

Chapter 14

Geometrical Optics and Image Formation

First Edition

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science

AGH University of Krakow

Physics 2

Physics for Technical and Engineering Students -
Volume II

Electricity, Magnetism, Waves, Optics, Relativity,
Quantum Physics, and Nuclear Physics

1 September 2026

Contents

Preface	iv
Conventions and notation	v
14 Geometrical Optics and Image Formation	1
14.1 Motivation, prerequisites and roadmap	1
14.2 From electromagnetic waves to light rays	2
14.2.1 What a ray represents	2
14.2.2 Fermat's principle as a unifying idea	3
14.2.3 Reflection, refraction and total internal reflection: a focused review	3
14.3 The language of image formation	4
14.3.1 Object points and image points	4
14.3.2 Why a virtual image is not imaginary	4
14.3.3 Paraxial rays and the small-angle approximation	5
14.4 Plane mirrors: the simplest imaging system	5
14.4.1 Lateral inversion	6
14.5 Spherical mirrors	7
14.5.1 Geometry, principal axis and focal point	7
14.5.2 Deriving the mirror equation from similar triangles	7
14.5.3 Principal rays for mirrors	8
14.5.4 Image classes for a concave mirror	8
14.5.5 Convex mirrors and wide fields of view	9
14.6 Refraction at plane and spherical surfaces	10
14.6.1 Apparent depth at a plane interface	10
14.6.2 A single spherical refracting surface	10
14.6.3 Paraxial lateral magnification at a refracting surface	12
14.6.4 Surface power and a reversed index step	12
14.7 Thin lenses: converging and diverging elements	12
14.7.1 What makes a lens thin?	12
14.7.2 The thin-lens equation	13
14.7.3 Lens-maker formula	13
14.7.4 Principal rays for thin lenses	14
14.7.5 Why two surfaces give one lens equation	15
14.8 Ray diagrams and image classification for thin lenses	16
14.8.1 Converging lens: object outside the focal length	16
14.8.2 Converging lens: object inside the focal length	16
14.8.3 Diverging lenses	17
14.8.4 A graphical way to understand the thin-lens equation	17
14.9 Multiple lenses and sequential imaging	18
14.9.1 Thin lenses in contact	20
14.10 Optical power and the engineering language of diopters	20
14.11 Pinhole imaging and the transition to lens imaging	21
14.11.1 Why use a lens?	21
14.12 Cameras, projectors and digital imaging sensors	22
14.12.1 A camera as a real-image system	22
14.12.2 Field of view and sensor size	22

14.12.3	Aperture and f-number	23
14.12.4	Projectors: the camera problem in reverse	23
14.12.5	Focus, zoom and perspective	24
14.12.6	Why close-up imaging needs more exposure	25
14.13	The human eye and corrective optics	25
14.13.1	Accommodation, near point and far point	25
14.13.2	Myopia and hyperopia	26
14.13.3	Beyond a scalar refractive error	27
14.14	Angular size and the simple magnifier	27
14.15	Compound microscopes	28
14.15.1	Sequential imaging description	28
14.15.2	Magnification, resolution and working distance	29
14.16	Astronomical telescopes	30
14.16.1	Why large objectives matter	30
14.16.2	Focal ratio and aperture	31
14.17	Optical fibres, prisms and ray guidance	31
14.17.1	Deriving the acceptance cone	32
14.18	Laboratory methods for measuring focal length	33
14.18.1	Direct object–screen method	33
14.18.2	Object–screen separation and the Bessel displacement method	34
14.18.3	Autocollimation idea	34
14.18.4	Measured quantities and model-dependent inferences	34
14.19	Computational ray tracing: from equations to an algorithm	35
14.19.1	Paraxial thin-lens ray tracing	35
14.19.2	Numerical experiment: sweeping object distance	36
14.20	Engineering interpretation: tolerances, alignment and image quality	36
14.20.1	Alignment and diagnostic signatures	37
14.21	A reliable solution and diagnostic method	38
14.22	Guided checks for the main chapter	38
14.23	Main chapter summary	39
14.24	Advanced Topics for Technical and Engineering Students	41
14.25	The paraxial approximation as a controlled expansion	41
14.25.1	Paraxial optics as linear optics	41
14.26	Newtonian form of the imaging equation	42
14.27	ABCD ray-transfer matrices	43
14.27.1	Translation matrix	43
14.27.2	Thin-lens matrix	43
14.27.3	Order matters	43
14.27.4	Magnification from the system matrix	44
14.28	Separated thin lenses and effective focal length	44
14.29	Principal planes and cardinal points	45
14.30	Refraction matrices and index-weighted ray coordinates	46
14.30.1	Translation through a medium	46
14.30.2	Refraction at a spherical surface	46
14.31	Thick-lens equivalent power and the Gullstrand relation	47
14.32	Afocal systems, beam expanders and telescope matrices	48
14.32.1	Finite-plane imaging with an afocal pair	49
14.33	Aperture stops, pupils and numerical aperture	49
14.33.1	Throughput and the optical invariant	50
14.33.2	Object-side and image-side cones	50
14.34	Radiance, etendue and why passive optics cannot create brightness	50
14.35	Geometrical aberrations	51

14.35.1	Spherical aberration	51
14.35.2	Coma	51
14.35.3	Astigmatism and field curvature	51
14.35.4	Distortion	51
14.35.5	Chromatic aberration	51
14.35.6	Lens bending as a design freedom	52
14.35.7	Aspheric surfaces	53
14.36	Exact ray tracing at a spherical interface	53
14.36.1	A reproducible spherical-aberration benchmark	54
14.37	Circle of confusion and depth of field	54
14.37.1	Why smaller apertures increase depth of field	55
14.37.2	Near and far acceptable object distances	55
14.38	Telecentric imaging and machine vision	56
14.39	Prisms: minimum deviation and dispersion	57
14.39.1	Angular dispersion	57
14.40	Fermat's principle and the paraxial imaging condition	58
14.41	Bessel method derived and uncertainty propagated	58
14.41.1	Uncertainty propagation	59
14.41.2	Shared readings and systematic errors	60
14.42	Extended tutorial: uncertainty in image position	60
14.43	Sensor sampling and a preview of modulation transfer	61
14.44	Calibration models for real cameras	62
14.45	Extended tutorial: inverse optical design with constraints	62
14.45.1	Adding a depth-of-field constraint	63
14.45.2	Adding a resolution constraint	63
14.46	Extended laboratory project: characterize an unknown lens	63
14.46.1	Stage 1: focal length by direct conjugates	64
14.46.2	Stage 2: Bessel method	64
14.46.3	Stage 3: aperture dependence	64
14.46.4	Stage 4: chromatic test	64
14.46.5	Stage 5: uncertainty budget	64
14.47	Advanced synthesis: where geometrical optics ends	65
Problems and Selected Solutions		66
Bibliography and source note		74

Preface

Geometrical optics is one of the most successful approximation schemes in applied physics. It replaces a spatially extended electromagnetic wave by an idealized *ray* that indicates the local direction of energy propagation. This simplification is powerful whenever the wavelength is much smaller than the characteristic dimensions of the optical system and when diffraction is not the dominant effect. With that approximation, questions about cameras, mirrors, lenses, projectors, telescopes, microscopes, eyeglasses and optical sensors can be reduced to geometry, refraction, reflection and carefully defined sign conventions.

This chapter develops image formation as a mapping problem. Light emitted or scattered by an object point travels through an optical system. If the emerging rays actually converge, a real image is formed. If the emerging rays diverge but their backward extensions intersect, a virtual image is perceived. The distinction is physical and operational: a real image can, in principle, be projected onto a screen at the image plane, while a virtual image cannot be captured on a screen placed at the apparent image position because the rays do not physically pass through that point.

The chapter begins by connecting the ray approximation to the electromagnetic-wave picture developed earlier in Physics 2. Reflection, refraction, Fermat's principle and total internal reflection are reviewed only to the extent needed for image formation. Plane and spherical mirrors are then treated systematically, followed by refraction at spherical surfaces, thin lenses, lens combinations and optical power. The second half of the main chapter emphasizes systems: the eye, cameras, projectors, magnifiers, microscopes and telescopes. Laboratory and computational sections show how ray diagrams become quantitative measurements and numerical ray-tracing algorithms.

For Technical Physics, engineering, computer-science and applied-science students, geometrical optics is especially valuable because the same mathematical structure appears in imaging hardware, machine vision, photonics, detector design, remote sensing and instrument alignment. A ray diagram is therefore not merely a classroom sketch. It is a simplified engineering model whose validity, assumptions and failure modes must be understood.

The Guideline

For every imaging problem, separate four pieces of information before doing algebra: (1) the direction of light propagation, (2) the optical element and its focal properties, (3) the object location and orientation, and (4) the desired image information. Draw a quick ray diagram first. Then choose signs. The diagram is the physics; the sign convention is bookkeeping.

Pedagogical boundary

This chapter uses geometrical optics and the paraxial approximation. It does not attempt a full wave-optics derivation of diffraction, interference, resolution or polarization. Those effects become essential when apertures are comparable with the wavelength or when phase coherence matters. Interference is developed in Chapter 15 and diffraction and resolution in Chapter 16. Here, diffraction is mentioned only when it limits the validity of a ray model.

Conventions and notation

The notation in geometrical optics is compact, but the signs deserve explicit care. Throughout the chapter we use a *real-is-positive* convention for object and image distances. The symbol p denotes object distance and i denotes image distance. A real object has $p > 0$; a converging incident bundle has a virtual object at its would-be focus and $p < 0$. A real image has $i > 0$, whereas a virtual image has $i < 0$. The focal length f is positive for a converging optical element and negative for a diverging optical element. For mirrors and thin lenses with the same surrounding medium on both sides, the transverse magnification is

$$m = \frac{h_i}{h_o} = -\frac{i}{p},$$

where h_o and h_i are the signed object and image heights. In this convention, $m > 0$ means upright and $m < 0$ means inverted.

For spherical mirrors, $R > 0$ and $f > 0$ for a concave mirror, while $R < 0$ and $f < 0$ for a convex mirror. In the paraxial approximation,

$$f = \frac{R}{2}.$$

For thin lenses, $f > 0$ denotes a converging lens and $f < 0$ a diverging lens.

The radii in the lens-maker formula require a separate geometrical sign rule. When light travels from left to right, the radius R_j of the j th refracting surface is positive if its center of curvature lies to the right of that surface and negative if the center lies to the left. Thus an ordinary biconvex lens has $R_1 > 0$ and $R_2 < 0$. With a lens of refractive index n in air,

$$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right).$$

For a single spherical refracting surface separating indices n_1 and n_2 , we use

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{R},$$

with $p > 0$ for a real object on the incident side, $i > 0$ for a real image on the transmitted side, and $R > 0$ when the center of curvature lies on the transmitted side. This convention is compatible with the sketches and examples used below.

A symbol-and-unit reference table is collected in Appendix A. Solid ray segments represent propagating light; dashed ray segments are geometric continuations. Transverse scales may be expanded to make the ray relationships legible.

Caution

No sign convention is “the physics.” Different textbooks may define p , i , R or positive coordinate directions differently. A correct calculation remains correct if the convention is internally consistent. The dangerous situation is not using a different convention; it is switching conventions halfway through a problem.

Chapter 14

Geometrical Optics and Image Formation

14.1 Motivation, prerequisites and roadmap

The practical question of geometrical optics is simple to state: given an object and an optical system, where does the image appear, how large is it, and is it real or virtual? The simplicity of the question hides a rich modelling hierarchy. At the most fundamental level of this course, light is an electromagnetic wave. In many instruments, however, the wavelength is tiny compared with mirror diameters, lens curvatures, object distances and image distances. Under those conditions it is efficient to track the direction of propagation with rays.

The ray picture is particularly effective because reflection and refraction are local rules. At a mirror, an incident ray changes direction according to the law of reflection. At a transparent interface, Snell's law determines the new direction. An optical system is then a sequence of such direction changes. A lens is not a mysterious object that “makes an image” by itself; it is a shaped refracting system designed so that many rays from one object point leave the lens in directions that converge to one image point, at least approximately.

Learning goals

After studying this chapter, the student should be able to:

- explain the ray approximation and state when it is expected to work;
- distinguish object points, image points, real images and virtual images;
- apply reflection and refraction laws in an imaging context;
- construct ray diagrams for plane mirrors, spherical mirrors and thin lenses;
- derive and use the paraxial mirror equation and magnification relation;
- analyze refraction at a single spherical surface;
- use the thin-lens equation and lens-maker formula with an explicit sign convention;
- classify images produced by converging and diverging lenses;
- analyze two-lens and multi-lens systems sequentially;
- use optical power and diopters in engineering and vision-correction problems;
- explain the basic operation of the eye, camera, projector, magnifier, microscope and telescope;
- perform laboratory-style focal-length measurements and uncertainty estimates;
- implement a simple computational ray tracer and recognize the limits of paraxial models;
- extend the subject using ray-transfer matrices, principal planes and aberration concepts.

The chapter assumes that students remember basic trigonometry, similar triangles, small-angle approximations and the laws of reflection and refraction. It also assumes familiarity with the electromagnetic-wave interpretation of light from earlier Physics 2 material. No advanced calculus is required for the main chapter, although the enrichment section uses matrix methods and systematic

series approximations.

Figure 14.1 connects the wave description, ray rules and instrument models developed in this chapter.

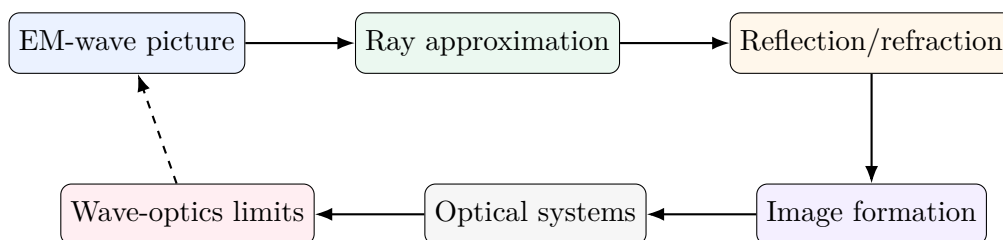


Figure 14.1: Roadmap of the chapter. Geometrical optics is obtained by simplifying the wave description, then applying local reflection and refraction rules to construct images and optical systems.

Key message

Geometrical optics is not a competing theory of light. It is an approximation to wave propagation that is extraordinarily efficient for many macroscopic optical systems.

14.2 From electromagnetic waves to light rays

14.2.1 What a ray represents

A light ray is an idealized curve whose tangent indicates the local direction of propagation. In a homogeneous isotropic medium, rays are straight lines. At an interface, their directions change according to reflection and refraction laws. In a slowly varying refractive-index field, rays can curve. The ray therefore represents directional information, not a narrow physical filament of light.

In electromagnetic language, a monochromatic plane wave has surfaces of constant phase called wavefronts. In a homogeneous isotropic medium the ray direction is normal to the wavefront. If the wavefront is approximately planar over the region of interest, neighboring rays are approximately parallel. If the wavefront is spherical, the rays are radial. This geometrical connection between wavefronts and rays explains why rays can summarize propagation without retaining the full electric- and magnetic-field oscillations.

The approximation becomes most trustworthy when the characteristic optical dimension L is much larger than the wavelength λ ,

$$\frac{\lambda}{L} \ll 1. \quad (14.1)$$

This criterion is qualitative rather than a universal threshold. The relevant L may be an aperture diameter, a radius of curvature, a feature size or a propagation distance associated with a sharp edge. When apertures become only a few wavelengths across, diffraction strongly modifies the ray prediction.

Guided checks

1. Why can a laser beam be represented by rays even though the light is a wave?
2. If an aperture is gradually narrowed, which effect eventually invalidates a purely geometrical prediction?
3. Does a ray carry an intrinsic thickness? If not, what does a bundle of rays represent?
4. In a homogeneous medium, what is the relation between a ray and a wavefront?

14.2.2 Fermat's principle as a unifying idea

A powerful formulation of geometrical optics is Fermat's principle: for fixed endpoints, the optical path along a physical ray is stationary under small admissible path variations. For a piecewise uniform medium, the optical path length is

$$\mathcal{L}_{\text{opt}} = \int n \, ds, \quad (14.2)$$

where n is refractive index and ds is an element of geometric path length. Here n is the phase refractive index at the chosen frequency. In a nondispersive stationary medium, $v = c/n$ and the travel time is $\mathcal{L}_{\text{opt}}/c$. In a dispersive medium, Fermat's principle at fixed frequency concerns phase optical path; pulse travel time involves the group index and must be distinguished from it.

For reflection from a plane mirror, the stationary-time construction gives equal incident and reflected angles. For refraction across a plane interface, it gives Snell's law,

$$n_1 \sin \theta_1 = n_2 \sin \theta_2. \quad (14.3)$$

These laws therefore fit into one geometrical principle. We will not use variational calculus in the main chapter, but the principle is valuable conceptually: the allowed ray paths follow from the optical geometry and material response. The stationary-path statement is a propagation law, not a claim that light makes a choice.

Caution

The phrase “least time” is useful but incomplete. The precise statement is that the optical travel time is stationary under small path variations. In common reflection and refraction problems the stationary path is indeed a minimum, but stationary paths can also be maxima or saddle-type extrema in more complicated systems.

14.2.3 Reflection, refraction and total internal reflection: a focused review

The law of reflection is

$$\theta_i = \theta_r, \quad (14.4)$$

with both angles measured from the local surface normal. For refraction, Eq. (14.3) determines the transmitted angle. When light passes from a higher-index medium n_1 to a lower-index medium n_2 , Snell's law predicts a critical angle

$$\theta_c = \sin^{-1} \left(\frac{n_2}{n_1} \right), \quad n_1 > n_2, \quad (14.5)$$

above which no propagating transmitted ray exists in the geometrical model. For a lossless interface with a sufficiently extended lower-index medium, the time-averaged power is totally reflected. An evanescent field exists beyond the boundary but carries no net power normal to an isolated planar interface. Absorption or a nearby second medium can change this conclusion.

Total internal reflection is central to optical fibers, prism systems and some compact imaging devices. It is also a reminder that an optical system may guide rays without forming an image. Image formation requires an additional property: rays from each object point must be redirected so that they converge to, or appear to diverge from, a corresponding image point.

Example 14.1: Critical angle in a polymer light guide

A polymer core has refractive index $n_1 = 1.49$ and is surrounded by a cladding of index $n_2 = 1.40$. Estimate the critical angle for total internal reflection at the core-cladding interface,

measured from the normal.

Solution. Since $n_1 > n_2$,

$$\theta_c = \sin^{-1} \left(\frac{1.40}{1.49} \right) \approx 70.0^\circ. \quad (14.6)$$

A ray inside the core that reaches the interface at an incidence angle larger than about 70° from the normal is totally internally reflected in the ideal geometrical model.

Reasonableness check. Because the two indices are close, the ratio n_2/n_1 is close to one, so the critical angle should be large. The result is consistent with that expectation.

14.3 The language of image formation

14.3.1 Object points and image points

An extended object can be understood as a collection of object points. Consider one point O . Many rays leave or scatter from O . After reflection or refraction by an optical system, the rays may intersect at a point I . If the actual rays intersect, I is a real image point. If the rays do not intersect but the backward extensions of the outgoing rays intersect, I is a virtual image point.

This point-to-point viewpoint is important. A good imaging system attempts to map every relevant object point to one image point. If rays from one object point fail to meet at a single image point, the image is blurred. Aberrations are therefore failures of perfect point-to-point mapping within geometrical optics, while diffraction produces an additional wave-based spreading even in an ideal aberration-free system.

Definition 14.1: Real and virtual image

A *real image* is located where the physical outgoing rays converge. A screen placed at that location can intercept the rays and display the image. A *virtual image* is located where backward extensions of outgoing rays intersect. The rays do not actually pass through the virtual image point, so a screen placed there does not capture that image.

Figure 14.2 contrasts physical ray intersection with intersection of backward extensions.

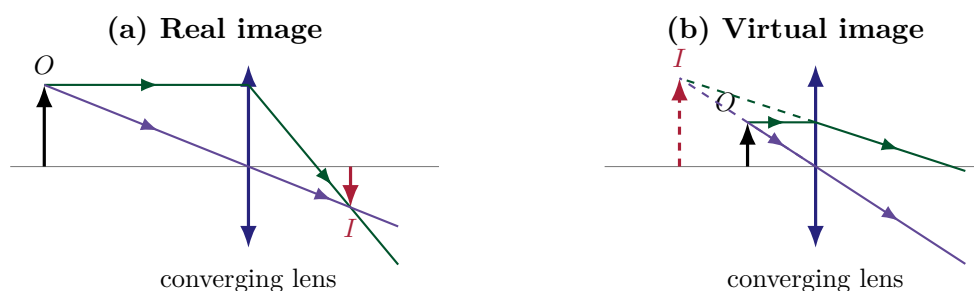


Figure 14.2: Two consistent thin-lens constructions. In (a) outgoing rays cross at the real image; in (b) only their dashed backward extensions cross. Solid ray arrows show propagation, and transverse dimensions are expanded for visibility.

14.3.2 Why a virtual image is not imaginary

A virtual image is observable because the eye interprets rays arriving at the pupil as if they traveled in straight lines from an apparent source. A bathroom mirror is the everyday example. Light from the face reflects from the mirror and reaches the eye. The brain extrapolates those rays backward and locates the apparent face behind the mirror. No light actually travels from behind the mirror.

toward the eye, but the image has a definite geometrical location and can be photographed by placing a camera in front of the mirror.

This distinction is often misunderstood. “Virtual” means that the rays do not physically converge at the image point. It does not mean the observation is subjective or unreal. The apparent position can be measured, calculated and used in optical-system design.

Guided checks

1. A camera photographs your reflection in a plane mirror. Which image is virtual, and where does the camera form a new real image?
2. Can a virtual image be seen by the eye? Can it be projected onto a screen located at its apparent position?
3. If outgoing rays are parallel, where is the image located in the idealized geometrical description?
4. What must be true of rays from one object point for an ideal stigmatic image point to exist?

14.3.3 Paraxial rays and the small-angle approximation

Real spherical mirrors and lenses are not perfect imaging elements for arbitrary rays. The simple equations used in introductory optics are obtained for rays close to the optical axis and making small angles with it. Such rays are called *paraxial*. In radians,

$$\sin \theta \approx \theta, \quad \tan \theta \approx \theta, \quad \cos \theta \approx 1, \quad (14.7)$$

when $|\theta| \ll 1$.

The approximation is not merely mathematical convenience. It defines the regime in which a spherical surface behaves approximately like an ideal imaging element. Rays far from the axis can focus at different longitudinal positions, producing spherical aberration. The advanced section quantifies this failure and connects it to lens design.

Key message

Whenever you use $1/f = 1/p + 1/i$ for a spherical mirror or thin lens, you are using an idealized paraxial model. The equation can be very accurate, but it is not exact for every ray through a real optical system.

14.4 Plane mirrors: the simplest imaging system

A plane mirror gives a clean introduction to virtual images because no curvature or focal length is required. Consider an object point at perpendicular distance p in front of the mirror. Two rays from the point reflect toward an observer. Extending the reflected rays backward produces an apparent intersection the same perpendicular distance behind the mirror. Thus

$$i = -p \quad (14.8)$$

in our real-is-positive convention.

The magnification is

$$m = -\frac{i}{p} = +1. \quad (14.9)$$

The image is upright and the same size as the object. The sign $i < 0$ identifies it as virtual.

Figure 14.3 locates the mirror image by extending two reflected rays.

