

Chapter 11

Electromagnetic Oscillations and Alternating Current

First Edition

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Preface

This chapter joins three familiar ideas: electric energy in a capacitor, magnetic energy in an inductor, and current in a closed circuit. Energy exchange between a capacitor and an inductor produces oscillation. Resistance removes energy; a sinusoidal source can replenish it. The resulting amplitude, phase and power depend on the driving frequency.

These ideas connect basic electromagnetism to transformers, filters, impedance measurements, resonant sensors and communication circuits. They also connect electrical circuits to mechanical and acoustic oscillators. The aim is to understand how storage, dissipation and driving combine, and to use that understanding to select a model before calculating.

The core requires elementary calculus, differential equations and real phasor geometry. The advanced layer develops complex impedance, filter design, power-factor correction, resonance measurements and numerical dynamics. Worked examples explain the physical meaning of calculations; the problems support independent study and tutorials.

The Guideline

Treat every AC-circuit problem as both an energy problem and a phase problem. First identify which elements store energy and which elements dissipate it. Then choose a consistent sinusoidal convention, draw the phasor relation or write the impedance equation, and only then insert numbers. In alternating-current analysis, a correct phase convention is part of the physics, not a cosmetic detail.

Pedagogical boundary

This chapter studies lumped-element circuits: the circuit dimensions are assumed small compared with the relevant electromagnetic wavelength, so one can speak of a single current around a series loop at a given time. Distributed transmission lines, antennas and full electromagnetic-wave propagation are not treated as main topics here. They are natural extensions after Maxwell's equations and electromagnetic waves have been developed.

Conventions and notation

This chapter primarily uses scalar circuit variables. The instantaneous charge on a capacitor plate is denoted by $q(t)$, the instantaneous current by $i(t)$, and the instantaneous potential difference by $v(t)$. Peak amplitudes are denoted by a subscript zero, for example V_0 and I_0 . Root-mean-square values are written as V_{rms} and I_{rms} .

The angular frequency is ω and the ordinary frequency is $f = \omega/(2\pi)$. The ideal LC angular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

For a sinusoidal source we usually write

$$v_s(t) = V_0 \cos(\omega t),$$

with the source voltage chosen as the phase reference. If the current is written as

$$i(t) = I_0 \cos(\omega t - \phi),$$

then $\phi > 0$ means that the current lags the source voltage, while $\phi < 0$ means that the current leads the source voltage. Some books use the opposite sign convention. The convention used here is stated explicitly whenever a formula depends on it.

Symbol	Meaning	SI unit or dimension
C	capacitance	$\text{F} = \text{C V}^{-1}$
L	inductance	$\text{H} = \text{V s A}^{-1}$
R	resistance	Ω
$q(t)$	instantaneous capacitor charge	C
$i(t)$	instantaneous current	A
$v(t)$	instantaneous potential difference	V
V_0, I_0	peak voltage and current amplitudes	V, A
$V_{\text{rms}}, I_{\text{rms}}$	root-mean-square voltage and current	V, A
ω	angular frequency of driving or oscillation	rad s^{-1}
f	ordinary frequency, $f = \omega/(2\pi)$	Hz
ω_0	natural angular frequency of an ideal LC circuit	rad s^{-1}
X_L	inductive reactance, ωL	Ω
X_C	capacitive reactance, $1/(\omega C)$	Ω
Z	impedance magnitude; $\mathcal{Z}$ denotes complex	Ω
	impedance in enrichment	
Q, α	quality factor; decay coefficient $R/(2L)$	$1; \text{s}^{-1}$
ϕ	phase angle between source voltage and current	rad or degree
P_{avg}	time-averaged power	W
N_p, N_s	primary and secondary transformer turns	dimensionless

For sinusoids, $V_{\text{rms}} = V_0/\sqrt{2}$ and $I_{\text{rms}} = I_0/\sqrt{2}$. Keep the convention consistent: $P_{\text{avg}} = V_{\text{rms}} I_{\text{rms}} \cos \phi = V_0 I_0 \cos \phi / 2$.

Chapter 11

Electromagnetic Oscillations and Alternating Current

11.1 Motivation and roadmap

A capacitor can store energy in an electric field. An inductor can store energy in a magnetic field. A resistor can convert ordered electrical energy into internal energy. These three facts, combined with Kirchhoff's loop rule and Faraday induction, generate the complete introductory theory of electromagnetic oscillations and alternating-current circuits.

The simplest new idea is the LC oscillator. Charge initially stored on a capacitor begins to flow through an inductor. As the capacitor discharges, the current increases and energy is transferred into the magnetic field of the inductor. Later the magnetic field collapses, the current continues in the same direction for a while, and the capacitor recharges with opposite polarity. In the ideal limit this cycle repeats forever. The circuit has a natural frequency determined only by L and C .

Real circuits contain resistance, so oscillations lose energy. Driven circuits are more important technologically: a sinusoidal source can maintain a steady oscillation. The amplitude of the steady current depends on frequency, and so does its phase relative to the source voltage. This frequency-dependent response is the origin of resonance curves, filters, radio tuning, impedance measurements and power-factor engineering.

Figure 11.1 summarizes the logical structure of the chapter.

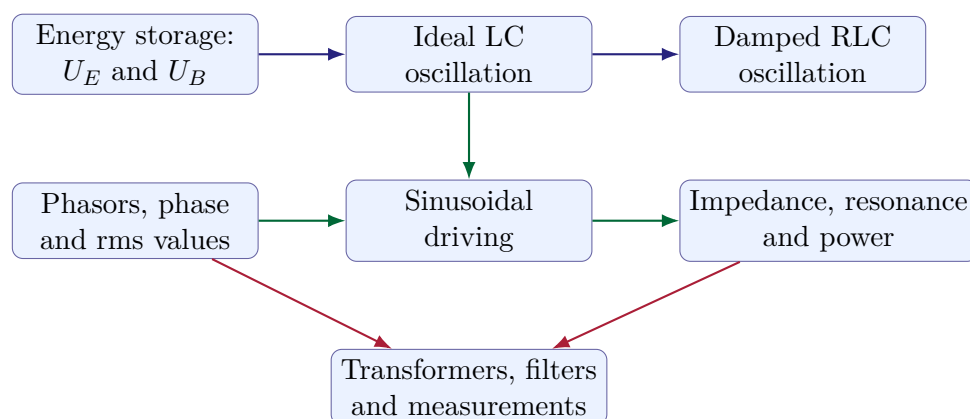


Figure 11.1: Conceptual roadmap for Chapter 11. The chapter begins with energy exchange in ideal LC circuits, adds damping, then develops the steady-state response of driven AC circuits using phase, phasors, impedance, power and resonance.

Learning goals

After studying this chapter, the student should be able to:

- explain the energy exchange between a capacitor and an inductor in an LC circuit;
- derive the ideal LC angular frequency $\omega_0 = 1/\sqrt{LC}$;
- compare an LC oscillator with a mass-spring oscillator;
- write and interpret the differential equation for a damped series RLC circuit;
- distinguish transient behaviour from steady-state sinusoidal response;
- convert between peak and rms values for sinusoidal voltage and current;
- define capacitive reactance, inductive reactance and impedance;
- determine whether current leads or lags voltage in simple AC circuits;
- construct and interpret phasor diagrams for R, L, C and series RLC circuits;
- calculate current amplitude, phase angle and average power in a driven series RLC circuit;
- identify resonance, bandwidth and quality factor at an introductory level;
- explain the ideal transformer voltage and current ratios and why transformers require changing magnetic flux;
- connect AC-circuit ideas to power systems, signal processing, filters, sensors and laboratory measurements.

Key message

The unifying idea of this chapter is that capacitors and inductors do not dissipate energy in the ideal model; they store and return it. A resistor removes energy. An AC source can continuously supply energy. The observable response is therefore controlled by a competition between storage, dissipation, driving frequency and phase.

11.2 Required prior knowledge

The chapter assumes that students can use Kirchhoff's loop rule, current continuity and the basic element laws for capacitors, resistors and inductors. The most important previous Physics 2 results are

$$q = Cv_C, \quad U_C = \frac{q^2}{2C} = \frac{1}{2}Cv_C^2, \quad (11.1)$$

$$v_R = iR, \quad P_R = i^2R, \quad (11.2)$$

$$v_L = L \frac{di}{dt}, \quad U_L = \frac{1}{2}Li^2. \quad (11.3)$$

These element laws use the passive sign convention: positive current enters the terminal labelled positive for the element voltage. Thus $v_L = L di/dt$ is a terminal voltage drop. The self-induced emf $-L di/dt$, recalled from Chapter 10, describes the opposing emf around the winding. They are compatible descriptions, not contradictory signs. All R , L and C values are positive and constant unless stated otherwise.

The chapter also uses familiar Physics 1 oscillator ideas. The mass-spring equation

$$m \frac{d^2x}{dt^2} + kx = 0 \quad (11.4)$$

has sinusoidal solutions with angular frequency $\omega = \sqrt{k/m}$. With damping and driving, the oscillator equation becomes

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos \omega t. \quad (11.5)$$

The series RLC circuit is mathematically parallel to this equation, with charge q playing a role analogous to displacement x .

Method

Before solving an RLC problem, identify the regime.

1. Is the circuit free or driven?
2. Is resistance neglected, small, or important?
3. Are you asked for transient motion or steady-state sinusoidal response?
4. Are given voltages peak values, rms values, or instantaneous values?
5. Is the requested answer a magnitude, a signed component, a phase angle or a time function?

Most errors in this chapter come from using a correct formula in the wrong regime.

11.3 Energy storage in capacitors and inductors

A capacitor with charge q and capacitance C has potential difference $v_C = q/C$ and stored electric energy

$$U_C = \frac{q^2}{2C} = \frac{1}{2}Cv_C^2. \quad (11.6)$$

An inductor carrying current i has stored magnetic energy

$$U_L = \frac{1}{2}Li^2. \quad (11.7)$$

These forms already contain the seeds of oscillation. The capacitor energy is quadratic in charge, while the inductor energy is quadratic in current. If charge can flow through the inductor, then a decrease in capacitor energy can appear as an increase in magnetic energy.

A resistor is qualitatively different. The instantaneous power dissipated in a resistor is

$$p_R = i^2R. \quad (11.8)$$

It is non-negative for either current direction. A resistor does not store energy and later return it to the circuit; it converts electrical energy into internal energy of the resistor and its surroundings. This asymmetry is the origin of damping.

Theoretical point 11.1: Energy balance for a loop with ideal elements

For ideal capacitors and inductors the stored energies are reversible. For an ideal resistor the power i^2R is dissipative. A source can add or remove energy depending on the sign of v_si . Therefore, in an RLC circuit, energy accounting gives a physical check on the differential equation.

Caution

Reactance is not resistance. A resistor removes energy from the ideal circuit model. An ideal inductor or capacitor can reduce the current amplitude in an AC circuit, but it does so by storing and returning energy with a phase shift, not by dissipating average power.

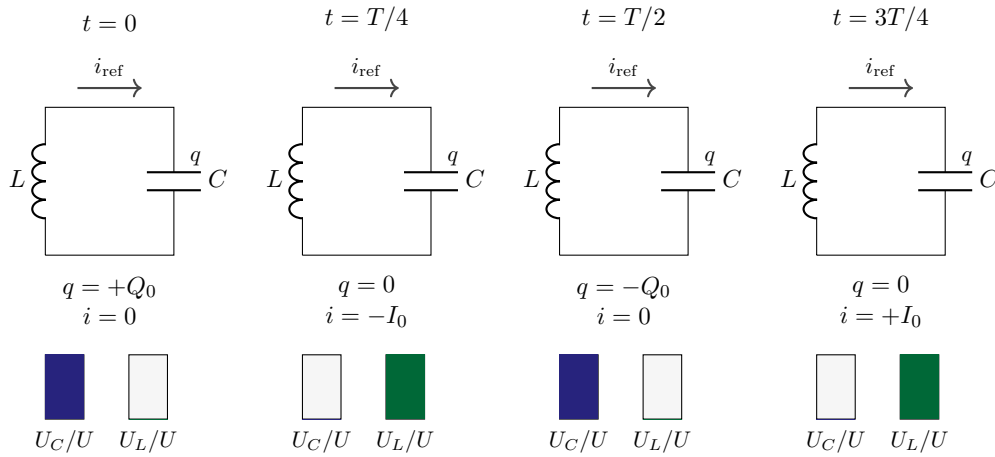


Figure 11.2: Four stages of an ideal closed LC loop starting with $q = +Q_0$ and $i = 0$. The same clockwise reference arrow is used in every panel; it enters the upper capacitor plate, so $i = dq/dt$. A negative listed current flows opposite to that arrow. The bars show energy fractions: charge polarity reverses after half a period, while electric energy returns to its maximum.

11.4 The ideal LC oscillator

Consider an ideal circuit containing only a capacitor C and an inductor L . Let $q(t)$ be the charge on one chosen capacitor plate, and let the loop current be

$$i(t) = \frac{dq}{dt} \quad (11.9)$$

with positive current directed into the plate whose signed charge is q . Figure 11.2 fixes this convention. A negative value of i simply means that the actual current flows opposite to the reference arrow. Applying Kirchhoff's loop rule gives

$$v_L + v_C = 0. \quad (11.10)$$

Using $v_C = q/C$ and $v_L = L di/dt = L d^2q/dt^2$ with the chosen passive convention, the free oscillation equation is

$$L \frac{d^2q}{dt^2} + \frac{q}{C} = 0. \quad (11.11)$$

Dividing by L gives

$$\frac{d^2q}{dt^2} + \omega_0^2 q = 0, \quad \omega_0 = \frac{1}{\sqrt{LC}}. \quad (11.12)$$

The general solution can be written as

$$q(t) = Q_0 \cos(\omega_0 t + \delta), \quad (11.13)$$

where Q_0 and δ are determined by initial conditions. The current is

$$i(t) = \frac{dq}{dt} = -\omega_0 Q_0 \sin(\omega_0 t + \delta). \quad (11.14)$$

Thus charge and current are a quarter cycle out of phase. When the capacitor charge magnitude is maximum, the current is zero. When the current magnitude is maximum, the capacitor charge is zero.

Definition 11.1: Ideal LC angular frequency

For an ideal circuit containing only an inductor L and a capacitor C , the natural angular frequency and ordinary frequency are

$$\omega_0 = \frac{1}{\sqrt{LC}}, \quad f_0 = \frac{1}{2\pi\sqrt{LC}}. \quad (11.15)$$

This result assumes ideal lumped elements, no resistance, no radiation loss and no external driving.

11.4.1 Energy conservation in the LC oscillator

The total stored energy in the ideal LC circuit is

$$U = U_C + U_L = \frac{q^2}{2C} + \frac{1}{2}Li^2. \quad (11.16)$$

Using Eqs. (11.13) and (11.14), and the relation $\omega_0^2 = 1/(LC)$, one finds

$$U_C(t) = \frac{Q_0^2}{2C} \cos^2(\omega_0 t + \delta), \quad (11.17)$$

$$U_L(t) = \frac{1}{2}L\omega_0^2 Q_0^2 \sin^2(\omega_0 t + \delta) = \frac{Q_0^2}{2C} \sin^2(\omega_0 t + \delta). \quad (11.18)$$

Therefore

$$U = \frac{Q_0^2}{2C} \quad (11.19)$$

is constant. The energy does not disappear at intermediate times; it changes form.

Guided checks

1. In an ideal LC circuit, when q is maximum, what is i ? Zero.
2. When i is maximum, what is q ? Zero.
3. Does the total energy oscillate in the ideal model? No. The electric and magnetic parts oscillate, but their sum is constant.
4. What assumptions are needed for $\omega_0 = 1/\sqrt{LC}$? Ideal lumped L and C , no resistance, no radiation and no driving.

Example 11.1: Natural frequency and peak current of an LC oscillator

A capacitor of capacitance $C = 0.50 \mu\text{F}$ is charged to $V_0 = 12 \text{ V}$ and then connected to an ideal inductor of inductance $L = 20 \text{ mH}$. Find the natural frequency and the maximum current.

Solution. The angular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(20 \times 10^{-3})(0.50 \times 10^{-6})}} = 1.0 \times 10^4 \text{ rad s}^{-1}. \quad (11.20)$$

Thus

$$f_0 = \frac{\omega_0}{2\pi} = 1.59 \times 10^3 \text{ Hz}. \quad (11.21)$$

The initial charge is $Q_0 = CV_0 = (0.50 \times 10^{-6})(12) = 6.0 \times 10^{-6} \text{ C}$. The maximum current is

$$I_0 = \omega_0 Q_0 = (1.0 \times 10^4)(6.0 \times 10^{-6}) = 6.0 \times 10^{-2} \text{ A}. \quad (11.22)$$

So the maximum current is 60 mA.

Energy check. Initially $U_C = (1/2)CV_0^2 = 3.6 \times 10^{-5} \text{ J}$. At maximum current, $U_L = (1/2)LI_0^2 = 3.6 \times 10^{-5} \text{ J}$. The agreement checks the calculation.

11.4.2 State variables and sign conventions in oscillating circuits

A circuit with a capacitor and an inductor has memory. The state of the ideal LC circuit at an instant is not determined only by the source voltage or by one algebraic equation. One needs a variable describing the capacitor charge and a variable describing the inductor current. These are analogous to position and velocity in mechanics. A convenient state choice is

$$\text{state at time } t = (q(t), i(t)). \quad (11.23)$$

The capacitor charge determines the electric energy and the current determines the magnetic energy. The pair also determines the future evolution in the ideal model.

The sign convention is chosen by selecting one capacitor plate as positive for q and one loop direction as positive for i . If the positive current direction carries positive charge onto the chosen positive plate, then $i = dq/dt$. If it carries positive charge away from that plate, then $i = -dq/dt$. Both choices can be correct, but they produce different intermediate signs. The final physical predictions are the same if the convention is used consistently.

Method

A safe sign-convention procedure.

1. Draw the capacitor and mark the plate whose charge is called q .
2. Draw an arrow for positive loop current.
3. Decide whether that current arrow increases or decreases q .
4. Write $i = dq/dt$ or $i = -dq/dt$ accordingly.
5. Apply Kirchhoff's loop rule using the same traversal direction.
6. Check the final equation: the free ideal LC equation must have the stable form $q'' + \omega_0^2 q = 0$.

Example 11.2: LC oscillator with nonzero initial charge and current

An ideal LC circuit has $L = 40 \text{ mH}$ and $C = 1.0 \mu\text{F}$. At $t = 0$ the capacitor charge is $q(0) = 2.0 \mu\text{C}$ and the current, with the convention $i = dq/dt$, is $i(0) = 30 \text{ mA}$. Find $q(t)$.

Solution. The angular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(40 \times 10^{-3})(1.0 \times 10^{-6})}} = 5.00 \times 10^3 \text{ rad s}^{-1}. \quad (11.24)$$

Write the general solution as

$$q(t) = A \cos \omega_0 t + B \sin \omega_0 t. \quad (11.25)$$

The first initial condition gives $A = q(0) = 2.0 \times 10^{-6} \text{ C}$. The current is

$$i(t) = \frac{dq}{dt} = -A\omega_0 \sin \omega_0 t + B\omega_0 \cos \omega_0 t. \quad (11.26)$$

At $t = 0$, $i(0) = B\omega_0$, so

$$B = \frac{i(0)}{\omega_0} = \frac{30 \times 10^{-3}}{5.00 \times 10^3} = 6.0 \times 10^{-6} \text{ C}. \quad (11.27)$$

Therefore

$$q(t) = (2.0 \times 10^{-6}) \cos(5.00 \times 10^3 t) + (6.0 \times 10^{-6}) \sin(5.00 \times 10^3 t) \quad \text{C.} \quad (11.28)$$

This can also be written in amplitude-phase form. The charge amplitude is

$$Q_0 = \sqrt{A^2 + B^2} = 6.32 \mu\text{C}. \quad (11.29)$$

The initial current means that the circuit is not starting from an extreme of capacitor charge.

11.4.3 Energy ellipse in the LC phase plane

Figure 11.3 connects the state of the oscillator to its conserved energy and the direction of time evolution.

