

Chapter 12

Maxwell's Equations and Magnetism in Matter

First Edition

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Preface

This chapter brings together the main laws of classical electromagnetism in their integral form and then applies the same physical thinking to magnetic matter. Earlier chapters introduced electric charge, electric field, electric flux, potential, current, magnetic field, magnetic force, Ampere's law, Faraday's law, inductance and electromagnetic energy. Those topics are not independent islands. Maxwell's equations show the structure that connects them. They are the compact classical statement that electric charges produce electric fields, that the classical theory used here has no magnetic-charge sources, that changing magnetic flux produces circulating electric fields, and that currents together with changing electric flux produce circulating magnetic fields.

The chapter has two equally important roles. First, it completes the conceptual architecture of classical electromagnetism and prepares the student for electromagnetic waves. Second, it explains why matter responds magnetically. A piece of iron, a transformer core, a hard-disk material, a Hall sensor package, a medical-imaging magnet and a small permanent magnet on a laboratory bench are all macroscopic systems whose behaviour is rooted in microscopic magnetic moments and collective alignment. We shall not develop a full quantum theory of magnetism here, but we shall use enough microscopic language to make diamagnetism, paramagnetism and ferromagnetism physically intelligible.

The target audience is Technical Physics, engineering, computer-science, applied-science and space-science students. The presentation therefore emphasises equations, modelling assumptions, measurement meaning and engineering interpretation. The goal is not to memorise four formulas in a table. The goal is to learn how to read an integral equation as a statement about sources, circulation, surfaces, boundaries and symmetries.

The chapter uses the established notation and visual style of this volume. The core develops integral laws and material response; the applied sections turn these ideas into measurements and design estimates. The advanced enrichment then supplies the local field equations and the material models needed for further study.

The Guideline

Read each Maxwell equation as a physical operation. Ask what surface or loop is being used, what field is being integrated, what source term appears on the right-hand side, and what assumptions about orientation and sign convention have been made. The integral sign is not decoration: it tells you what is being measured or summed over space.

Pedagogical boundary

This chapter treats Maxwell's equations mainly in integral form. A differential-form bridge is included in the advanced section because it is extremely useful for later theoretical work and numerical modelling. However, the main chapter does not require a full vector-calculus course. The magnetic-material discussion introduces magnetization and susceptibility qualitatively and semi-quantitatively, but it does not attempt a microscopic quantum theory of solids.

Conventions and notation

Vectors are bold. Oriented line and area elements are $d\boldsymbol{\ell}$ and $d\mathbf{A}$. The symbol $\oint$ denotes a closed integral; the differential element distinguishes a path from a surface. Flux and circulation are different operations:

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A}, \quad \Phi_B = \int_S \mathbf{B} \cdot d\mathbf{A}, \quad \text{circulation} = \oint_C \mathbf{F} \cdot d\boldsymbol{\ell}.$$

For an open surface bounded by C , curl the fingers of the right hand along positive traversal: the thumb gives the positive normal. A closed surface instead has an outward normal everywhere.

Symbol	Meaning	SI unit
$\mathbf{E}$	electric field	V/m
$\mathbf{B}$	magnetic field (magnetic flux density)	T
Φ_E	electric flux	V m
Φ_B	magnetic flux	Wb
q_{enc}	net charge inside a closed surface	C
I_{enc}	total charge current through an oriented surface	A
I_d	vacuum displacement current, $\varepsilon_0 d\Phi_E/dt$	A
ε_0	vacuum permittivity	F/m
μ_0	vacuum permeability	H/m
c	speed of light in vacuum	m/s
$\mathbf{m}$	magnetic dipole moment	A m ²
$\mathbf{M}$	magnetization: dipole moment per volume	A/m
$\mathbf{H}$	auxiliary magnetic field, $\mathbf{B}/\mu_0 - \mathbf{M}$	A/m
χ_m	susceptibility in a linear isotropic model	dimensionless
μ_r	relative permeability in a linear isotropic model	dimensionless

Note

$\mathbf{B}$ appears in the Lorentz force and magnetic flux. $\mathbf{H}$ is introduced with susceptibility; its field equation is developed in the advanced section. The term “displacement current” denotes a contribution to magnetic circulation. In a vacuum capacitor gap, it does not mean a stream of charge crossing the gap.

For calculations, use $\varepsilon_0 \simeq 8.854 \times 10^{-12}$ F/m, $\mu_0 \simeq 4\pi \times 10^{-7}$ H/m and $c \simeq 3.00 \times 10^8$ m/s. In the current SI, $c = 299\,792\,458$ m/s is exact; μ_0 and ε_0 are experimentally determined and obey $\mu_0 \varepsilon_0 c^2 = 1$ [4].

Chapter 12

Maxwell's Equations and Magnetism in Matter

12.1 Motivation and roadmap

Electromagnetism has a special place in Physics 2 because it begins with simple laboratory observations and ends with a unified field theory. At the start of the volume we used electric charge and Coulomb's law to describe forces between charged particles. We then introduced the electric field, electric flux, electric potential, capacitors, currents, circuits, magnetic fields, magnetic forces, magnetic fields produced by currents and electromagnetic induction. The present chapter is the synthesis point: the four Maxwell equations organise those ideas into one structure.

A useful way to read Maxwell's equations is to sort them into two families. Two equations are flux equations: they tell us about the net field passing through a closed surface. Two equations are circulation equations: they tell us about the line integral of a field around a closed loop. One flux equation has electric charge as a source. The magnetic flux equation has no magnetic charge source in the classical theory. One circulation equation says that changing magnetic flux generates a circulating electric field. The other says that charge current and changing electric flux generate a circulating magnetic field.

Figure 12.1 summarises the logic. The left side recalls earlier field laws. The middle shows the Maxwell synthesis. The right side points to the two directions developed after the synthesis: electromagnetic waves and magnetic matter.

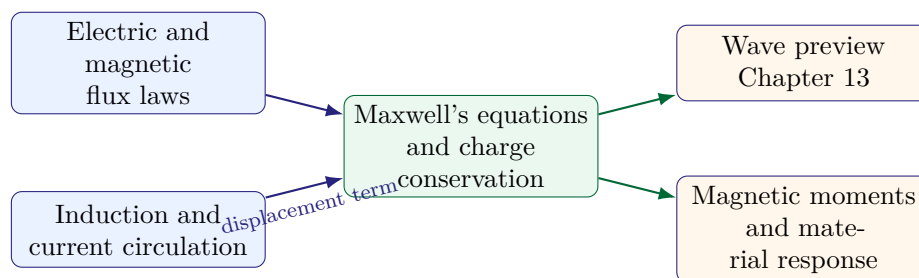


Figure 12.1: Roadmap: flux and circulation laws are unified before material response and the wave connection are developed. The detailed wave physics belongs to Chapter 13.

Learning goals

After studying this chapter, the student should be able to:

- state the four Maxwell equations in integral form and explain the physical meaning of each term;
- distinguish flux equations from circulation equations;
- explain why Ampere's original law is incomplete for time-dependent electric fields;

- derive and use the displacement current $I_d = \epsilon_0 d\Phi_E/dt$;
- apply the Ampere-Maxwell law to the field between charging capacitor plates;
- explain the statement that magnetic monopoles are absent in classical electromagnetism;
- connect Maxwell's equations to charge conservation and the electromagnetic-wave preview;
- define magnetic dipole moment and magnetization at an introductory level;
- distinguish diamagnetic, paramagnetic and ferromagnetic response;
- interpret domains, remanence, coercivity and hysteresis qualitatively;
- identify engineering contexts where Maxwell equations and magnetic materials are essential;
- avoid common misconceptions about displacement current, field lines, magnetic poles and material notation.

Key message

Maxwell's equations are not four unrelated formulas. They are four complementary statements about how fields pass through surfaces, how fields circulate around loops, and how charges, currents and changing fields act as sources.

12.2 Required prior knowledge

This chapter assumes that the student is comfortable with vectors, dot products, right-hand-rule orientations, basic integrals and the idea of a field. From the previous Physics 2 chapters, the most important prerequisites are electric flux, Gauss's law, magnetic flux, Ampere's law, Faraday's law, Lenz's law and the behaviour of capacitors. The present chapter uses them together rather than separately.

Summary

Before starting the chapter, review the following ideas.

- The electric flux through a surface is $\Phi_E = \int \mathbf{E} \cdot d\mathbf{A}$.
- Gauss's law says $\oint \mathbf{E} \cdot d\mathbf{A} = q_{\text{enc}}/\epsilon_0$.
- The magnetic flux through a surface is $\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$.
- Faraday's law says $\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{\ell} = -d\Phi_B/dt$.
- Ampere's law for steady currents says $\oint \mathbf{B} \cdot d\mathbf{\ell} = \mu_0 I_{\text{enc}}$.
- A parallel-plate capacitor has approximately uniform electric field $E = Q/(\epsilon_0 A)$ between large plates when edge effects are neglected.

Caution

A closed curve C and an open surface S bounded by that curve are not the same object. Faraday's law and Ampere-Maxwell's law involve a line integral around C and a flux through a surface S whose boundary is C . Many sign and interpretation errors come from mixing these two geometric objects.

Guided checks

Check 1. What is the SI unit of magnetic flux?

Answer. The weber, $1 \text{ Wb} = 1 \text{ T m}^2$.

Check 2. What does a dot product with $d\mathbf{A}$ select?

Answer. The component of the field normal to the oriented surface element.

Check 3. In a line integral $\oint \mathbf{B} \cdot d\boldsymbol{\ell}$, what component of $\mathbf{B}$ contributes?

Answer. The component tangent to the chosen path in the direction of $d\boldsymbol{\ell}$.

12.3 Flux and circulation as the language of field laws

The integral form of Maxwell's equations uses two geometric operations. The first is flux, which measures how much of a vector field passes through a surface. The second is circulation, which measures how much of a vector field follows a closed path. A flux equation is naturally associated with sources and sinks. A circulation equation is naturally associated with rotational or vortex-like behaviour.

For a uniform field and a flat surface, the flux reduces to

$$\Phi = \mathbf{F} \cdot \mathbf{A} = FA \cos \theta, \quad (12.1)$$

where $\mathbf{A}$ is the area vector and θ is the angle between the field and the surface normal. For a non-uniform field or a curved surface, the surface must be divided into small elements:

$$\Phi = \int_S \mathbf{F} \cdot d\mathbf{A}. \quad (12.2)$$

The orientation of $d\mathbf{A}$ matters. Reversing the normal reverses the sign of the flux.

The circulation around a closed path C is

$$\Gamma_F = \oint_C \mathbf{F} \cdot d\boldsymbol{\ell}. \quad (12.3)$$

If the field tends to point along the direction of the path, the contribution is positive. If it points opposite, the contribution is negative. If it is perpendicular to the path, that part contributes zero.

Method

How to read an integral field law.

1. Identify whether the left-hand side is a flux integral or a circulation integral.
2. Draw the surface or loop and assign an orientation.
3. Identify the source term on the right-hand side.
4. Check whether the source is localised charge, conduction current, changing electric flux or changing magnetic flux.
5. Ask whether symmetry makes the integral simple. If not, the equation remains true but may not be an easy calculation tool.

Example 12.1: Flux and circulation for simple uniform fields

A uniform electric field $\mathbf{E} = E_0 \hat{\mathbf{z}}$ passes through a square surface of side a . The surface normal makes an angle θ with $\hat{\mathbf{z}}$. A separate circular loop of radius R lies in the xy -plane while a uniform magnetic field $\mathbf{B} = B_0 \hat{\mathbf{x}}$ is present. Find the electric flux through the square and the magnetic circulation around the circle.

Solution. The electric flux through the flat square is

$$\Phi_E = \mathbf{E} \cdot \mathbf{A} = E_0 a^2 \cos \theta. \quad (12.4)$$

Only the component of $\mathbf{E}$ normal to the surface contributes. For the circular line integral, the tangent direction changes continuously around the circle. The field $\mathbf{B} = B_0\hat{\mathbf{x}}$ is constant. The integral is

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mathbf{B} \cdot \oint_C d\boldsymbol{\ell} = \mathbf{B} \cdot \mathbf{0} = 0, \quad (12.5)$$

where the vector sum of all directed line elements around a closed curve is zero. A uniform field has no net circulation around a closed curve.

Interpretation. Nonzero flux through an open surface does not imply nonzero circulation around its boundary. Flux and circulation are different geometric probes of a field.

12.3.1 A loop orientation is a physical bookkeeping choice

For a planar loop in the xy -plane, choose its positive traversal counterclockwise as viewed from $+z$. The corresponding normal is $+\hat{\mathbf{z}}$. If a field points downward, its flux through this oriented surface is negative. The same physical field can therefore have positive or negative signed flux depending on the chosen orientation. Observable predictions must be invariant under changing both the loop and surface orientations together.

For a closed surface there is no boundary curve. Its standard normal is outward at every point. A flux calculation through the top and bottom of a box must consequently use opposite normals, even if the physical field has the same direction at both faces. These two situations are compared in Fig. 12.2.

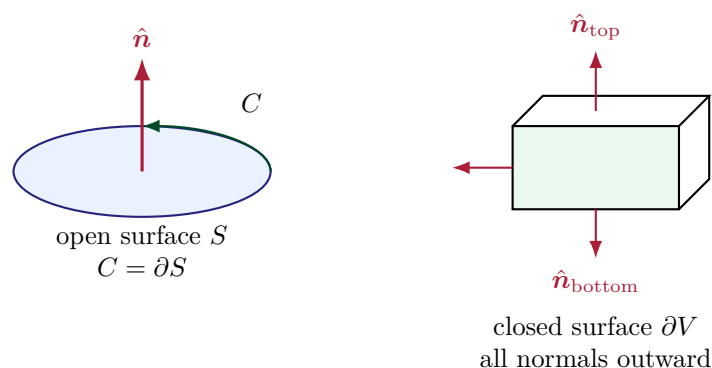


Figure 12.2: Orientation conventions. The open surface uses a normal tied to the positive loop direction. The closed surface uses outward normals. Arrows on a mathematical surface do not represent motion of the surface.

A useful distinction follows. A zero circulation around one particular loop does not prove that a field is conservative everywhere. Positive and negative contributions may cancel for that loop. Similarly, zero flux through one particular closed surface does not rule out positive and negative charge within it. To infer a local property, a statement must hold for appropriately arbitrary small loops or surfaces; this is the idea behind the differential equations developed later.

12.4 Gauss's law for electric fields

Gauss's law for electric fields is

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0}. \quad (12.6)$$

It states that the net electric flux through any closed surface S is determined by the net charge enclosed by that surface. The law is exact in classical electromagnetism. It does not require symmetry. Symmetry is required only if we want to use the law as an easy way to calculate $\mathbf{E}$.

The word “net” is essential. Positive charges contribute outward flux; negative charges contribute inward flux. Charges outside the closed surface can change the pattern of field lines crossing the surface locally, but they contribute zero net flux because any field line entering the surface from an external charge also leaves it.

Key message

Gauss’s law for electricity says that electric charge is a source of electric flux. It is not merely a shortcut for symmetric problems; it is one of the fundamental statements about the electric field.

Caution

A Gaussian surface is imaginary. It is a mathematical surface chosen by the modeller, not a physical shell that must exist in the laboratory. The enclosed charge is physical; the surface is a tool for organising the flux.

Example 12.2: Electric flux through a closed surface

A closed surface encloses three small charged beads with charges $+2.0 \text{ nC}$, -5.0 nC and $+1.0 \text{ nC}$. A fourth bead of charge $+20 \text{ nC}$ is outside the surface. Find the net electric flux through the closed surface.

Solution. The enclosed charge is the algebraic sum of only the charges inside:

$$q_{\text{enc}} = +2.0 \text{ nC} - 5.0 \text{ nC} + 1.0 \text{ nC} = -2.0 \text{ nC}. \quad (12.7)$$

The outside bead does not contribute to the net flux through the closed surface. Therefore

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{-2.0 \times 10^{-9} \text{ C}}{8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)} = -2.26 \times 10^2 \text{ N m}^2/\text{C}. \quad (12.8)$$

Interpretation. The negative sign means that the net electric flux is inward. It does not mean that the field points inward everywhere on the surface; it means that the inward contribution exceeds the outward contribution.

Guided checks

Check 1. A closed surface contains no net charge but lies in a strong external electric field. Must the electric field be zero on the surface?

Answer. No. The net flux is zero, but the field can be nonzero at many or all points.

Check 2. Does a negative enclosed charge make the flux magnitude negative?

Answer. No. Magnitude is non-negative. The signed flux is negative when the net flux is inward.

12.5 Gauss’s law for magnetism

The magnetic counterpart of Gauss’s law is

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0. \quad (12.9)$$

This equation says that the net magnetic flux through any closed surface is zero. In the classical theory used here, there are no isolated magnetic charges acting as sources or sinks of $\mathbf{B}$. Magnetic

field lines form closed loops or extend in such a way that every line entering a closed surface also leaves it.

Figure 12.3 illustrates flux continuity through a bar magnet. A bar magnet can appear to have a north pole and a south pole. However, cutting a bar magnet does not isolate a north pole from a south pole. Instead, each smaller piece has both poles. The flux continues through the material as well as through the surrounding space. Imagining a Gaussian surface through the magnet does not physically cut the sample or create a new pole at the mathematical surface. The equation does not say that magnetic fields are weak or absent. It says that their closed-surface flux has no net source term analogous to electric charge.

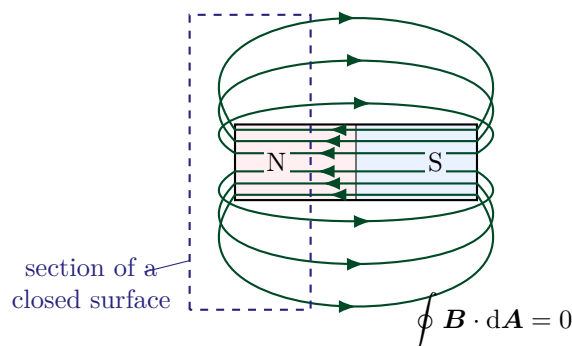


Figure 12.3: Continuous magnetic field loops, shown schematically. Outside the magnet the direction is N to S, and inside it is S to N. A closed Gaussian surface surrounding the N end also cuts the internal return flux. Its total flux is zero; the mathematical surface creates no new physical pole.

Key message

Gauss's law for magnetism is the mathematical statement that $\mathbf{B}$ has no isolated classical magnetic-charge sources. North and south poles of ordinary magnets are not separable magnetic charges in the way positive and negative electric charges can be separated.

Guided checks

Check 1. If a closed surface surrounds the north end of a bar magnet, is the net magnetic flux through that surface positive?

Answer. No. Some field lines leave the surface and others enter it; the net flux is zero.

Check 2. Would the discovery of isolated magnetic monopoles require modifying Eq. (12.9)?

Answer. Yes. A magnetic charge density would appear as a source term for magnetic flux. This is outside the classical treatment used in this course.

12.6 Faraday's law and electric circulation

Faraday's law in integral form is

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}. \quad (12.10)$$

Here C is a fixed closed curve and Φ_B is the magnetic flux through a fixed surface S whose boundary is C . The loop orientation and surface normal obey the right-hand rule. The minus sign is the mathematical expression of Lenz's law: the induced effect opposes the change in flux that produces it.

In electrostatics the electric field is conservative and $\oint \mathbf{E} \cdot d\boldsymbol{\ell} = 0$. Faraday's law shows that this statement is no longer true when the magnetic flux changes with time. A time-dependent magnetic

field can produce a circulating, nonconservative electric field even in empty space and even when no conducting wire is present. A wire placed along the loop can then guide charges and produce an induced current, but the induced electric field is the deeper object.

Caution

Do not identify Faraday's law only with a conducting loop. The conducting loop is a useful detector of the induced electric field. The field itself can exist in space whether or not the loop is made of metal.

Example 12.3: Induced electric field around a changing uniform magnetic field

A circular region of radius $R = 5.0$ cm contains a spatially uniform magnetic field perpendicular to the page. The field points out of the page ($+z$) and increases at the rate $dB/dt = 0.30$ T/s. Assuming cylindrical symmetry, find the magnitude of the induced electric field at radius $r = 3.0$ cm inside the region.

Model. Choose a circular path of radius $r < R$. Cylindrical symmetry makes the induced electric field tangential. Write its signed counterclockwise component as E_ϕ .