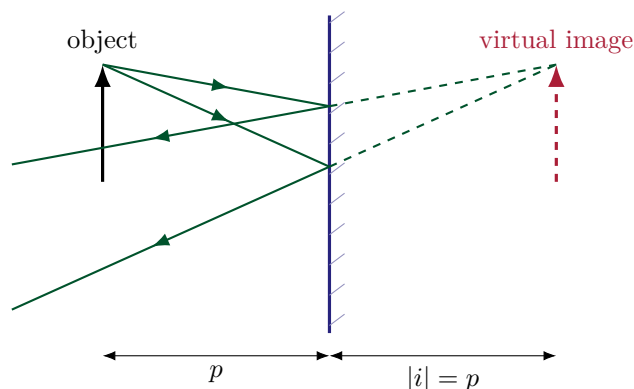


Figure 14.3: A plane mirror reverses the component normal to its surface. Both reflected rays obey equal incidence and reflection angles; their extensions locate the same virtual object point.

14.4.1 Lateral inversion

Plane mirrors are often said to “reverse left and right.” A more careful statement is that the mirror reverses the coordinate normal to its surface: front and back are interchanged in the image relation. The familiar left–right impression arises because humans mentally compare themselves with the image by imagining a rotation about a vertical axis. The mirror itself does not selectively exchange horizontal coordinates while preserving depth.

For engineering drawings and alignment systems, this distinction matters. A mirror transformation changes handedness. A sequence of reflections can restore or change image parity depending on the number and geometry of reflections. Periscopes and folded optical paths exploit this systematically.

Example 14.2: Minimum mirror height for viewing the full body

A person of height $H = 1.80$ m stands in front of a vertical plane mirror. Ignoring the finite size of the pupil, what minimum mirror height is required to see the full body? Does the answer depend on the person’s distance from the mirror?

Solution. A ray from the top of the head to the eye reflects from a point halfway in height between the eye and the top of the head. A ray from the feet to the eye reflects from a point halfway between the eye and the feet. The vertical separation between these two reflection points is one half of the person’s height. Therefore

$$H_{\text{mirror,min}} = \frac{H}{2} = 0.90 \text{ m.} \quad (14.10)$$

The result is independent of the person’s distance from the mirror because the relevant triangles remain similar as the person moves forward or backward.

Reasonableness check. A full-height mirror is not required. A mirror covering half the body height is sufficient if positioned correctly.

Method

For a plane-mirror geometry problem, replace the mirror by a virtual copy of the object or observer behind the mirror. Straight lines to the virtual point automatically encode the law of reflection and often turn a reflection problem into simple geometry.

14.5 Spherical mirrors

14.5.1 Geometry, principal axis and focal point

A spherical mirror is a portion of a spherical reflecting surface. A concave mirror has its reflecting surface on the inner side of the sphere and is converging for paraxial incident rays. A convex mirror reflects from the outer side and is diverging. The line through the vertex V and the center of curvature C is the principal axis. The distance VC is the radius of curvature R .

For paraxial rays incident parallel to the principal axis, a concave mirror reflects the rays toward a common focal point F located approximately halfway between V and C . A convex mirror produces diverging reflected rays whose backward extensions pass through a virtual focus behind the mirror. In the paraxial approximation,

$$f = \frac{R}{2}. \quad (14.11)$$

Figure 14.4 compares finite-height reflected rays with the paraxial focus.

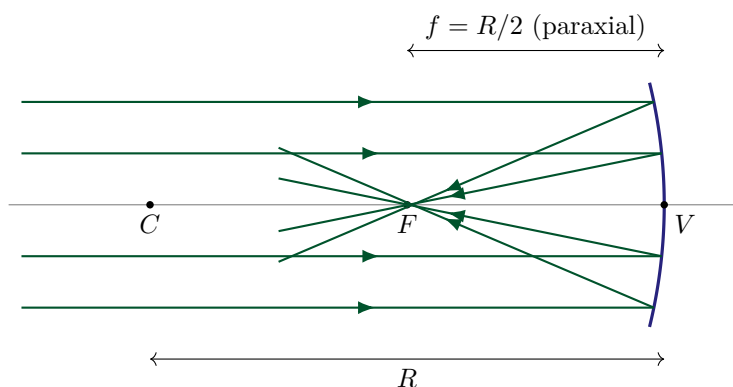


Figure 14.4: Reflection calculated at an actual spherical surface. Near-axis rays cross close to the paraxial focus F ; finite-height rays cross slightly nearer the vertex. The mirror, center and vertex share one consistent radius.

14.5.2 Deriving the mirror equation from similar triangles

Take an on-axis object and an on-axis real image of a concave mirror. A ray strikes the mirror at height y . To first order, the small positive angles made by the incident line, reflected line and normal with the axis have magnitudes

$$\alpha \simeq \frac{y}{p}, \quad \beta \simeq \frac{y}{i}, \quad \gamma \simeq \frac{y}{R}. \quad (14.12)$$

The normal passes through the center of curvature. Equal incidence and reflection angles give $\alpha + \beta = 2\gamma$, hence

$$\frac{y}{p} + \frac{y}{i} = \frac{2y}{R}. \quad (14.13)$$

The surface sag is of order y^2/R and is neglected at this order. Cancelling $y \neq 0$ gives

$$\boxed{\frac{1}{p} + \frac{1}{i} = \frac{2}{R} = \frac{1}{f}}. \quad (14.14)$$

The result is independent of ray height: this is precisely why an on-axis point has one paraxial image. Setting $p \rightarrow \infty$ identifies $f = R/2$. Signed distances extend the formula to virtual conjugates and convex mirrors.

For an extended object, use a ray from its tip reflected at the vertex. The equal-angle triangles give

$$\frac{h_i}{i} = -\frac{h_o}{p}, \quad m = \frac{h_i}{h_o} = -\frac{i}{p}. \quad (14.15)$$

The minus sign expresses inversion for real positive p and i , rather than an arbitrary algebraic addition.

Equation (14.14) is algebraically identical to the thin-lens equation, but the meaning of positive and negative sides differs because reflected light remains on the incident side of a mirror. In our convention, a concave mirror has $f > 0$ and can form real or virtual images; a convex mirror has $f < 0$ and, for a real object, forms a virtual upright reduced image.

Method

For spherical mirrors use the sequence: **sketch** $\rightarrow$ **classify mirror** $\rightarrow$ **assign f** $\rightarrow$ **solve** $1/f = 1/p + 1/i \rightarrow$ **compute** $m = -i/p \rightarrow$ **interpret signs**. Never interpret the numerical answer before checking whether the sign agrees with the ray sketch.

14.5.3 Principal rays for mirrors

Three paraxial rays are especially useful in a ray diagram for a spherical mirror:

1. a ray parallel to the principal axis reflects through the focal point for a concave mirror, or as if it came from the focal point for a convex mirror;
2. a ray directed through the focal point reflects parallel to the principal axis;
3. a ray directed through the center of curvature strikes the mirror normally and retraces its path.

Only two rays are needed to locate the image of one object point. The third provides a useful consistency check.

14.5.4 Image classes for a concave mirror

A concave mirror can produce several qualitatively different images depending on p/f :

- $p > 2f$: real, inverted, reduced image between f and $2f$;
- $p = 2f$: real, inverted, equal-size image at $2f$;
- $f < p < 2f$: real, inverted, enlarged image beyond $2f$;
- $p = f$: reflected rays are parallel and the ideal image is at infinity;
- $0 < p < f$: virtual, upright, enlarged image behind the mirror.

This list should not be memorized in isolation. It follows directly from the ray geometry and the signs in Eqs. (14.14) and (14.15).

Example 14.3: Concave mirror with a real enlarged image

A concave mirror has focal length $f = 20.0$ cm. A 3.0 cm tall object is placed $p = 30.0$ cm in front of the mirror. Find the image position, magnification and image height.

Solution. The mirror equation gives

$$\frac{1}{i} = \frac{1}{f} - \frac{1}{p} = \frac{1}{20.0} - \frac{1}{30.0} \text{ cm}^{-1} = \frac{1}{60.0} \text{ cm}^{-1}, \quad (14.16)$$

$$i = 60.0 \text{ cm}. \quad (14.17)$$

The positive image distance indicates a real image. The magnification is

$$m = -\frac{i}{p} = -\frac{60.0}{30.0} = -2.00, \quad (14.18)$$

so

$$h_i = mh_o = (-2.00)(3.0 \text{ cm}) = -6.0 \text{ cm}. \quad (14.19)$$

The image is real, inverted and twice as tall as the object.

Reasonableness check. The object lies between f and $2f$, so a real enlarged image beyond $2f$ is expected. Since $2f = 40 \text{ cm}$ and $i = 60 \text{ cm}$, the result has the correct qualitative behavior.

Example 14.4: Shaving mirror: object inside the focal length

A concave shaving mirror has focal length $f = 25 \text{ cm}$. A face is $p = 15 \text{ cm}$ from the mirror. Determine the image position and magnification.

Solution. Using the mirror equation,

$$\frac{1}{i} = \frac{1}{25} - \frac{1}{15} = -\frac{2}{75} \text{ cm}^{-1}, \quad (14.20)$$

so

$$i = -37.5 \text{ cm}. \quad (14.21)$$

The image is virtual and lies behind the mirror. Its magnification is

$$m = -\frac{i}{p} = +2.50. \quad (14.22)$$

Thus the image is upright and enlarged by a factor of 2.5.

Physical interpretation. A shaving mirror is useful precisely because an object inside the focal length produces an enlarged upright virtual image.

14.5.5 Convex mirrors and wide fields of view

For a convex mirror $f < 0$. With a real object $p > 0$, the mirror equation produces $i < 0$ and $0 < m < 1$. The image is therefore virtual, upright and reduced. The reduction allows a relatively large angular field to be represented in a compact mirror area, which is why convex mirrors are used for surveillance, road intersections and vehicle side mirrors.

Example 14.5: Convex safety mirror

A convex mirror has focal length $f = -0.60 \text{ m}$. A person stands $p = 3.0 \text{ m}$ in front of it. Find i and m .

Solution.

$$\frac{1}{i} = \frac{1}{f} - \frac{1}{p} = -1.667 - 0.333 = -2.000 \text{ m}^{-1}, \quad (14.23)$$

so $i = -0.500$ m. Then

$$m = -\frac{i}{p} = \frac{0.500}{3.0} = 0.167. \quad (14.24)$$

The image is virtual, upright and one sixth the object's linear size.

Guided checks

1. Why does a convex mirror always give $i < 0$ for a real object in this convention?
2. A concave mirror forms an upright image. What can you immediately conclude about the image being real or virtual?
3. What happens to i as $p \rightarrow f^+$ for a concave mirror?
4. Why does the paraxial mirror equation become less accurate for rays far from the principal axis?

14.6 Refraction at plane and spherical surfaces

14.6.1 Apparent depth at a plane interface

Before treating curved refracting surfaces, it is useful to examine a plane interface. A familiar effect is the apparent shallowness of an object under water. Rays from a submerged object refract away from the normal as they enter air. The observer extrapolates the rays backward along straight lines and therefore assigns the object a smaller apparent depth.

For small viewing angles, Snell's law becomes

$$n_1 \theta_1 \approx n_2 \theta_2. \quad (14.25)$$

Using similar triangles for an object in medium n_1 viewed from medium n_2 , one obtains

$$\frac{d_{\text{apparent}}}{d_{\text{real}}} \approx \frac{n_2}{n_1}. \quad (14.26)$$

For water viewed from air, $n_1 \approx 1.33$ and $n_2 \approx 1.00$, so the apparent depth is about 75% of the actual depth for paraxial viewing.

Example 14.6: Apparent depth of a sensor window

An optical marker is embedded 12.0 mm below a flat glass surface of refractive index $n = 1.50$. An observer views the marker nearly normally from air. Estimate the apparent depth.

Solution. Taking $n_1 = 1.50$ for glass and $n_2 = 1.00$ for air,

$$d_{\text{apparent}} \approx d_{\text{real}} \frac{n_2}{n_1} = 12.0 \text{ mm} \times \frac{1.00}{1.50} = 8.0 \text{ mm}. \quad (14.27)$$

The marker appears 4.0 mm closer to the surface.

Engineering note. Optical inspection through protective windows can therefore introduce systematic depth errors unless refraction is included in the calibration model.

14.6.2 A single spherical refracting surface

Many optical systems can be understood by first studying one curved interface. Suppose a spherical surface of radius R separates a medium of refractive index n_1 on the incident side from a medium

of index n_2 on the transmitted side. A point object lies on the optical axis at distance p . In the paraxial approximation, the image distance i satisfies

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{R}. \quad (14.28)$$

To see where the equation comes from, put the vertex at $z = 0$ and take positive z along transmitted light. At a small ray height y , the incident slope is $\theta_1 \simeq y/p$, the outgoing slope is $\theta_2 \simeq -y/i$, and the normal directed toward the transmitted side has slope $\phi \simeq -y/R$. Paraxial Snell refraction reads

$$n_1(\theta_1 - \phi) = n_2(\theta_2 - \phi), \quad (14.29)$$

$$n_2\theta_2 = n_1\theta_1 - \frac{n_2 - n_1}{R}y. \quad (14.30)$$

Inserting the slopes and cancelling y yields Eq. (14.28). The surface sag is second order in height and does not enter this first-order derivation. The signed slope method avoids guessing which small angles should be added or subtracted.

This equation is one of the most important bridges between Snell's law and lens imaging. A thin lens is essentially two spherical refracting surfaces placed close together.

The surface is converging or diverging depending on the index change and curvature. Curvature alone is not enough to determine the behavior. A convex surface can be converging for light going from a lower index to a higher index and behave differently if the index order is reversed. This is one reason why the terms “convex” and “converging” should not be treated as synonyms outside a specified medium.

Method

For spherical refraction, identify the incident medium n_1 , transmitted medium n_2 , real object distance p , and sign of R from the location of the center of curvature. Solve Eq. (14.28) for i and interpret $i > 0$ as a real transmitted-side image and $i < 0$ as a virtual incident-side image.

Example 14.7: Image formed by a convex glass surface

A small object in air is $p = 30.0$ cm from a convex spherical glass surface. The glass has refractive index $n_2 = 1.50$, air has $n_1 = 1.00$, and the surface radius is $R = +20.0$ cm. Find the paraxial image distance measured into the glass.

Solution.

$$\frac{1.00}{30.0} + \frac{1.50}{i} = \frac{0.50}{20.0}, \quad (14.31)$$

with distances in centimetres. Hence

$$\frac{1.50}{i} = 0.0250 - 0.0333 = -0.00833 \text{ cm}^{-1}, \quad (14.32)$$

so

$$i = -180 \text{ cm}. \quad (14.33)$$

The negative value means the rays remain divergent after refraction and appear to originate from a virtual image on the incident side.

Reasonableness check. The object is relatively close to the surface. Although the surface bends rays toward the normal on entering glass, that bending is not strong enough here to make the transmitted rays converge.

14.6.3 Paraxial lateral magnification at a refracting surface

For a spherical refracting surface, the transverse magnification follows from the paraxial geometry and Snell's law. With the sign conventions of this chapter,

$$m = \frac{h_i}{h_o} = -\frac{n_1 i}{n_2 p}. \quad (14.34)$$

For a ray crossing the vertex, the normal is axial. Snell's law gives $n_1(-h_o/p) = n_2(h_i/i)$, immediately yielding this result. In the plane-interface limit, $i = -n_2 p/n_1$, so $m = +1$: transverse dimensions are unchanged even though axial depth is rescaled. A single spherical surface also has unequal front and rear focal distances, $f_o = n_1 R/(n_2 - n_1)$ and $f_i = n_2 R/(n_2 - n_1)$.

The index ratio matters because the ray angles on the two sides are not related by equal-angle reflection. This differs from the simpler mirror and thin-lens result $m = -i/p$.

Guided checks

1. Why can a plane refracting surface produce an apparent depth change even though its radius is infinite?
2. In Eq. (14.28), what happens when $R \rightarrow \infty$?
3. Why is the word “convex” insufficient to decide whether a refracting surface is converging?
4. Which approximation converts Snell's law into a linear relation among the ray angles?

14.6.4 Surface power and a reversed index step

The quantity $\Phi_s = (n_2 - n_1)/R$ has units of inverse length. Its sign depends jointly on curvature and the index step; $\Phi_s = n_2/f_i$, rather than simply $1/f_i$ in a general transmitted medium.

Example 14.8: Glass to air at a convex boundary

A real object is inside glass of index 1.50, $p = 10.0$ cm before a surface leading to air. The center of curvature is on the air side, so $R = +20.0$ cm. Find the paraxial image distance.

Solution.

$$\frac{1.50}{10.0} + \frac{1.00}{i} = \frac{1.00 - 1.50}{20.0} = -0.0250. \quad (14.35)$$

Hence

$$\frac{1}{i} = -0.0250 - 0.150 = -0.175 \text{ cm}^{-1}, \quad (14.36)$$

so

$$i = -5.71 \text{ cm}. \quad (14.37)$$

The negative image distance means the refracted rays diverge as though from a virtual image on the incident side.

Figure 14.5 identifies the signed normal and ray slopes used in the spherical-surface derivation.

14.7 Thin lenses: converging and diverging elements

14.7.1 What makes a lens thin?

A lens has two refracting surfaces. The exact propagation through a real lens requires refraction at the first surface, translation through a finite thickness, and refraction at the second surface. In the *thin-lens approximation*, the lens thickness is small compared with the radii of curvature and the relevant object and image distances. We then treat the two refractions as occurring effectively in one plane.

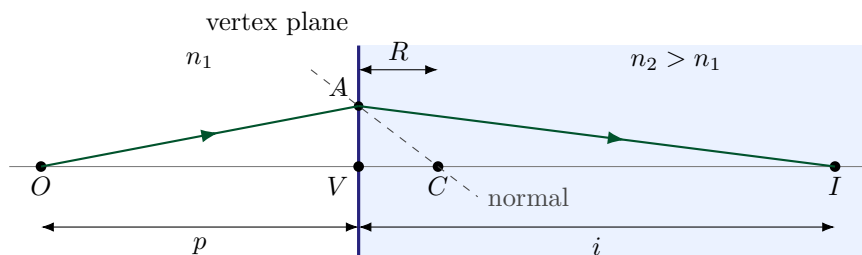


Figure 14.5: First-order spherical-refraction geometry with surface sag suppressed and transverse height exaggerated. The normal through A points toward C , with signed slope $-y/R$; it is not the horizontal normal of a physical plane interface. The vertex plane represents the paraxial expansion of the curved surface.

A converging lens has $f > 0$ in the surrounding medium. For a conventional glass lens in air, common biconvex and plano-convex shapes are converging. A diverging lens has $f < 0$; common biconcave and plano-concave shapes are diverging. Shape alone, however, does not determine power in every surrounding medium because the refractive-index contrast also matters.

Definition 14.2: Thin lens and focal points

An ideal thin lens is an optical element of negligible axial thickness that changes the direction of paraxial rays according to a single focal length f . Parallel incident rays emerge through the rear focal point for a converging lens, or diverge as if they originated from the front focal point for a diverging lens.

14.7.2 The thin-lens equation

For a thin lens in the paraxial approximation,

$$\boxed{\frac{1}{f} = \frac{1}{p} + \frac{1}{i}}. \quad (14.38)$$

The transverse magnification is

$$\boxed{m = \frac{h_i}{h_o} = -\frac{i}{p}}. \quad (14.39)$$

These expressions look identical to the mirror relations, but rays pass through a lens rather than returning to the incident side. A positive i therefore means a real image on the transmitted side of the lens.

The limiting behavior contains useful physical information. As $p \rightarrow \infty$, $i \rightarrow f$, so a distant object is imaged near the focal plane of a converging lens. As $p \rightarrow f^+$, $i \rightarrow +\infty$. As p moves just inside the focal length, i becomes large and negative, corresponding to an enlarged virtual image.

Key message

The focal length is not the distance at which every object is imaged. It is the image distance for an object at infinity in the paraxial model. Finite object distances generally produce image distances different from f .

14.7.3 Lens-maker formula

For a thin lens of refractive index n in air with spherical surface radii R_1 and R_2 ,

$$\boxed{\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)}. \quad (14.40)$$

For light traveling left to right, $R_j > 0$ when the center of curvature of surface j lies to its right. A biconvex lens therefore has $R_1 > 0$ and $R_2 < 0$, giving positive power for $n > 1$.

If the lens is immersed in a surrounding medium of index n_m , the relative index replaces n :

$$\frac{1}{f} = \left(\frac{n}{n_m} - 1 \right) \left(\frac{1}{R_1} - \frac{1}{R_2} \right). \quad (14.41)$$

This result explains an important laboratory observation: a transparent lens can become much weaker when immersed in a liquid whose refractive index is close to that of the lens material.

Example 14.9: Design focal length of a symmetric biconvex lens

A thin symmetric biconvex glass lens in air has refractive index $n = 1.52$ and surface radii $R_1 = +12.0$ cm and $R_2 = -12.0$ cm. Estimate its focal length.

Solution.

$$\frac{1}{f} = (1.52 - 1) \left(\frac{1}{12.0} - \frac{1}{-12.0} \right) \text{cm}^{-1} \quad (14.42)$$

$$= 0.52 \left(\frac{2}{12.0} \right) \text{cm}^{-1} = 0.0867 \text{cm}^{-1}, \quad (14.43)$$

so

$$f \approx 11.5 \text{ cm}. \quad (14.44)$$

The positive focal length confirms that the lens is converging in air.

Example 14.10: The same lens immersed in liquid

The lens in Example 14.9 is immersed in a transparent liquid with $n_m = 1.40$. Find the new focal length.

Solution. The relative index is $n/n_m = 1.52/1.40 = 1.0857$. Therefore

$$\frac{1}{f} = (1.0857 - 1) \left(\frac{2}{12.0 \text{ cm}} \right) \quad (14.45)$$

$$= 0.01429 \text{cm}^{-1}, \quad (14.46)$$

so

$$f \approx 70.0 \text{ cm}. \quad (14.47)$$

The lens is much weaker because the refractive-index contrast has been reduced.

Engineering interpretation. Optical power depends on the full material environment, not only on lens shape. Underwater cameras and immersed sensors must account for this effect.

14.7.4 Principal rays for thin lenses

For an ideal converging lens, three standard rays from the top of an object are convenient:

1. a ray parallel to the optical axis refracts through the far focal point;
2. a ray through the near focal point emerges parallel to the axis;
3. a ray through the optical center continues approximately undeviated in the thin-lens model.

For a diverging lens, the parallel incident ray emerges as if it originated from the near focal point, and a ray aimed toward the far focal point emerges parallel to the axis. The optical-center ray remains approximately straight.

Figure 14.6 checks that all three principal rays predict the same image.

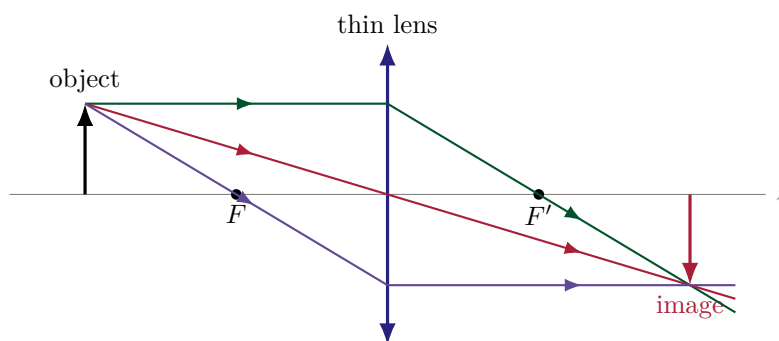


Figure 14.6: Three principal rays meet at the same image for $p = 2f$, $i = 2f$ and $m = -1$. The parallel ray passes through the rear focus; the ray through the front focus emerges horizontally; the central ray continues straight.

