

Chapter 13

Electromagnetic Waves

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Preface

Chapter 13 develops one of the central syntheses of classical physics: electric and magnetic fields can sustain one another and propagate through space as waves. Earlier chapters treated electric fields, magnetic fields, electromagnetic induction, inductance, alternating-current circuits, and Maxwell's equations largely in situations where the geometry of a device or circuit remained visible in the calculation. Electromagnetic waves require a conceptual enlargement. The fields themselves become dynamical physical entities that transport energy and momentum from one region of space to another, including across vacuum.

This chapter is written for technical, engineering, computer-science, space-science, and applied-science students who need both the field-theory foundation and the operational formulas used in technology. The main chapter therefore proceeds in three layers. First, Maxwell's equations are used to motivate and derive the vacuum wave equations. Second, plane-wave solutions are analyzed carefully: their phase, wavelength, frequency, propagation speed, field geometry, energy density, Poynting vector, intensity, and momentum transport are developed with explicit assumptions. Third, the wave picture is connected to polarization, reflection and refraction, total internal reflection, resonant cavities, antennas, sensing, communication, and optical systems.

The chapter deliberately keeps the distinction between exact statements and idealizations visible. A monochromatic plane wave of infinite transverse extent is an ideal mathematical model. Real transmitters emit finite beams or radiation patterns, real media may absorb and disperse, and practical antennas have near-field and far-field regions. The plane-wave model is nevertheless extraordinarily useful because many local properties of sufficiently extended waves and beams reduce to it, and because it exposes the structure enforced by Maxwell's equations with exceptional clarity.

All examples, diagrams, exercises, and computational tasks are written as original teaching material for this lecture-note series. The supplied textbooks have been compared for scientific coverage, derivations, visual explanations and problem-solving approaches at the calculus-based Physics II level. The wording, figures, problem statements, and instructional organization are independent.

The Guideline

For every electromagnetic-wave problem, identify five items before substituting numbers: (1) the propagation direction, (2) the electric-field direction, (3) the magnetic-field direction, (4) whether the stated field value is instantaneous, peak, or rms, and (5) whether the problem concerns instantaneous energy flow or time-averaged intensity. Most avoidable errors in this chapter come from skipping one of these checks.

Pedagogical boundary

The main chapter treats plane electromagnetic waves in vacuum and simple transparent, isotropic media; polarization with ideal polarizers; introductory reflection, refraction, total internal reflection, and Brewster-angle physics; and basic standing-wave/cavity ideas. A later optics treatment may revisit refraction, image formation, interference, diffraction, and resolving power in greater depth. The advanced enrichment part extends the present material to wave impedance, Fresnel coefficients at normal and oblique incidence, complex notation, dispersion, attenuation, waveguides, antenna radiation, polarization matrices, and a one-dimensional finite-difference time-domain model.

Conventions and notation

Vectors are printed in bold, for example $\mathbf{E}$ and $\mathbf{B}$. The Cartesian unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are equivalent to $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. The wave vector is $\mathbf{k} = k\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ denotes the propagation direction; it must not be confused with the Cartesian $\hat{\mathbf{k}}$ along z . Peak amplitudes have subscript 0 and rms values have subscript rms. For the standard traveling wave, propagation is along $+x$, the electric field is along y , and the magnetic field is along z . Angle measurements at interfaces are from the normal.

Numerical examples use rounded constants

$$c \simeq 2.998 \times 10^8 \text{ m s}^{-1}, \quad \varepsilon_0 \simeq 8.854 \times 10^{-12} \text{ F m}^{-1}, \quad \mu_0 \simeq 4\pi \times 10^{-7} \text{ H m}^{-1}.$$

In the SI, $c = 299\,792\,458 \text{ m s}^{-1}$ is exact. The quoted values of μ_0 and ε_0 are approximations, not exact defining values; their precision is more than sufficient here[4]. The vacuum relation $\mu_0\varepsilon_0c^2 = 1$ is retained throughout. Refractive indices, conductivities, efficiencies and source powers in examples are specified model inputs, not universal material constants.

Note
Prerequisites. Review sinusoidal waves, wavelength and frequency, superposition, vector cross products, energy and momentum from Physics I, and Faraday’s and Ampere–Maxwell laws from Chapters 10–12. Basic derivatives and integrals suffice for the core. Partial derivatives mean varying one coordinate while holding the others fixed; vector calculus and complex fields are developed in the optional enrichment part.

The core develops the field picture and observable quantities first. The enrichment sections are labeled 13.A and may be studied selectively. The final problem collection distinguishes core and enrichment tasks, and the notation dictionary and model-limit table provide a reference without interrupting the main narrative.

Chapter 13

Electromagnetic Waves

13.1 Motivation and roadmap

Faraday's law and the Ampere–Maxwell law couple spatial variation of one field to time variation of the other. Together they permit a disturbance to propagate away from its source, even through vacuum. The electric and magnetic parts evolve together as one electromagnetic field; they do not take turns creating each other. The source need not remain in mechanical contact with the receiver because the field itself carries energy and momentum.

This is the physical basis of radio communication, radar, microwave heating, remote sensing, optical fibers, laser ranging, wireless networks, solar-energy conversion, astronomical observation, and much of modern instrumentation. At very different frequencies, what we call radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X rays, and gamma rays are all manifestations of the same electromagnetic field theory.

The logical path of the chapter is shown in Fig. 13.1. We begin with Maxwell's equations and derive wave equations in empty space. We then study the plane-wave solutions, their geometry and speed, and the relation between the electric and magnetic amplitudes. From the fields we construct energy density, the Poynting vector, intensity, momentum flow, and radiation pressure. Polarization is introduced as a controlled degree of freedom. Finally, the wave model is connected to interfaces, total internal reflection, standing waves, cavities, antennas, and engineering systems.

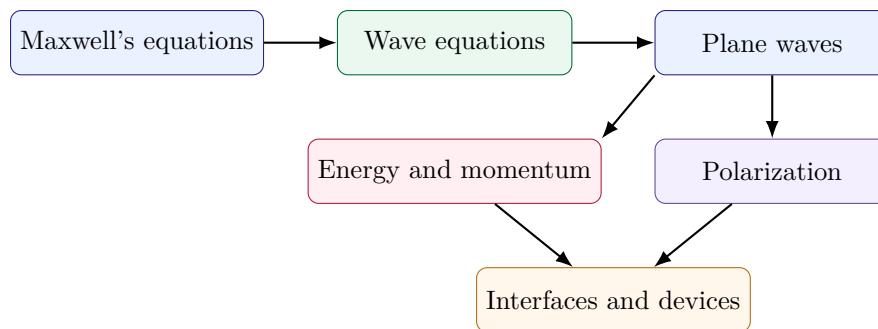


Figure 13.1: Conceptual roadmap. Maxwell's equations imply propagating fields. The plane-wave model exposes the geometry of the fields, while energy, momentum, polarization, and interface physics connect the field theory to measurement and technology.

Learning goals

After studying the main chapter, the student should be able to:

- explain qualitatively how Maxwell's equations permit self-propagating electromagnetic disturbances;
- derive the one-dimensional vacuum wave equation for a simple field geometry;
- write and interpret sinusoidal plane-wave fields and identify k , ω , f , λ , and phase;

- use the orthogonality and right-handed relation among $\mathbf{E}$, $\mathbf{B}$, and the propagation direction;
- derive and use $c = 1/\sqrt{\mu_0\epsilon_0}$ and $E_0 = cB_0$ in vacuum;
- distinguish instantaneous Poynting vector from time-averaged intensity;
- calculate energy density, intensity, power transfer, and inverse-square spreading for idealized sources;
- explain electromagnetic momentum transfer and calculate radiation pressure for ideal absorption and reflection;
- describe linear polarization and apply the one-half rule and Malus's law to sequences of ideal polarizers;
- apply Snell's law, the critical-angle condition, and Brewster's law with correct angle conventions;
- interpret standing electromagnetic waves and simple resonant-cavity conditions;
- connect the field theory to antennas, microwave systems, fibers, solar-energy systems, detectors, and communication technology.

Key message

Electromagnetic waves are not vibrations of a material medium. They are propagating configurations of electric and magnetic fields. In vacuum, their propagation speed is fixed by the electromagnetic constants ϵ_0 and μ_0 .

Guided checks

1. If an electromagnetic wave can travel through vacuum, what is oscillating?
2. Why does the existence of displacement current matter for wave propagation?
3. If $\mathbf{E}$ points along $+y$ and $\mathbf{B}$ along $+z$, in which direction does energy flow?
4. Which field determines the conventional polarization direction?

13.2 Maxwell's equations as the starting point

The electromagnetic-wave description is not an additional empirical rule appended to Maxwell's theory. It is a consequence of the theory. In integral form, Maxwell's equations in vacuum can be written as

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}, \quad (13.1)$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0, \quad (13.2)$$

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}, \quad (13.3)$$

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}. \quad (13.4)$$

The first two equations state the flux laws for electric and magnetic fields. Faraday's law, Eq. (13.3), connects a changing magnetic flux to a circulating electric field. The Ampère–Maxwell law, Eq. (13.4), connects conduction current and changing electric flux to a circulating magnetic field.

For a region of empty space with no free charge and no conduction current,

$$\rho = 0, \quad \mathbf{J} = 0. \quad (13.5)$$

The flux equations then imply no local free-charge source for $\mathbf{E}$ and no magnetic monopole source for $\mathbf{B}$. The two curl relations remain dynamically coupled: spatial variation of one field is related to time variation of the other.

At an introductory level, one can understand the coupling without vector calculus. Imagine a time-varying electric field in one region. Through the Ampere–Maxwell law it is associated with a curling magnetic field. If that magnetic field also varies, Faraday’s law associates it with a curling electric field. The coupled pattern can propagate away from the original source region. The more formal differential derivation is presented in the advanced enrichment part.

Theoretical point 13.1: Why Maxwell’s correction is essential

Without the displacement-current term $\mu_0\epsilon_0 d\Phi_E/dt$, the electromagnetic equations would not be mutually consistent for a charging capacitor and would not produce the correct symmetric coupling between time-varying electric and magnetic fields in empty space. Maxwell’s correction is therefore central both to local charge conservation and to the existence of the vacuum wave equation.

13.2.1 A one-dimensional derivation of the wave equation

Consider a simple plane-wave geometry in which the fields depend only on x and t ,

$$\mathbf{E}(x, t) = E_y(x, t)\hat{\mathbf{j}}, \quad \mathbf{B}(x, t) = B_z(x, t)\hat{\mathbf{k}}. \quad (13.6)$$

The coordinate dependence is an idealization of a wavefront that is uniform in y and z . A partial derivative such as $\partial E_y/\partial x$ therefore measures spatial change at a fixed time, whereas $\partial E_y/\partial t$ measures time change at a fixed position. Figure 13.2 fixes the loop orientations used below.

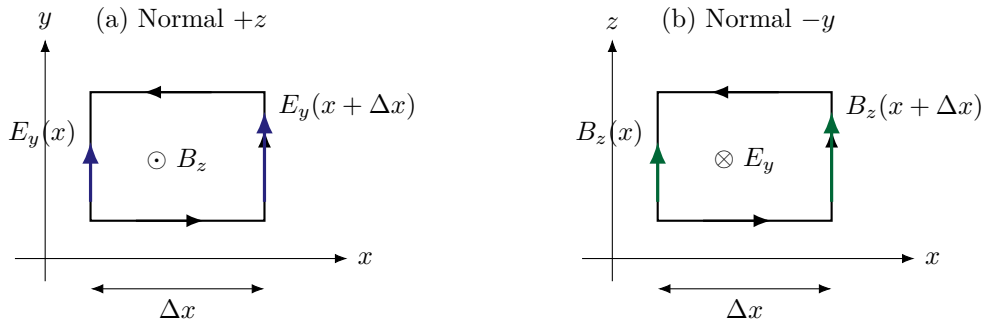


Figure 13.2: Rectangular loops of height ℓ , with positive traversal counterclockwise on the page. In (a), $+z$ is out of the page. In (b), the axes are x, z , so $+y$ points into the page and the outward loop normal is $-y$. The arrows illustrate a positive spatial field gradient, not a complete wave snapshot.

For the xy loop, choose the normal $+z$ and counterclockwise traversal. The horizontal sides contribute zero because $\mathbf{E}$ has only a y component. Thus, to leading order in the loop width,

$$\oint \mathbf{E} \cdot d\mathbf{\ell} = [E_y(x + \Delta x, t) - E_y(x, t)]\ell, \quad \Phi_B \simeq B_z \ell \Delta x.$$

Faraday’s law gives $[E_y(x + \Delta x, t) - E_y(x, t)]\ell = -\ell \Delta x \partial_t B_z$. Divide by $\ell \Delta x$ and let Δx tend to zero. For the xz loop with the same visible traversal, the normal is $-y$. Its circulation is $[B_z(x + \Delta x, t) - B_z(x, t)]\ell$ but its electric flux is $-E_y \ell \Delta x$. Ampere–Maxwell therefore supplies the second minus sign. The two results are: For the electric circulation,

$$\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \quad (13.7)$$

For the magnetic circulation,

$$\frac{\partial B_z}{\partial x} = -\mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t}. \quad (13.8)$$

These minus signs are fixed by the stated right-handed axes and field components. Reversing an integration loop reverses both sides of its integral law and does not change the resulting component equation.

Differentiate Eq. (13.7) with respect to x :

$$\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial}{\partial t} \left(\frac{\partial B_z}{\partial x} \right). \quad (13.9)$$

Substituting Eq. (13.8),

$$\boxed{\frac{\partial^2 E_y}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2}} \quad (13.10)$$

and the same procedure with the roles of the fields exchanged gives

$$\boxed{\frac{\partial^2 B_z}{\partial x^2} = \mu_0 \varepsilon_0 \frac{\partial^2 B_z}{\partial t^2}}. \quad (13.11)$$

These have the standard form of the one-dimensional wave equation,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2}. \quad (13.12)$$

Comparison gives

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c. \quad (13.13)$$

The experimentally known speed of light is therefore the propagation speed predicted by Maxwell's equations for electromagnetic disturbances in vacuum. This identification was historically decisive in recognizing light as an electromagnetic wave.

Caution

The wave equation does not say that the electric field “turns into” the magnetic field and then back again. The two fields coexist and are coupled. For an ideal plane wave they reach maxima and zeros simultaneously at a given point; they are in phase, not a quarter-period out of phase.

Guided checks

1. What assumptions were made when writing Eqs. (13.7) and (13.8)?
2. Why is the quantity $1/(\mu_0 \varepsilon_0)$ required to have units of speed squared?
3. What would be physically wrong with claiming that the wave speed is the drift speed of electrons in an antenna wire?

13.3 Sinusoidal plane waves: phase, wavelength, and frequency

A particularly important solution of the vacuum wave equations is the sinusoidal plane wave. For propagation along $+x$,

$$E_y(x, t) = E_0 \cos(kx - \omega t + \phi), \quad (13.14)$$

$$B_z(x, t) = B_0 \cos(kx - \omega t + \phi). \quad (13.15)$$

Every point in a plane perpendicular to the x axis has the same phase in the ideal model, which motivates the term *plane wave*. Surfaces of constant phase satisfy

$$kx - \omega t + \phi = \text{constant}. \quad (13.16)$$

Differentiating this condition gives

$$k \frac{dx}{dt} - \omega = 0, \quad \frac{dx}{dt} = \frac{\omega}{k}. \quad (13.17)$$

Thus the phase travels with speed

$$v_{\text{ph}} = \frac{\omega}{k}. \quad (13.18)$$

In vacuum, $v_{\text{ph}} = c$ and $\omega = ck$.

The sign in the phase determines the direction. A form $kx - \omega t$ travels in the $+x$ direction; a form $kx + \omega t$ travels in the $-x$ direction. This can be checked by following any fixed phase point. The initial phase ϕ shifts the waveform relative to the coordinate origin and time origin but does not alter the wavelength, frequency, or speed.

Method

To determine propagation direction from a sinusoidal field, set the phase equal to a constant and solve for how the position of that phase point changes with time. This method is more reliable than memorizing a sign rule.

Example 13.1: From frequency to wavelength and wave number

A microwave system operates at $f = 2.40$ GHz in air, which we approximate as vacuum for this calculation. Find the wavelength, angular frequency, and wave number.

Solution. With $c \simeq 2.998 \times 10^8 \text{ m s}^{-1}$,

$$\lambda = \frac{c}{f} = \frac{2.998 \times 10^8}{2.40 \times 10^9} = 1.249 \times 10^{-1} \text{ m} \approx 12.5 \text{ cm}. \quad (13.19)$$

The angular frequency is

$$\omega = 2\pi f = 1.51 \times 10^{10} \text{ rad s}^{-1}, \quad (13.20)$$

and

$$k = \frac{2\pi}{\lambda} \approx 50.3 \text{ rad m}^{-1}. \quad (13.21)$$

As a check, $\omega/k \approx 3.00 \times 10^8 \text{ m s}^{-1}$.

Example 13.2: Reading a wave from its phase

An electric field is written

$$\mathbf{E}(x, t) = 120 \cos(18x + 5.40 \times 10^9 t) \hat{\mathbf{j}} \text{ V m}^{-1}, \quad (13.22)$$

with x in metres and t in seconds. Determine the propagation direction, wavelength, frequency, and phase speed.

Solution. The phase has the form $kx + \omega t$, so constant phase moves toward decreasing x : the wave travels in the $-x$ direction. The wave number is $k = 18 \text{ rad m}^{-1}$, hence

$$\lambda = \frac{2\pi}{18} = 0.349 \text{ m}. \quad (13.23)$$

The angular frequency is $\omega = 5.40 \times 10^9 \text{ rad s}^{-1}$, so

$$f = \frac{\omega}{2\pi} = 8.59 \times 10^8 \text{ Hz.} \quad (13.24)$$

Finally,

$$v_{\text{ph}} = \frac{\omega}{k} = 3.00 \times 10^8 \text{ m s}^{-1}, \quad (13.25)$$

consistent with a vacuum electromagnetic wave.

Guided checks

1. What changes if the phase constant ϕ is increased by $\pi/2$?
2. Does doubling f in vacuum change the wave speed? What happens to λ ?
3. Why does k have dimensions of inverse length even though it is usually written in radians per metre?

13.4 Field geometry: transverse waves and the right-handed triad

The electric and magnetic fields of a vacuum plane wave are transverse to the propagation direction and perpendicular to one another. In the standard coordinate choice,

$$\mathbf{E} \parallel \hat{\mathbf{j}}, \quad \mathbf{B} \parallel \hat{\mathbf{k}}, \quad \text{propagation} \parallel \hat{\mathbf{i}}. \quad (13.26)$$

The orientation is fixed by the cross product,

$$\mathbf{E} \times \mathbf{B} \text{ points in the propagation direction.} \quad (13.27)$$

This is not merely a mnemonic. The same cross product appears in the Poynting vector and therefore gives the direction of electromagnetic energy transport.

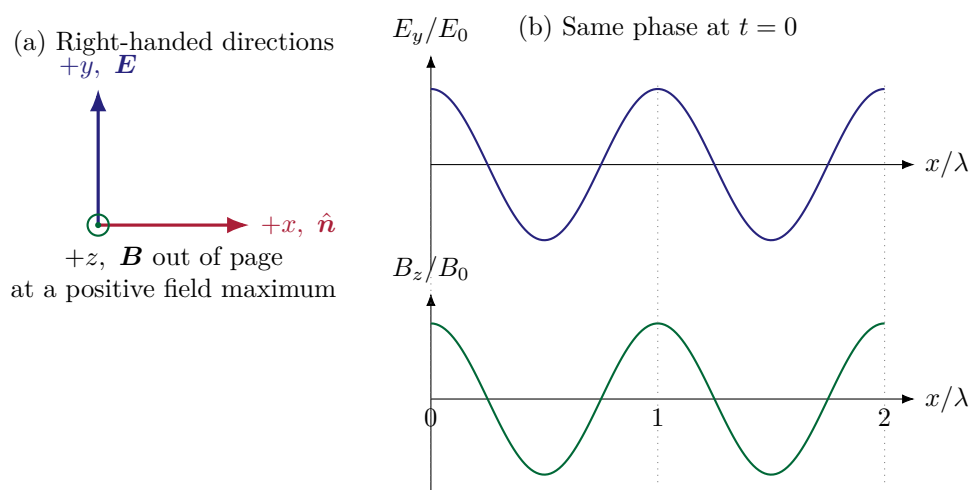


Figure 13.3: The directions and signed component graphs for $E_y = E_0 \cos(kx - \omega t)$ and $B_z = (E_0/c) \cos(kx - \omega t)$. The graphs use separate vertical axes; their maxima, minima and zeros coincide. Both vectors reverse together, so energy continues along $+x$.

Figure 13.3 separates a vector-direction diagram from graphs of signed components. A sinusoidal trace is not a trajectory of a particle or a wavy ray. It records field values at different positions at one instant; the fields exist throughout every transverse plane. For a traveling vacuum wave the

compact vector rule is

$$\mathbf{B} = \frac{1}{c} \hat{\mathbf{n}} \times \mathbf{E}. \quad (13.28)$$

It gives both the magnitude and sign and remains useful when propagation is reversed. The oscillatory part has no longitudinal component: Gauss's law applied to a thin box implies that E_x and B_x cannot vary along x in source-free space. Uniform static background fields may be superposed, but they are not the traveling disturbance. For the sinusoidal solution, substituting Eq. (13.14) into the coupled Maxwell relations gives

$$B_0 = \frac{k}{\omega} E_0 = \frac{E_0}{c}, \quad (13.29)$$

so

$$\boxed{E_0 = cB_0}. \quad (13.30)$$

The numerical value of c is very large, so a magnetic-field amplitude measured in tesla is much smaller numerically than the corresponding electric-field amplitude measured in volts per metre. This numerical difference does not mean that the magnetic part of the wave is energetically unimportant. As we shall show, the electric and magnetic contributions to the energy density of a vacuum plane wave are equal.

Example 13.3: Constructing the magnetic field from the electric field

A vacuum wave has

$$\mathbf{E}(x, t) = 300 \cos(kx - \omega t) \hat{\mathbf{j}} \text{ V m}^{-1}. \quad (13.31)$$

Write the magnetic field vector.

Solution. The phase shows propagation along $+x$. We require $\mathbf{E} \times \mathbf{B}$ to point along $+x$. Since $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$, the magnetic field points along $+z$. Its amplitude is

$$B_0 = \frac{E_0}{c} = \frac{300}{2.998 \times 10^8} = 1.00 \times 10^{-6} \text{ T}. \quad (13.32)$$

Therefore

$$\boxed{\mathbf{B}(x, t) = 1.00 \mu\text{T} \cos(kx - \omega t) \hat{\mathbf{k}}}. \quad (13.33)$$

The electric and magnetic fields have the same phase.

Caution

Do not use the right-hand rule for the Lorentz force $q\mathbf{v} \times \mathbf{B}$ in place of the plane-wave rule. For a plane wave, the relevant orientation is $\mathbf{E} \times \mathbf{B}$ in the propagation direction. These are different cross products used for different physical questions.

Guided checks

1. A wave travels along $+z$ and its electric field points instantaneously along $+x$. Which direction must $\mathbf{B}$ point at that instant?
2. If both $\mathbf{E}$ and $\mathbf{B}$ reverse sign half a period later, does the propagation direction reverse?
3. Why is E_0/B_0 a speed in SI units?

13.5 Electromagnetic spectrum and physical scale

The electromagnetic spectrum is a continuous range of frequencies and wavelengths. Figure 13.4 emphasizes the enormous range of scales and the narrow visible interval. Human labels such as

“radio”, “microwave”, “infrared”, “visible”, “ultraviolet”, “X ray”, and “gamma ray” are useful technological and historical categories, but there are no sharp universal boundaries in the field equations. The vacuum relation

$$\lambda f = c \quad (13.34)$$

connects wavelength and frequency throughout the spectrum.

What changes dramatically across the spectrum is the way radiation is generated, detected, and interacts with matter. Long-wavelength radio waves can be generated efficiently by macroscopic alternating currents in antennas. Infrared radiation is strongly associated with molecular and lattice excitations and thermal emission. Visible and ultraviolet radiation interact with electronic transitions. X rays and gamma rays involve still shorter wavelengths and higher photon energies, a connection that becomes central in quantum physics. The classical field description developed here remains valuable for propagation, polarization, interference, and energy transport whenever a wave model is appropriate.