Solution. The line integral is

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = E_\phi(2\pi r). \quad (12.11)$$

The magnetic flux through the circular path is $\Phi_B = B\pi r^2$, so

$$E_\phi(2\pi r) = -\frac{d}{dt}(B\pi r^2) = -\pi r^2 \frac{dB}{dt}. \quad (12.12)$$

The magnitude is therefore

$$E = \frac{r}{2} \left| \frac{dB}{dt} \right| = \frac{0.030 \text{ m}}{2} (0.30 \text{ T/s}) = 4.5 \times 10^{-3} \text{ V/m}. \quad (12.13)$$

Interpretation. The result is small because the field is changing slowly and the loop radius is only a few centimetres. Since $E_\phi < 0$, the circulation is clockwise as seen from $+z$. A conducting loop would carry an induced current whose field opposes the increase of the outward flux. The unit check is $\text{m T/s} = \text{V/m}$.

Guided checks

Check 1. If the magnetic flux through a loop is constant but nonzero, does Faraday's law predict a circulating electric field?

Answer. No. The right-hand side depends on the time derivative of the flux, not on the flux itself.

Check 2. Can $\oint \mathbf{E} \cdot d\boldsymbol{\ell}$ be nonzero even if no battery is present?

Answer. Yes. A changing magnetic flux can produce nonconservative electric circulation.

12.6.1 Outside the changing-field region and moving circuits

The circular induction example also illustrates the difference between a field at the observation point and flux enclosed by a loop. For the same axially symmetric field region, a loop of radius $r > R$ encloses flux $B\pi R^2$. With counterclockwise E_ϕ positive,

$$E_\phi(r) = -\frac{1}{2} \frac{dB}{dt} \begin{cases} r, & r \leq R, \\ R^2/r, & r \geq R. \end{cases} \quad (12.14)$$

The electric field outside can be nonzero even though the idealized magnetic field is zero there. The two expressions match at R . The sharp edge is a long-solenoid approximation: a finite solenoid has fringing and return flux. It does not mean that magnetic field lines physically end at the circular boundary.

The fixed-loop condition in Eq. (12.10) matters. If a wire loop moves or deforms with local velocity $\mathbf{u}$, a charge in it experiences a magnetic as well as an electric force. The general flux rule for a moving material circuit is

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{A}. \quad (12.15)$$

Thus a rotating loop can have nonzero emf in a static magnetic field. Its motional term is essential; it would be incorrect to infer a nonzero circulation of the electric field alone from its changing geometric flux. Chapter 10 develops motional emf. Here all Maxwell loop equations use fixed paths unless explicitly stated otherwise.

12.7 Ampere's law before Maxwell

For steady currents, Ampere's law is

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}. \quad (12.16)$$

The current I_{enc} is the signed current of moving charge through the chosen oriented surface. In the wire examples it is conduction current; in a source-explicit treatment of matter it must include bound as well as free currents. The law works beautifully for steady-current systems with high symmetry, such as long straight wires, ideal solenoids and toroids. It expresses a direct relation between electric current and magnetic circulation.

The two surfaces in Fig. 12.4 expose a limitation of Eq. (12.16). It assumes a steady current distribution. In a time-dependent situation, especially in a charging capacitor, the same boundary curve can be paired with different surfaces. One surface cuts through a wire and contains conduction current. Another bulges between the capacitor plates and contains no conduction current. Ampere's original law would then give two different answers for the same line integral, which is impossible.

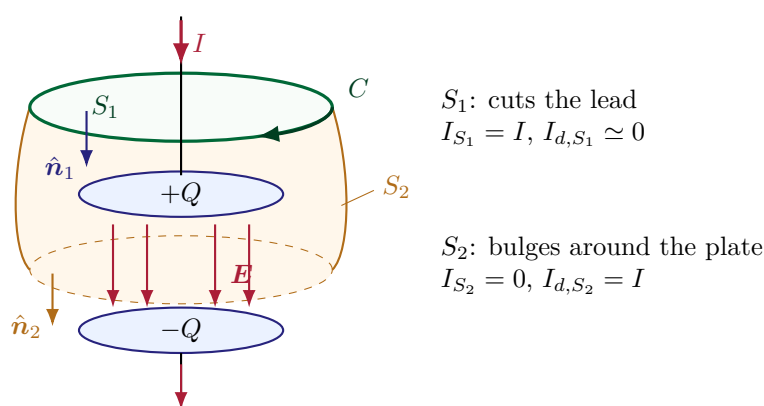


Figure 12.4: Two surfaces sharing the same oriented rim C . The flat disk S_1 is pierced by the lead. The bag-shaped S_2 passes around the positive plate and closes through the gap without crossing a conductor. In the ideal quasistatic model both yield $I_S + I_{d,S} = I$. The field arrows and surface normals point downward through the gap.

Key message

Ampere's original law is a steady-current law. Maxwell's correction is required when electric fields change with time. The correction is not optional; it is required for consistency with charge conservation and with the uniqueness of the line integral around a chosen loop.

12.8 Displacement current

Maxwell introduced the displacement current

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}. \quad (12.17)$$

This term has the same SI unit as current:

$$[\varepsilon_0 d\Phi_E/dt] = \left[\frac{\text{C}^2}{\text{N m}^2} \frac{\text{N m}^2/\text{C}}{\text{s}} \right] = \text{C/s} = \text{A}. \quad (12.18)$$

It is not a current of charges crossing empty space. It is the effective current associated with a changing electric flux, and it contributes to magnetic circulation.

For a vacuum parallel-plate capacitor with plate area A and negligible edge effects, the electric field between the plates is

$$E = \frac{Q}{\varepsilon_0 A}. \quad (12.19)$$

The electric flux between the plates is

$$\Phi_E = EA = \frac{Q}{\varepsilon_0}. \quad (12.20)$$

Therefore

$$I_d = \varepsilon_0 \frac{d}{dt} \left(\frac{Q}{\varepsilon_0} \right) = \frac{dQ}{dt} = I. \quad (12.21)$$

The displacement current between the plates equals the conduction current in the wire during charging, so the Ampere-Maxwell law gives the same magnetic circulation for any surface spanning the same boundary curve.

Example 12.4: Displacement current in a charging capacitor

A parallel-plate capacitor with circular plates is being charged by a constant current $I = 25 \text{ mA}$. Edge effects are neglected. What is the displacement current between the plates? If the plate area is $A = 40 \text{ cm}^2$, what is dE/dt between the plates?

Solution. For an ideal charging capacitor,

$$I_d = I = 25 \text{ mA} = 2.5 \times 10^{-2} \text{ A}. \quad (12.22)$$

The displacement current is also

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 A \frac{dE}{dt}, \quad (12.23)$$

so

$$\frac{dE}{dt} = \frac{I_d}{\varepsilon_0 A} = \frac{2.5 \times 10^{-2} \text{ A}}{(8.854 \times 10^{-12} \text{ F/m})(40 \times 10^{-4} \text{ m}^2)} = 7.1 \times 10^{11} \text{ V m}^{-1} \text{ s}^{-1}. \quad (12.24)$$

Interpretation. This large rate of change is not unusual because even moderate capacitor currents can change the field rapidly when the plate area is small.

Caution

The displacement current in the capacitor gap equals the conduction current in the ideal wire during charging, but this equality follows from charge conservation and the capacitor model. It does not mean that electrons jump across the empty gap.

Guided checks

Check 1. A capacitor is fully charged and disconnected. Is there displacement current between the plates?

Answer. No. The electric flux is then constant, so $d\Phi_E/dt = 0$.

Check 2. Why does I_d have the unit ampere?

Answer. Because $\epsilon_0\Phi_E$ has the unit coulomb, and its time derivative has the unit coulomb per second.

12.9 Ampere-Maxwell law

The completed circulation law for the magnetic field is

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}. \quad (12.25)$$

The first term contains the total current of moving charge through the chosen surface. The second contains the vacuum displacement current through that same surface. In a vacuum capacitor the former is zero in the gap. In matter, polarization and magnetization contribute bound currents and must not be omitted from the first term unless the law is rewritten in terms of auxiliary material fields.

The equation also reveals a deep symmetry. Faraday's law says that changing magnetic flux produces circulating electric field. Ampere-Maxwell's law says that changing electric flux produces circulating magnetic field. The signs and source structures are not perfectly symmetric because electric charge exists while isolated magnetic charge is absent in the classical theory. Still, the cross-coupling of changing fields is the essential step toward electromagnetic waves.

The circular field direction is shown in Fig. 12.5.

Example 12.5: Magnetic field inside a charging circular capacitor

A circular parallel-plate capacitor has plate radius $R = 6.0$ cm and is charged by a current $I = 0.80$ A. Neglect edge effects. Find the magnetic field magnitude at radius $r = 2.0$ cm between the plates.

Model. Choose a circular Amperian loop of radius $r < R$ centred on the capacitor axis. By cylindrical symmetry, $\mathbf{B}$ is tangential and has constant magnitude on the loop.

Solution. The electric flux through the loop is the fraction r^2/R^2 of the total flux between the plates. Therefore the displacement current enclosed by the loop is

$$I_{d,\text{enc}} = I \frac{\pi r^2}{\pi R^2} = I \frac{r^2}{R^2}. \quad (12.26)$$

Ampere-Maxwell's law gives

$$B(2\pi r) = \mu_0 I \frac{r^2}{R^2}, \quad (12.27)$$

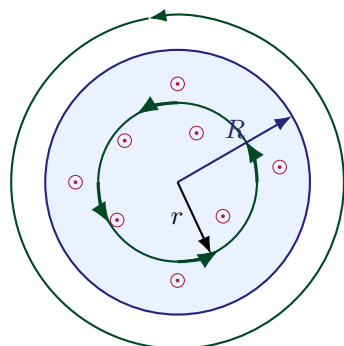
so

$$B = \frac{\mu_0 I r}{2\pi R^2}. \quad (12.28)$$

Substitution gives

$$B = \frac{(4\pi \times 10^{-7} \text{ T m/A})(0.80 \text{ A})(0.020 \text{ m})}{2\pi(0.060 \text{ m})^2} = 8.9 \times 10^{-7} \text{ T}. \quad (12.29)$$

Interpretation. The field increases linearly with r inside the plate radius because the enclosed displacement current grows as the area πr^2 while the circulation path length grows as $2\pi r$.



view from $+z$
 $\partial E_z / \partial t > 0$

$\odot$: increasing field toward viewer
 $B_\phi > 0$: counterclockwise

plate outline and concentric Amperian loops

Figure 12.5: End view of the ideal circular capacitor during increasing positive E_z . The dot symbols denote the direction of $\partial \mathbf{E} / \partial t$, and the green arrows show $\mathbf{B}$. Reversing the rate of change reverses the circulation.

Method

Using Ampere-Maxwell law in capacitor geometry.

1. Decide whether the chosen loop lies inside or outside the plate radius.
2. Use symmetry to argue that B is constant on a circular loop.
3. Compute the fraction of electric flux enclosed by the loop.
4. Replace $\epsilon_0 d\Phi_E / dt$ by the enclosed displacement current.
5. Solve $B(2\pi r) = \mu_0 I_{d, \text{enc}}$.

12.9.1 The complete radial profile and its limits

Let $+z$ point from the positively charging plate toward the negative plate. Take positive $I = dQ/dt$ to increase the positive plate charge, and positive B_ϕ to be counterclockwise as seen from $+z$. For the uniform vacuum gap model, the signed displacement-current density is

$$J_{d,z} = \epsilon_0 \frac{\partial E_z}{\partial t} = \frac{I}{\pi R^2}. \quad (12.30)$$

Inside the plate radius, the enclosed displacement current is $I r^2 / R^2$. Outside it, the idealized surface encloses the full current I . The two calculations combine as

$$B_\phi(r, t) = \frac{\mu_0 I(t)}{2\pi} \begin{cases} r/R^2, & 0 \leq r \leq R, \\ 1/r, & r \geq R. \end{cases} \quad (12.31)$$

A negative charging current reverses B_ϕ ; it does not produce a negative field magnitude. At the axis, symmetry requires $B_\phi = 0$. At the ideal plate edge both branches give $\mu_0 I / (2\pi R)$. Figure 12.6 shows how the competition between enclosed area and path length sets the profile.

The derivation has stronger assumptions than circular plates alone. It requires a small separation compared with plate radius, a nearly uniform gap field, an approximately axial feed and a sufficiently

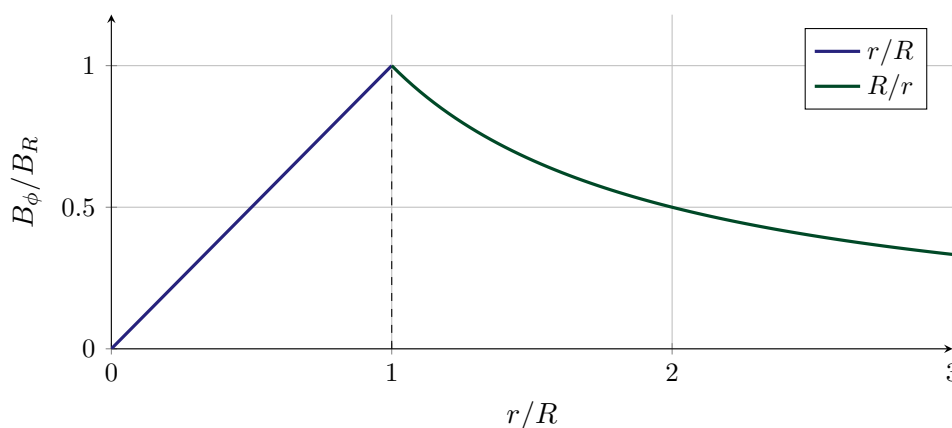


Figure 12.6: Calculated normalized profile for a positive current, with $B_R = \mu_0 I / (2\pi R)$. This is the uniform-gap, cylindrical, quasistatic model, not a full far-field solution for a finite capacitor and its return circuit.

symmetric return path. A nearby return wire can change the field on the Amperian circle. The integral law remains exact, but the step $\oint \mathbf{B} \cdot d\mathbf{\ell} = 2\pi r B_\phi$ would then fail.

There is also a time-scale condition. If L_{sys} is the relevant size of the circuit and the drive varies on time scale τ , a quasistatic approximation requires $L_{\text{sys}}/c \ll \tau$. For sinusoidal excitation this is $\omega L_{\text{sys}}/c \ll 1$. A spatially uniform $E_z(t)$ with the expression above is a leading approximation, not generally an exact time-dependent solution of both curl equations: a changing B_ϕ also requires a small spatial correction to $\mathbf{E}$ through Faraday's law. Arbitrarily rapid charging invalidates the model even if its algebra is performed correctly.

Example 12.6: A voltage-driven capacitor: current, phase and field

A vacuum capacitor has circular plates of radius 4.0 cm, separation 1.0 mm and prescribed voltage $V(t) = 100 \sin(2\pi \times 10^5 t)$ V. Find its capacitance, displacement-current amplitude, and magnetic-field amplitude at $r = 2.0$ cm. Assume a sufficiently symmetric feed and negligible fringing.

Strategy. The voltage determines $E_z = V/d$. Differentiating this field gives the displacement current; integrating it over the area inside the loop gives B_ϕ .

$$C = \frac{\epsilon_0 \pi R^2}{d} = 4.45 \times 10^{-11} \text{ F}, \quad (12.32)$$

$$I_d(t) = C \frac{dV}{dt} = \omega C V_0 \cos(\omega t), \quad (12.33)$$

$$I_{d,0} = (2\pi \times 10^5)(4.45 \times 10^{-11})(100) = 2.80 \times 10^{-3} \text{ A}, \quad (12.34)$$

$$B_0(r) = \frac{\mu_0 I_{d,0} r}{2\pi R^2} = 7.0 \times 10^{-9} \text{ T}. \quad (12.35)$$

The current and B_ϕ lead the voltage and electric field by one-quarter cycle. At maximum voltage the electric field is large but stationary instantaneously, so the ideal displacement current and the associated magnetic field vanish.

Checks. The plate ratio $d/R = 0.025$ supports the parallel-plate approximation away from the edge. Taking $L_{\text{sys}} = 0.10$ m gives $\omega L_{\text{sys}}/c \simeq 2.1 \times 10^{-4}$. The field is only a few nanotesla, so a direct measurement would require careful rejection of external fields and lead pickup. The large E/B ratio at this position is consistent: a capacitor near field is not a vacuum plane wave, and $E = cB$ does not apply.

12.9.2 One surface at a time

For a loop encircling a charging lead, suppose the wire-cutting surface has $I_{S_1} = I$ and negligible electric-flux derivative. A deformed surface with the same rim passes through the gap and has $I_{S_2} = 0$ and $I_{d,S_2} = I$. Both give circulation $\mu_0 I$, not $2\mu_0 I$. A third surface may cut only part of the relevant charge current and part of the changing electric flux. It is the complete sum on that one surface which is invariant. The general proof follows in Section 12.11.

12.9.3 When the electric field is not uniform

The relation $I_{d,\text{enc}} = I_d r^2 / R^2$ follows from a uniform displacement-current density. It is not a universal area rule. For an axially symmetric electric field $E_z(s, t)$ in a vacuum region, use an annulus of area $dA = 2\pi s ds$ to sum the flux:

$$I_{d,\text{enc}}(r, t) = 2\pi\epsilon_0 \int_0^r \frac{\partial E_z(s, t)}{\partial t} s ds, \quad (12.36)$$

$$B_\phi(r, t) = \frac{\mu_0\epsilon_0}{r} \int_0^r \frac{\partial E_z(s, t)}{\partial t} s ds. \quad (12.37)$$

The second line assumes no charge current crosses the chosen disk and that the full arrangement permits the stated magnetic symmetry. The integral weights the local field rate by annular area. One value of the total current constrains the field outside the entire source region in this ideal symmetry, but it does not uniquely determine the interior profile.

Example 12.7: The same total current with a different radial profile

In a quasistatic axisymmetric vacuum-field model, suppose that the instantaneous field rate is $\partial_t E_z = a(1 - s^2/R^2)$ for $0 \leq s \leq R$ and negligible beyond R . Derive the enclosed displacement current and magnetic field. Compare them with a uniform profile carrying the same total I_d . **Integrate the flux rate.** Equation (12.37) gives

$$\begin{aligned} I_{d,\text{enc}} &= 2\pi\epsilon_0 a \left(\frac{r^2}{2} - \frac{r^4}{4R^2} \right), \\ I_d(R) &= \frac{\pi\epsilon_0 a R^2}{2}, \\ B_\phi(r) &= \mu_0\epsilon_0 a \left(\frac{r}{2} - \frac{r^3}{4R^2} \right) = \frac{\mu_0 I_d}{2\pi R} (2x - x^3), \quad x = \frac{r}{R} \leq 1. \end{aligned}$$

For $r > R$, the total current is enclosed and $B_\phi = \mu_0 I_d / (2\pi r)$. At $r = R/2$, the nonuniform profile encloses fraction $2x^2 - x^4 = 7/16$ of the total current, compared with $1/4$ for a uniform profile. Its magnetic field there is larger by a factor $7/4$ for the same total current.

Find the maximum. The interior derivative is proportional to $2 - 3x^2$, so the maximum occurs at $x = \sqrt{2/3} \simeq 0.816$, slightly inside the source edge. At the edge, the normalized interior slope is $2 - 3 = -1$, matching the exterior slope of $1/x$. A sharp slope change at R is therefore a feature of the uniform truncated profile, not an unavoidable consequence of Ampere–Maxwell.

Interpretation and validity. This is a prescribed instantaneous flux-rate profile for testing the integral law. It is not claimed to be the field of ideal flat equipotential plates, which gave an approximately uniform field, nor a full time-dependent solution without specifying compatible sources and the other Maxwell equations. Real nonuniform profiles require consistent boundary and initial data.

The integral can also be read in reverse. If $B_\phi(r, t)$ is known smoothly and there is no conduction

current through the disk, differentiate rB_ϕ :

$$\frac{\partial E_z}{\partial t} = \frac{1}{\mu_0 \varepsilon_0 r} \frac{\partial}{\partial r} (rB_\phi). \quad (12.38)$$

This reconstructs a local field rate from the circulation profile within the model. Near $r = 0$, use the regular limiting form rather than dividing noisy data by a small radius. If $B_\phi = kr + O(r^3)$, the finite central rate is $2k/(\mu_0 \varepsilon_0)$. Differentiation also amplifies measurement noise, so this inverse operation is usually less robust than integrating a known electric-field profile.