Guided checks

1. Why is the optical-center ray only approximately undeviated for a real lens?
2. What sign of f identifies a diverging lens?
3. What happens to the power of a glass lens when the surrounding refractive index approaches the glass index?
4. A converging lens receives parallel rays from a very distant object. Where is the paraxial image?

14.7.5 Why two surfaces give one lens equation

A lens in air may be treated as two refracting surfaces. Let the first surface have radius R_1 and the second R_2 . The refractive index changes from 1 to n at the first surface and from n back to 1 at the second.

For the first surface,

$$\frac{1}{p} + \frac{n}{i_1} = \frac{n-1}{R_1}. \quad (14.48)$$

For a thin lens, the image produced by the first surface acts as the object for the second with approximately the opposite signed distance, so $p_2 \simeq -i_1$. The second surface gives

$$\frac{n}{p_2} + \frac{1}{i} = \frac{1-n}{R_2}. \quad (14.49)$$

Substituting $p_2 = -i_1$ and adding Eqs. (14.48) and (14.49) eliminates the intermediate image:

$$\frac{1}{p} + \frac{1}{i} = (n-1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right). \quad (14.50)$$

Identifying the right-hand side with $1/f$ yields

$$\boxed{\frac{1}{f} = (n-1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)}. \quad (14.51)$$

This derivation clarifies why the sign convention for the radii matters and why the thin-lens approximation is required: the separation of the two refracting surfaces has been neglected.

14.8 Ray diagrams and image classification for thin lenses

14.8.1 Converging lens: object outside the focal length

When $p > f$ for a converging lens, Eq. (14.38) gives $i > 0$. The image is real and inverted. Its size depends on the ratio of p to f . The cases are analogous to the concave mirror:

Object location	Image location	Type	Orientation	Relative size
$p > 2f$	$f < i < 2f$	real	inverted	reduced
$p = 2f$	$i = 2f$	real	inverted	equal
$f < p < 2f$	$i > 2f$	real	inverted	enlarged
$p = f$	$i \rightarrow \infty$	parallel rays	–	–
$0 < p < f$	$i < 0$	virtual	upright	enlarged

These are the operating regimes of cameras and projectors. A camera usually has objects much farther than $2f$, so the real sensor image is reduced. A projector places a small bright object just beyond the focal point, producing a large real image on a distant screen.

Example 14.11: Converging lens and sensor position

A camera lens is modeled as a thin converging lens with focal length $f = 50.0$ mm. It photographs an object $p = 2.00$ m away. Find the sensor distance i and the transverse magnification.

Solution. Use consistent units: $p = 2000$ mm. Then

$$\frac{1}{i} = \frac{1}{50.0} - \frac{1}{2000} = 0.01950 \text{ mm}^{-1}, \quad (14.52)$$

so

$$i = 51.28 \text{ mm}. \quad (14.53)$$

The magnification is

$$m = -\frac{51.28}{2000} = -0.0256. \quad (14.54)$$

The sensor image is real, inverted and about 2.56% of the object's linear size.

Reasonableness check. Because the object is much farther away than f , the image should lie only slightly beyond the focal plane. The result 51.28 mm is consistent with that limit.

14.8.2 Converging lens: object inside the focal length

For $0 < p < f$, the thin-lens equation gives $i < 0$ and the magnification is positive. The image is virtual, upright and enlarged. This is the operating regime of a simple magnifier. The eye receives diverging rays from the lens and interprets them as originating from an enlarged object on the same side of the lens as the physical object.

Example 14.12: Magnifying-lens geometry

A converging lens has focal length $f = 10.0$ cm. A small object is placed $p = 7.50$ cm from the lens. Find the image distance and transverse magnification.

Solution.

$$\frac{1}{i} = \frac{1}{10.0} - \frac{1}{7.50} = -0.03333 \text{ cm}^{-1}, \quad (14.55)$$

so $i = -30.0$ cm. Then

$$m = -\frac{-30.0}{7.50} = +4.00. \quad (14.56)$$

The virtual image is upright and four times the object's height.

14.8.3 Diverging lenses

For a diverging lens, $f < 0$. With a real object $p > 0$, Eq. (14.38) gives a negative image distance whose magnitude is less than the object distance. The image is virtual, upright and reduced. This behavior is useful in corrective eyeglasses for myopia, beam expansion systems and optical combinations where a negative element modifies the effective focal length.

Figure 14.7 constructs the reduced virtual image of a diverging lens.

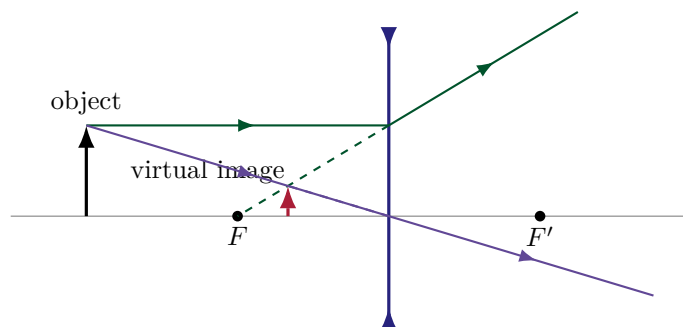


Figure 14.7: Diverging-lens construction with $p = 2|f|$: $i = -2|f|/3$ and $m = 1/3$. The outgoing green ray and its dashed extension lie on the same straight line through the virtual focus.

Example 14.13: Diverging lens

A diverging lens has focal length $f = -15.0$ cm. An object is $p = 30.0$ cm in front of it. Determine the image position and magnification.

Solution.

$$\frac{1}{i} = -\frac{1}{15.0} - \frac{1}{30.0} = -0.100 \text{ cm}^{-1}, \quad (14.57)$$

so $i = -10.0$ cm. The magnification is

$$m = -\frac{-10.0}{30.0} = +\frac{1}{3}. \quad (14.58)$$

The image is virtual, upright and reduced to one third of the object's height.

Caution

Do not decide whether an image is real or virtual from whether it looks upright or inverted alone. For the simple single-element cases in this chapter, upright images are commonly virtual, but the robust criterion is the sign and ray geometry: real images occur where physical rays converge.

14.8.4 A graphical way to understand the thin-lens equation

It is useful to normalize the thin-lens equation by f . Define $\xi = p/f$ and $\eta = i/f$. Then

$$1 = \frac{1}{\xi} + \frac{1}{\eta}, \quad \eta = \frac{\xi}{\xi - 1}. \quad (14.59)$$

For a converging lens with $f > 0$, the singularity at $\xi = 1$ corresponds to the object at the focal plane. For $\xi > 1$, $\eta > 0$ and the image is real. For $0 < \xi < 1$, $\eta < 0$ and the image is virtual. This dimensionless form separates the geometry from the absolute scale of the lens.

Guided checks

1. For a converging lens, why does $p = f$ correspond to $|i| \rightarrow \infty$?
2. A lens produces an upright image from a real object. If it is a single thin lens, what sign must i have?
3. Why is a projector object placed just outside the focal length rather than very far from the lens?
4. How does the sensor position change as a camera focuses from infinity to a nearby object?

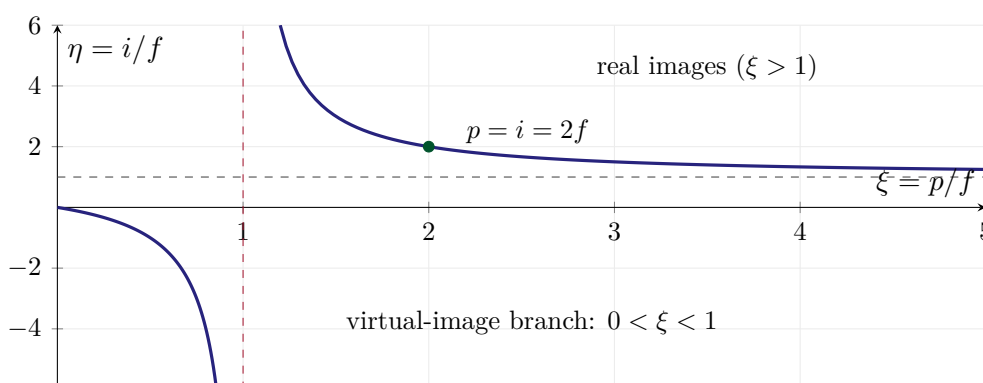


Figure 14.8: The normalized thin-lens equation for $f > 0$, plotted directly from $\eta = \xi/(\xi - 1)$. The two branches are not connected across $p = f$; dashed lines mark the vertical and horizontal asymptotes.

14.9 Multiple lenses and sequential imaging

Real optical instruments rarely contain a single ideal lens. The simplest extension is a sequence of thin lenses with a common optical axis. The central principle is straightforward: the image formed by one element becomes the object for the next. The geometry, however, must be handled carefully because the intermediate image may be real or virtual relative to the next lens.

For two thin lenses separated by distance d , suppose lens 1 forms an image at distance i_1 measured to its right when $i_1 > 0$. The object distance for lens 2 is then

$$p_2 = d - i_1 \quad (14.60)$$

for left-to-right propagation, including both real and virtual intermediate conjugates. If $i_1 > d$, the intermediate image would lie to the right of lens 2 in the absence of lens 2; it is therefore a virtual object for lens 2 and $p_2 < 0$ in an extended real-is-positive treatment. The second lens equation is

$$\frac{1}{f_2} = \frac{1}{p_2} + \frac{1}{i_2}. \quad (14.61)$$

The total transverse magnification is the product

$$m_{\text{tot}} = m_1 m_2 \cdots m_N. \quad (14.62)$$

Method

For a multi-lens system, never try to use one thin-lens equation for the entire system unless an equivalent focal length and principal-plane model has been established. Instead, solve element by element, carry the intermediate image location forward geometrically, and multiply magnifications.

Example 14.14: Two converging lenses

Two thin converging lenses are separated by $d = 40.0$ cm. Their focal lengths are $f_1 = 10.0$ cm and $f_2 = 20.0$ cm. An object is placed $p_1 = 30.0$ cm to the left of lens 1. Find the final image location relative to lens 2 and the total magnification.

Solution, lens 1.

$$\frac{1}{i_1} = \frac{1}{10.0} - \frac{1}{30.0} = \frac{1}{15.0} \text{ cm}^{-1}, \quad (14.63)$$

so $i_1 = 15.0$ cm. The intermediate image lies 15.0 cm to the right of lens 1. Therefore it is $p_2 = 40.0 - 15.0 = 25.0$ cm to the left of lens 2 and is a real object for lens 2.

Solution, lens 2.

$$\frac{1}{i_2} = \frac{1}{20.0} - \frac{1}{25.0} = 0.0100 \text{ cm}^{-1}, \quad (14.64)$$

so $i_2 = 100$ cm to the right of lens 2.

The magnifications are

$$m_1 = -\frac{15.0}{30.0} = -0.500, \quad m_2 = -\frac{100}{25.0} = -4.00, \quad (14.65)$$

thus

$$m_{\text{tot}} = (-0.500)(-4.00) = +2.00. \quad (14.66)$$

The final image is real, upright relative to the original object, and twice its height.

Interpretation. Each lens inverts its own object. Two inversions restore the original orientation.

Example 14.15: Intermediate image as a virtual object

Lens 1 has $f_1 = 12.0$ cm and lens 2 has $f_2 = 8.0$ cm. The lenses are $d = 20.0$ cm apart. A real object is $p_1 = 18.0$ cm to the left of lens 1. Find whether the intermediate image is a real or virtual object for lens 2.

Solution.

$$\frac{1}{i_1} = \frac{1}{12.0} - \frac{1}{18.0} = \frac{1}{36.0} \text{ cm}^{-1}, \quad (14.67)$$

so $i_1 = 36.0$ cm. But lens 2 is only 20.0 cm to the right of lens 1. The rays reaching lens 2 are already converging toward a point 16.0 cm to the right of lens 2. Therefore, for lens 2, the object is virtual and one may use

$$p_2 = d - i_1 = 20.0 - 36.0 = -16.0 \text{ cm}. \quad (14.68)$$

Now complete the second stage:

$$\frac{1}{i_2} = \frac{1}{8.0} - \frac{1}{-16.0} = \frac{3}{16} \text{ cm}^{-1}, \quad i_2 = 5.33 \text{ cm}. \quad (14.69)$$

The final image is real, at $z = 25.33$ cm measured from lens 1. The stage magnifications are $m_1 = -2$ and $m_2 = +1/3$, giving $m_{\text{tot}} = -2/3$. A positive second lens increases an already

converging bundle's convergence and brings the focus forward from $z = 36.0$ cm. There is no physical object or screen at the virtual-object location.

14.9.1 Thin lenses in contact

If two thin lenses are in contact, their powers add:

$$\frac{1}{f_{\text{eq}}} = \frac{1}{f_1} + \frac{1}{f_2}, \quad P_{\text{eq}} = P_1 + P_2. \quad (14.70)$$

This relation is widely used for approximate instrument design and for combining corrective lens powers. It is a special case of the ray-transfer matrix method developed in the advanced section. It is not valid for arbitrarily separated lenses without modification.

Guided checks

1. Why is the final magnification of a two-lens system the product $m_1 m_2$?
2. What does a negative object distance for the second lens mean physically?
3. When may two lens powers be added directly?
4. Why can two individually converging lenses form a system whose principal planes are not located at either lens?

Figure 14.9 shows how the second lens intercepts an already converging bundle.

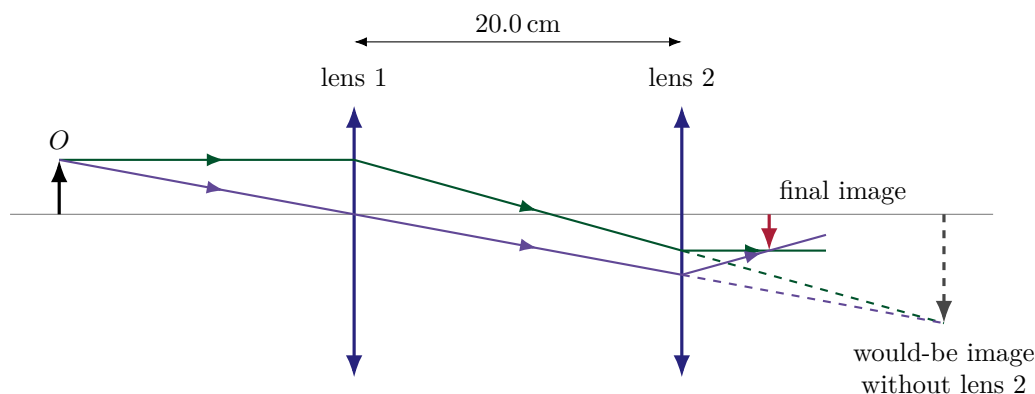


Figure 14.9: Geometry of Example 14.15. The rays reach lens 2 converging toward a point on its right. Dashed lines show the counterfactual continuation without lens 2, not physical rays in the completed system.

14.10 Optical power and the engineering language of diopters

Optical systems are often specified by power rather than focal length. The optical power of a thin lens is

$$P = \frac{1}{f}, \quad (14.71)$$

when f is measured in metres. The unit is the diopter,

$$1 \text{ D} = 1 \text{ m}^{-1}. \quad (14.72)$$

A $+4.0 \text{ D}$ lens has focal length $+0.250 \text{ m}$ and is converging. A -2.0 D lens has focal length -0.500 m and is diverging.

The diopter is convenient because powers of thin lenses in contact add linearly. Lens prescriptions for simple spherical refractive errors are therefore naturally expressed in diopters. In more complete

optometry, cylindrical power and axis are also needed to describe astigmatism, but the scalar spherical-power model is sufficient for the introductory treatment here.

Example 14.16: Combining corrective powers

A thin +2.50 D lens is placed in contact with a −0.75 D lens. Find the equivalent power and focal length.

Solution.

$$P_{\text{eq}} = +2.50 - 0.75 = +1.75 \text{ D.} \quad (14.73)$$

Thus

$$f_{\text{eq}} = \frac{1}{1.75} \text{ m} = 0.571 \text{ m.} \quad (14.74)$$

The positive equivalent power means the pair is converging.

Note

Power is not energy. In optics, “power” in diopters means inverse focal length. Optical radiant power, measured in watts, is a completely different quantity. The context must make the distinction clear.

14.11 Pinhole imaging and the transition to lens imaging

A pinhole camera forms an image without a lens. Rays from each object point that pass through a small aperture continue approximately in straight lines and project an inverted image onto a screen. If the object height is h_o at distance p from the pinhole and the screen is distance i behind it, similar triangles give

$$\frac{h_i}{h_o} = -\frac{i}{p}. \quad (14.75)$$

This resembles the magnification formula for a lens, but there is no focal length: the pinhole does not refract rays to make them converge.

If the aperture is too large, many rays from one object point reach different detector points, causing geometric blur. If the aperture is made extremely small, diffraction becomes important and again broadens the image. Therefore even the pinhole camera provides a natural bridge between geometrical and wave optics.

Example 14.17: Pinhole image size

A 1.8 m-tall object is $p = 6.0$ m from a pinhole camera whose screen is $i = 0.15$ m behind the pinhole. Find the image height.

Solution.

$$h_i = -h_o \frac{i}{p} = -(1.8) \frac{0.15}{6.0} \text{ m} = -0.045 \text{ m.} \quad (14.76)$$

The image is 4.5 cm high and inverted.

14.11.1 Why use a lens?

A lens allows a much larger aperture while still redirecting rays from one object point toward one image point. It therefore gathers far more light than a tiny pinhole while maintaining image formation. The price is that focus becomes conjugate-dependent and aberrations appear.

Key message

The pinhole camera selects rays; a lens redirects rays. Both can produce geometric images, but the lens permits much greater light collection because it actively brings a wider ray bundle from each object point toward the corresponding image point.

14.12 Cameras, projectors and digital imaging sensors

Geometrical optics becomes especially concrete when an optical system must form a real image on a physical surface. A camera sensor, photographic film, projection screen and many machine-vision detectors all impose the same requirement: rays from each object point must be brought to a corresponding image point at the detector plane. The thin-lens equation then becomes not merely a formula for an image position, but a design constraint linking focal length, object distance and sensor location.

14.12.1 A camera as a real-image system

A simplified camera contains a converging lens, an aperture stop and a photosensitive sensor. For an object at distance p and a sensor at distance i behind the lens,

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{i}. \quad (14.77)$$

For a very distant object, $p \gg f$ and therefore $i \simeq f$. Focusing on a nearer object requires moving the effective lens-sensor separation slightly farther from the focal plane. In compact camera modules this motion may be achieved by translating one lens group rather than the entire optical assembly.

Figure 14.10 places the sensor at the common intersection of rays from an object point.

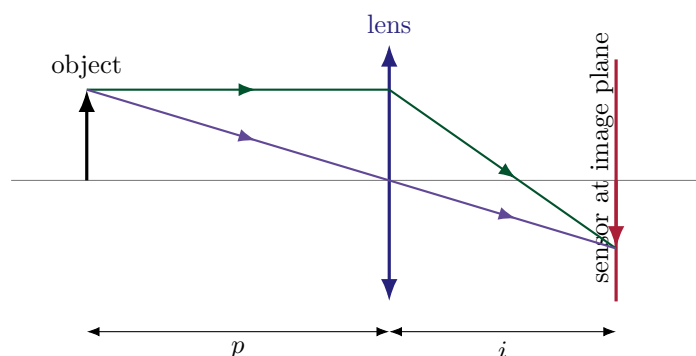


Figure 14.10: A focused camera places the sensor where the rays actually meet. This consistent finite-conjugate model has $i/p = 3/4$ and an inverted sensor image; the focal length is $f = pi/(p + i)$.

The inversion of the optical image is normally irrelevant in a digital camera because image orientation can be handled electronically. What matters physically is that neighboring object points be mapped reproducibly to neighboring sensor points with acceptable sharpness and distortion.

14.12.2 Field of view and sensor size

For a rectilinear thin-lens model focused approximately at infinity, a sensor dimension s subtends a full angular field of view

$$\Theta \simeq 2 \arctan\left(\frac{s}{2f}\right). \quad (14.78)$$

The equation is geometric: the sensor edge lies at transverse distance $s/2$ from the axis and approximately one focal length behind the lens. A shorter focal length therefore gives a larger field

of view for the same sensor size. A longer focal length gives a narrower field of view and a larger image scale for distant objects.

Example 14.18: Field of view of a small imaging sensor

A sensor has width $s = 13.2$ mm and is used with a lens of focal length $f = 8.8$ mm. Estimate the horizontal field of view for a distant scene.

Solution.

$$\Theta \simeq 2 \arctan\left(\frac{13.2}{2(8.8)}\right) \quad (14.79)$$

$$= 2 \arctan(0.750) \approx 73.7^\circ. \quad (14.80)$$

This is an ideal rectilinear projection estimate, not evidence that paraxial thin-lens accuracy extends to a 74° field. It does not specify edge sharpness, distortion or depth of field.

Reasonableness check. Since the sensor half-width is comparable to the focal length, an angle of several tens of degrees is expected.

14.12.3 Aperture and f-number

The aperture stop limits the bundle of rays that can pass through the system. If the effective entrance-pupil diameter is D_a , the f-number is

$$N = \frac{f}{D_a}. \quad (14.81)$$

For the same scene luminance and transmission, the irradiance at the image plane in a simple imaging model scales approximately as

$$E_{\text{image}} \propto \frac{1}{N^2}. \quad (14.82)$$

Thus changing from $N = 4$ to $N = 2$ increases the aperture diameter by a factor of two and the aperture area by a factor of four. This familiar “two-stop” intuition is geometric and photometric; real systems also contain transmission losses, vignetting and wavelength-dependent sensor response.

Caution

The f-number is not simply the physical diameter of a visible glass element. It uses the effective entrance pupil as seen from object space. In multi-element camera lenses, the entrance pupil can differ in size and location from the mechanical aperture.

Example 14.19: Aperture diameter from f-number

A 50.0 mm lens is set to $f/2.0$. Estimate the effective entrance-pupil diameter.

Solution.

$$D_a = \frac{f}{N} = \frac{50.0 \text{ mm}}{2.0} = 25.0 \text{ mm}. \quad (14.83)$$

At $f/4.0$ the effective diameter would be 12.5 mm, and the aperture area would be four times smaller.

14.12.4 Projectors: the camera problem in reverse

A projector places a small illuminated object—for example an LCD or DLP image plane—near the focal region of a converging lens and forms a larger real image on a screen. The same thin-lens

equation applies. Large magnification requires $|m| = i/p$ to be large, so the image distance is much larger than the object distance. The object therefore lies only slightly beyond the focal point.

Example 14.20: Projecting a small image onto a screen

A projector lens has $f = 8.0$ cm. The screen is $i = 3.20$ m from the lens. Find the required object distance and the magnitude of the lateral magnification.

Solution. Using consistent units, $i = 320$ cm,

$$\frac{1}{p} = \frac{1}{8.0} - \frac{1}{320} = 0.121875 \text{ cm}^{-1}, \quad (14.84)$$

so

$$p \approx 8.205 \text{ cm}. \quad (14.85)$$

The magnification is

$$m = -\frac{320}{8.205} \approx -39.0. \quad (14.86)$$

The image is real, inverted and about 39 times larger linearly. Projector electronics or optical layout can compensate for the inversion.