The ideal LC energy equation

$$\frac{q^2}{2C} + \frac{1}{2}Li^2 = U \quad (11.30)$$

can be rewritten as

$$\frac{q^2}{2CU} + \frac{i^2}{2U/L} = 1. \quad (11.31)$$

In the (q, i) phase plane, this is an ellipse. The oscillator moves around this ellipse periodically. A larger total energy corresponds to a larger ellipse. Underdamped resistance causes the trajectory to spiral inward because energy decreases; critically damped and overdamped trajectories approach the origin without repeated rotations. A sinusoidal source can maintain a nonzero steady-state trajectory after the transient has decayed.

This representation is useful in computation. If a numerical solution of an ideal LC oscillator shows a growing spiral or a decaying spiral when $R = 0$, the numerical method or time step is introducing artificial energy error.

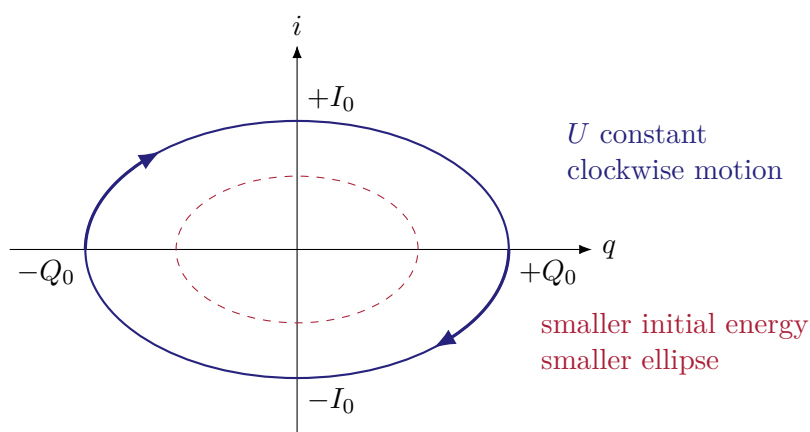


Figure 11.3: Constant-energy curves $q^2/Q_0^2 + i^2/I_0^2 = 1$, with $I_0 = \omega_0 Q_0$. For $i = q'$, the motion is clockwise: at $q = +Q_0$ one has $i = 0$ but $i' = -q/(LC) < 0$. The dashed curve represents another initial energy, not damping within a lossless trajectory.

Guided checks

1. What variables define the state of an ideal LC oscillator? The capacitor charge and inductor current.
2. What happens to the phase-plane ellipse if the total energy is increased? It becomes larger.

3. What would resistance do to the phase-plane curve? It would spiral inward because energy is dissipated.
4. Why is the phase-plane view useful for numerical work? It provides an energy-conservation diagnostic.

11.5 Mechanical analogy: LC circuit and mass-spring oscillator

The LC equation

$$L \frac{d^2 q}{dt^2} + \frac{1}{C} q = 0 \quad (11.32)$$

has the same mathematical structure as the mass-spring equation

$$m \frac{d^2 x}{dt^2} + kx = 0. \quad (11.33)$$

The correspondence is

$$q \leftrightarrow x, \quad i = \frac{dq}{dt} \leftrightarrow v = \frac{dx}{dt}, \quad L \leftrightarrow m, \quad \frac{1}{C} \leftrightarrow k. \quad (11.34)$$

The capacitor energy $q^2/(2C)$ corresponds to spring potential energy $kx^2/2$, while the inductor energy $Li^2/2$ corresponds to kinetic energy $mv^2/2$.

The analogy is powerful but not identical in every detail. Charge is not a position in space, current is not mechanical velocity, and inductance is not mass. The analogy is a mathematical and energetic guide. It tells us how to expect oscillation, damping, resonance and phase to behave.

Mechanical oscillator	Electromagnetic oscillator	Interpretation
Displacement x	Capacitor charge q	coordinate-like variable
Velocity $v = dx/dt$	Current $i = dq/dt$	rate of coordinate change
Mass m	Inductance L	inertia-like coefficient
Spring constant k	Reciprocal capacitance $1/C$	restoring coefficient
Kinetic energy $mv^2/2$	Magnetic energy $Li^2/2$	energy in motion/current
Spring energy $kx^2/2$	Electric energy $q^2/(2C)$	energy in displacement/charge separation
Viscous damping b	Resistance R	energy removal
Driving force $F_0 \cos \omega t$	Source emf $V_0 \cos \omega t$	external drive

Caution

The mechanical analogy should not be turned into a literal picture in which charge sloshes like a ball attached to a spring. The analogy is useful because the equations and energy structure match. Physical charge carriers in a wire have complicated microscopic motion superposed on a small drift.

11.6 Damped oscillations in a series RLC circuit

A real loop has resistance. For a series RLC circuit with no external source, Kirchhoff's loop rule gives

$$L \frac{di}{dt} + Ri + \frac{q}{C} = 0. \quad (11.35)$$

Using $i = dq/dt$, we obtain

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0. \quad (11.36)$$

This is the same structure as a damped mechanical oscillator. Dividing by L gives

$$\frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \omega_0^2 q = 0, \quad \omega_0 = \frac{1}{\sqrt{LC}}. \quad (11.37)$$

The damping strength is controlled by R/L . Define $\alpha = R/(2L)$, in s^{-1} . Whenever $\alpha < \omega_0$, the exact charge response is a sinusoid multiplied by a decreasing envelope:

$$q(t) = Q_0 e^{-Rt/(2L)} \cos(\omega_d t + \delta), \quad (11.38)$$

where the damped angular frequency is

$$\omega_d = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}. \quad (11.39)$$

This underdamped expression applies when $R < 2\sqrt{L/C}$. At larger resistance the circuit is critically damped or overdamped and no sinusoidal oscillation occurs.

Definition 11.2: Underdamped, critically damped and overdamped RLC response

For the free series RLC equation, the characteristic damping threshold is

$$R_c = 2\sqrt{\frac{L}{C}}. \quad (11.40)$$

If $R < R_c$, the response is underdamped and oscillatory. If $R = R_c$, the response is critically damped. If $R > R_c$, the response is overdamped and non-oscillatory. These categories describe the free transient, not the steady-state response under continuous sinusoidal driving.

Theoretical point 11.2: Energy decay in a free RLC circuit

Multiplying Eq. (11.36) by $i = dq/dt$ gives

$$\frac{d}{dt} \left(\frac{1}{2} L i^2 + \frac{q^2}{2C} \right) = -i^2 R. \quad (11.41)$$

The left-hand side is the rate of change of stored electromagnetic energy. The right-hand side is non-positive and vanishes momentarily when $i = 0$. The resistor drains the stored energy monotonically even though charge and current may still oscillate.

Example 11.3: Damping regime and damped frequency

A series RLC circuit has $L = 80 \text{ mH}$, $C = 2.0 \mu\text{F}$ and $R = 40 \Omega$. Determine whether the free response is underdamped, and find ω_0 and ω_d .

Solution. The undamped angular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(80 \times 10^{-3})(2.0 \times 10^{-6})}} = 2.50 \times 10^3 \text{ rad s}^{-1}. \quad (11.42)$$

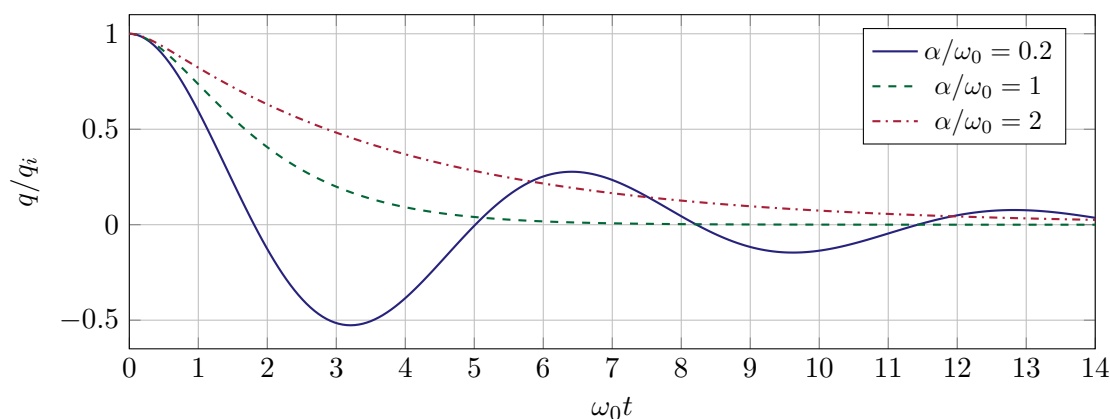


Figure 11.4: Exact normalized free responses for $q(0) = q_i > 0$, $i(0) = 0$. Underdamping permits repeated reversals; critical damping gives the fastest non-oscillatory return for this standard release; strong overdamping leaves a slower tail. All curves have zero initial slope.

The critical resistance is

$$R_c = 2\sqrt{\frac{L}{C}} = 2\sqrt{\frac{80 \times 10^{-3}}{2.0 \times 10^{-6}}} = 400 \Omega. \quad (11.43)$$

Since $R = 40 \Omega < R_c$, the response is underdamped. The damping factor is

$$\frac{R}{2L} = \frac{40}{2(80 \times 10^{-3})} = 250 \text{ s}^{-1}. \quad (11.44)$$

Therefore

$$\omega_d = \sqrt{(2.50 \times 10^3)^2 - (250)^2} = 2.49 \times 10^3 \text{ rad s}^{-1}. \quad (11.45)$$

The damping changes the frequency only slightly in this example, but it strongly affects the amplitude over many cycles.

Guided checks

1. Which term in $Lq'' + Rq' + q/C = 0$ represents energy dissipation? The Rq' term.
2. Does every RLC circuit oscillate freely? No. Oscillatory free response requires underdamping.
3. Does the resistor change the stored energy instantaneously into zero? No. It removes energy at rate $i^2 R$.
4. Can a heavily damped circuit still have a natural LC scale? Yes, $1/\sqrt{LC}$ remains a parameter, but the observable free response may be non-oscillatory.

11.6.1 Initial conditions and the three damping regimes

The envelope alone does not specify a discharge experiment. In particular, setting $q(t) = q_i e^{-\alpha t} \cos \omega_d t$ would give $i(0) = -\alpha q_i$, not zero. A capacitor initially charged to q_i and an inductor initially carrying i_i require both conditions. Substitution of $q = e^{st}$ into the differential equation gives

$$s^2 + 2\alpha s + \omega_0^2 = 0, \quad s_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}. \quad (11.46)$$

For the underdamped case the useful initial-value form is

$$q(t) = e^{-\alpha t} \left[q_i \cos \omega_d t + \frac{i_i + \alpha q_i}{\omega_d} \sin \omega_d t \right], \quad (11.47)$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}. \quad (11.48)$$

For $i_i = 0$, differentiating gives

$$i(t) = -\frac{q_i \omega_0^2}{\omega_d} e^{-\alpha t} \sin \omega_d t. \quad (11.49)$$

Thus the first current is negative for $q_i > 0$: it leaves the initially positive plate, as the circuit diagram predicts. The sine term in the charge solution is essential even when damping is small. It is the correction needed to satisfy the initial current.

At critical damping, $\alpha = \omega_0$, the two roots coincide. Two independent exponential solutions are no longer available, and the solution is

$$q(t) = e^{-\omega_0 t} [q_i + (i_i + \omega_0 q_i)t]. \quad (11.50)$$

For an initially charged capacitor with $i_i = 0$, this approaches zero without crossing it. With arbitrary initial current, however, a critically damped trajectory can cross zero once. “No oscillation” means no repeated periodic crossings; it is not a universal prohibition on overshoot for every initial state.

For overdamping let $\beta = \sqrt{\alpha^2 - \omega_0^2}$. Then

$$q(t) = A e^{(-\alpha+\beta)t} + B e^{(-\alpha-\beta)t}, \quad A + B = q_i, \quad (-\alpha + \beta)A + (-\alpha - \beta)B = i_i. \quad (11.51)$$

Both exponents are negative. For very large R , the slow root is approximately $-1/(RC)$ and the fast root approximately $-R/L$. A large resistance therefore suppresses ringing but can leave a long capacitive discharge tail. Figure 11.4 compares all three regimes for identical initial charge and zero initial current.

Example 11.4: A discharge initial condition and the first current maximum

Use $L = 80 \text{ mH}$, $C = 2.0 \text{ }\mu\text{F}$ and $R = 40 \text{ }\Omega$ from Example 11.3. The capacitor initially has 10 V and the current is zero. Find the first extremum of current and compare its magnitude with the lossless prediction.

Strategy. The free response is underdamped. Use the initial-value solution, rather than attaching an exponential to the undamped solution without checking its derivative. Here $q_i = 20 \text{ }\mu\text{C}$, $\alpha = 250 \text{ s}^{-1}$ and $\omega_d = 2487.47 \text{ rad s}^{-1}$.

An extremum of i occurs when

$$\omega_d \cos \omega_d t - \alpha \sin \omega_d t = 0. \quad (11.52)$$

The first positive time is

$$t_* = \frac{1}{\omega_d} \tan^{-1} \frac{\omega_d}{\alpha} = 0.591 \text{ ms}. \quad (11.53)$$

Substitution gives $i(t_*) = -43.1 \text{ mA}$. The lossless prediction would be $q_i \omega_0 = 50.0 \text{ mA}$. The magnitude is smaller because the resistor has already dissipated energy. At this instant $di/dt = 0$, so $v_L = 0$ and $q/C = -Ri$, giving $v_C = +1.73 \text{ V}$ before rounding the current. Unlike the ideal LC case, the capacitor is not yet empty when the current magnitude is greatest.

Check. The product Ri has units of voltage, and its sign opposes q/C in the free-loop equation. Taking $R \rightarrow 0$ recovers $t_* = \pi/(2\omega_0)$ and the lossless maximum current.

Caution

The exponential $e^{-\alpha t}$ is the exact charge-envelope factor in the linear underdamped model. The instantaneous total energy is not generally a pure exponential: $dU/dt = -Ri^2$ even vanishes at each current zero. For weak damping, an average over a cycle changes slowly and is approximately proportional to $e^{-2\alpha t}$. Values sampled at equal phases one damped period apart do scale exactly by $e^{-2\alpha T_d}$.

11.7 Sinusoidal quantities and rms values

A sinusoidal voltage can be written as

$$v(t) = V_0 \cos(\omega t + \alpha), \quad (11.54)$$

where V_0 is the peak value. The average value of this voltage over a complete cycle is zero. However, its ability to deliver heat in a resistor is not zero because power depends on v^2 or i^2 .

The root-mean-square value of a periodic quantity $x(t)$ over one period T is

$$x_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T x^2(t) dt}. \quad (11.55)$$

For a sinusoid $x(t) = X_0 \cos \omega t$, use $\cos^2 \theta = (1 + \cos 2\theta)/2$. The oscillatory term integrates to zero over $T = 2\pi/\omega$, giving $\langle x^2 \rangle = X_0^2/2$ and hence

$$x_{\text{rms}} = \frac{X_0}{\sqrt{2}}. \quad (11.56)$$

Therefore

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}, \quad I_{\text{rms}} = \frac{I_0}{\sqrt{2}}. \quad (11.57)$$

Definition 11.3: Rms value

The rms value of an AC current is the value of a DC current that would produce the same average heating power in a resistor. For a sinusoidal current,

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}}. \quad (11.58)$$

The rms value is not the average current; the average of a symmetric sinusoidal current over one full cycle is zero.

Example 11.5: Peak and rms values

A sinusoidal voltage source has rms voltage 230 V. What is the peak voltage? If it is connected to a purely resistive load of $R = 46 \Omega$, what are the rms current and average power?

Solution. The peak voltage is

$$V_0 = \sqrt{2}V_{\text{rms}} = \sqrt{2}(230) = 325 \text{ V}. \quad (11.59)$$

For a resistor, rms Ohm's law is

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{R} = \frac{230}{46} = 5.0 \text{ A}. \quad (11.60)$$

The average power is

$$P_{\text{avg}} = I_{\text{rms}}^2 R = (5.0)^2 (46) = 1.15 \times 10^3 \text{ W}. \quad (11.61)$$

Equivalently, $P_{\text{avg}} = V_{\text{rms}} I_{\text{rms}} = 1150 \text{ W}$ for a purely resistive load.

Caution

An rms value is not a safety guarantee. A sinusoidal voltage with 230 V rms has a peak magnitude of about 325 V. Insulation, breakdown, and semiconductor ratings often depend on peak or transient values, not only rms values.

11.7.1 What rms does and does not tell us

The heating interpretation assumes an ohmic resistor whose resistance is effectively constant over the cycle. The more general identity $\langle i^2 R \rangle = I_{\text{rms}}^2 R$ remains valid for any periodic current under that assumption; the factor $1/\sqrt{2}$ is specific to a sinusoid. A waveform with narrow peaks can have a modest rms value but a much greater peak current than a sinusoid with the same heating effect.

If a signal contains a DC component and a zero-mean AC part, $v(t) = V_{\text{dc}} + v_{\text{ac}}(t)$, then

$$V_{\text{rms}}^2 = V_{\text{dc}}^2 + V_{\text{ac,rms}}^2, \quad (11.62)$$

because the mean cross term $2V_{\text{dc}}\langle v_{\text{ac}} \rangle$ vanishes. An AC-coupled measurement removes the DC part and can therefore report a different rms quantity from a measurement of the full waveform.

Example 11.6: Equal heating does not imply equal peak voltage

A $100\ \Omega$ resistor receives a periodic voltage that is 12 V for one quarter of each cycle and zero otherwise. Find its mean voltage, rms voltage and average power. Compare it with a zero-mean sinusoid producing the same power.

Strategy. Apply the defining integral separately to the mean and to the mean square. The waveform is a unipolar pulse train, so a sinusoidal conversion factor is inappropriate.

$$\langle v \rangle = \frac{1}{4}(12 \text{ V}) = 3.0 \text{ V}, \quad (11.63)$$

$$V_{\text{rms}} = \sqrt{\frac{1}{4}(12 \text{ V})^2} = 6.0 \text{ V}, \quad (11.64)$$

$$P = \frac{V_{\text{rms}}^2}{R} = 0.36 \text{ W}. \quad (11.65)$$

The equal-heating sinusoid has peak voltage $6\sqrt{2} = 8.49 \text{ V}$, whereas the pulse train peaks at 12 V. Its AC-only rms value is $\sqrt{6^2 - 3^2} = 5.20 \text{ V}$. These are three different descriptions of one signal, answering different measurement questions.

11.8 AC response of a resistor, capacitor and inductor

Figure 11.5 summarizes the phase relations derived below.

Before combining elements, we examine the response of each ideal element to sinusoidal current or voltage.

11.8.1 Resistor

For a resistor,

$$v_R(t) = i(t)R. \quad (11.66)$$

If $i(t) = I_0 \cos \omega t$, then

$$v_R(t) = I_0 R \cos \omega t. \quad (11.67)$$

Voltage and current are in phase. The peak voltage is $V_{R0} = I_0 R$.

11.8.2 Inductor

For an inductor,

$$v_L(t) = L \frac{di}{dt}. \quad (11.68)$$

If $i(t) = I_0 \cos \omega t$, then

$$v_L(t) = -\omega L I_0 \sin \omega t = \omega L I_0 \cos \left(\omega t + \frac{\pi}{2} \right). \quad (11.69)$$

The inductor voltage leads the current by 90° . The peak voltage is

$$V_{L0} = I_0 X_L, \quad X_L = \omega L. \quad (11.70)$$

The quantity X_L is the inductive reactance.

11.8.3 Capacitor

For a capacitor,

$$i(t) = C \frac{dv_C}{dt}. \quad (11.71)$$

If $v_C(t) = V_{C0} \cos \omega t$, then

$$i(t) = -\omega C V_{C0} \sin \omega t = \omega C V_{C0} \cos \left(\omega t + \frac{\pi}{2} \right). \quad (11.72)$$

The current leads the capacitor voltage by 90° , or equivalently the capacitor voltage lags the current by 90° . The peak voltage can be written

$$V_{C0} = I_0 X_C, \quad X_C = \frac{1}{\omega C}. \quad (11.73)$$

The quantity X_C is the capacitive reactance.

Key message

At high frequency an ideal inductor has large reactance and an ideal capacitor has small reactance. At low frequency the opposite is true: an ideal inductor has small reactance and an ideal capacitor has large reactance. This opposite frequency dependence is the mathematical origin of resonance in an RLC circuit.