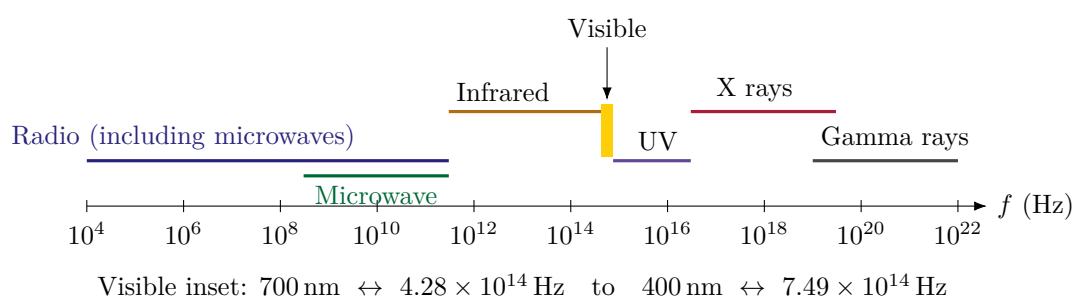


Figure 13.4: Illustrative spectral ranges on a logarithmic frequency axis. Microwaves are a subset of radio; X-ray and gamma-ray names overlap and can refer to production mechanism. Band endpoints are approximate conventions, not discontinuities in Maxwell theory. The visible band is narrow on this scale.

Example 13.4: Visible wavelength and frequency

Green light in vacuum has wavelength $\lambda = 530 \text{ nm}$. Find the frequency.

Solution.

$$f = \frac{c}{\lambda} = \frac{2.998 \times 10^8}{530 \times 10^{-9}} = 5.66 \times 10^{14} \text{ Hz.} \quad (13.35)$$

The optical frequency is therefore hundreds of terahertz. Electronic instruments normally do not follow the carrier oscillation directly; optical detectors typically respond to time-averaged power or to quantum absorption events.

Key message

The same Maxwell equations describe electromagnetic propagation across the entire classical spectrum. The names of spectral bands mainly reflect source technology, detector technology, and interaction with matter, not different fundamental field laws.

Guided checks

1. Which has the larger wavelength: a 100 MHz radio signal or a 10 GHz microwave signal?
2. Does the vacuum speed of a wave depend on whether it is radio, visible, or X-ray radiation?
3. Why are logarithmic axes useful when displaying the electromagnetic spectrum?

13.6 Propagation speed and refractive index in simple media

Vacuum is the cleanest setting for electromagnetic waves because the field equations contain only the universal constants ϵ_0 and μ_0 . In matter, the electromagnetic field interacts with charged constituents of the material. The collective response changes the relationship between the macroscopic fields and can modify the phase speed. At the introductory level, a transparent, isotropic, nonmagnetic medium is often described by a refractive index n defined by

$$n = \frac{c}{v}, \quad (13.36)$$

where v is the phase speed in the medium. For ordinary transparent materials at optical frequencies, n is typically greater than one. The frequency of a monochromatic wave remains fixed when it crosses a stationary interface, while the wavelength changes because

$$v = \lambda_{\text{med}} f. \quad (13.37)$$

Therefore

$$\lambda_{\text{med}} = \frac{\lambda_0}{n}, \quad (13.38)$$

where $\lambda_0 = c/f$ is the vacuum wavelength associated with the same frequency.

Equation (13.36) should not be interpreted as saying that light repeatedly stops and starts between atoms. The macroscopic wave speed emerges from coherent interaction between the electromagnetic field and the material response. A detailed microscopic theory requires more physics than is needed here and, in dispersive media, the refractive index can depend on frequency.

Example 13.5: Wavelength in glass

A monochromatic laser has vacuum wavelength 632.8 nm. It enters a transparent glass with refractive index $n = 1.52$. Find the frequency and the wavelength in the glass.

Solution. The frequency is determined by the source and remains unchanged at the stationary interface:

$$f = \frac{c}{\lambda_0} = \frac{2.998 \times 10^8}{632.8 \times 10^{-9}} = 4.74 \times 10^{14} \text{ Hz}. \quad (13.39)$$

The phase speed in the glass is $v = c/n$, so the wavelength is

$$\lambda_{\text{glass}} = \frac{\lambda_0}{n} = \frac{632.8 \text{ nm}}{1.52} = 416 \text{ nm}. \quad (13.40)$$

The color perceived after the light exits back into air is associated primarily with its frequency, not with the shorter wavelength it had while inside the glass.

Caution

At an interface, do not change the frequency merely because the refractive index changes. For a stationary boundary and a monochromatic source, the frequency is continuous across the boundary. The phase speed and wavelength change together.

Guided checks

1. A wave enters a medium with $n = 1.5$. By what factor does its phase speed change relative to vacuum?
2. If the frequency is unchanged, by what factor does the wavelength change?
3. Why is it dangerous to treat a single constant refractive index as exact over the entire

electromagnetic spectrum?

13.7 Electromagnetic energy density

Electric and magnetic fields store energy. In vacuum, the local energy densities are

$$u_E = \frac{1}{2}\epsilon_0 E^2, \quad u_B = \frac{B^2}{2\mu_0}. \quad (13.41)$$

For a vacuum plane wave, $E = cB$ and $c^2 = 1/(\mu_0\epsilon_0)$. Substituting $B = E/c$ into the magnetic energy density gives

$$u_B = \frac{1}{2\mu_0} \frac{E^2}{c^2} = \frac{1}{2\mu_0} E^2 \mu_0 \epsilon_0 = \frac{1}{2} \epsilon_0 E^2 = u_E. \quad (13.42)$$

Thus, at every point and instant in an ideal vacuum plane wave,

$$\boxed{u_E = u_B}. \quad (13.43)$$

The total electromagnetic energy density is therefore

$$u = u_E + u_B = \epsilon_0 E^2 = \frac{B^2}{\mu_0}. \quad (13.44)$$

For a sinusoidal wave with peak electric amplitude E_0 ,

$$\langle u \rangle = \epsilon_0 \langle E^2 \rangle = \frac{1}{2} \epsilon_0 E_0^2 = \epsilon_0 E_{\text{rms}}^2. \quad (13.45)$$

This equality of electric and magnetic energy densities is a plane-wave result in vacuum. One should not generalize it blindly to arbitrary electromagnetic field configurations. Near an antenna, inside a capacitor, or close to an inductor, one kind of field energy may dominate locally. Equal partition is a special property of the propagating ideal plane wave.

Example 13.6: Energy density of a radio wave

A plane radio wave in vacuum has rms electric field $E_{\text{rms}} = 20.0 \text{ V m}^{-1}$. Determine the time-averaged total energy density and the time-averaged electric contribution.

Solution. The average total energy density is

$$\langle u \rangle = \epsilon_0 E_{\text{rms}}^2 = (8.854 \times 10^{-12})(20.0)^2 = 3.54 \times 10^{-9} \text{ J m}^{-3}. \quad (13.46)$$

The electric and magnetic contributions are equal, so

$$\langle u_E \rangle = \langle u_B \rangle = \frac{1}{2} \langle u \rangle = 1.77 \times 10^{-9} \text{ J m}^{-3}. \quad (13.47)$$

The energy density is small on a macroscopic volumetric scale, but a continuously propagating wave can transfer substantial power because it moves at nearly $3 \times 10^8 \text{ m s}^{-1}$.

Key message

For a vacuum plane wave, the electric and magnetic fields contribute equally to the local electromagnetic energy density. This equality is a consequence of $E = cB$ together with $c^2 = 1/(\mu_0\epsilon_0)$.

Guided checks

1. If the electric amplitude doubles, by what factor does the average energy density change?
2. Can $u_E = u_B$ be assumed for a static electric field near a charged capacitor? Why not?
3. Show dimensionally that $\epsilon_0 E^2$ has units of J m^{-3} .

13.8 Poynting vector: local electromagnetic energy flow

Energy density answers the question “how much electromagnetic energy is present per unit volume?” To describe transport, we need a quantity that measures the rate at which electromagnetic energy crosses a surface. In vacuum, this role is played by the Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (13.48)$$

Its magnitude has units of power per unit area,

$$[S] = \text{W m}^{-2}, \quad (13.49)$$

and its direction gives the local direction of electromagnetic energy transport.

For a plane wave in vacuum, $\mathbf{E} \perp \mathbf{B}$, so

$$S = \frac{EB}{\mu_0}. \quad (13.50)$$

Using $B = E/c$,

$$S = \frac{E^2}{\mu_0 c}. \quad (13.51)$$

Using $1/(\mu_0 c) = \epsilon_0 c$,

$$S = c\epsilon_0 E^2 = cu. \quad (13.52)$$

For a plane wave, the energy flux therefore equals the local energy density multiplied by the propagation speed. This has the same structure as a general transport relation: a density moving at a speed produces a flux.

For the sinusoidal field $E = E_0 \cos(kx - \omega t)$,

$$S(x, t) = \frac{E_0^2}{\mu_0 c} \cos^2(kx - \omega t). \quad (13.53)$$

The Poynting vector does not oscillate symmetrically between positive and negative values for a single travelling wave because $\cos^2$ is nonnegative. Both fields reverse every half-cycle, so their cross product retains the same propagation direction.

The swept-volume construction in Fig. 13.5 provides an independent check: in time Δt , a locally uniform unidirectional field carries energy $uA_{\perp}c\Delta t$ across the area. Dividing by area and time recovers $S = cu$. This argument applies to a traveling wave, not to energy sloshing inside a standing wave. The instantaneous electromagnetic power crossing an oriented surface A is

$$P(t) = \int_A \mathbf{S} \cdot d\mathbf{A}. \quad (13.54)$$

For a uniform plane wave normally incident on a flat area A ,

$$P(t) = S(t)A. \quad (13.55)$$

If the surface normal makes an angle α with the propagation direction and the field is uniform over the surface,

$$P(t) = S(t)A \cos \alpha. \quad (13.56)$$

The factor $A \cos \alpha$ is the projected area perpendicular to the energy flow.

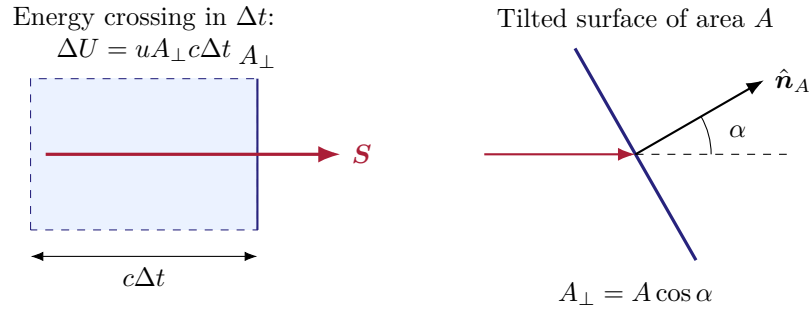


Figure 13.5: Energy density times transported volume gives the energy crossing a perpendicular area. For a tilted flat receiver, the flux uses the projected area. The surface normal $\hat{n}_A$ is a geometrical choice distinct from the wave direction.

Example 13.7: Instantaneous Poynting vector

At one location in a vacuum plane wave, the instantaneous fields are

$$\mathbf{E} = 150 \hat{\mathbf{j}} \text{ V m}^{-1}, \quad \mathbf{B} = 5.00 \times 10^{-7} \hat{\mathbf{k}} \text{ T.} \quad (13.57)$$

Find the instantaneous Poynting vector.

Solution. The direction is

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}. \quad (13.58)$$

The magnitude is

$$S = \frac{EB}{\mu_0} = \frac{(150)(5.00 \times 10^{-7})}{4\pi \times 10^{-7}} = 59.7 \text{ W m}^{-2}. \quad (13.59)$$

Thus

$$\boxed{\mathbf{S} = 59.7 \hat{\mathbf{i}} \text{ W m}^{-2}.} \quad (13.60)$$

The ratio $E/B = 3.00 \times 10^8 \text{ m s}^{-1}$, confirming consistency with a vacuum plane wave.

Caution

The Poynting vector is generally an instantaneous quantity. Intensity is normally the time average of the energy flux over many oscillation cycles. Writing $I = S$ without clarifying whether an average is intended can conceal a factor of two in sinusoidal-wave calculations.

13.9 Intensity, rms fields, and power transfer

For a sinusoidal plane wave, the quantity most often measured over times long compared with an optical or radio period is the time-averaged Poynting magnitude. This defines the intensity,

$$I \equiv \langle S \rangle. \quad (13.61)$$

Because the cycle average of $\cos^2$ is $1/2$,

$$\boxed{I = \frac{E_0^2}{2\mu_0 c} = \frac{cB_0^2}{2\mu_0}.} \quad (13.62)$$

In terms of rms fields,

$$\boxed{I = c\epsilon_0 E_{\text{rms}}^2 = \frac{c}{\mu_0} B_{\text{rms}}^2.} \quad (13.63)$$

A useful alternative follows from the impedance of free space $Z_0 = \mu_0 c \approx 376.7 \Omega$:

$$I = \frac{E_{\text{rms}}^2}{Z_0}. \quad (13.64)$$

The impedance interpretation is developed more fully in the advanced part.

Method

When intensity is given and a field amplitude is requested, first ask whether the requested field is peak or rms. Use $E_0 = \sqrt{2\mu_0 c I}$ for the peak field and $E_{\text{rms}} = \sqrt{I/(c\epsilon_0)}$ for the rms field. State explicitly which convention you used.

Example 13.8: Field amplitudes from measured intensity

A plane electromagnetic wave in vacuum has average intensity $I = 800 \text{ W m}^{-2}$. Find E_0 , E_{rms} , and B_0 .

Solution. From Eq. (13.62),

$$E_0 = \sqrt{2\mu_0 c I}. \quad (13.65)$$

Numerically,

$$E_0 = \sqrt{2(4\pi \times 10^{-7})(2.998 \times 10^8)(800)} \approx 7.76 \times 10^2 \text{ V m}^{-1}. \quad (13.66)$$

Then

$$E_{\text{rms}} = \frac{E_0}{\sqrt{2}} \approx 5.49 \times 10^2 \text{ V m}^{-1}, \quad (13.67)$$

and

$$B_0 = \frac{E_0}{c} \approx 2.59 \times 10^{-6} \text{ T}. \quad (13.68)$$

A direct check using $I = c\epsilon_0 E_{\text{rms}}^2$ gives the same intensity.

Example 13.9: Power intercepted by a solar panel

A flat panel of area $A = 1.80 \text{ m}^2$ receives radiation of intensity $I = 900 \text{ W m}^{-2}$. The panel normal is 25° from the propagation direction. What electromagnetic power is incident on the panel?

Solution. The projected area is $A \cos 25^\circ$, so

$$P_{\text{inc}} = IA \cos 25^\circ = (900)(1.80) \cos 25^\circ \approx 1.47 \times 10^3 \text{ W}. \quad (13.69)$$

This is incident electromagnetic power, not electrical output. An actual photovoltaic conversion efficiency would have to be applied separately.

Guided checks

1. If I is quadrupled, by what factor does E_0 change?
2. Why is the electrical output of a solar panel generally smaller than $IA \cos \alpha$?
3. If a detector reports an rms electric field, why should one not substitute it directly into the peak-field formula $I = E_0^2/(2\mu_0 c)$ without interpretation?

13.10 Geometric spreading and the inverse-square law

A monochromatic traveling plane wave in a homogeneous lossless medium has constant cycle-averaged intensity: the model has no geometric spreading or absorption. A pulsed plane wave can instead have an intensity envelope that varies with time and position. Radiation from a localized source generally spreads over a growing area. If a source radiates total time-averaged power P_s isotropically into free space, conservation of energy gives

$$P_s = I(r) 4\pi r^2, \quad (13.70)$$

so

$$I(r) = \frac{P_s}{4\pi r^2}. \quad (13.71)$$

The intensity decreases as $1/r^2$. Since intensity is proportional to the square of field amplitude, the far-field electric and magnetic amplitudes decrease approximately as $1/r$,

$$E_0 \propto \frac{1}{r}, \quad B_0 \propto \frac{1}{r}. \quad (13.72)$$

Real antennas are not usually isotropic. Their radiation patterns redistribute power with direction. Equation (13.71) remains a useful reference model and becomes exact for an ideal isotropic point radiator. Antenna gain and directional patterns are treated in the advanced enrichment part.

Example 13.10: Intensity and field at two distances

An ideal isotropic source radiates $P_s = 100$ W. Find the intensity at $r = 5.0$ m and $r = 20.0$ m. Also determine the ratio of electric-field amplitudes at the two distances.

Solution. At 5.0 m,

$$I_5 = \frac{100}{4\pi(5.0)^2} = 0.318 \text{ W m}^{-2}. \quad (13.73)$$

At 20.0 m,

$$I_{20} = \frac{100}{4\pi(20.0)^2} = 1.99 \times 10^{-2} \text{ W m}^{-2}. \quad (13.74)$$

The distance increased by a factor of four, so the intensity fell by a factor of $4^2 = 16$. Since $E_0 \propto \sqrt{I}$,

$$\frac{E_0(20)}{E_0(5)} = \sqrt{\frac{I_{20}}{I_5}} = \frac{1}{4}. \quad (13.75)$$

The field amplitude therefore decreases as $1/r$ in this far-field idealization.

Caution

Do not apply the inverse-square law automatically near an antenna, inside a waveguide, inside a resonant cavity, or to a well-collimated beam over a short distance. The $1/r^2$ intensity law assumes three-dimensional geometric spreading from a localized source in the far field without absorption.

13.11 Electromagnetic momentum and radiation pressure

Electromagnetic waves carry not only energy but also momentum. For a collimated wave packet in vacuum, with all propagation directions effectively parallel, a useful relation is

$$p = \frac{U}{c}, \quad (13.76)$$

where U is the electromagnetic energy carried in the packet and p is the magnitude of its momentum in the propagation direction. In a continuous beam, momentum transport produces a force when radiation is absorbed, reflected, or otherwise redirected.

For a normally incident beam with time-averaged intensity I , a perfectly absorbing surface receives momentum per unit time per unit area I/c . The radiation pressure is therefore

$$p_{\text{rad}} = \frac{I}{c} \quad (\text{perfect absorption, normal incidence}). \quad (13.77)$$

If the radiation is perfectly reflected back along its incident direction, the momentum change is twice as large,

$$p_{\text{rad}} = \frac{2I}{c} \quad (\text{perfect reflection, normal incidence}). \quad (13.78)$$

The corresponding force on area A is $F = p_{\text{rad}}A$ when the intensity is uniform.

Figure 13.6 uses momentum balance rather than a new force law. The dimensions are consistent: $[I/c] = \text{W m}^{-2}/\text{m s}^{-1} = \text{N m}^{-2}$. For a neutral absorber, the electric field can drive charges transversely; the accompanying magnetic force can give a forward momentum transfer. The complete force is obtained from the Lorentz force or Maxwell stress, so electrical neutrality does not imply absence of radiation pressure. These pressures are small for ordinary laboratory intensities but become important in precision optics, high-power laser systems, optical trapping, spacecraft concepts such as solar sails, and astrophysical environments.

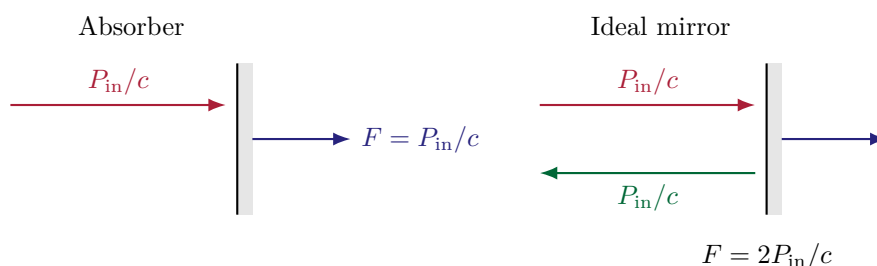


Figure 13.6: Normal-incidence momentum bookkeeping in vacuum. The arrows beside the beams label momentum per unit time. An absorber removes the incident momentum; a mirror also produces an equal outgoing momentum in the opposite direction. Forces are directed into the targets; arrow lengths are schematic.

Example 13.11: Radiation force on an absorbing sensor

A laser beam of power $P = 5.0 \text{ W}$ is fully absorbed by a black target. Assume the beam is normally incident and all the laser power reaches the target. Find the average force.

Solution. The total momentum delivered per unit time is P/c , so

$$F = \frac{P}{c} = \frac{5.0}{2.998 \times 10^8} = 1.67 \times 10^{-8} \text{ N}. \quad (13.79)$$

This is only about 17 nN, illustrating why radiation-pressure measurements often require sensitive force detection.

Example 13.12: Radiation pressure on a reflective sail

A perfectly reflecting sail of area 20 m^2 is illuminated normally by radiation of intensity $1.00 \times 10^3 \text{ W m}^{-2}$. Find the ideal radiation force.

Solution. For perfect reflection,

$$p_{\text{rad}} = \frac{2I}{c} = \frac{2.00 \times 10^3}{2.998 \times 10^8} = 6.67 \times 10^{-6} \text{ Pa.} \quad (13.80)$$

Hence

$$F = p_{\text{rad}} A = (6.67 \times 10^{-6})(20) = 1.33 \times 10^{-4} \text{ N.} \quad (13.81)$$

The force is small, but in space it can act continuously for long times without propellant.

Guided checks

1. Why does perfect reflection produce twice the normal-incidence pressure of perfect absorption?
2. If a beam power is fixed but the illuminated area is doubled, what happens to radiation pressure and total force for uniform absorption?
3. Why can a very small force still be useful for a solar sail over long times?

Example 13.13: A detector with partial reflection and absorption

A normally incident beam has power $P = 2.0 \text{ W}$. A target reflects 30% of the incident power directly backward, absorbs 50%, and transmits 20% forward without changing direction. Assume the incoming, reflected and transmitted beams are all in vacuum, and neglect recoil from later thermal emission. Estimate the force on the target due to radiation momentum.

Solution. Treat each power fraction separately. Absorbed power contributes P_{abs}/c , reflected power contributes $2P_{\text{refl}}/c$, and the idealized forward-transmitted portion produces no net longitudinal momentum transfer in this simple bookkeeping model. Hence

$$F = \frac{0.50P}{c} + \frac{2(0.30P)}{c} \quad (13.82)$$

$$= \frac{1.10P}{c} = \frac{2.20}{2.998 \times 10^8} = 7.34 \times 10^{-9} \text{ N.} \quad (13.83)$$

This decomposition is a useful first model for real optical surfaces. More detailed treatments must account for refraction, angular redistribution, and material momentum conventions when those effects matter.

13.12 How electromagnetic radiation is generated

The plane-wave solutions derived earlier answer an important mathematical question: once a propagating electromagnetic disturbance exists in free space, how can the fields evolve consistently with Maxwell's equations? A second question is equally important for engineering: *how is the disturbance launched in the first place?* The essential physical answer is that time-dependent charge and current distributions generate time-dependent electromagnetic fields. In the radiation zone, the part of those fields that falls as $1/r$ can carry energy away from the source as an electromagnetic wave.

A static charge produces an electrostatic field, and a steady current produces a magnetostatic field. Neither situation radiates continuously. A changing source can radiate, although cancellations between different parts of an extended source can suppress its radiation. In particular, accelerated charge is the microscopic origin of electromagnetic radiation. The word “accelerated” includes charges that reverse direction, oscillate, or move around a curved path. An alternating current in

an antenna is therefore a macroscopic realization of many charges undergoing repeated acceleration.

This statement is more subtle than “a moving charge radiates.” A charge in uniform rectilinear motion does not, in its own inertial frame, constitute an oscillating source. It possesses electromagnetic fields, but the far-field radiation mechanism is associated with acceleration and with the time dependence of the source distribution. For introductory purposes, the most useful source model is an oscillating electric dipole.