Guided checks

1. Why is a camera sensor normally close to one focal length behind the lens when photographing a distant object?
2. Why does a shorter focal length give a wider field of view for fixed sensor size?
3. Why does reducing the f-number increase image-plane irradiance in a simple geometric model?
4. In a projector with large magnification, why must the object be placed close to, but outside, the focal point?

14.12.5 Focus, zoom and perspective

Students often mix three operations that have different physical meanings.

Focusing

Focusing adjusts the relationship between object distance, effective focal length and sensor plane so that the desired object plane is sharply imaged. The main effect is to satisfy the conjugate condition.

Zooming

Optical zoom changes effective focal length, usually by moving lens groups relative to one another. For a fixed sensor size, changing focal length changes field of view and image scale. A zoom lens is therefore a variable multi-element optical system, not simply a lens translated toward or away from the sensor.

Changing perspective

Perspective depends primarily on camera position relative to the three-dimensional scene. If the camera is moved closer and the focal length is changed to preserve the same framing of one subject, the relative sizes and angular positions of foreground and background objects change because the

viewpoint changed. Focal length controls framing from a given position; camera position controls perspective geometry.

Example 14.21: Same framing, different viewpoint

A portrait fills the same sensor height with a short lens used close to the subject and a long lens used farther away. Why can facial perspective look different even if the final image size on the sensor is the same?

Answer. The two photographs are taken from different positions. Ratios of distances from the camera to the nose, ears and background differ, so the angular geometry of the three-dimensional scene changes. The lens focal length is adjusted mainly to restore framing after the viewpoint has changed.

This distinction is important in computer vision, photogrammetry and technical imaging because geometric reconstruction depends on the camera center and orientation, not only on the nominal lens focal length.

14.12.6 Why close-up imaging needs more exposure

For a simple thin lens with an aperture at the lens and equal entrance and exit pupil diameters, a sensor at distance i sees a cone with $\tan u_i = D_a/(2i)$. Define the working f-number by $N_{\text{eff}} = i/D_a$. For real imaging, $i = f(1 + |m|)$, so

$$N_{\text{eff}} = N(1 + |m|). \quad (14.87)$$

At unit magnification the same aperture gives twice the working f-number and approximately one quarter of the image irradiance compared with infinity focus for the same source radiance. This result assumes a simple pupil geometry and negligible losses; compound lenses require the pupil magnification. It also explains why the $1/N^2$ exposure rule must be used with care at close range.

14.13 The human eye and corrective optics

The human eye is an adaptive optical system rather than a single thin lens. Most of the refraction occurs at the air–cornea interface because the refractive-index contrast is large there. The crystalline lens then provides adjustable additional power. Together they form a real, inverted image on the retina. Neural processing determines our perceived orientation; the fact that the retinal image is inverted is not a defect of the optical system.

14.13.1 Accommodation, near point and far point

For a relaxed normal eye, distant objects can be focused on the retina without strong muscular adjustment. To focus on nearer objects, ciliary muscles alter the shape and optical power of the crystalline lens. This process is called *accommodation*. The nearest point that can be brought into sharp focus with maximum comfortable accommodation is the near point. The far point is the farthest point seen sharply with the eye relaxed.

For introductory calculations it is common to use a conventional reference near-point distance

$$D = 25 \text{ cm}, \quad (14.88)$$

but this is a pedagogical reference value rather than a universal biological constant. Actual near points depend strongly on age and individual vision.

Figure 14.11 shows the equivalent optical power bringing an object point to the retina.

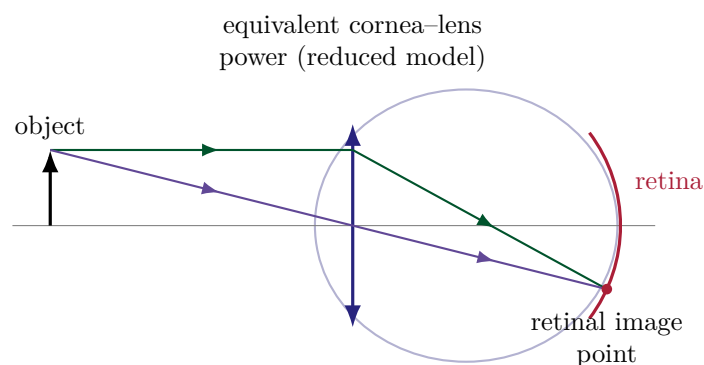


Figure 14.11: Reduced-eye ray model: the cornea and crystalline lens are represented by one equivalent first-order element. The image point lies on the retinal boundary. The outline is illustrative, not anatomical, and the transverse scale is expanded.

14.13.2 Myopia and hyperopia

In *myopia* (short-sightedness), a relaxed eye has too much optical power for its axial length, so parallel rays from a distant object would come to focus in front of the retina. A diverging corrective lens can reduce the total power and make a distant object appear as a virtual image at the uncorrected eye's far point.

In *hyperopia* (long-sightedness), the relaxed eye has insufficient power for its axial length. A converging corrective lens can supply additional power. Presbyopia is the age-related loss of accommodation; its correction is often discussed in terms of the additional positive power required for comfortable near viewing.

Example 14.22: Correcting myopia from the far point

A myopic eye has a far point of 80.0 cm. Estimate the thin-lens power required for spectacles that allow a very distant object to be seen clearly. Assume the spectacle lens is close enough to the eye that the vertex distance can be neglected.

Strategy. The corrective lens must take rays from an object at infinity and form a virtual image at the eye's far point, $i = -0.800$ m.

Solution. For $p \rightarrow \infty$,

$$\frac{1}{f} = \frac{1}{i} = -\frac{1}{0.800 \text{ m}}, \quad (14.89)$$

so

$$f = -0.800 \text{ m}, \quad P = -1.25 \text{ D}. \quad (14.90)$$

The negative power correctly identifies a diverging lens.

Example 14.23: Reading correction for a receded near point

A person can focus no closer than 75 cm without correction. What thin-lens power, placed close to the eye, allows an object at 25 cm to appear at the person's near point?

Solution. The real object is at $p = 0.250$ m and the desired virtual image is at $i = -0.750$ m:

$$\frac{1}{f} = \frac{1}{0.250} + \frac{1}{-0.750} = 4.000 - 1.333 = 2.667 \text{ m}^{-1}. \quad (14.91)$$

Hence

$$f = 0.375 \text{ m}, \quad P = +2.67 \text{ D}. \quad (14.92)$$

A positive-power converging lens is required.

Caution

These examples use the thin-lens approximation and neglect the distance between spectacle lens and eye. In clinical optics, vertex distance, astigmatism, binocular vision and other physiological factors can matter. The calculations here illustrate geometrical-optics principles; they are not a prescription procedure.

Guided checks

1. Why is a diverging lens used to correct simple myopia?
2. In a reading correction, why is the image intentionally virtual?
3. Why must a corrective spectacle lens be analyzed before the eye, rather than treating the spectacle image as the final retinal image?
4. Which part of the eye changes optical power during accommodation?

14.13.3 Beyond a scalar refractive error

The simple eye model treats refractive error as a single scalar power, but real eyes are more complex.

Astigmatism

In astigmatism, optical power differs with meridian. A point object may not focus to a single point at one axial location. Corrective prescriptions therefore include cylindrical power and an axis angle in addition to spherical power. The detailed clinical formalism is beyond this chapter, but the optical concept is accessible: rotational symmetry of the focusing system has been broken.

Presbyopia

Presbyopia is the age-related reduction in accommodative ability. The far-point correction and near-addition problem should be separated conceptually. A person may have clear distance vision but insufficient accommodation for a near task, requiring positive additional power for reading.

Angular resolution of the eye

The eye's ability to distinguish fine detail depends on pupil size, diffraction, aberrations, retinal sampling and neural processing. Therefore visual acuity cannot be predicted from thin-lens focal length alone. This provides a useful warning against overextending geometrical optics.

14.14 Angular size and the simple magnifier

A microscope or telescope does not merely make an image physically larger. What matters to an observer is often the *angular size* of the viewed object or image at the eye. The angular magnification of an optical instrument compares the viewing angle obtained with the instrument to a reference viewing angle without it.

Consider an object of height h at the conventional near-point distance D . Without a lens and for small angles,

$$\theta_0 \simeq \frac{h}{D}. \quad (14.93)$$

A converging magnifying lens of focal length f can be used with the object inside its focal length to produce a larger upright virtual image. If the final image is at infinity, the eye is relaxed and

$$\theta \simeq \frac{h}{f}, \quad (14.94)$$

so the angular magnification is

$$M_\theta \equiv \frac{\theta}{\theta_0} \simeq \frac{D}{f} \quad (\text{relaxed eye}). \quad (14.95)$$

If the final virtual image is instead arranged at the near point, the familiar paraxial result is

$$M_\theta \simeq 1 + \frac{D}{f}. \quad (14.96)$$

The near-point formula assumes the eye is close to the lens. With $i = -D$, the lens equation gives $1/p = 1/f + 1/D$. Since the angle at the lens is approximately h/p , comparison with h/D gives $M_\theta = D/p = 1 + D/f$. This explains both the extra unit and the accommodation requirement.

If the eye is a distance e behind the lens, the virtual image at the eye's near point lies $D - e$ in front of the lens. Then $i = -(D - e)$ and the same geometry gives $M_\theta = 1 + (D - e)/f$. The familiar formula therefore has a specific viewing geometry; it is not solely a property of the glass.

Example 14.24: Angular magnification of a hand lens

A magnifying glass has focal length $f = 5.0$ cm. Estimate its angular magnification for relaxed viewing using $D = 25$ cm.

Solution.

$$M_\theta \simeq \frac{25}{5.0} = 5.0. \quad (14.97)$$

If the final image were arranged at the standard near point, the idealized value would be approximately $1 + 25/5 = 6$.

Key message

Lateral magnification and angular magnification answer different questions. Lateral magnification compares physical image and object sizes. Angular magnification compares viewing angles at the eye. Optical instruments that assist vision are usually best understood in terms of angular magnification.

14.15 Compound microscopes

A compound microscope uses two principal lens groups. The objective, with short focal length, forms a magnified real intermediate image of a nearby object. The eyepiece then acts approximately as a magnifier for that intermediate image. The two stages therefore perform different optical tasks.

14.15.1 Sequential imaging description

For a quantitative design problem, the most reliable method is to use the thin-lens equation twice. Let the objective form an intermediate image at distance i_o with magnification

$$m_o = -\frac{i_o}{p_o}. \quad (14.98)$$

The intermediate image becomes the object for the eyepiece. The eyepiece is usually adjusted so that the final image is virtual, often at infinity for relaxed viewing. The overall visual effect is then characterized by the product of the objective's lateral magnification and the eyepiece's angular magnification.

For a microscope adjusted for a relaxed eye, a common approximate expression is

$$M \simeq -\frac{L}{f_o} \frac{D}{f_e}, \quad (14.99)$$

where $L = i_o - f_o$ is the distance from the objective's rear focal point to the intermediate image, which is at the eyepiece's front focal plane for relaxed viewing. Indeed $i_o = f_o + L$, $p_o = f_o(f_o + L)/L$, and $m_o = -L/f_o$ exactly within the ideal thin-lens model. Thus the lens separation is $f_o + L + f_e$, not L . Some texts use the physical lens separation as an approximate tube length; then the approximation requires that separation to be large compared with both focal lengths. Here the optical definition of L is used consistently.

Figure 14.12 traces the objective image through the eyepiece for relaxed viewing.

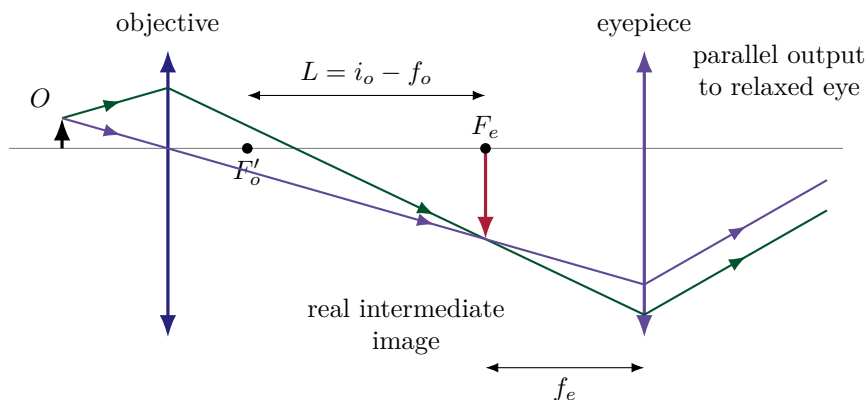


Figure 14.12: A finite-tube microscope adjusted for relaxed viewing. Both rays are propagated continuously through the same intermediate image and leave the eyepiece parallel to each other. Their angle is generally nonzero. The optical tube length L is measured between the marked focal planes.

Example 14.25: Approximate microscope magnification

A microscope has an objective with $f_o = 4.0$ mm, an eyepiece with $f_e = 25$ mm, and an effective tube length $L = 160$ mm. Estimate the relaxed-eye angular magnification using $D = 250$ mm.

Solution.

$$M \simeq -\frac{160}{4.0} \frac{250}{25} = -(40)(10) = -400. \quad (14.100)$$

The negative sign represents inversion relative to the original object in this sign convention. The magnitude indicates the angular enlargement in the idealized paraxial model.

Caveat. A larger nominal magnification does not automatically reveal more detail. Resolution is ultimately limited by diffraction, aberrations, detector sampling or the observer's eye. Diffraction-limited resolution belongs to the wave-optics chapters.

14.15.2 Magnification, resolution and working distance

The approximate microscope magnification

$$M \simeq -\frac{L}{f_o} \frac{D}{f_e} \quad (14.101)$$

can be made numerically large by choosing very short focal lengths. But magnification alone does not determine how much detail is resolved. If the objective fails to transmit distinct information about two nearby object points, a stronger eyepiece merely enlarges the same blurred intermediate image.

This distinction leads to the concept of *empty magnification*: increasing apparent size without increasing resolved detail. The detailed resolution criterion depends on wavelength and numerical aperture and is therefore treated in wave optics. Geometrical optics contributes by defining the image scale and ray collection cone.

Working distance

High-power objectives often have short working distance, meaning the front surface lies close to the specimen. Mechanical access, sample thickness and immersion media therefore become part of the optical design. An objective specification is not just its nominal magnification.

Field number and usable image circle

The eyepiece and intermediate image geometry limit the angular or transverse field that can be viewed without severe aberration or vignetting. In microscopy and imaging systems, usable field is an engineering property distinct from axial magnification.

Caution

Never judge a microscope, telescope or camera lens from magnification alone. Image quality also depends on aperture, aberrations, diffraction, field coverage, detector or eye sampling, illumination and alignment.

14.16 Astronomical telescopes

A telescope is designed primarily to increase the angular size of a distant object. For a distant source, the objective forms a real image near its focal plane. The eyepiece is then used as a magnifier for that image. In a simple Keplerian refracting telescope adjusted so that the final image is at infinity, the separation of the objective and eyepiece is approximately

$$L \simeq f_o + f_e, \quad (14.102)$$

and the angular magnification is

$$M_\theta = -\frac{f_o}{f_e}. \quad (14.103)$$

The negative sign indicates image inversion in the simple Keplerian arrangement.

Example 14.26: Designing a simple refracting telescope

An objective has focal length $f_o = 800$ mm and the eyepiece has $f_e = 20.0$ mm. Estimate the angular magnification and tube length for relaxed viewing.

Solution.

$$M_\theta = -\frac{800}{20.0} = -40.0, \quad (14.104)$$

and

$$L \simeq 800 + 20 = 820 \text{ mm}. \quad (14.105)$$

The telescope gives an angular magnification of magnitude 40 and an inverted image. Terrestrial telescopes commonly add erecting optics if an upright view is desired.

14.16.1 Why large objectives matter

The focal length controls image scale and angular magnification, but the objective diameter controls additional performance properties. A larger aperture gathers more light from faint sources and, as will be developed in wave optics, can also improve diffraction-limited angular resolution. These are distinct roles. A small high-power telescope can have large angular magnification yet a dim or poorly resolved image.

Reflecting telescopes use mirrors rather than a large refracting objective lens. This avoids chromatic aberration of the objective and makes very large apertures mechanically practical. The basic paraxial image-formation language of focal length and real intermediate image remains applicable.

Guided checks

1. In a compound microscope, which element first forms a real image?
2. Why is the eyepiece best described using angular rather than lateral magnification?
3. Why can a telescope with very large nominal angular magnification still reveal little additional detail?
4. Why is the objective focal length much larger than the eyepiece focal length in a Keplerian telescope with $|M_\theta| \gg 1$?

14.16.2 Focal ratio and aperture

For a telescope objective of diameter D_o and focal length f_o , the focal ratio is

$$N_o = \frac{f_o}{D_o}. \quad (14.106)$$

The telescope's angular magnification with a simple eyepiece is

$$|M_\theta| = \frac{f_o}{|f_e|}. \quad (14.107)$$

These are independent design quantities. Two telescopes can have the same focal ratio but different aperture and focal length, or the same magnification with different eyepieces.

Example 14.27: Two telescope configurations

Telescope A has $f_o = 1000$ mm and $D_o = 100$ mm. Telescope B has $f_o = 500$ mm and $D_o = 50$ mm.

Both have

$$N_o = 10. \quad (14.108)$$

With a 25 mm eyepiece, their angular magnifications are respectively

$$|M_A| = 40, \quad |M_B| = 20. \quad (14.109)$$

The equal focal ratio does not imply equal magnification or equal light-gathering area. Telescope A has four times the objective area of B.

Aperture diameter also enters diffraction-limited angular resolution, which is postponed to wave optics. Thus the objective diameter carries information not contained in focal ratio alone.

Figure 14.13 shows the common focal plane and the change in field angle through a Keplerian telescope.

14.17 Optical fibres, prisms and ray guidance

Although image formation is the central theme of this chapter, the same ray approximation underlies many optical transport systems. In a step-index optical fibre, a higher-index core is surrounded by a lower-index cladding. Rays that reach the core-cladding boundary at sufficiently large incidence angle undergo total internal reflection and remain guided. The complete electromagnetic description includes modes, dispersion and wavelength-dependent propagation, but the ray picture provides useful engineering intuition for multimode fibres.

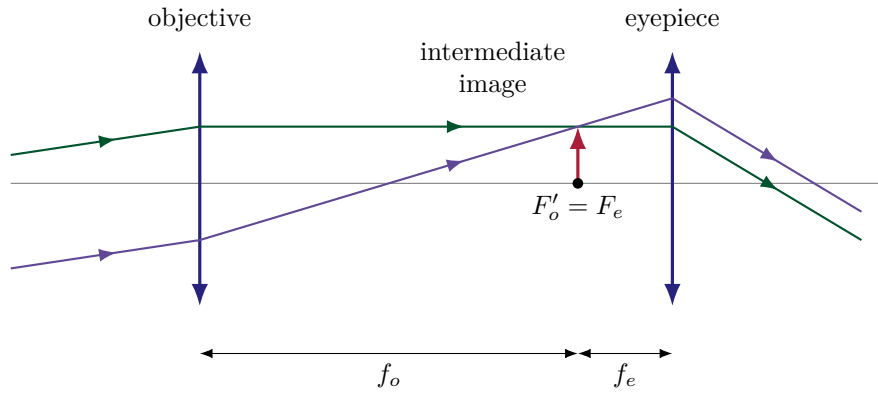


Figure 14.13: Keplerian telescope with a shared focal plane. A parallel input bundle at one field angle becomes a parallel output bundle at the opposite angular sign. The illustrated focal-length ratio is $f_o/f_e = 4$, so $M_\theta = -4$.

For a simple fibre in air with core index n_1 and cladding index $n_2 < n_1$, the acceptance cone is characterized by the numerical aperture

$$\text{NA} = \sqrt{n_1^2 - n_2^2} \quad (14.110)$$

when the external medium is air. More generally, if the external refractive index is n_0 ,

$$n_0 \sin \theta_{\max} = \sqrt{n_1^2 - n_2^2}. \quad (14.111)$$

These formulas follow from Snell's law at the fibre entrance and the critical-angle condition at the core-cladding interface.

Example 14.28: Acceptance angle of a step-index fibre

A fibre has $n_1 = 1.48$ and $n_2 = 1.46$ and is used in air. Estimate its numerical aperture and maximum acceptance half-angle.

Solution.

$$\text{NA} = \sqrt{1.48^2 - 1.46^2} = \sqrt{2.1904 - 2.1316} \quad (14.112)$$

$$= \sqrt{0.0588} = 0.242. \quad (14.113)$$

Thus

$$\theta_{\max} = \sin^{-1}(0.242) \approx 14.0^\circ. \quad (14.114)$$

The full acceptance cone has angular width about 28° in this idealized meridional-ray model.

Prisms exploit refraction and total internal reflection to redirect rays. Because refractive index depends on wavelength, a prism can also separate colors. The detailed wavelength dependence and interference effects are outside the present chapter, but the central engineering message is important: geometrical optics can describe ray routing even when the material's microscopic electromagnetic response is hidden inside a measured refractive index.

14.17.1 Deriving the acceptance cone

Let ϕ be the internal ray angle to the axis and let θ be the external entrance angle. At a flat entrance face normal to the axis, $n_0 \sin \theta = n_1 \sin \phi$. At the straight side wall the incidence angle is $90^\circ - \phi$, so total internal reflection requires $\cos \phi > n_2/n_1$. Combining the limiting equalities gives Eq. (14.111). This is a meridional-ray result for a straight step-index fibre; skew rays and full

electromagnetic modes require additional analysis. If the calculated right-hand side exceeds n_0 , the entrance-angle bound saturates at 90° , rather than giving an impossible sine.

Figure 14.14 distinguishes the entrance angle from the incidence angle at the core boundary.

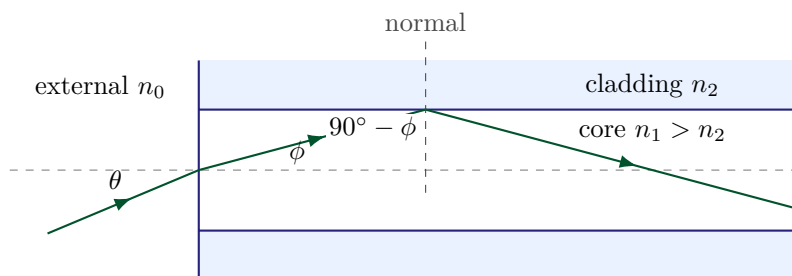


Figure 14.14: A meridional ray in a straight step-index fibre. The entrance angle is measured from the axis; the wall-incidence angle is measured from the vertical normal. These are complementary internal angles. The drawn refraction is consistent with $n_0 = 1$, $n_1 = 1.50$.

14.18 Laboratory methods for measuring focal length

A good optics chapter should connect formulas to measurable procedures. Focal length is not merely a catalog number; it can be inferred from object and image distances, from collimated light, from autocollimation, or from displacement methods. Each approach has different systematic uncertainties.

14.18.1 Direct object–screen method

For a converging lens, place an illuminated object at a measured distance $p > f$ and move a screen until a sharp image is obtained at distance i . The focal length follows from

$$f = \frac{pi}{p + i}. \quad (14.115)$$

Because a real laboratory lens has finite thickness, the correct reference planes are principal planes rather than necessarily the outer glass surfaces. For a thin classroom lens this difference may be small compared with other uncertainties, but it should be recognized.

Example 14.29: Focal length from laboratory distances

A lens forms a sharp real image when an object is $p = (30.0 \pm 0.2)$ cm from the lens and the screen is $i = (20.0 \pm 0.2)$ cm from the lens. Find the nominal focal length.