Method

An integral law is not automatically a field formula. First determine whether symmetry fixes the field direction and makes its tangential or normal component constant on the integration path or surface. Next integrate only the enclosed source. If symmetry is absent, Maxwell's law still constrains the integral, but pulling a single field magnitude outside it is unjustified. A source total and a loop length alone do not reconstruct a general vector field.

12.10 The four Maxwell equations in integral form

For fixed integration surfaces and paths, the source-explicit Maxwell equations are the following. Here q_{enc} and I_{enc} include all charge and charge-transport current, respectively. They include bound sources in matter; they are not automatically just the free charge and wire current:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\varepsilon_0}, \quad (12.39)$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0, \quad (12.40)$$

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}, \quad (12.41)$$

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}. \quad (12.42)$$

Law and geometry	Source term and physical meaning
Electric Gauss law: closed surface	$q_{\text{enc}}/\varepsilon_0$: net enclosed charge determines net electric flux.
Magnetic Gauss law: closed surface	Zero: magnetic flux has no magnetic-charge source in this theory.
Faraday law: fixed loop	$-d\Phi_B/dt$: changing magnetic flux determines electric circulation.
Ampere–Maxwell law: fixed loop	$\mu_0(I_{\text{enc}} + I_d)$: charge current and changing electric flux determine magnetic circulation.

The Lorentz force law, $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, describes how these fields act on a charge. Maxwell's equations determine field constraints and evolution; source motion, material response, and initial or boundary data complete a physical problem.

Key message

The two Gauss laws are about closed-surface flux. The Faraday and Ampere–Maxwell laws are about closed-loop circulation. Remembering this distinction prevents most first-pass mistakes.

Example 12.8: Choosing the relevant Maxwell equation

For each situation, identify the Maxwell equation that most directly describes the phenomenon.

- (a) A point charge produces net electric flux through a surrounding sphere.
- (b) A changing magnetic field drives current in a conducting ring.
- (c) A steady current in a long straight wire produces circular magnetic field lines.
- (d) A closed surface around a magnet has zero net magnetic flux.
- (e) The field in the gap of a charging capacitor contributes to magnetic circulation.

Solution.

- (a) Gauss's law for electricity.
- (b) Faraday's law.
- (c) Ampere-Maxwell law, reducing to Ampere's law when fields are steady.
- (d) Gauss's law for magnetism.
- (e) Ampere-Maxwell law through the displacement-current term.

Interpretation. This exercise is simple but important. Before doing algebra, identify the physical operation: electric flux, magnetic flux, electric circulation or magnetic circulation.

12.11 Charge conservation and why Maxwell's correction is necessary

Charge conservation can be expressed without vector calculus. Let V be a fixed volume, Q_V its charge, and ∂V its outward-oriented closed boundary. The net outward charge current is

$$I_{\text{out}} = \oint_{\partial V} \mathbf{J} \cdot d\mathbf{A} = -\frac{dQ_V}{dt}. \quad (12.43)$$

A positive outward current reduces the charge inside. Gauss's law, differentiated for this fixed volume, gives

$$\epsilon_0 \frac{d}{dt} \oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{dQ_V}{dt}. \quad (12.44)$$

Adding these equations yields

$$\oint_{\partial V} \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{A} = 0. \quad (12.45)$$

The combination in parentheses has no net outward flux through any closed surface, even when charge accumulates inside. The displacement term compensates for the nonzero net outward charge current; it does not replace charge conservation.

Now take two fixed surfaces S_1 and S_2 with the same oriented boundary C . Give both surfaces the normal associated with the same traversal of C . Together, S_1 and the oppositely oriented S_2 form a closed surface. Equation (12.45) therefore implies

$$I_{S_1} + I_{d,S_1} = I_{S_2} + I_{d,S_2}. \quad (12.46)$$

The individual terms may differ, but their sum cannot. This is the precise statement needed by Ampere–Maxwell's law. On the idealized wire-cutting surface the charge-current contribution dominates; on the gap surface the displacement contribution supplies the same result. We never add the wire current on one surface to the displacement current on a different surface.

This argument proves that the Maxwell term restores consistency with charge conservation and surface independence. Conservation alone does not uniquely determine every possible field equation; the physical theory is also established by experiment. A local derivation using divergence and curl is reserved for the advanced enrichment.

Theoretical point 12.1: Consistency with conservation laws

A good field equation is not only a rule for calculating fields. It must also respect conservation laws. Maxwell's correction to Ampere's law is one of the clearest examples in introductory physics where mathematical consistency and physical conservation force a new term into an equation.

Guided checks

Check 1. In a charging capacitor, is charge accumulating on the plates?

Answer. Yes. The plate charge changes with time, so the electric field and electric flux between the plates change.

Check 2. Why is a current-only version of Ampere's law suspicious in a capacitor gap?

Answer. Because no conduction charge crosses the gap, but the magnetic field around the circuit cannot abruptly vanish depending on which spanning surface we choose.

12.12 The route to electromagnetic waves

The time-dependent curl equations form a coupled evolution system. In a source-free region they admit propagating disturbances of the electric and magnetic fields. This does not mean that one field switches on first and creates the other after a delay, nor that every time-dependent field is a freely travelling wave. In a vacuum plane wave the two fields are simultaneous parts of one solution, with their spatial derivatives tied to each other's time derivatives.

In vacuum, where $q_{\text{enc}} = 0$ and $I_{\text{enc}} = 0$, the equations become

$$\oint \mathbf{E} \cdot d\mathbf{A} = 0, \quad \oint \mathbf{B} \cdot d\mathbf{A} = 0, \quad (12.47)$$

$$\oint \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}, \quad \oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}. \quad (12.48)$$

The advanced section sketches how the differential form leads to a wave equation with speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \quad (12.49)$$

This is the speed of light in vacuum. The equality of this electromagnetic wave speed with the measured speed of light was one of the great unifications of physics.

Pedagogical boundary

The full electromagnetic-wave chapter will develop wave propagation, polarization, energy transport, radiation pressure, reflection and refraction. Here the purpose is more modest: to show why the displacement-current term is the conceptual gateway from circuits and induction to light as an electromagnetic wave.

Example 12.9: Checking the dimensions of the electromagnetic wave speed

Show that $1/\sqrt{\mu_0 \epsilon_0}$ has the dimension of speed.

Solution. Use $[\mu_0] = \text{N/A}^2$ and $[\epsilon_0] = \text{C}^2/(\text{N m}^2)$. Since $1 \text{ A} = 1 \text{ C/s}$,

$$[\mu_0 \epsilon_0] = \frac{\text{N}}{\text{A}^2} \frac{\text{C}^2}{\text{N m}^2} = \frac{\text{C}^2}{\text{A}^2 \text{m}^2} = \frac{\text{s}^2}{\text{m}^2}. \quad (12.50)$$

Therefore

$$\left[\frac{1}{\sqrt{\mu_0 \epsilon_0}} \right] = \text{m/s}. \quad (12.51)$$

Interpretation. Dimensional consistency alone does not prove the wave equation, but it confirms that the constants in Maxwell's equations naturally define a speed.

12.13 Magnetic dipoles and magnetic moment

Magnetic matter is best approached by first recalling the magnetic dipole. A small current loop with current I and area vector $\mathbf{A}$ has magnetic dipole moment

$$\mathbf{m} = I\mathbf{A}. \quad (12.52)$$

The direction is given by the right-hand rule: curl the fingers in the direction of the current and the thumb points along $\mathbf{m}$. A magnetic dipole in a magnetic field experiences a torque

$$\boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}, \quad (12.53)$$

with potential energy

$$U = -\mathbf{m} \cdot \mathbf{B}. \quad (12.54)$$

The energy is lowest when the dipole moment aligns with the magnetic field.

At the microscopic level, atoms and electrons can carry magnetic moments. Some moments arise from orbital motion, and some are intrinsic spin magnetic moments. A full explanation is quantum mechanical, but the classical current-loop picture gives a useful introductory language: magnetic materials respond because many microscopic magnetic moments reorient, oppose, or collectively align in applied fields.

Example 12.10: Torque on a small current loop

A rectangular loop has area $A = 3.0 \times 10^{-4} \text{ m}^2$ and carries current $I = 0.20 \text{ A}$. Its magnetic moment makes an angle 40° with a uniform field $B = 0.50 \text{ T}$. Find the torque magnitude and the energy difference between this orientation and perfect alignment.

Solution. The magnetic moment magnitude is

$$m = IA = (0.20)(3.0 \times 10^{-4}) = 6.0 \times 10^{-5} \text{ A m}^2. \quad (12.55)$$

The torque magnitude is

$$\tau = mB \sin \theta = (6.0 \times 10^{-5})(0.50) \sin 40^\circ = 1.9 \times 10^{-5} \text{ N m}. \quad (12.56)$$

The potential energy at angle θ is $U(\theta) = -mB \cos \theta$, and at alignment it is $U(0) = -mB$. The energy difference is

$$\Delta U = U(\theta) - U(0) = mB(1 - \cos \theta) = (6.0 \times 10^{-5})(0.50)(1 - \cos 40^\circ) = 7.0 \times 10^{-6} \text{ J}. \quad (12.57)$$

Interpretation. The field tends to rotate the loop toward alignment, reducing the magnetic potential energy.

Guided checks

Check 1. When is the torque on a magnetic dipole in a uniform field zero?

Answer. When $\mathbf{m}$ is parallel or antiparallel to $\mathbf{B}$. Parallel alignment is stable; antiparallel alignment is unstable.

Check 2. Does a current loop in a uniform magnetic field necessarily experience a net force?

Answer. No. It can experience a torque with zero net force if the field is uniform.

12.13.1 Why a uniform field gives torque without a net force

For a small planar loop in a spatially uniform field, the force on a wire element is $d\mathbf{F} = I d\mathbf{\ell} \times \mathbf{B}$. Since the field can be taken outside the line integral,

$$\mathbf{F}_{\text{net}} = I \left(\oint d\mathbf{\ell} \right) \times \mathbf{B} = 0. \quad (12.58)$$

The displacement vectors around a closed loop sum to zero. The forces do not vanish individually: opposite sides can experience equal and opposite forces at different positions. Those forces form a couple.

For a rectangular loop with sides a and b , take an axis through the centre parallel to the b -sides, which are perpendicular to $\mathbf{B}$. Those two sides carry forces of magnitude IbB , separated by perpendicular lever arm $a \sin \theta$, where θ is the angle between the loop normal and the field. Their torque is $IabB \sin \theta = IAB \sin \theta$. The remaining pair supplies no torque about that axis. This gives the magnitude of $\mathbf{m} \times \mathbf{B}$ and makes clear why the relevant area is the area enclosed by the loop, not the wire's cross-sectional area. For N identical turns, the torques add and $m = NIA$.

The signed generalized torque that reduces θ is $\tau_\theta = -mB \sin \theta$. Using $\tau_\theta = -dU/d\theta$, integration gives $U = -mB \cos \theta$ up to a constant. At $\theta = 0$, a small angular displacement increases energy quadratically; at $\theta = \pi$ it decreases energy. Both orientations have zero instantaneous torque, but only the first is stable in this externally imposed field model.

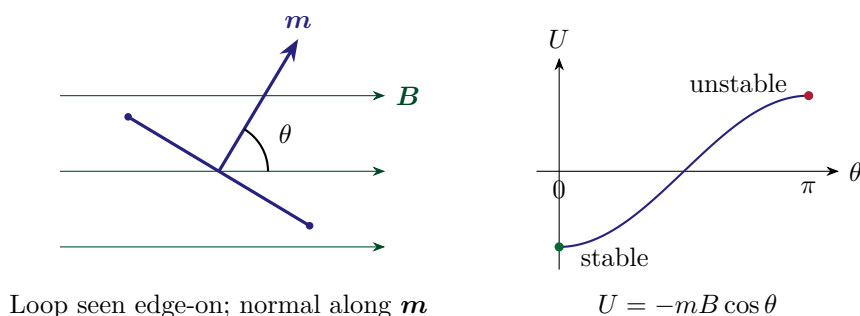


Figure 12.7: The magnetic moment is normal to the loop plane. The orientation energy distinguishes stable alignment from unstable anti-alignment even though both have zero torque. The right-hand plot is schematic with a common positive scale mB .

Figure 12.7 also separates two kinds of motion. Rotation changes θ and can lower energy even when the net translational force is zero. Translation in a nonuniform field can change B itself. For a fixed permanent dipole whose orientation is held constant while it samples a slowly varying external field,

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}). \quad (12.59)$$

In one dimension, with $\mathbf{m}$ parallel to the locally positive axial field, this reduces to $F_z = m dB_z/dz$. It is a gradient force. A locally uniform approximation may therefore be sufficient for torque but insufficient to predict translational attraction to a magnet.

Example 12.11: A small permanent dipole in a field gradient

A small rigid magnet has fixed moment $m = 2.0 \times 10^{-2} \text{ A m}^2$ and mass 5.0 g. On the axis of an external field, its moment is constrained to point along $+z$. Near its position the axial field is $B_z = B_* + gz$, with $g = 0.80 \text{ T/m}$. Estimate its axial force and compare its magnitude with the weight.

Model. The magnet is small compared with the spatial scale of the field, and its moment is assumed independent of position. The expression specifies the field on an axis, not a globally uniform transverse field with $\partial_z B_z \neq 0$; off-axis components are needed to satisfy magnetic Gauss law.

Calculation. $F_z = mg = (2.0 \times 10^{-2})(0.80) = 1.6 \times 10^{-2} \text{ N}$. To avoid confusing the moment symbol with mass, write the weight explicitly as $(0.0050 \text{ kg})(9.81 \text{ m/s}^2) = 4.9 \times 10^{-2} \text{ N}$. The magnetic force is about 0.33 of the weight. It points toward increasing B_z . This comparison alone does not establish stable levitation; full spatial force balance and stability are separate questions.

Limit. An induced moment that changes with field does not generally have the same potential energy as a fixed permanent dipole. Its response must be included when integrating the work. Substituting a field-dependent m into $-mB$ without that analysis can give a factor-of-two error.

12.13.2 From individual dipoles to a measured material moment

A magnetization is a vector average over a volume containing many microscopic constituents. If n identical moments per volume have an average projection $\langle m_z \rangle$ along a measurement axis, then $M_z = n\langle m_z \rangle$. A sample with large moments can have small M_z because their projections cancel. The macroscopic average therefore conveys different information from the size of one atomic moment.

For a uniformly magnetized specimen, the total moment is $\mathbf{m}_{\text{sample}} = V\mathbf{M}$. In a uniform external field, its torque can be used to infer a component of that moment; in a calibrated gradient, a force measurement can do so. Such measurements require knowledge of orientation, background forces and the internal magnetic state. Ferromagnetic samples may rotate their domains while being measured, so a fixed-moment formula needs justification. The domain model later in the chapter illustrates one source of that complication.

12.14 Magnetization and the magnetic response of matter

The magnetization $\mathbf{M}$ of a material is magnetic dipole moment per unit volume:

$$\mathbf{M} = \frac{\Delta \mathbf{m}}{\Delta V} \quad (12.60)$$

in the continuum limit where the volume is large compared with atomic scales but small compared with the macroscopic sample. Magnetization is measured in amperes per metre. It describes how strongly the microscopic moments in a material are aligned on average.

In a simple introductory classification, magnetic materials are divided into diamagnetic, paramagnetic and ferromagnetic materials. Diamagnetic materials develop a weak magnetization opposite to an applied magnetic field. Paramagnetic materials develop a weak magnetization in the direction of an applied field because permanent microscopic moments partially align. Ferromagnetic materials can exhibit strong collective alignment and can remain magnetized even after the applied field is removed.

Figure 12.8 compares net response directions; the auxiliary field $\mathbf{H}$ used there is defined immediately below.

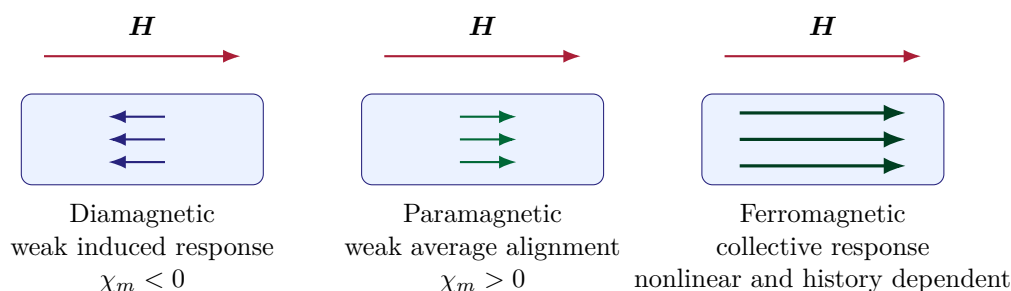


Figure 12.8: Qualitative directions of net magnetization. The arrows inside each sample represent the average $\mathbf{M}$, not a literal alignment of every atomic moment. The simple dia- and paramagnetic pictures assume weak, isotropic response; ferromagnetic response cannot generally be represented by a single constant χ_m .

Definition 12.1: Magnetization

The magnetization $\mathbf{M}$ is the magnetic dipole moment per unit volume of a material. It is a macroscopic field obtained by averaging microscopic magnetic moments over a small but not atomically small volume.

Key message

Magnetic matter is not explained by visible currents in wires. It is explained by microscopic magnetic moments and their averaged alignment, opposition or collective ordering inside the material.

12.14.1 Defining susceptibility before using its sign

The field within a specimen is not generally the field that would be present if the specimen were removed. The specimen itself contributes to the field. To describe material response consistently, define the auxiliary field

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \quad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}). \quad (12.61)$$

Both $\mathbf{H}$ and $\mathbf{M}$ have unit A/m, whereas $\mathbf{B}$ has unit tesla. The definition does not say that $\mathbf{H}$ is always the externally applied field. They coincide in some useful ideal geometries; sample shape can make them quite different.

For a linear, isotropic, reversible response at fixed temperature we define

$$\mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H}, \quad \mu = \mu_0(1 + \chi_m), \quad \mu_r = 1 + \chi_m. \quad (12.62)$$

The sign of χ_m refers to the alignment of $\mathbf{M}$ with the internal $\mathbf{H}$. For an ordinary weakly responding sample it also describes its alignment relative to the externally imposed field to good approximation. These statements assume a scalar susceptibility; in an anisotropic medium the response can depend on direction.

For a long solenoid completely filled by such a material, end effects are negligible and $\mathbf{H} \simeq n\mathbf{I}$, where n is the number of turns per unit length. Hence $\mathbf{B} \simeq \mu_0(1 + \chi_m)n\mathbf{I}$. This is a useful special case, not permission to multiply every vacuum field by a universal μ_r . The advanced section derives the corresponding field equations and explains the role of sample shape.

Response	Physical origin in the introductory model	Macroscopic signature
Diamagnetic	Field-induced electronic response	Small negative χ_m for ordinary diamagnets; no remanence.
Paramagnetic	Partial alignment of permanent moments, competing with thermal disorder	Positive weak-field response; typically no remanence at equilibrium.
Ferromagnetic	Collective ordering caused by exchange interactions	Spontaneous local magnetization below T_C ; domains, saturation and possible hysteresis.

These are principal classes, not an exhaustive taxonomy. Antiferromagnets have oppositely ordered sublattices whose moments can cancel; ferrimagnets have unequal opposing sublattice moments and a net magnetization. Many ferrites used in cores are ferrimagnetic. Their high electrical resistivity can help suppress eddy currents, a property separate from their magnetic order.

Example 12.12: Magnetic susceptibility from measured magnetization

In a linear material the magnetization is measured as $M = 120 \text{ A/m}$ when the auxiliary field magnitude is $H = 4.0 \times 10^4 \text{ A/m}$. Estimate the magnetic susceptibility and classify the sign of the response.

Solution. For a linear isotropic material,

$$M = \chi_m H. \quad (12.63)$$

Therefore

$$\chi_m = \frac{M}{H} = \frac{120}{4.0 \times 10^4} = 3.0 \times 10^{-3}. \quad (12.64)$$

The susceptibility is positive, so the response is paramagnetic-like in sign if the material is genuinely linear in this range.

Interpretation. The numerical value is small. A strong ferromagnetic response would not usually be described by a small constant susceptibility over a wide field range.

12.15 Diamagnetism

Diamagnetism is a weak magnetic response in which the induced magnetization opposes the applied magnetic field. It appears in all materials, although it can be hidden by stronger paramagnetic or ferromagnetic effects. An applied field changes the electronic states and their orbital contribution to magnetism, producing an induced moment opposite to the field. This equilibrium response can persist in a static applied field; it must not be confused with transient eddy currents or described as a response only to dB/dt .