Guided checks

1. In a pure resistor, are voltage and current in phase? Yes.
2. In a pure inductor, does voltage lead or lag current? Voltage leads current by 90° .
3. In a pure capacitor, does current lead or lag voltage? Current leads voltage by 90° .
4. What happens to X_L if frequency doubles? It doubles.

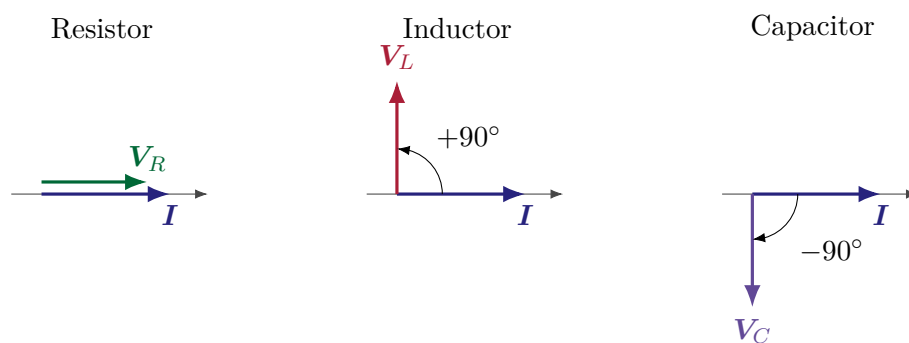


Figure 11.5: Element voltage phasors relative to current. Positive phase is counterclockwise. Inductor voltage leads; capacitor voltage lags. Current and voltage use independent drawing scales, since they have different units. The phasor arrows describe phase relationships, not spatial directions in a wire.

5. What happens to X_C if frequency doubles? It is halved.

Example 11.7: Pure inductor driven by a sinusoidal voltage

An ideal inductor $L = 0.50 \text{ H}$ is connected to $v(t) = 20 \cos(100t) \text{ V}$. Find the current amplitude and phase in steady state.

Solution. For an inductor, $v = L di/dt$. The reactance is

$$X_L = \omega L = (100)(0.50) = 50 \Omega. \quad (11.74)$$

The current amplitude is

$$I_0 = \frac{V_0}{X_L} = \frac{20}{50} = 0.40 \text{ A}. \quad (11.75)$$

In a pure inductor, current lags voltage by 90° . Therefore

$$i(t) = 0.40 \cos(100t - \pi/2) \text{ A}. \quad (11.76)$$

A direct derivative check confirms this: differentiating gives $di/dt = 40 \cos(100t)$ in A s^{-1} , and multiplying by $L = 0.50 \text{ H}$ gives the $20 \cos(100t)$ voltage.

Example 11.8: Pure capacitor driven by a sinusoidal voltage

An ideal capacitor $C = 25 \mu\text{F}$ is connected to $v(t) = 10 \cos(200t) \text{ V}$. Find the current.

Solution. The capacitive reactance is

$$X_C = \frac{1}{\omega C} = \frac{1}{(200)(25 \times 10^{-6})} = 200 \Omega. \quad (11.77)$$

The current amplitude is

$$I_0 = \frac{V_0}{X_C} = \frac{10}{200} = 0.050 \text{ A}. \quad (11.78)$$

For a capacitor, current leads voltage by 90° , so

$$i(t) = 0.050 \cos(200t + \pi/2) \text{ A}. \quad (11.79)$$

Equivalently, from $i = C dv/dt$,

$$i(t) = C[-200(10) \sin(200t)] = -0.050 \sin(200t) = 0.050 \cos(200t + \pi/2). \quad (11.80)$$

Note

A perfectly lossless inductor driven by a sinusoidal voltage can retain an arbitrary constant current set by its initial condition. The phasor solution represents the zero-mean sinusoidal component. A small series resistance removes the constant offset after a transient; the limit of that steady state gives the usual pure-inductor formula.

11.9 Phasors and the idea of impedance

A sinusoidal source supplies the energy lost in resistance. Before introducing a shortcut for the periodic response, write the full time-domain equation:

$$Lq'' + Rq' + \frac{q}{C} = v_s(t), \quad v_s(t) = V_0 \cos \omega t. \quad (11.81)$$

Its solution is a sum $q = q_{\text{tr}} + q_{\text{ss}}$. The transient q_{tr} solves the free equation and enforces the initial charge and current. For $R > 0$ it eventually decays. The sinusoidal steady response q_{ss} oscillates at the source frequency ω , which need not equal either ω_0 or ω_d . Phasors determine this steady response only. They do not determine the transient immediately after switching.

For a high- Q underdamped circuit, the envelope time is $1/\alpha = 2L/R$, so waiting only one or two source periods may be insufficient. For strong overdamping the slow characteristic root instead determines the settling time. These distinctions matter when an oscilloscope trace is used to test an impedance calculation.

A phasor is a rotating-vector representation of a sinusoidal quantity. For the cosine convention used here, the physical voltage or current is the horizontal projection of a counterclockwise rotating phasor. These arrows live in a mathematical phase plane, not physical space; voltage and current arrows also have different units. In steady-state AC analysis all phasors rotate at the same angular frequency ω , so their relative angles are constant. The diagram can therefore be drawn as a static vector diagram in phase space.

For a series circuit, the same current passes through each element. It is natural to choose the current phasor as the horizontal reference. The resistor voltage phasor is horizontal, the inductor voltage phasor points upward, and the capacitor voltage phasor points downward. The source voltage phasor is the vector sum

$$\mathbf{V}_s = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C. \quad (11.82)$$

In terms of amplitudes,

$$V_0^2 = V_{R0}^2 + (V_{L0} - V_{C0})^2. \quad (11.83)$$

Since $V_{R0} = I_0 R$, $V_{L0} = I_0 X_L$ and $V_{C0} = I_0 X_C$, the current amplitude is

$$I_0 = \frac{V_0}{Z}, \quad Z = \sqrt{R^2 + (X_L - X_C)^2}. \quad (11.84)$$

The phase angle between source voltage and current is

$$\tan \phi = \frac{X_L - X_C}{R}. \quad (11.85)$$

With our convention $i(t) = I_0 \cos(\omega t - \phi)$ and $v_s(t) = V_0 \cos \omega t$, positive ϕ means current lags voltage. For a passive series circuit with $R > 0$, choose $-\pi/2 < \phi < \pi/2$. A phase measured in degrees must be converted to radians before being added to ωt in a time function.

Definition 11.4: Reactance and impedance of a series RLC circuit

For sinusoidal steady-state motion at angular frequency ω ,

$$X_L = \omega L, \quad X_C = \frac{1}{\omega C}, \quad (11.86)$$

and the impedance magnitude of a series RLC circuit is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}. \quad (11.87)$$

This formula applies to the sinusoidal steady state of an ideal lumped series circuit. It is not the transient solution immediately after switching.

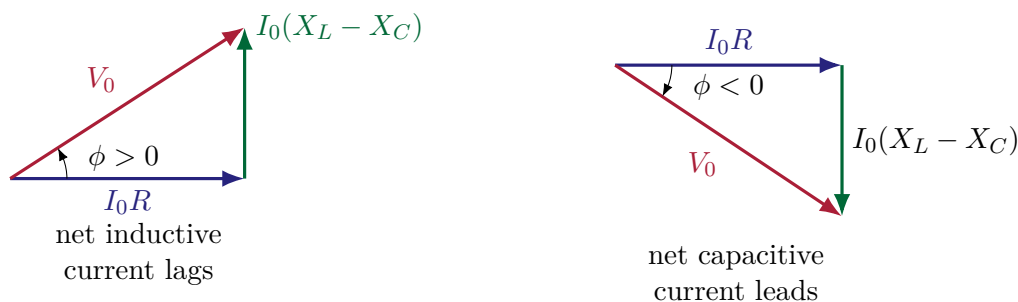


Figure 11.6: Head-to-tail addition of resistor and net reactive voltages. The two panels use the current direction as the horizontal phase reference. The vertical component can be positive or negative; its sign fixes the interpretation of the phase angle.

Method
Series RLC phasor method.

1. Compute $X_L = \omega L$ and $X_C = 1/(\omega C)$.
2. Compare X_L and X_C to determine whether the circuit is net inductive or net capacitive.
3. Compute $Z = \sqrt{R^2 + (X_L - X_C)^2}$.
4. Compute $I_0 = V_0/Z$ or $I_{\text{rms}} = V_{\text{rms}}/Z$.
5. Compute $\tan \phi = (X_L - X_C)/R$ and interpret the sign.
6. Use rms values for average power and peak values for instantaneous sinusoidal functions.

Example 11.9: Current amplitude and phase in a series RLC circuit

A series circuit has $R = 30 \Omega$, $L = 0.20 \text{ H}$ and $C = 50 \mu\text{F}$. It is driven by $v_s(t) = 120 \cos(400t) \text{ V}$. Find the current amplitude and phase.

Solution. The angular frequency is $\omega = 400 \text{ rad s}^{-1}$. The reactances are

$$X_L = \omega L = (400)(0.20) = 80 \Omega, \quad (11.88)$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(400)(50 \times 10^{-6})} = 50 \Omega. \quad (11.89)$$

The circuit is net inductive because $X_L > X_C$. The impedance magnitude is

$$Z = \sqrt{30^2 + (80 - 50)^2} = 42.4 \Omega. \quad (11.90)$$

Thus

$$I_0 = \frac{V_0}{Z} = \frac{120}{42.4} = 2.83 \text{ A.} \quad (11.91)$$

The phase angle is

$$\tan \phi = \frac{80 - 50}{30} = 1, \quad \phi = 45^\circ. \quad (11.92)$$

With the convention $i(t) = I_0 \cos(\omega t - \phi)$, the current is

$$i(t) = 2.83 \cos(400t - \pi/4) \text{ A.} \quad (11.93)$$

The current lags the source voltage because the circuit is net inductive.

Example 11.10: A capacitive series RLC response

A series circuit has $R = 50 \Omega$, $L = 40 \text{ mH}$ and $C = 10 \mu\text{F}$. It is driven at $\omega = 1000 \text{ rad s}^{-1}$ with rms voltage 24 V . Find I_{rms} and state whether the current leads or lags.

Solution. The reactances are

$$X_L = (1000)(40 \times 10^{-3}) = 40 \Omega, \quad X_C = \frac{1}{(1000)(10 \times 10^{-6})} = 100 \Omega. \quad (11.94)$$

Thus $X_L - X_C = -60 \Omega$, so the circuit is net capacitive. The impedance is

$$Z = \sqrt{50^2 + (-60)^2} = 78.1 \Omega. \quad (11.95)$$

Therefore

$$I_{\text{rms}} = \frac{24}{78.1} = 0.307 \text{ A.} \quad (11.96)$$

The phase angle satisfies

$$\tan \phi = \frac{-60}{50} = -1.20, \quad \phi = -50.2^\circ. \quad (11.97)$$

Since $\phi < 0$ in the convention $i(t) = I_0 \cos(\omega t - \phi)$, the current leads the source voltage by 50.2° .

11.9.1 Building a phasor diagram by hand

Figure 11.6 shows the voltage sums for inductive and capacitive cases.

To build a phasor diagram for a series RLC circuit, choose the current as the reference. Draw V_R along the current. Draw V_L upward and V_C downward. The net reactive voltage is the difference between V_L and V_C . Then add the resistor voltage and the net reactive voltage as perpendicular components. The source voltage is the hypotenuse.

This construction is more reliable than memorizing whether to add or subtract angles. It also gives a visual reason why the source voltage may be much smaller than the individual inductor and capacitor voltages near resonance.

Example 11.11: Hand phasor construction from rms data

In a series RLC circuit the measured rms current is 0.20 A . At the driving frequency the reactances are $X_L = 160 \Omega$ and $X_C = 60 \Omega$, and $R = 75 \Omega$. Find the rms voltage across each

element and the source rms voltage.

Solution. The resistor voltage is

$$V_R = I_{\text{rms}} R = (0.20)(75) = 15 \text{ V}. \quad (11.98)$$

The inductor and capacitor voltages are

$$V_L = (0.20)(160) = 32 \text{ V}, \quad V_C = (0.20)(60) = 12 \text{ V}. \quad (11.99)$$

The net reactive voltage is $V_L - V_C = 20 \text{ V}$ upward in the phasor diagram. Therefore

$$V_s = \sqrt{15^2 + 20^2} = 25 \text{ V}. \quad (11.100)$$

The phase angle is $\tan \phi = 20/15$, so $\phi = 53.1^\circ$. The circuit is net inductive and the current lags the source voltage.

11.10 Power in alternating-current circuits

The instantaneous power delivered to a circuit is

$$p(t) = v(t)i(t). \quad (11.101)$$

Choose positive i entering the driven load at its positive terminal. Then $v_s i$ is the power delivered by the source to the load. Multiplying Eq. (11.81) by $i = q'$ establishes the instantaneous balance

$$v_s i = \frac{d}{dt} \left(\frac{q^2}{2C} + \frac{Li^2}{2} \right) + Ri^2. \quad (11.102)$$

Thus the source can replenish stored energy, supply resistive heating, or receive returned energy. Over one period of steady motion the change in stored energy is zero, leaving $\langle v_s i \rangle = R \langle i^2 \rangle$.

For $v_s = V_0 \cos \omega t$ and $i = I_0 \cos(\omega t - \phi)$, the product-to-sum identity gives

$$p(t) = \frac{V_0 I_0}{2} [\cos \phi + \cos(2\omega t - \phi)]. \quad (11.103)$$

The first term is constant; the second averages to zero. Its frequency is twice the source frequency, since both voltage and current reverse sign during a cycle. Negative values of p mean energy is returning to the source, not negative resistive heating. Figure 11.7 illustrates this distinction.

For a sinusoidal source voltage $v(t) = V_0 \cos \omega t$ and current $i(t) = I_0 \cos(\omega t - \phi)$, the average power over one cycle is

$$P_{\text{avg}} = \frac{1}{2} V_0 I_0 \cos \phi. \quad (11.104)$$

Equivalently,

$$P_{\text{avg}} = V_{\text{rms}} I_{\text{rms}} \cos \phi. \quad (11.105)$$

The factor $\cos \phi$ is called the power factor. In a pure resistor, $\phi = 0$ and all source energy delivered over a cycle is dissipated as heat. In a pure inductor or pure capacitor, $\phi = \pm 90^\circ$ and the average power is zero in the ideal model. Energy moves back and forth between the source and the reactive element but no net energy is dissipated over a full cycle.

For a series RLC circuit, the average power dissipated in the resistor is also

$$P_{\text{avg}} = I_{\text{rms}}^2 R. \quad (11.106)$$

This is consistent with Eq. (11.105) because $\cos \phi = R/Z$.

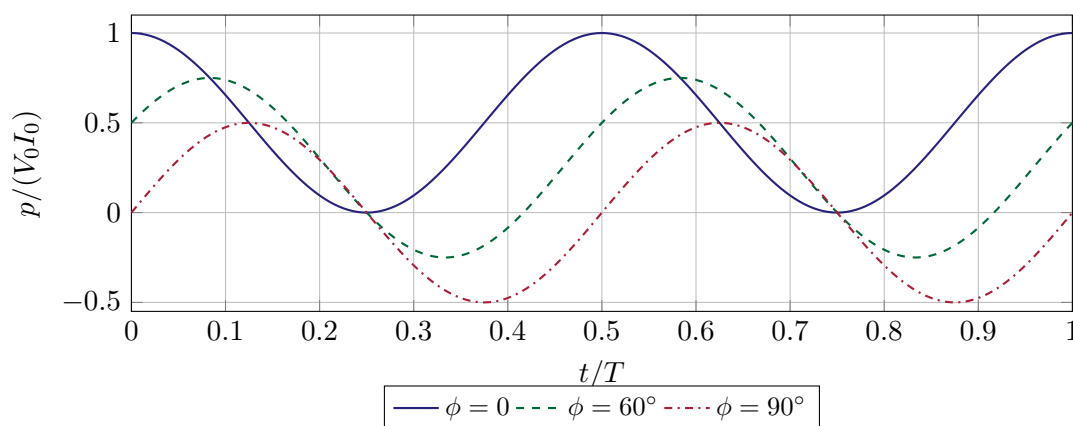


Figure 11.7: Calculated instantaneous power for equal peak-product normalization. The mean levels are $1/2$, $1/4$ and zero, respectively. The reactive cases contain intervals of energy return; the resistor case is non-negative throughout.

Definition 11.5: Power factor

The power factor of a sinusoidal steady-state circuit is

$$\cos \phi = \frac{P_{\text{avg}}}{V_{\text{rms}} I_{\text{rms}}}. \quad (11.107)$$

It measures the fraction of the apparent power $V_{\text{rms}} I_{\text{rms}}$ that is converted into average real power. A low power factor means that large rms currents may flow while relatively little average power is delivered to useful or dissipative loads.

Example 11.12: Average power away from resonance

Use the circuit in Example 11.9. Find the rms current and average power.

Solution. The voltage amplitude is $V_0 = 120 \text{ V}$, so

$$V_{\text{rms}} = \frac{120}{\sqrt{2}} = 84.9 \text{ V}. \quad (11.108)$$

The current amplitude was $I_0 = 2.83 \text{ A}$, so

$$I_{\text{rms}} = \frac{2.83}{\sqrt{2}} = 2.00 \text{ A}. \quad (11.109)$$

The phase angle is 45° , hence

$$P_{\text{avg}} = V_{\text{rms}} I_{\text{rms}} \cos \phi = (84.9)(2.00) \cos 45^\circ = 120 \text{ W}. \quad (11.110)$$

The resistor check gives

$$I_{\text{rms}}^2 R = (2.00)^2 (30) = 120 \text{ W}. \quad (11.111)$$

The equality confirms that only the resistor dissipates average power in the ideal RLC model.

Caution

The quantities $V_{\text{rms}} I_{\text{rms}}$ and P_{avg} are not generally the same. Their ratio is controlled by the power factor. Confusing apparent power with average real power is a common and practically important AC-circuit error.

Guided checks

1. What is the average power in an ideal pure capacitor driven sinusoidally? Zero.
2. Can the instantaneous power in an ideal inductor be nonzero? Yes; energy is exchanged with the source, but the cycle average is zero.
3. In a series RLC circuit, where is average power dissipated? In the resistor.
4. What does $\cos \phi = 0.6$ mean? Only 60 percent of the apparent power is real average power.

Example 11.13: Inferring loss and reactance from measurements

A black-box load connected to a sinusoidal source draws 1.0 A rms at 10 V rms, but the average power is only 4.0 W. Find the impedance magnitude, power factor and equivalent series resistance.

Solution. The impedance magnitude is

$$|Z| = \frac{V_{\text{rms}}}{I_{\text{rms}}} = 10 \Omega. \quad (11.112)$$

The power factor is

$$\cos \phi = \frac{P}{VI} = \frac{4.0}{(10)(1.0)} = 0.40. \quad (11.113)$$

The equivalent series resistance is the part responsible for average power:

$$R_{\text{eq}} = \frac{P}{I_{\text{rms}}^2} = 4.0 \Omega. \quad (11.114)$$

The remaining impedance magnitude is reactive. Its magnitude is $\sqrt{10^2 - 4^2} = 9.17 \Omega$, but the sign requires phase information.

11.11 Resonance in a driven series RLC circuit

Figure 11.8 compares currents for three resistances at the same source voltage.

For a sinusoidal source of fixed voltage amplitude and frequency-independent R , L and C , the current amplitude in a driven series RLC circuit is

$$I_0(\omega) = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}. \quad (11.115)$$

The denominator is smallest when

$$\omega L = \frac{1}{\omega C}. \quad (11.116)$$

Thus the resonance angular frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}. \quad (11.117)$$

At resonance the impedance magnitude is $Z = R$, the phase angle is zero, and the current amplitude is $I_0 = V_0/R$.

Resonance does not mean that the capacitor and inductor disappear. At resonance their voltage amplitudes are equal and opposite in phase, so their sum in the loop phasor equation cancels. The individual voltages across L and C can be much larger than the source voltage when resistance is small. This is one reason resonant circuits are useful and also one reason they must be designed carefully.