Solution.

$$f = \frac{(30.0)(20.0)}{30.0 + 20.0} \text{ cm} = 12.0 \text{ cm}. \quad (14.116)$$

If the quoted errors are independent standard uncertainties,

$$\sigma_f^2 = \left[\frac{i^2}{(p+i)^2} \sigma_p \right]^2 + \left[\frac{p^2}{(p+i)^2} \sigma_i \right]^2, \quad (14.117)$$

so $\sigma_f = \sqrt{(0.16 \times 0.2)^2 + (0.36 \times 0.2)^2} \text{ cm} = 0.079 \text{ cm}$. Report $f = (12.00 \pm 0.08) \text{ cm}$ under these assumptions. On one optical rail the two distances can share a lens-position error, so independence must be checked; the advanced measurement discussion gives the covariance correction.

14.18.2 Object–screen separation and the Bessel displacement method

If the object and screen are fixed a distance L apart, a converging lens can produce a sharp real image at two different lens positions provided $L > 4f$. If the separation of the two lens positions is d , then

$$f = \frac{L^2 - d^2}{4L}. \quad (14.118)$$

This result is derived in the advanced section. The displacement cancels a fixed offset in the position indicator of a thin lens. Finite principal-plane separation still changes the relation between the physical object–screen distance and the sum of the optical conjugate distances; Bessel’s method does not remove thick-lens model error automatically.

14.18.3 Autocollimation idea

Place a plane mirror behind a converging lens and an object near the front focal plane. Light passes through the lens, reflects from the mirror, and returns through the lens. When the returning image coincides with the object, the object is at the front focal plane in the paraxial approximation. Autocollimation is widely useful in optical alignment because coincidence can often be judged more accurately than two independent distances.

Method

Laboratory focal-length workflow

1. identify the optical axis and keep object, lens and screen at a common height;
2. begin with p comfortably larger than the estimated focal length;
3. search for the sharpest image rather than the brightest image;
4. record the reference point used for all distance measurements;
5. repeat the measurement at several object distances;
6. compare focal-length estimates and look for systematic trends;
7. report uncertainty and discuss whether the thin-lens approximation is adequate.

Guided checks

1. Why can a real image be captured on a screen but a virtual image cannot be captured by simply placing a screen at the apparent image location?
2. Why is it useful to repeat focal-length measurements at several object distances?
3. What practical ambiguity arises because a real lens has finite thickness?
4. In the Bessel method, why must L be larger than $4f$?

14.18.4 Measured quantities and model-dependent inferences

A disciplined laboratory record distinguishes directly measured quantities from derived quantities.

Quantity	Typical measurement	Main caution
Object distance p	ruler or optical rail	choose and document the lens reference plane
Image distance i	screen position	“best focus” has finite judgment uncertainty
Focal length f	derived from p, i or Bessel method	model assumes paraxial conditions

Quantity	Typical measurement	Main caution
Image height h_i	ruler, sensor pixels	calibration and perspective can matter
Aperture diameter	caliper or known iris setting	entrance pupil may differ from mechanical opening
Field of view	angular target or known scene geometry	distortion affects edge mapping
Distortion	calibration grid	target geometry and camera pose must be known
Chromatic focal shift	best focus versus wavelength	source bandwidth and sensor spectral response matter

A good report should state which columns are measurements and which are model-dependent inferences. This distinction is transferable to every area of experimental physics.

14.19 Computational ray tracing: from equations to an algorithm

Ray tracing is an excellent bridge between geometrical optics and computational science. A ray may be represented by a position and a direction. The program repeatedly finds the next interface, computes the local surface normal, applies reflection or refraction, and propagates the ray to the next surface. Even a simple two-dimensional code reveals why sign conventions are bookkeeping devices: the geometry itself determines the direction of each ray.

14.19.1 Paraxial thin-lens ray tracing

For a first computational model, represent a ray by its height y and small slope $\theta \simeq dy/dz$. Free propagation over a distance d changes the ray according to

$$y_{\text{out}} = y_{\text{in}} + d\theta_{\text{in}}, \quad \theta_{\text{out}} = \theta_{\text{in}}. \quad (14.119)$$

An ideal thin lens changes the slope but not the height:

$$y_{\text{out}} = y_{\text{in}}, \quad \theta_{\text{out}} = \theta_{\text{in}} - \frac{y_{\text{in}}}{f}. \quad (14.120)$$

These two rules are the computational content of the ray-transfer matrices introduced later.

```
# Pseudocode: paraxial thin-lens ray tracing
y, theta = y0, theta0
for element in optical_system:
    if element.type == "space":
        y = y + element.length * theta
    if element.type == "thin_lens":
        theta = theta - y / element.focal_length
save(y, theta)
```

A useful student exercise is to launch several rays from one object point with different initial slopes. If the detector plane is at the correct image distance, the rays should intersect at the same height to numerical precision within this ideal first-order model. Moving the detector away from that plane produces a finite blur bundle.

14.19.2 Numerical experiment: sweeping object distance

Set $f = 10$ cm and vary the object distance from $p = 30$ cm to $p = 5$ cm, sampling on both sides of $p = f$ but excluding that singular point. For each value, compute

$$i = \frac{fp}{p-f}, \quad m = -\frac{f}{p-f}. \quad (14.121)$$

The numerical output illustrates three regimes:

1. $p > f$: $i > 0$, giving a real image;
2. $p \rightarrow f^+$: $i \rightarrow +\infty$;
3. $p < f$: $i < 0$, giving a virtual image.

The divergence at $p = f$ is not a numerical failure; it expresses the ideal thin-lens prediction that outgoing rays are parallel.

Key message

Computational ray tracing should reproduce the analytic formulas in simple cases before it is trusted for more complicated systems. Verification against a known thin-lens or mirror result is the optical analogue of a unit test in software engineering.

14.20 Engineering interpretation: tolerances, alignment and image quality

Ideal formulas assume perfect surfaces, exact alignment and paraxial rays. Real optical engineering adds tolerances. A lens may be displaced sideways (decentered), tilted, separated incorrectly from another element, or manufactured with a radius slightly different from its nominal value. The sensor may be tilted relative to the image plane. These errors can shift focus, introduce asymmetry or produce field-dependent blur.

A useful first-order design habit is to identify which parameter most strongly affects the required image plane. Differentiating the thin-lens equation gives a local sensitivity relation. From

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \quad (14.122)$$

with fixed f ,

$$-\frac{dp}{p^2} - \frac{di}{i^2} = 0, \quad (14.123)$$

so

$$\frac{di}{dp} = -\frac{i^2}{p^2} = -m^2. \quad (14.124)$$

Near unit magnification, moving the object by a small distance moves the image by a comparable distance in the opposite direction. At large magnification, the image-plane sensitivity can be much larger.

Example 14.30: Focus sensitivity at macro magnification

A system operates at $m = -2.0$. Estimate the first-order image-plane shift caused by moving the object 0.50 mm farther from the lens while the focal length remains fixed.

Solution. Equation (14.124) gives

$$di \simeq -m^2 dp = -(4)(0.50 \text{ mm}) = -2.0 \text{ mm}. \quad (14.125)$$

The image plane moves toward the lens by about 2.0 mm. This explains why close-up imaging can require precise focusing mechanics.

Note

Image quality is broader than focus. Even if the paraxial image plane is placed correctly, aberrations, diffraction, scattering, sensor sampling and motion can reduce sharpness. Geometrical optics predicts the ideal mapping of rays; later chapters add wave-optical limitations.

14.20.1 Alignment and diagnostic signatures

The optical axis is straightforward for an ideal rotationally symmetric lens or mirror: it is the line through the centers of curvature or symmetry centers. In a real multi-element system, each component can have its own mechanical and optical axis. Decenter and tilt create misalignment aberrations and shift the chief rays.

Simple alignment procedure

A laboratory alignment can proceed hierarchically:

1. establish a reference rail or beam direction;
2. center each lens or aperture on the nominal beam height;
3. make the element planes approximately perpendicular to the axis where required;
4. use back-reflections or autocollimation to refine tilt;
5. verify that the image does not walk laterally as focus is adjusted;
6. test several field points rather than only the center.

Decenter and tilt signatures

A centered defocus produces an approximately symmetric blur for an on-axis point. Decenter and tilt can produce asymmetric blur, image displacement and field-dependent errors. This diagnostic distinction is useful before any sophisticated optical model is attempted.

Example 14.31: Diagnosing sensor tilt

A flat calibration target is sharply focused on the left side of an image when the right side is slightly blurred. Moving the sensor or focus makes the right side sharp while the left becomes blurred. One plausible cause is relative tilt between the target plane, lens conjugate plane and sensor plane. Field curvature can produce a superficially similar symptom, so rotating or translating the target and comparing best-focus surfaces can help distinguish them.

14.21 A reliable solution and diagnostic method

The most serious errors combine a correct formula with an incorrect physical model: using the wrong reference plane, mistaking a virtual object for a real one, or applying a paraxial equation to marginal rays. Keep a sign-and-position table for each optical element and use limiting cases as independent checks.

Method

The draw–predict–calculate–check method

1. **Draw:** show the optical axis, element, object, relevant focal points and direction of light.
2. **Predict:** decide qualitatively whether the image should be real or virtual, upright or inverted, enlarged or reduced.
3. **Calculate:** apply the chosen sign convention consistently.
4. **Check:** compare the algebraic signs and magnitudes with the ray diagram, dimensions and limiting cases.

14.22 Guided checks for the main chapter

Guided checks

1. A real object approaches a converging lens from far away. Describe how the real image position changes as the object moves toward the focal point from outside.
2. Why does a diverging lens with a real object not form a real image by itself?
3. A mirror produces $m = +0.40$. What can be inferred about orientation and relative size? What additional information is needed to identify the mirror uniquely?
4. A lens has $f = -20$ cm and a real object at $p = 30$ cm. Before calculating, predict the sign of i and m .
5. Why can a plane mirror be regarded as the $R \rightarrow \infty$ limit of a spherical mirror?
6. Why is the lens-maker equation sensitive to the sign assigned to R_1 and R_2 ?
7. In a two-lens system, what physical situation produces a negative object distance for lens 2?
8. Why does a camera need a focusing mechanism even though the focal length of its lens may remain almost constant?
9. What is the difference between “magnification 5” for a photograph and “angular magnification 5” for a hand lens?
10. Which optical limitations are fundamentally outside pure geometrical optics and require a wave description?

Brief feedback

The real image recedes as p approaches f from above; a negative lens with a real object gives $1/i = 1/f - 1/p < 0$. A magnification $+0.40$ means upright and reduced; for a single spherical mirror and a real object it identifies a convex mirror. For $f = -20$ cm and $p = 30$ cm, $i = -12$ cm and $m = +0.40$. A plane mirror has zero power but still reflects, so $i = -p$. Lens-maker radii encode the two surface normals; a virtual second object describes an already converging incident bundle. A camera must follow the changing conjugate distance. Lateral and angular magnifications compare lengths and angles respectively. Interference and finite-aperture diffraction require wave amplitudes and phase.

14.23 Main chapter summary

Summary

- Geometrical optics represents light by rays and is most reliable when the wavelength is much smaller than the dimensions over which the optical geometry varies.
- Fermat's principle and Snell's law provide a physical basis for ray paths through media. Reflection, refraction and total internal reflection determine how rays are redirected.
- A real image is formed where rays physically converge. A virtual image is formed where backward extensions of rays converge. Both are physically meaningful optical constructions.
- Plane mirrors form upright virtual images of equal size. Spherical mirrors obey the mirror equation in the paraxial approximation, with $f = R/2$.
- Refraction at a spherical surface is described by $n_1/p + n_2/i = (n_2 - n_1)/R$ with a declared sign convention.
- Thin lenses obey $1/f = 1/p + 1/i$ and $m = -i/p$ in the paraxial approximation. The lens-maker equation connects focal length with refractive index and surface curvatures.
- In multi-lens systems, the image produced by one element becomes the object for the next. Overall lateral magnification is the product of the stage magnifications.
- Optical power $P = 1/f$ is measured in diopters when f is in metres. Powers add directly only for thin lenses in contact.
- Cameras and projectors are real-image systems. Magnifiers, microscope eyepieces and telescope eyepieces are understood primarily through angular magnification.
- The eye is an adaptive optical system whose cornea and crystalline lens focus a real image on the retina. Simple refractive errors can be modelled with corrective thin lenses.
- Laboratory and computational ray tracing make the sign convention operational: geometry, rather than memorized signs, determines where rays go.
- Real optical systems also involve apertures, aberrations, diffraction, tolerances and detector limitations. The advanced section develops several of these ideas while wave-optical limits are reserved for later chapters.

Formula summary

The mirror/lens equations below are paraxial; the thin lens has the same medium on both sides. The field-of-view estimate assumes rectilinear distant-object projection. The fibre relation assumes a straight step-index guide and an accessible entrance cone. Other validity conditions are given beside each derivation.

Snell:	$n_1 \sin \theta_1 = n_2 \sin \theta_2,$
critical angle:	$\sin \theta_c = \frac{n_2}{n_1} \quad (n_1 > n_2),$
mirror/lens:	$\frac{1}{f} = \frac{1}{p} + \frac{1}{i},$
spherical mirror:	$f = \frac{R}{2},$
lateral magnification:	$m = \frac{h_i}{h_o} = -\frac{i}{p},$
spherical refraction:	$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{R},$
lens maker in air:	$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right),$
optical power:	$P = \frac{1}{f} \quad [P] = \text{D} = \text{m}^{-1},$
f-number:	$N = \frac{f}{D_a},$
field of view:	$\Theta \simeq 2 \arctan \left(\frac{s}{2f} \right),$
simple magnifier:	$M_\theta \simeq \frac{D}{f} \quad (\text{relaxed eye}),$
Keplerian telescope:	$M_\theta = -\frac{f_o}{f_e}, \quad L \simeq f_o + f_e,$
fibre numerical aperture:	$n_0 \sin \theta_{\max} = \sqrt{n_1^2 - n_2^2}.$

Take-home message

Draw first, calculate second. The ray diagram carries the physical meaning; the equations make it quantitative. A sign convention is useful only when it faithfully encodes the geometry shown in the diagram.

14.24 Advanced Topics for Technical and Engineering Students

The following modules extend the core with first-order system design, nonparaxial ray tracing, pupils, aberrations, measurement uncertainty and detector limits. Sections on matrices and reduced angles form one sequence; the measurement and calibration modules can be read independently after the core. This material is optional for the basic Physics II course.

14.25 The paraxial approximation as a controlled expansion

The word “paraxial” means close to the optical axis both in position and in direction. It is not a binary property of a ray; it is an approximation whose accuracy improves as the relevant angles decrease. For $|\theta| \ll 1$ rad,

$$\sin \theta = \theta - \frac{\theta^3}{6} + \mathcal{O}(\theta^5), \quad (14.126)$$

$$\tan \theta = \theta + \frac{\theta^3}{3} + \mathcal{O}(\theta^5), \quad (14.127)$$

$$\cos \theta = 1 - \frac{\theta^2}{2} + \mathcal{O}(\theta^4). \quad (14.128)$$

Replacing $\sin \theta$ and $\tan \theta$ by θ discards relative corrections of order θ^2 . For example, at $\theta = 0.10$ rad, the difference between $\tan \theta$ and θ is about 0.3%; at 0.30 rad it is already several percent.

Example 14.32: Quantifying a small-angle approximation

Compare $\tan \theta$ with θ for $\theta = 5^\circ$ and 20° .

Solution. In radians, $5^\circ = 0.08727$ and $20^\circ = 0.3491$.

$$\tan 5^\circ = 0.08749, \quad \frac{\tan \theta - \theta}{\theta} \approx 0.25\%, \quad (14.129)$$

$$\tan 20^\circ = 0.3640, \quad \frac{\tan \theta - \theta}{\theta} \approx 4.27\%. \quad (14.130)$$

The same optical formula can therefore be excellent at one numerical aperture and visibly inaccurate at another.

14.25.1 Paraxial optics as linear optics

The deeper advantage of the paraxial approximation is that ray propagation becomes linear in ray height and angle. Nonlinear trigonometric relations are replaced by linear transformations. That is why complex first-order optical systems can be represented by 2×2 matrices. It is also why the superposition of small deviations and tolerance sensitivities becomes tractable.

Guided checks

1. Why are corrections to $\sin \theta \simeq \theta$ cubic in the angle while the relative error is quadratic?
2. Why does paraxial optics naturally lead to linear algebra?
3. Which rays in a wide-aperture lens are most likely to violate the approximation?

14.26 Newtonian form of the imaging equation

The Gaussian lens equation is usually written with distances measured from the lens or principal planes. An alternative form becomes elegant when distances are measured from the focal points. Let a real object lie a distance x beyond the front focal point and let the real image lie a distance x' beyond the rear focal point. Then

$$p = f + x, \quad i = f + x'. \quad (14.131)$$

Substituting into

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \quad (14.132)$$

gives

$$\frac{1}{f+x} + \frac{1}{f+x'} = \frac{1}{f}. \quad (14.133)$$

After multiplication and cancellation,

$$\boxed{xx' = f^2}. \quad (14.134)$$

This is the Newtonian form of the conjugate relation. It makes the symmetry between object and image especially visible. If the object lies only a small distance x beyond the front focal plane, the image lies a large distance $x' = f^2/x$ beyond the rear focal plane.

The magnification can also be expressed as

$$m = -\frac{i}{p} = -\frac{f+x'}{f+x}. \quad (14.135)$$

Using $xx' = f^2$, one can show the equivalent forms

$$m = -\frac{f}{x} = -\frac{x'}{f} \quad (14.136)$$

for the corresponding signed focal-coordinate convention.

Example 14.33: Newtonian conjugates

A positive thin lens has $f = 50$ mm. An object is placed $x = 25$ mm beyond the front focal point. Find the focal-coordinate image distance x' and the ordinary object and image distances.

Solution.

$$x' = \frac{f^2}{x} = \frac{(50 \text{ mm})^2}{25 \text{ mm}} = 100 \text{ mm}. \quad (14.137)$$

Therefore

$$p = f + x = 75 \text{ mm}, \quad i = f + x' = 150 \text{ mm}. \quad (14.138)$$

A direct check gives

$$\frac{1}{75} + \frac{1}{150} = \frac{1}{50} \text{ mm}^{-1}. \quad (14.139)$$

The magnification is $m = -150/75 = -2$, consistent with $m = -f/x = -50/25 = -2$.

Note

The Newtonian form is not a different physical law. It is the same first-order imaging relation expressed using focal points as coordinate origins. It can be especially convenient in laboratory optics and multi-element system analysis.

14.27 ABCD ray-transfer matrices

First-order optical systems can be represented compactly by matrices. In a region of constant refractive index, describe a paraxial ray by the column vector

$$\mathbf{r} = \begin{pmatrix} y \\ \theta \end{pmatrix}, \quad (14.140)$$

where y is ray height and θ is the small angle to the axis. An optical subsystem transforms the ray according to

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}. \quad (14.141)$$

The matrix is often called an ABCD matrix.

14.27.1 Translation matrix

Free propagation through distance d gives

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}}_{T(d)} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}. \quad (14.142)$$

14.27.2 Thin-lens matrix

A thin lens of focal length f gives

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}}_{L(f)} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}. \quad (14.143)$$

14.27.3 Order matters

If a ray first passes through subsystem M_1 and then through M_2 , the total transformation is

$$M_{\text{tot}} = M_2 M_1. \quad (14.144)$$

Matrix multiplication is not commutative, which is physically appropriate: a lens followed by a propagation distance is not the same operation as propagation followed by the lens.

Example 14.34: Recovering the thin-lens imaging equation from matrices

An object plane is a distance p before a thin lens and an image plane is a distance i after it. Show that the condition for imaging is the thin-lens equation.

Solution. The total matrix from object plane to image plane is

$$M = T(i)L(f)T(p). \quad (14.145)$$

Multiplying,

$$M = \begin{pmatrix} 1 - i/f & p + i - pi/f \\ -1/f & 1 - p/f \end{pmatrix}. \quad (14.146)$$

For a point in the object plane to map to a point in the image plane independent of the initial ray angle, the coefficient multiplying θ_1 in y_2 must vanish. Therefore $B = 0$:

$$p + i - \frac{pi}{f} = 0. \quad (14.147)$$

Dividing by pi gives

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f}. \quad (14.148)$$

The imaging equation is therefore equivalent to the matrix condition $B = 0$.

14.27.4 Magnification from the system matrix

When $B = 0$,

$$y_2 = Ay_1, \quad (14.149)$$

so the lateral magnification is simply

$$m = A. \quad (14.150)$$

For the object–lens–image system above, $A = 1 - i/f = -i/p$ after use of the imaging equation, reproducing the familiar result.

Key message

ABCD matrices do not replace geometrical optics; they organize it. Each matrix is a first-order ray rule. The power of the method comes from composing many simple rules without re-deriving the geometry for every system.

14.28 Separated thin lenses and effective focal length

Consider two thin lenses of focal lengths f_1 and f_2 separated by distance d in air. The matrix from just before lens 1 to just after lens 2 is

$$M = L(f_2)T(d)L(f_1). \quad (14.151)$$

Multiplication gives

$$M = \begin{pmatrix} 1 - d/f_1 & d \\ -\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}\right) & 1 - d/f_2 \end{pmatrix}. \quad (14.152)$$

For a system in air, the effective focal length F is related to the matrix element C by $C = -1/F$, so

$$\boxed{\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}}. \quad (14.153)$$

When $d \rightarrow 0$, the familiar power-addition formula is recovered.

Example 14.35: Effective focal length of separated lenses

Two positive lenses have $f_1 = 100$ mm, $f_2 = 50$ mm and separation $d = 20$ mm. Find the effective focal length.

Solution.

$$\frac{1}{F} = \frac{1}{100} + \frac{1}{50} - \frac{20}{(100)(50)} \text{ mm}^{-1} \quad (14.154)$$

$$= 0.010 + 0.020 - 0.004 = 0.026 \text{ mm}^{-1}. \quad (14.155)$$

Thus

$$F = 38.46 \text{ mm}. \quad (14.156)$$

If the lenses were in contact, the result would be 33.33 mm. The separation therefore changes the first-order power appreciably.

14.29 Principal planes and cardinal points

A multi-element system has an effective focal length, but the rear lens surface is generally not the origin from which that length is measured. This matters immediately when placing a detector. Let the matrix from a front reference plane V_1 to a rear reference plane V_2 , in air, be

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad AD - BC = 1, \quad C \neq 0. \quad (14.157)$$

Factor it into translations and one equivalent lens:

$$M = T(b)L(F)T(a), \quad F = -\frac{1}{C}. \quad (14.158)$$

Multiplication gives $A = 1 - b/F$ and $D = 1 - a/F$, hence

$$a = F(1 - D) = \frac{D - 1}{C}, \quad b = F(1 - A) = \frac{A - 1}{C}. \quad (14.159)$$

The front principal plane H_1 is at signed distance a to the right of V_1 . The rear principal plane H_2 is at signed distance $-b$ relative to V_2 . The equivalent lens maps the two principal planes with unit lateral magnification. These are reference planes of the first-order mapping, not places where all physical rays suddenly bend.

A parallel input ray has output height Ay and slope Cy . It reaches the axis a distance

$$\text{BFL} = -\frac{A}{C} = AF \quad (14.160)$$

after V_2 . This back focal distance differs from F unless $A = 1$. With V_1 at $z = 0$ and V_2 at $z = d$, object and image distances are $p_H = a - z_O$ and $i_H = z_I - (d - b)$; they obey $1/p_H + 1/i_H = 1/F$.