Diamagnetic susceptibility is negative in the linear regime:

$$\chi_m < 0. \quad (12.65)$$

For most ordinary diamagnets $|\chi_m|$ is small. The material is weakly repelled from regions of stronger magnetic field. This weak repulsion is usually not obvious in everyday experience unless strong fields, sensitive balances or special materials are used.

Caution

Diamagnetism does not mean that the material has no magnetic response. It means that the induced response is opposite to the applied field and usually weak. A material can be diamagnetic even if it is not a permanent magnet.

12.16 Paramagnetism

Paramagnetic materials contain microscopic magnetic moments that tend to align with an applied field. Thermal motion randomises the moment directions, so the alignment is only partial unless the field is strong or the temperature is low. In the simplest linear regime,

$$\chi_m > 0 \quad (12.66)$$

and small. The magnetization is along the applied field.

A common introductory model is Curie's law,

$$\chi_m = \frac{C}{T}, \quad (12.67)$$

where C is a material-dependent Curie constant and T is the absolute temperature. This relation is not universal, but it expresses an important competition: applied field tends to order the moments, while thermal agitation tends to disorder them.

Example 12.13: A Curie-law estimate

A paramagnetic salt has susceptibility $\chi_m = 3.0 \times 10^{-3}$ at $T = 300$ K in a temperature range where Curie's law is a reasonable approximation. Estimate its susceptibility at 150 K.

Solution. Curie's law gives $\chi_m T = C$, so

$$\chi_m(150 \text{ K}) = \chi_m(300 \text{ K}) \frac{300}{150} = (3.0 \times 10^{-3})(2) = 6.0 \times 10^{-3}. \quad (12.68)$$

Interpretation. Lower temperature reduces thermal disorder and permits stronger alignment in the same applied field. The result should not be extrapolated blindly to very low temperature, where interactions and saturation may matter.

Guided checks

Check 1. Which response has negative susceptibility in the simplest linear description?

Answer. Diamagnetism.

Check 2. Why does temperature matter for paramagnetism?

Answer. Thermal agitation competes with the field's tendency to align microscopic moments.

12.16.1 Why the field does not align every atomic moment

For a permanent moment of magnitude m , the energy separation between parallel and antiparallel orientations in a field of magnitude B is $2mB$. Compare this with the thermal energy scale $k_B T$. The dimensionless ratio

$$x = \frac{mB}{k_B T} \quad (12.69)$$

measures the importance of magnetic alignment for a weakly interacting moment. When $x \ll 1$, thermal fluctuations strongly compete with the field and the average polarization is small. This does

not mean that a fixed fraction of the atoms is perfectly aligned while all the others are unaffected. It is an average over an orientation or quantum-state distribution.

Example 12.14: An atomic energy-scale check

Take $m = 9.27 \times 10^{-24} \text{ A m}^2$, approximately one Bohr magneton, $B = 1.0 \text{ T}$ and $T = 300 \text{ K}$. Estimate x and the parallel–antiparallel energy difference.

Solution. With $k_B \simeq 1.381 \times 10^{-23} \text{ J/K}$,

$$k_B T = 4.14 \times 10^{-21} \text{ J}, \quad (12.70)$$

$$2mB = 1.85 \times 10^{-23} \text{ J}, \quad (12.71)$$

$$x = 2.24 \times 10^{-3}. \quad (12.72)$$

The magnetic orientation energy is much smaller than the thermal scale. Strong alignment of independent atomic moments at room temperature is therefore not expected at this field. The estimate explains why a dilute paramagnet responds weakly, while the strong response of a ferromagnet requires collective interactions.

Model boundary. Spin is intrinsic angular momentum, not rotation of a tiny solid sphere. Atomic orbital states are not classical planetary trajectories. The dipole-energy model is useful without taking either picture literally. A two-state statistical model in the advanced enrichment turns this scale estimate into a quantitative magnetization law.

Curie’s law describes the low-field limit of a simple localized-moment model. It is not a law for every paramagnetic solid: conduction-electron paramagnetism and strong interactions can give different temperature dependences. A measured $\chi_m(T)$ is therefore a test of a model, not merely a curve to which C/T must be fitted.

12.17 Ferromagnetism, domains and hysteresis

Ferromagnetism is qualitatively different from ordinary dia- and paramagnetism. In a ferromagnet, interactions between neighbouring microscopic moments favour collective alignment. Below a characteristic Curie temperature, large groups of moments can align in domains. Each domain is strongly magnetized, but an unmagnetized sample can contain many domains oriented in different directions so that the net magnetization is small.

When an external magnetic field is applied, domains favourably oriented with the field can grow, and moments inside domains can rotate. When the field is removed, the material may retain part of its magnetization. This retained magnetization is called remanence. A reverse field may be needed to bring the magnetization back to zero; the required reverse field is related to the coercivity. The resulting B – H or M – H curve is hysteretic: the state depends on the history.

Figure 12.10 isolates the effect of changing domain volumes.

Example 12.15: Magnetization from partial domain alignment

A small ferromagnetic sample consists, in a simplified model, of two domain populations. A fraction 0.70 of the volume has magnetization magnitude $M_s = 8.0 \times 10^5 \text{ A/m}$ along $+x$, and the remaining fraction has the same magnitude along $-x$. Estimate the net magnetization.

Solution. The volume-averaged magnetization is

$$M_x = (0.70)M_s + (0.30)(-M_s) = 0.40M_s. \quad (12.73)$$

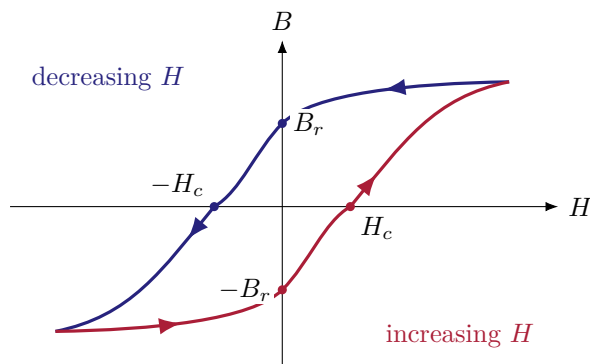


Figure 12.9: Schematic closed B - H hysteresis loop with its physical traversal. Remanent flux densities lie on the vertical axis; coercive fields defined by $B = 0$ lie on the horizontal axis. The curve is illustrative, not measured data or a quantitative constitutive model.

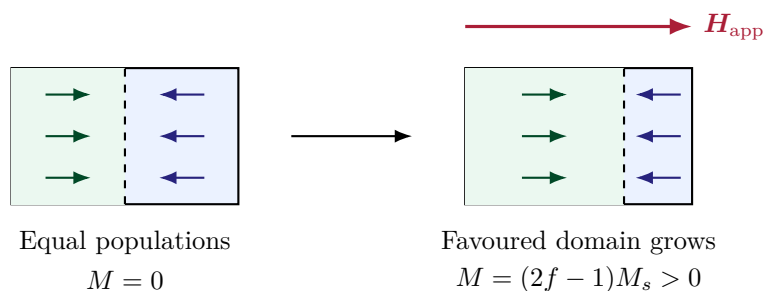


Figure 12.10: Two-domain volume model, with moments along $\pm x$. A field can move a wall so that the favourable volume fraction f grows. The drawing illustrates domain-wall motion; real domain patterns also involve rotation, anisotropy and pinning.

Thus

$$M_x = 0.40(8.0 \times 10^5) = 3.2 \times 10^5 \text{ A/m.} \quad (12.74)$$

Interpretation. Each domain is strongly magnetized, but the net sample magnetization is a volume-weighted vector average. Domain redistribution can therefore change macroscopic magnetization without changing the saturation magnetization inside each domain.

Example 12.16: Energy dissipated in a hysteresis cycle

A ferromagnetic core is cycled through a hysteresis loop. In a simplified measurement the area enclosed by the B - H loop is estimated as 120 J/m^3 per cycle. The core volume is $8.0 \times 10^{-5} \text{ m}^3$ and it is cycled at 50 Hz . Estimate the power dissipated as hysteresis loss.

Solution. The energy lost per unit volume per cycle is the loop area:

$$u_{\text{loss}} = 120 \text{ J/m}^3. \quad (12.75)$$

The energy lost per cycle in the whole core is

$$E_{\text{cycle}} = u_{\text{loss}} V = (120)(8.0 \times 10^{-5}) = 9.6 \times 10^{-3} \text{ J.} \quad (12.76)$$

At 50 cycles per second, the power loss is

$$P = f E_{\text{cycle}} = 50(9.6 \times 10^{-3}) = 0.48 \text{ W.} \quad (12.77)$$

Interpretation. Hysteresis loss converts electromagnetic energy into internal energy. Low-loss magnetic materials and suitable operating flux reduce hysteresis loss. Insulated laminations

primarily reduce the separate eddy-current contribution; they are not a general cure for hysteresis.

Guided checks

Check 1. Can a ferromagnet have domains and still show zero net magnetization?

Answer. Yes. Strongly magnetized domains can be oriented differently so that their macroscopic sum is small.

Check 2. What does remanence mean?

Answer. It is magnetization that remains after the applied field is reduced to zero.

Check 3. Why is hysteresis important in AC transformer cores?

Answer. Repeated cycling dissipates energy each cycle, producing heat and reducing efficiency.

12.17.1 Reading a hysteresis loop without confusing its axes

Figure 12.9 uses B on the vertical axis and H on the horizontal axis. Starting from a large positive field, decrease H along the upper branch. At $H = 0$ the remaining flux density is B_r , the remanent flux density. A reverse field $H = -H_c$ makes $B = 0$ on that branch. Continuing to negative field and then increasing it completes the loop. The arrows encode this history; the same value of H does not specify a unique B .

The coercive field defined by $B = 0$ need not equal the intrinsic coercive field defined by $M = 0$. Since $B = \mu_0(H + M)$, the condition $B = 0$ at nonzero H still allows nonzero M . Likewise, magnetization saturation is not a strict upper bound on B . When $M \simeq M_s$, increasing H still increases $B \simeq \mu_0(H + M_s)$ with differential slope close to μ_0 . A quoted engineering saturation flux density is an operating criterion for the knee of a curve, not an absolute ceiling imposed by Maxwell's equations.

Domains explain why local order and total magnetization differ. Exchange favours neighbouring moment alignment, while domain formation can reduce external stray-field energy at a cost in wall energy. Defects and anisotropy impede wall motion. Those irreversible changes produce history dependence. Above the Curie temperature, long-range spontaneous ferromagnetic order is lost; this does not mean that the exchange interaction itself abruptly vanishes or that all microscopic moments disappear.

12.17.2 Why the loop area is an energy density

Consider a thin toroidal core of volume $V_c = A\ell$, carrying a uniformly distributed winding of N turns. Neglect leakage, winding loss and spatial variation of the core field. Then $H\ell = NI$ and the applied terminal voltage needed to oppose the induced emf is $v = NA \, dB/dt$. The magnetic power delivered per core volume is

$$\frac{vi}{V_c} = \frac{NA(dB/dt)(H\ell/N)}{A\ell} = H \frac{dB}{dt}. \quad (12.78)$$

Over a closed cycle the magnetic state returns to its starting point, so the net supplied magnetic energy becomes loss:

$$w_{\text{loss}} = \oint H \, dB, \quad P_{\text{loss}} = fV_c w_{\text{loss}}. \quad (12.79)$$

The physical traversal gives positive work for a passive material; reversing the recorded point order reverses the signed integral. The magnitude is the enclosed geometric area. Its units are $A/m \times T = J/m^3$. For an M - H graph an extra factor μ_0 is needed, since $\oint H \, dB = \mu_0 \oint H \, dM$.

At sufficiently low frequency, the quasistatic loop characterizes hysteretic loss. A measured dynamic loop can also contain eddy-current and other rate-dependent contributions. Multiplying a measured dynamic area by frequency gives the corresponding total cycle loss within the measurement model; adding a separately estimated eddy-current loss to that result can double count. Even for nominal hysteresis loss, $P \propto f$ presumes an unchanged loop, temperature and flux excursion.

12.18 Engineering applications: geometry, frequency and material choice

A field law becomes an engineering model only after geometry, material response and operating conditions are specified. Three examples show what this means: a voltage-driven core, unwanted coupling between circuits, and low-frequency magnetic shielding. They use the same equations already developed, but require decisions about which idealizations remain adequate.

12.18.1 Flux swing and saturation in a transformer core

For a winding that links a nearly uniform core flux, the terminal voltage across the magnetic part of the winding is

$$v(t) = N \frac{d\Phi_B}{dt} \simeq NA \frac{dB}{dt}. \quad (12.80)$$

This is a passive terminal-voltage convention: the applied voltage balances the oppositely directed induced emf. Integrating gives

$$B(t) - B(t_0) = \frac{1}{NA} \int_{t_0}^t v(t') dt'. \quad (12.81)$$

The flux swing is therefore set by volt-seconds, not directly by permeability. Permeability and hysteresis determine the magnetizing current needed to produce that swing. A DC imbalance in the voltage integral can progressively shift the operating point toward saturation.

For a sinusoidal flux density $B(t) = B_{pk} \sin \omega t$, the voltage is $v(t) = NA\omega B_{pk} \cos \omega t$. Its rms value is

$$V_{rms} = \frac{2\pi}{\sqrt{2}} f N A B_{pk} \simeq 4.44 f N A B_{pk}. \quad (12.82)$$

The factor 4.44 is specific to a sinusoid. Using it for a square wave or treating an rms voltage as a peak voltage leads to an incorrect flux estimate.

Example 12.17: A low-voltage core design check

A winding has $N = 500$ turns on a core with effective area $A = 2.0 \times 10^{-4} \text{ m}^2$. It is driven by a sinusoidal 12 V rms magnetic winding voltage at 50 Hz. A specified design criterion is $|B| \leq 1.2 \text{ T}$. Estimate the peak flux density and the minimum frequency allowed by this criterion at the same voltage.

Assumptions. All turns link the same flux; winding drops, leakage and waveform distortion are neglected. The given 1.2 T is a hypothetical operating limit, not a universal material constant.

$$B_{pk} = \frac{12}{4.44(50)(500)(2.0 \times 10^{-4})} = 0.54 \text{ T}, \quad (12.83)$$

$$f_{min} = \frac{12}{4.44(500)(2.0 \times 10^{-4})(1.2)} \simeq 23 \text{ Hz}. \quad (12.84)$$

At 25 Hz the peak flux would be about 1.08 T, twice its value at 50 Hz. At 20 Hz it would exceed the stated criterion. The flux itself has amplitude $AB_{pk} = 1.08 \times 10^{-4} \text{ Wb}$ at 50 Hz.

Interpretation. A lower frequency can be more demanding for saturation at fixed voltage, even though it reduces the number of hysteresis cycles per second. Increasing permeability does not change Eq. (12.82); it changes the magnetizing current. A complete design also checks heating, temperature dependence and winding insulation.

12.18.2 Hysteresis and eddy currents are separate loss mechanisms

A narrow quasistatic hysteresis loop favours repeated magnetization with low irreversible magnetic work. High electrical resistivity or thin insulated laminations reduce currents induced within the core by changing flux. These design choices solve different parts of the loss problem.

One elementary calculation makes the lamination argument quantitative. Consider a conducting sheet of thickness a along x , with an approximately uniform sinusoidal magnetic field $B_z(t)$ parallel to its broad faces. Away from edges, Faraday's law admits an induced component

$$E_y(x, t) = -x \frac{dB_z}{dt}, \quad -a/2 < x < a/2, \quad (12.85)$$

where the symmetry choice makes $E_y = 0$ at the mid-plane. The return currents close near the sheet edges. With electrical conductivity σ_c , the local heating is $p = \sigma_c E_y^2$. Averaging over thickness and a cycle gives

$$\langle p \rangle = \frac{\sigma_c a^2}{12} \left\langle \left(\frac{dB_z}{dt} \right)^2 \right\rangle = \frac{\sigma_c a^2 \omega^2 B_{pk}^2}{24}. \quad (12.86)$$

Thus halving the sheet thickness reduces this idealized loss density by a factor of four at fixed frequency and flux density. The result assumes negligible skin effect, nearly uniform B and negligible perturbation of B by the eddy currents. It is a scaling model, not a universal core-loss formula. The sheets must be insulated from one another and oriented appropriately; merely cutting metal into layers without interrupting current paths does not guarantee the reduction.

12.18.3 Unwanted inductive and capacitive coupling

For a stationary wiring loop in an approximately uniform field, the induced emf is $-A \cos \theta dB/dt$. Reducing the enclosed area reduces pickup. A twisted pair can cancel flux contributions from successive small loops when the external field varies little over the twist pitch. It does not perfectly cancel arbitrary nearby, nonuniform fields.

A different mechanism acts between conductors separated by insulation. Their stray capacitance supports a current scale $I = C_s dV/dt$. No conduction path through the insulating gap is required. The induced receiving voltage depends on the receiving impedance, so a current estimate alone is not an estimate of the final noise voltage.

Example 12.18: Two coupling mechanisms on the same scale

A single stationary loop of area 20 cm^2 is exposed normally to $B(t) = 50 \mu\text{T} \cos(2\pi \times 10^4 t)$. Separately, a stray capacitance $C_s = 10 \text{ pF}$ couples a 5.0 V amplitude sinusoidal signal at 1.0 MHz into a receiver. Estimate the induced loop-emf amplitude and the capacitive-current amplitude.

Calculation.

$$\mathcal{E}_0 = A\omega B_0 = (2.0 \times 10^{-3})(2\pi \times 10^4)(50 \times 10^{-6}) = 6.3 \text{ mV}, \quad (12.87)$$

$$I_{s,0} = \omega C_s V_0 = (2\pi \times 10^6)(10 \times 10^{-12})(5.0) = 0.31 \text{ mA}. \quad (12.88)$$

Reducing the loop area by a factor of 100 reduces its predicted emf to $63 \mu\text{V}$. If the capacitive

receiver is approximately a $100\ \Omega$ resistance and its voltage is small compared with the driving voltage, its voltage amplitude is about 31 mV. This last estimate must be revised if the receiver impedance significantly loads the coupling capacitor.

Check. $\omega C_s R \simeq 0.0063 \ll 1$, so the assumed small receiving voltage is self-consistent. The examples demonstrate why loop area and electric-field coupling both matter in sensitive electronics.

12.18.4 Magnetic shielding and the importance of geometry

A high-permeability shell can divert low-frequency magnetic flux around a protected region, but it does not destroy flux. Open ends, seams, apertures and saturation can strongly affect the interior field. A finite permeability alone does not determine the shielding factor. Conducting shielding instead relies on induced currents for time-varying fields, so its effectiveness depends on frequency and conductivity as well as shape.

A saturation check estimates flux density in the shield itself, where flux can be concentrated. Increasing the available cross-sectional area can lower that flux density and help avoid saturation. It would be incorrect to claim that thickness can never help a saturated shield. Conversely, a high-permeability shell that works in a weak field need not work unchanged in a much stronger field. Permanent magnets, transformer cores and shields all use magnetic materials, but they impose different requirements on remanence, coercivity, loss and geometry.

12.19 From field equations to measurements and numerical checks

A measurement should identify the directly observed quantity before inferring a field. A search coil measures induced voltage; a current shunt measures a voltage proportional to current; a Hall probe measures a calibrated field component. The following activities connect these observables to Maxwell's equations, and distinguish numerical consistency from experimental validation.

12.19.1 Search-coil amplitude, polarity and uncertainty

Let the positive coil normal define positive flux and the associated positive winding traversal. With the voltage leads defined consistently, an open-circuit search coil gives

$$v_s(t) = -N_s A_s \cos \theta \frac{dB(t)}{dt} \quad (12.89)$$

for a spatially uniform field and fixed angle θ between the field and normal. Reversing the leads changes the displayed sign. For $B = B_0 \sin \omega t$, the amplitude is $v_{s,0} = N_s A_s \omega B_0 |\cos \theta|$. A stationary coil has no sustained voltage in a strictly static field, but rotating it or changing the field allows flux changes to be measured.

The measured voltage approximates the open-circuit emf only when the instrument impedance is large compared with the coil impedance over the frequency range used. Coil resistance, self-inductance, parasitic capacitance, probe loading and nonuniform field all affect the calibration. Integrating the voltage reconstructs changes in flux, but not an unknown constant starting flux.