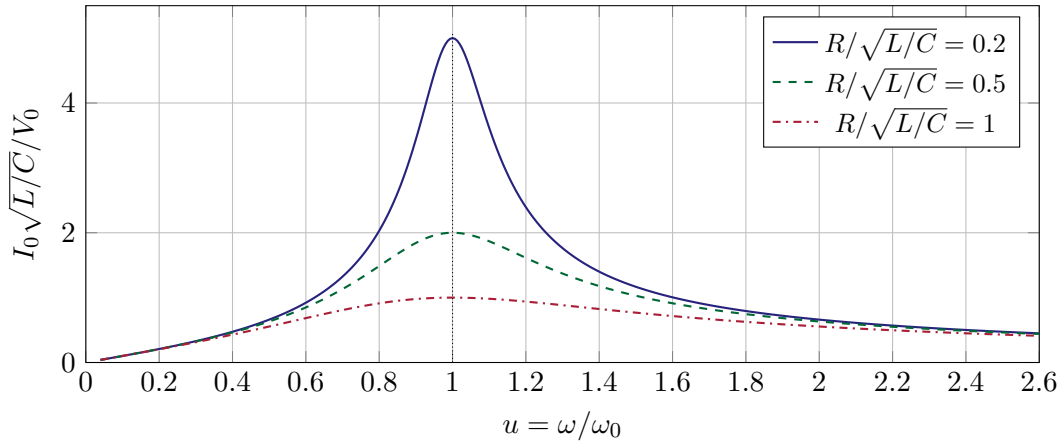


Figure 11.8: Calculated current responses for the same L , C and source amplitude. The normalization is common to all three curves, so their different peak heights are meaningful. Decreasing resistance increases the peak and narrows the bandwidth; each current maximum remains at ω_0 .

11.11.1 Half-power frequencies and quality factor

At resonance the average resistor power is $P_{\max} = V_{\text{rms}}^2/R$. At another frequency the ratio is

$$\frac{P(\omega)}{P_{\max}} = \frac{R^2}{R^2 + (\omega L - 1/\omega C)^2}. \quad (11.118)$$

The half-power points therefore satisfy $|\omega L - 1/(\omega C)| = R$. The current there is $I_{\max}/\sqrt{2}$, not $I_{\max}/2$. Solving the two quadratic equations gives

$$\omega_1 = \frac{\sqrt{R^2 + 4L/C} - R}{2L}, \quad \omega_2 = \frac{\sqrt{R^2 + 4L/C} + R}{2L}. \quad (11.119)$$

Both roots are positive; the smaller root is below ω_0 . Subtraction and multiplication yield

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L}, \quad \omega_1\omega_2 = \omega_0^2. \quad (11.120)$$

These are exact results for the stated ideal series model, even when the peak is broad. On a linear frequency axis the half-power points are not generally symmetric about ω_0 ; ω_0 is their geometric mean. Their approximate locations $\omega_{1,2} \simeq \omega_0 \mp R/(2L)$ require $R/L \ll \omega_0$.

Definition 11.6: Series-resonator quality factor

For constant positive R , L and C , define

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}. \quad (11.121)$$

All equalities are exact for this series current response. High Q means a narrow fractional bandwidth. The free response is underdamped for $Q > 1/2$, since $\alpha = \omega_0/(2Q)$. A high- Q approximation additionally requires $Q \gg 1$.

At driven current resonance, write $i = I_0 \cos \omega_0 t$ and $q = (I_0/\omega_0) \sin \omega_0 t$. The stored energy is then constant:

$$U = \frac{1}{2} L I_0^2, \quad E_{\text{loss/cycle}} = \frac{1}{2} R I_0^2 \frac{2\pi}{\omega_0}. \quad (11.122)$$

Consequently $2\pi U/E_{\text{loss/cycle}} = \omega_0 L/R = Q$. The source supplies the loss while the two fields exchange energy. Applying a per-cycle loss definition to a *freely decaying* oscillation instead requires care, because its energy is changing during the cycle; the simple identification is then a weak-damping approximation.

Example 11.14: Resonance frequency and current at resonance

A series RLC circuit has $L = 25 \text{ mH}$, $C = 0.40 \mu\text{F}$ and $R = 12 \Omega$. It is driven by a source with rms voltage 10 V . Find the resonance frequency and the rms current at resonance.

Solution. The angular resonance frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(25 \times 10^{-3})(0.40 \times 10^{-6})}} = 1.00 \times 10^4 \text{ rad s}^{-1}. \quad (11.123)$$

The ordinary frequency is

$$f_0 = \frac{\omega_0}{2\pi} = 1.59 \times 10^3 \text{ Hz}. \quad (11.124)$$

At resonance $Z = R$, so

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{R} = \frac{10}{12} = 0.833 \text{ A}. \quad (11.125)$$

The power factor is unity at resonance, and the average power delivered is

$$P_{\text{avg}} = I_{\text{rms}}^2 R = (0.833)^2 (12) = 8.33 \text{ W}. \quad (11.126)$$

Example 11.15: Large capacitor and inductor voltages at resonance

For the resonant circuit of Example 11.14, find the rms voltage across the inductor and capacitor at resonance.

Solution. At resonance,

$$X_L = \omega_0 L = (1.00 \times 10^4)(25 \times 10^{-3}) = 250 \Omega, \quad (11.127)$$

and $X_C = 250 \Omega$ as well. The rms current is 0.833 A , so

$$V_{L,\text{rms}} = I_{\text{rms}} X_L = (0.833)(250) = 208 \text{ V}, \quad (11.128)$$

$$V_{C,\text{rms}} = I_{\text{rms}} X_C = (0.833)(250) = 208 \text{ V}. \quad (11.129)$$

The source rms voltage is only 10 V , but the individual reactive voltages are much larger. They are opposite in phase in the loop sum and cancel each other. This example illustrates why resonance is not the same as absence of L and C .

Guided checks

1. At resonance in a series RLC circuit, what is $X_L - X_C$? Zero.
2. Is the capacitor absent at resonance? No. Its voltage can be large, but it cancels the inductor voltage in the phasor sum.
3. What is the phase angle at resonance? Zero for an ideal series RLC circuit.
4. What happens to the resonance peak when R is reduced? It becomes taller and narrower.
5. Why can resonance be useful for tuning? The circuit responds strongly near one frequency and less strongly away from it.

11.11.2 Voltage magnification and component ratings

The ratio of a reactive element voltage at resonance to the source voltage is closely related to the quality factor. In a series RLC circuit at resonance,

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{R}. \quad (11.130)$$

The inductor rms voltage is

$$V_{L,\text{rms}} = I_{\text{rms}}\omega_0 L = V_{\text{rms}} \frac{\omega_0 L}{R} = QV_{\text{rms}}. \quad (11.131)$$

Similarly, $V_{C,\text{rms}} = QV_{\text{rms}}$. Thus a high- Q resonator can place voltages across components that are many times larger than the source voltage.

Example 11.16: Checking a capacitor voltage rating

A series resonant circuit has $Q = 18$ and is driven at resonance by a 6.0 V rms source. Estimate the rms voltage across the capacitor. If its permitted sinusoidal operating voltage at this frequency is 50 V rms, does the estimate satisfy that limit?

Solution. At resonance the capacitor voltage magnitude is, in the ideal model,

$$V_{C,\text{rms}} = QV_{\text{source,rms}} = (18)(6.0) = 108 \text{ V}. \quad (11.132)$$

This exceeds the 50 V rms rating. The stated operating limit is exceeded, and a real circuit may also experience transients or tolerances that make the voltage larger.

11.11.3 Designing for a target resonance frequency

The resonance condition $\omega_0 = 1/\sqrt{LC}$ can be used as a design equation. If L is selected first, then

$$C = \frac{1}{\omega_0^2 L}. \quad (11.133)$$

If C is selected first, then

$$L = \frac{1}{\omega_0^2 C}. \quad (11.134)$$

In engineering practice one must also consider component tolerances. If L and C have tolerances, the resonance frequency is uncertain. For small fractional uncertainties,

$$\frac{\Delta f_0}{f_0} \simeq \frac{1}{2} \left(\frac{\Delta L}{L} + \frac{\Delta C}{C} \right) \quad (11.135)$$

for worst-case estimates when the errors shift in the same direction. For independent random uncertainties, quadrature estimates may be more appropriate.

Example 11.17: Choosing a capacitor for an audio-frequency resonator

A designer has a $L = 100 \text{ mH}$ inductor and wants a resonance frequency of $f_0 = 1.00 \text{ kHz}$. What capacitance is required?

Solution. The angular frequency is

$$\omega_0 = 2\pi f_0 = 6.283 \times 10^3 \text{ rad s}^{-1}. \quad (11.136)$$

Thus

$$C = \frac{1}{\omega_0^2 L} = \frac{1}{(6.283 \times 10^3)^2 (0.100)} = 2.53 \times 10^{-7} \text{ F}. \quad (11.137)$$

The required capacitance is approximately $0.253 \mu\text{F}$.

11.12 Transformers

An AC voltage applied to a coil establishes a changing magnetic flux. A second coil linked by that flux acquires an induced emf even though there is no conducting connection between the windings.

In the ideal transformer model, all turns link the same flux $\Phi(t)$, winding resistance and leakage flux are neglected, and the magnetizing current needed to establish the core flux is negligible. The core remains in its linear operating range.

Faraday's law gives emf magnitudes proportional to the number of turns. With corresponding terminal polarities on the two windings,

$$v_p(t) = N_p \frac{d\Phi}{dt}, \quad v_s(t) = N_s \frac{d\Phi}{dt}. \quad (11.138)$$

The positive signs here describe terminal voltage drops opposing the induced emfs. Reversing one winding's terminal reference reverses its signed voltage; it does not change the magnitude ratio. Thus, for peak or rms magnitudes used consistently,

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}. \quad (11.139)$$

Figure 11.9 shows the two electrically separate circuits and their common magnetic path.

For a resistive secondary load, lossless power balance gives $V_p I_p = V_s I_s$, where all four quantities are rms values. Hence

$$\frac{I_s}{I_p} = \frac{N_p}{N_s}. \quad (11.140)$$

These are current magnitudes; the primary current enters the transformer and the secondary current leaves it toward the load. The current ratio also holds for a reactive load in the ideal model, because the load current is referred to the primary by the inverse turns ratio. The primary then has the same load phase angle. In that case VI is *apparent* power, and the conserved average real power is

$$P_p = V_p I_p \cos \phi_p = V_s I_s \cos \phi_s = P_s, \quad \phi_p = \phi_s. \quad (11.141)$$

A transformer does not correct an arbitrary load's power factor merely by changing voltage. In a real transformer, magnetizing current and losses alter the input phase and current; efficiency alone is insufficient to determine primary rms current unless the input power factor is also known.

When the secondary is loaded, its current produces a magnetic effect opposing the applied flux change. The primary draws additional current that approximately balances this effect, so the flux is still mainly set by the applied voltage and frequency. This explains how changing the secondary load changes the primary current. In the ideal open-secondary limit the output current and the load component of primary current both vanish, while a finite secondary voltage may remain.

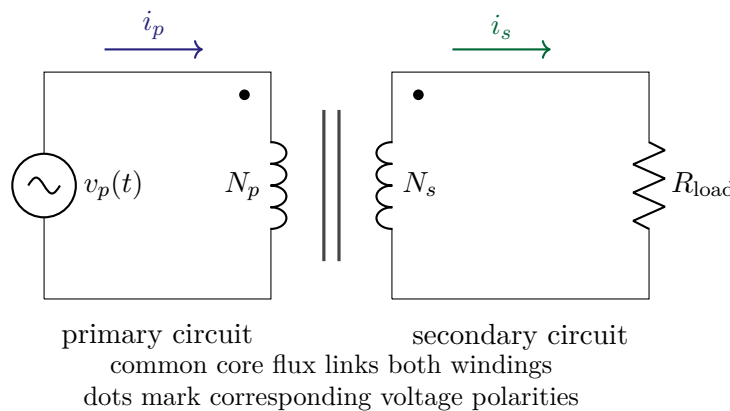


Figure 11.9: Ideal transformer with a resistive load. The two windings are magnetically coupled and electrically isolated; the parallel lines denote the core. Current enters the dotted primary terminal and leaves the dotted secondary terminal toward the load. The winding symbols are schematic and do not encode a numerical turns ratio.

Caution

An ordinary transformer is intended for a changing applied voltage with suitable volt-seconds. A steady *current* produces steady flux and no sustained secondary emf. A steady nonzero *voltage*, however, cannot produce a bounded steady state in the ideal lossless winding: $v_p = N_p d\Phi/dt$ makes the flux ramp. In a real core this leads toward saturation and excessive winding current. It is therefore misleading to describe a DC-voltage connection as simply an inactive, harmless transformer.

Example 11.18: Step-down transformer for a low-voltage device

An ideal transformer is used to step down from 230 V rms to 12 V rms. The primary has $N_p = 1150$ turns. Find the required number of secondary turns. If the secondary supplies 2.0 A rms, find the primary current.

Solution. The turns ratio must satisfy

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}, \quad (11.142)$$

so

$$N_s = N_p \frac{V_s}{V_p} = 1150 \frac{12}{230} = 60. \quad (11.143)$$

The ideal power balance gives

$$V_p I_p = V_s I_s, \quad (11.144)$$

so

$$I_p = \frac{V_s I_s}{V_p} = \frac{(12)(2.0)}{230} = 0.104 \text{ A}. \quad (11.145)$$

The low-voltage side has larger current, while the high-voltage side has smaller current.

11.12.1 Transformers and transmission-line loss estimates

For a given transmitted power P , the line current is approximately $I = P/V$ if the power factor is near unity. The resistive line loss is $P_{\text{loss}} = I^2 R_{\text{line}}$. Increasing the transmission voltage reduces the current and therefore reduces the line loss quadratically. This is the basic reason transformers are central to AC power distribution.

Example 11.19: Why high voltage reduces line loss

A single-phase load receives $P = 20 \text{ kW}$ at unity power factor through a line with total outgoing-and-return resistance $R_{\text{line}} = 2.0 \Omega$. Compare losses when the receiving-end voltage is 500 V rms and 5000 V rms. Assume that the sending voltage is adjusted to maintain each receiving voltage.

Solution. At 500 V,

$$I_1 = \frac{P}{V} = \frac{20 \times 10^3}{500} = 40 \text{ A}, \quad (11.146)$$

so

$$P_{\text{loss},1} = I_1^2 R_{\text{line}} = (40)^2 (2.0) = 3.2 \text{ kW}. \quad (11.147)$$

At 5000 V,

$$I_2 = \frac{20 \times 10^3}{5000} = 4.0 \text{ A}, \quad (11.148)$$

so

$$P_{\text{loss},2} = (4.0)^2 (2.0) = 32 \text{ W}. \quad (11.149)$$

Raising voltage by a factor of ten reduces current by a factor of ten and resistive line loss by a factor of one hundred in this simplified estimate.

11.12.2 Load resistance referred to the primary

An ideal transformer can make a load resistance appear different to the source. If a load R_L is connected to the secondary, then the primary sees an effective resistance

$$R_{\text{in}} = \left(\frac{N_p}{N_s} \right)^2 R_L. \quad (11.150)$$

This result follows from the voltage and current ratios:

$$R_{\text{in}} = \frac{V_p}{I_p} = \frac{(N_p/N_s)V_s}{(N_s/N_p)I_s} = \left(\frac{N_p}{N_s} \right)^2 \frac{V_s}{I_s}. \quad (11.151)$$

The idea of reflected impedance is important in power transfer and impedance matching.

Example 11.20: Reflected resistance

An ideal transformer has $N_p/N_s = 5$. A $4.0\ \Omega$ load is connected to the secondary. What resistance is seen at the primary?

Solution.

$$R_{\text{in}} = \left(\frac{N_p}{N_s} \right)^2 R_L = 5^2(4.0) = 100\ \Omega. \quad (11.152)$$

The transformer does not change the physical resistor, but it changes the voltage-current ratio seen by the source.

11.12.3 Why voltage and frequency must be considered together

For sinusoidal core flux $\Phi(t) = \Phi_{\text{max}} \sin \omega t$, Faraday's law gives $V_{p0} = \omega N_p \Phi_{\text{max}}$. If the effective core area is A_c and flux is approximately uniform,

$$V_{p,\text{rms}} = \frac{2\pi}{\sqrt{2}} f N_p A_c B_{\text{max}} \simeq 4.44 f N_p A_c B_{\text{max}}. \quad (11.153)$$

The factor 4.44 is specific to sinusoidal flux. Its units check directly: $\text{Hz m}^2 \text{T} = \text{Wb/s} = \text{V}$. At fixed applied voltage, reducing frequency increases the required flux density. The turns ratio alone therefore cannot tell us whether a transformer will operate within its core limits.

Example 11.21: A transformer flux-density check

A hypothetical transformer primary has $N_p = 600$ turns and effective core area $A_c = 4.0\ \text{cm}^2$. It is driven at 24 V rms and 50 Hz. Estimate the peak flux density, and determine what happens if the frequency is reduced to 25 Hz at the same voltage.

Strategy. Convert the area to $4.0 \times 10^{-4}\ \text{m}^2$ and use the sinusoidal flux model, neglecting winding voltage drop.

$$B_{\text{max}} = \frac{24}{4.44(50)(600)(4.0 \times 10^{-4})} = 0.450\ \text{T}. \quad (11.154)$$

At half the frequency this becomes 0.900 T. Whether that is acceptable depends on the actual core material and specified operating limit; it cannot be inferred from the turns ratio. Maintaining the original B_{max} at 25 Hz requires reducing the voltage to 12 V rms.

11.13 Measurement, calibration and engineering inference

A resonance measurement tests both the circuit model and the measurement arrangement. The useful observables are not only the location of a peak, but also its height, width and phase. Taken together they can identify inductance, resistance and departures from a simple lumped model.

Use a low-voltage sinusoidal generator and a series circuit with a known, approximately noninductive monitor resistor R_m . The inductor contributes winding resistance r_L ; the generator has source resistance R_s . The distinction between source emf and measured terminal voltage is essential. Figure 11.10 defines the relevant nodes.

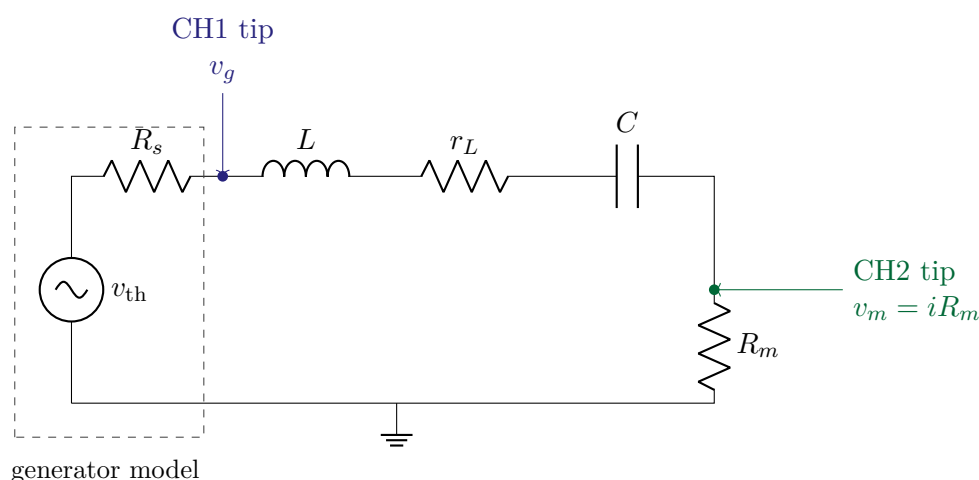


Figure 11.10: A series RLC measurement with a low-side current monitor. Both oscilloscope channel returns connect to the common bottom node. CH1 measures the loaded generator terminal voltage, not the ideal internal emf v_{th} . The monitor is part of the circuit and adds damping.

For ordinary oscilloscope channels with a common return, connecting their return clips to different circuit nodes would join those nodes electrically. The low-side monitor makes both measured voltages refer to the same bottom node. A floating element voltage instead requires an appropriate differential measurement. Use the low-voltage circuit shown for the activity; the power-system examples elsewhere are calculations, not bench wiring instructions.