Example 14.36: Where should the sensor actually go?

Two thin lenses in air have $f_1 = 60 \text{ mm}$, $f_2 = 40 \text{ mm}$ and separation $d = 15 \text{ mm}$. Find the distant-object sensor position relative to lens 2.

Strategy. Compute the system matrix, then use the outgoing parallel-ray bundle. An effective focal length alone does not locate the sensor relative to the housing.

$$M = \begin{pmatrix} 0.750 & 15.0 \text{ mm} \\ -0.0354167 \text{ mm}^{-1} & 0.625 \end{pmatrix}. \quad (14.161)$$

Thus $F = 28.235 \text{ mm}$, $a = 10.588 \text{ mm}$, and $b = 7.059 \text{ mm}$. The rear principal plane lies

7.059 mm before lens 2, so

$$\text{BFL} = F - b = 21.176 \text{ mm.} \quad (14.162)$$

Placing the sensor one effective focal length behind lens 2 would miss focus by 7.059 mm. The direct ray check $Ay + \text{BFL}Cy = 0$ confirms the result for every paraxial input height y .

For equal refractive indices at input and output, nodal and principal points coincide. When the two exterior indices differ, front and rear focal lengths are unequal and the reduced-coordinate treatment is needed. Afocal systems have $C = 0$ and cannot be assigned a finite F or this finite-principal-plane factorization; use their full matrix directly.

Figure 14.15 separates focal distance from principal planes and back focal distance from the rear vertex.

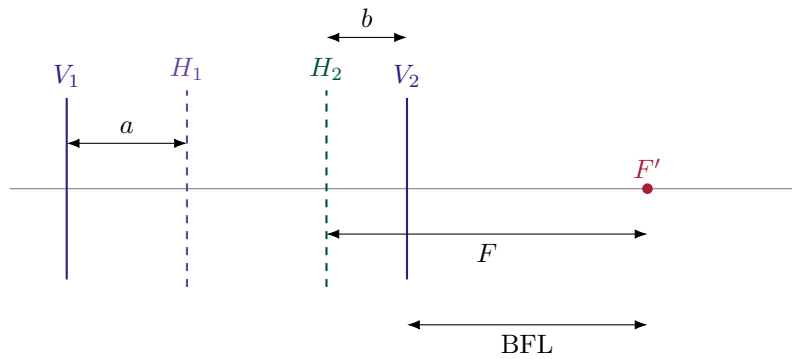


Figure 14.15: Reference-plane schematic, not to scale. The effective rear focal length begins at H_2 , while the mechanical back focal distance begins at V_2 . Their difference is the principal-plane offset b .

14.30 Refraction matrices and index-weighted ray coordinates

The ray vector (y, θ) is convenient when the refractive index is the same throughout a sequence of spaces. At refracting interfaces, an alternative coordinate

$$\mathbf{q} = \begin{pmatrix} y \\ n\theta \end{pmatrix} \quad (14.163)$$

keeps the matrix algebra especially simple. The second component is sometimes called the reduced angle.

14.30.1 Translation through a medium

If a ray travels a physical distance d in refractive index n , then

$$y_2 = y_1 + d\theta_1 = y_1 + \frac{d}{n}(n\theta_1). \quad (14.164)$$

Therefore

$$T_n(d) = \begin{pmatrix} 1 & d/n \\ 0 & 1 \end{pmatrix}. \quad (14.165)$$

14.30.2 Refraction at a spherical surface

For a spherical interface of radius R from n_1 to n_2 , the paraxial refraction rule can be written

$$n_2\theta_2 = n_1\theta_1 - \Phi y, \quad (14.166)$$

where the surface power is

$$\Phi = \frac{n_2 - n_1}{R}. \quad (14.167)$$

Thus

$$S(\Phi) = \begin{pmatrix} 1 & 0 \\ -\Phi & 1 \end{pmatrix}. \quad (14.168)$$

The determinant remains one. This formulation allows a thick lens to be built from a surface matrix, a translation through glass, and a second surface matrix.

Example 14.37: Matrix of a thick biconvex lens

A lens of index $n = 1.50$ in air has $R_1 = +50$ mm, $R_2 = -50$ mm and center thickness $t = 10$ mm. Construct the first-order reduced-coordinate matrix from just before the first surface to just after the second.

Solution. The first surface power is

$$\Phi_1 = \frac{1.50 - 1.00}{50} = 0.0100 \text{ mm}^{-1}, \quad (14.169)$$

while the second is

$$\Phi_2 = \frac{1.00 - 1.50}{-50} = 0.0100 \text{ mm}^{-1}. \quad (14.170)$$

Translation through the glass is

$$T_{1.5}(10) = \begin{pmatrix} 1 & 6.6667 \text{ mm} \\ 0 & 1 \end{pmatrix}. \quad (14.171)$$

The complete matrix is

$$M = S(\Phi_2)T_{1.5}(10)S(\Phi_1). \quad (14.172)$$

Carrying out the multiplication gives approximately

$$M = \begin{pmatrix} 0.9333 & 6.6667 \text{ mm} \\ -0.01933 \text{ mm}^{-1} & 0.9333 \end{pmatrix}. \quad (14.173)$$

Because the surrounding medium is air, the effective power is $\Phi_{\text{eq}} = -C \simeq 0.01933 \text{ mm}^{-1}$, corresponding to

$$F \simeq 51.7 \text{ mm}. \quad (14.174)$$

The thin-lens approximation would give $f = 50.0$ mm, so the finite thickness shifts the first-order result by several percent in this example.

14.31 Thick-lens equivalent power and the Gullstrand relation

For two refracting surfaces of powers Φ_1 and Φ_2 separated by thickness t in index n , multiplication of the reduced-coordinate matrices yields the equivalent power

$$\Phi = \Phi_1 + \Phi_2 - \frac{t}{n} \Phi_1 \Phi_2. \quad (14.175)$$

This is a form of the Gullstrand equation. It is the refracting-surface analogue of the separated-thin-lens power formula. The correction term shows explicitly how finite thickness changes the total first-order power.

For a lens in air,

$$\Phi_1 = \frac{n-1}{R_1}, \quad \Phi_2 = \frac{1-n}{R_2}. \quad (14.176)$$

Substitution gives the thick-lens lens-maker form

$$\frac{1}{F} = (n-1) \left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)t}{nR_1R_2} \right]. \quad (14.177)$$

The last term disappears as $t \rightarrow 0$.

Caution

The effective focal length of a thick lens is not, by itself, enough to locate a sensor or object. The principal planes are displaced from the physical surfaces, so front focal distance, back focal distance and effective focal length are generally different quantities.

For the symmetric lens of Example 14.37, $A = D = 0.93333$ and $F = 51.724$ mm. Equations (14.159) give $a = b = 3.448$ mm. Thus the back focal distance is 48.276 mm, even though the effective focal length is greater than 50 mm. Effective focal length and distance from the last glass surface can change in different directions.

14.32 Afocal systems, beam expanders and telescope matrices

An afocal system takes a collimated input beam into a collimated output beam. Its effective focal length is infinite and its net first-order power is zero. In matrix language, an afocal system in the same medium on both sides has

$$C = 0. \quad (14.178)$$

For two positive lenses separated by $d = f_1 + f_2$, Eq. (14.153) gives

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{f_1 + f_2}{f_1 f_2} = 0. \quad (14.179)$$

This is the Keplerian telescope condition for relaxed viewing.

The transverse scaling of a collimated beam can be obtained from the ray matrix. For a Keplerian beam expander with input lens focal length f_1 and output lens focal length f_2 , the beam-diameter ratio in the ideal paraxial limit is

$$\left| \frac{D_{\text{out}}}{D_{\text{in}}} \right| = \left| \frac{f_2}{f_1} \right|. \quad (14.180)$$

The angular divergence scales inversely, illustrating conservation of the first-order optical invariant.

Example 14.38: Keplerian beam expander

A 10 mm focal-length positive lens is followed by a 50 mm positive lens, separated by 60 mm. A collimated beam enters the short lens. Find the ideal beam expansion ratio.

Solution. The spacing equals $f_1 + f_2$, so the system is afocal. The diameter increases by

$$\frac{D_{\text{out}}}{D_{\text{in}}} = \frac{50}{10} = 5. \quad (14.181)$$

The output angular divergence is reduced by the corresponding factor in the first-order model.

A Galilean beam expander uses one negative and one positive lens and can be shorter because the lens separation is approximately $f_2 - |f_1|$. Unlike the Keplerian arrangement, it does not form a real internal focus, which can be advantageous for high optical powers where an internal focus would create high intensity.

14.32.1 Finite-plane imaging with an afocal pair

A relay lens system transfers an image from one plane to another. A simple idealized arrangement uses two identical positive lenses of focal length f separated by $2f$, with the object plane one focal length before the first lens and the output plane one focal length after the second. Rays from each object point leave the first lens collimated and are focused by the second lens. The two-lens pair is afocal for collimated input, yet the complete object-plane-to-image-plane system also images finite conjugates.

For equal focal lengths, the transverse magnification is exactly -1 within the ideal paraxial model for the stated focal planes. Such layouts are important because the shared rear/front focal plane midway between the lenses can host a spatial filter. The collimated bundles between lenses are tilted differently for different object points; collimated does not mean that every ray is parallel to the optical axis.

Figure 14.16 traces the parallel inter-lens bundle of the finite-conjugate relay.

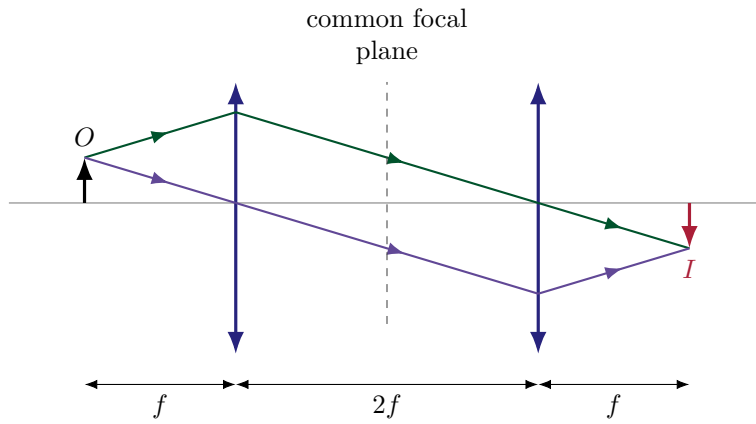


Figure 14.16: A calculated $4f$ relay with $m = -1$. Rays from the off-axis point form a parallel, tilted bundle between the lenses. The dashed middle plane is the common focal plane, not an extra refracting element.

Indeed the complete relay has

$$T(f)L(f)T(2f)L(f)T(f) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (14.182)$$

The condition $B = 0$ confirms imaging, while $C = 0$ confirms zero net first-order power. The two conditions refer to different properties and can hold simultaneously.

14.33 Aperture stops, pupils and numerical aperture

An optical system does not accept every geometrically possible ray. Mechanical mounts and finite lens diameters restrict the ray bundles. The element that most directly limits the cone of rays from an on-axis object point is the *aperture stop*. Its image through the optical elements before it is the *entrance pupil*; its image through elements after it is the *exit pupil*.

The f-number $N = f/D_{\text{ep}}$ uses the entrance-pupil diameter D_{ep} for a distant object. Another common measure is the numerical aperture,

$$\text{NA} = n \sin u, \quad (14.183)$$

where u is the marginal-ray half-angle in a medium of refractive index n . For a single thin lens with the stop at the lens, focused on a distant object in air, the image-space cone has

$$\text{NA} \simeq \sin u \simeq u \simeq \frac{D_{\text{ep}}}{2f} = \frac{1}{2N}. \quad (14.184)$$

For a compound system one must use the exit-pupil diameter and its distance from the image to determine the image-space cone. Substituting the entrance-pupil diameter into a local image-space triangle without justification can be wrong. Object-space and image-space numerical apertures are different quantities, especially at finite conjugates. At high aperture, retain the sine definition and use a model beyond first-order optics.

14.33.1 Throughput and the optical invariant

There is a trade-off between ray-bundle width and angular spread. In first-order lossless optics, the Lagrange invariant is conserved. One common form for two paraxial rays is

$$H = n(y_1\theta_2 - y_2\theta_1) = \text{constant}. \quad (14.185)$$

For imaging systems this expresses a fundamental constraint: one cannot arbitrarily compress both spatial extent and angular extent using passive ideal optics. The more general phase-space concept is related to optical *etendue*.

Note

Conservation of optical throughput is one reason a passive optical system cannot create arbitrary brightness by “concentrating light” without a corresponding change in angular spread or area. The exact radiometric statement is developed in more advanced optics courses.

14.33.2 Object-side and image-side cones

For a pair of conjugate planes, the two-ray invariant implies, in magnitude,

$$|y_o|n_o u_o \simeq |y_i|n_i u_i, \quad \text{NA}_o \simeq |m| \text{NA}_i. \quad (14.186)$$

Here u_o, u_i are positive half-angles. Using signed m in an equality between two positive cone apertures would introduce a false sign for an inverted image. This relation is paraxial; its extension with exact sines requires the appropriate sine-condition imaging model.

For an ideal visual telescope whose objective is the aperture stop, the eyepiece images that objective into an exit pupil of diameter $D_{\text{exit}} = D_o/|M_\theta|$. If the observer’s pupil is smaller, some collected light is lost. This explains why changing the eyepiece changes brightness at the eye as well as angular size.

14.34 Radiance, etendue and why passive optics cannot create brightness

A common misconception is that a lens can increase the intrinsic brightness of an extended source without limit by concentrating its rays. A lens can certainly concentrate optical power into a smaller area, as sunlight focused by a magnifying glass demonstrates. However, the angular spread increases correspondingly. In lossless geometrical optics, phase-space volume represented by etendue is conserved.

For a bundle crossing area A within solid angle Ω in a medium of index n , a simple small-angle form of etendue is

$$G = \int_A \int_\Omega n^2 \cos \theta \, d\Omega \, dA \simeq n^2 A \Omega. \quad (14.187)$$

More precise definitions include directional cosine factors. Conservation of etendue implies that a passive optical system cannot simultaneously reduce the beam area and its angular spread arbitrarily. More precisely, if L_e is radiance (power per projected area per solid angle), then L_e/n^2 is conserved along rays in an ideal lossless system. Radiance itself is conserved when the refractive index is

unchanged. Fresnel losses, absorption and clipping can reduce transmitted radiance or accepted etendue; they cannot increase brightness through passive propagation.

Example 14.39: Concentrating a solar image

A converging lens forms a small image of the Sun. Why does this not violate conservation of radiance?

Answer. The lens reduces the illuminated area at the image plane but fills that area with a range of ray angles corresponding to different points across the solar disk and the finite aperture. Spatial concentration is accompanied by angular spread. The lens redistributes the available phase-space bundle; it does not create radiance.

This principle is important in illumination design, concentrator photovoltaics, fibre coupling and detector optics.

14.35 Geometrical aberrations

An aberration is a departure of a real optical system from ideal first-order imaging. Aberrations are not all caused by poor manufacturing; many arise even from perfectly manufactured spherical surfaces because the paraxial approximation is only approximate.

14.35.1 Spherical aberration

For a spherical mirror or spherical lens, marginal rays far from the axis generally intersect the axis at a different location from paraxial rays. This produces spherical aberration. Stopping down the aperture removes the most marginal rays and can reduce the blur, but it also reduces light throughput and eventually makes diffraction more important.

14.35.2 Coma

For off-axis object points, different zones of a lens can produce an asymmetric comet-shaped blur. This is coma. It becomes important in wide-field imaging and astronomical systems.

14.35.3 Astigmatism and field curvature

For an off-axis bundle, tangential and sagittal rays can focus at different axial locations, producing astigmatism. Even when local images are sharp, the locus of best focus may be curved rather than planar, producing field curvature. A flat electronic sensor therefore imposes a practical design requirement not automatically satisfied by a simple lens.

14.35.4 Distortion

Distortion changes image geometry without necessarily making each local point image blurry. Barrel distortion reduces magnification toward the field edge; pincushion distortion increases it. Digital calibration can correct much of the geometric mapping if the distortion is stable and well characterized.

14.35.5 Chromatic aberration

The refractive index of glass depends on wavelength, so the lens-maker equation implies a wavelength-dependent focal length:

$$\frac{1}{f(\lambda)} = [n(\lambda) - 1] \left(\frac{1}{R_1} - \frac{1}{R_2} \right). \quad (14.188)$$

Blue and red light can therefore focus at different axial positions. Reflecting systems do not have this particular refractive chromatic aberration, although coatings and detectors still have wavelength-dependent behavior.

Example 14.40: First-order change of focal length with refractive index

A thin lens has fixed curvatures, so $1/f = (n - 1)K$ where K is constant. Show the first-order fractional focal-length change caused by a small index change dn .

Solution. Since

$$f = \frac{1}{(n - 1)K}, \quad (14.189)$$

logarithmic differentiation gives

$$\frac{df}{f} = -\frac{dn}{n - 1}. \quad (14.190)$$

Thus a higher refractive index produces a shorter focal length for fixed surface geometry. The relation also explains why dispersion leads directly to longitudinal chromatic aberration.

Guided checks

1. Why can reducing the aperture improve geometrical sharpness?
2. Why does reducing the aperture too far eventually fail to improve resolution indefinitely?
3. Which aberration changes local magnification across the field without necessarily making a point image fuzzy?
4. Why are reflecting telescope objectives naturally free of ordinary refractive chromatic aberration?

14.35.6 Lens bending as a design freedom

The lens-maker equation shows that focal length depends on surface curvatures through the combination $1/R_1 - 1/R_2$. Many different pairs of radii can therefore produce the same paraxial focal length. This freedom is called lens bending: the designer can redistribute curvature between the two surfaces while approximately preserving first-order power.

For a lens in air with prescribed power $1/f$ and refractive index n , define a shape factor

$$q = \frac{R_2 + R_1}{R_2 - R_1} \quad (14.191)$$

in one common convention. Different q values correspond to different bending of the same approximate power. Aberrations can depend strongly on bending, so two lenses with identical focal length may perform differently for a given object and image conjugate.

Example 14.41: Two lenses with the same paraxial power

Take glass with $n = 1.50$ and target focal length $f = 20.0$ cm. A symmetric biconvex lens with $R_1 = +20.0$ cm and $R_2 = -20.0$ cm satisfies

$$\frac{1}{f} = 0.50 \left(\frac{1}{20} - \frac{1}{-20} \right) = 0.050 \text{ cm}^{-1}. \quad (14.192)$$

A plano-convex lens with one plane surface requires

$$0.050 = 0.50 \frac{1}{R}, \quad (14.193)$$

so $R = 10.0$ cm. Both lenses have the same ideal thin-lens focal length, but their ray aberrations and mechanical properties are not identical.

This distinction helps explain why selecting an optical component by focal length alone can be insufficient in an engineering design.

14.35.7 Aspheric surfaces

Spherical surfaces are easy to manufacture and analyze, but they are not uniquely privileged by geometrical optics. An aspheric surface departs deliberately from a sphere to control aberrations. A rotationally symmetric conic surface may be written in sag form

$$z(r) = \frac{\kappa r^2}{1 + \sqrt{1 - (1 + k)\kappa^2 r^2}}, \quad (14.194)$$

where $\kappa = 1/R$ is vertex curvature and k is the conic constant. Additional polynomial terms can be added for more general aspheres.

The equation is included not for memorization but to make an engineering point: real optical design treats surface shape as a design degree of freedom. A spherical surface is one member of a larger family. Appropriate aspheric surfaces can reduce spherical aberration or allow fewer elements, though they may increase manufacturing and metrology demands.

A paraboloidal mirror is a classic example. Parallel rays incident along its axis reflect to a common geometric focus exactly in ideal ray optics, unlike a spherical mirror for which only paraxial rays share the same focal point.

14.36 Exact ray tracing at a spherical interface

Paraxial matrix optics cannot predict spherical aberration because it has deliberately linearized the geometry. To study aberrations computationally, one must trace rays with the exact surface geometry and exact Snell law.

For a spherical surface in a meridional (z, y) plane with center $(z_c, 0)$ and radius $R_s > 0$, a ray is written parametrically as

$$\mathbf{x}(t) = \mathbf{x}_0 + t\hat{\mathbf{s}}, \quad (14.195)$$

where $\hat{\mathbf{s}}$ is a unit direction. The intersection satisfies

$$[z(t) - z_c]^2 + y(t)^2 = R_s^2. \quad (14.196)$$

This is a quadratic equation for t . Retain only forward roots that lie on the intended spherical cap and inside the clear aperture, then choose the nearest valid intersection. A lens surface is usually a cap, not an entire sphere; blindly choosing the nearest sphere root can select the wrong physical surface. At the intersection point, the surface normal is proportional to

$$\mathbf{n} = \mathbf{x}_{\text{int}} - (z_c, 0). \quad (14.197)$$

Normalize and orient the normal $\hat{\mathbf{n}}$ toward the incident medium, so that $c_i = -\hat{\mathbf{n}} \cdot \hat{\mathbf{s}} \geq 0$. Put $\eta = n_1/n_2$ and

$$k_t = 1 - \eta^2(1 - c_i^2). \quad (14.198)$$

If $k_t < 0$, there is total internal reflection. Otherwise the transmitted unit direction is

$$\hat{\mathbf{t}} = \eta\hat{\mathbf{s}} + (\eta c_i - \sqrt{k_t})\hat{\mathbf{n}}. \quad (14.199)$$

The reflected direction is $\hat{\mathbf{r}} = \hat{\mathbf{s}} + 2c_i\hat{\mathbf{n}}$. These formulas conserve the appropriate tangential component and avoid inverse-trigonometric branch ambiguity. At normal incidence, $\hat{\mathbf{s}} = -\hat{\mathbf{n}}$ and the transmitted direction is unchanged. Roundoff values just below zero may be clipped only within a declared numerical tolerance.

Method

Robust numerical interface algorithm

1. find the nearest forward intersection of the ray with the optical surface;
2. compute and normalize the local surface normal;
3. orient the normal consistently with the chosen incident and transmitted media;
4. compute the incidence angle from a dot product;
5. test the total-internal-reflection condition before evaluating the transmitted angle;
6. construct the reflected or refracted direction;
7. offset the new ray origin by a tiny numerical tolerance or advance to the next surface to avoid re-intersecting the same boundary because of roundoff;
8. verify the routine against plane-interface and paraxial limits.

The exact algorithm is more computationally demanding than ABCD propagation but remains elementary enough for a short Python project. It also introduces software-engineering issues such as root selection, floating-point tolerances and regression testing.

14.36.1 A reproducible spherical-aberration benchmark

Consider parallel light in air entering a spherical glass surface with $n = 1.50$, vertex $z = 0$ and center at $z = R > 0$. At height y , the surface point is $z_s = R - \sqrt{R^2 - y^2}$. Let $\alpha = \arcsin(y/R)$. Snell's law gives the refracted angle to the normal $r = \arcsin[\sin \alpha/n]$, so the ray turns toward the axis by $\alpha - r$. Its axial intercept is

$$z_x = z_s + \frac{y}{\tan(\alpha - r)}. \quad (14.200)$$

For $y \rightarrow 0$, this tends to $nR/(n - 1) = 3R$. At finite height it is smaller. This analytic nonparaxial test is more informative than testing only the axial ray, which is undeviated and has no unique axial crossing.

y/R	z_x/R	Shift from paraxial focus
0.10	2.993	−0.22%
0.30	2.939	−2.04%
0.50	2.824	−5.86%

The data follow Eq. (14.200); they are calculated model values, not experimental measurements. An exact ray tracer should reproduce them before it is used to optimize a multi-surface lens.

