

Chapter 15

Interference of Light

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Preface

Interference reveals the role of optical phase. A beam can be divided, sent along different optical paths and recombined. The electric fields add first; the detector then measures a time-averaged intensity. The resulting pattern carries information about wavelength, refractive index, displacement, film thickness, surface shape and coherence.

The chapter develops this idea from field superposition and the distinction between geometric path, optical path and phase. Young’s double-slit experiment introduces fringe positions and intensity. Thin films, anti-reflection coatings and Newton’s rings extend the phase accounting to reflections and material paths. The Michelson interferometer connects these principles to precision measurement.

The intended readers are students of Technical Physics, engineering, computer science, space science and applied science. Camera sampling, refractive-index sensing, thin-film metrology, vibration sensitivity and numerical fitting connect the ideal models to measurements. Advanced enrichment develops coherence functions, multilayer optics, Fourier methods and noise limits after the introductory core.

The Guideline

For every interference problem, identify four items before writing a condition such as “ $m\lambda$ ” or “ $(m + \frac{1}{2})\lambda$ ”:

1. which waves are being compared;
2. the optical-path difference accumulated during propagation;
3. any additional phase changes introduced at reflections or optical elements;
4. whether the observable is a maximum, a minimum, or a specified phase shift.

This four-step bookkeeping exposes the source of most sign and half-wavelength errors.

Pedagogical boundary

The core uses the far-field and near-normal regimes appropriate to introductory Physics 2. Finite-slit diffraction appears only to explain a realistic interference envelope. A full treatment of diffraction, gratings and resolving power belongs to the following wave-optics chapter. Multiple-beam interference and quantitative coherence are reserved for advanced enrichment.

Conventions and notation

The vacuum wavelength is denoted by λ_0 when it is important to distinguish it from the wavelength in a medium. In a transparent, nonmagnetic medium of refractive index n , the frequency remains unchanged while the phase velocity and wavelength become

$$v = \frac{c}{n}, \quad \lambda = \frac{\lambda_0}{n}.$$

The vacuum wave number is $k_0 = 2\pi/\lambda_0$, and the wave number in the medium is $k = nk_0 = 2\pi/\lambda$. Angular frequency is $\omega = 2\pi f$.

For geometric path length L , the *optical path length* (OPL) and the propagation phase difference are

$$\begin{aligned} \text{OPL} &= \int n \, ds \quad (= nL \text{ in a uniform medium}), \\ \Delta_{\text{OPL}} &= \text{OPL}_2 - \text{OPL}_1, \\ \Delta\phi_{\text{prop}} &= k_0 \Delta_{\text{OPL}} = \frac{2\pi}{\lambda_0} \Delta_{\text{OPL}}. \end{aligned}$$

Any source phase, reflection phase or phase introduced by an optical component must be added separately.

For a monochromatic plane wave with fixed polarization we may represent one electric-field component as

$$E(z, t) = E_0 \cos(kz - \omega t + \phi_0).$$

The measurable optical intensity is a time average. In a transparent medium we use the proportionality $I \propto E_0^2$ whenever the same medium is used for the waves being compared. When absolute intensities are required for a plane wave in a nonmagnetic medium,

$$I = \frac{1}{2} n c \varepsilon_0 E_0^2.$$

The main geometrical symbols are d (slit separation), a (slit width), D (screen distance), y (screen coordinate), t (film thickness) and R (lens radius of curvature). Appendix A gives the complete dictionary.

Note
Path difference has units of length; phase difference is dimensionless and is normally expressed in radians. Their conversion involves wavelength. When different media are present, use optical path to calculate propagation phase.

Note
Core thin-film formulas assume transparent, nonabsorbing, nonmagnetic dielectrics and near-normal incidence unless stated otherwise. Oblique incidence, absorption and polarization-dependent effects require the Fresnel amplitudes and phases developed in the advanced section.

Chapter 15

Interference of Light

15.1 Motivation and roadmap

Geometrical optics treats light as rays. That approximation is extraordinarily useful for mirrors, lenses and image formation, but it deliberately suppresses the phase of the electromagnetic wave. Interference restores that phase information. Two beams that would follow perfectly sensible ray paths can produce very different detected intensities depending on whether their electric fields arrive in phase, out of phase, or somewhere in between.

The practical consequence is powerful. A change in optical path of only a fraction of a wavelength can move a fringe from bright to dark. Visible wavelengths are of order hundreds of nanometres, so a fringe pattern can respond to extremely small displacements, thickness variations or refractive-index changes. Interferometry therefore converts phase into a macroscopic pattern that can be photographed, counted and fitted.