There are two valid response models, depending on the measured input:

$$I_{\text{rms}} = \frac{V_{\text{th,rms}}}{\sqrt{(R_s + r_L + R_m)^2 + (X_L - X_C)^2}}, \quad (11.155)$$

$$\frac{V_{m,\text{rms}}}{V_{g,\text{rms}}} = \frac{R_m}{\sqrt{(r_L + R_m)^2 + (X_L - X_C)^2}}. \quad (11.156)$$

The first describes current for fixed internal source amplitude. The second is a measured voltage ratio across the external load and does not contain R_s . Dividing the two measured amplitudes corrects for changes in generator terminal voltage. Mixing the numerator from one model with the resistance of the other would give an incorrect Q .

Method

Estimate f_0 and a plausible bandwidth first. Sweep broadly to locate the peak, then sample finely enough that several points lie across each side of the bandwidth. Record V_g , V_m and the signed delay of v_m relative to v_g at every frequency. Use V_m/R_m for current, and use V_m/V_g when fitting the external-load response. Half power means an amplitude ratio of $1/\sqrt{2}$ relative to the peak of the same measured response. Repeat at a lower drive amplitude: a changing peak location or shape can indicate heating or nonlinear inductance.

11.13.1 Source and winding resistance

A real signal generator can often be modelled as an ideal sinusoidal voltage source in series with an internal resistance R_s . If the load is a series RLC circuit with resistance R , the total series resistance in the amplitude formula becomes

$$R_{\text{tot}} = R + R_s + R_{\text{coil}} + \cdots \quad (11.157)$$

At resonance the current is therefore

$$I_0 = \frac{V_0}{R_{\text{tot}}}, \quad (11.158)$$

not V_0/R if R denotes only the external resistor. Ignoring source resistance leads to an overestimate of resonance current and quality factor.

Example 11.22: Effect of source resistance on peak current

A resonant circuit has an external resistor $R = 10 \Omega$ and is driven by a generator with internal resistance $R_s = 50 \Omega$. The generator open-circuit peak voltage is 6.0 V. Estimate the peak current at resonance.

Solution. The total series resistance seen by the ideal source is approximately

$$R_{\text{tot}} = 10 + 50 = 60 \Omega. \quad (11.159)$$

At resonance,

$$I_0 = \frac{6.0}{60} = 0.10 \text{ A}. \quad (11.160)$$

If the internal resistance had been ignored, one would predict 0.60 A, six times too large. This is a common laboratory discrepancy.

11.13.2 Oscilloscope phase measurements

In a laboratory, phase can be measured by comparing two sinusoidal traces on an oscilloscope. If the period is T and the time shift between corresponding peaks is Δt , then the phase difference is

$$\Delta\phi = 2\pi \frac{\Delta t}{T} \quad (11.161)$$

in radians, or

$$\Delta\phi = 360^\circ \frac{\Delta t}{T} \quad (11.162)$$

in degrees. The sign depends on which trace reaches its peak first.

Measuring current often requires an indirect method. In a series RLC circuit, the voltage across a known resistor is proportional to the current. The resistor voltage can therefore be used as a current monitor, provided the resistor is sufficiently noninductive over the frequency range of interest.

Example 11.23: Phase from oscilloscope traces

Two sinusoidal traces have period $T = 0.80 \text{ ms}$. The current-monitor trace reaches its maximum 0.10 ms after the source-voltage trace. Find the phase of current relative to voltage.

Solution. The magnitude of the phase shift is

$$\Delta\phi = 360^\circ \frac{0.10}{0.80} = 45^\circ. \quad (11.163)$$

Since the current-monitor trace peaks later, the current lags the source voltage by 45° . In the convention of this chapter, $\phi = +45^\circ$.

11.13.3 Uncertainty in a phase measurement

If phase is determined from a time shift,

$$\phi = 360^\circ \frac{\Delta t}{T}, \quad (11.164)$$

then uncertainty in Δt contributes directly to phase uncertainty:

$$\Delta\phi \simeq 360^\circ \frac{\Delta(\Delta t)}{T} \quad (11.165)$$

if the period is known much more accurately than the time shift. At high frequency the period is small, so the same absolute time uncertainty produces a larger phase uncertainty.

Example 11.24: Phase uncertainty estimate

An oscilloscope measurement at $f = 10$ kHz has time-shift uncertainty $\Delta(\Delta t) = 1.0 \mu\text{s}$. Estimate the phase uncertainty.

Solution. The period is

$$T = \frac{1}{f} = 100 \mu\text{s}. \quad (11.166)$$

Thus

$$\Delta\phi = 360^\circ \frac{1.0}{100} = 3.6^\circ. \quad (11.167)$$

This is acceptable for many introductory measurements but not for precision impedance metrology.

11.13.4 Inferring an inductance from resonance

A common laboratory method for estimating an unknown inductance is to connect it in series or parallel with a known capacitor and locate a resonance frequency. In the ideal series model,

$$f_0 = \frac{1}{2\pi\sqrt{LC}}. \quad (11.168)$$

If C is known and f_0 is measured, the inductance estimate is

$$L = \frac{1}{(2\pi f_0)^2 C}. \quad (11.169)$$

This method is attractive because frequency can often be measured accurately. However, the inferred inductance depends on the effective capacitance in the circuit, including stray capacitance from leads, probes and the coil itself.

Example 11.25: Inferring an unknown inductance

A coil is connected in a series resonant circuit with a capacitor marked $C = 100$ nF. The measured resonance frequency is $f_0 = 3.20$ kHz. Estimate the inductance.

Solution. The angular frequency is

$$\omega_0 = 2\pi f_0 = 2.01 \times 10^4 \text{ rad s}^{-1}. \quad (11.170)$$

Thus

$$L = \frac{1}{\omega_0^2 C} = \frac{1}{(2.01 \times 10^4)^2 (100 \times 10^{-9})} = 2.48 \times 10^{-2} \text{ H}. \quad (11.171)$$

The ideal estimate is $L \simeq 24.8 \text{ mH}$. If the capacitor tolerance is 10%, the inductance estimate cannot be more accurate than the capacitance value unless the actual capacitance is measured.

Guided checks

1. What quantity is directly measured in this method? The resonance frequency.
2. What known component is needed? A capacitor of known effective capacitance.
3. What nonideality can bias the result? Stray capacitance, coil resistance, core losses or source/probe loading.
4. Does a sharp peak make the capacitor tolerance disappear? No. A sharp peak helps locate frequency, but component uncertainty remains.

11.13.5 Inferring resistance from resonance width

The width of a resonance curve contains information about damping. For an ideal series RLC circuit,

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L}. \quad (11.172)$$

Thus, if L is known and the half-power frequencies are measured, the effective series resistance can be estimated:

$$R = L\Delta\omega = 2\pi L\Delta f. \quad (11.173)$$

This resistance includes the external resistor, source resistance, inductor winding resistance and any other effective series loss that contributes over the frequency range.

Example 11.26: Estimating effective resistance from bandwidth

A series resonant circuit with $L = 15 \text{ mH}$ has half-power frequencies $f_1 = 4.65 \text{ kHz}$ and $f_2 = 5.05 \text{ kHz}$. Estimate the effective series resistance.

Solution. The bandwidth is

$$\Delta f = f_2 - f_1 = 0.40 \text{ kHz} = 400 \text{ Hz}. \quad (11.174)$$

Therefore

$$R = 2\pi L\Delta f = 2\pi(15 \times 10^{-3})(400) = 37.7 \Omega. \quad (11.175)$$

This is an effective resistance. If an external resistor of 30Ω was installed, the remaining about 8Ω could come from winding resistance and other losses.

11.13.6 Uncertainty and tolerance in resonance measurements

Suppose the resonance frequency is computed from measured inductance and capacitance:

$$f_0 = \frac{1}{2\pi\sqrt{LC}}. \quad (11.176)$$

For small relative uncertainties, logarithmic differentiation gives

$$\frac{\delta f_0}{f_0} = -\frac{1}{2} \frac{\delta L}{L} - \frac{1}{2} \frac{\delta C}{C}. \quad (11.177)$$

The signs tell the direction of change: increasing L or C decreases the resonance frequency. For uncertainty magnitudes, one often writes a conservative estimate as

$$\left| \frac{\Delta f_0}{f_0} \right| \lesssim \frac{1}{2} \left| \frac{\Delta L}{L} \right| + \frac{1}{2} \left| \frac{\Delta C}{C} \right|. \quad (11.178)$$

This is useful in the laboratory because component tolerances may be several percent.

Example 11.27: Tolerance estimate for resonance frequency

An inductor has tolerance 5% and a capacitor has tolerance 10%. Estimate the worst-case fractional uncertainty in f_0 .

Solution. Using the conservative estimate,

$$\frac{\Delta f_0}{f_0} \lesssim \frac{1}{2}(0.05) + \frac{1}{2}(0.10) = 0.075. \quad (11.179)$$

The resonance frequency may differ by about 7.5% from the nominal value in the worst-case estimate. This is large enough that a frequency sweep should cover a wider interval than the nominal bandwidth alone.

11.13.7 A reproducible frequency-sweep analysis

A model-based calculation can be checked with a spreadsheet or a short program. Use $\omega = 2\pi f$ and calculate the signed net reactance before taking its magnitude. The following activity models the external load with $R_m = 20\ \Omega$, $r_L = 10\ \Omega$, $L = 10\ \text{mH}$ and $C = 1.00\ \mu\text{F}$. At each frequency the measured terminal voltage is adjusted to 1.00 V rms; these are calculated values, not experimental data.

f (Hz)	I_{rms} (mA)	$V_{m,\text{rms}}$ (V)	ϕ (degrees)
1000	9.91	0.198	-72.7
1300	19.8	0.395	-53.6
1591.55	33.3	0.667	0.0
1900	21.5	0.429	+49.9
2400	11.2	0.223	+70.4

The phase signs test the physics more strongly than amplitudes alone: below resonance the current leads and above resonance it lags. From the peak ratio $H_{\text{max}} = V_m/V_g = 2/3$, the external total resistance is $R_m/H_{\text{max}} = 30\ \Omega$, revealing $10\ \Omega$ beyond the monitor. The model predicts

$$f_1 = 1370.6\ \text{Hz}, \quad f_2 = 1848.1\ \text{Hz}, \quad Q = 3.33, \quad \Delta f = 477.5\ \text{Hz}. \quad (11.180)$$

A fit to the full amplitude and phase curve is preferable to reading a noisy maximum alone. If the fitted resistance changes systematically with frequency, a constant-loss model may be inadequate. It is also possible for a nominal capacitor error to shift the peak while leaving the width nearly unchanged.

```
# Pseudocode; SI units, positive frequencies only
Rload = Rmonitor + Rcoil
for f in frequencies:
    w = 2*pi*f
    X = w*L - 1/(w*C)
    Z = sqrt(Rload**2 + X**2)
    Irms = Vterminal_rms/Z
    monitor_ratio = Rmonitor/Z
    phase = atan2(X, Rload)
```

For independent standard uncertainties u_L and u_C , linear propagation gives

$$\frac{u_f}{f_0} = \frac{1}{2} \sqrt{\left(\frac{u_L}{L}\right)^2 + \left(\frac{u_C}{C}\right)^2}. \quad (11.181)$$

A tolerance interval is not automatically a standard uncertainty; a statistical model for the component distribution is needed to convert between them. The earlier 7.5% estimate for 5% and 10% tolerances is a first-order bound. Exact extrema are $f_{\max}/f_{\text{nom}} = 1/\sqrt{0.95 \cdot 0.90} = 1.0815$ and $f_{\min}/f_{\text{nom}} = 1/\sqrt{1.05 \cdot 1.10} = 0.9305$. The asymmetry matters when choosing a sweep range near a tolerance limit.

11.14 Advanced Topics for Technical and Engineering Students

The core chapter can be studied without complex numbers. The following enrichment extends the same physical models using complex amplitudes, transfer functions and time integration. It is intended for students who want to design measurements or connect the course to electronics and scientific computing.

We retain Z for a non-negative impedance magnitude and use $\mathcal{Z}$ for a complex impedance, so $Z = |\mathcal{Z}|$. The complex amplitudes introduced below are *peak* phasors unless rms phasors are explicitly identified. A real sinusoidal voltage is always recovered at the end; complex notation is an algebraic representation of amplitude and phase.

11.14.1 Complex notation for sinusoidal steady state

A real sinusoidal quantity can be represented as the real part of a complex exponential:

$$v(t) = \Re \left[\tilde{V} e^{i\omega t} \right], \quad (11.182)$$

where $\tilde{V}$ is a complex amplitude. If $\tilde{V} = V_0 e^{i\alpha}$, then $v(t) = V_0 \cos(\omega t + \alpha)$. Differentiation with respect to time becomes multiplication by $i\omega$:

$$\frac{d}{dt} \left(\tilde{V} e^{i\omega t} \right) = i\omega \tilde{V} e^{i\omega t}. \quad (11.183)$$

This is the central simplification. Differential equations in sinusoidal steady state become algebraic equations for complex amplitudes.

For a resistor,

$$\tilde{V}_R = R \tilde{I}. \quad (11.184)$$

For an inductor,

$$\tilde{V}_L = i\omega L \tilde{I}. \quad (11.185)$$

For a capacitor,

$$\tilde{I} = i\omega C \tilde{V}_C, \quad \tilde{V}_C = \frac{1}{i\omega C} \tilde{I}. \quad (11.186)$$

Thus the complex impedances are

$$\mathcal{Z}_R = R, \quad \mathcal{Z}_L = i\omega L, \quad \mathcal{Z}_C = \frac{1}{i\omega C} = -\frac{i}{\omega C}. \quad (11.187)$$

Theoretical point 11.3: Complex impedance is phasor geometry in algebraic form

The imaginary unit in $\mathcal{Z}_L = i\omega L$ and $\mathcal{Z}_C = 1/(i\omega C)$ does not mean the physical voltage is imaginary. It records a phase shift of 90° . The measurable voltage is the real part of the complex representation after the algebra is complete.

For a series RLC circuit the total complex impedance is

$$\mathcal{Z}_{\text{series}} = R + i \left(\omega L - \frac{1}{\omega C} \right). \quad (11.188)$$

Its magnitude is

$$|\mathcal{Z}_{\text{series}}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2}, \quad (11.189)$$

and its phase is

$$\arg \mathcal{Z}_{\text{series}} = \tan^{-1} \left(\frac{\omega L - 1/(\omega C)}{R} \right). \quad (11.190)$$

These are exactly the phasor results from the main chapter.

Example 11.28: Complex impedance calculation

A series circuit has $R = 100\ \Omega$, $L = 50\text{ mH}$ and $C = 2.0\ \mu\text{F}$ at angular frequency $\omega = 2000\text{ rad s}^{-1}$. Find the complex impedance.

Solution. The inductive and capacitive reactances are

$$X_L = \omega L = (2000)(0.050) = 100\ \Omega, \quad (11.191)$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(2000)(2.0 \times 10^{-6})} = 250\ \Omega. \quad (11.192)$$

Thus

$$\mathcal{Z} = 100 + i(100 - 250) = 100 - i150\ \Omega. \quad (11.193)$$

The impedance magnitude is $|\mathcal{Z}| = 180\ \Omega$, and the phase is $\arg \mathcal{Z} = -56.3^\circ$. The negative phase of impedance means that the circuit is net capacitive; current leads the source voltage.

11.14.2 Driven RLC differential equation and steady-state trial solution

For a driven series RLC circuit with source voltage $v_s(t) = V_0 \cos \omega t$, the charge equation is

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = V_0 \cos \omega t. \quad (11.194)$$

The complete solution is the sum of a transient solution and a particular steady-state solution. The transient depends on initial conditions and decays when $R > 0$. After sufficient time, the steady-state response remains at the driving frequency.

Using complex notation, write the source as the real part of $\tilde{V}e^{i\omega t}$ and the charge as the real part of $\tilde{Q}e^{i\omega t}$. Substitution into Eq. (11.194) gives

$$\left(-\omega^2 L + i\omega R + \frac{1}{C}\right) \tilde{Q} = \tilde{V}. \quad (11.195)$$

Since $\tilde{I} = i\omega \tilde{Q}$, this can be written as

$$\tilde{V} = \left[R + i \left(\omega L - \frac{1}{\omega C} \right) \right] \tilde{I}. \quad (11.196)$$

The bracket is the complex impedance. This derivation shows where impedance comes from: it is the algebraic form of the driven differential equation in sinusoidal steady state.

11.14.3 Which observable is resonant?

A resonant frequency must be attached to a specified response. The free oscillation frequency, the driven current peak, and the maximum capacitor voltage need not coincide. This is also the precise limit of the mechanical analogy: charge corresponds to displacement, while current corresponds to velocity. A driven mechanical displacement resonance should therefore be compared with the charge response, not automatically with the current response.

Let $u = \omega/\omega_0$ and $Q = \omega_0 L/R$. The capacitor voltage and charge amplitudes follow from $V_{C0} = I_0/(\omega C)$:

$$\frac{V_{C0}}{V_0} = \frac{1}{\sqrt{(1-u^2)^2 + u^2/Q^2}}, \quad q_{\text{amp}} = CV_{C0}. \quad (11.197)$$

Minimizing the squared denominator gives

$$\frac{d}{du} [(1-u^2)^2 + u^2/Q^2] = 2u[2(u^2-1) + Q^{-2}] = 0. \quad (11.198)$$

A nonzero-frequency maximum exists only for $Q > 1/\sqrt{2}$, at

$$\omega_C = \omega_0 \sqrt{1 - \frac{1}{2Q^2}} = \sqrt{\omega_0^2 - 2\alpha^2}. \quad (11.199)$$

For $Q \leq 1/\sqrt{2}$ the capacitor amplitude decreases from its low-frequency limit without a resonant maximum. Nevertheless the current response still has a maximum at ω_0 for every finite $R > 0$. A circuit with $1/2 < Q \leq 1/\sqrt{2}$ can even ring freely while having no capacitor-voltage peak in a frequency scan.

The inductor amplitude has the same denominator but numerator u^2 . For $Q > 1/\sqrt{2}$ its peak occurs at

$$\omega_L = \frac{\omega_0}{\sqrt{1 - 1/(2Q^2)}}. \quad (11.200)$$

Figure 11.11 places the two voltage peaks on opposite sides of the current resonance. Component-rating checks over a sweep should therefore examine the complete voltage responses, not only the current-resonance point.

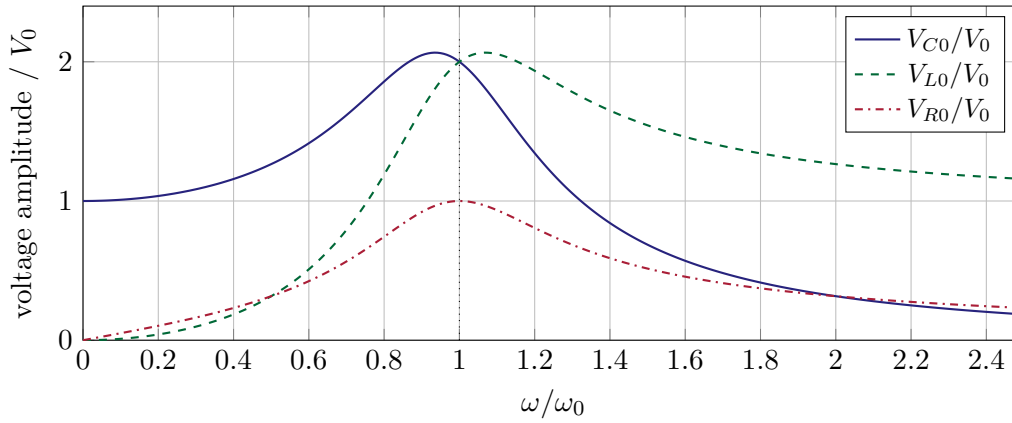


Figure 11.11: Different driven observables for the same series RLC circuit with $Q = 2$. The resistor voltage, and therefore current, peaks at $u = 1$; the capacitor and inductor voltages peak below and above it. All curves are calculated from the same impedance model.

Example 11.29: Four characteristic frequencies of one circuit

Let $L = 10 \text{ mH}$, $C = 1.0 \text{ } \mu\text{F}$ and $R = 50 \text{ } \Omega$. Distinguish the undamped frequency, free ringdown frequency and the two reactive-voltage peak frequencies.