13.12.1 Oscillating dipole picture

Imagine two conducting arms separated by a small gap and connected to a sinusoidal source. During one half-cycle, positive charge accumulates near one end and negative charge near the other. During the next half-cycle, the polarity reverses. The dipole moment therefore oscillates approximately as

$$\mathbf{p}(t) = p_0 \cos(\omega t) \hat{\mathbf{z}}. \quad (13.84)$$

The changing electric field is accompanied by a changing magnetic field. Close to the source, much of the field energy is stored temporarily and returned to the source; farther away, a component of the field has the geometry and phase relation of a traveling wave and transports energy outward.

A complete derivation of dipole radiation requires solving Maxwell’s equations with sources and is beyond the main mathematical level of this chapter. Nevertheless, several reliable qualitative conclusions are useful now:

- the radiation field is transverse in the far zone;
- its amplitude decreases approximately as $1/r$, so its intensity decreases as $1/r^2$;
- the radiation is not isotropic for a simple linear dipole: it is strongest broadside to the dipole axis and vanishes ideally along the axis;
- the radiation frequency is the driving frequency of the oscillating source in the ideal monochromatic case;
- the polarization is closely related to the orientation of the radiating current or dipole.

Key message

An antenna does not create a radio wave by “sending current through empty space.” The alternating source drives charges in the conductor; their time-dependent charge and current distributions create time-dependent electric and magnetic fields. The far-field part of those fields detaches, in the practical sense of transporting energy irreversibly away from the source, as electromagnetic radiation.

13.12.2 Near field and radiation field

The words *near field* and *far field* describe different spatial regimes around a radiating source. In the near field, the geometry depends strongly on the antenna dimensions and on position, and energy can oscillate back and forth between electric and magnetic storage. Simple relations such as $E = cB$, $I \propto 1/r^2$, and a fixed plane-wave impedance should not be applied blindly very close to the source. In the far field, the radiative terms dominate, the wavefront is locally nearly planar, $\mathbf{E}$ and $\mathbf{B}$ are transverse, and the familiar plane-wave relations become accurate locally.

The boundary between regimes is not a universal single radius. It depends on wavelength and antenna size. For a large aperture of characteristic dimension D , an engineering estimate for the beginning of the far field often scales as D^2/λ . The precise criterion belongs to antenna engineering;

the conceptual point is that “distance from the source” should always be compared with the physical dimensions and wavelength of the source.

Caution

Do not use the phrase “the electric field creates the magnetic field, which then creates the electric field” as if the two fields were separate causes chasing one another through space. Maxwell’s equations form one coupled field system. Time and spatial derivatives of the fields are related simultaneously. The wave is a self-consistent solution of the coupled equations, not a two-step mechanical relay.

13.12.3 Antennas as energy transducers

A transmitting antenna converts electrical energy supplied by a source into electromagnetic energy propagating through space. A receiving antenna performs the reverse operation: an incident electromagnetic wave drives charges in a conductor and produces a measurable voltage or current. For ordinary linear reciprocal antenna systems, reciprocity is a powerful engineering idea: the same geometry that favors radiation in certain directions often favors reception from those directions.

For a short linear receiving element, the induced response is strongly dependent on polarization. A vertically oriented antenna responds strongly to a vertically polarized electric field and weakly, ideally zero in the simplest model, to a horizontally polarized one. This provides a direct macroscopic demonstration that polarization refers to the direction of the electric field.

Example 13.14: Polarization mismatch in a receiving antenna

A linearly polarized radio wave has its electric field vertical. An ideal short receiving dipole is rotated so that its axis makes an angle $\theta = 60^\circ$ with the vertical. If the received power is proportional to the square of the electric-field component along the dipole, what fraction of the maximum power is received?

Solution. The field component along the antenna is

$$E_{\parallel} = E_0 \cos \theta. \quad (13.85)$$

Power is proportional to $E_{\parallel}^2$, so the normalized received power is

$$\frac{P}{P_{\max}} = \cos^2 \theta = \cos^2 60^\circ = \frac{1}{4}. \quad (13.86)$$

Only 25% of the maximum ideal power is received. This is mathematically the same projection principle that later appears in Malus’s law for optical polarizers.

Guided checks

1. Why does a steady direct current in an ideal straight wire not behave like a continuously radiating radio transmitter?
2. In the radiation zone, if the electric-field amplitude halves when distance doubles, how does the intensity change?
3. Why does antenna orientation provide evidence that the electric field, not the propagation direction, defines linear polarization?
4. Which plane-wave assumptions become questionable very close to an antenna?

13.13 Polarization: orientation of the transverse electric field

Because electromagnetic waves are transverse, the field oscillations can have different orientations within the plane perpendicular to propagation. This geometric degree of freedom is called *polarization*. Figure 13.7 shows the special case of a fixed transverse line. Polarization is not an additional force or a separate kind of light; it describes the time evolution of the transverse electric-field vector at a fixed point in space.

Suppose a wave propagates along $+x$. The electric field must lie in the yz plane. The most general monochromatic transverse field can be written as two perpendicular components,

$$E_y(x, t) = E_{0y} \cos(kx - \omega t), \quad (13.87)$$

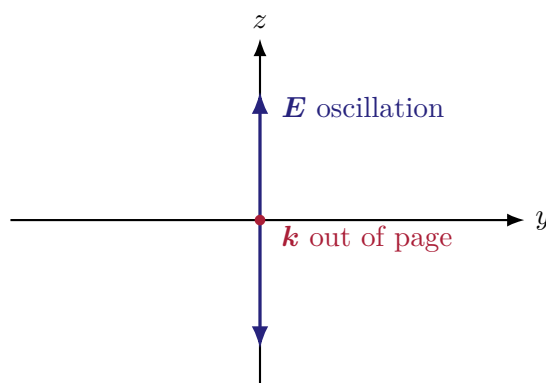
$$E_z(x, t) = E_{0z} \cos(kx - \omega t + \delta), \quad (13.88)$$

where δ is the relative phase. The path traced by the tip of $\mathbf{E}$ at fixed x determines the polarization state.

In the main chapter we focus on linear polarization. If the two components are in phase or exactly out of phase, their ratio is constant and the field oscillates along one fixed line in the transverse plane. Special choices such as $E_{0z} = 0$ give polarization along one coordinate axis. Circular and elliptical polarization are developed in the advanced section.

Definition 13.1: Linear polarization

A monochromatic electromagnetic wave is linearly polarized when, at a fixed point in space, its electric-field vector oscillates back and forth along a fixed line perpendicular to the propagation direction.



head-on view of a wave propagating toward the observer

Figure 13.7: Head-on representation of a linearly polarized wave. Polarization specifies the direction along which the electric field oscillates in the transverse plane.

13.13.1 Unpolarized light

Thermal sources and many ordinary light sources contain a huge number of emitters whose microscopic phases and orientations vary rapidly. On time scales long compared with the microscopic fluctuations, no single fixed transverse direction describes the electric field. Such light is called *unpolarized*. This does not mean that the instantaneous electric field vanishes; rather, the orientation varies statistically so that no preferred linear direction persists in the time average.

The distinction between coherent and incoherent descriptions matters here. More precisely, unpolarized light has equal mean powers in any two orthogonal transverse components and zero mutual coherence between them. A deterministic circularly polarized wave also has no preferred

linear axis in its time-averaged intensity, but has a fixed relative phase and is fully polarized. A rotating linear analyzer alone therefore cannot distinguish circularly polarized from unpolarized light; a phase-retarding element is needed as well. One should not model unpolarized light as a single deterministic sinusoid whose axis slowly rotates unless that model is explicitly justified. In elementary intensity calculations, however, the statistical result needed for an ideal linear polarizer is simple: the average of $\cos^2 \theta$ over all transverse orientations is $1/2$.

13.13.2 Ideal linear polarizers

An ideal linear polarizer transmits the electric-field component parallel to its transmission axis and removes the perpendicular component. If a linearly polarized incident field of amplitude E_0 makes an angle θ with the transmission axis, the transmitted field amplitude is

$$E_{0,\text{out}} = E_0 \cos \theta. \quad (13.89)$$

Because intensity is proportional to the square of amplitude, the transmitted intensity obeys Malus's law,

$$I = I_0 \cos^2 \theta. \quad (13.90)$$

For unpolarized incident light, the first ideal polarizer transmits half the incident intensity,

$$I_1 = \frac{1}{2} I_{\text{unpol}}. \quad (13.91)$$

The output is then linearly polarized along the first polarizer's transmission axis, so Malus's law can be applied to any subsequent polarizer.

Method

For a sequence of ideal polarizers, work from left to right. At each stage, write down two quantities: the intensity entering the element and the polarization direction leaving it. Use the one-half rule only when the incident light at that stage is unpolarized; use $\cos^2 \theta$ when the incident light is already linearly polarized.

For two crossed ideal polarizers, the first transmits $I_0/2$ from an unpolarized input and the second transmits zero. The next example tests whether inserting an additional absorbing element must always reduce the final output.

Example 13.15: Three-polarizer transmission

The same unpolarized beam with $I_0 = 600 \text{ W m}^{-2}$ passes through three ideal polarizers. Their transmission axes are at 0° , 45° , and 90° relative to vertical. Find the final intensity.

Solution. After the first polarizer,

$$I_1 = \frac{1}{2} I_0 = 300 \text{ W m}^{-2}. \quad (13.92)$$

After the 45° polarizer,

$$I_2 = I_1 \cos^2 45^\circ = 150 \text{ W m}^{-2}. \quad (13.93)$$

The light leaving the second polarizer is polarized at 45° . The angle to the final horizontal axis is again 45° , hence

$$I_3 = I_2 \cos^2 45^\circ = 75 \text{ W m}^{-2}. \quad (13.94)$$

Inserting the intermediate polarizer therefore increases the transmitted intensity from zero to 75 W m^{-2} . There is no violation of energy conservation: each polarizer removes part of the beam, but the intermediate element also changes the polarization direction of the transmitted component.

Caution

Malus's law acts on *intensity* as $\cos^2 \theta$, because the electric-field amplitude is projected with one factor of $\cos \theta$ and intensity is proportional to amplitude squared. Do not write $I = I_0 \cos \theta$.

Guided checks

1. Which field vector is used to define the polarization of an electromagnetic wave?
2. Why does an ideal polarizer reduce unpolarized light to half its original intensity on average?
3. After a polarizer, is the output polarization determined by the input polarization or by the polarizer axis?
4. Explain qualitatively why a 45° polarizer inserted between crossed polarizers can make the final beam brighter rather than dimmer.

13.14 Reflection and refraction as wave phenomena

When an electromagnetic wave reaches an interface between two materials, the fields must satisfy electromagnetic boundary conditions. In general, part of the wave energy is reflected and part is transmitted. In the geometric-optics limit, where the wavelength is small compared with the dimensions of the optical system and wavefront curvature can be represented locally by rays, these processes are summarized by the laws of reflection and refraction.

The angle of incidence θ_1 , angle of reflection θ_r , and angle of refraction θ_2 are all measured from the normal to the interface, not from the surface. The law of reflection is

$$\boxed{\theta_r = \theta_1.} \quad (13.95)$$

For transparent isotropic media characterized by refractive indices n_1 and n_2 , Snell's law is

$$\boxed{n_1 \sin \theta_1 = n_2 \sin \theta_2.} \quad (13.96)$$

The refractive index is related to the phase speed in a simple transparent medium by

$$n = \frac{c}{v}. \quad (13.97)$$

Frequency is fixed by the source and remains continuous across a stationary interface in the elementary model. Therefore a change in speed is accompanied by a change in wavelength:

$$f_2 = f_1, \quad \lambda_2 = \frac{v_2}{f} = \lambda_1 \frac{n_1}{n_2}. \quad (13.98)$$

The explicit normal and ray geometry in Fig. 13.8 make the angle convention visible before calculation.

13.14.1 Why the ray bends

At a stationary planar boundary, the field matching must hold at every point and at every time. Let s denote distance along the interface. The incident phase restricted to it is $k_1 s \sin \theta_1 - \omega_1 t + \phi_i$; the transmitted phase is $k_2 s \sin \theta_2 - \omega_2 t + \phi_t$. A fixed phase difference along the whole interface requires

$$\omega_1 = \omega_2, \quad k_1 \sin \theta_1 = k_2 \sin \theta_2. \quad (13.99)$$

Since $k_j = n_j \omega / c$, the second condition is precisely Snell's law. The reflected wave is in the incident medium, so the same argument gives $\theta_r = \theta_1$. These laws describe the directions, not the reflected

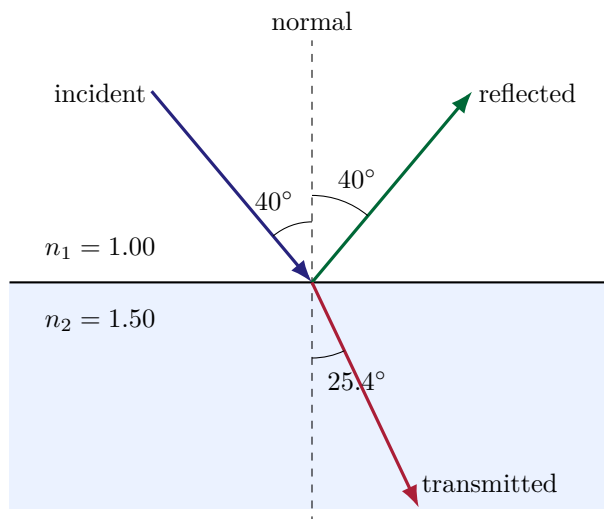


Figure 13.8: Reflection and refraction for the air–glass example. The ray directions are calculated from $\theta_r = \theta_i$ and Snell's law. Angles are measured from the normal; the transmitted ray bends toward it.

and transmitted powers; the latter require amplitude boundary conditions. This derivation is exact for plane waves at the stated uniform interface. Using a ray to represent a finite beam introduces the additional geometrical-optics approximation.

Snell's law may be motivated by several equivalent viewpoints. In the wavefront picture, different parts of a wavefront enter the second medium at different times, so the propagation direction changes when the phase speed changes. In Fermat's principle, the physical path makes the travel time stationary. In the full electromagnetic description, refraction follows from phase matching along the interface together with the dispersion relations in the two media.

For $n_2 > n_1$, the wave slows and bends toward the normal, so $\theta_2 < \theta_1$. For $n_2 < n_1$, it speeds up and bends away from the normal, so $\theta_2 > \theta_1$ until the limit of total internal reflection is reached.

Example 13.16: Refraction from air into glass

A narrow laser beam travels from air ($n_1 = 1.00$) into glass with $n_2 = 1.50$. The incidence angle is 40.0° . Find the refracted angle.

Solution. Snell's law gives

$$\sin \theta_2 = \frac{n_1}{n_2} \sin \theta_1 = \frac{1.00}{1.50} \sin 40.0^\circ = 0.4285. \quad (13.100)$$

Therefore

$$\theta_2 = \sin^{-1}(0.4285) = 25.4^\circ. \quad (13.101)$$

The beam bends toward the normal, consistent with entering the higher-index medium.

Caution

The refractive index is generally frequency dependent. Writing $v = c/n$ is a phase-velocity statement for the frequency under consideration. It does not imply that a material has one universal wave speed for every electromagnetic frequency, nor should it be used to conclude that information necessarily travels at c/n in a strongly dispersive medium.

Guided checks

1. Which quantities remain unchanged when a monochromatic wave crosses a stationary transparent interface: frequency, wavelength, speed, direction?
2. If $n_2 > n_1$, does the refracted ray bend toward or away from the normal?
3. Why must optical angles be measured from the normal when using Snell's law?
4. If an interface is parallel to the incident wavefront, what incidence angle does the ray have and is there any directional bending?

13.15 Total internal reflection and optical confinement

A particularly useful consequence of Snell's law occurs when a wave travels from a medium of higher refractive index to one of lower index, $n_1 > n_2$. As the incidence angle increases, the refracted angle increases even faster. At one special incidence angle, the refracted ray would propagate along the interface, $\theta_2 = 90^\circ$. This incidence angle is the *critical angle* θ_c :

$$n_1 \sin \theta_c = n_2, \quad (13.102)$$

so

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right), \quad n_1 > n_2. \quad (13.103)$$

For $\theta_1 > \theta_c$, Snell's law would require $\sin \theta_2 > 1$, so no propagating refracted ray exists in the lower-index medium. The wave undergoes total internal reflection (TIR).

Figure 13.9 compares the three regimes at the same lossless glass–air interface. In the ideal semi-infinite second medium, TIR means unit power reflectance and zero time-averaged energy flux normal to the boundary on its low-index side.

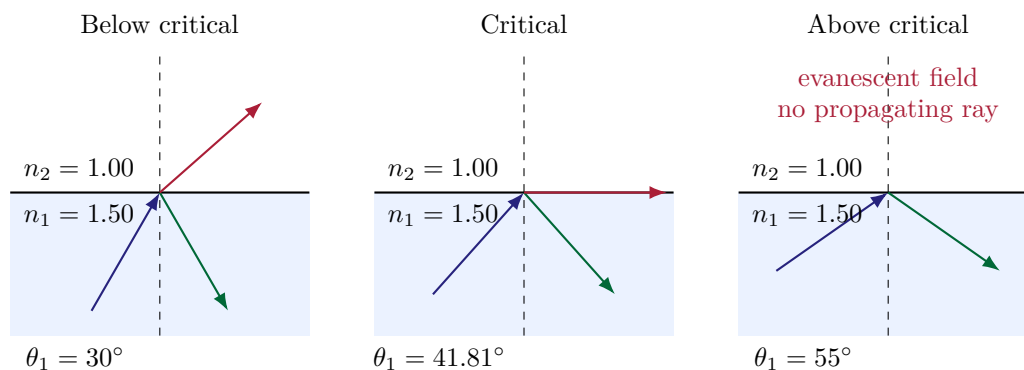


Figure 13.9: Below critical incidence a propagating transmitted ray exists; at $\theta_c = \sin^{-1}(1/1.5) = 41.81^\circ$ its limiting direction is along the surface. Above critical incidence, a reflected wave and an evanescent field satisfy the boundary conditions. Arrow lengths do not represent power fractions.

The phrase “total internal reflection” does not mean that the field is exactly zero on the low-index side. Electromagnetic boundary conditions require an evanescent field that penetrates a short distance into the second medium and decays exponentially away from the interface. The main chapter does not derive that field, but remembering its existence prevents an overly mechanical picture of rays bouncing from a hard wall.

13.15.1 Optical fibers

A simple step-index optical fiber has a core of refractive index n_1 surrounded by cladding of slightly lower index n_2 . Rays entering within an appropriate acceptance cone meet the core-cladding

boundary above the critical angle and remain confined by repeated total internal reflection in the geometrical-optics picture. Real fibers are better described as waveguides supporting electromagnetic modes, especially when the core dimensions are comparable with wavelength, but TIR provides a valuable introductory interpretation.

Fiber communication illustrates several themes of this chapter at once. Electromagnetic waves transport energy and information; refractive index controls phase speed and confinement; polarization and dispersion influence signal quality; and engineering design requires matching wavelength, material loss, source properties, detector response, and allowable bandwidth.

Example 13.17: Critical angle in a glass-air interface

Light travels inside glass with $n_1 = 1.52$ toward a glass-air boundary, where $n_2 = 1.00$. Find the critical angle.

Solution.

$$\theta_c = \sin^{-1}\left(\frac{1.00}{1.52}\right) = 41.1^\circ. \quad (13.104)$$

For incidence angles larger than 41.1° , measured inside the glass from the normal, the ray undergoes total internal reflection in the ideal lossless interface model.

Example 13.18: Core-cladding critical angle

A step-index fiber has core index $n_1 = 1.480$ and cladding index $n_2 = 1.460$. Find the critical angle at the core-cladding boundary.

Solution.

$$\theta_c = \sin^{-1}\left(\frac{1.460}{1.480}\right) = 80.6^\circ. \quad (13.105)$$

The large angle measured from the normal means that a guided ray meets the boundary at a shallow grazing angle relative to the surface. This is typical when the core and cladding indices differ only slightly.

Method

Before using the critical-angle formula, check the direction of travel. Total internal reflection is possible only when the incident wave is in the higher-index medium. If $n_1 < n_2$, the ratio n_2/n_1 exceeds unity and the formula correctly signals that no real critical angle exists for that direction.

Guided checks

1. Why is total internal reflection impossible for light going from air directly into ordinary glass?
2. At the critical angle, what is the direction of the refracted ray in the ray picture?
3. Does TIR imply an identically zero electromagnetic field in the lower-index medium?
4. Why can a small index contrast between fiber core and cladding still provide optical confinement?

13.16 Polarization by reflection and Brewster's angle

Reflection coefficients depend on polarization because the electromagnetic boundary conditions act differently on electric-field components parallel and perpendicular to the plane of incidence.

At one special angle of incidence, the reflected component whose electric field lies in the plane of incidence vanishes for ideal transparent dielectric media. The reflected wave is then linearly polarized perpendicular to the plane of incidence. This angle is the *Brewster angle*.

For light incident from refractive index n_1 onto n_2 , the Brewster condition is

$$\theta_B + \theta_2 = 90^\circ. \quad (13.106)$$

Combining this with Snell's law gives

$$n_1 \sin \theta_B = n_2 \cos \theta_B, \quad (13.107)$$

and hence

$$\boxed{\tan \theta_B = \frac{n_2}{n_1}.} \quad (13.108)$$

This relation assumes the usual nonmagnetic dielectric situation and refers to the angle measured from the normal.

Polarizing sunglasses exploit the fact that glare reflected from approximately horizontal surfaces such as water or roads is often partially polarized. A properly oriented absorbing polarizer reduces the dominant electric-field component of the reflected glare while transmitting a useful fraction of other light.

Example 13.19: Brewster angle for an air-glass interface

Find the Brewster angle for light incident from air ($n_1 = 1.00$) onto glass with $n_2 = 1.50$. Also find the refracted angle at this incidence.

Solution.

$$\theta_B = \tan^{-1}\left(\frac{1.50}{1.00}\right) = 56.3^\circ. \quad (13.109)$$

At Brewster incidence the reflected and refracted rays are perpendicular, so

$$\theta_2 = 90^\circ - 56.3^\circ = 33.7^\circ. \quad (13.110)$$

A direct substitution into Snell's law confirms the same result.

Caution

Brewster's law does not say that *all* reflection disappears. At the Brewster angle, one polarization component has zero reflection in the ideal dielectric model; the perpendicular component is still reflected. Unpolarized incident light therefore produces a polarized reflected beam, not generally zero reflected power.

Guided checks

1. What happens geometrically to the reflected and refracted rays at the Brewster angle?
2. If n_2 increases while n_1 is fixed, does θ_B increase or decrease?
3. Why can reflected glare be partially polarized even when the incident sunlight is unpolarized?
4. What common misunderstanding results from saying merely "there is no reflection at Brewster's angle"?

Example 13.20: Polarizer plus reflecting interface

Unpolarized light of intensity I_0 first passes through an ideal linear polarizer. The resulting beam is incident from air onto glass ($n = 1.50$) at the Brewster angle. The polarizer is oriented so that the transmitted electric field is perpendicular to the plane of incidence. What can be concluded, without using Fresnel coefficients, about (a) the intensity immediately after the polarizer and (b) whether the reflected beam is zero?

Solution. The first ideal polarizer transmits half of the unpolarized intensity,

$$I_1 = \frac{1}{2}I_0. \quad (13.111)$$

At Brewster incidence, the reflection coefficient vanishes for the electric-field component *parallel* to the plane of incidence, not for the perpendicular component. The beam has deliberately been polarized perpendicular to that plane, so it is generally reflected. Therefore one cannot conclude that the reflected intensity is zero. Quantifying it requires the oblique-incidence Fresnel coefficients in the advanced part, which give $R_s \simeq 0.148$ here and reflected intensity $\simeq 0.074I_0$.

This example illustrates why remembering only the formula for θ_B is not enough; the polarization statement attached to Brewster's angle is essential.

13.17 Standing electromagnetic waves and resonant cavities

Traveling electromagnetic waves transport energy through space, but boundaries can cause waves to reflect and interfere with one another. The superposition of waves traveling in opposite directions can produce a standing-wave pattern. This is the electromagnetic analogue of standing waves on strings and in air columns, with an important difference: the oscillating quantities are electric and magnetic fields.