Example 12.19: Inferring a field with a search coil

A coil has $N_s = 200$ turns, $A_s = (1.50 \pm 0.03) \times 10^{-4} \text{ m}^2$ and its normal is aligned with a sinusoidal field at $f = (60.0 \pm 0.1) \text{ Hz}$. The measured voltage amplitude is $(0.226 \pm 0.004) \text{ V}$. Treat the quoted uncertainties as independent standard uncertainties and neglect loading. Find B_0 .

Solution. The amplitude, rather than an instantaneous signed voltage, gives

$$B_0 = \frac{v_{s,0}}{2\pi f N_s A_s} = \frac{0.226}{2\pi(60.0)(200)(1.50 \times 10^{-4})} = 1.998 \times 10^{-2} \text{ T}. \quad (12.90)$$

For a counted integer number of turns, the fractional standard uncertainty is

$$\left(\frac{u(B_0)}{B_0}\right)^2 = \left(\frac{0.004}{0.226}\right)^2 + \left(\frac{0.1}{60.0}\right)^2 + \left(\frac{0.03}{1.50}\right)^2, \quad (12.91)$$

so $u(B_0)/B_0 = 0.0268$ and $B_0 = (20.0 \pm 0.5) \text{ mT}$.

Alignment check. At exact nominal alignment the first derivative of $\cos \theta$ vanishes. A 5° unknown tilt reduces the signal by $1 - \cos 5^\circ \simeq 0.0038$, a systematic effect of about 0.38%. Away from alignment, angle uncertainty contributes approximately $|\tan \theta|u(\theta)$ to the fractional uncertainty. Do not simply add independent standard uncertainties linearly, and do not report the central value with more precision than its uncertainty supports.

12.19.2 Reconstructing a toroidal hysteresis loop

A thin toroidal specimen keeps most flux within the core and reduces the demagnetizing correction. A drive winding has N_p turns. A high-impedance sense winding has N_s turns, and the core has effective area A_c and mean magnetic path length ℓ . Under the uniform-core approximation,

$$H(t) \simeq \frac{N_p I(t)}{\ell}, \quad B(t) = B(t_0) - \frac{1}{N_s A_c} \int_{t_0}^t v_s(t') dt'. \quad (12.92)$$

The sign of v_s follows the chosen sense-winding orientation. The measurement chain in Fig. 12.11 emphasizes that H and B are inferred from two synchronized signals.

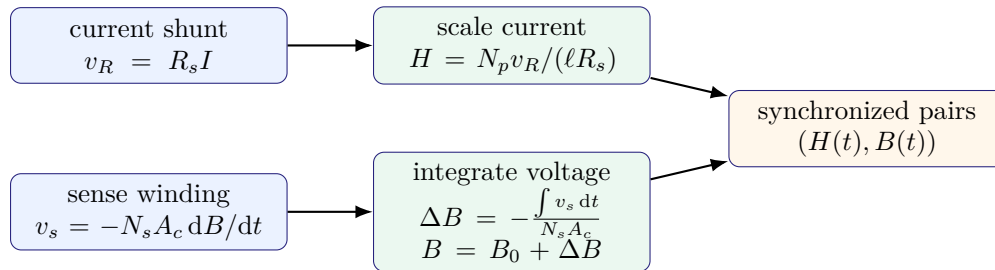


Figure 12.11: Measurement logic for a toroidal loop. A primary current determines the approximate H , while integrated secondary emf determines changes in B . The two signals require calibrated gain, known polarity and a common time base.

A constant voltage offset causes an artificial drift in the integrated B . In a reproducible periodic state, one may use cycle closure to diagnose or remove a small offset, after checking that the physical cycle truly is stationary. A phase delay between the current and voltage channels can create a false loop area even for a reversible specimen. Gain, phase and integrator response must therefore be calibrated; a visually attractive loop is not evidence of valid magnetic loss data.

For synchronized samples ordered in time, a trapezoidal cycle integral is

$$w_{\text{loss}} \simeq \sum_{j=0}^{n-1} \frac{H_{j+1} + H_j}{2} (B_{j+1} - B_j), \quad (H_n, B_n) = (H_0, B_0). \quad (12.93)$$

Do not sort the points by H : that destroys their traversal order and mixes the two branches. For a deliberately ideal rectangular loop with corners (H_c, B_r) , $(-H_c, B_r)$, $(-H_c, -B_r)$, $(H_c, -B_r)$,

this expression gives $4H_c B_r$. This is synthetic geometry for checking a calculation, not measured material data.

A useful experiment varies frequency while holding the same peak flux density and temperature as closely as possible. Compare loop areas per cycle, not just average powers. If area changes with frequency, a single quasistatic hysteresis parameter does not describe the data. Averaging repeated synchronized cycles can reduce random noise, but smoothing must not erase real switching structure or introduce a channel delay.

12.19.3 A reproducible displacement-current calculation

Consider a vacuum capacitor with $A = 2.0 \times 10^{-3} \text{ m}^2$ and $d = 1.0 \text{ mm}$, initially uncharged. Let $I(t) = I_0 e^{-t/\tau}$ for $t \geq 0$, with $I_0 = 10 \mu\text{A}$ and $\tau = 20 \mu\text{s}$. Integration gives

$$Q(t) = I_0 \tau (1 - e^{-t/\tau}), \quad (12.94)$$

$$E(t) = \frac{I_0 \tau}{\epsilon_0 A} (1 - e^{-t/\tau}), \quad (12.95)$$

$$I_d(t) = \epsilon_0 A \frac{dE}{dt} = I_0 e^{-t/\tau}. \quad (12.96)$$

The limiting values are $Q_\infty = 2.0 \times 10^{-10} \text{ C}$, $E_\infty = 1.13 \times 10^4 \text{ V/m}$ and $V_\infty = 11.3 \text{ V}$. These scales keep the example consistent with a modest laboratory capacitor rather than requiring a huge ideal field. The prescribed pulse remains a model; a real source has finite rise time and parasitic impedances.

The following complete Python example uses only the standard library. It integrates a sampled current, reconstructs the field, and compares a centred derivative with the current at interior points. It prints a maximum error normalized to I_0 , rather than a relative error that would become unstable in the small-current tail.

```
from math import exp

eps0 = 8.854e-12
area = 2.0e-3
I0, tau = 1.0e-5, 20.0e-6
for points_per_tau in (10, 20, 40, 80):
    dt = tau / points_per_tau
    n = 5 * points_per_tau
    t = [j * dt for j in range(n + 1)]
    I = [I0 * exp(-s / tau) for s in t]
    Q = [0.0]
    for j in range(1, n + 1):
        Q.append(Q[-1] + 0.5 * dt * (I[j-1] + I[j]))
    E = [q / (eps0 * area) for q in Q]
    Id = [eps0 * area * (E[j+1] - E[j-1]) / (2*dt)
          for j in range(1, n)]
    error = max(abs(Id[j-1] - I[j]) / I0
                for j in range(1, n))
    print(points_per_tau, f"{error:.6e}")
```

The errors are approximately 2.264×10^{-3} , 5.946×10^{-4} , 1.524×10^{-4} and 3.858×10^{-5} . Halving the step approaches a factor-four reduction, consistent with a second-order method. The precise first ratio differs slightly because the first retained point moves closer to $t = 0$ as the grid is refined.

For this combination of trapezoidal integration and centred differentiation on a uniform grid, direct algebra gives

$$I_{d,j} = \frac{I_{j-1} + 2I_j + I_{j+1}}{4}. \quad (12.97)$$

The reconstruction is therefore a mildly smoothed current and is not identically equal to each sample. Substituting the exponential yields

$$\frac{I_{d,j}}{I_j} = \frac{1 + \cosh(\Delta t/\tau)}{2} = 1 + \frac{1}{4} \left(\frac{\Delta t}{\tau} \right)^2 + \dots \quad (12.98)$$

This derivation explains the error rather than just observing it. For measured data, differentiation amplifies noise and ever smaller time steps need not improve the result. Comparing independently measured current and field is a physics test; integrating and then differentiating the same current is a test of numerical consistency only.

Guided checks

Check. If a reconstructed hysteresis loop acquires a large area when one oscilloscope channel is delayed, has the material loss necessarily increased?

Answer. No. A timing error changes which H and B values are paired and can create an artificial area. Likewise, numerical agreement between quantities generated from the same assumed model does not independently validate that model.

12.19.4 An end-to-end loop measurement calculation

The separate conversions from voltage to current and from voltage integral to flux can hide large scale errors. The following calculation joins the measurement chain to its energy interpretation and gives checks that can be made before fitting a material model.

Example 12.20: From recorded signals to H , B and integrator drift

A toroidal core has mean path $\ell = 0.25$ m and area $A_c = 1.0 \times 10^{-4}$ m². Its drive winding has $N_p = 250$ turns and the search winding has $N_s = 50$. A shunt $R_s = 0.50 \Omega$ records a peak current-channel voltage $v_{R,\text{pk}} = 0.10$ V. During one interval the signed integrated search voltage is $\int v_s dt = -1.5 \times 10^{-3}$ V s. Determine the drive-field amplitude and the change in B . Then estimate drift from an unnoticed search-channel offset of 1.0 mV over a 20 ms cycle.

Scale the current channel. $I_{\text{pk}} = v_{R,\text{pk}}/R_s = 0.20$ A, so

$$H_{\text{pk}} = \frac{N_p I_{\text{pk}}}{\ell} = \frac{250(0.20)}{0.25} = 200 \text{ A/m}.$$

This uses the thin-toroid approximation. A thick toroid has a radial H variation proportional to $1/r$, and one mean path is then an approximation to a spatially varying material state.

Integrate the sense channel with its sign. For the stated winding convention,

$$\Delta B = -\frac{\int v_s dt}{N_s A_c} = -\frac{-1.5 \times 10^{-3}}{50(1.0 \times 10^{-4})} = +0.30 \text{ T}.$$

This is a change, not an absolute field. A negative voltage integral gives an increase of positive flux under the chosen polarity. Since V s = Wb, division by area yields tesla.

Diagnose the offset. The false change over one cycle would be

$$\Delta B_{\text{offset}} = -\frac{v_{\text{off}} T}{N_s A_c} = -\frac{(10^{-3})(0.020)}{0.0050} = -4.0 \times 10^{-3} \text{ T}.$$

Ten cycles would accumulate -0.040 T if the offset were left uncorrected. Relative to a 0.30 T swing, this is large enough to distort remanence and apparent closure. Offset estimation should use a verified periodic state or an independently measured instrument baseline. Subtracting whatever value makes an arbitrary transient close would erase physical information.

12.19.5 Incremental response and large-signal response

An inductor used as a small-signal component need not experience a full major hysteresis loop. Suppose a thin toroidal core is biased near a working point (H_b, B_b) and undergoes a sufficiently small reversible excursion. Locally,

$$\delta B \simeq \mu_{\text{inc}} \delta H, \quad \mu_{\text{inc}} = \left. \frac{dB}{dH} \right|_{\text{working point}}. \quad (12.99)$$

Using $\delta H = N\delta I/\ell$ and flux linkage $\lambda = NAB$ gives

$$L_{\text{inc}} = \frac{d\lambda}{dI} \simeq \frac{N^2 A}{\ell} \mu_{\text{inc}}. \quad (12.100)$$

The slope dB/dH differs from the secant ratio B/H . A core may have a large B at its bias point while its incremental slope, and thus its small-signal inductance, is small. Near magnetization saturation, increasing H adds little M , so the incremental permeability approaches the vacuum contribution in the simplest limit. This explains why a biased inductor can lose inductance before a naive B/H calculation suggests a problem.

If the small excursion traces a minor hysteresis loop, one real derivative is no longer enough to describe both storage and loss. The response depends on amplitude, frequency and preparation history. Instrument specifications must therefore state test conditions. A single number labelled “permeability” or “inductance” cannot automatically be transferred from a low-amplitude bench test to a strongly driven device.

Example 12.21: A measured slope determines an incremental inductance

A thin toroid has $N = 100$, $A = 1.0 \times 10^{-4} \text{ m}^2$ and $\ell = 0.20 \text{ m}$. Around a bias point, a reversible change $\delta H = 5.0 \text{ A/m}$ produces $\delta B = 2.0 \text{ mT}$. Estimate L_{inc} and the associated current change.

The measured slope is $\mu_{\text{inc}} = 4.0 \times 10^{-4} \text{ H/m}$, corresponding to $\mu_{r,\text{inc}} \simeq 318$. Therefore $L_{\text{inc}} = 2.0 \text{ mH}$. The current change is $\delta I = \ell \delta H / N = 0.010 \text{ A}$. Independently, $\delta \lambda = N A \delta B = 2.0 \times 10^{-5} \text{ Wb}$ and $\delta \lambda / \delta I = 2.0 \text{ mH}$. The agreement checks the winding factors and units. The result describes this small excursion; it does not predict a full-amplitude flux swing, a major loop area or the energy of an arbitrary biased hysteretic state.

12.20 Advanced Topics for Technical and Engineering Students

The integral laws are the core of this chapter. This section develops a connected extension: local equations describe what happens at each point; a free-bound source separation makes matter manageable; interface and shape effects then explain practical magnetic devices. Finally, energy conservation provides an independent check on the field picture. Vector calculus is introduced where it is used. These topics enrich the core and need not be prerequisites for the earlier exercises.

12.20.1 From integral statements to differential equations

A flux law applies to every sufficiently small volume, not just to a convenient sphere or cylinder. This is what allows it to become a local statement. The *divergence* of a vector field is its outward flux per unit volume in the limit of a shrinking volume. The divergence theorem makes this precise:

$$\oint_{\partial V} \mathbf{F} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{F} dV. \quad (12.101)$$

For the electric field, $q_{\text{enc}} = \int_V \rho dV$, so the integral of $\nabla \cdot \mathbf{E} - \rho/\varepsilon_0$ vanishes for every volume. For continuous fields inside a region, the integrand must vanish point by point. A singular sheet of charge requires an interface condition instead of an ordinary pointwise derivative on the sheet.

The *curl* measures circulation per unit area, including its orientation. Stokes's theorem relates it to a loop:

$$\oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}. \quad (12.102)$$

For a fixed surface, time differentiation can be taken inside the flux integral. Applying the two theorems to the four integral equations gives

Formula summary

Source-explicit differential Maxwell equations

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \quad \nabla \cdot \mathbf{B} = 0, \quad (12.103)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (12.104)$$

Here ρ and $\mathbf{J}$ are total charge and charge-current densities. Their units are C/m³ and A/m². At the microscopic level these include all charges; for spatially averaged fields they include both free and bound sources consistently averaged over the same scale. The partial time derivative holds position fixed. None of these equations requires a constant permeability for a ferromagnet.

Divergence is a scalar, whereas curl is a vector. For example, $\nabla \cdot \mathbf{B} = 0$ constrains the sum of three spatial derivatives; it does not say that every component of $\mathbf{B}$ is uniform. Likewise a zero curl does not require a field to vanish. Boundary conditions and sources still determine its magnitude.

12.20.2 Local charge conservation as a compatibility condition

Taking the divergence of the magnetic curl equation and using $\nabla \cdot (\nabla \times \mathbf{B}) = 0$ gives

$$0 = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) = \mu_0 \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right), \quad (12.105)$$

$$\therefore \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (12.106)$$

Positive outward current divergence means that charge density decreases. If displacement current were omitted, the curl equation would impose $\nabla \cdot \mathbf{J} = 0$ at all times and would forbid local charge accumulation. A capacitor provides a direct physical counterexample to that restriction.

The same reasoning explains how a constraint is propagated. If the two curl equations and charge conservation hold, then $\partial_t(\nabla \cdot \mathbf{E} - \rho/\epsilon_0) = 0$. Thus an electric Gauss-law constraint that is satisfied initially remains satisfied. The curl equations do not repair an incorrectly specified initial charge–field relation. Numerical field solvers must respect this compatibility rather than treating the four equations as unrelated formulas.

Example 12.22: Reading the continuity equation locally

In a small region, the current density is $\mathbf{J} = ax\hat{x}$, with $a = 2.0 \text{ A/m}^3$. Determine the initial rate of change of charge density and check it with a cube of side L whose left face is at $x = 0$. **Strategy.** Compute the net outflow, not just the current at one point. Since $\partial J_x / \partial x = a$, the continuity equation gives $\partial \rho / \partial t = -a = -2.0 \text{ C/(m}^3 \text{ s)}$. On the cube, only the right face contributes outward current: $J_x(L)L^2 = aL^3$. Therefore $dQ/dt = -aL^3$ and, after division by L^3 , the same local result follows. The field pattern is an instantaneous local specification; it need not remain unchanged as charge accumulates elsewhere.

12.20.3 Polarization, magnetization and free sources

A dielectric contains positive and negative bound charge whose relative displacement produces electric dipoles. Its polarization $\mathbf{P}$ is dipole moment per volume, in C/m^2 . Magnetic dipoles give the magnetization $\mathbf{M}$. Spatial and temporal variations of these dipole densities can be represented by bound sources:

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad \mathbf{J}_b = \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}. \quad (12.107)$$

A nonuniform polarization leaves uncompensated bound charge; changing polarization transports bound charge locally. Magnetization corresponds to microscopic circulating currents. At a material boundary there are also surface sources: $\sigma_b = \mathbf{P} \cdot \hat{n}$ and $\mathbf{K}_b = \mathbf{M} \times \hat{n}$, with the normal outward from that material. These expressions are macroscopic descriptions; they do not imply that electrons follow classical rigid orbits.

Write $\rho = \rho_f + \rho_b$ and $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$, and define

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}. \quad (12.108)$$

Substitution into the source-explicit equations, followed by collecting the polarization and magnetization terms, yields

Formula summary

Macroscopic Maxwell equations

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \cdot \mathbf{B} = 0, \quad (12.109)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \partial_t \mathbf{D}. \quad (12.110)$$

The corresponding integral magnetic law is $\oint \mathbf{H} \cdot d\boldsymbol{\ell} = I_{f,\text{enc}} + d\Phi_D/dt$. A material displacement-current density means $\partial_t \mathbf{D}$ in this formulation. It includes the vacuum term and the polarization current. Adding the polarization current again would double count it.

Constitutive relations are additional physical information. A simple linear, isotropic, instantaneous model has $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, with constant positive ϵ and μ . Real materials can be anisotropic, dispersive, nonlinear and history dependent. In particular, a hysteretic B – H curve is not described by one constant μ . The auxiliary fields reorganize the sources; they do not remove the need to measure or model matter.

Example 12.23: Which displacement current in a dielectric?

A homogeneous nonmagnetic dielectric with $\epsilon_r = 3.0$ fills a parallel-plate capacitor. The uniform field rises at $1.5 \times 10^8 \text{ V}/(\text{m s})$ over area $8.0 \times 10^{-4} \text{ m}^2$. Assume negligible leakage and fringing.

Calculation. The material displacement current is

$$I_D = A \frac{dD}{dt} = 3\epsilon_0 A \frac{dE}{dt} = 3.19 \mu\text{A}.$$

The vacuum term is $I_{d,0} = 1.06 \mu\text{A}$; the remaining $2.13 \mu\text{A}$ is polarization current. In the gap, the $\mathbf{H}$ equation uses I_D and no free conduction current. The source-explicit $\mathbf{B}$ equation uses $I_{d,0} + I_{\text{polarization}}$. Since $\mathbf{M} = 0$, both descriptions give the same circulation of $\mathbf{B} = \mu_0 \mathbf{H}$.

Interpretation. Bound charges shift within the dielectric while free charges arrive at its electrodes. It is incorrect to infer the lead current from $\epsilon_0 A dE/dt$ alone. The unit $(\text{F/m})\text{m}^2\text{V}/(\text{m s}) = \text{A}$ checks both calculations.

12.20.4 Conduction, displacement and a relaxation time

For a homogeneous linear medium with conductivity σ , permittivity ϵ , and $E(t) = E_0 \cos \omega t$, the free conduction and material displacement current densities are

$$J_c = \sigma E_0 \cos \omega t, \quad J_D = -\epsilon \omega E_0 \sin \omega t. \quad (12.111)$$

Their amplitude ratio is $\epsilon \omega / \sigma$, and their phases differ by a quarter cycle. The amplitudes therefore do not add arithmetically: the total amplitude is $E_0 \sqrt{\sigma^2 + (\epsilon \omega)^2}$. A lossy dielectric can have significant values of both. The ratio alone does not determine propagation or heating in a whole device, which also depend on geometry and material response.