Solution. One obtains $\omega_0 = 10^4 \text{ rad s}^{-1}$, $Q = 2$ and $\alpha = 2500 \text{ s}^{-1}$. Therefore

$$\omega_d = \omega_0 \sqrt{1 - 1/(4Q^2)} = 9682 \text{ rad s}^{-1}, \quad (11.201)$$

$$\omega_C = 9354 \text{ rad s}^{-1}, \quad \omega_L = 10690 \text{ rad s}^{-1}. \quad (11.202)$$

The current peak remains at 10000 rad s^{-1} . Thus a capacitor-voltage probe and a current monitor locate different peaks in the same damped circuit.

11.14.4 The lossless resonant limit is a transient problem

If $R = 0$ and $\omega = \omega_0$, the finite sinusoidal steady-state formula ceases to exist. With an initially uncharged capacitor and zero current, the exact resonantly driven solution is

$$q(t) = \frac{V_0}{2L\omega_0} t \sin \omega_0 t, \quad (11.203)$$

$$i(t) = \frac{V_0}{2L\omega_0} \sin \omega_0 t + \frac{V_0}{2L} t \cos \omega_0 t. \quad (11.204)$$

Direct substitution in $Lq'' + q/C = V_0 \cos \omega_0 t$ verifies the result and both initial conditions. The amplitude grows with time because the ideal source continuously adds energy without any loss. This is not a finite steady current that has somehow become infinite at an instant. For every real circuit, loss, source limits or nonlinear behaviour eventually modifies this ideal prediction.

11.14.5 Transfer functions and filters

A transfer function is the ratio of an output complex amplitude to an input complex amplitude. For example, consider a series RC circuit driven by input voltage $\tilde{V}_{\text{in}}$, with output measured across the capacitor. The complex voltage divider gives

$$H_C(\omega) = \frac{\tilde{V}_C}{\tilde{V}_{\text{in}}} = \frac{Z_C}{R + Z_C} = \frac{1}{1 + i\omega RC}. \quad (11.205)$$

The magnitude is

$$|H_C(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}. \quad (11.206)$$

This is a low-pass filter: low frequencies pass to the capacitor output, while high frequencies are attenuated.

If the output is measured across the resistor instead, the transfer function is

$$H_R(\omega) = \frac{\tilde{V}_R}{\tilde{V}_{\text{in}}} = \frac{R}{R + Z_C} = \frac{i\omega RC}{1 + i\omega RC}. \quad (11.207)$$

Its magnitude is

$$|H_R(\omega)| = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}, \quad (11.208)$$

which is high-pass behaviour.

Key message

A filter is not a device that knows what is signal and what is noise. It is a frequency-dependent linear system. It passes or attenuates components according to their frequency. Whether this is useful depends on the physical information carried by different frequency bands.

Example 11.30: RC cutoff frequency

An RC low-pass filter has $R = 1.0 \text{ k}\Omega$ and $C = 0.10 \text{ }\mu\text{F}$. Find the cutoff angular frequency and cutoff frequency.

Solution. The cutoff occurs at $\omega_c RC = 1$, so

$$\omega_c = \frac{1}{RC} = \frac{1}{(1.0 \times 10^3)(0.10 \times 10^{-6})} = 1.0 \times 10^4 \text{ rad s}^{-1}. \quad (11.209)$$

Thus

$$f_c = \frac{\omega_c}{2\pi} = 1.59 \text{ kHz.} \quad (11.210)$$

At this frequency the output amplitude across the capacitor is reduced by a factor $1/\sqrt{2}$ relative to the low-frequency limit.

11.14.6 Series RLC as a band-pass response

Figure 11.12 shows how the quality factor changes the normalized transmission bandwidth.

If the output of a series RLC circuit is taken across the resistor, then $\tilde{V}_R = \tilde{I}R$ and

$$H_R(\omega) = \frac{\tilde{V}_R}{\tilde{V}_{\text{in}}} = \frac{R}{R + i\left(\omega L - \frac{1}{\omega C}\right)}. \quad (11.211)$$

The magnitude is

$$|H_R(\omega)| = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}. \quad (11.212)$$

This function is largest at resonance, where $\omega = \omega_0$. At much lower frequencies the capacitor reactance is large, current is small and the resistor output is small. At much higher frequencies the inductor reactance is large, current is again small. The circuit therefore acts as a band-pass filter around resonance.

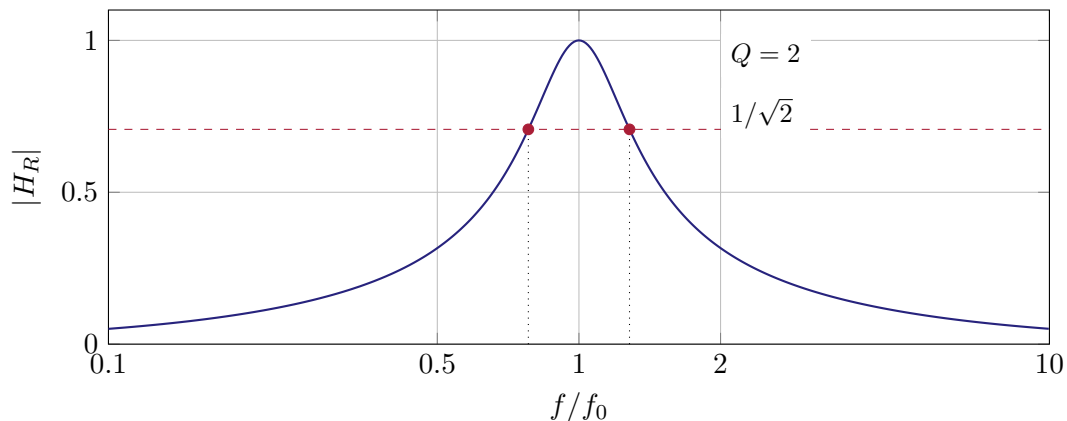


Figure 11.12: Calculated series band-pass amplitude $|H_R| = [1 + Q^2(u - 1/u)^2]^{-1/2}$ for $Q = 2$. The half-power points are $u_1 = 0.7808$ and $u_2 = 1.2808$. They are symmetric on this logarithmic axis because $u_1 u_2 = 1$, although their arithmetic midpoint is not 1.

11.14.7 Filter topology, loading and a design constraint

A component is not intrinsically a low-pass or high-pass filter. The source impedance, circuit connections and choice of output determine the transfer function. Figure 11.13 makes the output terminals explicit. The ideal divider formulas assume an ideal voltage source and a measuring load with effectively infinite input impedance.

For the low-pass circuit, the phase is $\arg H_C = -\tan^{-1}(\omega RC)$. The gain is near one and the phase near zero for $\omega RC \ll 1$; at cutoff the gain is $1/\sqrt{2}$ and the phase is -45° ; far above cutoff the gain falls as $1/(\omega RC)$. When plotted as $20 \log_{10} |H|$ in decibels, this last behaviour has slope -20 dB per decade. The cutoff is a reference point on a smooth curve, not a boundary at which a frequency suddenly disappears.

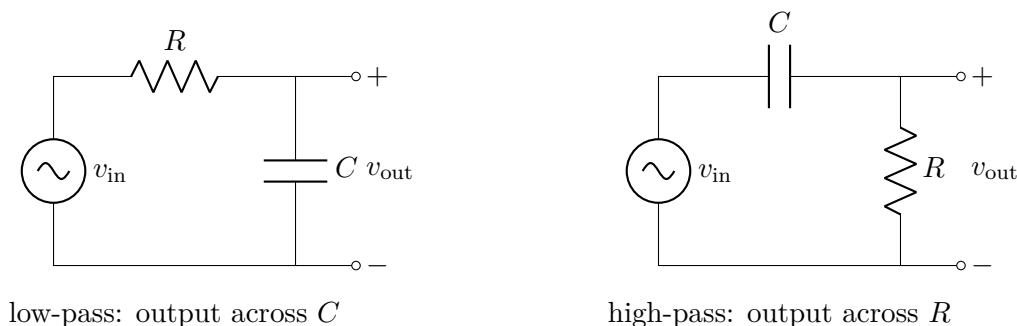


Figure 11.13: Two RC filters using the same components but different placement relative to the common return. The output is defined between the indicated terminals. A finite load across those terminals changes the voltage divider.

If a finite resistance R_{load} loads the capacitor output, and a source resistance R_s is in series with the filter resistor, set $R_a = R + R_s$. Parallel combination of the load and capacitor gives

$$H(\omega) = \frac{R_{\text{load}}}{R_a + R_{\text{load}}} \frac{1}{1 + i\omega C(R_a \parallel R_{\text{load}})}, \quad R_a \parallel R_{\text{load}} = \frac{R_a R_{\text{load}}}{R_a + R_{\text{load}}}. \quad (11.213)$$

Loading reduces the low-frequency gain and changes the pole frequency. A formula derived for an unloaded divider should not be applied unchanged to a low-impedance sensor input.

Example 11.31: Preserving a signal while suppressing a noise band

An unloaded RC low-pass filter must have amplitude gain at least 0.99 at 20 Hz and at most 0.10 at 2.0 kHz. Find the allowable cutoff interval. Then choose C for $R = 10 \text{ k}\Omega$ and $f_c = 200 \text{ Hz}$. **Strategy.** Translate each requirement into an inequality using $|H(f)| = [1 + (f/f_c)^2]^{-1/2}$. The signal requirement places a *lower* bound on f_c , while the noise requirement places an *upper* bound:

$$f_c \geq \frac{20}{\sqrt{0.99^2 - 1}} = 140 \text{ Hz}, \quad (11.214)$$

$$f_c \leq \frac{2000}{\sqrt{0.10^2 - 1}} = 201 \text{ Hz}. \quad (11.215)$$

The interval is nonempty, so a first-order filter can meet these ideal amplitude specifications. For $f_c = 200 \text{ Hz}$,

$$C = \frac{1}{2\pi R f_c} = 79.6 \text{ nF}. \quad (11.216)$$

The gains are 0.995 and 0.0995. At 20 Hz the phase is still $-\tan^{-1}(0.1) = -5.71^\circ$, so amplitude fidelity does not imply zero timing distortion. The small margin at the noise requirement also means that actual component tolerances must be included before accepting a design.

Loading check. If a $10 \text{ k}\Omega$ load is connected to the output and $R_s = 0$, the low-frequency gain becomes $1/2$ and the cutoff relative to that gain becomes 400 Hz. The original 0.99 signal-gain specification then fails. A higher input-impedance load or a buffer is needed.

The name “half-power cutoff” for a voltage transfer function refers to the squared gain reaching one half when comparing powers in the same reference resistance. An ideal capacitor itself consumes zero average power at all frequencies; its voltage falling to $1/\sqrt{2}$ does not mean it dissipates half the input power.

11.14.8 Power-factor correction

Many practical loads, especially motors and transformers, are inductive. The current lags the voltage, and the power factor can be significantly below one. For fixed real power and rms voltage, a lower power factor requires a larger rms current:

$$I_{\text{rms}} = \frac{P_{\text{avg}}}{V_{\text{rms}} \cos \phi}. \quad (11.217)$$

Larger current increases transmission losses and requires larger conductors and switchgear. Power-factor correction uses capacitors or controlled power electronics to reduce the phase difference between voltage and current seen by the supply.

At an introductory level, one can understand the correction as adding a capacitive current component that partially cancels the inductive reactive current component. The load still consumes the required real power, but the supply current is reduced.

Example 11.32: Current reduction by power-factor correction

A single-phase load consumes $P = 5.0 \text{ kW}$ from a 230 V rms supply. Initially its power factor is 0.60 . After correction its power factor is 0.95 . Compare the supply currents.

Solution. Initially,

$$I_1 = \frac{P}{V_{\text{rms}} \cos \phi_1} = \frac{5000}{(230)(0.60)} = 36.2 \text{ A}. \quad (11.218)$$

After correction,

$$I_2 = \frac{5000}{(230)(0.95)} = 22.9 \text{ A}. \quad (11.219)$$

The real load power and supply voltage are held fixed; only the reactive current drawn from the supply is reduced. If line resistance is fixed, resistive line losses are reduced by the factor $(I_2/I_1)^2 \simeq 0.40$.

11.14.9 Sizing a shunt correction capacitor

Let P be fixed real load power and let $\phi_1 > 0$ describe a lagging load. Resolve its rms current into an in-phase component $I_{\parallel} = P/V$ and a lagging quadrature component $I_{\perp} = (P/V) \tan \phi_1$. A capacitor connected *in parallel* with the load contributes a leading current $I_C = \omega CV$. The corrected source current therefore has

$$I_{\perp,2} = \frac{P}{V} \tan \phi_1 - \omega CV. \quad (11.220)$$

To obtain a chosen remaining lag ϕ_2 , set $I_{\perp,2} = (P/V) \tan \phi_2$, giving

$$C = \frac{P(\tan \phi_1 - \tan \phi_2)}{\omega V^2}. \quad (11.221)$$

Here V is the rms voltage across both branches. The reactive power $\mathcal{Q} = P \tan \phi$ is measured in var; the calligraphic symbol distinguishes it from resonator quality factor Q and capacitor-charge amplitude Q_0 . The capacitor supplies reactive power $-\omega CV^2$, but ideally no real average power.

For the 5.0 kW example above, take a hypothetical sinusoidal 50 Hz supply. Then $\tan \phi_1 = 1.333$ and $\tan \phi_2 = 0.3287$, so the ideal shunt capacitance is

$$C = \frac{5000(1.333 - 0.3287)}{2\pi(50)(230)^2} = 302 \mu\text{F}. \quad (11.222)$$

The source-current reduction follows the phasor construction in Fig. 11.14. This is a sinusoidal linear-load calculation. With harmonic distortion, the true power factor is $P/(V_{\text{rms}} I_{\text{rms}})$ and need not equal a single $\cos \phi$; a capacitor correcting the fundamental phase alone need not correct the full waveform.

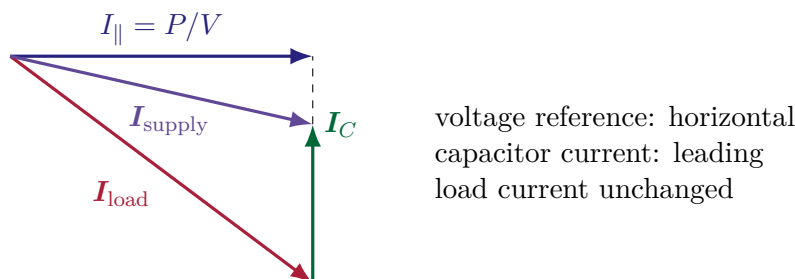


Figure 11.14: Current addition for a shunt correction capacitor. It reduces the lagging quadrature current supplied by the source while the load itself still draws the same current at the fixed voltage. Current vectors, not their magnitudes, must be added.

11.14.10 Nonideal components

Ideal circuit elements are models. A real inductor has winding resistance, parasitic capacitance and often a magnetic core with losses. At high frequency, the winding resistance can increase due to skin effect. A real capacitor has equivalent series resistance, leakage resistance, dielectric absorption and voltage limits. A real resistor has parasitic inductance and capacitance. A real source has internal impedance and maximum current or power ratings.

The ideal model is still valuable because it identifies the dominant behaviour and provides a baseline. The engineering question is not whether ideal formulas are exactly true; they are not. The engineering question is whether nonideal corrections are small enough for the intended purpose, or whether they must be included in the design model.

Caution

Component labels are not complete physical descriptions. A device sold as an inductor may behave like an inductor only over a certain frequency range and current range. Outside that range parasitic capacitance, core saturation, heating or breakdown can dominate.

11.14.11 Probe loading

A measuring instrument is part of the circuit. An oscilloscope probe has an input resistance and capacitance. At low frequencies the large input resistance may make loading negligible. At high frequencies the input capacitance can create a reactance

$$X_{C,\text{probe}} = \frac{1}{\omega C_{\text{probe}}}. \quad (11.223)$$

If this reactance is comparable to circuit impedances, the measurement changes the circuit. This is especially important near resonant nodes with high impedance.

Example 11.33: Probe capacitance estimate

An oscilloscope probe has input capacitance $C_p = 15 \text{ pF}$. Estimate its capacitive reactance at 1.0 MHz .

Solution. The angular frequency is $\omega = 2\pi \times 10^6 \text{ rad s}^{-1}$. Thus

$$X_{C,p} = \frac{1}{\omega C_p} = \frac{1}{(2\pi \times 10^6)(15 \times 10^{-12})} = 1.06 \times 10^4 \Omega. \quad (11.224)$$

A $10.6 \text{ k}\Omega$ reactance may be negligible in a low-impedance circuit but significant in a high-impedance resonant measurement.

11.14.12 Equivalent series resistance and resonator loss

A real capacitor can dissipate energy in its dielectric. One simple model is an ideal capacitor with a small equivalent series resistance, often called ESR. In a resonant circuit this ESR adds to the total series resistance and lowers Q :

$$Q = \frac{\omega_0 L}{R_{\text{total}}}. \quad (11.225)$$

Even a small ESR can matter when the intended external resistance is very small.

Example 11.34: Effect of capacitor ESR on quality factor

A resonant circuit has $L = 5.0 \text{ mH}$, $C = 0.20 \mu\text{F}$ and external resistance $R = 1.0 \Omega$. The capacitor ESR at the operating frequency is 0.80Ω , and the inductor winding resistance is 1.2Ω . Estimate Q .

Solution. The resonance angular frequency is

$$\omega_0 = \frac{1}{\sqrt{(5.0 \times 10^{-3})(0.20 \times 10^{-6})}} = 3.16 \times 10^4 \text{ rad s}^{-1}. \quad (11.226)$$

The total series resistance estimate is

$$R_{\text{total}} = 1.0 + 0.80 + 1.2 = 3.0 \Omega. \quad (11.227)$$

Therefore

$$Q = \frac{\omega_0 L}{R_{\text{total}}} = \frac{(3.16 \times 10^4)(5.0 \times 10^{-3})}{3.0} = 52.7. \quad (11.228)$$

Ignoring ESR and winding resistance would give a much larger and unrealistic Q .

11.14.13 Parallel resonance and the choice of source

Series elements share a current and their impedances add. Parallel elements share a voltage and their *admittances* add. Admittance is the inverse complex impedance, $\mathcal{Y} = 1/\mathcal{Z}$, measured in siemens. For an ideal parallel R_p , L and C network,

$$\mathcal{Y} = \frac{1}{R_p} + i \left(\omega C - \frac{1}{\omega L} \right). \quad (11.229)$$

At $\omega_0 = 1/\sqrt{LC}$ the inductive and capacitive *branch currents* cancel at the input. The input impedance is then maximum, $|\mathcal{Z}| = R_p$, unlike a series resonator's minimum R .

With a fixed voltage source, the source-current amplitude is minimized at parallel resonance; the internal reactive currents can still be large. With a fixed current source, the terminal-voltage amplitude is maximized. “Resonance gives a large current” is therefore incomplete without a topology, source and measured quantity.

This parallel model has free voltage equation

$$Cv'' + \frac{1}{R_p}v' + \frac{v}{L} = 0, \quad (11.230)$$

so its decay coefficient is $\alpha_p = 1/(2R_p C)$ and its quality factor is $Q_p = \omega_0 R_p C$. Increasing a parallel loss resistance reduces damping, whereas increasing series resistance increases damping. A coil with winding resistance is not an ideal inductor in parallel with R_p ; its effective parallel model must be derived or fitted over the frequency range of interest.

11.14.14 Maximum power transfer is not maximum efficiency

Represent a source at one frequency by an ideal rms phasor $\tilde{V}_{\text{th,rms}}$ and series impedance $\mathcal{Z}_s = R_s + iX_s$ with $R_s > 0$. Let the adjustable load be $\mathcal{Z}_\ell = R_\ell + iX_\ell$. The average load power is

$$P_\ell = \frac{|\tilde{V}_{\text{th,rms}}|^2 R_\ell}{(R_s + R_\ell)^2 + (X_s + X_\ell)^2}. \quad (11.231)$$

First maximize with respect to the load reactance: $X_\ell = -X_s$. Then differentiate $R_\ell/(R_s + R_\ell)^2$ with respect to R_ℓ . Its derivative has the sign of $R_s - R_\ell$, so the maximum occurs at $R_\ell = R_s$. Thus

$$\mathcal{Z}_\ell = \mathcal{Z}_s^*, \quad P_{\ell,\text{max}} = \frac{|\tilde{V}_{\text{th,rms}}|^2}{4R_s}. \quad (11.232)$$

This is *conjugate matching*, subject to independent adjustment of both load components. At that operating point the same current dissipates equal powers in the source resistance and load resistance: their combined resistive efficiency is 50%. Power distribution usually seeks small source loss, rather than maximum power extraction from a fixed internal resistance.