The conceptual chain of this chapter is

superposition of fields $\longrightarrow$ phase difference $\longrightarrow$ intensity pattern $\longrightarrow$ precision measurement.

The same chain will later reappear in diffraction, Fourier optics and quantum interference.

Learning goals

After studying this chapter, the student should be able to:

- distinguish geometric path, optical path and phase, and explain the role of coherence;
- derive the two-wave intensity law, extrema and fringe visibility;
- derive angular and small-angle Young fringe positions and predict their dependence on wavelength, slit separation, distance and refractive index;
- distinguish the interference fringes from the finite-slit diffraction envelope;
- account for propagation and reflection phases in thin films;
- calculate quarter-wave coating thickness and explain the amplitude-matching condition;
- relate Newton-ring radii to film thickness, lens curvature and imperfect contact;
- use Michelson fringe counts to infer displacement, wavelength or refractive index;
- design laboratory or computational measurements and estimate their uncertainty.

15.1.1 Required prior knowledge

The chapter assumes familiarity with sinusoidal waves, wavelength, frequency, angular frequency, phase, the superposition principle, trigonometric identities, small-angle approximations and the basic electromagnetic-wave relation $v = f\lambda$. From earlier optics, students should recall refractive index, Snell's law and the distinction between reflection and transmission at an interface. The main algebraic tool is the addition of sinusoidal waves with a phase difference.

Key message

Interference is not an additional force acting on light. It is the consequence of adding electromagnetic fields and then computing an intensity from the resulting field. The “bright” and “dark” regions are therefore statements about energy detection after superposition, not about one beam being physically blocked by the other.

15.2 From electromagnetic waves to the superposition principle

Consider two linearly polarized monochromatic waves of the same frequency arriving at a detector point. If their polarization directions are parallel, one Cartesian component can be written

$$E_1(t) = E_{01} \cos(\phi_1 - \omega t), \quad (15.1)$$

$$E_2(t) = E_{02} \cos(\phi_2 - \omega t). \quad (15.2)$$

Maxwell’s equations are linear in ordinary transparent optical media to an excellent approximation, so the total electric field is

$$E(t) = E_1(t) + E_2(t). \quad (15.3)$$

The detector does not normally follow the electric field oscillation itself because visible-light frequencies are of order 10^{14} – 10^{15} Hz. Instead, it responds to a time average proportional to $\langle E^2 \rangle$.

We use the common convention $E_j(t) = E_{0j} \cos(\phi_j - \omega t)$, consistent with $E(z, t) = E_0 \cos(kz - \omega t + \phi_0)$. A longer optical path therefore adds positive propagation phase to ϕ_j . Reversing the overall time convention is possible, but all phase differences must then be changed consistently.

The scalar intensity model applies to waves in the same detection medium, with parallel polarization and common or nearly common propagation direction, as in paraxial interference. For arbitrary crossing angles, energy flux must instead be found from the total Poynting vector. Figure 15.1 illustrates field addition within the scalar model.

Squaring the total field gives

$$E^2 = E_1^2 + E_2^2 + 2E_1E_2. \quad (15.4)$$

The first two terms lead to the intensities I_1 and I_2 that would be observed from the individual beams. The third term is the interference term.

To evaluate it, use

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)], \quad (15.5)$$

$$\langle E_j^2 \rangle = \frac{1}{2} E_{0j}^2, \quad \langle E_1 E_2 \rangle = \frac{1}{2} E_{01} E_{02} \cos(\phi_2 - \phi_1). \quad (15.6)$$

The term at frequency 2ω averages to zero over many optical periods. Multiplication by the common factor $nc\varepsilon_0$ converts these field averages into intensity, with $I_j = \frac{1}{2} nc\varepsilon_0 E_{0j}^2$. This is why the interference term involves $\sqrt{I_1 I_2}$ rather than $I_1 I_2$. Its time average depends on the phase difference

$$\Delta\phi = \phi_2 - \phi_1. \quad (15.7)$$

For waves with a stable phase relation and the same polarization,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \Delta\phi. \quad (15.8)$$

This equation is the central intensity relation for two-beam interference.