Figure 14.17 shows the axial focus moving toward the surface as the ray height increases.

14.37 Circle of confusion and depth of field

A detector plane can be exactly conjugate to only one object plane in the ideal thin-lens model. Objects slightly nearer or farther are not focused to mathematical points at that detector plane; instead they produce small blur circles. If the blur diameter is below an application-dependent acceptable value c , the object may still appear sufficiently sharp. The range of object distances satisfying this criterion is the *depth of field*.

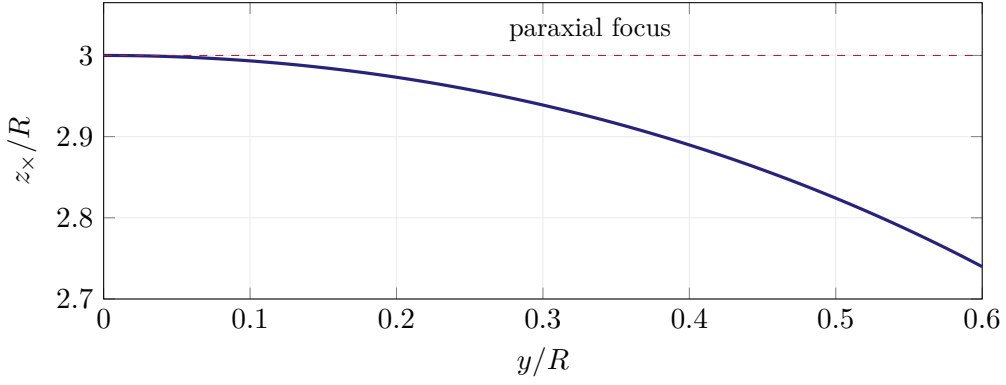


Figure 14.17: Exact axial intercept for parallel rays refracted from air into $n = 1.50$ glass at one spherical surface, calculated from Eq. (14.200). The nonzero height dependence is spherical aberration; a first-order matrix would predict the horizontal value $3R$.

Consider a thin lens of clear aperture diameter D_a . A point object whose exact image would lie a distance i' behind the lens is observed on a detector fixed at distance i . Similar triangles give an approximate blur diameter

$$c \simeq D_a \left| 1 - \frac{i}{i'} \right|. \quad (14.201)$$

The equation makes several trends transparent. A smaller aperture diameter produces a smaller defocus blur. A detector moved closer to the true image plane also reduces the blur. The acceptable c depends on sensor pixel pitch, display size, viewing distance and the intended scientific or photographic task.

14.37.1 Why smaller apertures increase depth of field

For a fixed permitted blur circle, Eq. (14.201) tolerates a larger mismatch between i and i' when D_a is smaller. Therefore more object planes appear acceptably sharp. This is the geometrical origin of increased depth of field at larger f-number.

Caution

Depth of field is not a sharp physical boundary. It depends on an adopted blur criterion. Furthermore, very large f-number eventually increases diffraction blur. The optimum aperture is therefore application dependent and combines geometrical and wave-optical effects.

14.37.2 Near and far acceptable object distances

Let the detector be focused at object distance $p_0 > f$, so $i_0 = fp_0/(p_0 - f)$. For a second real object distance p , substitution in Eq. (14.201) gives

$$c(p) = D_a i_0 \left| \frac{1}{p} - \frac{1}{p_0} \right|. \quad (14.202)$$

For an allowed blur $c_{\max}$, define $\epsilon = c_{\max}/(D_a i_0)$, which has units of inverse length. The acceptable reciprocal-distance interval is centered on $1/p_0$, giving

$$p_{\text{near}} = \frac{1}{1/p_0 + \epsilon}, \quad p_{\text{far}} = \frac{1}{1/p_0 - \epsilon}. \quad (14.203)$$

The far endpoint is infinite when $\epsilon \geq 1/p_0$. The range is symmetric in inverse distance, not in distance. Depth of *focus*, by contrast, describes allowable detector displacement for a fixed object; the same cone geometry gives $|\Delta i| \lesssim c_{\max} i_0 / D_a$.

Example 14.42: An aperture constraint from a moving inspection target

A thin lens has $f = 50.0$ mm and is focused at $p_0 = 1000$ mm. Take $N = 8$, a stop at the lens, and an allowed geometrical blur $c_{\max} = 0.020$ mm. What is the object-distance range?

Solution. $i_0 = 52.632$ mm and $D_a = f/N = 6.25$ mm. Then $\epsilon = 6.08 \times 10^{-5} \text{ mm}^{-1}$, giving

$$p_{\text{near}} = 943 \text{ mm}, \quad p_{\text{far}} = 1065 \text{ mm}. \quad (14.204)$$

The detector tolerance is about ± 0.168 mm. These numbers describe defocus only; diffraction, aberration and pixel integration must fit within the remaining image-quality budget. Halving the aperture diameter would halve the defocus blur but also reduce ideal throughput by four.

Figure 14.18 relates the detector blur diameter to the defocused ray cone.

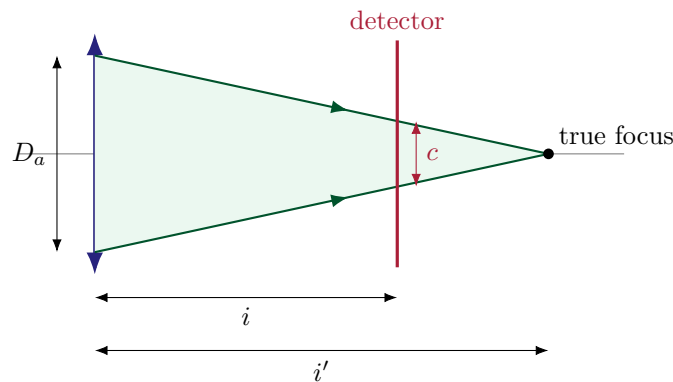


Figure 14.18: Defocus cone for an on-axis point and a stop at the thin lens. At the fixed detector the cone has diameter $c = D_a |1 - i/i'|$. Only geometrical defocus is shown.

14.38 Telecentric imaging and machine vision

In measurement systems, ordinary perspective can be undesirable because the apparent size of an object changes with object distance. A telecentric system is designed so that selected chief rays are parallel to the optical axis in object space, image space, or both. In an object-space telecentric system, modest axial displacement of an object does not strongly change its measured magnification, making such systems useful for dimensional inspection.

A simple first-order route to object-space telecentricity is to place the aperture stop near the back focal plane of the objective group. The entrance pupil is then effectively at infinity, and chief rays in object space are nearly parallel to the axis. This does not eliminate all aberrations or depth-of-field issues, but it controls perspective-related magnification changes.

Example 14.43: Why telecentricity helps metrology

Suppose two identical cylindrical parts are displaced slightly along the optical axis in a conventional camera. Their image diameters differ because their magnifications differ. In an ideal object-space telecentric system, the chief rays are parallel in object space and the first-order transverse magnification is approximately insensitive to this axial shift. Thus a threshold-based edge measurement is less sensitive to imperfect part placement.

Engineering interpretation. Telecentricity trades optical complexity and often reduced field or light throughput for more stable dimensional measurements.

14.39 Prisms: minimum deviation and dispersion

A prism provides a clean application of Snell's law and geometric constraints. Let the prism apex angle be A . If the internal refraction angles are r_1 and r_2 , geometry gives

$$r_1 + r_2 = A. \quad (14.205)$$

If the external incidence and emergence angles are i_1 and i_2 , the total deviation is

$$\delta = i_1 + i_2 - A. \quad (14.206)$$

At minimum deviation the path is symmetric:

$$i_1 = i_2 = i, \quad r_1 = r_2 = \frac{A}{2}. \quad (14.207)$$

Since

$$\delta_{\min} = 2i - A, \quad (14.208)$$

we have

$$i = \frac{A + \delta_{\min}}{2}. \quad (14.209)$$

Applying Snell's law for a prism in air yields

$$n = \frac{\sin [(A + \delta_{\min})/2]}{\sin(A/2)}. \quad (14.210)$$

This relation provides a practical refractometry method.

Example 14.44: Refractive index from minimum deviation

A prism has apex angle $A = 60.0^\circ$ and minimum deviation $\delta_{\min} = 40.0^\circ$. Estimate its refractive index.

Solution.

$$n = \frac{\sin 50.0^\circ}{\sin 30.0^\circ} = \frac{0.7660}{0.5000} = 1.532. \quad (14.211)$$

If the minimum deviation is measured at several wavelengths, the wavelength dependence $n(\lambda)$ can be mapped experimentally.

14.39.1 Angular dispersion

Because n depends on wavelength, so does the deviation. For a thin prism with small apex angle and small deviations,

$$\delta \simeq (n - 1)A. \quad (14.212)$$

Differentiating with respect to wavelength gives

$$\frac{d\delta}{d\lambda} \simeq A \frac{dn}{d\lambda}. \quad (14.213)$$

This is a geometric-optics route to spectral separation. Resolving two nearby wavelengths also depends on aperture and diffraction, so a complete spectrometer treatment belongs partly to wave optics.

14.40 Fermat's principle and the paraxial imaging condition

Fermat's principle states that the physical optical path is stationary with respect to nearby allowed paths. In a homogeneous medium the optical path length is n times the geometric path length. For a ray traveling from a point $(0, y_o)$ to a thin lens at axial coordinate $z = p$, crossing the lens at height y , then reaching $(p + i, y_i)$, the geometric path lengths are

$$\ell_1 = \sqrt{p^2 + (y - y_o)^2}, \quad (14.214)$$

$$\ell_2 = \sqrt{i^2 + (y_i - y)^2}. \quad (14.215)$$

For paraxial rays,

$$\sqrt{a^2 + u^2} \simeq a + \frac{u^2}{2a}. \quad (14.216)$$

For real conjugates $p, i > 0$ in air, retain the quadratic terms and the ideal-lens optical-path delay:

$$\mathcal{L}(y) = \text{constant} + \frac{(y - y_o)^2}{2p} + \frac{(y_i - y)^2}{2i} - \frac{y^2}{2f}. \quad (14.217)$$

Stationarity at a crossing height y gives

$$\frac{d\mathcal{L}}{dy} = y \left(\frac{1}{p} + \frac{1}{i} - \frac{1}{f} \right) - \frac{y_o}{p} - \frac{y_i}{i} = 0. \quad (14.218)$$

For all admitted heights to connect the same object and image points, both the coefficient of y and the remaining constant must vanish:

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f}, \quad \frac{y_i}{y_o} = -\frac{i}{p}. \quad (14.219)$$

Thus equal optical path to this order yields both the imaging equation and its magnification, not merely a single favored ray.

This perspective is valuable because it bridges geometrical and wave optics. In wave language, a thin lens imposes a quadratic phase. In ray language, the gradient of that phase redirects the ray. The same first-order imaging equation emerges from both viewpoints.

Pedagogical boundary

The full wave-optical derivation of a lens as a quadratic phase transformer requires complex field amplitudes and diffraction integrals. Here the stationary-path argument is used only to show the conceptual continuity between ray optics and the wave description that follows later.

14.41 Bessel method derived and uncertainty propagated

Let an object and screen be separated by fixed distance L . For a lens position that forms a sharp image,

$$p + i = L \quad (14.220)$$

and

$$\frac{1}{p} + \frac{1}{i} = \frac{1}{f} \implies pi = fL. \quad (14.221)$$

Therefore p and i are the two roots of

$$x^2 - Lx + fL = 0. \quad (14.222)$$

The roots are

$$x_{\pm} = \frac{L \pm \sqrt{L^2 - 4fL}}{2}. \quad (14.223)$$

Two real distinct lens positions exist only if

$$L^2 - 4fL > 0 \implies L > 4f. \quad (14.224)$$

The separation of the two positions is

$$d = \sqrt{L^2 - 4fL}. \quad (14.225)$$

Solving for f gives

$$\boxed{f = \frac{L^2 - d^2}{4L}}. \quad (14.226)$$

14.41.1 Uncertainty propagation

If L and d are independent measurements with standard uncertainties σ_L and σ_d , then to first order

$$\sigma_f^2 \simeq \left(\frac{\partial f}{\partial L} \sigma_L \right)^2 + \left(\frac{\partial f}{\partial d} \sigma_d \right)^2. \quad (14.227)$$

Since

$$f = \frac{L}{4} - \frac{d^2}{4L}, \quad (14.228)$$

we obtain

$$\frac{\partial f}{\partial L} = \frac{1}{4} + \frac{d^2}{4L^2}, \quad (14.229)$$

$$\frac{\partial f}{\partial d} = -\frac{d}{2L}. \quad (14.230)$$

Hence

$$\boxed{\sigma_f \simeq \sqrt{\left(\frac{1}{4} + \frac{d^2}{4L^2} \right)^2 \sigma_L^2 + \left(\frac{d}{2L} \right)^2 \sigma_d^2}}. \quad (14.231)$$

Example 14.45: Bessel focal length with uncertainty

An experiment gives $L = (100.0 \pm 0.2) \text{ cm}$ and $d = (60.0 \pm 0.2) \text{ cm}$. Estimate f and its propagated uncertainty.

Solution.

$$f = \frac{100.0^2 - 60.0^2}{4(100.0)} \text{ cm} = 16.0 \text{ cm}. \quad (14.232)$$

The derivatives are

$$\frac{\partial f}{\partial L} = \frac{1}{4} + \frac{60^2}{4(100)^2} = 0.34, \quad (14.233)$$

$$\left| \frac{\partial f}{\partial d} \right| = \frac{60}{200} = 0.30. \quad (14.234)$$

Thus

$$\sigma_f \simeq \sqrt{(0.34 \times 0.2)^2 + (0.30 \times 0.2)^2} \text{ cm} \quad (14.235)$$

$$\approx 0.091 \text{ cm}. \quad (14.236)$$

A sensible result is therefore $f = (16.00 \pm 0.09) \text{ cm}$ if the stated uncertainties represent one-standard-deviation independent errors and systematic effects are negligible.

14.41.2 Shared readings and systematic errors

If L and d share calibrated readings, add $2(\partial f/\partial L)(\partial f/\partial d) \text{Cov}(L, d)$ to the variance. For direct conjugates $p = z_\ell - z_o$ and $i = z_s - z_\ell$, independent rail-position errors imply $\text{Cov}(p, i) = -\sigma_{z_\ell}^2$. A lens-position error increases one distance while decreasing the other; treating the derived distances as independent can therefore give the wrong uncertainty.

Near $L = 4f$, the two Bessel positions merge. Their separation is then difficult to measure, and a fixed small σ_d is not a realistic assumption. A formally small derivative $\partial f/\partial d$ at $d = 0$ does not guarantee a practical precision advantage. Report both repeatability and systematic effects such as finite thickness, aberration and ruler calibration.

14.42 Extended tutorial: uncertainty in image position

For a thin lens,

$$i = \frac{fp}{p-f}. \quad (14.237)$$

If f and p have independent uncertainties, first-order propagation gives

$$\sigma_i^2 \simeq \left(\frac{\partial i}{\partial f} \sigma_f \right)^2 + \left(\frac{\partial i}{\partial p} \sigma_p \right)^2. \quad (14.238)$$

The derivatives are

$$\frac{\partial i}{\partial f} = \frac{p^2}{(p-f)^2}, \quad (14.239)$$

$$\frac{\partial i}{\partial p} = -\frac{f^2}{(p-f)^2}. \quad (14.240)$$

Therefore

$$\sigma_i \simeq \frac{1}{|p-f|^2} \sqrt{p^4 \sigma_f^2 + f^4 \sigma_p^2} \quad (14.241)$$

for the stated assumptions.

The denominator reveals a key sensitivity: when p is close to f , small parameter uncertainties produce very large image-distance uncertainties. This is not merely an algebraic inconvenience. It reflects the physical fact that nearly collimated output rays intersect only very far away, so a small angular change shifts the crossing point dramatically.

Example 14.46: Uncertain focus near the focal point

A lens has $f = (100.0 \pm 0.5) \text{ mm}$ and an object is at $p = (120.0 \pm 0.5) \text{ mm}$. Estimate i and σ_i .

Solution.

$$i = \frac{(100)(120)}{20} = 600 \text{ mm}. \quad (14.242)$$

The derivatives are

$$\frac{\partial i}{\partial f} = \frac{120^2}{20^2} = 36, \quad (14.243)$$

$$\frac{\partial i}{\partial p} = -\frac{100^2}{20^2} = -25. \quad (14.244)$$

Thus

$$\sigma_i \simeq \sqrt{(36 \times 0.5)^2 + (25 \times 0.5)^2} \text{ mm} \approx 21.9 \text{ mm}. \quad (14.245)$$

The large propagated uncertainty is a genuine consequence of operating close to the focal condition.

First-order uncertainty propagation requires $\sqrt{\sigma_p^2 + \sigma_f^2}$ to be small compared with $|p - f|$ for independent parameters. If the uncertainty range crosses $p = f$, the image-distance distribution can have very long tails and mix real and virtual images. A symmetric “mean $\pm$ standard deviation” description may then be misleading. Propagating the outgoing vergence $1/i = 1/f - 1/p$, or reporting intervals separately on each side of the singularity, better represents the physics.

14.43 Sensor sampling and a preview of modulation transfer

A digital sensor samples the optical image on a grid of finite pixels. Even if the optical system were perfect, a detector with pixel pitch a cannot represent arbitrarily fine spatial variation. A one-dimensional sinusoidal image intensity

$$I(x) = I_0 [1 + C \cos(2\pi\nu x)] \quad (14.246)$$

has spatial frequency ν and contrast C . The Nyquist sampling frequency associated with pixel pitch a is

$$\nu_{\text{Nyq}} = \frac{1}{2a}. \quad (14.247)$$

Spatial frequencies above this threshold cannot be uniquely represented by uniform point sampling and can alias into lower frequencies.

The optical system itself also reduces contrast at high spatial frequency because of aberrations and diffraction. The modulation transfer function (MTF) quantifies the ratio of image modulation to object modulation as a function of spatial frequency. A complete MTF theory belongs beyond introductory geometrical optics because diffraction is fundamental, but the concept is essential for engineering: sharpness depends on the combined optical transfer and sensor sampling, not on focal length alone.

Example 14.47: Sensor Nyquist frequency

A sensor has pixel pitch $a = 4.0 \mu\text{m}$. Find its one-dimensional Nyquist frequency in line pairs per millimetre.

Solution.

$$\nu_{\text{Nyq}} = \frac{1}{2(4.0 \times 10^{-3} \text{ mm})} = 125 \text{ mm}^{-1}. \quad (14.248)$$

Thus the sampling grid alone cannot uniquely represent arbitrarily higher spatial frequencies. Whether the optical system transmits significant contrast near this limit is a separate question.

14.44 Calibration models for real cameras

Computer vision often separates the ideal pinhole projection from lens distortion and sensor calibration. Choose a computer-vision coordinate convention with positive Z toward the scene and a virtual upright image plane in front of the projection center. A point (X, Y, Z) , $Z > 0$, maps to normalized coordinates

$$x = \frac{X}{Z}, \quad y = \frac{Y}{Z}. \quad (14.249)$$

The sensor coordinates are then

$$u = f_x x + c_x, \quad v = f_y y + c_y, \quad (14.250)$$

where f_x and f_y are effective projection-scale parameters measured in pixels and (c_x, c_y) is the principal point. The physical sensor image behind the lens is inverted; the displayed positive-coordinate convention already absorbs that reversal. At finite focus the projection scale relates to the image conjugate and pixel pitch, and should not automatically be identified with catalog focal length divided by pixel pitch.

A common radial distortion model writes

$$x_d = x \left(1 + k_1 r^2 + k_2 r^4 + \cdots \right), \quad (14.251)$$

$$y_d = y \left(1 + k_1 r^2 + k_2 r^4 + \cdots \right), \quad (14.252)$$

with $r^2 = x^2 + y^2$. The coefficients are empirical calibration parameters. This model is not a substitute for physical lens design, but it is extremely useful when the task is accurate geometric reconstruction rather than explaining every optical surface.

Method

Camera-calibration workflow

1. image a target with accurately known geometry at several orientations;
2. detect the target control points in pixel coordinates;
3. fit the intrinsic parameters and distortion coefficients by minimizing reprojection error;
4. examine residual vectors across the image rather than reporting only one average error;
5. validate on independent images or poses;
6. distinguish geometric calibration from photometric calibration and from deblurring.

This connection is particularly important for computer-science students: geometrical optics provides the physical origin of projection, while calibration turns an imperfect real imaging system into a quantitative measurement device.

14.45 Extended tutorial: inverse optical design with constraints

Many engineering problems are inverse problems. Instead of being given a lens and asked where its image appears, one is given a desired image and asked what optical system can produce it.

Suppose a detector has usable width s , the target field width is W , and the object distance is approximately p . The required magnitude of lateral magnification is

$$|m| = \frac{s}{W}. \quad (14.253)$$

For a real inverted image,

$$i = |m|p. \quad (14.254)$$

Then the first-order focal length is

$$f = \frac{pi}{p+i} = \frac{|m|}{1+|m|}p. \quad (14.255)$$

This provides a first design estimate.

Example 14.48: Machine-vision inverse design

A 16 mm sensor width must cover a 240 mm object field at approximately 600 mm object distance. Estimate the required image distance and focal length.

Solution.

$$|m| = \frac{16}{240} = 0.06667. \quad (14.256)$$

Thus

$$i = 0.06667(600 \text{ mm}) = 40.0 \text{ mm}, \quad (14.257)$$

and

$$f = \frac{(600)(40)}{640} \text{ mm} = 37.5 \text{ mm}. \quad (14.258)$$

A commercial focal length near this value would be a logical starting point. The design would then be refined using the true working-distance definition, sensor aspect ratio, distortion tolerance, aperture and depth-of-field requirements.

14.45.1 Adding a depth-of-field constraint

If the target surface can move axially, an aperture requirement appears. A smaller aperture increases geometrical depth of field but reduces irradiance. A longer exposure can compensate for reduced irradiance only if motion blur and throughput permit. Thus focal length, aperture, illumination, exposure time and mechanical tolerances form a coupled design problem.

14.45.2 Adding a resolution constraint

The sensor must sample the smallest required feature with sufficient pixels, and the optical image must preserve corresponding spatial frequencies. Merely achieving the correct field of view is not enough. This is where geometrical design connects to MTF, diffraction and detector sampling.

For the example above, suppose axial target motion is ± 10 mm and the allowed defocus blur is 0.010 mm. With $p_0 = 600$ mm and $i_0 = 40$ mm, the nearer extreme $p = 590$ mm gives the stricter condition

$$D_a \leq \frac{0.010}{40|1/590 - 1/600|} \text{ mm} = 8.85 \text{ mm}, \quad (14.259)$$

so $N \geq 37.5/8.85 = 4.24$. This is only a geometrical lower bound on f-number. With $4 \mu\text{m}$ pixels, the object-space sampling pitch is $a/|m| = 60 \mu\text{m}$; a sinusoidal object period must exceed $120 \mu\text{m}$ to lie below Nyquist. Real contrast and reliable feature detection require margin beyond that necessary sampling condition. The image scale, aperture and detector now form one quantitative design, rather than three unrelated specifications.

14.46 Extended laboratory project: characterize an unknown lens

An unknown positive lens can be characterized at several levels of sophistication. The following project emphasizes the distinction between repeatability, model error and systematic uncertainty.

