories.

A3: Force and power for the sliding rod

Choose the same convention as Figure 10.4. The external flux is $-B\ell x$ and $\dot{x} = v > 0$, giving

$$\mathcal{E} = +B\ell v = (0.60)(0.50)(3.0) = 0.90 \text{ V}.$$

The resistance-dominated current is $I = 0.90/0.90 = 1.0$ A, counterclockwise. In the rod it is upward; crossing that direction with the into-page field gives a leftward force. Its magnitude is $I\ell B = 0.30$ N.

For constant velocity, the external force is 0.30 N rightward and supplies $P_{\text{mech}} = Fv = 0.90$ W. The resistor dissipates $I^2R = 0.90$ W, so the budget closes. If a solution obtains a rightward magnetic force for this initially unenergised resistive circuit, it would predict heat and mechanical gain without the required energy source. Reversing the field reverses the current but preserves the drag direction and power magnitude.

A4: Two pulses and a nonzero heat integral

Use counterclockwise traversal and the out-of-page normal. The entry duration is $w/v = 0.20/1.5 = 0.1333$ s. The leading edge reaches the far boundary at $d/v = 0.400$ s and the trailing edge leaves at $(d + w)/v = 0.5333$ s. The emf magnitude during either crossing is $Bwv = 0.24$ V.

Therefore the emf is -0.24 V for $0 < t < 0.1333\text{ s}$, zero for $0.1333 < t < 0.400\text{ s}$, and $+0.24\text{ V}$ for $0.400 < t < 0.5333\text{ s}$. It is zero before and after the passage. Boundary values at the ideal jumps do not affect the integrals.

The pulse current magnitude is $0.24/0.40 = 0.60\text{ A}$, giving power 0.144 W over total crossing time 0.2667 s . Thus $Q = 0.0384\text{ J}$. The signed voltage areas cancel: $\int \mathcal{E} dt = 0$, consistent with zero change between initial and final linkage. The heat integral instead involves $\mathcal{E}^2/R$ and is positive. Self-inductance would change the current pulse shapes and redistribute heat in time; the specified zero-inductance approximation is essential to the rectangular current waveform.

A7: A transient with a complete energy ledger

The time constant is $\tau_L = L/R = 0.050\text{ s}$ and $I_\infty = V/R = 3.0\text{ A}$. At $t = 0.10\text{ s} = 2\tau_L$,

$$I = 3(1 - e^{-2}) = 2.594\text{ A}, \quad v_L = 24e^{-2} = 3.248\text{ V}.$$

Then $\dot{I} = v_L/L = 8.120\text{ A/s}$. The resistor drop is $RI = 20.752\text{ V}$, and the two drops sum to 24 V .

The source work is an integral because current is changing:

$$W_s = \frac{24^2}{8} \left[0.10 - 0.050(1 - e^{-2}) \right] = 4.087\text{ J}.$$

The field energy is $U_B = \frac{1}{2}(0.40)(2.594)^2 = 1.346\text{ J}$. With zero initial field energy, conservation gives $Q_R = W_s - U_B = 2.741\text{ J}$. Direct integration of RI^2 yields the same answer.

The final eventual field energy is $U_\infty = 1.80\text{ J}$, which is neither the present field energy nor the heat already dissipated. The source remains connected in this problem, so heat continues to accumulate even after U_B has nearly stopped changing. This explicitly distinguishes transient storage from ongoing power consumption.

A8: Discharge energy and its time scale

The closed discharge path gives $\tau_L = 2.0/5.0 = 0.40\text{ s}$. At 0.50 s ,

$$I = 1.2e^{-1.25} = 0.3438\text{ A}, \quad U_B = \frac{1}{2}(2.0)(0.3438)^2 = 0.1182\text{ J}.$$

Initially $U_0 = \frac{1}{2}(2.0)(1.2)^2 = 1.44\text{ J}$. Since no source is present, the heat already released is $1.44 - 0.1182 = 1.3218\text{ J}$. The missing energy is precisely what remains stored.

For the energy threshold, set $U_B/U_0 = \exp(-2t/\tau_L) = 0.10$, yielding $t = (\tau_L/2) \ln 10 = 0.4605\text{ s}$. The current at that time is $\sqrt{0.10}$ of its initial value, not 0.10 . A factor of two in the exponential is required because energy is quadratic in current.