Consider two plane waves of equal electric-field amplitude E_0 and frequency, one traveling in $+x$ and one in $-x$, polarized along y :

$$E_+(x, t) = E_0 \cos(kx - \omega t), \quad (13.112)$$

$$E_-(x, t) = -E_0 \cos(kx + \omega t), \quad (13.113)$$

where the relative sign has been chosen to represent a boundary condition appropriate to a tangential electric-field node at a conducting surface. Their sum can be written using trigonometric identities as

$$E_y(x, t) = 2E_0 \sin(kx) \sin(\omega t). \quad (13.114)$$

The spatial factor $\sin(kx)$ fixes nodes and antinodes. Adjacent electric-field nodes are separated by $\lambda/2$.

The associated magnetic standing wave is shifted spatially by a quarter wavelength: where the electric-field amplitude is zero, the magnetic-field amplitude can be maximal, and vice versa. Energy therefore oscillates locally between electric and magnetic forms. In an ideal lossless closed cavity, the time-averaged net energy flow through the walls vanishes even though the local instantaneous Poynting vector need not vanish everywhere at every instant.

For the backward wave the magnetic field must have the opposite direction relative to its electric field. The two magnetic components are consequently

$$B_+ = \frac{E_0}{c} \cos(kx - \omega t), \quad B_- = -\frac{E_0}{c} \cos(kx + \omega t),$$

and their sum is

$$B_z(x, t) = \frac{2E_0}{c} \cos(kx) \cos(\omega t). \quad (13.115)$$

At a fixed position away from nodes, the total electric and magnetic fields are a quarter cycle apart, although each constituent traveling wave has in-phase fields. At the conducting surface $x = 0$, tangential E_y vanishes while tangential B_z can be nonzero. The surface current supports that magnetic field. Figure 13.10 shows the different spatial nodes in the fundamental mode.

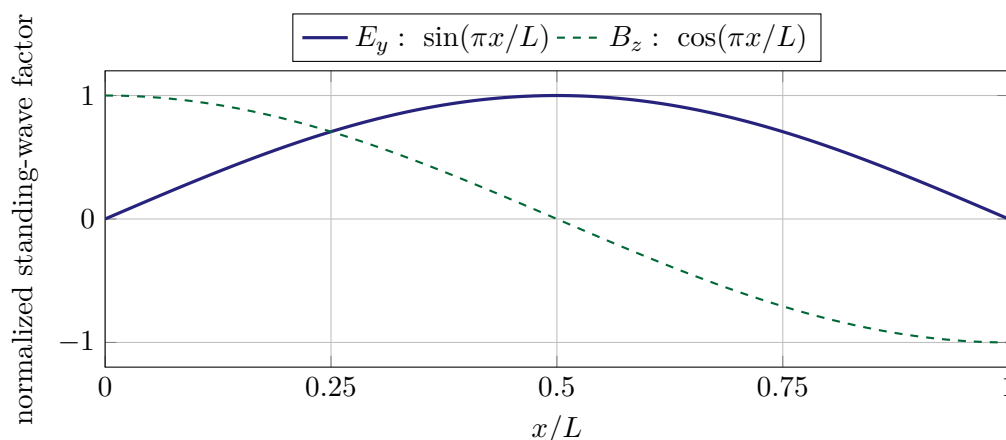


Figure 13.10: Fundamental spatial factors between conducting planes at $x = 0, L$. The actual fields also contain the different time factors in Eqs. (13.114) and (13.115); the curves are not simultaneous peak fields. Electric nodes lie at the walls and the magnetic node lies at the center.

Use the full fields, not $S = cu$, for the standing wave:

$$u(x, t) = 2\varepsilon_0 E_0^2 [\sin^2(kx) \sin^2(\omega t) + \cos^2(kx) \cos^2(\omega t)], \quad (13.116)$$

$$S_x(x, t) = c\varepsilon_0 E_0^2 \sin(2kx) \sin(2\omega t). \quad (13.117)$$

Thus $\langle S_x \rangle = 0$ at every point while $\langle u \rangle = \varepsilon_0 E_0^2$ is nonzero and spatially uniform. Energy flows alternately in opposite directions during a cycle. Integrating u over an integer number of half wavelengths gives a time-independent total energy: the electric and magnetic totals exchange energy while their sum remains fixed. Zero average transport therefore does not imply zero stored energy.

Example 13.21: Energy in a standing mode

An ideal one-dimensional vacuum mode occupies effective transverse area $A = 0.0100 \text{ m}^2$ and length $L = 0.120 \text{ m}$. Each of its two traveling-wave components has peak electric amplitude $E_0 = 200 \text{ V m}^{-1}$. Ignore edge effects. Find the maximum electric field and the total stored energy.

Strategy. Add fields before computing energy; the standing-wave peak is twice a constituent amplitude. Integrate Eq. (13.116) over the fundamental mode, using $\int_0^L \sin^2(\pi x/L) dx = \int_0^L \cos^2(\pi x/L) dx = L/2$.

$$E_{\text{max}} = 2E_0 = 400 \text{ V m}^{-1},$$

$$U = \varepsilon_0 E_0^2 AL = (8.854 \times 10^{-12})(200)^2(0.0100)(0.120) = 4.25 \times 10^{-10} \text{ J}.$$

The time factors combine as $\sin^2 \omega t + \cos^2 \omega t = 1$. The dimensions are energy density times volume. Substituting the standing-wave peak into a traveling-wave intensity formula would instead assign a fictitious net transmitted power to a field with $\langle S_x \rangle = 0$.

13.17.1 Resonance condition in a one-dimensional cavity

For two perfectly conducting parallel walls separated by distance L , a simple one-dimensional mode must have electric-field nodes at both walls. The allowed wavelengths satisfy

$$L = m \frac{\lambda_m}{2}, \quad m = 1, 2, 3, \dots, \quad (13.118)$$

so

$$\lambda_m = \frac{2L}{m}, \quad f_m = \frac{mc}{2L} \quad (13.119)$$

in vacuum. Real microwave cavities are three-dimensional and have richer TE and TM mode structures, but this one-dimensional result captures the resonance logic.

Example 13.22: Fundamental resonance of a cavity

Two large conducting plates form a simplified one-dimensional vacuum cavity of separation $L = 0.120$ m. Find the fundamental resonance frequency and the spacing between adjacent electric-field nodes in that mode.

Solution. For $m = 1$,

$$f_1 = \frac{c}{2L} = \frac{2.998 \times 10^8}{0.240} = 1.25 \times 10^9 \text{ Hz}. \quad (13.120)$$

The fundamental wavelength is $\lambda_1 = 2L = 0.240$ m. Adjacent electric nodes are separated by $\lambda_1/2 = 0.120$ m, exactly the cavity length for the fundamental.

13.17.2 Microwave ovens as a qualitative illustration

A domestic microwave oven is not a perfect textbook cavity, but it provides a familiar example of electromagnetic resonance, reflection, absorption, and interference. Microwave radiation is injected into a metallic enclosure. Reflections form a complicated field pattern with spatial maxima and minima. Food absorbs part of the electromagnetic energy, with losses depending on its dielectric properties and composition. A rotating turntable or mode-stirring geometry helps average the spatial nonuniformity experienced by the food.

One should avoid the oversimplified claim that microwaves “heat water by matching a natural resonance frequency of the water molecule.” The dominant heating in typical food is dielectric loss in a complex material, not a single sharp molecular resonance selected by the appliance frequency. The exact response depends on frequency, temperature, composition, and material structure.

Guided checks

1. What distinguishes a standing electromagnetic wave from a single traveling plane wave in terms of spatial nodes?
2. Why are electric and magnetic energy densities not necessarily maximal at the same position in a standing-wave cavity?
3. Why is the simple formula $f_m = mc/(2L)$ a model rather than a complete description of a real microwave oven?
4. What happens to a cavity resonance if the effective wave speed decreases because a dielectric material fills the cavity uniformly?

13.18 Engineering systems built from electromagnetic-wave principles

Electromagnetic-wave theory is valuable because the same small set of field principles reappears in systems that, at first sight, seem unrelated. A Wi-Fi link, a radar set, a fiber-optic connection, a microwave sensor, a solar-energy system, an infrared camera, and a radio telescope all manipulate electromagnetic fields. Their engineering details differ, but each system must address generation, propagation, energy transport, interaction with matter, detection, noise, bandwidth, geometry, and ultimately information extraction.

13.18.1 Wireless communication: from voltage to field and back

A simplified wireless link contains the chain

$$\begin{aligned} \text{information} &\rightarrow \text{electrical signal} \rightarrow \text{transmitting antenna} \rightarrow \text{EM wave} \\ &\rightarrow \text{receiving antenna} \rightarrow \text{electrical signal} \rightarrow \text{information.} \end{aligned} \quad (13.121)$$

The source waveform controls amplitude, frequency, phase, or combinations of these quantities. The antenna converts guided electrical energy into radiated fields. During propagation, the wave may spread geometrically, reflect from surfaces, diffract around obstacles, interfere with other paths, and be absorbed by materials. The receiver converts a small fraction of the incident field back into an electrical signal.

This chain explains why a field description is more useful than thinking only about charge motion inside the transmitter. Once radiation leaves the antenna, Maxwell's equations determine its propagation through the environment. The receiver responds to the local field that reaches its position, not to a direct mechanical action by the transmitter's electrons.

For digital communication, the electromagnetic wave is a physical carrier of information. The present chapter does not develop modulation or coding theory, but it provides the quantities needed to understand the physical layer: carrier frequency, wavelength, polarization, intensity, phase, reflection, and propagation speed.

13.18.2 Radar and time-of-flight measurement

Radar uses transmitted electromagnetic pulses or modulated continuous waves and analyzes radiation returned from a target. In the simplest pulse picture, if a pulse travels through air at approximately c , reflects from a target, and returns after time delay Δt , the range is

$$R \simeq \frac{c\Delta t}{2}. \quad (13.122)$$

The factor $1/2$ accounts for the outward and return paths. Real radar additionally depends on antenna gain, scattering properties, bandwidth, signal processing, noise, and target motion.

Example 13.23: Radar range from a measured delay

A short radar pulse returns $36.0 \mu\text{s}$ after transmission. Approximate the target range, neglecting atmospheric refractive corrections.

Solution.

$$R = \frac{c\Delta t}{2} = \frac{(2.998 \times 10^8)(36.0 \times 10^{-6})}{2} = 5.40 \times 10^3 \text{ m.} \quad (13.123)$$

The target is approximately 5.40 km away. The calculation is simple, but a practical range measurement also requires identifying the returned pulse against noise and possible reflections from other objects.

13.18.3 Solar-energy conversion

At the surface of a photovoltaic module, the electromagnetic quantity immediately relevant to available incident power is irradiance, measured in W m^{-2} . The power intercepted by a flat panel is

$$P_{\text{in}} = IA \cos \alpha, \quad (13.124)$$

where α is the angle between the panel normal and the propagation direction. The electrical output is smaller because of reflection, incomplete absorption, carrier losses, temperature effects, and conversion efficiency. The wave viewpoint therefore connects naturally to an engineering energy budget.

13.18.4 Optical communication

An optical fiber confines light over long distances while maintaining sufficiently low loss and manageable dispersion for communication. Information is carried by modulating an optical source and detecting the received light with a photodetector. At the introductory level, four chapter ideas are central:

- confinement is related to refractive-index structure and, in the ray picture, total internal reflection;
- the optical carrier frequency is extremely high, allowing large available communication bandwidths;
- attenuation reduces optical power and must be included in a link budget;
- dispersion changes pulse shapes and limits how closely data pulses can be packed without distortion.

The advanced section develops numerical aperture and a simple link-budget viewpoint.

13.18.5 Sensing and imaging across the electromagnetic spectrum

Different wavelength bands interact with matter differently. Radio and microwave systems are useful for communication and radar; infrared measurements are strongly connected to thermal emission and molecular responses; visible light is central to human vision and conventional imaging; ultraviolet, X-rays, and gamma rays involve progressively higher photon energies and, at sufficiently high energies, ionization becomes important. A complete treatment of photon energy $E_\gamma = hf$ belongs to the quantum-physics part of the course, but the classical electromagnetic-wave description remains essential for propagation, polarization, interference, diffraction, and many detector-system calculations.

The electromagnetic spectrum should therefore be regarded as a continuum of the same field phenomenon, not as unrelated categories of radiation. The labels “radio,” “microwave,” “infrared,” and so on identify useful ranges of frequency and typical source/detector technologies rather than boundaries at which Maxwell’s equations change form.

13.18.6 Astronomy and remote observation

Almost all astronomical information arriving at Earth from outside the Solar System is carried by electromagnetic radiation, apart from other messengers such as neutrinos, cosmic rays, and gravitational waves. Measuring wavelength, intensity, polarization, and time variability reveals information about distant sources. Radio telescopes and optical telescopes differ dramatically in hardware because wavelength affects antenna or aperture dimensions, detector technology, atmospheric transmission, and achievable angular resolution; nevertheless, both measure electromagnetic fields or the energy they deliver.

Key message

Engineering applications are not separate from the theory. They are organized combinations of the same physical quantities: field amplitude, frequency, wavelength, phase, polarization, intensity, momentum flow, and boundary response. Learning to translate a device description into those quantities is a central problem-solving skill.

Guided checks

1. Why does a radar range formula contain a factor of $1/2$?
2. Which chapter quantities would you expect to matter when orienting a receiving antenna?
3. Why is irradiance, rather than electric-field amplitude alone, often the most convenient quantity for a solar-panel power estimate?
4. What common physical principle links an optical-fiber communication system and a microwave wireless link even though their frequencies differ by many orders of magnitude?

13.19 Laboratory and computational connections

Electromagnetic waves are especially well suited to connecting analytic theory with measurement and computation. Many quantities in this chapter can be observed directly or inferred from simple measurements: frequency with a spectrum analyzer, wavelength from standing-wave spacing, polarization from angle-dependent transmission, intensity from detector power and aperture area, and refractive index from refraction angles. Computation complements these measurements by allowing students to visualize vector fields and explore parameter dependence beyond hand calculations.

13.19.1 Laboratory activity A: microwave polarization

A transmitter and receiver operating in the microwave range can make polarization visible on a laboratory scale. If the transmitter produces a linearly polarized field and the receiver is rotated through an angle θ , a simple model predicts

$$P_r(\theta) = P_{r,\max} \cos^2 \theta, \quad (13.125)$$

provided the receiver output is proportional to received power and multipath reflections are small. A practical experiment can proceed as follows:

1. Align transmitter and receiver for maximum signal and define this as $\theta = 0$.
2. Record the detector reading at angles such as $0^\circ, 10^\circ, \dots, 90^\circ$.
3. Normalize the readings by the maximum value.
4. Plot normalized power versus $\cos^2 \theta$.
5. Examine residuals from a straight line and discuss possible causes: finite antenna cross-polarization, reflections from walls, detector offsets, alignment errors, and nonlinearity.

This transforms Malus-like projection from an equation into an experimental model that can be tested.

For a nonideal analyzer use $V(\theta) = b + A \cos^2(\theta - \theta_0)$ over its calibrated linear range. Measure the source-blocked background separately. The fitted b can contain both background and finite extinction; subtracting a dark signal does not prove the remaining minimum must be zero. The angle offset θ_0 is a parameter, not a reason to force the maximum to a nominal zero. Fit repeated angles and inspect residuals before attributing a small discrepancy to a failure of Malus's law.

Example 13.24: An apparent extinction floor

A calibrated detector with output proportional to optical power reads 1.02 V at maximum transmission, 0.06 V at minimum, and 0.02 V with the source blocked. Assume the background is additive and stable. Estimate the analyzer's observed minimum-to-maximum power ratio, and explain what it establishes.

Solution. Subtract the same background from both measurements:

$$\frac{P_{\min}}{P_{\max}} = \frac{0.06 - 0.02}{1.02 - 0.02} = 0.040.$$

The observed extinction ratio is $P_{\max}/P_{\min} = 25$, or $10 \log_{10} 25 = 14.0$ dB. It describes this complete optical and detector arrangement. Imperfect source polarization, stray light, angular error and the analyzer can all contribute; the three readings alone do not isolate the analyzer's intrinsic extinction.

13.19.2 Laboratory activity B: measuring wavelength with a standing wave

A movable probe along a microwave line or cavity can locate successive minima or maxima of a standing-wave pattern. If the mean spacing of adjacent minima is d , then

$$\lambda = 2d, \quad f = \frac{v}{\lambda}. \quad (13.126)$$

In air, $v \approx c$ is often adequate for an introductory estimate. Repeated measurements of several node spacings reduce the effect of position-reading uncertainty.

Suppose N successive minima are indexed from 0 to $N - 1$. Rather than computing each adjacent difference separately, a more robust method is to fit

$$x_m = x_0 + m \frac{\lambda}{2} \quad (13.127)$$

with a straight line. The slope provides $\lambda/2$ and regression diagnostics reveal inconsistent points. This is an elementary example of using data analysis to estimate a physical parameter.

13.19.3 Laboratory activity C: Snell's law and refractive index

If light travels from air into an unknown transparent material, measurements of θ_1 and θ_2 satisfy approximately

$$\sin \theta_1 = n \sin \theta_2, \quad (13.128)$$

where $n_{\text{air}} \simeq 1$. A plot of $\sin \theta_1$ against $\sin \theta_2$ should therefore be linear with slope n . Using the whole data set makes systematic deviations easier to see. Ordinary least squares assumes negligible uncertainty in its horizontal coordinate; when both angles are uncertain, fit the angle relation with their uncertainties or use an errors-in-variables fit. A free intercept is a diagnostic, not proof that an angular zero error has been removed.

Example 13.25: Standing-wave diagnosis from measured maxima

A probe finds electric-field maxima at positions $x = 1.2, 7.3, 13.4, 19.5$ cm along a microwave apparatus. Assuming these are consecutive antinodes of a simple standing wave, estimate the wavelength and frequency in air.

Solution. Consecutive antinodes are separated by $\lambda/2$. The spacings are all approximately 6.1 cm, so

$$\frac{\lambda}{2} \simeq 6.1 \text{ cm}, \quad \lambda \simeq 12.2 \text{ cm} = 0.122 \text{ m}. \quad (13.129)$$

Then

$$f \simeq \frac{c}{\lambda} = \frac{2.998 \times 10^8}{0.122} \simeq 2.46 \text{ GHz}. \quad (13.130)$$

The repeated equal spacing is itself evidence that the simple standing-wave model is reasonable for these measurements.

13.19.4 Computational activity A: animate a plane wave

A minimal simulation can evaluate

$$E_y(x, t) = E_0 \cos(kx - \omega t), \quad (13.131)$$

$$B_z(x, t) = \frac{E_0}{c} \cos(kx - \omega t) \quad (13.132)$$

on a grid. Students can plot both components at successive times and verify:

- equal phase of E and B for an ideal vacuum plane wave;
- the amplitude ratio $E/B = c$;
- motion of fixed-phase points with speed ω/k ;
- reversal of propagation direction when the phase changes to $kx + \omega t$ and the sign of B_z is reversed;
- the direction of $\mathbf{E} \times \mathbf{B}$.

A useful extension is to plot the instantaneous energy density and Poynting vector. Students should observe that energy flow oscillates in magnitude for a monochromatic sinusoid even though its cycle average is constant and positive.

13.19.5 Computational activity B: polarizer sequences

For a series of ideal linear polarizers at angles $\alpha_1, \alpha_2, \dots, \alpha_N$, an initially unpolarized beam follows

$$I_N = \frac{I_0}{2} \prod_{j=2}^N \cos^2(\alpha_j - \alpha_{j-1}). \quad (13.133)$$

A simple program can scan an intermediate polarizer angle between crossed polarizers and display the final intensity. For three polarizers at 0, α , and 90° ,

$$I_3 = \frac{I_0}{2} \cos^2 \alpha \sin^2 \alpha = \frac{I_0}{8} \sin^2(2\alpha), \quad (13.134)$$

which reaches its maximum at $\alpha = 45^\circ$. This makes a good example of numerical parameter scanning confirming an analytic result.

13.19.6 Computational activity C: ray tracing through planar interfaces

A basic two-dimensional ray tracer can represent a ray by a starting point and direction. At each interface, the program computes the surface normal, finds the incidence angle, applies Snell's law,

and determines whether a refracted direction exists. If not, it reflects the ray according to the law of reflection. Even this elementary algorithm teaches several habits important in computational physics:

- normalize direction vectors;
- distinguish geometry from optical material parameters;
- enforce angle conventions consistently;
- check for numerical domain errors when an attempted $\sin \theta_2$ exceeds unity;
- validate code against analytically simple cases such as normal incidence.

Example 13.26: Uncertainty in field amplitude inferred from intensity

A detector measures a plane-wave intensity $I = (120 \pm 6) \text{ W m}^{-2}$. Assuming the vacuum plane-wave model and that the uncertainty is small and approximately symmetric, estimate E_0 and its uncertainty.

Solution. From

$$E_0 = \sqrt{2\mu_0 c I}, \quad (13.135)$$

the logarithmic differential gives

$$\frac{\delta E_0}{E_0} \simeq \frac{1}{2} \frac{\delta I}{I}. \quad (13.136)$$

The relative intensity uncertainty is $6/120 = 0.050$, so the relative amplitude uncertainty is approximately 0.025. Numerically,

$$E_0 = \sqrt{2(4\pi \times 10^{-7})(2.998 \times 10^8)(120)} \approx 301 \text{ V m}^{-1}. \quad (13.137)$$

Thus

$$\delta E_0 \approx 0.025(301) \approx 8 \text{ V m}^{-1}, \quad (13.138)$$

so a suitable result is

$$E_0 = (301 \pm 8) \text{ V m}^{-1}. \quad (13.139)$$

The square-root relation reduces the relative uncertainty by a factor of two in this one-variable approximation.

Method

A computational or experimental result should be tested against physics before it is trusted. Check units, limiting cases, expected symmetry, order of magnitude, conservation laws, and at least one case with a known analytic solution. A smooth plot is not evidence that a model or program is correct.

Guided checks

1. Why does fitting many standing-wave node positions usually give a more stable wavelength estimate than using one adjacent pair?
2. In a Malus-law experiment, what physical effects could produce nonzero detector output at 90° ?
3. Why should a ray-tracing code explicitly test whether the Snell-law solution requires $\sin \theta_2 > 1$?

4. If intensity has a 4% relative uncertainty, what is the approximate relative uncertainty of E_0 inferred from it when other uncertainties are neglected?

13.19.7 From electromagnetic field to instrument reading

A textbook problem often gives E_0 or I directly. An experiment rarely does. Instead, an instrument converts electromagnetic energy into another measurable quantity: voltage, current, temperature rise, charge, digital counts, or a frequency-domain spectrum. Understanding this measurement chain is essential for applied work.

Antenna and RF receiver chain

A receiving antenna responds to the incident field and makes available an RF voltage or power at its terminals. A practical receiver then uses matching networks, filters, amplifiers, mixers, analog-to-digital converters, and digital signal processing. Each stage has bandwidth, gain, noise, and saturation limits. Thus a statement such as “the receiver measured the electric field” usually means that a calibrated chain converted an electrical output back into an estimate of field strength.

The effective aperture relation

$$P_r = I A_e \quad (13.140)$$

illustrates this conversion at the power level. To infer I , one must know the receiving antenna response for the frequency, direction, and polarization of interest.

Optical power detector

A photodiode converts absorbed optical power into electrical current through a quantum process. In an introductory calibration model one may write

$$I_d = \mathcal{R} P_{\text{opt}}, \quad (13.141)$$

where $\mathcal{R}$ is responsivity in A/W. The value of $\mathcal{R}$ depends on wavelength and device properties. If the detector is connected to a transimpedance amplifier, the measured voltage may be proportional to I_d and therefore to optical power over a suitable linear range.

Even though the microscopic detection process is quantum, the incident wave can still be characterized classically by Poynting flux when the measurement involves large numbers of photons. This coexistence of classical propagation and quantum detection is common in optics.

Thermal detector

A bolometric or thermal detector absorbs electromagnetic power and measures a property that changes with temperature. Such detectors can be broadband because they need not rely on a particular electronic transition. Their response is often slower than that of direct electronic detectors because heat capacity and thermal conductance introduce time constants.

A simple lumped thermal model is

$$C \frac{d\Delta T}{dt} + G \Delta T = P_{\text{abs}}(t), \quad (13.142)$$

where C is heat capacity and G is thermal conductance to the environment. This equation is not needed for the core wave chapter, but it demonstrates how electromagnetic power becomes an experimentally accessible temperature signal.