For constant σ and ϵ , combine free-charge continuity with $\nabla \cdot \mathbf{D} = \rho_f$:

$$\partial_t \rho_f = -\nabla \cdot (\sigma \mathbf{E}) = -\frac{\sigma}{\epsilon} \rho_f, \quad \rho_f(t) = \rho_f(0) e^{-t/\tau_q}, \quad \tau_q = \frac{\epsilon}{\sigma}. \quad (12.112)$$

This describes the decay of excess free charge in the homogeneous bulk, not the disappearance of conserved charge. Charge flows to other regions, often to surfaces. Spatially varying coefficients, externally maintained sources and non-Ohmic response need a more detailed treatment. Since $\epsilon \omega / \sigma = \omega \tau_q$, the comparison of currents also compares the drive timescale with bulk charge relaxation.

12.20.5 Boundary conditions from shrinking surfaces and loops

Maxwell's equations relate fields on either side of an interface. Let $\hat{\mathbf{n}}$ point from medium 1 into medium 2. A thin pillbox has outward normal $+\hat{\mathbf{n}}$ on its upper face and $-\hat{\mathbf{n}}$ on its lower face. As its height tends to zero, the side flux vanishes for bounded fields. Applying the two Gauss laws gives

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f, \quad \hat{\mathbf{n}} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0. \quad (12.113)$$

Thus normal B is continuous even when permeability jumps. Normal D is continuous only if no free surface charge is present.

A thin rectangular loop straddles the interface, with its long sides traversed in opposite directions (Fig. 12.12). Its enclosed flux tends to zero with its height if the field time derivatives remain bounded. Faraday's law then requires equal tangential E . A free surface-current sheet contributes finite enclosed current even in this limit, so the magnetic condition is

$$\hat{\mathbf{n}} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0, \quad \hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f. \quad (12.114)$$

The vector $\mathbf{K}_f$ is surface current per unit width, in A/m. Its sign is fixed by the declared normal. The conditions stated here assume ordinary interfaces without imposed singular magnetic sources or singular flux sheets.

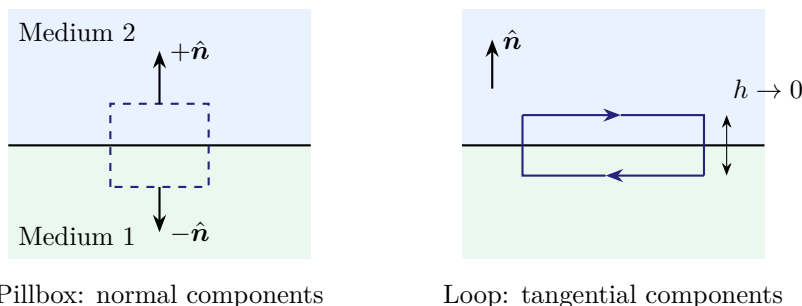


Figure 12.12: Two distinct limiting constructions at the same interface. The pillbox isolates a difference of normal fluxes; the loop isolates a difference of tangential circulations. Opposite normals and opposite traversal directions generate the differences in the boundary conditions.

Example 12.24: How a magnetic field changes direction at an interface

Two linear isotropic media with $\mu_1 = \mu_0$ and $\mu_2 = 4\mu_0$ meet at a plane with no free surface current. Immediately in medium 1, $B_{1n} = 0.20$ mT and $B_{1t} = 0.10$ mT. Find the field immediately in medium 2.

Strategy. Preserve B_n and H_t , not both components of B . Hence $B_{2n} = 0.20$ mT and $B_{2t} = (\mu_2/\mu_1)B_{1t} = 0.40$ mT. The magnitude is 0.447 mT. Angles measured from the normal satisfy

$$\tan \theta_2 = \frac{\mu_2}{\mu_1} \tan \theta_1, \quad \theta_1 = 26.6^\circ, \quad \theta_2 = 63.4^\circ.$$

The field in the higher-permeability medium is more nearly tangent to the interface for these local components. This boundary relation alone does not determine the whole field geometry or establish that a proposed global configuration is realizable. No Snell-law formula for travelling waves has been used.

12.20.6 Bound currents and the field of a magnetized cylinder

For uniform axial magnetization $\mathbf{M} = M\hat{\mathbf{z}}$ inside a long cylinder, $\nabla \times \mathbf{M} = 0$ in the bulk. On the curved surface, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so

$$\mathbf{K}_b = M\hat{\mathbf{z}} \times \hat{\mathbf{r}} = M\hat{\boldsymbol{\phi}}. \quad (12.115)$$

There is no bound surface current on the flat ends because their normals are parallel or antiparallel to $\mathbf{M}$. The curved-surface current has exactly the geometry of an ideal solenoid with surface current $nI = M$. Far from the ends,

$$\mathbf{B} \simeq \mu_0 M\hat{\mathbf{z}}, \quad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} \simeq 0. \quad (12.116)$$

This is a useful warning: $H = 0$ does not imply $B = 0$ in a permanently magnetized body. The equation $\mathbf{M} = \chi_m \mathbf{H}$ with a fixed finite χ_m describes a particular reversible material model, not a permanent magnet on an arbitrary hysteresis branch.

A finite cylinder has end fields and a demagnetizing field. Even though no physical magnetic charges have been introduced, effective pole pictures can sometimes reproduce the field mathematically. They must not be mistaken for violations of $\nabla \cdot \mathbf{B} = 0$: flux remains continuous through the material and space.

12.20.7 Shape changes the measured susceptibility

For a uniformly magnetized ellipsoid driven along a principal axis, a particularly useful magnetostatic result is

$$H_{\text{int}} = H_{\text{appl}} - N_d M, \quad (12.117)$$

where N_d is a dimensionless demagnetizing factor. In the SI convention used here, the three principal factors sum to one. A sphere has $N_d = 1/3$, a long needle along its axis approaches zero, and a thin plate along its normal approaches one. Other shapes generally have nonuniform internal fields, so one factor is then an approximation.

For an intrinsic linear susceptibility $M = \chi_m H_{\text{int}}$, substitution and rearrangement give

$$M = \frac{\chi_m}{1 + N_d \chi_m} H_{\text{appl}}, \quad \chi_{\text{apparent}} = \frac{M}{H_{\text{appl}}} = \frac{\chi_m}{1 + N_d \chi_m}. \quad (12.118)$$

A large intrinsic response is therefore limited by shape in a measurement referenced to the applied field. For positive $\chi_m \rightarrow \infty$ and $N_d > 0$, $\chi_{\text{apparent}} \rightarrow 1/N_d$. This is not microscopic saturation; it is a consequence of the opposing internal field.

Example 12.25: The same linear material in two shapes

A hypothetical reversible material has $\chi_m = 9.0$. Compare a sphere and a very long axial rod in an applied field $H_{\text{appl}} = 100 \text{ A/m}$, assuming the linear model remains valid. For the sphere, $M = 9(100)/(1 + 9/3) = 225 \text{ A/m}$ and $H_{\text{int}} = 100 - 225/3 = 25 \text{ A/m}$. Thus $B = \mu_0(H_{\text{int}} + M) = 0.314 \text{ mT}$. For the long rod, $N_d \simeq 0$, so $M \simeq 900 \text{ A/m}$, $H_{\text{int}} \simeq 100 \text{ A/m}$ and $B \simeq 1.26 \text{ mT}$. The fourfold difference arises from geometry despite identical constitutive parameters. Using $M = \chi_m H_{\text{appl}}$ for the sphere would miss the entire effect.

12.20.8 A magnetic circuit with an air gap

A core with mean material path length ℓ_c , cross-section A , relative permeability μ_r , and one narrow gap of length g carries an N -turn winding. Assume a linear core, negligible leakage, nearly uniform flux and a gap narrow enough that fringing can be neglected. Flux continuity makes approximately the same B pass through the core and gap. Ampere's law for H then gives

$$NI = H_c \ell_c + H_g g = \frac{B \ell_c}{\mu_0 \mu_r} + \frac{B g}{\mu_0}, \quad (12.119)$$

$$B = \frac{\mu_0 NI}{g + \ell_c / \mu_r}, \quad (12.120)$$

$$\Phi = BA = \frac{NI}{\mathcal{R}}, \quad \mathcal{R} = \frac{\ell_c}{\mu_0 \mu_r A} + \frac{g}{\mu_0 A}, \quad (12.121)$$

$$L = \frac{N\Phi}{I} = \frac{N^2}{\mathcal{R}}. \quad (12.122)$$

The reluctance $\mathcal{R}$ has units A/Wb; N counts turns and is dimensionless. These formulas use the flux linked by every turn. Leakage would require distinguishing that linked flux from flux elsewhere.

For a linear, nondissipative material, magnetic energy density is $u = B^2/(2\mu)$. The gap-to-core energy ratio is consequently

$$\frac{U_g}{U_c} = \frac{B^2 Ag/(2\mu_0)}{B^2 A \ell_c/(2\mu_0 \mu_r)} = \frac{\mu_r g}{\ell_c}. \quad (12.123)$$

A small gap can store most of the energy because its permeability is much lower. It also reduces inductance and reduces sensitivity to uncertain core permeability. Hysteretic energy storage needs a path-dependent treatment; the formula $B^2/(2\mu)$ must not be applied with an arbitrary instantaneous ferromagnetic permeability.

Example 12.26: An energy-storage inductor

Take $N = 200$, $I = 0.50$ A, $A = 1.0 \times 10^{-4}$ m², $\ell_c = 0.20$ m, $g = 1.0$ mm and $\mu_r = 1000$. Determine the flux, inductance and division of stored energy in the idealized model.

The effective length is $g + \ell_c/\mu_r = 1.20 \times 10^{-3}$ m, giving $B = 0.105$ T, $\Phi = 1.05 \times 10^{-5}$ Wb and $L = 4.19$ mH. The fields are $H_c = 83.3$ A/m and $H_g = 8.33 \times 10^4$ A/m. Their contributions to NI are 16.7 and 83.3 ampere-turns, which sum to the required 100.

The gap holds $U_g = 0.436$ mJ and the core $U_c = 0.0873$ mJ. Their sum, 0.524 mJ, agrees with $LI^2/2$. The ratio is five, as Eq. (12.123) predicts. With no gap the same constant- μ_r model would predict $B = 0.628$ T and six times the inductance. Whether that prediction is physically allowed requires the actual material curve. This estimate is a model check, not a guarantee against saturation or heating.

12.20.9 A two-state microscopic model of paramagnetism

Consider n independent moments per unit volume. Each moment has only two allowed projections, $+m$ and $-m$, along a uniform field. This is a simplified quantum two-state model, not a classical freely rotating dipole. The two energies are $-mB$ and $+mB$. At thermal equilibrium, the Boltzmann probabilities are

$$p_+ = \frac{e^x}{e^x + e^{-x}}, \quad p_- = \frac{e^{-x}}{e^x + e^{-x}}, \quad x = \frac{mB}{k_B T}. \quad (12.124)$$

Consequently the mean moment is $m(p_+ - p_-)$ and

$$M = nm \tanh x, \quad M_s = nm. \quad (12.125)$$

The tanh response in Fig. 12.13 tends to $\pm M_s$ at large $|x|$. In weak fields, $\tanh x \simeq x$, so $M \simeq nm^2 B/(k_B T)$. If the susceptibility is small enough that $B \simeq \mu_0 H$, and interactions and demagnetization are negligible,

$$\chi_m \simeq \frac{\mu_0 nm^2}{k_B T}. \quad (12.126)$$

This derives an inverse-temperature law for a specified microscopic model. A classical freely rotating moment has a different coefficient, involving $1/3$; general quantum angular momentum gives another model-dependent factor. These coefficients must not be mixed.

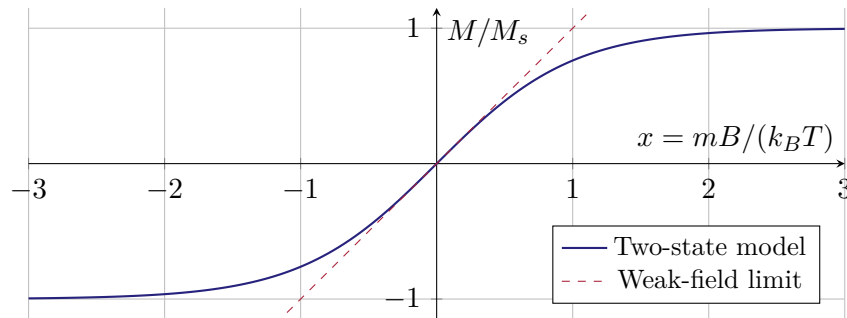


Figure 12.13: Competition between field energy and thermal energy for independent two-state moments. The linear approximation is reliable only near the origin; it cannot predict the saturation region. This curve is reversible and is not a ferromagnetic hysteresis loop.

At high temperature or small field, nearly equal populations give a small net magnetization even though individual moments are not small. At large x , the higher-energy state is depleted. A classical sketch with every arrow fully aligned at ordinary room-temperature laboratory fields would therefore misrepresent a typical weak paramagnet.

Independent moments do not produce spontaneous ferromagnetic order or hysteresis. A phenomenological interacting model may yield a Curie–Weiss form $\chi = C/(T - \Theta)$ in an appropriate high-temperature range, but Θ is an interaction parameter, and the expression must not be extrapolated through its apparent divergence as a complete theory. Domain formation, pinning and critical behavior require further physics. The core chapter uses Curie’s law only where its assumptions are reasonable.

12.20.10 Energy conservation and the Poynting vector

The fields in a charging capacitor carry energy as well as satisfying flux and circulation laws. In vacuum or the source-explicit formulation, take the dot product of Ampere–Maxwell with $\mathbf{E}/\mu_0$, and of Faraday with $\mathbf{B}/\mu_0$. Subtract using

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}).$$

This yields the local energy balance

$$\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}, \quad (12.127)$$

$$u = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}, \quad \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (12.128)$$

The Poynting vector has units W/m². Positive $\mathbf{J} \cdot \mathbf{E}$ is work per unit time and volume done by the field on matter. An integral over a volume gives

$$-\oint_{\partial V} \mathbf{S} \cdot d\mathbf{A} = \frac{d}{dt} \int_V u \, dV + \int_V \mathbf{J} \cdot \mathbf{E} \, dV. \quad (12.129)$$

Net inward energy flow supplies both increasing field energy and work on charges. In material descriptions, separating stored field energy from material energy requires constitutive assumptions; replacing ε_0 and μ_0 by arbitrary frequency-dependent constants is not a general derivation of a stored-energy formula.

Example 12.27: Energy enters a vacuum capacitor through its side

For the ideal circular capacitor, let $\mathbf{E} = E\hat{\mathbf{z}}$ increase with $\dot{E} > 0$. At the rim, $B_\phi(R) = \mu_0 \varepsilon_0 R \dot{E}/2$. Because $\hat{\mathbf{z}} \times \hat{\boldsymbol{\phi}} = -\hat{\mathbf{r}}$, the energy flux points radially inward:

$$S_r = -\frac{EB_\phi}{\mu_0} = -\frac{\varepsilon_0 R}{2} E \dot{E}.$$

Through side area $2\pi R d$, the inward power is

$$P_{\text{in}} = \varepsilon_0 \pi R^2 d E \dot{E} = \frac{d}{dt} \left(\frac{1}{2} \varepsilon_0 E^2 \pi R^2 d \right) = (Ed)(\varepsilon_0 \pi R^2 \dot{E}) = VI.$$

Thus the field description agrees with circuit power and capacitor energy $CV^2/2$. No conduction current crosses the vacuum gap. The result uses the leading quasistatic fields: small magnetic-energy and nonuniform-field corrections are beyond that approximation. On discharge, $E\dot{E} < 0$ and the flux reverses. The direction follows from vectors, not from the sign of E alone.

12.20.11 A controlled preview of waves

In source-free vacuum, $\rho = 0$ and $\mathbf{J} = 0$. Taking the curl of Faraday’s law and using Ampere–Maxwell gives

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \partial_t^2 \mathbf{E}. \quad (12.130)$$

The identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ and electric Gauss law then give

$$\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \partial_t^2 \mathbf{E}, \quad \nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \partial_t^2 \mathbf{B}. \quad (12.131)$$

Each component satisfies a wave equation with speed $c = 1/\sqrt{\mu_0 \varepsilon_0}$. A candidate pair $\mathbf{E} = E_0 \cos(kx - \omega t) \hat{\mathbf{y}}$, $\mathbf{B} = B_0 \cos(kx - \omega t) \hat{\mathbf{z}}$ satisfies both curl equations only when $\omega = ck$ and $E_0 = cB_0$. Its energy flux is along $+x$.

The second-order wave equations are necessary consequences of Maxwell's equations; an arbitrary pair of their solutions need not satisfy the first-order curl and divergence constraints. In a plane wave, E and B oscillate in phase. Near a driven capacitor, they need not have that amplitude ratio or phase relation. Chapter 13 develops wave polarization, intensity, propagation and optical applications. Here the derivation closes the argument that Maxwell's correction changes the dynamical possibilities of the theory.

12.20.12 Superconductors: an instructive limit of magnetic response

A material in a Meissner state excludes magnetic flux from its bulk apart from a thin penetration region. It can be modeled on macroscopic scales by $B \simeq 0$, giving $M \simeq -H$ locally, a much stronger response than ordinary weak diamagnetism. Persistent screening currents provide the required surface-current pattern.

Perfect conductivity by itself says that an ideal conductor has $E = 0$ in its interior; Faraday's law then preserves any pre-existing interior flux. It does not by itself demand expulsion of flux when a material is cooled. The Meissner effect is therefore additional physical behavior. Nor do all superconductors exclude all applied flux in every superconducting state: type-II superconductors can admit quantized vortices in a mixed state above their lower critical field, while remaining superconducting. Critical fields, temperature and sample geometry set the applicable regime. These distinctions prevent a useful limiting example from turning into an incorrect universal material law.

Take-home message

The source-explicit equations, the macroscopic D - H equations and magnetic-circuit approximations are different levels of description of the same electromagnetic accounting. Decide which sources are explicit, state the material law, and check geometry and timescale before choosing a formula. Charge conservation and energy conservation then provide independent consistency checks.

12.21 Chapter summary

Summary

- Maxwell's equations in integral form describe electric flux, magnetic flux, electric circulation and magnetic circulation.
- Gauss's law for electricity states that electric charge is the source of net electric flux through a closed surface.
- Gauss's law for magnetism states that the net magnetic flux through any closed surface is zero in classical electromagnetism.
- Faraday's law states that changing magnetic flux produces a circulating electric field.
- Ampere-Maxwell law states that total charge current and changing electric flux produce circulating magnetic field.
- The displacement current $I_d = \varepsilon_0 d\Phi_E/dt$ is not charge crossing a vacuum gap; it is the effective current associated with changing electric flux.

- Maxwell's correction makes Ampere's law consistent with a charging capacitor and with charge conservation.
- The coupled field equations admit electromagnetic waves with vacuum speed $c = 1/\sqrt{\mu_0\epsilon_0}$.
- Magnetic materials are described macroscopically by magnetization, the magnetic dipole moment per unit volume.
- Diamagnetic response is weak and opposite to the applied field; paramagnetic response is weak and along the applied field; ferromagnetic response can be strong, collective and hysteretic.
- Ferromagnetic domains, remanence, coercivity and hysteresis are central to permanent magnets, transformer cores and magnetic storage.

Formula summary

Fixed paths and surfaces; total sources in the first four laws

$$\begin{aligned}\oint_S \mathbf{E} \cdot d\mathbf{A} &= q_{\text{enc}}/\epsilon_0, & \oint_S \mathbf{B} \cdot d\mathbf{A} &= 0, \\ \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} &= -\frac{d\Phi_B}{dt}, & I_d &= \epsilon_0 \frac{d\Phi_E}{dt}, \\ \oint_C \mathbf{B} \cdot d\boldsymbol{\ell} &= \mu_0 \left(I_{\text{enc}} + \epsilon_0 \frac{d\Phi_E}{dt} \right).\end{aligned}$$

Dipoles and matter

$$\begin{aligned}\mathbf{m} &= N I \mathbf{A}, & \boldsymbol{\tau} &= \mathbf{m} \times \mathbf{B}, \\ U &= -\mathbf{m} \cdot \mathbf{B}, & \mathbf{M} &= \Delta \mathbf{m} / \Delta V, \\ \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}), & \mathbf{M} &= \chi_m \mathbf{H} \quad (\text{linear model}).\end{aligned}$$

$$w_{\text{loss}} = \oint H dB, \quad P_{\text{loss}} = f V_c w_{\text{loss}}, \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

The loop-area result assumes a closed magnetic cycle and consistent B – H data. The dipole force/energy model uses an externally imposed field; $\mathbf{m} = N I \mathbf{A}$ assumes N equivalent planar turns. Material displacement current in the D – H formulation is $d\Phi_D/dt$.