Example 11.35: Matching a reactive source at one frequency

A source has open-circuit rms voltage 10 V and internal impedance $(40 + i30) \Omega$. What adjustable load maximizes its average power, and what power is delivered?

Solution. Choose $\mathcal{Z}_\ell = (40 - i30) \Omega$. The reactances cancel, the total resistance is 80Ω , and $I_{\text{rms}} = 0.125 \text{ A}$. Hence

$$P_\ell = (0.125)^2 (40) = 0.625 \text{ W}, \quad (11.233)$$

consistent with $10^2/(4 \cdot 40)$. An equal 0.625 W is dissipated in the source resistance. This narrow-frequency match need not remain a match when the driving frequency changes.

11.14.15 Ringdown, bandwidth and exact logarithmic decrement

An underdamped circuit carries information about loss even after its source is removed. Successive peaks of the same polarity are separated by the damped period $T_d = 2\pi/\omega_d$. Because the charge solution is an exponential times a periodic function, peak amplitudes satisfy exactly

$$\frac{A_{n+N}}{A_n} = e^{-\alpha N T_d}, \quad \delta = \frac{1}{N} \ln \frac{|A_n|}{|A_{n+N}|} = \alpha T_d. \quad (11.234)$$

The use of peaks of the *same polarity* avoids a factor-of-two ambiguity. Adjacent absolute extrema are separated by $T_d/2$, and must be counted accordingly.

Since $\omega_d^2 = \omega_0^2 - \alpha^2$ and $Q = \omega_0/(2\alpha)$,

$$\delta = \frac{2\pi\alpha}{\omega_d}, \quad Q = \sqrt{\left(\frac{\pi}{\delta}\right)^2 + \frac{1}{4}}. \quad (11.235)$$

Only when $\delta \ll 2\pi$ can one neglect $1/4$ and use $Q \simeq \pi/\delta$. The charge-envelope e-folding time $\tau_A = 1/\alpha = 2L/R$ gives the exact parameter relation $Q = \omega_0 \tau_A/2$. The high- Q replacement $\omega_0 \simeq \omega_d$ is often what enters a measured ringdown estimate.

Example 11.36: Loss measured in the time domain

A low-voltage resonator has same-polarity capacitor-voltage peaks decreasing from 4.0 V to 1.0 V over 12 damped periods. The measured period is 0.200 ms. Infer Q , α , ω_0 and the

series-model bandwidth.

Solution. The logarithmic decrement is $\delta = \ln 4/12 = 0.11552$. Hence

$$Q = \sqrt{(\pi/0.11552)^2 + 1/4} = 27.20, \quad (11.236)$$

$$\alpha = \delta/T_d = 577.6 \text{ s}^{-1}, \quad (11.237)$$

$$\omega_0 = \sqrt{(2\pi/T_d)^2 + \alpha^2} = 3.1421 \times 10^4 \text{ rad s}^{-1}. \quad (11.238)$$

The bandwidth is $\Delta f = 2\alpha/(2\pi) = 183.9 \text{ Hz}$. A frequency scan of the *same terminated circuit* should agree. Disconnecting a generator can change the damping resistance; disagreement between ringdown and driven bandwidth may therefore be a real change of circuit, not a failure of the oscillator model.

Uncertainty. Peaks near the noise floor should be excluded or fitted with a model of the noise. Fitting a straight line to $\ln |A_n|$ versus peak time uses many cycles and can reveal whether one exponential envelope is adequate.

11.14.16 Numerical dynamics: testing the physics of an algorithm

Time integration is useful when a circuit is switched, driven by a pulse, or observed before reaching sinusoidal steady state. The state equations are

$$q' = i, \quad i' = \frac{v_s(t) - Ri - q/C}{L}. \quad (11.239)$$

A simulation must reproduce both phase and energy behaviour. Merely displaying a smooth sinusoid does not establish accuracy.

Consider forward Euler for the free ideal LC circuit:

$$q_{n+1} = q_n + hi_n, \quad i_{n+1} = i_n - h\omega_0^2 q_n. \quad (11.240)$$

Substituting these updates into $U_n = q_n^2/(2C) + Li_n^2/2$ cancels the mixed terms and gives

$$U_{n+1} = (1 + h^2\omega_0^2)U_n. \quad (11.241)$$

Energy therefore grows for every nonzero time step. Reducing h slows the error but does not make forward Euler energy-stable for an undamped oscillator. At 100 steps per period, one period multiplies the energy by $[1 + (2\pi/100)^2]^{100} \simeq 1.483$: almost a 48% error despite a seemingly fine grid.

An instructive alternative is the implicit midpoint method. Write the state increment using the arithmetic mean of the old and new states and evaluate the source at the time midpoint:

$$\frac{q_{n+1} - q_n}{h} = \frac{i_{n+1} + i_n}{2} = \bar{i}, \quad (11.242)$$

$$L \frac{i_{n+1} - i_n}{h} = v_s(t_n + h/2) - R\bar{i} - \frac{q_{n+1} + q_n}{2C}. \quad (11.243)$$

Multiplying the second equation by $h\bar{i}$ and using the first yields

$$U_{n+1} - U_n = h v_s(t_n + h/2)\bar{i} - hR\bar{i}^2. \quad (11.244)$$

Thus the ideal free oscillator conserves its quadratic energy exactly in exact arithmetic; a damped free oscillator loses energy with the correct sign. This is a discrete energy balance, not a guarantee that every aspect of the waveform is exact.

The following Python function solves the two linear midpoint equations directly. It needs no numerical library. The source callable takes time in seconds and returns volts.

```

from math import pi, sqrt

def midpoint(q, current, t, h, L, C, R, source):
    a = h/2
    b = h/(2*L*C)
    d = 1 + h*R/(2*L)
    rhs_q = q + a*current
    rhs_i = ((1 - h*R/(2*L))*current - b*q
             + h*source(t + h/2)/L)
    next_i = (rhs_i - b*rhs_q)/(d + a*b)
    next_q = rhs_q + a*next_i
    return next_q, next_i

L, C, R = 0.01, 1e-6, 0.0
period = 2*pi*sqrt(L*C)
h = period/100
q, current = C*1.0, 0.0
initial_energy = q*q/(2*C)
max_relative_error = 0.0
for n in range(100*100):
    q, current = midpoint(q, current, n*h, h,
                          L, C, R, lambda t: 0.0)
    energy = q*q/(2*C) + L*current*current/2
    max_relative_error = max(max_relative_error,
                              abs(energy/initial_energy - 1))

print(max_relative_error)

```

For the ideal oscillator this code should return an energy error close to floating-point roundoff. Nevertheless its numerical angular frequency is

$$\omega_{\text{num}} = \frac{2}{h} \tan^{-1} \left(\frac{\omega_0 h}{2} \right) = \omega_0 \left[1 - \frac{(\omega_0 h)^2}{12} + \dots \right]. \quad (11.245)$$

At 100 steps per period the relative frequency error is about -0.0329% , accumulating about -0.207 rad of phase error over 100 physical periods. Halving h reduces this leading phase error by approximately four. Energy and phase are independent numerical diagnostics.

Method

For a driven test, set $R = 20 \Omega$, $L = 10 \text{ mH}$, $C = 1.0 \mu\text{F}$ and $v_s(t) = \cos(\omega_0 t) \text{ V}$. Begin with $q = i = 0$. The steady current amplitude is $V_0/R = 50 \text{ mA}$ and its phase relative to the source is zero. Allow several decay times $2L/R = 1.0 \text{ ms}$ before fitting the final cycles. Repeat with half the time step; compare the fitted amplitude and phase, and verify the integrated form of Eq. (11.102). For strongly overdamped circuits, also resolve the fast scale L/R and integrate long enough to see the slow scale RC .

11.14.17 Resonant sensing and small perturbations

A small added capacitance changes the resonance frequency. If the original capacitance is C and an additional parallel capacitance ΔC is added, the new frequency is

$$f' = \frac{1}{2\pi\sqrt{L(C + \Delta C)}}. \quad (11.246)$$

For $\Delta C \ll C$,

$$\frac{f' - f_0}{f_0} \simeq -\frac{1}{2} \frac{\Delta C}{C}. \quad (11.247)$$

This is the basis of many capacitive sensing ideas: a small environmental change modifies capacitance and shifts a frequency.

Example 11.37: Small capacitance perturbation

A resonant sensor has nominal capacitance $C = 100$ pF. A nearby object increases the effective capacitance by 2.0 pF. Estimate the fractional frequency shift.

Solution. Since $\Delta C/C = 2.0/100 = 0.020$,

$$\frac{\Delta f}{f_0} \simeq -\frac{1}{2}(0.020) = -0.010. \quad (11.248)$$

The resonance frequency decreases by approximately 1.0%.

11.14.18 Inductive sensing and loss

An inductive sensor may detect the presence of a material by a change in inductance. In a simple resonant model, increasing L decreases the resonance frequency:

$$\frac{\Delta f}{f_0} \simeq -\frac{1}{2} \frac{\Delta L}{L}. \quad (11.249)$$

However, real materials may also introduce losses, changing R and therefore changing the resonance amplitude and width. A complete sensor model may use both frequency shift and quality-factor change.

Example 11.38: Frequency shift from inductance change

A coil in a resonant circuit has inductance increased by 6% when a sample is inserted. Estimate the resonance frequency shift if capacitance is unchanged.

Solution. For small changes,

$$\frac{\Delta f}{f_0} \simeq -\frac{1}{2} \frac{\Delta L}{L} = -0.5(0.06) = -0.03. \quad (11.250)$$

The resonance frequency decreases by about 3%. If the resonance also becomes broader, the sample has likely introduced additional loss.

11.14.19 Tuning range and frequency selectivity

A tunable resonant circuit can select a narrow frequency band. If the inductance is fixed and the capacitance is variable, the resonance frequency changes as

$$f_0 = \frac{1}{2\pi\sqrt{LC}}. \quad (11.251)$$

Increasing the capacitance lowers the resonance frequency. Decreasing the capacitance raises it. The sharpness of selection depends on Q .

Example 11.39: Variable capacitor range

An ideal tuner uses $L = 250 \mu\text{H}$ and a variable capacitor from $C_{\min} = 50 \text{ pF}$ to $C_{\max} = 500 \text{ pF}$. Estimate the resonance frequency range.

Solution. At the maximum capacitance,

$$f_{\min} = \frac{1}{2\pi\sqrt{(250 \times 10^{-6})(500 \times 10^{-12})}} = 450 \text{ kHz}. \quad (11.252)$$

At the minimum capacitance,

$$f_{\max} = \frac{1}{2\pi\sqrt{(250 \times 10^{-6})(50 \times 10^{-12})}} = 1.42 \text{ MHz}. \quad (11.253)$$

The ideal tuning range is approximately 0.45 to 1.42 MHz.

11.14.20 From a sinusoid to a periodic waveform

For a linear circuit, superposition applies to the complete time-dependent equations. A periodic input can therefore be decomposed into a DC term and sinusoidal harmonics. Each harmonic is multiplied by the transfer function at its own frequency. One must retain the phase of that transfer function as well as its magnitude to reconstruct the output.

For a bipolar square wave of amplitude V_a , zero mean and fundamental angular frequency ω , choose the time origin so that

$$v_{\text{in}}(t) = \frac{4V_a}{\pi} \sum_{m=1,3,5,\dots} \frac{\sin(m\omega t)}{m}. \quad (11.254)$$

Away from the jumps the Fourier series converges to the input waveform; at a jump it takes the midpoint value. Passing it through an unloaded RC low-pass filter gives

$$v_{\text{out}}(t) = \frac{4V_a}{\pi} \sum_{m=1,3,5,\dots} \frac{|H_C(m\omega)|}{m} \sin[m\omega t + \arg H_C(m\omega)]. \quad (11.255)$$

If $\omega RC = 1$, the fundamental output amplitude is $0.9003V_a$. The third-harmonic amplitude is $0.1342V_a$, compared with $0.4244V_a$ in the input. The relative suppression of harmonics explains the rounding of edges. A very low cutoff suppresses the fundamental too; smoothing and signal preservation are competing requirements.

A complementary exact check uses the time-domain RC equation $RC v'_{\text{out}} + v_{\text{out}} = v_{\text{in}}$. Within each half-cycle of constant input, the output approaches that input exponentially. In periodic steady state the positive and negative output extrema are $\pm V_b$, with

$$V_b = V_a \frac{1 - e^{-T/(2RC)}}{1 + e^{-T/(2RC)}} = V_a \tanh\left(\frac{T}{4RC}\right). \quad (11.256)$$

To derive this, evolve from $-V_b$ toward $+V_a$ for a half-period and set the final value equal to $+V_b$. This result is a useful test of both a truncated Fourier sum and a time-domain simulation.

The same linear decomposition explains rms values. Orthogonality of distinct harmonics makes the mean-square voltage equal to the sum of their individual mean squares, plus the square of any DC component. For voltage and current containing many harmonics, average real power is

$$P = V_{\text{dc}}I_{\text{dc}} + \sum_{n \geq 1} V_{n,\text{rms}}I_{n,\text{rms}} \cos(\alpha_n - \beta_n), \quad (11.257)$$

where α_n and β_n are the voltage and current phases of the n th harmonic. This is why a single phase angle is insufficient for a distorted waveform. Rectification by diodes is nonlinear and lies outside this linear superposition model; semiconductor devices are developed later in the book.

11.15 Common mistakes and how to avoid them

Caution

Common mistake 1: using DC intuition for AC elements. In AC steady state, a capacitor can carry sinusoidal current even though no charge crosses the dielectric. An inductor can have a voltage across it even if its resistance is small. Use the time-dependent element laws, not only DC limiting cases.

Caution

Common mistake 2: adding voltage amplitudes as ordinary scalars. In a series RLC circuit, V_0 is not generally $V_{R0} + V_{L0} + V_{C0}$. The element voltages have different phases. Add phasors or use the impedance triangle.

Caution

Common mistake 3: assuming resonance means infinite current. In the ideal formula, the current at resonance is V_0/R . If R is zero and the source is ideal, the simple steady-state model breaks down. Real circuits have resistance, source limitations, losses, heating and breakdown limits.

Caution

Common mistake 4: treating transformer current ratios as the same as voltage ratios. In an ideal transformer, voltage scales as turns ratio, while current scales inversely. A step-up transformer increases voltage but decreases current for the same ideal power.

Summary

Diagnostic checklist.

1. Have I identified whether I am using peak, rms or instantaneous values?
2. Have I stated the phase convention?
3. Have I computed reactances at the actual driving frequency?
4. Have I interpreted the sign of $X_L - X_C$?
5. Have I used phasor addition rather than scalar addition of out-of-phase voltages?
6. Have I checked whether the problem is asking for transient or steady-state behaviour?
7. Have I checked dimensions of ω_0 , X_L , X_C , Z and power?

11.16 Chapter summary

The ideal LC circuit is the electromagnetic analogue of a simple harmonic oscillator. Its charge obeys

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0, \quad (11.258)$$

so its natural angular frequency is $\omega_0 = 1/\sqrt{LC}$. The capacitor and inductor exchange energy between electric and magnetic forms. In the ideal model the total stored energy is constant.

Adding resistance gives a damped RLC oscillator. Resistance removes energy at rate i^2R . The free response can be underdamped, critically damped or overdamped. In the underdamped model the

charge envelope decays exactly as $e^{-Rt/(2L)}$.

For sinusoidal steady-state AC circuits, rms values measure heating equivalence and phasors encode phase. A resistor voltage is in phase with current, an inductor voltage leads current by 90° , and a capacitor voltage lags current by 90° . Reactances are $X_L = \omega L$ and $X_C = 1/(\omega C)$.

For a driven series RLC circuit,

$$Z = \sqrt{R^2 + (X_L - X_C)^2}, \quad I_0 = \frac{V_0}{Z}, \quad \tan \phi = \frac{X_L - X_C}{R}. \quad (11.259)$$

The average power is $P_{\text{avg}} = V_{\text{rms}} I_{\text{rms}} \cos \phi = I_{\text{rms}}^2 R$. At resonance $X_L = X_C$, the impedance is minimized, the phase angle is zero and the current is largest for a given source amplitude.

Transformers use changing magnetic flux to transfer energy between circuits. In the ideal model,

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}, \quad \frac{I_s}{I_p} = \frac{N_p}{N_s}. \quad (11.260)$$

Real transformers and real resonant circuits require attention to losses, heating, insulation and nonideal elements.

Take-home message

The most important conceptual transition in this chapter is from resistance-only thinking to phase-aware thinking. AC circuits cannot be understood only by asking how large a voltage or current is. One must also ask when it occurs in the cycle relative to the other quantities.

11.17 Formula summary

Formula summary

Energy storage

$$U_C = \frac{q^2}{2C} = \frac{1}{2} C v_C^2, \quad U_L = \frac{1}{2} L i^2. \quad (11.261)$$

Ideal LC oscillator

$$L \frac{d^2 q}{dt^2} + \frac{q}{C} = 0, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad f_0 = \frac{1}{2\pi\sqrt{LC}}. \quad (11.262)$$

Damped series RLC free response

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0, \quad \omega_d = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}. \quad (11.263)$$

The expression for ω_d describes oscillation only when $R < 2\sqrt{L/C}$.

Rms values for sinusoids

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}, \quad I_{\text{rms}} = \frac{I_0}{\sqrt{2}}. \quad (11.264)$$

Reactance and impedance

$$X_L = \omega L, \quad X_C = \frac{1}{\omega C}, \quad Z = \sqrt{R^2 + (X_L - X_C)^2}. \quad (11.265)$$

Series RLC current and phase

$$I_0 = \frac{V_0}{Z}, \quad I_{\text{rms}} = \frac{V_{\text{rms}}}{Z}, \quad \tan \phi = \frac{X_L - X_C}{R}. \quad (11.266)$$

Average power

$$P_{\text{avg}} = \frac{1}{2} V_0 I_0 \cos \phi = V_{\text{rms}} I_{\text{rms}} \cos \phi = I_{\text{rms}}^2 R. \quad (11.267)$$

Series resonance and bandwidth at fixed source voltage

$$\Delta\omega = R/L, \quad Q = \frac{\omega_0 L}{R} = \frac{f_0}{\Delta f}, \quad \omega_1 \omega_2 = \omega_0^2. \quad (11.268)$$

At current resonance, $V_{L,\text{rms}} = V_{C,\text{rms}} = Q V_{s,\text{rms}}$.

Ideal transformer

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}, \quad \frac{I_s}{I_p} = \frac{N_p}{N_s}. \quad (11.269)$$

11.18 Problems for independent study and tutorials

The following 44 problems progress from conceptual reasoning to modelling. Unless stated otherwise, use ideal linear lumped elements, a fixed-amplitude sinusoidal voltage source and the sign convention $i = q'$. Core problems require no complex arithmetic. The last group draws on the advanced enrichment. Selected detailed solutions follow; try the problem before reading its solution.

Conceptual questions: C1–C8

- C1.** In Fig. 11.2, explain why current continues at $T/4$ even though the capacitor voltage is zero. Determine the signs of q , i and i' just after that instant.
- C2.** An LC capacitor returns to its initial electric energy after half a period. Has the circuit returned to its initial state? Explain using polarity and current direction.
- C3.** A free RLC discharge has $i(0) = 0$ but nonzero stored energy. Show that the instantaneous energy-loss rate is initially zero even though the later response is damped.
- C4.** A student says, “The average AC current is zero, so its heating must be zero.” Identify the invalid interchange of operations.
- C5.** Use Fig. 11.5 to explain why the resistor and inductor voltage amplitudes in a series RL circuit cannot generally be added as scalars. Is Kirchhoff’s loop rule violated?
- C6.** At current resonance, $V_{C,\text{rms}} = V_{L,\text{rms}} = 100\text{ V}$ while the source is 5 V rms. Explain where the voltage cancellation occurs and whether either reactive component dissipates real average power.
- C7.** Explain why a turns ratio alone cannot establish transformer power rating or acceptable operating frequency. Contrast a constant-current excitation with a constant-voltage excitation.
- C8.** A frequency scan gives a current maximum even though the free circuit is overdamped. Explain why these observations are compatible, and identify the source-frequency assumptions needed.