Dynamic range and saturation

Every instrument has a range over which its calibration is meaningful. At too low a signal, noise and offsets dominate. At too high a signal, amplifiers, photodiodes, mixers, or digitizers may saturate. A reliable experiment therefore reports not only a measured value but also whether the instrument operated in its calibrated linear region.

Example 13.27: From photodetector current to optical intensity

A photodiode has responsivity $\mathcal{R} = 0.60 \text{ A/W}$ at the wavelength used. It produces a photocurrent $I_d = 1.8 \text{ mA}$ when uniformly illuminated over an active area $A = 3.0 \text{ mm}^2$. Neglect reflection and calibration uncertainty. Estimate the incident optical power and intensity.

Solution.

$$P_{\text{opt}} = \frac{I_d}{\mathcal{R}} = \frac{1.8 \times 10^{-3}}{0.60} = 3.0 \times 10^{-3} \text{ W}. \quad (13.143)$$

The active area is $3.0 \times 10^{-6} \text{ m}^2$, so

$$I = \frac{P}{A} = 1.0 \times 10^3 \text{ W m}^{-2}. \quad (13.144)$$

If one then wanted the equivalent plane-wave electric amplitude, the additional vacuum-plane-wave model would give $E_0 = \sqrt{2Z_0 I}$. That last step is an inference from a model, not a quantity measured directly by the photodiode.

Caution

Never confuse detector output units with field units. A voltage displayed by an RF detector, a current from a photodiode, or digital camera counts are not automatically volts per metre or watts per square metre. The conversion requires calibration and a model of the receiving geometry.

13.20 Main chapter summary

Summary

- Maxwell's equations predict electromagnetic waves. In source-free vacuum, both electric and magnetic fields satisfy wave equations with speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

- A plane wave is transverse. For a wave traveling along $\hat{n}$, the vectors $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular and $\mathbf{E} \times \mathbf{B}$ points along the direction of energy transport.
- In a sinusoidal vacuum plane wave, the fields are in phase and their amplitudes satisfy $E_0 = cB_0$.
- Electric and magnetic field energy densities are equal instantaneously for an ideal vacuum plane wave. The Poynting vector

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

describes electromagnetic energy flux.

- The time-averaged intensity of a sinusoidal plane wave is

$$I = \frac{E_0^2}{2\mu_0 c} = \frac{cB_0^2}{2\mu_0} = \frac{E_{\text{rms}}^2}{\mu_0 c}.$$

- Electromagnetic waves carry momentum. At normal incidence, ideal absorption produces pressure I/c and ideal reflection produces $2I/c$.
- Time-dependent charge and current distributions generate radiation. Oscillating currents in antennas provide a practical source; the far field is locally well approximated by a transverse wave.
- Linear polarization specifies a fixed electric-field oscillation direction. Ideal polarizers project the field, giving Malus's law $I = I_0 \cos^2 \theta$ for linearly polarized incident light and the one-half rule for initially unpolarized light.
- Reflection and refraction at interfaces are described in the ray limit by $\theta_r = \theta_1$ and Snell's law $n_1 \sin \theta_1 = n_2 \sin \theta_2$.
- Total internal reflection is possible only for propagation from higher to lower refractive index and begins above $\theta_c = \sin^{-1}(n_2/n_1)$.
- At Brewster incidence, any nonzero reflected wave is fully polarized perpendicular to the plane of incidence in the ideal dielectric model, with $\tan \theta_B = n_2/n_1$.
- Reflections from boundaries can create standing waves and cavity resonances. Boundary conditions select discrete mode frequencies.
- The same principles organize engineering systems in communication, radar, optical fibers, microwave technology, solar energy, sensing, and astronomy.

Formula summary

For a sinusoidal plane wave in vacuum,

$$\begin{array}{lll}
 k = \frac{2\pi}{\lambda}, & \omega = 2\pi f, & c = \frac{\omega}{k} = \lambda f, \\
 c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}, & E_0 = cB_0, & E_{\text{rms}} = \frac{E_0}{\sqrt{2}}, \\
 u_E = \frac{1}{2}\epsilon_0 E^2, & u_B = \frac{B^2}{2\mu_0}, & \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}, \\
 I = \frac{E_0^2}{2\mu_0 c} = \frac{cB_0^2}{2\mu_0}, & p_{\text{abs}} = \frac{I}{c}, & p_{\text{refl}} = \frac{2I}{c}.
 \end{array}$$

For ideal polarization and interfaces,

$$\begin{array}{ll}
 I = I_0 \cos^2 \theta, & n_1 \sin \theta_1 = n_2 \sin \theta_2, \\
 \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \quad (n_1 > n_2), & \tan \theta_B = \frac{n_2}{n_1}.
 \end{array}$$

For simple geometric spreading and a one-dimensional cavity,

$$I(r) = \frac{P}{4\pi r^2} \quad (\text{isotropic far-field source}), \quad f_m = \frac{mc}{2L}.$$

Always attach the stated assumptions to each formula.

13.21 Advanced Topics for Technical and Engineering Students

This enrichment part develops material that is not required for the first pass through electromagnetic waves but is highly valuable for strong students in technical physics, electrical engineering, photonics, computer science, space technology, and applied physics. The emphasis is not on collecting additional formulas. Instead, the aim is to show how the introductory results arise from the full vector structure of Maxwell's equations and how those results extend into numerical modeling and engineering design.

The advanced material uses partial derivatives, vector-calculus identities, complex notation, and elementary differential-equation reasoning. Each section states its assumptions explicitly. Students should be able to separate idealized results from approximations used in practical systems.

Pedagogical boundary

The advanced section goes beyond the minimum Physics II syllabus in mathematical depth. It remains within classical electromagnetism and introductory wave engineering. Relativistic field transformations, quantum electrodynamics, microscopic dielectric-response theory, and full antenna/waveguide design are outside the present scope.

13.22 Differential Maxwell equations and the three-dimensional wave equation

The integral forms of Maxwell's equations describe fluxes and circulations over finite curves and surfaces. For field analysis, the local differential forms are often more powerful. In vacuum with charge density ρ and current density $\mathbf{J}$,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0, \quad (13.145)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \quad (13.146)$$

In a source-free region, $\rho = 0$ and $\mathbf{J} = 0$. Taking the curl of Faraday's law gives

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}). \quad (13.147)$$

Using the Ampère-Maxwell equation in vacuum,

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (13.148)$$

The vector identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (13.149)$$

then reduces the left-hand side because $\nabla \cdot \mathbf{E} = 0$ in source-free vacuum. The result is

$$\boxed{\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.} \quad (13.150)$$

Repeating the procedure for $\mathbf{B}$ yields

$$\boxed{\nabla^2 \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.} \quad (13.151)$$

Comparison with the standard wave equation identifies the propagation speed $c = 1/\sqrt{\mu_0 \epsilon_0}$.

This derivation demonstrates why Gauss's law matters even though it contains no explicit time derivative. The divergence condition removes the longitudinal term $\nabla(\nabla \cdot \mathbf{E})$ in source-free space. All four Maxwell equations cooperate to determine the structure of the wave.

Theoretical point 13.2: Why transversality follows from Gauss's law

Consider a monochromatic plane wave

$$\mathbf{E}(\mathbf{r}, t) = \Re \left\{ \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \right\}. \quad (13.152)$$

For its complex representation,

$$\nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}. \quad (13.153)$$

Source-free Gauss's law requires this to vanish for all $\mathbf{r}$ and t , so

$$\boxed{\mathbf{k} \cdot \mathbf{E}_0 = 0.} \quad (13.154)$$

Thus the electric field is perpendicular to the propagation vector. The same argument gives $\mathbf{k} \cdot \mathbf{B}_0 = 0$.

Substituting the plane-wave form into Faraday's law gives

$$i\mathbf{k} \times \mathbf{E}_0 = i\omega \mathbf{B}_0, \quad (13.155)$$

so

$$\boxed{\mathbf{B}_0 = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}_0 = \frac{1}{c} \hat{\mathbf{n}} \times \mathbf{E}_0.} \quad (13.156)$$

This compact relation contains the geometry, magnitude ratio, and handedness of a vacuum plane wave.

Example 13.28: Recovering the magnetic field from a vector plane wave

A vacuum wave has

$$\mathbf{E} = E_0 \cos(kz - \omega t)(\cos 30^\circ \hat{\mathbf{x}} + \sin 30^\circ \hat{\mathbf{y}}). \quad (13.157)$$

Find the associated magnetic field.

Solution. The phase $kz - \omega t$ indicates propagation along $+z$, so $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Using Eq. (13.156),

$$\mathbf{B} = \frac{E_0}{c} \cos(kz - \omega t) \hat{\mathbf{z}} \times (\cos 30^\circ \hat{\mathbf{x}} + \sin 30^\circ \hat{\mathbf{y}}) \quad (13.158)$$

$$= \frac{E_0}{c} \cos(kz - \omega t) (\cos 30^\circ \hat{\mathbf{y}} - \sin 30^\circ \hat{\mathbf{x}}). \quad (13.159)$$

The magnetic field is rotated by 90° in the transverse plane and has amplitude E_0/c .

Guided checks

1. Which term in the vector identity would allow a longitudinal electric component if $\nabla \cdot \mathbf{E}$ were nonzero?
2. Starting from $\mathbf{B}_0 = (1/\omega)\mathbf{k} \times \mathbf{E}_0$, recover $B_0 = E_0/c$.
3. Why is the plane-wave propagation direction naturally represented by $\mathbf{k}$ rather than only by the scalar wave number k ?

13.23 Poynting's theorem: field energy and work on matter

The formula for $\mathbf{S}$ becomes more informative when embedded in a local conservation law. In vacuum with specified charge/current sources, dot the Ampère–Maxwell equation with $\mathbf{E}/\mu_0$ and Faraday's

equation with $\mathbf{B}/\mu_0$:

$$\begin{aligned}\frac{1}{\mu_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}) &= \mathbf{J} \cdot \mathbf{E} + \varepsilon_0 \mathbf{E} \cdot \partial_t \mathbf{E}, \\ \frac{1}{\mu_0} \mathbf{B} \cdot (\nabla \times \mathbf{E}) &= -\frac{1}{\mu_0} \mathbf{B} \cdot \partial_t \mathbf{B}.\end{aligned}$$

Subtract the first line from the second. The vector identity $\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B})$ gives

$$\boxed{\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}}, \quad u = \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}. \quad (13.160)$$

Here $\mathbf{J} \cdot \mathbf{E}$ is work per unit volume per unit time done by the field on charges. For Ohmic matter it is positive and becomes heating; in a driven source it can be negative, supplying energy to the field. Every term has units W m^{-3} .

Integrating over a fixed volume and applying the divergence theorem,

$$\frac{d}{dt} \int_V u \, dV + \oint_{\partial V} \mathbf{S} \cdot d\mathbf{A} = - \int_V \mathbf{J} \cdot \mathbf{E} \, dV. \quad (13.161)$$

The surface normal points outward. Positive outward flux reduces stored field energy unless sources replenish it. A source-free traveling pulse can leave a region, reducing its field energy without local absorption. In a steady absorber the cycle-averaged stored energy is constant, and net inward electromagnetic power equals work on matter. In a lossless closed cavity both the net wall flux and work vanish, conserving total energy even while $u(x, t)$ varies locally.

For a homogeneous nondispersive linear medium, a corresponding macroscopic result uses $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ and $u = (\varepsilon E^2 + \mu H^2)/2$. In absorbing or dispersive media the stored energy includes material dynamics; inserting a complex or frequency-dependent ε into that simple instantaneous expression is not justified.

Example 13.29: Checking a detector's energy budget

A detector receives 12.0 mW, reflects 1.0 mW and transmits 2.0 mW. During a short transient its stored electromagnetic energy increases at 0.5 mW. Find the instantaneous cycle-averaged power transferred to its material degrees of freedom.

Solution. The net outward flux is $1.0 + 2.0 - 12.0 = -9.0$ mW. Equation (13.161) then gives $P_{\text{matter}} = 9.0 - 0.5 = 8.5$ mW. In steady state the storage term vanishes and the material receives 9.0 mW. Absorbed electromagnetic power may subsequently become electrical output, heat or other internal energy; this balance does not itself specify a conversion efficiency.

13.24 Complex notation, phasors, and wave impedance

Complex exponentials are not new physics; they are an efficient representation of sinusoidal solutions. A real field can be written as the real part of a complex field,

$$\mathbf{E}(\mathbf{r}, t) = \Re \left\{ \tilde{\mathbf{E}}(\mathbf{r}) e^{-i\omega t} \right\}. \quad (13.162)$$

For a plane wave propagating along $+z$,

$$\tilde{\mathbf{E}}(z) = \mathbf{E}_0 e^{ikz}. \quad (13.163)$$

Time differentiation becomes multiplication by $-i\omega$, and spatial differentiation of a plane-wave factor becomes multiplication by $i\mathbf{k}$. This converts differential equations into algebraic vector equations for monochromatic fields.

13.24.1 Intrinsic impedance of free space

For a vacuum plane wave,

$$\frac{E_0}{H_0} = Z_0, \quad (13.164)$$

where $\mathbf{H} = \mathbf{B}/\mu_0$ in vacuum. Therefore

$$Z_0 = \frac{E_0}{B_0/\mu_0} = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 376.7 \Omega. \quad (13.165)$$

The ohm unit is not accidental: E/H has units $(\text{V/m})/(\text{A/m}) = \Omega$. Free space behaves, for a traveling wave, as a medium with a characteristic ratio of electric to magnetic field strength.

The time-averaged Poynting vector for phasors is conveniently written

$$\langle \mathbf{S} \rangle = \frac{1}{2} \Re (\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*), \quad (13.166)$$

where $*$ denotes complex conjugation. For a lossless plane wave with $\mathbf{E}$ and $\mathbf{H}$ in phase,

$$I = \frac{E_0^2}{2Z_0} = \frac{Z_0 H_0^2}{2}. \quad (13.167)$$

13.24.2 Impedance matching as a preview

In a homogeneous isotropic lossless medium with real positive ϵ, μ , the same wave derivation gives

$$v = \frac{1}{\sqrt{\mu\epsilon}}, \quad \mathbf{B} = \frac{1}{v} \hat{\mathbf{n}} \times \mathbf{E}, \quad Z = \sqrt{\frac{\mu}{\epsilon}}, \quad I = \frac{E_0^2}{2Z}.$$

For a nonmagnetic dielectric $\mu \simeq \mu_0$, so $Z \simeq Z_0/n$. Consequently $I \simeq nE_0^2/(2Z_0)$, rather than the vacuum formula with the same E_0 . In this simple nondispersive model the electric and magnetic energy densities are equal for a single traveling wave, and $I = v\langle u \rangle$. Reflection depends on Z , so refractive index alone is insufficient in a general magnetic medium.

When a traveling wave encounters a boundary, reflection is strongly influenced by the mismatch between wave impedances of the two media. This is closely analogous to reflections on transmission lines. At normal incidence between lossless nonmagnetic dielectrics, the electric-field reflection coefficient can be expressed in terms of refractive indices or impedances. The next section derives the result explicitly.

Caution

Complex amplitudes are calculation tools. Measured electric and magnetic fields are real. Never report an imaginary electric field as if it were a directly measurable quantity; take the real part after the algebra, or work consistently with phasor magnitudes and phases.

13.25 Fresnel coefficients at dielectric interfaces

The main chapter used reflection and refraction without calculating how much power is reflected. Fresnel coefficients provide that information. The simplest derivation is for normal incidence between two homogeneous, isotropic, lossless, nonmagnetic dielectrics.

Let a wave travel from medium 1 to medium 2. At the interface $z = 0$, write the electric fields as

$$E_i(z, t) = E_{0i} \cos(k_1 z - \omega t), \quad (13.168)$$

$$E_r(z, t) = E_{0r} \cos(-k_1 z - \omega t), \quad (13.169)$$

$$E_t(z, t) = E_{0t} \cos(k_2 z - \omega t). \quad (13.170)$$

The tangential electric field must be continuous, so at the interface

$$E_{0i} + E_{0r} = E_{0t}. \quad (13.171)$$

For each traveling wave, $H = E/Z$, but the reflected magnetic field reverses sign because its propagation direction is reversed. Continuity of the tangential magnetic field gives

$$\frac{E_{0i}}{Z_1} - \frac{E_{0r}}{Z_1} = \frac{E_{0t}}{Z_2}. \quad (13.172)$$

Define the amplitude coefficients

$$r \equiv \frac{E_{0r}}{E_{0i}}, \quad t \equiv \frac{E_{0t}}{E_{0i}}. \quad (13.173)$$

Solving Eqs. (13.171) and (13.172) yields

$$\boxed{r = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad t = \frac{2Z_2}{Z_1 + Z_2}}. \quad (13.174)$$

For nonmagnetic dielectrics, $Z = Z_0/n$, so

$$\boxed{r = \frac{n_1 - n_2}{n_1 + n_2}, \quad t = \frac{2n_1}{n_1 + n_2}}. \quad (13.175)$$

The sign of r describes the phase of the reflected electric field. If $n_2 > n_1$, then $r < 0$, corresponding to a π phase reversal of the reflected electric field in this convention.

Power coefficients must be constructed from the Poynting flux, not by squaring t blindly. For lossless nonmagnetic media,

$$R = |r|^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2, \quad (13.176)$$

and

$$T = \frac{n_2}{n_1} |t|^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}. \quad (13.177)$$

For the ideal lossless interface,

$$R + T = 1. \quad (13.178)$$

Example 13.30: Normal-incidence reflection from glass

Light in air ($n_1 = 1.00$) strikes glass ($n_2 = 1.50$) normally. Find the electric-field reflection coefficient and the reflected power fraction.

Solution.

$$r = \frac{1.00 - 1.50}{1.00 + 1.50} = -0.200. \quad (13.179)$$

Thus the reflected electric field has 20% of the incident amplitude and is phase reversed. The reflected power fraction is

$$R = r^2 = 0.0400. \quad (13.180)$$

About 4.0% of the incident power is reflected at a single ideal air-glass interface at normal incidence. The transmitted power fraction is $T = 0.960$.

Example 13.31: Why $T \neq t^2$ across unequal indices

For the same air-glass interface, evaluate t and compare t^2 with the actual transmitted power fraction.

Solution.

$$t = \frac{2n_1}{n_1 + n_2} = \frac{2}{2.5} = 0.800. \quad (13.181)$$

Hence $t^2 = 0.640$, which is not the power transmittance. The correct expression includes the ratio of wave impedances, equivalently n_2/n_1 :

$$T = \frac{n_2}{n_1} t^2 = (1.50)(0.640) = 0.960. \quad (13.182)$$

The difference occurs because equal electric-field amplitudes do not imply equal energy flux in media with different impedances.

Guided checks

1. What physical boundary conditions were used to derive the normal-incidence Fresnel coefficients?
2. What does a negative electric-field reflection coefficient mean physically?
3. Why must transmitted power be computed from the Poynting vector rather than simply from t^2 ?
4. Verify algebraically that Eqs. (13.176) and (13.177) sum to unity.

13.25.1 Oblique incidence and the Brewster zero

Let the interface be $z = 0$ and the plane of incidence be xz . For s polarization all electric amplitudes are referred to $+y$. For p polarization choose the in-plane unit vector $\hat{e}_p = \hat{y} \times \hat{n}$ separately for each ray. The reflected p basis then has its tangential direction reversed relative to the incident basis at normal incidence. This is a convention for signed amplitudes, not a physical difference in reflectance.

For lossless nonmagnetic dielectrics, the two tangential boundary conditions give

$$\begin{aligned} 1 + r_s &= t_s, & n_1 \cos \theta_i (1 - r_s) &= n_2 \cos \theta_t t_s, \\ \cos \theta_i (1 - r_p) &= \cos \theta_t t_p, & n_1 (1 + r_p) &= n_2 t_p. \end{aligned}$$

Solving each pair yields

$$r_s = \frac{n_1 \cos \theta_i - n_2 \cos \theta_t}{n_1 \cos \theta_i + n_2 \cos \theta_t}, \quad r_p = \frac{n_2 \cos \theta_i - n_1 \cos \theta_t}{n_2 \cos \theta_i + n_1 \cos \theta_t}, \quad (13.183)$$

$$t_s = \frac{2n_1 \cos \theta_i}{n_1 \cos \theta_i + n_2 \cos \theta_t}, \quad t_p = \frac{2n_1 \cos \theta_i}{n_2 \cos \theta_i + n_1 \cos \theta_t}. \quad (13.184)$$

The power coefficients for either polarization are

$$R_{s,p} = |r_{s,p}|^2, \quad T_{s,p} = \frac{n_2 \cos \theta_t}{n_1 \cos \theta_i} |t_{s,p}|^2, \quad R_{s,p} + T_{s,p} = 1. \quad (13.185)$$

The cosine factors compare flux through the *same interface area*, not areas perpendicular to different ray directions. The expressions above use real propagating angles; TIR requires a complex transmitted normal wave number and cannot be handled by inserting a real θ_t beyond 90° .

Setting the numerator of r_p to zero and combining with Snell's law gives $\theta_i + \theta_t = 90^\circ$ and $\tan \theta_B = n_2/n_1$ for unequal positive indices. If the indices are equal there is no interface reflection

at any angle, rather than one distinguished Brewster angle. Figure 13.11 shows how the p zero differs from the behavior of s polarization.

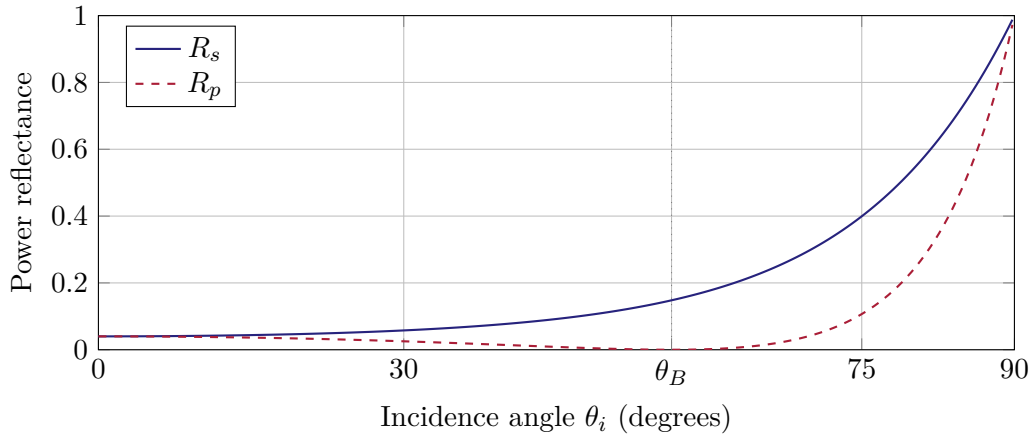


Figure 13.11: Calculated Fresnel power reflectances for $n_1 = 1.00$, $n_2 = 1.50$. Both begin at 0.04 at normal incidence. Only R_p vanishes at $\theta_B = 56.31^\circ$; both tend to unity at grazing incidence. An unpolarized beam has $R = (R_s + R_p)/2$.

13.26 Polarization ellipse, Jones vectors, and polarization optics

Linear polarization is one member of a larger family. Consider a monochromatic wave propagating along $+z$ with transverse components

$$E_x(t) = E_{0x} \cos \omega t, \quad (13.186)$$

$$E_y(t) = E_{0y} \cos(\omega t - \delta) \quad (13.187)$$

(at one fixed position, with an unimportant common phase omitted). Eliminating time gives the polarization ellipse

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\frac{E_x E_y}{E_{0x} E_{0y}} \cos \delta = \sin^2 \delta. \quad (13.188)$$

Special cases are especially important:

- $\delta = 0$ or π : linear polarization;
- $E_{0x} = E_{0y}$ and $|\delta| = \pi/2$: circular polarization;
- all other nondegenerate combinations: elliptical polarization.

The sign convention used to label left- and right-circular polarization varies between communities depending on viewing direction and time dependence. For this reason, technical work should state the convention rather than relying on an unlabeled “left/right” word alone.