Take-home message

Maxwell's equations teach a disciplined way of thinking: surfaces measure flux, loops measure circulation, charges and currents are sources, and changing fields are sources too. Magnetic materials add one further layer: matter contains microscopic magnetic moments whose collective response shapes the macroscopic field.

12.22 Exercises

The 46 problems below progress from interpretation to calculation, design and advanced modeling. Use the rounded constants in the conventions section. State the surface or loop, orientation, material model and approximation before substituting numbers. Labels C, A and E belong to the core and applied sections; Ch problems use the advanced enrichment. A dagger † marks a problem with a detailed solution in the next section. The remaining problems have concise checks or assessment criteria after those solutions.

12.22.1 Conceptual questions

- C1.** A closed surface has zero electric flux. Give two physically different charge or field configurations consistent with this observation. What can be concluded about net enclosed charge, and what cannot be concluded about the field on the surface?
- C2.** A Gaussian surface encloses the north end of a bar magnet but intersects the magnet. Explain where the compensating inward magnetic flux crosses the surface. Would physically cutting the magnet isolate magnetic charge?
- C3.** [†] Two surfaces share the same oriented circular boundary near a charging capacitor. One intersects the lead and the other passes through the vacuum gap. Explain why the magnetic circulation is the same, why the separate source terms need not be, and why $I + I_d = 2I$ is an incorrect conclusion here.
- C4.** Is the induced electric field around a changing magnetic field present when the conducting loop is removed? Explain the difference between electric circulation, emf and current, including what additional information is needed to find the current.
- C5.** A loop translates through a static nonuniform magnetic field. Which term in the moving-circuit emf can be nonzero even though $\partial \mathbf{B} / \partial t = 0$? Explain why the fixed-loop form of Faraday's law cannot be used unchanged for this moving boundary.
- C6.** At an instant, the net current through one Ampere–Maxwell surface is zero and its electric flux is constant. Does zero magnetic circulation prove that $\mathbf{B} = 0$ everywhere on the loop? Supply a counterexample.
- C7.** Compare equilibrium diamagnetism with eddy-current screening. Explain why a diamagnetic response can persist under a constant applied field, while ordinary resistive eddy currents from a past field change decay.
- C8.** Why can an unmagnetized ferromagnetic specimen contain domains with large local magnetization? Describe how domain-wall motion can change the sample's net moment without increasing the magnitude of each domain's magnetization.
- C9.** [†] On a B – H loop, identify remanent flux density and the coercive field defined by $B = 0$. Explain why the magnetization need not vanish at that coercive field and why saturation of M does not make B strictly constant.
- C10.** Compare the desired hysteresis behavior for a repeatedly cycled transformer core and a permanent magnet resisting demagnetization. Name one reason why loop area alone is insufficient to select a practical material.
- C11.** Under what assumptions is $M = \chi_m H$ useful? Why is neither the sign of χ_m nor one quoted permeability sufficient to describe the history of a ferromagnet?
- C12.** Explain why $E_0 = cB_0$ is appropriate for a vacuum travelling plane wave but is not a general relation between electric and magnetic fields near a driven capacitor. State one feature of geometry or phase that distinguishes the two situations.

12.22.2 Analytical and numerical problems

- A1.** A closed surface encloses charges $+4.0 \text{ nC}$, -1.5 nC and -3.0 nC . Find its net outward electric flux. Explain why its shape is irrelevant to this result.
- A2.** [†] A uniform field B_z is confined approximately to a cylindrical region of radius $R = 4.0 \text{ cm}$ and decreases at 0.40 T/s . Assume axial symmetry. Find the signed counterclockwise electric

circulation and E_ϕ at $r = 2.0$ cm and $r = 8.0$ cm. Check continuity at R and explain the direction.

- A3.** A fixed circular loop of radius 0.10 m has its positive normal along $+z$. A uniform field is $B_z(t) = B_0 + \alpha t^2$, where $\alpha = 0.30$ T/s². Find the signed emf at $t = 2.0$ s. If the loop resistance is $5.0\ \Omega$ and its self-inductance is negligible, find its signed current.
- A4.** A vacuum parallel-plate capacitor has area 5.0×10^{-3} m² and uniform $\dot{E}_z = 2.0 \times 10^9$ V/(m s). Find the displacement current through a full-gap surface with positive normal $+z$. How does the answer change if that normal is reversed?
- A5.** [†] A circular vacuum capacitor of radius $R = 5.0$ cm is charged with a positive displacement current of 0.40 A along $+z$. Under the ideal symmetric approximation, find B_ϕ at $r = 1.0$, 5.0 and 10.0 cm. Locate the maximum field and explain why applying the full enclosed current at the first radius is wrong.
- A6.** [†] Let $V(t) = V_0 \sin \omega t$ across an ideal vacuum capacitor with $R = 3.0$ cm, $d = 0.50$ mm, $V_0 = 20$ V and $f = 100$ kHz. Find C , the current amplitude, and the magnetic-field amplitude at $r = R/2$. State their phases relative to V . For apparatus size $L_{\text{sys}} = 0.10$ m, estimate $\omega L_{\text{sys}}/c$.
- A7.** The current charging an initially uncharged vacuum capacitor is $I(t) = I_0 e^{-t/\tau}$ for $t \geq 0$. Derive $Q(t)$, $E(t)$ and $V(t)$ for area A and gap d . Check $\epsilon_0 A \dot{E} = I(t)$ and explain why the field approaches a nonzero constant while the current vanishes.
- A8.** A coil has 50 turns, area 2.0×10^{-4} m² per turn and current 0.15 A. Find its magnetic moment. In a uniform 0.80 T field at 30° to that moment, find its torque magnitude and potential energy. What work must an external agent do to rotate it quasistatically from alignment to anti-alignment at fixed current?
- A9.** A uniformly magnetized sample of volume 1.2×10^{-6} m³ has moment 3.6×10^{-4} A m². Its internal H is 2.0×10^4 A/m and it obeys a linear isotropic model. Find M , χ_m , μ_r and B .
- A10.** A Curie-law paramagnet has $\chi_m = 1.2 \times 10^{-3}$ at 300 K. Estimate χ_m at 200 K and the ratio of magnetizations at fixed internal H . Explain why extrapolation to $T = 0$ is not justified.
- A11.** In a two-domain population, a volume fraction f points along $+z$ with magnitude M_s and the rest along $-z$. Derive the net M_z . Evaluate M_z for $f = 0.65$ and $M_s = 8.0 \times 10^5$ A/m. What value of f gives $M_z = -0.20 M_s$?
- A12.** [†] A quasistatic B – H loop has area 80 J/m³ per cycle in a core of volume 2.5×10^{-4} m³. Estimate the hysteresis power at 60 Hz if this area remains representative. A dynamic loop measured at that frequency instead has area 110 J/m³. What loss does its area imply, and why must the two results not simply be added?
- A13.** Calculate c from the rounded μ_0 and ϵ_0 in this chapter. A vacuum plane wave has electric amplitude 120 V/m. Find its magnetic amplitude and the direction of propagation if $\mathbf{E}$ points along $+y$ and $\mathbf{B}$ along $+z$ at a positive field maximum.
- A14.** A steady axial current I flows in a long straight wire. Derive the magnetic field outside the wire from Ampere–Maxwell, explicitly stating the symmetry argument and the reason the displacement term vanishes. Why is the same circular symmetry not generally exact for a short wire segment with a nearby return conductor?

12.22.3 Engineering, measurement and computation

- E1.** [†] A transformer winding has $N = 400$ turns and a nearly uniform core area $A = 2.0 \times 10^{-4} \text{ m}^2$. Its sinusoidal voltage is $12 \text{ V}_{\text{rms}}$. Neglect winding drop. Find B_{pk} at 50 Hz and the minimum frequency consistent with a chosen 1.0 T limit. If a switching error produces a net 0.020 V s per cycle, calculate the flux-density drift per cycle before nonlinear effects alter the waveform.
- E2.** A laminated conductor satisfies the thin-sheet assumptions of the eddy-current model. How does its predicted average eddy loss change when sheet thickness is halved and frequency doubled at fixed B_{pk} ? Repeat at fixed sinusoidal winding voltage for a fixed core and winding, provided saturation is avoided. Explain which assumption is most likely to fail as frequency rises.
- E3.** Two candidate cores of equal volume have quasistatic loop areas 40 and 200 J/m^3 . For $V_c = 3.0 \times 10^{-5} \text{ m}^3$ and $f = 400 \text{ Hz}$, compare hysteresis estimates. List the additional measurements needed before claiming that the lower-area material has lower total operating loss.
- E4.** [†] A search coil has $N_s = 100$, area $(2.00 \pm 0.04) \times 10^{-4} \text{ m}^2$ and measured peak voltage $(0.126 \pm 0.002) \text{ V}$ at $f = (50.0 \pm 0.1) \text{ Hz}$. With the field normal to the coil, estimate B_{pk} and its combined uncertainty, treating the stated uncertainties as independent standard uncertainties. What systematic error results if the normal is actually tilted by 10° but the analyst assumes perfect alignment?
- E5.** Plan a toroidal B – H measurement. Specify the two recorded signals, the conversions to H and B , the sign test, the need for synchronization and one check for integrator drift. Explain why an unknown integration constant matters for remanence but not for the area of a closed loop.
- E6.** [†] One cycle has the ordered (H, B) pairs: $(100, 0.4)$, $(-100, 0.4)$, $(-100, -0.4)$, $(100, -0.4)$, $(100, 0.4)$, in A/m and tesla. Use the trapezoidal line integral to estimate the dissipated energy density. Calculate the average power in $2.0 \times 10^{-5} \text{ m}^3$ at 50 Hz. Explain why sorting these points by H destroys the required information.
- E7.** Run the capacitor code in Sec. 12.19.3. Extend the refinement to 160 points per τ , record the error, and compute the ratio to the 80-point result. Derive the asymptotic order from the expression involving $\cosh(\Delta t/\tau)$. State what this exercise tests and what it does not test experimentally.
- E8.** Add deterministic measurement noise to the electric field in that calculation: $E_j^{\text{noisy}} = E_j + 10^{-3} E_\infty \sin(0.7j)$. Use the same centered derivative for 10, 20, 40 and 80 points per τ . Report maximum errors normalized by I_0 . Explain why finer sampling of this specified sample-to-sample noise can increase derivative error. This noise prescription changes its physical frequency with the grid; identify that modeling limitation.
- E9.** [†] A circuit loop of area 5.0 cm^2 is normal to a sinusoidal stray field of amplitude $20 \mu\text{T}$ at 30 kHz. Estimate the induced peak emf. Separately, a stray capacitance $C_s = 5.0 \text{ pF}$ couples a 3.0 V peak signal at 2.0 MHz into a 200Ω resistive input. Estimate the input voltage and check the small- $\omega C_s R$ approximation. Propose one geometric mitigation for each coupling path.
- E10.** Plot $B(r)/B(R)$ from $r = 0$ to $3R$ for the ideal capacitor. Mark its maximum and compare the two one-sided slopes at R . Explain why a real finite-plate field need not have the sharp slope change of the ideal model. State where the neglected return current would alter the far-field prediction.

- E11.** A shield for a slowly varying external magnetic field has high initial permeability but a narrow region with a much smaller cross-section. Explain why that region may limit performance even if the quoted permeability is large. Identify what field or flux estimate and material data would be needed to assess saturation. Compare this flux-routing mechanism with eddy-current screening at higher frequencies.
- E12.** A toroidal loss measurement contains $B(t) = B_0 \sin \omega t$ and a reversible $H(t) = H_0 \sin \omega t$. The recorded B channel is accidentally delayed so that $B_{\text{rec}} = B_0 \sin(\omega t - \phi)$. Derive the apparent magnitude of $\oint H dB_{\text{rec}}$ and evaluate it for $H_0 = 100 \text{ A/m}$, $B_0 = 0.20 \text{ T}$ and $\phi = 2^\circ$. What does the true loop area equal?

12.22.4 Advanced challenges

- Ch1.** Starting with the two curl equations, derive the vacuum wave equation for $\mathbf{B}$. Use the divergence equation explicitly. For the plane-wave pair in the advanced section, check both curl equations and derive $E_0 = cB_0$ and $\omega = ck$; explain why solving only one wave equation is insufficient.
- Ch2.** [†] A weakly conducting linear dielectric has $\epsilon_r = 4.0$, $\sigma = 2.0 \times 10^{-4} \text{ S/m}$ and is driven at $\omega = 1.0 \times 10^7 \text{ rad/s}$ with field amplitude $E_0 = 100 \text{ V/m}$. Find the conduction and material displacement current-density amplitudes, their ratio, the total current-density amplitude and its phase relative to E . Find the bulk charge-relaxation time and compare it with $1/\omega$.
- Ch3.** At a plane interface with normal $+\hat{z}$ from medium 1 to 2, there is a free sheet current $\mathbf{K}_f = 20\hat{x} \text{ A/m}$. In medium 1, $H_{1y} = 5.0 \text{ A/m}$. Find H_{2y} with the correct sign. If $B_{1z} = 0.30 \text{ mT}$, what is B_{2z} ? State which field components these data do not determine.
- Ch4.** A cylinder has uniform axial magnetization $M = 8.0 \times 10^5 \text{ A/m}$ and radius $a = 5.0 \text{ mm}$. Find its bound volume and surface currents, including on the ends. For a very long cylinder estimate interior B and H . Explain why a finite cylinder needs a correction even if M were constrained to remain uniform.
- Ch5.** [†] A spherical, reversible, linear sample has measured $M/H_{\text{appl}} = 1.5$. Infer its intrinsic susceptibility using $N_d = 1/3$. Find its magnetization and internal field at $H_{\text{appl}} = 200 \text{ A/m}$. Explain why interpreting the apparent ratio as intrinsic susceptibility would be wrong.
- Ch6.** [†] An ideal gapped core has $N = 300$, $A = 1.5 \times 10^{-4} \text{ m}^2$, $\ell_c = 0.30 \text{ m}$, $\mu_r = 1500$ and $g = 0.80 \text{ mm}$. Find its inductance and the current producing $B = 0.20 \text{ T}$. Compute the gap and core energies and compare their sum with $LI^2/2$. Derive the logarithmic sensitivity $\partial \ln L / \partial \ln \mu_r$ at fixed geometry.
- Ch7.** [†] A two-state paramagnet has number density $n = 1.0 \times 10^{28} \text{ m}^{-3}$ and moment $m = 9.27 \times 10^{-24} \text{ A m}^2$. At $B = 1.0 \text{ T}$ and $T = 300 \text{ K}$, compute x , M_s , M and the weak-response susceptibility estimate. Use $k_B = 1.380649 \times 10^{-23} \text{ J/K}$. Estimate the field required for $M = 0.5M_s$ within this model and explain the limitations of that extrapolation.
- Ch8.** For a quasistatic circular vacuum capacitor, derive the radial Poynting flux at any $r < R$ and its inward power through a coaxial cylindrical sidewall of length d . Show that it equals the rate of electric-energy increase inside that radius. On discharge, analyze the sign. State why this is an approximate energy balance for the assumed leading fields.

12.23 Selected detailed solutions

The solutions emphasize the choice of law, geometry and model. A correct numerical value without those choices is not a complete field argument. Extra digits are retained in intermediate calculations

for verification; the final precision is limited by the input data.

12.23.1 C3: one boundary, two surfaces

Identify the invariant. The circulation $\oint_C \mathbf{B} \cdot d\mathbf{\ell}$ depends on the field evaluated along C . Replacing a spanning surface does not change that curve or its field values. The right side of Ampere–Maxwell must therefore have the same value for both surfaces, with normals tied to the same loop orientation.

Account for each surface separately. In the ideal charging-capacitor picture, the surface through the lead carries approximately the wire current I ; its displacement contribution is negligible in the quasistatic wire model. The surface through the vacuum gap carries no charge current and has $I_d = \epsilon_0 d\Phi_E/dt = dQ/dt = I$. Thus each evaluation gives $\mu_0 I$, not $2\mu_0 I$. The equality refers to sums evaluated on individual surfaces, not to adding contributions obtained on different surfaces.

Check beyond the ideal picture. If the surfaces include regions with appreciable time-dependent electric fields, both contributions may be nonzero on either surface. The equality of their sums still follows from charge conservation and electric Gauss law, Eq. (12.46). The vacuum gap distinguishes absence of charge transport from absence of electromagnetic effects; it does not break the conservation of charge.

12.23.2 C9: what crosses zero on a hysteresis loop?

Read the axes first. On a B – H plot, remanent flux density is the intercept $B = \pm B_r$ at $H = 0$ after an appropriate magnetization history. A coercive field defined on that plot has magnitude H_c at $B = 0$ on the return branch. At the crossing $H = -H_c$, the relation $B = \mu_0(H + M)$ requires $M = +H_c$. Magnetization is therefore not zero there. An M – H loop reaches $M = 0$ at an intrinsic coercive field that can differ from H_c .

Distinguish two meanings of saturation. Once the material magnetization approaches M_s , further alignment has little room to increase M . Nevertheless $B \simeq \mu_0(H + M_s)$ continues to increase with slope approximately μ_0 . A schematic whose high-field B branches become perfectly horizontal can conceal this distinction. Practical core specifications often use a chosen saturation flux density or knee as an operating limit, but that engineering designation is not a mathematical ceiling on B .

12.23.3 A2: induced electric field inside and outside a changing flux region

Choose counterclockwise traversal as viewed from $+z$. Faraday’s law gives $2\pi r E_\phi = -\dot{\Phi}_B$. The flux-changing area is πr^2 inside the cylinder and πR^2 outside it. Therefore

$$E_\phi = \begin{cases} -r\dot{B}_z/2, & r \leq R, \\ -R^2\dot{B}_z/(2r), & r \geq R. \end{cases}$$

Since $\dot{B}_z = -0.40$ T/s, both values of E_ϕ are positive. At $r = 0.020$ m,

$$\oint \mathbf{E} \cdot d\mathbf{\ell} = -(\pi \times 0.020^2)(-0.40) = 5.03 \times 10^{-4} \text{ V}, \quad E_\phi = 4.00 \times 10^{-3} \text{ V/m}.$$

At $r = 0.080$ m, the linked changing flux has already reached its full value:

$$\oint \mathbf{E} \cdot d\mathbf{\ell} = -(\pi \times 0.040^2)(-0.40) = 2.01 \times 10^{-3} \text{ V}, \quad E_\phi = 4.00 \times 10^{-3} \text{ V/m}.$$

The equal field magnitudes at these two radii are specific to the numbers chosen; the circulations differ because the circumferences differ. At $r = R$, both expressions give $E_\phi = 8.00 \times 10^{-3}$ V/m, so the field is continuous in the ideal model. A conducting test loop would carry a counterclockwise current tending to support the decreasing $+z$ magnetic flux. The units m T/s = V/m confirm the result.

12.23.4 A5: a capacitor field profile requires enclosed current

Symmetry and source. Under axial symmetry, B_ϕ is constant on the circular integration path and its line integral is $2\pi r B_\phi$. For $r < R$, only fraction r^2/R^2 of the uniform displacement-current density is enclosed. For $r > R$, the whole ideal gap current is enclosed. Thus

$$B_\phi(r) = \frac{\mu_0 I}{2\pi} \begin{cases} r/R^2, & r \leq R, \\ 1/r, & r \geq R. \end{cases}$$

With $R = 0.050$ m and $I = 0.40$ A the fields are

r	Enclosed displacement current	B_ϕ
0.010 m	0.016 A	$0.32 \mu\text{T}$
0.050 m	0.40 A	$1.60 \mu\text{T}$
0.100 m	0.40 A	$0.80 \mu\text{T}$

The maximum occurs at the ideal edge because the interior field rises linearly while the exterior field decreases as $1/r$. At the first radius, using the full current would overestimate the field by $(R/r)^2 = 25$. Positive axial displacement current produces positive, counterclockwise B_ϕ as viewed from $+z$.