A9: A signed induced field inside and outside

Let $\hat{\phi}$ point counterclockwise viewed from $+z$. For $r < R$, circulation is $2\pi r E_\phi = -\pi r^2 \dot{B}$; outside, it is $2\pi r E_\phi = -\pi R^2 \dot{B}$. Therefore

$$E_\phi(0.020) = -0.012\text{ V/m}, \quad E_\phi(0.050) = -0.030\text{ V/m}, \quad E_\phi(0.10) = -0.015\text{ V/m}.$$

All fields circulate clockwise. At the boundary either formula gives the same value. The outer circle encloses the entire changing magnetic flux even though B on that circle is negligible. Replacing the flux by $B(r)\pi r^2$ there would incorrectly give zero; the flux requires the integral over the spanning disk.

A12: Current reversal takes time

During the first stage, $\tau_L = 0.040$ s and $I_\infty = 2.0$ A. Thus the switching current is

$$I_s = 2(1 - e^{-2}) = 1.7293 \text{ A.}$$

It is continuous at reversal. Immediately after the source becomes -5.0 V, the circuit equation gives

$$\dot{I}(0^+) = \frac{-5.0 - 5.0I_s}{0.20} = -68.23 \text{ A/s.}$$

Here zero denotes elapsed time since reversal. The new asymptotic current is -1.0 A, so

$$I(t') = -1 + (I_s + 1)e^{-t'/0.040} \quad \text{in amperes.}$$

Setting $I = 0$ gives $t' = 0.040 \ln(I_s/(1 \text{ A}) + 1) = 0.04016$ s. The dimensionless logarithm has been written explicitly. Before this crossing the current remains positive while its derivative is negative. This is a direct example of an emf reversing before the current does.

D2: Radial integration before differentiation

Use annular area elements, $dA = 2\pi r dr$. Flux through the full disk is

$$\Phi_B = 2\pi B_0(t) \int_0^a \left(1 - \frac{r^2}{a^2}\right) r dr = \frac{\pi a^2 B_0(t)}{2}.$$

The positive-normal emf is $\mathcal{E} = -\pi a^2 \dot{B}_0/2$. The average field is one half of the central field; this is why $B_0 \pi a^2$ fails.

For a circular path with $r < a$, integrate only to r :

$$\Phi_B(r) = \pi B_0 \left(r^2 - \frac{r^4}{2a^2}\right), \quad E_\phi(r) = -\dot{B}_0 \left(\frac{r}{2} - \frac{r^3}{4a^2}\right).$$

For $r > a$, the enclosed flux is fixed by the full disk, giving $E_\phi = -a^2 \dot{B}_0/(4r)$. At $r = a$ both give $-a \dot{B}_0/4$. The inner field vanishes at the axis as symmetry requires. Increasing positive B_0 gives a clockwise field throughout. As in the uniform-solenoid calculation, these local field values also assume cylindrical symmetry and no added electrostatic field.

D4: Deriving the growth heat without an analogy

Let $I_\infty = V/R$, $x = t/\tau_L$ and $P_\infty = V^2/R$. Source power is $P_s = P_\infty(1 - e^{-x})$ and resistor power is $P_R = P_\infty(1 - 2e^{-x} + e^{-2x})$. Integrating from zero gives

$$\begin{aligned} W_s &= P_\infty \tau_L [x - 1 + e^{-x}], \\ Q_R &= P_\infty \tau_L [x - 2(1 - e^{-x}) + \tfrac{1}{2}(1 - e^{-2x})]. \end{aligned}$$

Since $P_\infty \tau_L = LI_\infty^2 = 2U_\infty$, at $x = 1$ their ratios to U_∞ are 0.7358 and 0.3362. Their difference is $0.3996 = (1 - e^{-1})^2$, the field-energy fraction.

At large x , $W_s \simeq P_\infty(t - \tau_L)$ and $Q_R \simeq P_\infty(t - 3\tau_L/2)$, so their difference approaches $P_\infty \tau_L/2 = U_\infty$. Each integrated quantity grows without bound, while the difference is finite. This is why a charging-capacitor-style equal-energy rule cannot be asserted for resistor heat accumulated during indefinitely driven RL growth.

L1: Integrating a discrete pickup trace

With $\Delta t = 0.010$ s, the trapezoidal area is

$$\int V dt \simeq \Delta t \left[\frac{1}{2}V_0 + V_1 + V_2 + V_3 + \frac{1}{2}V_4 \right] = (0.010)(0.20 + 0.40 + 0.20) = 0.0080 \text{ V s}.$$

Thus $\Delta\lambda = -0.0080$ Wb. The coil factor is $NA = 0.050 \text{ m}^2$, giving $\Delta B = -0.160$ T for a uniform aligned field. This integration is exact for the piecewise-linear synthetic trace specified by the samples; it need not be exact for an unknown waveform between real samples.