The ellipse equation follows without a new physical assumption. Write $X = E_x/E_{0x} = \cos \omega t$ and $Y = E_y/E_{0y} = \cos \omega t \cos \delta + \sin \omega t \sin \delta$. Squaring $Y - X \cos \delta = \sin \omega t \sin \delta$ and replacing $\sin^2 \omega t$ by $1 - X^2$ gives Eq. (13.188). If either component amplitude vanishes, use the original component equations rather than dividing by zero. Figure 13.12 compares three nonzero-component cases.

13.26.1 Jones-vector representation

For a fully polarized monochromatic wave, the complex transverse amplitudes can be collected into a Jones vector,

$$\mathbf{J} = \begin{pmatrix} E_{0x} e^{i\phi_x} \\ E_{0y} e^{i\phi_y} \end{pmatrix}. \quad (13.189)$$

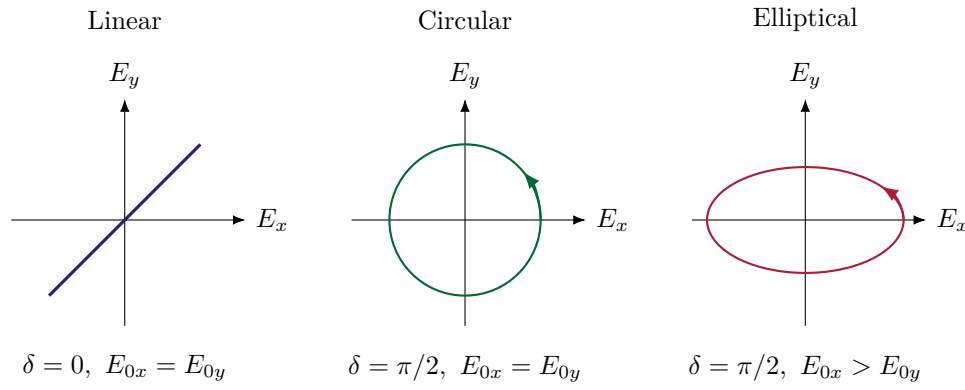


Figure 13.12: Paths of the electric-field tip at a fixed position. The circular and elliptical arrows show increasing time for $E_x = E_{0x} \cos \omega t$, $E_y = E_{0y} \sin \omega t$, with $+z$ out of the page. These are polarization paths, not particle or ray trajectories.

An overall phase does not change the polarization ellipse, so only the amplitude ratio and relative phase are physically relevant for polarization.

An ideal linear polarizer whose transmission axis makes angle θ with the x axis can be represented by the Jones matrix

$$P(\theta) = \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix}. \quad (13.190)$$

Applying the matrix to an input Jones vector projects the field onto the transmission axis. Malus's law follows immediately when the input is linearly polarized.

A quarter-wave plate is an element that introduces a relative phase shift of $\pi/2$ between two orthogonal principal axes. In a coordinate system aligned with those axes, an ideal quarter-wave plate can be represented as

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad (13.191)$$

up to an overall phase. Sending linearly polarized light at 45° to these axes gives equal components with a quarter-cycle phase difference and therefore circular polarization in the ideal case.

Example 13.32: Creating circular polarization with a quarter-wave plate

A linearly polarized input has Jones vector

$$\mathbf{J}_{\text{in}} = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (13.192)$$

It passes through the quarter-wave plate of Eq. (13.191). Find the output Jones vector and identify the polarization type.

Solution.

$$\mathbf{J}_{\text{out}} = Q \mathbf{J}_{\text{in}} = \frac{E_0}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}. \quad (13.193)$$

The transverse components have equal amplitudes and a relative phase of $\pi/2$, so the field tip traces a circle. The wave is circularly polarized; its handedness depends on the time-dependence and viewing convention.

13.26.2 Stokes parameters as measurable polarization descriptors

Jones calculus assumes a fully coherent polarized field. Real optical beams can be partially polarized or incoherent. Stokes parameters provide a more general intensity-based description. With one common convention,

$$S_0 = I_x + I_y, \quad (13.194)$$

$$S_1 = I_x - I_y, \quad (13.195)$$

$$S_2 = I_{45} - I_{135}, \quad (13.196)$$

$$S_3 = I_R - I_L. \quad (13.197)$$

Here the intensities are measured in selected linear and circular polarization bases. Define the R analyzer here as transmitting the normalized Jones vector $(1, i)^T/\sqrt{2}$ with $e^{-i\omega t}$ time dependence for propagation along $+z$; the L analyzer transmits $(1, -i)^T/\sqrt{2}$. This operational definition fixes the sign of S_3 without relying on a viewing convention. A fully polarized beam satisfies

$$S_1^2 + S_2^2 + S_3^2 = S_0^2, \quad (13.198)$$

while partially polarized light satisfies a smaller magnitude. The degree of polarization can be defined as

$$\mathcal{P} = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0}, \quad 0 \leq \mathcal{P} \leq 1. \quad (13.199)$$

This connects polarization physics directly to measurement and data analysis.

Note

The Stokes convention for the sign of S_3 and the naming of circular handedness varies among optics, astronomy, and radio-engineering communities. Always document the convention used by an instrument or software package before comparing data sets.

13.26.3 A retarder between crossed polarizers

A birefringent plate of thickness d produces relative phase $\Delta = 2\pi(n_y - n_x)d/\lambda_0$ between its principal transverse axes. Here the refractive indices are evaluated at the source frequency and absorption and reflection losses are neglected. With input polarization at 45° , the Jones vector after the plate is $E_0(1, e^{i\Delta})^T/\sqrt{2}$. Projection onto the crossed analyzer $(1, -1)^T/\sqrt{2}$ gives amplitude $E_0(1 - e^{i\Delta})/2$, hence

$$\frac{I_{\text{out}}}{I_{\text{in}}} = \sin^2(\Delta/2). \quad (13.200)$$

For $\Delta = 0$ the output is dark, for $\Delta = \pi/2$ it transmits half, and for $\Delta = \pi$ all the input intensity passes the analyzer in the ideal model. The plate changes phase, while the analyzer converts that change to an intensity signal. This is the basis of photoelastic stress imaging when stress changes the birefringence, and a useful idealization for electrically controlled polarization devices. Recovering a stress from a measured brightness also requires the material's stress-optic calibration and resolving the periodic ambiguity of the sine squared.

13.27 Dispersion, phase velocity, and group velocity

In a nondispersive vacuum plane wave, the relation $\omega = ck$ is linear and all Fourier components propagate at the same speed c . In a material, the refractive index generally depends on angular frequency: $n = n(\omega)$. The dispersion relation becomes

$$k(\omega) = \frac{n(\omega)\omega}{c}. \quad (13.201)$$

The phase velocity is

$$v_p = \frac{\omega}{k} = \frac{c}{n(\omega)}. \quad (13.202)$$

To see which speed controls an envelope, superpose two equal-amplitude waves with nearby wave numbers and frequencies. The cosine addition identity gives

$$\cos(k_1 z - \omega_1 t) + \cos(k_2 z - \omega_2 t) = 2 \cos\left(\frac{\Delta k z - \Delta \omega t}{2}\right) \cos(\bar{k} z - \bar{\omega} t),$$

where bars denote arithmetic means and deltas denote differences. The rapid carrier has speed $\bar{\omega}/\bar{k}$; the slowly varying beat envelope has speed $\Delta\omega/\Delta k$. In the narrowband limit, this ratio tends to the derivative below. A localized pulse contains many frequencies, but the same first-order expansion of $\omega(k)$ gives its envelope speed; higher-order terms distort the pulse. A narrowband wave packet is described by an envelope multiplying a carrier. The envelope propagates approximately at the group velocity

$$v_g = \frac{d\omega}{dk}. \quad (13.203)$$

Using $k = n(\omega)\omega/c$,

$$\frac{dk}{d\omega} = \frac{1}{c} \left(n + \omega \frac{dn}{d\omega} \right), \quad (13.204)$$

so

$$v_g = \frac{c}{n + \omega dn/d\omega}. \quad (13.205)$$

The quantity

$$n_g = n + \omega \frac{dn}{d\omega} \quad (13.206)$$

is called the group index in this simple treatment.

13.27.1 Pulse broadening

A real pulse contains a range of frequencies. If different spectral components have different group delays, the pulse shape changes as it propagates. This is one source of dispersion in optical communication. In fibers, both material dispersion and waveguide dispersion contribute. The practical consequence is that pulses can broaden and overlap, limiting the achievable data rate for a given distance unless dispersion is controlled or compensated.

Example 13.33: Group velocity from a simple index model

At a particular optical frequency, a material has $n = 1.50$ and $\omega dn/d\omega = 0.030$. Estimate its phase velocity and group velocity.

Solution.

$$v_p = \frac{c}{1.50} = 0.667c. \quad (13.207)$$

The group index is

$$n_g = 1.50 + 0.030 = 1.53, \quad (13.208)$$

so

$$v_g = \frac{c}{1.53} = 0.654c. \quad (13.209)$$

The two velocities are close but not identical. The difference is a direct signature of dispersion.

Caution

Phase velocity, group velocity, signal velocity, and energy velocity are not universally interchangeable. In weakly dispersive transparent media, group velocity often approximates pulse-envelope and energy transport well. In strongly dispersive or absorbing media, more careful analysis is required. Statements about causality must be made using the full signal response, not a single anomalous value of v_p or v_g .

Example 13.34: Fiber interface and pulse travel time

A simplified optical fiber has a core refractive index $n = 1.47$ and physical length $L = 2.00$ km. Ignore dispersion and assume the relevant group speed can be approximated by c/n . Estimate the one-way travel time. Then explain why the statement is an approximation.

Solution. The approximate propagation speed is

$$v \simeq \frac{c}{n}. \quad (13.210)$$

Hence

$$t = \frac{L}{v} \simeq \frac{nL}{c} = \frac{(1.47)(2.00 \times 10^3)}{2.998 \times 10^8} = 9.81 \mu\text{s}. \quad (13.211)$$

The approximation treats the refractive index as effectively constant and uses the simple phase-index relation for a pulse. In a dispersive waveguide, pulse delay is governed by group velocity and includes both material and waveguide dispersion. The estimate is nevertheless useful for order-of-magnitude timing.

13.28 Lossy media, attenuation, and skin depth

Assume a homogeneous isotropic Ohmic conductor with real positive σ, ε, μ , no imposed bulk free charge and $\mathbf{J} = \sigma \mathbf{E}$. The curl equations yield

$$\nabla^2 \mathbf{E} = \mu \varepsilon \partial_t^2 \mathbf{E} + \mu \sigma \partial_t \mathbf{E}. \quad (13.212)$$

For $\mathbf{E} \propto e^{i(\tilde{k}z - \omega t)}$ this becomes $\tilde{k}^2 = \mu \varepsilon \omega^2 + i \mu \sigma \omega$. When $\sigma \gg \omega \varepsilon$, displacement current is small compared with conduction current and the decaying branch is

$$\tilde{k} \simeq (1 + i) \sqrt{\frac{\mu \sigma \omega}{2}} = \frac{1 + i}{\delta}.$$

Thus the field has both attenuation $e^{-z/\delta}$ and phase accumulation $e^{iz/\delta}$. The electric and magnetic fields are generally out of phase in a conductor; $E = cB$ is not applicable. The skin depth is an *amplitude* e-folding length, so the corresponding power attenuation is $e^{-2z/\delta}$ for a single wave in this homogeneous model.

A real material may absorb electromagnetic energy. In a conductor, the electric field drives current and energy is converted into thermal energy. A monochromatic field penetrating a good conductor decays approximately exponentially with depth,

$$E(z) \propto e^{-z/\delta}, \quad (13.213)$$

where δ is the skin depth. For a good conductor satisfying $\sigma \gg \omega \varepsilon$,

$$\delta \simeq \sqrt{\frac{2}{\mu \sigma \omega}}. \quad (13.214)$$

Here σ is conductivity and μ is magnetic permeability. The corresponding current density is concentrated within several skin depths of the surface at high frequency.

This phenomenon explains why high-frequency conductor design differs from DC design. Increasing frequency reduces the effective cross-sectional area carrying current, increasing AC resistance and loss. Surface plating, conductor geometry, and material choice can therefore matter strongly in RF and microwave hardware.

Example 13.35: Order-of-magnitude skin depth in copper

Estimate the skin depth of copper at $f = 1.00$ MHz using $\sigma = 5.8 \times 10^7 \text{ S m}^{-1}$ and $\mu \simeq \mu_0$.

Solution. With $\omega = 2\pi f$,

$$\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}} \quad (13.215)$$

$$= \sqrt{\frac{2}{(4\pi \times 10^{-7})(5.8 \times 10^7)(2\pi \times 10^6)}} \quad (13.216)$$

$$\approx 6.6 \times 10^{-5} \text{ m} = 66 \mu\text{m}. \quad (13.217)$$

At 1 MHz, current is already concentrated within a thin surface region compared with the radius of many ordinary wires.

13.28.1 Complex refractive index

Optical attenuation is often represented by a complex refractive index

$$\tilde{n} = n + i\kappa \quad (13.218)$$

with sign convention tied to the chosen exponential time dependence. The real part controls phase accumulation and the imaginary part controls attenuation. If a field varies as $e^{i(kz - \omega t)}$ and the complex wave number is written $\tilde{k} = k + i\alpha_E$, then the field amplitude decays as $e^{-\alpha_E z}$. Intensity decays twice as fast in the exponent because it is proportional to amplitude squared.

A common engineering attenuation model is

$$I(z) = I_0 e^{-\alpha_I z}, \quad (13.219)$$

where α_I is an intensity attenuation coefficient. The corresponding loss in decibels over length z is

$$L_{\text{dB}} = 10 \log_{10} \frac{I_0}{I(z)}. \quad (13.220)$$

Decibel notation is developed more systematically in the link-budget section.

13.29 Antenna patterns, directivity, and the far-field viewpoint

The isotropic source used in inverse-square examples is a useful reference but not a realizable antenna pattern. Real antennas redistribute radiated power with direction. In the far field, a radiation pattern describes this angular dependence.

For an ideal short electric dipole aligned with the z axis, the far-field intensity has the qualitative form

$$I(r, \theta) \propto \frac{\sin^2 \theta}{r^2}, \quad (13.221)$$

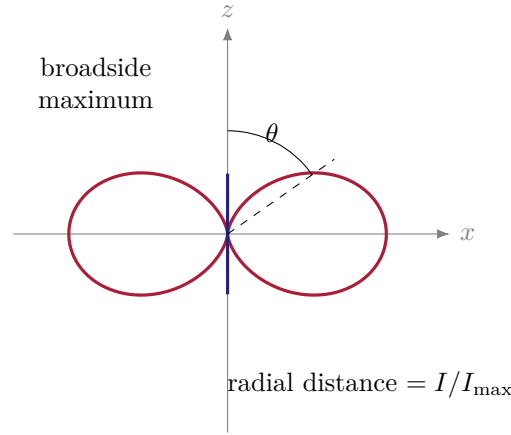


Figure 13.13: Normalized polar section $I/I_{\max} = \sin^2 \theta$ for a short dipole along z . The plotting angle is measured from $+z$, so both axis directions are nulls and the maxima lie along $\pm x$. The plotted radius represents relative power density, not electric-field amplitude.

where θ is measured from the dipole axis. Radiation is zero on the ideal axis ($\theta = 0, \pi$) and maximal in the equatorial plane ($\theta = 90^\circ$). Revolving this polar section around the dipole axis produces a toroidal radiation pattern in three dimensions.

The angle convention is essential in Fig. 13.13: measuring the polar angle from the horizontal axis without changing the sine to a cosine would rotate the lobes incorrectly by 90° .

13.29.1 Radiation intensity and directivity

At large distance r , define radiation intensity $U(\theta, \phi)$ as power per unit solid angle,

$$U(\theta, \phi) = r^2 I(r, \theta, \phi). \quad (13.222)$$

The total radiated power is

$$P_{\text{rad}} = \int U(\theta, \phi) d\Omega. \quad (13.223)$$

The average radiation intensity is $P_{\text{rad}}/(4\pi)$. Directivity is defined by

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{P_{\text{rad}}/(4\pi)}, \quad (13.224)$$

and the maximum directivity is

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}. \quad (13.225)$$

For a short dipole $U = U_{\max} \sin^2 \theta$ and $d\Omega = \sin \theta d\theta d\phi$. Therefore

$$P_{\text{rad}} = 2\pi U_{\max} \int_0^\pi \sin^3 \theta d\theta = \frac{8\pi}{3} U_{\max}, \quad D_0 = \frac{3}{2}.$$

This explicit normalization checks the power pattern and illustrates why increasing angular concentration need not increase total radiated power. The intensity I is power per area; $U(\theta, \phi)$ here is power per solid angle, distinct from the symbol U used for energy elsewhere. Directivity describes angular concentration, not efficiency. Antenna gain additionally accounts for losses. If radiation efficiency is η , then a simplified relation is

$$G = \eta D. \quad (13.226)$$

Example 13.36: Intensity in the main direction of a directive antenna

A transmitter delivers $P_t = 10 \text{ W}$ to an antenna whose gain in a given direction is $G = 6.0$ in linear units. Ignoring environmental reflections and absorption, estimate the far-field power density at $r = 100 \text{ m}$ in that direction.

Solution. Relative to an isotropic radiator, gain multiplies the power density:

$$I = \frac{P_t G}{4\pi r^2} = \frac{(10)(6.0)}{4\pi(100)^2} = 4.77 \times 10^{-4} \text{ W m}^{-2}. \quad (13.227)$$

The result is a far-field free-space estimate; actual received power also depends on receiver aperture, polarization, multipath, and losses.

13.29.2 Effective aperture of a receiving antenna

A receiving antenna intercepts only part of the incident wave. Its effective aperture A_e relates incident power density to available received power under matched polarization and impedance conditions:

$$P_r = I A_e. \quad (13.228)$$

For a reciprocal antenna in free space,

$$A_e = \frac{G_r \lambda^2}{4\pi}. \quad (13.229)$$

This relation reveals an important scale: at fixed gain, a lower-frequency antenna has a larger effective aperture because λ is larger.

13.30 Free-space link budgets and decibels

Combining transmitting gain, geometric spreading, and receiving effective aperture gives the Friis transmission equation for an ideal line-of-sight far-field link:

$$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi R} \right)^2. \quad (13.230)$$

Here P_t is the power accepted at the transmitting antenna terminals and G_t, G_r include radiation efficiency, so it must not be replaced by total radiated power while retaining the same gain factors. The assumptions are substantial: free-space propagation, matched polarization, aligned antennas, far-field conditions, impedance matching, and no additional atmospheric, cable, pointing, or multipath losses.

The factor

$$\left(\frac{4\pi R}{\lambda} \right)^2 \quad (13.231)$$

is called the free-space path-loss factor in linear power units. Its decibel form is

$$L_{\text{FS,dB}} = 20 \log_{10} \left(\frac{4\pi R}{\lambda} \right). \quad (13.232)$$

13.30.1 Why decibels are useful

For a power ratio P_2/P_1 , define

$$L_{\text{dB}} = 10 \log_{10} \left(\frac{P_2}{P_1} \right). \quad (13.233)$$

A factor of 10 in power corresponds to 10 dB; a factor of 2 corresponds to approximately 3.01 dB. When voltage or field amplitudes are compared under the same impedance, the square relation between power and amplitude gives

$$20 \log_{10} \left| \frac{E_2}{E_1} \right|. \quad (13.234)$$

Decibels turn multiplication of gains and losses into addition, which is convenient for engineering budgets.

Example 13.37: Idealized 2.4 GHz link

A transmitter supplies accepted input power $P_t = 100$ mW at $f = 2.40$ GHz. Both antennas have gain $G_t = G_r = 2.0$ in linear units. Estimate the ideal free-space received power at $R = 50.0$ m.

Solution. The wavelength is

$$\lambda = \frac{c}{f} = 0.125 \text{ m}. \quad (13.235)$$

Then

$$P_r = (0.100)(2.0)(2.0) \left(\frac{0.125}{4\pi(50.0)} \right)^2 \quad (13.236)$$

$$\approx 1.58 \times 10^{-8} \text{ W}. \quad (13.237)$$

This is about 15.8 nW. A real indoor wireless link may differ greatly because walls and multipath violate the simple free-space assumptions.

Example 13.38: Free-space path loss in decibels

For the same 2.40 GHz frequency and 50.0 m distance, estimate the free-space path loss.

Solution.

$$L_{\text{FS,dB}} = 20 \log_{10} \left(\frac{4\pi(50.0)}{0.125} \right) \approx 74.0 \text{ dB}. \quad (13.238)$$

The result is not “energy destroyed by empty space.” It represents geometric dilution between a transmitting radiation pattern and a finite receiving aperture.

Caution

A link budget expressed in dB is still a physical model. Adding numbers in dB does not excuse missing mechanisms. Cable loss, polarization mismatch, antenna misalignment, atmospheric attenuation, multipath fading, connector loss, and receiver noise must be included when they matter.

13.31 Optical-fiber acceptance angle and numerical aperture

The main chapter introduced total internal reflection at the core-cladding boundary. A second design question is: which external rays can enter the fiber and still become guided in the ray approximation?

Figure 13.14 distinguishes the entrance angle, measured from the fiber axis, from the incidence angle at the side wall.

Let the outside medium have refractive index n_0 , the core n_1 , and the cladding n_2 , with $n_1 > n_2$. Consider a meridional ray entering the flat fiber face at angle θ_a to the axis. Snell’s law at the

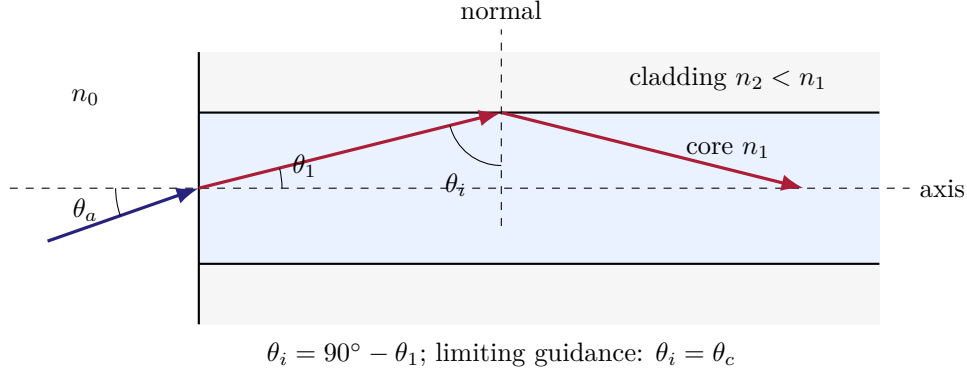


Figure 13.14: Meridional-ray geometry for a straight step-index fiber. The sketch exaggerates the angular separation for readability; the symbolic angle relations, not its dimensions, determine the acceptance condition. Reflection at the side wall preserves the angle to the fiber axis.

entrance gives

$$n_0 \sin \theta_a = n_1 \sin \theta_1. \quad (13.239)$$

At the core-cladding boundary, the incidence angle relative to the normal is $90^\circ - \theta_1$. At the limiting acceptance condition, this equals the critical angle θ_c , where

$$\sin \theta_c = \frac{n_2}{n_1}. \quad (13.240)$$

Therefore

$$\cos \theta_1 = \frac{n_2}{n_1}. \quad (13.241)$$

Using $\sin^2 \theta_1 + \cos^2 \theta_1 = 1$,

$$n_1^2 \sin^2 \theta_1 = n_1^2 - n_2^2. \quad (13.242)$$

Combining with Eq. (13.239) yields

$$n_0 \sin \theta_a = \sqrt{n_1^2 - n_2^2}. \quad (13.243)$$

The quantity

$$\text{NA} \equiv n_0 \sin \theta_a = \sqrt{n_1^2 - n_2^2} \quad (13.244)$$

is the numerical aperture for this simple step-index ray model. For coupling from air, $n_0 \simeq 1$, so $\text{NA} \simeq \sin \theta_a$. This expression for a limiting acceptance angle assumes $\sqrt{n_1^2 - n_2^2} \leq n_0$. If the right side exceeds n_0 , every externally accessible meridional angle satisfies the ideal TIR condition; one must cap the external half-angle at 90° , not take an inverse sine of a number greater than one. Fresnel and mode-overlap losses may still prevent efficient coupling.