Analytical and quantitative problems: P1–P16

- P1.** A $2.0\text{ }\mu\text{F}$ capacitor charged to 8.0 V is connected to a 50 mH ideal inductor with zero initial current. Find ω_0 , f_0 , total energy and peak current.
- P2.** An ideal loop has $L = 25\text{ mH}$ and $C = 4.0\text{ }\mu\text{F}$. Initially $q = 0$ and $i = +20\text{ mA}$. Find $q(t)$, its amplitude, and the time of its first positive maximum. Check the initial energy.

- P3.** A free loop has $L = 0.10 \text{ H}$, $C = 4.0 \mu\text{F}$ and $R = 50 \Omega$, with $q_i = 8.0 \mu\text{C}$ and $i_i = 0$. Determine the damping regime and write the complete initial-value solution. Evaluate $q'(0)$ and $U(0)$.
- P4.** Starting from the characteristic equation, show that for $R \gg 2\sqrt{L/C}$ the two decay rates approach $-R/L$ and $-1/(RC)$. Identify the slow process physically.
- P5.** A periodic voltage is $+5.0 \text{ V}$ for 40% of each period and -5.0 V for the remaining 60%. Find its mean, total rms and AC-only rms values, and the average power in 100Ω .
- P6.** Use differentiation to find the current through a $10 \mu\text{F}$ capacitor driven by $v_C = 6 \cos(500t) \text{ V}$. State the current amplitude, phase and units of every coefficient.
- P7.** A series circuit has $R = 40 \Omega$, $L = 0.15 \text{ H}$ and $C = 20 \mu\text{F}$. For $v_s = 100 \cos(500t) \text{ V}$, find both reactances, Z , ϕ , $i(t)$ and average power. Verify power with two independent formulas.
- P8.** A series circuit carries 0.20 A rms , with $R = 60 \Omega$, $X_L = 150 \Omega$ and $X_C = 70 \Omega$. Draw a labelled voltage triangle and find the source voltage and power factor.
- P9.** For $L = 10 \text{ mH}$, $C = 1.0 \mu\text{F}$ and $R = 200 \Omega$, find the exact half-power frequencies, bandwidth and Q . Compare the arithmetic and geometric means of the half-power frequencies. Classify the free response.
- P10.** For the curves in Fig. 11.8, calculate the peak ordinate and the normalized bandwidth $\Delta\omega/\omega_0$ for each resistance. Explain why the plotted curves have unequal heights.
- P11.** A series resonator has $Q = 15$, $f_0 = 2.0 \text{ kHz}$ and $L = 20 \text{ mH}$. Find R , C and the resonant capacitor peak voltage for a 2.0 V rms source. Compare with a stated permissible sinusoidal peak of 40 V .
- P12.** An ideal transformer has $N_s/N_p = 1/20$ and primary voltage 240 V rms . A 6.0Ω secondary load is connected. Find both currents, secondary voltage, real power and input resistance.
- P13.** A real transformer delivers 60 W at efficiency 0.92 . Its primary voltage is 230 V rms and input power factor 0.80 . Find input power and primary rms current. Explain why efficiency alone would not give the current.
- P14.** A single-phase unity-power-factor load receives 10 kW at 1000 V rms through an outgoing-and-return line resistance of 4.0Ω . Find line current, line loss and sending-end voltage. Repeat for a receiving-end voltage of 5000 V at the same delivered power.
- P15.** A 400-turn primary has core area 6.0 cm^2 and is supplied by a sinusoidal 24 V rms source at 50 Hz . Find the peak core flux density. What voltage at 40 Hz would retain the same flux density?
- P16.** Show that the two frequencies with equal current amplitudes in an ideal series RLC circuit on opposite sides of resonance obey $\omega_a\omega_b = \omega_0^2$. State why constant source amplitude matters.

Measurement, engineering and computation: E1–E10

- E1.** A generator has open-circuit voltage 2.0 V rms and $R_s = 50 \Omega$. It drives a series resonator with $R_m = 20 \Omega$ and coil resistance 10Ω . At resonance find current, terminal voltage and monitor voltage. Which resistance follows from the ratio V_m/V_g ?
- E2.** For the arrangement in Fig. 11.10, explain what electrical connection is introduced by attaching the CH2 return to the top of R_m while CH1 remains grounded at the bottom node. Propose a valid measurement configuration.

- E3.** Write a frequency-scan algorithm for $L = 10 \text{ mH}$, $C = 1.0 \mu\text{F}$ and total series resistance 30Ω . Locate both half-power points numerically. Refine the grid until their difference changes by less than 1%; compare with the exact bandwidth.
- E4.** A current-monitor trace peaks $(0.100 \pm 0.002) \text{ ms}$ after the terminal-voltage trace. The period is $(0.800 \pm 0.004) \text{ ms}$. Treat uncertainties as independent standard uncertainties. Find ϕ , its uncertainty, and the sign of net reactance.
- E5.** With known capacitance $C = 100 \text{ nF}$, a series current resonance is measured at 3.20 kHz . Infer L . If $u_C/C = 2.0\%$ and $u_f/f = 0.50\%$ independently, find u_L/L .
- E6.** A known $L = 15 \text{ mH}$ resonator has half-power frequencies 4.65 and 5.05 kHz . Infer R , f_0 , Q and C under the constant-series-loss model. Identify one measurement change that would invalidate a comparison with a free ringdown.
- E7.** A current scan at fixed open-circuit source voltage has the expected peak frequency but a peak half as high as predicted. Give two distinct explanations and one additional measurement that would distinguish them.
- E8.** A sensor has $C = 100 \text{ pF}$ and acquires an additional parallel 2.0 pF . Compare the exact and linearized fractional frequency shifts. Discuss why a loss change could affect the ease of detecting this shift.
- E9.** Derive exact upper and lower resonance frequencies for components with independent bounds $L = L_0(1 \pm 0.05)$ and $C = C_0(1 \pm 0.10)$. Explain why these bounds do not define a standard uncertainty without further assumptions.
- E10.** Design a simulation comparison for the free ideal LC circuit using forward Euler and implicit midpoint. Record both relative energy error and phase error after 10 and 100 periods. Explain why one diagnostic cannot replace the other.

Advanced challenges: A1–A10

- A1.** Derive the series RLC impedance from the driven charge equation using a complex exponential. Recover the real-valued current amplitude and phase convention.
- A2.** An RC low-pass filter with $R = 10 \text{ k}\Omega$, $C = 100 \text{ nF}$ is loaded by $R_{\text{load}} = 20 \text{ k}\Omega$. The source resistance is negligible. Find the low-frequency gain, cutoff relative to that gain, and gain and phase at 100 Hz .
- A3.** Derive the capacitor-voltage resonance condition and explain how a free underdamped circuit can lack a capacitor-voltage peak in its driven response.
- A4.** Derive H_R for the series RLC resistor output, and show that its gain is reduced at resonance if the numerator uses only an external monitor resistor but the denominator contains additional loss resistance.
- A5.** A sinusoidal single-phase load consumes 2.0 kW at 200 V rms and 50 Hz with lagging power factor 0.80 . Find the ideal shunt capacitance for unity power factor, and compare supply currents before and after correction.
- A6.** A parallel circuit has $R_p = 1.0 \text{ k}\Omega$, $L = 10 \text{ mH}$ and $C = 1.0 \mu\text{F}$. At resonance a source holds 2.0 V rms across it. Find the source current and each reactive branch current. Explain why the source current is small.
- A7.** A source has $\mathcal{Z}_s = (50 + i20) \Omega$ and $V_{\text{th,rms}} = 12 \text{ V}$. Find the conjugate-matched load and maximum real power. If the load must be purely resistive, derive and evaluate the optimum

load resistance.

- A8.** Starting from the midpoint updates, prove Eq. (11.244). Explain what is and is not established by this discrete energy identity.
- A9.** Same-polarity ringdown peaks decrease by a factor of e in two damped periods. Use the exact decrement relation to determine Q , and quantify the error in $Q \simeq \pi/\delta$.
- A10.** A symmetric square wave with amplitude V_a and period T drives an unloaded RC low-pass filter. Derive the periodic output extrema and find their limits when $RC \ll T$ and when $RC \gg T$. Interpret both limits physically.

11.19 Selected detailed solutions

These solutions emphasize model choice and checks. Rounded numerical values are reported after carrying sufficient precision through intermediate calculations.

Example 11.40: P1: frequency and energy of a released LC loop

Strategy. The current starts at zero, so the initial energy is entirely electric. Use the period relation for the time scale and conservation of energy for the current amplitude.

$$\omega_0 = [(0.050)(2.0 \times 10^{-6})]^{-1/2} = 3.162 \times 10^3 \text{ rad s}^{-1}, \quad (11.270)$$

$$f_0 = \omega_0/(2\pi) = 503 \text{ Hz}, \quad (11.271)$$

$$U = \frac{1}{2}CV_i^2 = 64.0 \mu\text{J}, \quad (11.272)$$

$$I_0 = V_i\sqrt{C/L} = 50.6 \text{ mA}. \quad (11.273)$$

The initial charge is $q_i = 16.0 \mu\text{C}$, so $\omega_0 q_i$ gives the same current. The dimensional identity $\text{HF} = \text{s}^2$ checks the angular frequency. The signed current begins negative because the initially positive plate discharges; I_0 is a non-negative amplitude.

Example 11.41: P2: an oscillator starting with magnetic energy

Strategy. Both initial conditions are specified, and an assumed cosine with zero phase would fail $q(0) = 0$. Write $q = A \cos \omega_0 t + B \sin \omega_0 t$.

Here $\omega_0 = 3162.28 \text{ rad s}^{-1}$, $A = 0$ and $B = i_i/\omega_0 = 6.325 \mu\text{C}$. Thus

$$q(t) = (6.325 \mu\text{C}) \sin[(3162.28 \text{ s}^{-1})t], \quad i(t) = (20.0 \text{ mA}) \cos \omega_0 t. \quad (11.274)$$

The first positive charge maximum occurs at $t = \pi/(2\omega_0) = 0.497 \text{ ms}$. Initially $U = Li_i^2/2 = 5.00 \mu\text{J}$; at that charge maximum, $q_{\text{max}}^2/(2C) = 5.00 \mu\text{J}$. The quarter-period exchange therefore works in either direction, depending on the initial state.

Example 11.42: P3: damping with a zero initial current

The critical resistance is $R_c = 2\sqrt{0.10/(4.0 \times 10^{-6})} = 316.2 \Omega$. Since $50 \Omega < R_c$, the response is underdamped. The parameters are

$$\omega_0 = 1581.14 \text{ rad s}^{-1}, \quad \alpha = 250 \text{ s}^{-1}, \quad \omega_d = 1561.25 \text{ rad s}^{-1}. \quad (11.275)$$

Using Eq. (11.47), with t in seconds,

$$q(t) = (8.0 \mu\text{C})e^{-250t} [\cos(1561.25t) + 0.16013 \sin(1561.25t)]. \quad (11.276)$$

The derivative at zero contains $-250 + 0.16013(1561.25) = 0$ to the precision shown. Omitting the sine term would impose the wrong current. The initial energy is $q_i^2/(2C) = 8.0 \mu\text{J}$. Although $dU/dt = 0$ at the instant of release, the nonzero initial acceleration $i'(0) = -q_i/(LC)$ produces current immediately afterward and hence resistive loss.

Example 11.43: P5: mean, rms and DC subtraction

Weight each level by its fraction of the period:

$$\langle v \rangle = 0.4(5) + 0.6(-5) = -1.0 \text{ V}, \quad (11.277)$$

$$\langle v^2 \rangle = 0.4(25) + 0.6(25) = 25 \text{ V}^2. \quad (11.278)$$

Thus $V_{\text{rms}} = 5.0 \text{ V}$ and $V_{\text{ac,rms}} = \sqrt{25 - 1} = 4.90 \text{ V}$. The average resistor power is $25/100 = 0.250 \text{ W}$. Using only the mean would predict 0.010 W and discard the AC heating. Using $5/\sqrt{2}$ would be another error: this two-level waveform is not sinusoidal.

Example 11.44: P7: a capacitive response checked by power

At $\omega = 500 \text{ rad s}^{-1}$,

$$X_L = 75 \Omega, \quad X_C = 100 \Omega, \quad Z = \sqrt{40^2 + (-25)^2} = 47.1699 \Omega. \quad (11.279)$$

The phase is $\phi = \tan^{-1}(-25/40) = -0.55860 \text{ rad} = -32.0^\circ$. Therefore

$$i(t) = (2.120 \text{ A}) \cos(500t + 0.55860), \quad (11.280)$$

where t is in seconds. The current leads. For power, $I_{\text{rms}} = 2.120/\sqrt{2} = 1.499 \text{ A}$, so $P = I_{\text{rms}}^2 R = 89.9 \text{ W}$. Independently,

$$P = \frac{1}{2} V_0 I_0 \cos \phi = \frac{1}{2} (100)(2.120) \frac{40}{47.1699} = 89.9 \text{ W}. \quad (11.281)$$

The factor $1/2$ belongs here because the second calculation uses peak amplitudes.

Example 11.45: P9: exact bandwidth for a broad resonance

The LC scale is $\omega_0 = 10^4 \text{ rad s}^{-1}$ and $Q = \omega_0 L/R = 0.5$. Applying the exact quadratic roots,

$$\omega_1 = 4142.14 \text{ rad s}^{-1}, \quad f_1 = 659.24 \text{ Hz}, \quad (11.282)$$

$$\omega_2 = 24142.14 \text{ rad s}^{-1}, \quad f_2 = 3842.34 \text{ Hz}. \quad (11.283)$$

The bandwidth is 20000 rad s^{-1} , or 3183.10 Hz , exactly equal to R/L in angular-frequency units. The arithmetic midpoint is $14142.14 \text{ rad s}^{-1}$, whereas the geometric mean is 10000 rad s^{-1} . A symmetric narrow-peak approximation would be poor here. The free circuit is critically damped because $Q = 1/2$; this does not remove the maximum of the driven current response.

Example 11.46: P12: transformer loading and input resistance

The secondary voltage is $V_s = 240/20 = 12.0$ V rms. The resistive load draws $I_s = 12/6 = 2.00$ A rms and consumes $P_s = 24.0$ W. In the ideal model $I_p = I_s/20 = 0.100$ A rms, so $V_p I_p = 24.0$ W as required.

The resistance referred to the primary is

$$R_{\text{in}} = (N_p/N_s)^2 R_{\text{load}} = 20^2(6.0) = 2400 \Omega. \quad (11.284)$$

Checking $V_p/R_{\text{in}} = 0.100$ A closes the calculation. A step-down voltage transformation implies a step-up current magnitude on the load side; it does not imply increased power.

Example 11.47: P14: receiving voltage, sending voltage and loss

The specified voltage is at the receiving load, so $I = P_{\text{load}}/V_{\text{recv}}$ is exact under the stated unity-power-factor assumption. At 1000 V, $I = 10.0$ A, $P_{\text{line}} = I^2 R = 400$ W and $V_{\text{send}} = V_{\text{recv}} + IR = 1040$ V.

At 5000 V, $I = 2.00$ A, $P_{\text{line}} = 16.0$ W and $V_{\text{send}} = 5008$ V. Increasing the receiving voltage by five reduces the line loss by 25. In each case $V_{\text{send}} I = P_{\text{load}} + P_{\text{line}}$. The sending voltage cannot be substituted for the receiving voltage while retaining the same power in $I = P/V$ without changing the meaning of the calculation.

Example 11.48: E1: what the oscilloscope actually measures

At resonance the total resistance in the generator-emf model is $50 + 20 + 10 = 80 \Omega$. Hence $I_{\text{rms}} = 2.0/80 = 25.0$ mA. The source's internal resistor drops 1.25 V rms, leaving $V_g = 0.750$ V across the external circuit. The monitor reads $V_m = IR_m = 0.500$ V.

The ratio $V_m/V_g = 2/3$ gives an external resistance $R_m/(2/3) = 30 \Omega$, not 80Ω . In contrast $V_m/V_{\text{th}} = 1/4$ gives the full 80Ω when the numerator is R_m . This is why the reference voltage must be identified before interpreting the peak height or bandwidth.

Example 11.49: E4: signed phase with uncertainty

The current-monitor signal peaks later, so $\phi > 0$. The phase is $360^\circ(0.100/0.800) = 45.0^\circ$, indicating net inductive behaviour.

For independent small standard uncertainties, differentiate $\phi = 360^\circ \Delta t/T$ and add the squared contributions:

$$u_\phi = |\phi| \sqrt{\left(\frac{u_{\Delta t}}{\Delta t}\right)^2 + \left(\frac{u_T}{T}\right)^2} = 45.0^\circ \sqrt{0.02^2 + 0.005^2} = 0.93^\circ. \quad (11.285)$$

The result is $\phi = (45.0 \pm 0.9)^\circ$ with the stated standard-uncertainty interpretation. Correlated timing errors from the same trace would require a covariance term rather than the independent-error formula.

Example 11.50: E6: extracting a series model from a resonance width

The width is 400 Hz, so

$$R = 2\pi L \Delta f = 2\pi(0.015)(400) = 37.70 \Omega. \quad (11.286)$$

The peak frequency follows from the geometric mean:

$$f_0 = \sqrt{4650 \cdot 5050} = 4845.87 \text{ Hz}, \quad Q = f_0/\Delta f = 12.11. \quad (11.287)$$

Then $C = [(2\pi f_0)^2 L]^{-1} = 71.9 \text{ nF}$. These are effective model parameters, conditional on the specified inductance and a frequency-independent series resistance. If the source resistance contributes to the scan but disappears when the source is disconnected for ringdown, the two measurements probe different damping. Keep the termination the same or include its change explicitly.

Example 11.51: A2: a loaded RC divider

The low-frequency capacitor is effectively open, so the two resistors form an ordinary divider with gain $20/(10 + 20) = 2/3$. The time constant is determined by the resistance seen by the capacitor with the ideal source set to zero:

$$\tau = (10 \text{ k}\Omega \parallel 20 \text{ k}\Omega)(100 \text{ nF}) = 0.6667 \text{ ms}. \quad (11.288)$$

Thus $f_c = 1/(2\pi\tau) = 238.7 \text{ Hz}$ relative to the reduced low-frequency gain. At 100 Hz,

$$|H| = \frac{2/3}{\sqrt{1 + (100/238.7)^2}} = 0.6149, \quad \arg H = -22.73^\circ. \quad (11.289)$$

Loading has changed both the passband gain and the cutoff. The original unloaded cutoff 159.2 Hz would not describe this circuit.

Example 11.52: A5: shunt correction from current components

Initially $I_1 = P/(V \cos \phi) = 2000/(200 \cdot 0.80) = 12.5 \text{ A}$. Its in-phase component is $P/V = 10.0 \text{ A}$ and its lagging component is $10 \tan(\cos^{-1} 0.80) = 7.50 \text{ A}$. The capacitor must supply 7.50 A leading, so

$$C = \frac{7.50}{2\pi(50)(200)} = 119.4 \mu\text{F}. \quad (11.290)$$

The supply then carries 10.0 A while the load branch still carries 12.5 A. The capacitor current does not subtract directly from the load-current magnitude; it cancels its quadrature component. With fixed line resistance, the line-loss ratio becomes $(10/12.5)^2 = 0.64$.

Example 11.53: A9: exact and approximate ringdown quality factors

The same-polarity amplitude ratio is e over two periods, so $\delta = (\ln e)/2 = 0.5$. Equation (11.235) gives

$$Q = \sqrt{(\pi/0.5)^2 + 1/4} = 6.3030. \quad (11.291)$$

The weak-damping estimate is $Q \simeq 2\pi = 6.2832$, which underestimates the exact value by 0.315%. Even when the numerical difference is small, stating the approximation matters: the period used to count cycles is the damped period, not the undamped one.

11.20 Bridge to the next chapter

This chapter has treated circuits as lumped systems. The next stage in Physics 2 is to connect oscillating circuits to electromagnetic fields in space. A changing electric field can be associated with magnetic effects, and a changing magnetic field can be associated with electric effects. Chapter 12 develops Maxwell's equations and the magnetic response of matter; Chapter 13 develops electromagnetic waves. The oscillating currents and voltages of this chapter are therefore not an isolated topic: they are the circuit-level doorway to wave propagation, radiation, antennas and modern communication.

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