An unremoved positive offset of 0.0050 V across 0.040 s adds 2.0×10^{-4} V s to the area. It biases ΔB by -0.0040 T, making the inferred decrease too large. Baseline correction should use stationary parts of the record and propagate the uncertainty of that correction.

L4: A false oscillation produced by the algorithm

Euler decay obeys $I_n/I_0 = (1 - h)^n$. The first four values after the initial point are

h	I_1/I_0	I_2/I_0	I_3/I_0	I_4/I_0
0.2	0.8	0.64	0.512	0.4096
1.2	-0.2	0.04	-0.008	0.0016
2.2	-1.2	1.44	-1.728	2.0736

The $h = 0.2$ case decays monotonically but is not exact: after four steps the physical value is $e^{-0.8} = 0.4493$. The $h = 1.2$ case is numerically stable because the multiplier magnitude is below one, yet it reverses current spuriously at every step. The $h = 2.2$ case is unstable because the multiplier magnitude exceeds one. The physical solution e^{-nh} stays positive and decays monotonically in every case. Stability, qualitative fidelity and quantitative accuracy are separate tests.

E2: Where the energy goes in a clamped discharge

The current follows Equation (10.62) until zero. Its stopping time is

$$t_{\text{off}} = \frac{0.10}{4.0} \ln \left(1 + \frac{(4.0)(2.0)}{20} \right) = 8.412 \text{ ms}.$$

Initially the ideal inductor drop is $v_L = -(R_w I_0 + V_c) = -28$ V, using the old current direction as positive. The clamp absorbs $V_c I$ while it conducts. Integrating the current gives

$$\int_0^{t_{\text{off}}} I dt = \frac{L I_0}{R_w} - \frac{V_c}{R_w} t_{\text{off}}.$$

Thus $Q_c = V_c \int I dt = 0.15882$ J. The initial field energy is $U_0 = \frac{1}{2}(0.10)(2.0)^2 = 0.200$ J, leaving $Q_w = 0.04118$ J as winding heat.

Both contributions are positive and sum to U_0 . Integrating the current formula beyond t_{off} would yield negative current and corrupt this budget because the unidirectional clamp is then off. A component must be rated for its pulse energy and peak power as well as its voltage; here the initial clamp power is 40 W in the idealised model.

E4: Coupling from two series measurements

Adding and subtracting the series relations gives

$$\frac{L_+ + L_-}{2} = L_1 + L_2 = 50 \text{ mH}, \quad M = \frac{L_+ - L_-}{4} = 14 \text{ mH}.$$

The measured sum therefore agrees with the individual inductances. Since $\sqrt{L_1 L_2} = 20 \text{ mH}$, the coupling magnitude is $k = 0.70$. Substitution yields $50 + 28 = 78 \text{ mH}$ and $50 - 28 = 22 \text{ mH}$ as required.

A proposal $M = 25 \text{ mH}$ exceeds the energy bound. To see the failure explicitly, choose $I_1 = -(M/L_1)I_2$. The completed-square expression leaves $U_B = \frac{1}{2}(L_2 - M^2/L_1)I_2^2$, whose coefficient is negative for those values. A passive linear magnetic-storage model cannot have negative field energy for an allowed current combination.

E8: Average field and orbit field play different roles

At fixed radius $p = er_0 B_{\text{orb}}$, so $\dot{p} = er_0 \dot{B}_{\text{orb}}$. Faraday's law and the electron charge give the tangential momentum gain $\dot{p} = e\dot{\Phi}_B/(2\pi r_0)$. Equating them yields $\dot{\bar{B}} = 2\dot{B}_{\text{orb}}$.

For the given ramp, $\dot{\bar{B}} = 1.6 \text{ T/s}$ and $|E_\phi| = (r_0/2)\dot{\bar{B}} = 0.12 \text{ V/m}$. The emf magnitude around the orbit is $2\pi(0.15)(0.12) = 0.1131 \text{ V}$, corresponding to an electron gain of 0.1131 eV per revolution when the field is nearly constant over that revolution.

Integration gives $\bar{B} - 2B_{\text{orb}} = C$. Only if the injection/reference fields satisfy $C = 0$ can one write $\bar{B} = 2B_{\text{orb}}$ throughout the ramp. A uniform ramp over the whole disk has equal average and orbit fields and cannot maintain this fixed orbit while accelerating. The condition constrains the spatial field profile as well as its time dependence.

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