Example 13.39: Numerical aperture of a step-index fiber

A fiber has $n_1 = 1.480$ and $n_2 = 1.460$ and is used in air. Estimate its numerical aperture and maximum acceptance half-angle.

Solution.

$$\text{NA} = \sqrt{1.480^2 - 1.460^2} = 0.242. \quad (13.245)$$

Hence

$$\theta_a = \sin^{-1}(0.242) = 14.0^\circ. \quad (13.246)$$

Only rays entering within approximately 14° of the axis are guided in this simple meridional-ray picture.

13.31.1 Ray model versus modal model

The numerical-aperture derivation is intuitive but incomplete for a real optical fiber. Maxwell's equations with cylindrical boundary conditions produce discrete guided modes. Whether a fiber is single-mode or multimode depends on wavelength, core radius, and index contrast. A useful dimensionless parameter is the normalized frequency

$$V = \frac{2\pi a}{\lambda_0} \sqrt{n_1^2 - n_2^2} = \frac{2\pi a}{\lambda_0} \text{NA}, \quad (13.247)$$

for a step-index fiber of core radius a in the standard weak-guidance context. A full modal derivation requires Bessel functions and is beyond this chapter, but the parameter shows how geometry and wavelength combine.

13.32 Waveguides and cutoff: waves constrained by conducting boundaries

A waveguide confines electromagnetic waves using conducting or dielectric boundaries. Unlike an infinite plane wave in vacuum, a guided field must satisfy boundary conditions on the guide walls. These conditions quantize the transverse spatial variation and lead to cutoff frequencies.

For a rectangular metallic waveguide of dimensions a along x and b along y , propagating along z , modes are labeled TE_{mn} or TM_{mn} . In a vacuum-filled ideal guide, the cutoff frequency is

$$f_{c,mn} = \frac{c}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}. \quad (13.248)$$

For a standard rectangular guide with $a > b$, TE denotes $E_z = 0$ and TM denotes $H_z = 0$. TE indices obey $m, n \geq 0$ but cannot both be zero; TM indices require $m, n \geq 1$. A hollow single-conductor guide has no TEM mode with both longitudinal components zero. For $a > b$, the dominant mode is TE_{10} , with

$$f_{c,10} = \frac{c}{2a}. \quad (13.249)$$

Propagation requires $f > f_c$. Below cutoff a single decaying mode in a semi-infinite ideal guide carries no net axial average power. A finite below-cutoff segment can nevertheless transmit through evanescent coupling; the absence of a freely propagating mode does not assert zero transmission through every finite structure.

The longitudinal propagation constant is

$$\beta = \sqrt{k^2 - k_c^2}, \quad (13.250)$$

where $k = \omega/c$ and k_c is set by the transverse boundary conditions. Therefore the phase velocity along the guide is

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - (f_c/f)^2}}, \quad (13.251)$$

which exceeds c . The group velocity is

$$v_g = c\sqrt{1 - (f_c/f)^2}, \quad (13.252)$$

and for the ideal guide

$$v_p v_g = c^2. \quad (13.253)$$

The superluminal phase velocity does not transmit information faster than light; the group velocity associated with a narrowband signal remains below c .

Example 13.40: Cutoff of a rectangular waveguide

An air-filled rectangular waveguide has broad dimension $a = 2.29$ cm. Estimate the TE_{10} cutoff frequency. Would a 10.0 GHz signal propagate in the ideal model?

Solution.

$$f_c = \frac{c}{2a} = \frac{2.998 \times 10^8}{2(2.29 \times 10^{-2})} = 6.55 \text{ GHz.} \quad (13.254)$$

Since $10.0 \text{ GHz} > 6.55 \text{ GHz}$, the TE_{10} mode can propagate.

Example 13.41: Phase and group velocity in a guide

For the guide above, evaluate v_p and v_g at $f = 10.0$ GHz.

Solution. The ratio is $f_c/f = 0.655$. Therefore

$$v_g = c\sqrt{1 - 0.655^2} = 0.756c, \quad (13.255)$$

and

$$v_p = \frac{c}{0.756} = 1.32c. \quad (13.256)$$

Their product is c^2 within rounding. The result illustrates why phase velocity alone must not be interpreted as a signal speed.

Guided checks

1. Why can a conducting waveguide have a minimum frequency for propagation even though free-space electromagnetic waves have no such cutoff?
2. What boundary-condition idea determines the transverse mode numbers m and n ?
3. Why does $v_p > c$ in an ideal waveguide not violate relativity?

13.33 Three-dimensional resonant cavities

A resonant cavity is a finite region in which boundary conditions allow only discrete standing electromagnetic modes. It can be regarded as a waveguide closed at both ends. Cavities are central to microwave oscillators, filters, particle accelerators, frequency standards, sensors, and many laboratory instruments.

For an ideal rectangular vacuum cavity with side lengths a , b , and d , a useful modal-frequency expression is

$$f_{mnp} = \frac{c}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}. \quad (13.257)$$

For TE modes relative to z ($E_z = 0$), $m, n \geq 0$ with $(m, n) \neq (0, 0)$ and $p \geq 1$. For TM modes relative to z ($H_z = 0$), $m, n \geq 1$ and $p \geq 0$. These restrictions follow from vanishing tangential electric field on every conducting wall. For example, TE_{101} and TM_{110} are allowed, whereas a putative TE_{100} cavity mode is not. The one-dimensional parallel-plane model ignores lateral walls and should not be substituted unchanged for a closed three-dimensional cavity.

13.33.1 Quality factor

A real cavity loses energy through finite wall resistance, dielectric absorption, radiation leakage, and coupling ports. The quality factor Q measures energy storage relative to loss:

$$Q = \omega_0 \frac{U}{P_{\text{loss}}}, \quad (13.258)$$

where U is the time-averaged stored energy at resonance and P_{loss} is average power dissipated. For a weakly damped isolated resonance, one also has approximately

$$Q \simeq \frac{f_0}{\Delta f}, \quad (13.259)$$

where Δf is the full width at half maximum of the power resonance under the usual conditions. A bandwidth measured through coupling ports normally yields the *loaded* quality factor Q_L , including coupling losses. If Q_0 describes internal dissipation and Q_{ext} describes energy escaping through the ports, $Q_L^{-1} = Q_0^{-1} + Q_{\text{ext}}^{-1}$. After the drive is removed, $dU/dt = -\omega_0 U/Q_L$, hence $U(t) = U(0)e^{-\omega_0 t/Q_L}$. The energy decay time is Q_L/ω_0 ; the field-amplitude decay time is twice as long. This distinction connects the bandwidth measurement to a time-domain ring-down test.

Example 13.42: Cavity quality factor from bandwidth

A microwave cavity resonates at $f_0 = 9.80$ GHz with measured half-power bandwidth $\Delta f = 4.0$ MHz. Estimate the loaded Q .

Solution.

$$Q \simeq \frac{9.80 \times 10^9}{4.0 \times 10^6} = 2.45 \times 10^3. \quad (13.260)$$

A larger Q would correspond to a narrower resonance and slower energy decay, all else equal.

13.34 Radiation pressure beyond normal incidence

For a beam striking a surface obliquely, both the projected intercepted power and the normal component of momentum transfer matter. Let the incidence angle θ be measured from the surface normal. A beam of intensity I crossing an actual surface area A intercepts power

$$P_{\text{inc}} = IA \cos \theta. \quad (13.261)$$

For perfect absorption, the normal incident momentum delivered per unit time is

$$F_n = \frac{P_{\text{inc}}}{c} \cos \theta = \frac{IA}{c} \cos^2 \theta. \quad (13.262)$$

Thus the normal pressure on the actual surface is

$$p_n = \frac{I}{c} \cos^2 \theta \quad (\text{ideal absorption}). \quad (13.263)$$

For ideal specular reflection, the normal momentum reverses, doubling the normal force:

$$p_n = \frac{2I}{c} \cos^2 \theta. \quad (13.264)$$

These formulas neglect absorption/reflection mixtures and assume a flat surface large enough to intercept the local beam. Absorption also transfers tangential momentum: $F_t = IA \cos \theta \sin \theta/c$. Its total force is along the incident beam. For specular reflection the tangential photon or wave momentum is unchanged, so the ideal reflecting force is purely normal. These are incident-radiation forces; recoil from subsequent thermal emission is a separate contribution.

Example 13.43: Tilted solar sail

A perfectly reflecting sail of area $A = 100 \text{ m}^2$ receives sunlight of intensity $I = 1.36 \times 10^3 \text{ W m}^{-2}$. The sail normal is tilted by 30° relative to the incident beam. Estimate the force normal to the sail.

Solution.

$$F_n = \frac{2IA}{c} \cos^2 30^\circ \quad (13.265)$$

$$= \frac{2(1.36 \times 10^3)(100)}{2.998 \times 10^8} (0.75) \quad (13.266)$$

$$= 6.80 \times 10^{-4} \text{ N}. \quad (13.267)$$

The force is below a millinewton but can act continuously without propellant.

13.35 One-dimensional FDTD: Maxwell equations on a computer grid

Finite-difference time-domain (FDTD) methods solve Maxwell's equations by replacing derivatives with finite differences on a spatial and temporal grid. The method is widely used because it evolves the fields directly in time and can handle pulses, complex geometries, and material boundaries. A full three-dimensional solver is a substantial software project, but the one-dimensional case reveals the essential structure.

Consider a wave depending only on x with components $E_y(x, t)$ and $H_z(x, t)$. In a homogeneous medium,

$$\frac{\partial E_y}{\partial x} = -\mu \frac{\partial H_z}{\partial t}, \quad (13.268)$$

$$\frac{\partial H_z}{\partial x} = -\varepsilon \frac{\partial E_y}{\partial t}. \quad (13.269)$$

A staggered Yee grid stores electric and magnetic samples offset in space and time. Let

$$E_i^n \equiv E_y(i\Delta x, n\Delta t), \quad (13.270)$$

and place H at half indices. A centered finite-difference update can be written schematically as

$$H_{i+1/2}^{n+1/2} = H_{i+1/2}^{n-1/2} - \frac{\Delta t}{\mu \Delta x} (E_{i+1}^n - E_i^n), \quad (13.271)$$

followed by

$$E_i^{n+1} = E_i^n - \frac{\Delta t}{\varepsilon \Delta x} (H_{i+1/2}^{n+1/2} - H_{i-1/2}^{n+1/2}). \quad (13.272)$$

The staggered arrangement mirrors the curl coupling: spatial differences of one field update the time evolution of the other.

13.35.1 Courant stability condition

An explicit FDTD scheme is stable only if the time step is sufficiently small. In one dimension for a homogeneous medium of wave speed $v = 1/\sqrt{\mu\varepsilon}$, the Courant condition is

$$S_c \equiv \frac{v\Delta t}{\Delta x} \leq 1. \quad (13.273)$$

The dimensionless number S_c is the Courant number. Choosing Δt too large causes unphysical numerical growth even when the physical Maxwell problem is stable.

13.35.2 Spatial resolution and numerical dispersion

Stability is not the same as accuracy. A grid with only a few points per wavelength badly represents sinusoidal fields and introduces numerical dispersion. In practical introductory simulations, one commonly chooses many grid cells per minimum wavelength of interest. Convergence should be checked by repeating the calculation with smaller Δx and Δt while respecting the Courant condition.

Method

A trustworthy FDTD study should include four validation tests: (1) a known wave speed in homogeneous space, (2) amplitude behavior under grid refinement, (3) reflection from a simple boundary with known expectation, and (4) an energy or Poynting-flux check where feasible. Numerical stability alone is not validation.

13.35.3 Numerical dispersion and a reproducible propagation test

Inserting staggered Fourier components with numerical frequency Ω and numerical wave number κ into the two updates gives

$$\sin^2\left(\frac{\Omega\Delta t}{2}\right) = S_c^2 \sin^2\left(\frac{\kappa\Delta x}{2}\right). \quad (13.274)$$

For small grid spacings $\sin z \simeq z$ recovers $\Omega = v\kappa$. For finite spacing, the numerical phase speed depends on wavelength. If $S_c > 1$, some spatial modes require $|\sin(\Omega\Delta t/2)| > 1$, so Ω cannot remain real: exponentially growing solutions appear. The endpoint $S_c = 1$ gives exact phase speed for resolved 1D homogeneous modes on the physical branch, but is not a general accuracy guarantee for interfaces or multidimensional grids.

The following complete NumPy experiment uses dimensionless coordinates $\xi = x/L_*$, $\tau = ct/L_*$ and fields $e = E_y/E_*$, $h = Z_0 H_z/E_*$. A periodic smooth Gaussian pulse starts at $\xi = 0.25$ and moves toward $+x$. The initial magnetic samples are evaluated at their actual staggered time $-\Delta\tau/2$ and position $\xi_{i+1/2}$, not copied from the electric array. Periodic boundaries model a repeating domain; this is a propagation test, not an open antenna.

```
import numpy as np
def run(N, S=0.9):
    dx = 1.0/N
    dt = S*dx
    x = np.arange(N)*dx
    def pulse(q):
        d = (q - 0.25 + 0.5) % 1.0 - 0.5
        return np.exp(-(d/0.04)**2)
    e = pulse(x)
    h = pulse(x + dx/2 + dt/2) # time -dt/2
    steps = round(0.30/dt)
    for n in range(steps):
        h -= S*(np.roll(e, -1) - e)
        e -= S*(h - np.roll(h, 1))
    tau = steps*dt
    exact = pulse(x - tau)
    error = np.sqrt(np.mean((e - exact)**2))
    return tau, error
for N in (200, 400, 800):
    print(N, run(N))
```

The electric-field error should fall approximately by a factor of four when N doubles at fixed Courant number: the centered scheme is second order for this smooth solution. Compare with the

analytic pulse at the *actual* final time, which changes slightly because the step count is an integer. Dividing an apparent position shift by a requested rather than actual elapsed time can disguise the convergence.

At the final step, e and h still have different time stamps. Before computing a local Poynting flux, interpolate the fields to a common position and time, or use an explicitly staggered discrete flux. A naive product of equal-index arrays is not the physical $E_y H_z$ at one spacetime point. For driven or inhomogeneous problems, specify material parameters, source placement and boundary conditions before changing the update. Fixed-zero tangential electric fields represent reflecting conducting ends; open-domain models require absorbing boundary treatment. The homogeneous Courant bound must use the largest wave speed in the domain. Absorbing layers and dispersive materials require their own stability analysis beyond the present scheme.

Example 13.44: Choosing an FDTD time step

A one-dimensional vacuum grid has $\Delta x = 1.00$ mm. What is the largest Courant-limited time step? Give a conservative choice using $S_c = 0.90$.

Solution. The stability limit is

$$\Delta t_{\max} = \frac{\Delta x}{c} = \frac{1.00 \times 10^{-3}}{2.998 \times 10^8} = 3.34 \times 10^{-12} \text{ s.} \quad (13.275)$$

For $S_c = 0.90$,

$$\Delta t = 0.90 \Delta t_{\max} = 3.00 \text{ ps.} \quad (13.276)$$

This small time step illustrates the computational cost of resolving high-frequency electromagnetic fields over large physical domains.

Guided checks

1. Why are electric and magnetic samples staggered in a Yee-grid update?
2. What failure occurs if $v\Delta t/\Delta x$ exceeds the one-dimensional stability limit?
3. Why can a stable simulation still be inaccurate?
4. What physical artifact is produced by simply setting the field to zero at the edge of a finite computational domain?

13.36 Measurement, calibration, and uncertainty

In practice, the electromagnetic field is rarely known directly. Instruments respond through calibrated transfer functions. A photodiode produces current; a microwave detector produces voltage; an antenna produces an RF signal; a power meter integrates energy over an aperture and bandwidth. The experimental task is therefore an inverse problem: infer the physical field quantity from the instrument response.

13.36.1 Calibration model

Suppose a detector output voltage V_d is approximately linear in incident power P over a specified range,

$$V_d = aP + b, \quad (13.277)$$

where a is sensitivity and b is an offset. Calibration measurements using known powers determine a and b . An unknown power is then inferred from

$$P = \frac{V_d - b}{a}. \quad (13.278)$$

If the active detector area is known and uniformly illuminated, intensity may be estimated by $I = P/A$.

The model should be checked for saturation, background offsets, dark current, frequency dependence, polarization response, and spatial nonuniformity. A single conversion factor is meaningful only inside the range and conditions for which it was calibrated.

13.36.2 Propagation of independent uncertainties

For a quantity $q = q(x_1, x_2, \dots)$ with small independent standard uncertainties, a common first-order estimate is

$$\sigma_q^2 \simeq \sum_i \left(\frac{\partial q}{\partial x_i} \right)^2 \sigma_{x_i}^2. \quad (13.279)$$

For $I = P/A$,

$$\left(\frac{\sigma_I}{I} \right)^2 \simeq \left(\frac{\sigma_P}{P} \right)^2 + \left(\frac{\sigma_A}{A} \right)^2 \quad (13.280)$$

when P and A uncertainties are independent. If $E_0 = \sqrt{2ZI}$ and uncertainty in Z is negligible,

$$\frac{\sigma_{E_0}}{E_0} \simeq \frac{1}{2} \frac{\sigma_I}{I}. \quad (13.281)$$

Example 13.45: Power-density uncertainty from detector calibration

A calibrated sensor measures $P = (2.00 \pm 0.04)$ mW over an effective area $A = (4.00 \pm 0.08)$ cm². Treat the uncertainties as independent standard uncertainties. Estimate I and its uncertainty.

Solution. Convert the area:

$$A = 4.00 \times 10^{-4} \text{ m}^2. \quad (13.282)$$

Then

$$I = \frac{2.00 \times 10^{-3}}{4.00 \times 10^{-4}} = 5.00 \text{ W m}^{-2}. \quad (13.283)$$

Both relative uncertainties are 2.0%. Therefore

$$\frac{\sigma_I}{I} = \sqrt{0.020^2 + 0.020^2} = 0.0283. \quad (13.284)$$

Thus

$$\sigma_I = (0.0283)(5.00) = 0.14 \text{ W m}^{-2}, \quad (13.285)$$

so

$$I = (5.00 \pm 0.14) \text{ W m}^{-2}. \quad (13.286)$$

13.36.3 Systematic effects

Statistical uncertainty is only part of measurement quality. Systematic effects can dominate: detector calibration drift, misalignment, polarization mismatch, reflections from nearby objects, finite beam size, imperfect knowledge of active area, or frequency-dependent response. Repeating a measurement many times does not automatically remove such biases. A professional report should therefore separate repeatability from calibration and modeling uncertainty whenever possible.

13.37 Advanced synthesis: one field theory, many models

The advanced topics can be organized by the hierarchy of models shown below.

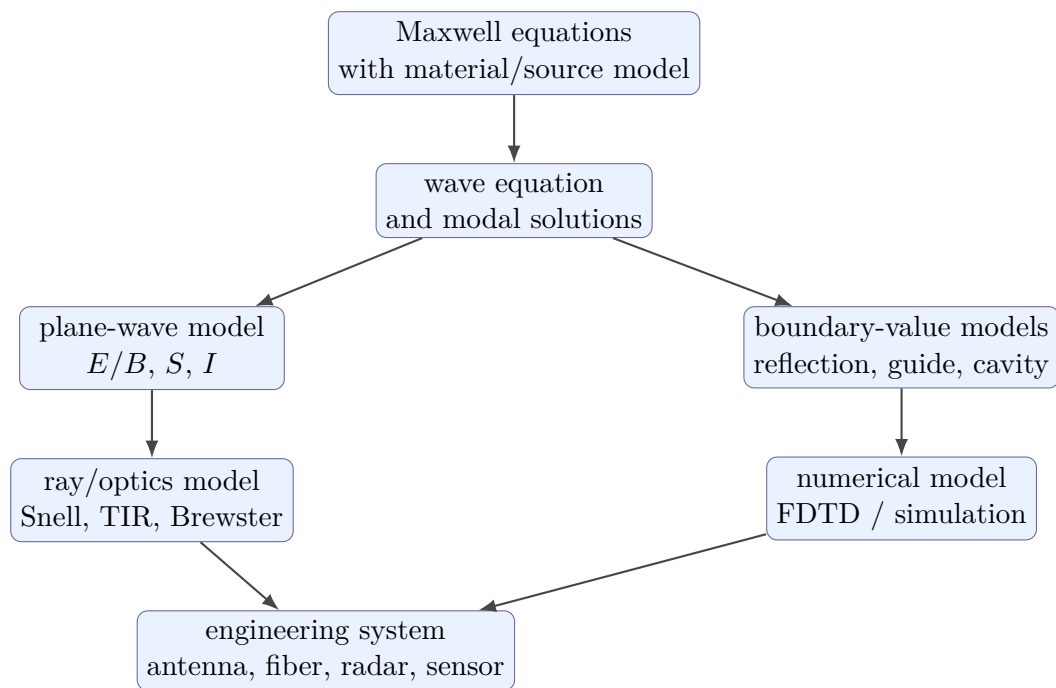


Figure 13.15: Hierarchy of models used in electromagnetic-wave analysis. Simpler models are powerful when their assumptions are satisfied; Maxwell’s equations provide the common foundation.

Figure 13.15 places these approximations in relation to one another. A useful calculation chooses one branch with explicit assumptions rather than combining incompatible formulas. The model hierarchy suggests a professional workflow:

1. identify the physical scale, frequency range, geometry, sources, and materials;
2. choose the simplest model whose assumptions match the problem;
3. derive or select the relevant equations rather than mixing formulas from incompatible regimes;
4. compute the desired observable and attach units;
5. validate by limits, conservation, and comparison with a simpler case;
6. refine the model only if the required accuracy demands it.

Summary

The enrichment section extended the chapter from elementary plane waves to vector wave equations, complex phasors, wave impedance, Fresnel reflection, general polarization states, dispersion, attenuation, antenna patterns, free-space links, optical-fiber acceptance, waveguide cutoff, cavity quality factor, oblique radiation pressure, FDTD simulation, and measurement uncertainty. These are not separate subjects: each is a controlled extension of Maxwell’s equations under particular source, material, boundary, and approximation choices.

13.38 Problems and selected solutions

Problems C, A1–A13, N1–N3, L and E1–E2/E7 use the core; A14–A16, N4–N6, E3–E6 and H use the enrichment. Each problem states a distinct task rather than a required sequence of homework assignments. They are grouped by style so that students can practise conceptual reasoning, analytical calculation, computation, measurement thinking, and engineering modeling. Unless otherwise stated,

use

$$c \simeq 2.998 \times 10^8 \text{ m s}^{-1}, \quad \mu_0 \simeq 4\pi \times 10^{-7} \text{ H m}^{-1}, \quad \varepsilon_0 \simeq 8.854 \times 10^{-12} \text{ F m}^{-1}.$$

State assumptions, keep SI units through the calculation, and distinguish peak from rms field amplitudes.

Conceptual questions

- C1.** A sinusoidal electromagnetic wave travels along $+z$. Its electric field points instantaneously along $+x$. Determine the magnetic-field direction and explain the right-hand-rule reasoning.
- C2.** Why does the numerical equality $1/\sqrt{\mu_0\varepsilon_0} = c$ suggest a unification of optics with electricity and magnetism?
- C3.** A student says that electromagnetic waves require a material medium because every wave must “shake something.” Give a field-based response.
- C4.** In a vacuum plane wave, why are the electric and magnetic energy densities equal at each instant?
- C5.** A sinusoidal wave has zero electric and magnetic field at a particular instant and point. Does that imply that its time-averaged intensity is zero? Explain.
- C6.** A beam is spread uniformly over twice the area with the same total power. What happens to intensity and field amplitude?
- C7.** Two ideal polarizers are crossed. Why can inserting a third polarizer between them increase the transmitted intensity?
- C8.** When light enters a higher-index transparent medium, which changes: frequency, wavelength, phase speed? Explain.
- C9.** At Brewster incidence, which reflected polarization component vanishes in the ideal dielectric model?
- C10.** Why does an ideal standing wave not have a single constant direction of net energy transport like a traveling plane wave?