14.46.1 Stage 1: focal length by direct conjugates

Measure at least five object–image pairs (p, i) and calculate

$$f_k = \frac{p_k i_k}{p_k + i_k}. \quad (14.260)$$

Report the mean and scatter. Plot $1/i$ versus $1/p$ using

$$\frac{1}{i} = \frac{1}{f} - \frac{1}{p}. \quad (14.261)$$

The ideal slope is -1 and the intercept is $1/f$. A fitted slope significantly different from -1 can signal systematic distance definitions, data errors or breakdown of the thin-lens model.

14.46.2 Stage 2: Bessel method

Choose several object–screen separations $L > 4f$ and measure the lens-position difference d . Calculate f from Eq. (14.118). Compare with the direct-conjugate result.

14.46.3 Stage 3: aperture dependence

Use an adjustable aperture or masks to compare the sharpest-image position and blur quality for narrow and wide ray bundles. A shift or strong change in sharpness can reveal spherical aberration or alignment issues. Do not interpret the smallest visible blur solely as a focal-length change; image quality and first-order focus are related but distinct.

14.46.4 Stage 4: chromatic test

If safe narrow-band sources or color filters are available, compare best focus for two wavelength bands. A measurable shift is evidence of longitudinal chromatic aberration. Quantitative interpretation requires knowledge of the lens material and spectral bandwidth.

14.46.5 Stage 5: uncertainty budget

Separate at least the following contributions:

- ruler or rail reading resolution;
- uncertainty in defining the lens reference plane;
- subjective uncertainty in best focus;
- object and screen alignment;
- repeatability over multiple trials;
- model error from finite thickness or aberrations.

The final result should not report more significant digits than the uncertainty justifies.

In the reciprocal-distance plot, both axes have measurement uncertainty, often correlated through the lens position. An ordinary least-squares fit that treats $1/p$ as exact can bias the slope. Use a fit accounting for both errors, or fit the raw rail positions with their uncertainty model. A slope different from -1 is a diagnostic clue, not by itself proof of a failed thin-lens model.

14.47 Advanced synthesis: where geometrical optics ends

Geometrical optics is powerful precisely because it discards phase and wavelength from the local propagation rule. That simplification has limits. Several important observations require a wave description:

- a finite aperture produces diffraction, so an ideal point object never forms a mathematical point image;
- coherent beams can interfere, producing intensity patterns not predicted by independent-ray addition;
- polarization concerns the vector electromagnetic field, not merely ray direction;
- subwavelength structures can invalidate a simple refractive-index ray model;
- resolution is ultimately constrained by wavelength and aperture, even if all geometrical aberrations are corrected.

The correct intellectual transition is not that geometrical optics is “wrong.” Rather, it is an asymptotic approximation to wave optics. When the wavelength is small compared with relevant system dimensions, rays emerge naturally as approximate directions of energy propagation. Chapter 15 develops interference, and Chapter 16 treats diffraction and resolution, adding wave phase and finite-aperture effects to the geometrical framework built here.

Take-home message

First-order optics tells us where an ideal image should be and how large it should be. Higher-order geometrical optics tells us how real rays depart from that ideal. Wave optics tells us why even a perfectly corrected finite-aperture system has a nonzero image blur. These are complementary levels of description, not competing theories.

Problems and Selected Solutions

Unless stated otherwise, use the declared real-is-positive distances, paraxial mirrors and thin lenses in air. State the reference plane, predict the image type, and report units. Problems 1–36 concern the core and applied material; Problems 37–50 use advanced modules. Drawings should distinguish physical rays, virtual extensions and counterfactual paths.

Conceptual reasoning

1. What experimental observation distinguishes a real image from a virtual image using only a movable screen?
2. Can a virtual image be photographed? Explain carefully why the answer is not contradictory.
3. A concave mirror receives parallel rays close to its axis. Why is the focal point at approximately $R/2$ rather than at the center of curvature?
4. Why can a converging lens form either a real or virtual image, while a diverging lens with a real object forms only a virtual image?
5. What changes physically when a positive lens is immersed in a liquid whose refractive index is closer to that of the lens material?
6. Why can a multi-lens system have a virtual intermediate object even though the original object is real?
7. Why does stopping down a camera generally increase geometrical depth of field?
8. Why is the statement “a smaller aperture always produces a sharper image” incomplete?
9. What does it mean physically for an optical system to be afocal?

Calculation, instruments and measurement

10. A +3.00 D thin lens views a real object 50.0 cm away. Find i , m and the image type.
11. A concave mirror has $R = 60.0$ cm. An object is 20.0 cm from the mirror. Find the image location and magnification.
12. A convex mirror has focal length -25.0 cm. Find the image of an object 40.0 cm away.
13. An object 3.0 cm high is placed 30.0 cm from a +12.0 cm lens. Find image distance and height.
14. A thin symmetric biconvex lens in air has $R_1 = +20.0$ cm, $R_2 = -20.0$ cm and $n = 1.50$. Find f and P .
15. A thin symmetric biconvex lens has $R_1 = +20.0$ cm, $R_2 = -20.0$ cm and $n = 1.50$. Immerse it in water of index 1.33. Find the new focal length and compare it with the value in air.
16. A spherical interface separates air from glass of index 1.50. Its radius is $R = +10.0$ cm and an object lies 30.0 cm in front. Find the image distance.

17. A coin lies 2.00 m below a water surface. Using $n = 1.333$, estimate its apparent depth when viewed nearly normally from air.
18. A 35 mm camera lens images an object 2.00 m away. Find the image distance and compare it with f .
19. A sensor width is 24.0 mm. Estimate horizontal field of view for $f = 24$ mm, 50 mm and 100 mm.
20. A 100 mm lens operates at $f/4$. Find the entrance-pupil diameter. By what factor does ideal image-plane irradiance change if the lens is opened to $f/2$?
21. A projector lens with $f = 5.0$ cm forms a real image on a screen 5.0 m away. Find the object distance and magnification.
22. A myopic eye has far point 50 cm. Estimate the spectacle power needed for distant vision when vertex distance is neglected.
23. A person has near point 100 cm. Estimate the additional lens power needed to view an object at 25 cm as a virtual image at the near point.
24. A simple magnifier has $f = 4.0$ cm. Find the relaxed-eye angular magnification using $D = 25$ cm.
25. A Keplerian telescope uses $f_o = 1.20$ m and $f_e = 30$ mm. Find angular magnification and approximate separation for relaxed viewing.
26. A microscope has $f_o = 5.0$ mm, $f_e = 20$ mm and effective tube length 160 mm. Estimate the relaxed-eye magnification.
27. A fibre has $n_1 = 1.50$ and $n_2 = 1.47$ in air. Find numerical aperture and acceptance half-angle.
28. Two lenses have $f_1 = +15$ cm and $f_2 = +10$ cm and are separated by 35 cm. An object is 50 cm before lens 1. Find the final image position relative to lens 2 and total magnification.
29. In a focal-length experiment, $p = 40.0$ cm and $i = 20.0$ cm. Find f . Repeat for $p = 30.0$ cm and $i = 24.0$ cm. Discuss whether the two results are mutually consistent to the precision shown.
30. In a Bessel experiment, $L = 120$ cm and the two lens positions differ by $d = 80$ cm. Find f .
31. For $f = 15$ cm, state the strict condition on object–screen separation for two distinct Bessel positions. What happens at the boundary value? Explain why that boundary is an infimum, not an attained minimum for two distinct positions.
32. A lens of focal length 20 cm images an object with $m = -3$. Determine p and i .
33. A camera operates at magnification $m = -0.10$. If the object moves 5.0 mm farther away, estimate the first-order image-plane shift using Eq. (14.124).
34. Design a thin-lens camera that maps a 200 mm object width to a 20 mm sensor width at $p = 1.0$ m. Find the required i and f .
35. Using Figure 14.8, classify the images at $p/f = 0.5, 1.5, 2$ and 4 . Then calculate i/f and m . Explain why the plot must not be joined across $p/f = 1$.
36. A point object lies 4.0 cm before a converging lens of focal length 2.0 cm and has height 1.0 cm. A student draws its image at $i = 3.0$ cm with height -0.75 cm. Draw the central and parallel principal rays, identify the incompatible part of the drawing, and correct the image.

Advanced, engineering and computational problems

37. Derive $B = 0$ as the imaging condition for $T(i)L(f)T(p)$, recover $m = A$, and verify the determinant. State the units of each matrix entry.
38. Two lenses in air have $f_1 = 80$ mm and $f_2 = -120$ mm. Plot equivalent power and effective focal length for $0 \leq d \leq 100$ mm. Solve analytically for the afocal spacing and explain whether it lies in this interval.
39. A thick lens in air has $n = 1.60$, $R_1 = +40$ mm, $R_2 = -60$ mm and thickness 8 mm. Find its reduced-coordinate matrix, effective focal length, principal-plane offsets and back focal distance. Compare its focal length with the thin-lens result.
40. A Galilean beam expander uses $f_1 = -20$ mm and $f_2 = +100$ mm. Find the afocal spacing, matrix, beam-diameter expansion and angular scaling. Explain why there is no real internal focus.
41. A prism in air has apex angle 60° and refractive index 1.600. Calculate its minimum deviation. Starting from $r_1 + r_2 = A$, explain the symmetric ray geometry used.
42. A 50.0 mm lens with its stop at the lens is focused at $p_0 = 1000$ mm. The target may move between 950 and 1050 mm and the allowed geometrical blur is 0.020 mm. Find the largest aperture diameter and smallest f-number meeting this condition; identify the more restrictive extreme.
43. Write a paraxial ray tracer for $f = 100$ mm and $p = 300$ mm. Launch at least five slopes from the same off-axis object point and show that the output heights agree at the predicted image plane. Repeat for a detector displaced by 5 mm.
44. Implement Eq. (14.199) with sphere-cap intersection. Reproduce the three axial-intercept values in the spherical-aberration benchmark. Test the normal-incidence, equal-index and total-internal-reflection limits, and explain why a single axial-ray test is insufficient.
45. For $L = (100.0 \pm 0.2)$ cm and $d = (60.0 \pm 0.2)$ cm, compare analytical and Monte Carlo propagation of independent Gaussian standard uncertainties in the Bessel method. Then explain how a common ruler scale error changes the uncertainty model.
46. A sensor has pixel pitch $2.4 \mu\text{m}$. Find its Nyquist frequency. If an image has modulation at 300 lp/mm, explain whether aliasing is possible and why correcting geometric distortion cannot restore missing optical contrast.
47. An image system has $m = -0.20$, object-side index $n_o = 1$ and image-side index $n_i = 1.5$. If the object-side paraxial half-angle is $u_o = 0.015$ rad, estimate u_i . Explain why the aperture relation uses $|m|$, and state what radiance quantity is invariant across the index change.
48. For $f = (50.0 \pm 0.2)$ mm, evaluate i and its first-order standard uncertainty for $p = (200.0 \pm 0.5)$ mm and for $p = (60.0 \pm 0.5)$ mm. Explain when this linear propagation ceases to be meaningful near $p = f$.
49. Design an observation that distinguishes sensor tilt from field curvature using a flat calibration grid and a sequence of focus positions. State what changes upon rotating the sensor or target and what remains centered on the optical axis.
50. Build a two-lens matrix program from translation and thin-lens elements. Verify contact-power addition, the $4f$ relay matrix and the afocal condition. Given $M_0 = \begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix}$ from object

plane to the last element, show that a final translation $i = -B_0/D_0$ gives an image when $D_0 \neq 0$, and interpret the exceptional cases.

Selected detailed solutions

Solution to Problem 10: a +3.00 D lens

The focal length is

$$f = \frac{1}{3.00 \text{ m}^{-1}} = 0.333 \text{ m}. \quad (14.262)$$

With $p = 0.500 \text{ m}$,

$$\frac{1}{i} = \frac{1}{0.333} - \frac{1}{0.500} \approx 3.00 - 2.00 = 1.00 \text{ m}^{-1}, \quad (14.263)$$

so $i \approx 1.00 \text{ m}$ and

$$m = -\frac{1.00}{0.500} = -2.00. \quad (14.264)$$

The image is real, inverted and twice as high as the object.

Solution to Problem 12: convex mirror

With $f = -25.0 \text{ cm}$ and $p = 40.0 \text{ cm}$,

$$\frac{1}{i} = \frac{1}{f} - \frac{1}{p} = -0.0400 - 0.0250 = -0.0650 \text{ cm}^{-1}, \quad (14.265)$$

so

$$i = -15.38 \text{ cm}, \quad m = -\frac{-15.38}{40.0} = +0.3846. \quad (14.266)$$

The image is virtual, upright and reduced, as expected for a convex mirror.

Solution to Problem 14: symmetric biconvex lens

For $R_1 = +20.0 \text{ cm}$, $R_2 = -20.0 \text{ cm}$ and $n = 1.50$,

$$\frac{1}{f} = (0.50) \left(\frac{1}{20.0} - \frac{1}{-20.0} \right) \text{ cm}^{-1} \quad (14.267)$$

$$= (0.50)(0.100) = 0.0500 \text{ cm}^{-1}. \quad (14.268)$$

Thus $f = 20.0 \text{ cm} = 0.200 \text{ m}$ and $P = +5.00 \text{ D}$.

Solution to Problem 17: apparent depth

For near-normal viewing from air, the apparent depth is approximately

$$d_{\text{app}} = d_{\text{real}} \frac{n_{\text{air}}}{n_{\text{water}}} = 2.00 \text{ m} \frac{1}{1.333} \approx 1.50 \text{ m}. \quad (14.269)$$

The coin appears closer to the surface by about 0.50 m.

Solution to Problem 22: myopia

A distant object corresponds to $p \rightarrow \infty$. To make a virtual image at the far point $i = -0.500 \text{ m}$,

$$f = i = -0.500 \text{ m}, \quad (14.270)$$

so the required idealized power is

$$P = -2.00 \text{ D}. \quad (14.271)$$

Solution to Problem 28: two-lens system

For lens 1, $f_1 = 15$ cm and $p_1 = 50$ cm:

$$\frac{1}{i_1} = \frac{1}{15} - \frac{1}{50} = \frac{7}{150} \text{ cm}^{-1}, \quad (14.272)$$

so

$$i_1 = 21.43 \text{ cm}, \quad m_1 = -0.4286. \quad (14.273)$$

Lens 2 is 35 cm to the right, so

$$p_2 = 35.0 - 21.43 = 13.57 \text{ cm}. \quad (14.274)$$

Then

$$\frac{1}{i_2} = \frac{1}{10} - \frac{1}{13.57} \approx 0.02632 \text{ cm}^{-1}, \quad (14.275)$$

so

$$i_2 \approx 38.0 \text{ cm}. \quad (14.276)$$

The second magnification is

$$m_2 = -\frac{38.0}{13.57} \approx -2.80, \quad (14.277)$$

and therefore

$$m_{\text{tot}} \approx (+1.20). \quad (14.278)$$

The final image is real and upright relative to the original object, with about 20% greater height.

Solution to Problem 30: Bessel method

With $L = 120$ cm and $d = 80$ cm,

$$f = \frac{120^2 - 80^2}{4(120)} \text{ cm} = \frac{14400 - 6400}{480} \text{ cm} = 16.67 \text{ cm}. \quad (14.279)$$

The condition $L > 4f$ is satisfied because $120 > 66.7$ cm.

Solution to Problem 32: image distances from magnification

Given $m = -3$ and $m = -i/p$, we have $i = 3p$. With $f = 20$ cm,

$$\frac{1}{20} = \frac{1}{p} + \frac{1}{3p} = \frac{4}{3p}, \quad (14.280)$$

so

$$p = \frac{80}{3} \text{ cm} = 26.67 \text{ cm}, \quad i = 80.0 \text{ cm}. \quad (14.281)$$

Solution to Problem 34: inverse camera design

The required magnitude of magnification is $20/200 = 0.10$, and a real image gives $m = -0.10$. Thus

$$i = 0.10p = 0.10(1000 \text{ mm}) = 100 \text{ mm}. \quad (14.282)$$

The focal length is

$$f = \frac{pi}{p+i} = \frac{(1000)(100)}{1100} \text{ mm} = 90.9 \text{ mm}. \quad (14.283)$$

This is a first-order starting point; a real design would include working-distance definitions referenced to principal planes, field of view, aperture and tolerances.

Solution to Problem 39: thick lens and back focus

For $n = 1.60$, $R_1 = +40$ mm, $R_2 = -60$ mm and $t = 8$ mm,

$$\Phi_1 = \frac{0.60}{40} = 0.0150 \text{ mm}^{-1}, \quad (14.284)$$

$$\Phi_2 = \frac{1 - 1.60}{-60} = 0.0100 \text{ mm}^{-1}. \quad (14.285)$$

The Gullstrand relation gives

$$\Phi = 0.0150 + 0.0100 - \frac{8}{1.60}(0.0150)(0.0100) \quad (14.286)$$

$$= 0.0250 - 0.000750 = 0.02425 \text{ mm}^{-1}. \quad (14.287)$$

Thus

$$F = 41.24 \text{ mm}. \quad (14.288)$$

The corresponding thin-lens estimate would be

$$\Phi_{\text{thin}} = 0.0250 \text{ mm}^{-1}, \quad f_{\text{thin}} = 40.0 \text{ mm}. \quad (14.289)$$

The finite thickness changes the effective focal length by about 3.1% in this example.

The full matrix is $M = \begin{pmatrix} 0.925 & 5.00 \text{ mm} \\ -0.02425 \text{ mm}^{-1} & 0.950 \end{pmatrix}$. Thus $a = 2.062$ mm, $b = 3.093$ mm and $\text{BFL} = AF = 38.14$ mm. Checking $AD - BC = 0.87875 + 0.12125 = 1$ verifies the first-order invariant but does not, by itself, prove that the curvatures were entered correctly.

Solution to Problem 40: Galilean beam expander

Let $f_1 = -20$ mm and $f_2 = +100$ mm. For an afocal pair,

$$d = f_1 + f_2 = 80 \text{ mm}. \quad (14.290)$$

The ideal beam expansion magnitude is

$$\left| \frac{f_2}{f_1} \right| = 5. \quad (14.291)$$

Because the first lens is negative, no real internal focus appears between the elements in the ideal Galilean arrangement.

Direct multiplication gives $M = \begin{pmatrix} 5 & 80 \text{ mm} \\ 0 & 0.20 \end{pmatrix}$. For a collimated input with $\theta = 0$, heights scale by 5; a small field angle scales by 0.20. The positive angular scaling preserves orientation. The nonzero B does not contradict afocality: $C = 0$ is the afocal condition, whereas $B = 0$ applies to conjugate planes.

Solution to Problem 41: prism minimum deviation

At minimum deviation the ray path is symmetric: $r_1 = r_2$ and the entrance and exit angles are equal. Since $r_1 + r_2 = A$, each internal angle is $A/2$. The total deviation is $\delta = \theta_1 + \theta_2 - A$, so each external angle is $(A + \delta_{\min})/2$. The refraction law then gives the index relation below. For an equilateral prism, $A = 60^\circ$. At minimum deviation,

$$n = \frac{\sin[(A + \delta_{\min})/2]}{\sin(A/2)}. \quad (14.292)$$

With $n = 1.600$,

$$\sin\left(\frac{60^\circ + \delta_{\min}}{2}\right) = 1.600 \sin 30^\circ = 0.800. \quad (14.293)$$

Therefore

$$\frac{60^\circ + \delta_{\min}}{2} = \sin^{-1}(0.800) \approx 53.13^\circ, \quad (14.294)$$

so

$$\delta_{\min} \approx 46.26^\circ. \quad (14.295)$$

Solution to Problem 42: aperture budget

The nominal image distance is $i_0 = 50(1000)/(1000 - 50) = 52.632$ mm. The near extreme has the larger vergence error:

$$\begin{aligned} |1/950 - 1/1000| &= 5.2632 \times 10^{-5} \text{ mm}^{-1}, \\ |1/1050 - 1/1000| &= 4.7619 \times 10^{-5} \text{ mm}^{-1}. \end{aligned}$$

Thus $c_{\max} = D_a i_0 |1/p - 1/p_0|$ implies $D_a \leq 7.22$ mm and $N \geq 6.93$. A nominal $f/8$ setting satisfies this geometrical criterion; it must still be checked for diffraction, aberration and illumination constraints. The units cancel correctly in $N = f/D_a$, while $D_a i_0 \Delta(1/p)$ has units of length.

Notation dictionary

Symbol	Meaning	SI unit
p	object distance, with sign defined by the adopted real-is-positive convention	m
i	image distance	m
f	focal length	m
R, R_1, R_2	signed radius or radii of curvature	m
h_o, h_i	object and image height	m
m	lateral magnification h_i/h_o	dimensionless
n, n_1, n_2	refractive index	dimensionless
θ_1, θ_2	incidence and refraction angles measured from the normal	rad or deg
θ_c	critical angle for total internal reflection	rad or deg
P	optical power $1/f$	D = m ⁻¹
D_a	aperture or entrance-pupil diameter, according to context	m
N	f-number f/D_a	dimensionless
Θ	field-of-view angle	rad or deg
M_θ	angular magnification	dimensionless
D	conventional near-point reference distance, often 25 cm in examples	m
NA	numerical aperture	dimensionless
A, B, C, D	elements of an ABCD ray-transfer matrix	mixed dimensions
y	paraxial ray height	m
θ	paraxial ray angle or slope	rad
L	optical bench separation, tube length or system length according to context	m
d	separation or displacement according to context	m
c	acceptable circle-of-confusion diameter	m

Compact formula sheet

Use the stated sign conventions and paraxial assumptions. The relation $m = -i/p$ applies to mirrors and thin lenses with equal exterior indices. For a single refracting surface use $m = -n_1 i / (n_2 p)$ instead. Lens-maker, power-sum and separated-lens formulas below assume air.

Formula summary

$$n_1 \sin \theta_1 = n_2 \sin \theta_2,$$

$$\sin \theta_c = \frac{n_2}{n_1} \quad (n_1 > n_2),$$

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{i},$$

$$m = -\frac{i}{p} = \frac{h_i}{h_o},$$

$$f_{\text{mirror}} = \frac{R}{2},$$

$$\frac{n_1}{p} + \frac{n_2}{i} = \frac{n_2 - n_1}{R},$$

$$\frac{1}{f} = (n - 1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right),$$

$$P = \frac{1}{f},$$

$$P_{\text{contact}} = P_1 + P_2,$$

$$\frac{1}{f_{\text{separated}}} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2},$$

$$N = \frac{f}{D_a},$$

$$\Theta \simeq 2 \arctan \left(\frac{s}{2f} \right),$$

$$M_{\theta, \text{magnifier}} \simeq \frac{D}{f} \quad (\text{relaxed eye}),$$

$$M_{\theta, \text{telescope}} = -\frac{f_o}{f_e},$$

$$\text{NA} = n \sin u,$$

$$f_{\text{Bessel}} = \frac{L^2 - d^2}{4L},$$

$$T(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix},$$

$$L(f) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}.$$

Bibliography

- [1] D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 12th ed., Volume 2, Wiley. Chapter 34 (Images), especially Sections 34.1–34.6.
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th ed., Cengage Learning. Chapter 36 (Image Formation), with Chapter 35 for reflection, refraction and prisms.
- [3] S. J. Ling, J. Sanny, and W. Moebs, *University Physics, Volume 2*, OpenStax, Rice University. Section 16.2 supports the electromagnetic-wave background. Geometrical optics belongs to Volume 3, Chapter 2.
- [4] A. Tsagkaropoulos, “Reflection-refraction” and “Prism and Deflexion Angle,” TikZ.net (2021), <https://tikz.net/reflection-refraction/> and <https://tikz.net/prism1/>.