Validity. Real plate edges and return leads disturb the ideal geometry. This calculation is a useful local quasistatic approximation; it is not a prediction that $B \propto 1/r$ persists arbitrarily far from a complete finite charging circuit.

12.23.5 A6: voltage drive fixes the current phase

Choose the constitutive relation. A vacuum parallel-plate capacitor has

$$C = \frac{\varepsilon_0 \pi R^2}{d} = \frac{(8.854 \times 10^{-12})\pi(0.030)^2}{5.0 \times 10^{-4}} = 5.01 \times 10^{-11} \text{ F}.$$

Charge is $Q = CV_0 \sin \omega t$ and current is $I = \omega CV_0 \cos \omega t$. With $\omega = 2\pi \times 10^5 \text{ s}^{-1}$, its amplitude is $6.29 \times 10^{-4} \text{ A} = 0.629 \text{ mA}$.

At $r = R/2$, Eq. (12.31) gives

$$B_{\text{pk}} = \frac{\mu_0 I_{\text{pk}}}{4\pi R} = 2.10 \times 10^{-9} \text{ T} = 2.10 \text{ nT}.$$

The magnetic field is in phase with the current and leads V by a quarter cycle for the stated conventions. It is largest when V crosses zero and vanishes at voltage extrema. This is not the in-phase E – B relation of a freely travelling plane wave.

The apparatus parameter is $\omega L_{\text{sys}}/c = 2.09 \times 10^{-4} \ll 1$, supporting a quasistatic treatment of propagation across the apparatus. This check alone does not eliminate fringe fields or return-path asymmetry. Finally, $\mu_0 I/r$ has units tesla, and the small value is plausible for a sub-milliampere current at centimeter scales.

12.23.6 A12: cycle energy, power and double counting

An area in the B – H plane has units $\text{T A/m} = \text{J/m}^3$. Multiply the quasistatic area by core volume to get energy per cycle:

$$W_{\text{qs}} = 80(2.5 \times 10^{-4}) = 0.020 \text{ J}, \quad P_{\text{hyst}} \simeq 60W_{\text{qs}} = 1.20 \text{ W}.$$

The finite-frequency measured area gives instead

$$P_{\text{dynamic}} = 60(110)(2.5 \times 10^{-4}) = 1.65 \text{ W}.$$

A correctly measured dynamic loop already contains the magnetic cycle loss represented by that measurement, including frequency-dependent broadening. It is not a separate loss to be added to 1.20 W. Under comparable waveform, amplitude, temperature and calibration, the difference 0.45 W is an estimate of additional dynamic loss, but the area alone does not uniquely divide that difference among eddy currents and other mechanisms. Winding copper loss is not generally included in the core loop area and must be accounted for separately in a device power budget.

12.23.7 E1: volt-seconds and core flux

For a sinusoidal nearly uniform core field, $V_{\text{rms}} = 4.44fNAB_{\text{pk}}$. At 50 Hz,

$$B_{\text{pk}} = \frac{12}{4.44(50)(400)(2.0 \times 10^{-4})} = 0.676 \text{ T}.$$

At the chosen 1.0 T limit, $f_{\text{min}} = 33.8 \text{ Hz}$. Reducing frequency at fixed voltage increases flux swing; it does not simply make the core easier to drive. The exact sinusoidal coefficient is $\sqrt{2}\pi$, so use of 4.44 introduces only rounding here.

A nonzero net voltage integral adds a flux offset each cycle:

$$\Delta B = \frac{1}{NA} \int_{\text{cycle}} v \, dt = \frac{0.020}{(400)(2.0 \times 10^{-4})} = 0.25 \text{ T}.$$

The sign follows the signed volt-seconds. This offset accumulates on top of the alternating flux; a device can enter saturation even if its first-cycle alternating swing was acceptable. Once nonlinear current and circuit voltage drops matter, the simple prescribed-voltage calculation no longer describes the complete circuit. A flux-density limit also needs allowance for temperature, tolerances and the actual material curve.

12.23.8 E4: search-coil calibration and uncertainty

The measured voltage amplitude is $v_{\text{pk}} = N_s A_s \omega B_{\text{pk}}$ for a normal, uniform sinusoidal field. Therefore

$$B_{\text{pk}} = \frac{0.126}{100(2.00 \times 10^{-4})(2\pi \times 50.0)} = 0.02005 \text{ T}.$$

For independent small standard uncertainties, linear propagation in the product and quotient gives

$$\left(\frac{u_B}{B}\right)^2 = \left(\frac{0.002}{0.126}\right)^2 + \left(\frac{0.04}{2.00}\right)^2 + \left(\frac{0.1}{50.0}\right)^2 = 6.56 \times 10^{-4}.$$

Thus $u_B/B = 0.0256$ and the result can be reported as $B_{\text{pk}} = (20.1 \pm 0.5) \text{ mT}$ with the stated uncertainty interpretation. The integer turn count is treated as exact; coil loading, field nonuniformity and sensor calibration would require additional terms if significant.

With a tilt θ , the voltage is proportional to $B \cos \theta$. Ignoring a 10° tilt underestimates the true amplitude by $1 - \cos 10^\circ = 1.52\%$. This is a systematic geometric bias, not a random error that disappears by repeating the same tilted measurement. The corrected field would be the inferred value divided by $\cos 10^\circ$.

12.23.9 E6: integrating around a loop in time order

The trapezoidal line integral is

$$w \simeq \sum_i \frac{H_i + H_{i+1}}{2} (B_{i+1} - B_i).$$

The first and third edges have constant B , so their contributions vanish. Along the second edge, $H = -100 \text{ A/m}$ while $\Delta B = -0.8 \text{ T}$, giving $+80 \text{ J/m}^3$. The fourth edge contributes another $+80 \text{ J/m}^3$. Hence $w = 160 \text{ J/m}^3$, agreeing with the rectangle estimate $4H_c B_r$.

For the specified core and frequency,

$$P = f w V_c = 50(160)(2.0 \times 10^{-5}) = 0.160 \text{ W}.$$

Reversing traversal changes the sign of the line integral; the physically dissipated energy for the actual passive cycle is positive. Taking an absolute value is appropriate after confirming the orientation, not as a substitute for checking sensor polarity. Sorting by H interleaves points from different branches and destroys the chronology that distinguishes increasing and decreasing field histories. Numerical loop integration needs the cycle order and explicit closure.

12.23.10 E9: two different coupling paths

For the magnetic path, Faraday's law gives $\mathcal{E}_{\text{pk}} = A\omega B_{\text{pk}}$ under uniform normal exposure. Converting 5.0 cm^2 to $5.0 \times 10^{-4} \text{ m}^2$,

$$\mathcal{E}_{\text{pk}} = (5.0 \times 10^{-4})(2\pi \times 3.0 \times 10^4)(20 \times 10^{-6}) = 1.88 \text{ mV}.$$

Reducing the loop area by routing outgoing and return conductors close together reduces this emf. The circuit's actual noise current depends on its impedance.

For capacitive coupling, the small-coupling current amplitude is $I_{\text{pk}} \simeq \omega C_s V_s = 0.188 \text{ mA}$. Into 200Ω , it gives $V_{\text{in, pk}} \simeq 37.7 \text{ mV}$. The approximation is controlled by $\omega C_s R = 0.0126 \ll 1$. In the ideal series-capacitor/resistor model, the exact voltage magnitude is

$$|V_{\text{in}}| = V_s \frac{\omega C_s R}{\sqrt{1 + (\omega C_s R)^2}},$$

differing by less than 0.01% from the estimate. Increasing separation from the aggressor or inserting a suitably connected conducting screen reduces the coupling capacitance. Reducing magnetic loop area is useful but does not by itself remove this electric coupling. Peak and rms amplitudes must be handled consistently when comparing the mechanisms.

12.23.11 Ch2: displacement and conduction are in quadrature

The permittivity is $\varepsilon = 4\varepsilon_0 = 3.54 \times 10^{-11} \text{ F/m}$. With $E = E_0 \cos \omega t$,

$$\begin{aligned} J_c(t) &= 0.0200 \cos \omega t \text{ A/m}^2, \\ J_D(t) &= -0.0354 \sin \omega t \text{ A/m}^2. \end{aligned}$$

The amplitude ratio is 1.77. A negative sine term added to a cosine can be written as a cosine with a positive phase advance:

$$J_{\text{total}} = J_0 \cos(\omega t + \delta), \quad J_0 = \sqrt{0.0200^2 + 0.0354^2} = 0.0407 \text{ A/m}^2, \quad \delta = \tan^{-1}(1.77) = 60.5^\circ.$$

Arithmetic addition of the two amplitudes would incorrectly give 0.0554 A/m^2 . The displacement contribution is reversible field/polarization response in this simple constant- ε model, whereas the Ohmic term has positive average heating $\langle J_c E \rangle = \sigma E_0^2 / 2 = 1.0 \text{ W/m}^3$.

The relaxation time is $\tau_q = \varepsilon / \sigma = 1.77 \times 10^{-7} \text{ s}$, compared with $1/\omega = 1.00 \times 10^{-7} \text{ s}$. Their ratio is again 1.77. Both currents must be retained. This uses frequency-independent real ε and σ ; dispersive dielectric loss would require a more complete constitutive model.

12.23.12 Ch5: removing a shape effect from susceptibility

Let $\chi_a = M/H_{\text{appl}}$. From $\chi_a = \chi_m / (1 + N_d \chi_m)$,

$$\chi_a + N_d \chi_a \chi_m = \chi_m, \quad \chi_m = \frac{\chi_a}{1 - N_d \chi_a}.$$

With $\chi_a = 1.5$ and $N_d = 1/3$, $\chi_m = 3.0$. At $H_{\text{appl}} = 200 \text{ A/m}$ the measured magnetization is $M = 300 \text{ A/m}$. The internal field is $H_{\text{int}} = 200 - 300/3 = 100 \text{ A/m}$, so $M/H_{\text{int}} = 3.0$, as required.

The applied field is what would be present without the specimen; the specimen's own demagnetizing field reduces the internal field in this geometry. A susceptibility inferred by dividing by the wrong field consequently underestimates the intrinsic response here. The inversion also becomes sensitive to uncertainty as $N_d\chi_a$ approaches one. A fitted $\chi_a > 1/N_d$ cannot describe this positive, constant, reversible susceptibility model and would signal a failed assumption or inconsistent data rather than an arbitrarily large positive χ_m .

12.23.13 Ch6: gap energy and sensitivity to permeability

The effective magnetic length is

$$\ell_{\text{eff}} = g + \ell_c/\mu_r = 0.80 \times 10^{-3} + 0.30/1500 = 1.00 \times 10^{-3} \text{ m}.$$

Thus

$$L = \frac{\mu_0 N^2 A}{\ell_{\text{eff}}} = 1.70 \times 10^{-2} \text{ H},$$

$$I = \frac{B\ell_{\text{eff}}}{\mu_0 N} = 0.531 \text{ A}.$$

The magnetic energies are calculated over actual core and gap volumes:

$$U_g = \frac{B^2 A g}{2\mu_0} = 1.91 \text{ mJ},$$

$$U_c = \frac{B^2 A \ell_c}{2\mu_0 \mu_r} = 0.477 \text{ mJ}.$$

Their sum is 2.39 mJ, equal to $LI^2/2$ using unrounded values. The ratio $U_g/U_c = 4$ shows that 80% of the model's magnetic energy lies in the gap.

For the sensitivity, differentiate $\ln L = \ln(\mu_0 N^2 A) - \ln(g + \ell_c/\mu_r)$ with respect to $\ln \mu_r$:

$$\frac{\partial \ln L}{\partial \ln \mu_r} = \frac{\ell_c/\mu_r}{g + \ell_c/\mu_r} = 0.20.$$

A small 10% permeability change therefore produces approximately a 2% inductance change at this operating point. Without a gap the sensitivity would be one. The gap trades lower inductance and greater required ampere-turns for reduced material sensitivity and greater energy-storage capacity before reaching a chosen B limit. Fringing and nonlinear permeability bound the accuracy of this ideal circuit calculation.

12.23.14 Ch7: thermal disorder keeps the response small

The dimensionless energy ratio is

$$x = \frac{(9.27 \times 10^{-24})(1.0)}{(1.380649 \times 10^{-23})(300)} = 2.238 \times 10^{-3}.$$

The saturation magnetization is $M_s = nm = 9.27 \times 10^4 \text{ A/m}$, but the actual magnetization is only

$$M = M_s \tanh x \simeq M_s x = 2.07 \times 10^2 \text{ A/m}.$$

The weak-response susceptibility is $\chi_m \simeq \mu_0 nm^2/(k_B T) = 2.61 \times 10^{-4}$. It is small enough to support the approximation $B \simeq \mu_0 H$ used to infer it. The relative error of replacing $\tanh x$ by x is

approximately $x^2/3 = 1.67 \times 10^{-6}$ at these parameters, much less than the precision of the physical model.

For half saturation, $\tanh x = 0.5$ implies $x = \operatorname{artanh}(0.5) = \frac{1}{2} \ln 3 = 0.5493$. The formal field is

$$B = \frac{k_B T}{m} (0.5493) \simeq 2.45 \times 10^2 \text{ T}.$$

This large number expresses how weak the assumed moment–field energy is compared with thermal energy at room temperature. It is not a recommendation for an experiment or a universal saturation field: actual level structure, interactions and material changes may invalidate the two-state independent-moment model long before such an extrapolation is useful.

12.23.15 Checks and assessment criteria for the remaining problems

These are deliberately shorter than the preceding solutions. A complete submission should still show reasoning, signs and assumptions.

C1. Empty space with a nonzero uniform field has zero closed-surface flux; a surface enclosing equal and opposite charges also has zero net flux. Only the net enclosed charge is fixed by the observed flux.

C2. Flux enters through the surface portion inside the magnet and exits through the surrounding region. Cutting the magnet produces two dipoles in the ordinary magnetic-material picture.

C4–C6.

The induced electric field does not require a wire. Current requires impedance and a closed conducting path. A moving loop adds $\mathbf{u} \times \mathbf{B}$ to the emf integrand. A uniform nonzero magnetic field has zero circulation around a closed loop and is a counterexample to inferring a zero field from one zero integral.

C7–C8.

Equilibrium diamagnetism is a static material response; resistive eddy currents require a driving change and decay without it. Opposed domain moments can cancel macroscopically; shifting their volume fractions changes the average moment.

C10–C12.

Soft cores favor low cycle loss and easy reversal; permanent magnets require resistance to demagnetization. Frequency, temperature, saturation, conductivity and geometry also matter. Constant scalar susceptibility assumes a linear isotropic reversible range. The capacitor is a near-field driven configuration, with geometry-dependent amplitudes and generally different phase relations from a travelling plane wave.

A1. $q_{\text{enc}} = -0.5 \text{ nC}$, giving $\Phi_E = -56 \text{ N m}^2/\text{C}$ to two significant figures.

A3. $\mathcal{E} = -2\pi r^2 \alpha t = -0.0377 \text{ V}$; $I = -7.54 \text{ mA}$ with positive current counterclockwise as seen from $+z$.

A4. $I_d = 8.85 \times 10^{-5} \text{ A}$, approximately $89 \mu\text{A}$. Reversing the normal reverses the signed result.

A7. $Q = I_0 \tau (1 - e^{-t/\tau})$, $E = Q/(\epsilon_0 A)$ and $V = Ed$. Differentiation gives $I_d = I$. The ideal model retains charge without leakage after the current decays.

A8. $m = 1.50 \times 10^{-3} \text{ A m}^2$, $\tau = 6.0 \times 10^{-4} \text{ N m}$ and $U = -1.04 \times 10^{-3} \text{ J}$. The quasistatic external work is $\Delta U = 2mB = 2.40 \times 10^{-3} \text{ J}$.

A9. $M = 300 \text{ A/m}$, $\chi_m = 0.015$, $\mu_r = 1.015$ and $B = 2.55 \times 10^{-2} \text{ T}$.

A10. $\chi_m(200 \text{ K}) = 1.8 \times 10^{-3}$ and $M(200)/M(300) = 1.5$. The weak-field, thermally disordered regime cannot be extended to zero temperature by this formula.

A11. $M_z = (2f - 1)M_s = 2.4 \times 10^5 \text{ A/m}$; $f = 0.40$ gives $-0.20M_s$.

A13–A14.

$c \simeq 2.998 \times 10^8 \text{ m/s}$, $B_0 = 4.00 \times 10^{-7} \text{ T}$ and propagation along $+x$. In the long-wire steady problem, $2\pi rB = \mu_0 I$; the symmetry and steady-field assumptions are both essential.

E2. At fixed B_{pk} , $p_e \propto a^2 f^2$ gives no net change. At fixed sinusoidal voltage, $B_{\text{pk}} \propto 1/f$, giving a factor $1/4$. Nonuniform penetration, frequency-dependent response and proximity effects can invalidate the thin-sheet approximation.

E3. The estimates are 0.48 and 2.4 W. Compare dynamic losses at the actual frequency, waveform and flux swing, thermal behavior, conductivity/lamination construction and usable saturation range.

E5. Record winding current and search-coil voltage. Use $H = N_p I / \ell$ and signed voltage integration for B . Check polarity, relative delay and cycle closure. A constant B offset does not alter dB in the area integral, but moves its remanence intercept.

E7. At 160 points per τ , the normalized error is about 9.705×10^{-6} ; the preceding error divided by this one is about 3.98. The leading error is proportional to $(\Delta t)^2$.

E8. The added derivative error has the form $10^{-3}(\tau/\Delta t) \sin(0.7) \cos(0.7j)$. Its amplitude grows in proportion to points per τ . Maximum combined normalized errors are approximately 0.007459, 0.013281, 0.025890 and 0.051565 for the stated grids. The exact maxima include the numerical integration error as well as the noise term.

E10. The normalized curve is x for $x = r/R \leq 1$ and $1/x$ for $x \geq 1$. Its slopes at the ideal edge are $+1$ and -1 . Fringing smooths the sharp model transition; a complete return circuit changes the distant field.

E11. Flux concentration can drive the narrow region into a nonlinear low-incremental-permeability range. Estimate its $B = \Phi/A$ and compare with the appropriate material curve at the operating temperature and frequency. Eddy-current screening instead depends on time variation and conductive current paths.

E12. The apparent area magnitude is $\pi H_0 B_0 |\sin \phi| = 2.19 \text{ J/m}^3$; the true area is zero. This is an instrumentation artifact in the stipulated reversible model.

Ch1. Use $\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$ and $\nabla \cdot \mathbf{B} = 0$ to obtain $\nabla^2 \mathbf{B} = \mu_0 \varepsilon_0 \partial_t^2 \mathbf{B}$. The plane-wave curl equations give $kE_0 = \omega B_0$ and $kB_0 = \mu_0 \varepsilon_0 \omega E_0$.

Ch3. $\hat{z} \times [(H_{2y} - H_{1y})\hat{y}] = -(H_{2y} - H_{1y})\hat{x}$. Hence $H_{2y} = -15 \text{ A/m}$ and $B_{2z} = 0.30 \text{ mT}$. The other tangential component is constrained by the zero y component of sheet current, but its absolute value is not supplied; normal H requires material information.

Ch4. $\mathbf{J}_b = 0$ in the uniform bulk; $\mathbf{K}_b = M\hat{\phi}$ on the side and zero on the ends. For the long-cylinder interior, $B \simeq 1.01 \text{ T}$, $H \simeq 0$. Finite ends produce a demagnetizing field.

Ch8. $S_r(r) = -\varepsilon_0 r E \dot{E}/2$. Inward power is $\varepsilon_0 \pi r^2 dE \dot{E}$, equal to the leading electric-energy rate in that volume. It reverses during discharge. Magnetic-energy and field-nonuniformity corrections belong to higher orders of the quasistatic approximation.

Bibliography

- [1] D. Halliday, R. Resnick and J. Walker, *Fundamentals of Physics*, Volume 2, 12th ed., Wiley, 2022. Chapter 32, especially Secs. 32.1–32.8.
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th ed., Brooks/Cole, 2014. Sec. 30.6 and Secs. 34.1–34.2.
- [3] S. J. Ling, J. Sanny and W. Moebs, *University Physics*, Volume 2, OpenStax, 2016. Secs. 12.7 and 16.1, available at <https://openstax.org/details/books/university-physics-volume-2>.
- [4] NIST, *2022 CODATA Recommended Values of the Fundamental Physical Constants*, complete table, <https://physics.nist.gov/cuu/pdf/all.pdf>. Used for SI status and numerical constants.