Analytical problems

- A1.** A plane wave is described by $E_y = 250 \cos(12.0x - 3.60 \times 10^9 t) \text{ V m}^{-1}$. Determine its propagation direction, phase speed, frequency, wavelength, and corresponding B_z expression if the wave is in vacuum.
- A2.** A wave propagates along $-y$ with electric field polarized along $+z$. Write a consistent sinusoidal magnetic field and verify the propagation direction with $\mathbf{E} \times \mathbf{B}$.
- A3.** A vacuum wave has peak electric field $E_0 = 420 \text{ V m}^{-1}$. Find B_0 , E_{rms} , average intensity, and average total energy density.
- A4.** An isotropic source radiates $P = 60 \text{ W}$. Find the intensity at $r = 2.0 \text{ m}$ and $r = 8.0 \text{ m}$ and compare the field amplitudes at the two positions.
- A5.** A circular detector of radius 2.50 cm is perpendicular to a beam of intensity 300 W m^{-2} . Find the intercepted power. Repeat if the detector normal is tilted by 45° .
- A6.** A 10.0 W beam is fully absorbed. Find the radiation force. Repeat for ideal retroreflection.

- A7.** Unpolarized light of intensity I_0 passes through ideal polarizers at 0° , 30° , and 90° . Express the final intensity as a fraction of I_0 .
- A8.** Three polarizers have axes at 0° , α , and 90° . Derive the value of α that maximizes transmission of initially unpolarized light and calculate the maximum fraction.
- A9.** A beam travels from glass $n = 1.52$ to air with incidence angle 30.0° . Determine whether a refracted ray exists and find its angle if it does.
- A10.** Find the critical angle for a core-cladding interface with $n_1 = 1.50$ and $n_2 = 1.47$.
- A11.** Find the Brewster angle for incidence from air onto water with $n = 1.33$. Find the refracted angle and verify that the reflected and refracted rays are perpendicular.
- A12.** A 633 nm laser enters glass of index 1.62. Find its frequency, speed, and wavelength in glass.
- A13.** A one-dimensional vacuum cavity of length $L = 8.0$ cm has conducting ends. Calculate the first four resonance frequencies using the simple node model.
- A14.** At normal incidence, derive the reflected power fraction R for an air-dielectric boundary with refractive index n . Evaluate it for $n = 2.00$.
- A15.** A material has $n = 1.60$ and $\omega dn/d\omega = 0.080$ at a given frequency. Find phase and group velocities.
- A16.** A rectangular air-filled waveguide has $a = 4.0$ cm. Find the TE_{10} cutoff frequency and the group velocity at a frequency $1.50f_c$.

Numerical and computational problems

- N1.** Write a plane-wave visualizer for both directions along x . Test $\mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E}/c$ and both one-dimensional Maxwell equations, rather than dividing by a field that vanishes at nodes. Explain why reversing the phase direction alone fails.
- N2.** Construct a program that accepts a list of ideal polarizer angles and returns the transmitted intensity for initially unpolarized light. Validate it using crossed polarizers and the 0° - 45° - 90° case.
- N3.** Write a ray-tracing routine for a stack of parallel dielectric layers. The code should apply Snell's law and report total internal reflection when no propagating refracted ray exists.
- N4.** Implement the one-dimensional FDTD updates in Eqs. (13.271) and (13.272). Launch a Gaussian pulse and measure its propagation speed.
- N5.** In an FDTD calculation, repeat the same physical problem with 20, 40, and 80 grid cells per wavelength. Quantify the convergence of measured phase speed.
- N6.** Write a program that computes R and T at normal incidence from refractive indices. Verify numerically that $R + T = 1$ over $1 \leq n_1, n_2 \leq 3$ for lossless media.

Laboratory and data-analysis problems

- L1.** An illustrative microwave-polarization data set contains normalized powers 1.00, 0.75, 0.49, 0.24, 0.03 at analyzer angles 0° , 30° , 45° , 60° , 80° . Compute residuals from $\cos^2 \theta$. If each normalized power has independent standard uncertainty 0.02, is any point a significant outlier? Explain why one should not presume an outlier exists.

- L2.** An illustrative standing-wave data set gives successive minima at 1.4, 7.6, 13.7, 19.9, 26.0 cm. Estimate wavelength and frequency. Explain how a linear fit improves the result.
- L3.** A detector calibration gives $V_d = (2.00 \text{ V/W})P + 0.015 \text{ V}$. An unknown beam gives $V_d = 0.215 \text{ V}$. Find the optical power and state one condition under which the calibration would be invalid.
- L4.** In a Snell-law experiment, the fitted slope of $\sin \theta_1$ versus $\sin \theta_2$ is 1.487 ± 0.012 . Interpret the result and state the role of the intercept as a diagnostic.
- L5.** A power sensor reads $P = (5.00 \pm 0.10) \text{ mW}$ through an aperture of area $(1.00 \pm 0.02) \text{ cm}^2$. Estimate intensity and its uncertainty using independent-error propagation.

Engineering and applied problems

- E1.** A radar receives an echo $80 \mu\text{s}$ after transmission. Estimate target range. If timing uncertainty is $0.2 \mu\text{s}$, estimate range uncertainty.
- E2.** A solar panel of area 2.0 m^2 receives 900 W m^{-2} at 40° to its normal. If conversion efficiency is 22%, estimate electrical output power under this simplified model.
- E3.** A fiber has $n_1 = 1.465$ and $n_2 = 1.455$. Find its numerical aperture and acceptance half-angle in air. Discuss why connector misalignment matters.
- E4.** Use the Friis equation to estimate received power for $P_t = 1.0 \text{ W}$, $G_t = 4$, $G_r = 2$, $f = 900 \text{ MHz}$, and $R = 1.0 \text{ km}$. State at least four real-world losses omitted.
- E5.** A receiving antenna is rotated 35° relative to the incident linear polarization. Assuming ideal polarization projection, estimate the polarization-mismatch power factor and express it in dB.
- E6.** A sensor must infer plane-wave E_0 from measured intensity over the range 0.1 to 1000 W m^{-2} . Determine the corresponding E_0 range in vacuum and discuss detector dynamic range.
- E7.** An anti-glare optical system is intended to suppress horizontally polarized reflected light. Explain how the polarizer axis should be oriented and why a mistaken 90° orientation would defeat the design.

Challenge problems

- H1.** For a monochromatic plane wave, derive $\mathbf{B}_0 = (1/\omega)\mathbf{k} \times \mathbf{E}_0$ and then derive the dispersion relation $k = \omega/c$ using the Ampère-Maxwell equation.
- H2.** Derive the normal-incidence Fresnel coefficients from tangential E and H continuity for arbitrary real impedances Z_1 and Z_2 . Show energy conservation for lossless media.
- H3.** Starting with $E_x = E_{0x} \cos \omega t$ and $E_y = E_{0y} \cos(\omega t - \delta)$, eliminate time to obtain the polarization-ellipse equation.
- H4.** From the TE_{10} dispersion relation $\beta^2 = (\omega/c)^2 - (\pi/a)^2$, derive both phase and group velocities and show $v_p v_g = c^2$.
- H5.** Analyze a 0° - α - 90° three-polarizer system with Jones matrices rather than scalar Malus-law steps. Show that both methods agree for the final intensity.
- H6.** For an ideal one-dimensional FDTD scheme, insert trial solutions $E_i^n = E_0 e^{i(i\kappa\Delta x - n\Omega\Delta t)}$ and derive the numerical dispersion relation. Discuss how it approaches the continuum relation as $\Delta x, \Delta t \rightarrow 0$.

Brief answers to conceptual problems

C1–C3

$\mathbf{B}$ points along $+y$ because $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$. The Maxwell speed matches the independently observed light speed. Electric and magnetic fields are the quantities that vary; a material displacement is not required.

C4–C6

Substitution of $B = E/c$ gives $u_B = u_E$ for this traveling wave. A zero at one instant does not make the cycle average zero. At fixed power, doubling illuminated area halves intensity and reduces amplitude by $1/\sqrt{2}$.

C7–C8

The middle polarizer projects the field onto a new direction before the final analyzer. Frequency remains fixed at a stationary interface while wavelength and phase speed change.

C9–C10

The reflected p component vanishes at Brewster incidence in the stated dielectric model. A standing wave contains opposite traveling energy fluxes of equal mean magnitude, leaving zero net average flux but nonzero energy density.

Selected solutions

The following solutions illustrate the expected level of reasoning. They are not intended to replace independent work on the full problem set.

Example 13.46: Selected solution A1: reading a wave equation

For

$$E_y = 250 \cos(12.0x - 3.60 \times 10^9 t) \text{ V m}^{-1},$$

identify the wave quantities.

Solution. Compare with $E_0 \cos(kx - \omega t)$. Thus

$$E_0 = 250 \text{ V m}^{-1}, \quad k = 12.0 \text{ rad m}^{-1}, \quad \omega = 3.60 \times 10^9 \text{ rad s}^{-1}. \quad (13.287)$$

The minus sign indicates propagation along $+x$. The phase speed is

$$v = \frac{\omega}{k} = 3.00 \times 10^8 \text{ m s}^{-1}, \quad (13.288)$$

consistent with vacuum. The frequency and wavelength are

$$f = \frac{\omega}{2\pi} = 5.73 \times 10^8 \text{ Hz}, \quad \lambda = \frac{2\pi}{k} = 0.524 \text{ m}. \quad (13.289)$$

If $\mathbf{E}$ is along $+y$ and propagation is $+x$, then $\mathbf{B}$ must be along $+z$, with

$$B_0 = \frac{250}{c} = 8.34 \times 10^{-7} \text{ T}. \quad (13.290)$$

Example 13.47: Selected solution A4: inverse-square scaling

An isotropic source emits 60 W. Compare $r = 2 \text{ m}$ and $r = 8 \text{ m}$.

Solution.

$$I_2 = \frac{60}{4\pi(2)^2} = 1.19 \text{ W m}^{-2}, \quad (13.291)$$

while

$$I_8 = \frac{60}{4\pi(8)^2} = 7.46 \times 10^{-2} \text{ W m}^{-2}. \quad (13.292)$$

The distance increases by a factor of four, so intensity decreases by $4^2 = 16$, consistent with the numbers. Since $E_0 \propto \sqrt{I}$, the field amplitude decreases by a factor of four.

Example 13.48: Selected solution A8: optimum middle polarizer

Find the optimum α for axes 0° , α , 90° and initially unpolarized light.

Solution.

$$I_f = \frac{I_0}{2} \cos^2 \alpha \cos^2(90^\circ - \alpha) = \frac{I_0}{2} \cos^2 \alpha \sin^2 \alpha. \quad (13.293)$$

Use $\sin 2\alpha = 2 \sin \alpha \cos \alpha$:

$$I_f = \frac{I_0}{8} \sin^2 2\alpha. \quad (13.294)$$

The maximum is obtained when $|\sin 2\alpha| = 1$, giving $\alpha = 45^\circ$ in the interval 0 to 90° . The maximum transmitted fraction is

$$\frac{I_f}{I_0} = \frac{1}{8}. \quad (13.295)$$

Example 13.49: Selected solution A9: refraction versus TIR

Light travels from $n_1 = 1.52$ to air at 30° .

Solution. First find the critical angle:

$$\theta_c = \sin^{-1} \left(\frac{1.00}{1.52} \right) = 41.1^\circ. \quad (13.296)$$

Since $30^\circ < 41.1^\circ$, refraction occurs. Snell's law gives

$$\sin \theta_2 = 1.52 \sin 30^\circ = 0.760, \quad (13.297)$$

so

$$\theta_2 = 49.5^\circ. \quad (13.298)$$

The ray bends away from the normal, as expected when entering the lower-index medium.

Example 13.50: Selected solution A14: normal-incidence reflectance

For incidence from air onto a dielectric of index n , derive R .

Solution. At normal incidence for nonmagnetic lossless media,

$$r = \frac{1 - n}{1 + n}. \quad (13.299)$$

Therefore

$$R = |r|^2 = \left(\frac{n - 1}{n + 1} \right)^2. \quad (13.300)$$

For $n = 2.00$,

$$R = \left(\frac{1}{3}\right)^2 = 0.111. \quad (13.301)$$

Approximately 11.1% of the incident power is reflected at the single interface.

Example 13.51: Selected solution A16: waveguide cutoff and group velocity

An air-filled rectangular guide has $a = 4.0$ cm. Find TE_{10} cutoff and v_g at $1.50f_c$.

Solution.

$$f_c = \frac{c}{2a} = \frac{2.998 \times 10^8}{0.080} = 3.75 \text{ GHz}. \quad (13.302)$$

At $f = 1.50f_c$,

$$v_g = c\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = c\sqrt{1 - \left(\frac{2}{3}\right)^2} = 0.745c. \quad (13.303)$$

The signal-envelope propagation is therefore substantially slower than c near cutoff.

Example 13.52: Selected solution L5: uncertainty of measured intensity

A sensor measures $P = (5.00 \pm 0.10)$ mW through area $A = (1.00 \pm 0.02)$ cm².

Solution. Convert $A = 1.00 \times 10^{-4}$ m² and $P = 5.00 \times 10^{-3}$ W. Then

$$I = \frac{P}{A} = 50.0 \text{ W m}^{-2}. \quad (13.304)$$

Each relative uncertainty is 2%, so

$$\frac{\sigma_I}{I} = \sqrt{0.02^2 + 0.02^2} = 0.0283. \quad (13.305)$$

Thus

$$I = (50.0 \pm 1.4) \text{ W m}^{-2}. \quad (13.306)$$

Example 13.53: Selected solution E1: radar range uncertainty

An echo delay is (80.0 ± 0.2) μs .

Solution.

$$R = \frac{c\Delta t}{2} = \frac{(2.998 \times 10^8)(80.0 \times 10^{-6})}{2} = 12.0 \text{ km}. \quad (13.307)$$

Because R is proportional to Δt ,

$$\delta R = \frac{c \delta t}{2} = 30 \text{ m}. \quad (13.308)$$

Hence the timing contribution is approximately $R = (12.0 \text{ km}) \pm 30 \text{ m}$, before systematic range biases are considered.

Example 13.54: Selected solution E3: fiber numerical aperture

For $n_1 = 1.465$ and $n_2 = 1.455$ in air,

$$\text{NA} = \sqrt{n_1^2 - n_2^2} = \sqrt{1.465^2 - 1.455^2} = 0.171. \quad (13.309)$$

The acceptance half-angle is

$$\theta_a = \sin^{-1}(0.171) = 9.84^\circ. \quad (13.310)$$

A connector that launches light outside this angular range couples poorly into guided rays in the simple model. Real coupling additionally depends on mode-field overlap and alignment of the optical axes.

Example 13.55: Selected solution E5: polarization mismatch in dB

For a 35° polarization mismatch,

$$\eta_{\text{pol}} = \cos^2 35^\circ = 0.671. \quad (13.311)$$

The corresponding power loss relative to perfect alignment is

$$10 \log_{10}(0.671) = -1.73 \text{ dB}. \quad (13.312)$$

Thus ideal polarization mismatch alone reduces received power by about 1.7 dB.

Example 13.56: Selected solution H1: plane-wave dispersion from Maxwell equations

Start with

$$\mathbf{B}_0 = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}_0. \quad (13.313)$$

For a source-free monochromatic wave, the Ampère-Maxwell equation gives

$$i\mathbf{k} \times \mathbf{B}_0 = -i\omega\mu_0\varepsilon_0\mathbf{E}_0. \quad (13.314)$$

Substitute the first relation:

$$\frac{1}{\omega} \mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) = -\omega\mu_0\varepsilon_0\mathbf{E}_0. \quad (13.315)$$

Using

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) = \mathbf{k}(\mathbf{k} \cdot \mathbf{E}_0) - k^2\mathbf{E}_0 = -k^2\mathbf{E}_0 \quad (13.316)$$

for a transverse wave, we obtain

$$\frac{k^2}{\omega} = \omega\mu_0\varepsilon_0. \quad (13.317)$$

Hence

$$\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c. \quad (13.318)$$

This derivation simultaneously uses the curl equations and transversality.

Example 13.57: Selected solution H2: Fresnel coefficients from impedances

At normal incidence, tangential-field continuity gives

$$E_i + E_r = E_t \quad (13.319)$$

and

$$\frac{E_i - E_r}{Z_1} = \frac{E_t}{Z_2}. \quad (13.320)$$

Set $r = E_r/E_i$ and $t = E_t/E_i$. Then

$$1 + r = t, \quad \frac{1 - r}{Z_1} = \frac{t}{Z_2}. \quad (13.321)$$

Eliminating t gives

$$r = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad t = \frac{2Z_2}{Z_1 + Z_2}. \quad (13.322)$$

For real lossless impedances, the incident, reflected, and transmitted time-averaged power densities are proportional to E^2/Z . Therefore

$$R = r^2, \quad T = \frac{Z_1}{Z_2} t^2. \quad (13.323)$$

Substitution yields $R + T = 1$, confirming conservation of energy.

Example 13.58: Selected solution H6: numerical dispersion

Use trial fields $E_i^n = E_a e^{i(\kappa i \Delta x - \Omega n \Delta t)}$ and $H_{i+1/2}^{n+1/2} = H_a e^{i[\kappa(i+1/2)\Delta x - \Omega(n+1/2)\Delta t]}$. Subtracting exponential factors at neighboring samples gives

$$\begin{aligned} \sin(\Omega \Delta t / 2) H_a &= \frac{\Delta t}{\mu \Delta x} \sin(\kappa \Delta x / 2) E_a, \\ \sin(\Omega \Delta t / 2) E_a &= \frac{\Delta t}{\varepsilon \Delta x} \sin(\kappa \Delta x / 2) H_a. \end{aligned}$$

Multiplication and cancellation yield $\sin^2(\Omega \Delta t / 2) = S_c^2 \sin^2(\kappa \Delta x / 2)$. Both sides are dimensionless. Using $\sin z \simeq z$ gives $\Omega^2 = v^2 \kappa^2$ as the grid is refined. For $S_c < 1$ on the physical branch and $q = \kappa \Delta x \ll 1$,

$$\frac{v_{\text{ph,num}}}{v} = \frac{2}{S_c q} \sin^{-1}[S_c \sin(q/2)] = 1 - \frac{1 - S_c^2}{24} q^2 + O(q^4).$$

The leading phase error is second order and negative. At $S_c > 1$, short spatial wavelengths admit complex Ω , explaining numerical growth without any physical source of energy.

Bridge to geometrical and wave optics

Geometrical optics uses rays when wavelength is small compared with device dimensions. Wave optics restores phase to explain interference and diffraction. Both retain the boundary conditions, polarization and energy-flow principles developed here.

13.39 Notation dictionary for electromagnetic waves

Symbol	Meaning	SI unit
$\mathbf{E}$	Electric field vector	V m^{-1}
$\mathbf{B}$	Magnetic flux-density vector	T
$\mathbf{H}$	Magnetic field strength	A m^{-1}
E_0, B_0	Peak field amplitudes for a sinusoidal wave	$\text{V m}^{-1}, \text{T}$
$E_{\text{rms}}, B_{\text{rms}}$	Root-mean-square field amplitudes	$\text{V m}^{-1}, \text{T}$

f	Frequency	Hz
T	Period	s
ω	Angular frequency, $2\pi f$	rad s^{-1}
λ	Wavelength	m
k	Wave number, $2\pi/\lambda$	rad m^{-1}
$\mathbf{k}$	Wave vector, direction of phase propagation	rad m^{-1}
c	Speed of light in vacuum	m s^{-1}
v_p	Phase velocity	m s^{-1}
v_g	Group velocity	m s^{-1}
n	Refractive index	dimensionless
Z	Wave impedance	Ω
Z_0	Vacuum wave impedance	Ω
u_E, u_B	Electric and magnetic energy densities	J m^{-3}
$\mathbf{S}$	Poynting vector	W m^{-2}
I	Time-averaged intensity	W m^{-2}
p_{rad}	Radiation pressure	Pa
P	Power	W
R, T	Power reflectance and transmittance	dimensionless
r, t	Field-amplitude reflection/transmission coefficients	dimensionless
θ_c	Critical angle for TIR	rad or degree
θ_B	Brewster angle	rad or degree
Q	Resonator quality factor	dimensionless
σ	Electrical conductivity in the advanced section	S m^{-1}
δ	Skin depth	m
A_e	Effective receiving aperture	m^2
G, D	Antenna gain and directivity	dimensionless
NA	Optical-fiber numerical aperture	dimensionless

Note

The symbol T is used contextually for wave period and, in the Fresnel section, for power transmittance. The distinction is normally clear from the surrounding equations. The symbol δ denotes relative polarization phase or skin depth in separate contexts; U denotes energy or antenna power per solid angle. In software or data tables, use less ambiguous names such as T_{period} and $\mathcal{T}$ if both appear together.

13.40 Validity conditions and model limits

This appendix collects the assumptions attached to the most frequently used formulas in the chapter. It is intended as a quick reference when solving problems or checking simulation inputs. A formula should be treated as a compact model statement: its validity conditions are part of the formula even when they are not written on the same line.

Relation	Main assumptions	Do not use it blindly when
$E_0 = cB_0$	traveling plane wave in vacuum; transverse fields	near an antenna, in a standing wave, in a material without using the appropriate phase-speed/impedance relations, or for unrelated static fields

Relation	Main assumptions	Do not use it blindly when
$\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$	electromagnetic fields in vacuum	a macroscopic material requires a constitutive description; energy-flow bookkeeping can require care in dispersive or lossy media
$I = E_0^2/(2Z_0)$	sinusoidal vacuum plane wave; E_0 is peak amplitude	the field is rms, nonsinusoidal, strongly modulated, standing, evanescent, or not locally plane
$I = P/(4\pi r^2)$	isotropic source or isotropic-equivalent model; far field; no absorption	directive antennas, guided modes, near fields, cavities, obstacles, or strong environmental losses dominate
$p_{\text{rad}} = I/c$	perfect absorption; normal incidence	the surface reflects, transmits, is oblique, or the momentum direction changes in a more complicated way
$p_{\text{rad}} = 2I/c$	ideal retroreflection; normal incidence	reflectivity is below unity or reflection is not directly back along the incident direction
$I = I_0 \cos^2 \theta$	incident light is already linearly polarized; ideal linear analyzer	the incident light is unpolarized, partially polarized, depolarized, or the polarizer has finite extinction and loss
$I_1 = I_0/2$	unpolarized incident light; ideal first linear polarizer	the input already has a preferred polarization or the polarizer has nonideal transmission
$n_1 \sin \theta_1 = n_2 \sin \theta_2$	stationary uniform planar interface; homogeneous isotropic media; propagating plane-wave directions	anisotropy, spatially varying interfaces or finite-aperture diffraction require a fuller description
$\theta_c = \sin^{-1}(n_2/n_1)$	$n_1 > n_2$; ideal interface	propagation is from lower to higher index, or absorption/anisotropy changes the simple ray interpretation
$\tan \theta_B = n_2/n_1$	transparent nonmagnetic dielectric interface; ideal Fresnel model	strong absorption or magnetic response makes the simple expression inadequate
$R = ((n_1 - n_2)/(n_1 + n_2))^2$	normal incidence; lossless nonmagnetic media	incidence is oblique, media are absorbing, or thin-film interference between multiple interfaces matters
$\text{NA} = \sqrt{n_1^2 - n_2^2}$	step-index fiber; meridional-ray, weakly guiding introductory picture; external air when interpreted as $\sin \theta_a$	modal coupling, waveguide dispersion, bending loss, or non-step-index profiles are important

Relation	Main assumptions	Do not use it blindly when
$f_{c,10} = c/(2a)$	ideal air/vacuum-filled rectangular metallic waveguide; TE ₁₀	dielectric loading, finite conductivity, nonrectangular geometry, or higher-mode interactions matter
$Q \simeq f_0/\Delta f$	isolated weakly damped resonance; standard half-power bandwidth definition	overlapping resonances, strong coupling, nonlinear response, or non-Lorentzian line shapes occur
$\delta \simeq \sqrt{2/(\mu\sigma\omega)}$	good conductor, $\sigma \gg \omega\epsilon$	conductivity is low, permeability is strongly frequency dependent, or microscopic/nonlocal material effects matter
Friis equation	far-field, line-of-sight free space; matched polarization and impedance; known antenna gains	multipath, obstruction, atmospheric loss, near-field coupling, or strong scattering dominate
FDTD Courant condition $v\Delta t/\Delta x \leq 1$	standard explicit one-dimensional Yee-type update	multidimensional grids have stricter geometry-dependent limits; stability still does not guarantee accuracy

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