For equal intensities, $I_1 = I_2 = I_0$, Eq. (15.8) becomes

$$I = 2I_0(1 + \cos \Delta\phi) = 4I_0 \cos^2\left(\frac{\Delta\phi}{2}\right). \quad (15.9)$$

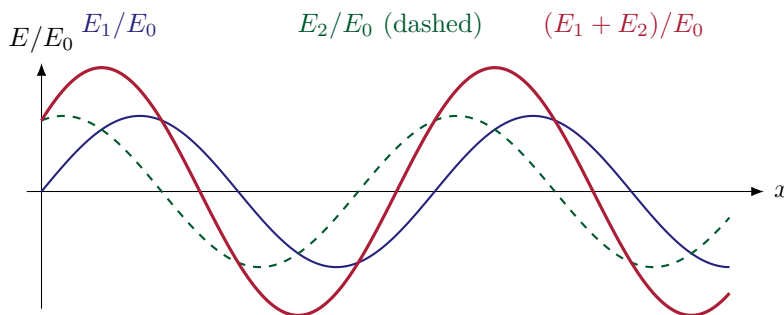


Figure 15.1: Field addition at a fixed instant, for equal-amplitude sinusoids with phase separation 70° . The horizontal coordinate has an arbitrary common scale. The red curve is calculated as the sum, not as a sum of intensities.

The maximum intensity is therefore $4I_0$, not $2I_0$. There is no violation of energy conservation: the interference pattern redistributes energy spatially. Regions above the incoherent sum are compensated by regions below it.

Guided checks

1. Two equal beams each have intensity I_0 . What are I for $\Delta\phi = 0, \pi/2$ and π ?
2. Why is it incorrect to obtain the interference pattern by adding the two intensities first?
3. If one beam is blocked, which term in Eq. (15.8) disappears automatically?

Caution

The formula $I = 4I_0 \cos^2(\Delta\phi/2)$ assumes equal beam intensities, common polarization and sufficient coherence. It is not the correct universal formula for any two light beams.

15.3 Phase, path difference and optical path length

A wave accumulates phase as it propagates. In a uniform medium,

$$\phi_{\text{prop}} = kL = \frac{2\pi}{\lambda}L = \frac{2\pi}{\lambda_0}nL. \quad (15.10)$$

Thus two waves of the same vacuum wavelength that travel geometric lengths L_1 and L_2 in the same medium have propagation phase difference

$$\Delta\phi_{\text{prop}} = \frac{2\pi}{\lambda}(L_2 - L_1). \quad (15.11)$$

If they travel through different materials, one must instead compare optical paths:

$$\Delta\phi_{\text{prop}} = \frac{2\pi}{\lambda_0} \left(\int_2 n \, ds - \int_1 n \, ds \right). \quad (15.12)$$

The full phase difference can contain additional terms,

$$\Delta\phi = \frac{2\pi}{\lambda_0} \Delta\text{OPL} + \Delta\phi_{\text{refl}} + \Delta\phi_{\text{src}} + \Delta\phi_{\text{elem}}. \quad (15.13)$$

Equation (15.13) is the “phase ledger” used throughout this chapter.

For two waves that begin in phase and experience no extra phase change, constructive interference occurs when

$$\Delta\text{OPL} = m\lambda_0, \quad m \in \mathbb{Z}, \quad (15.14)$$

and destructive interference occurs when

$$\Delta_{\text{OPL}} = \left(m + \frac{1}{2}\right) \lambda_0. \quad (15.15)$$

The familiar path conditions are therefore special cases of a more general phase condition.

Method

The phase-ledger method. For two interfering waves, write

$$\Delta\phi = \Delta\phi_{\text{prop}} + \Delta\phi_{\text{refl}} + \Delta\phi_{\text{other}}.$$

Then impose $\Delta\phi = 2\pi m$ for a maximum or $\Delta\phi = (2m + 1)\pi$ for a minimum. Do not memorize a thin-film formula before deciding how many reflection phase reversals are present.

Example 15.1: Changing the medium in one path

A two-beam interferometer uses light of vacuum wavelength 632.8 nm. One beam traverses once through a transparent plate of thickness 1.00 mm and refractive index 1.500, while the comparison beam traverses the same geometric distance in air, approximated by $n = 1$. Neglect reflection phases. Find the additional phase accumulated by the plate beam.

The optical-path increase is

$$\Delta_{\text{OPL}} = (n - 1)t = (0.500)(1.00 \times 10^{-3} \text{ m}) = 5.00 \times 10^{-4} \text{ m}.$$

Hence

$$\Delta\phi = \frac{2\pi}{632.8 \times 10^{-9}} (5.00 \times 10^{-4}) \approx 4.96 \times 10^3 \text{ rad}.$$

Equivalently, the plate introduces about

$$N = \frac{\Delta_{\text{OPL}}}{\lambda_0} \approx 790$$

wavelengths of additional optical path. The large phase change shows why interferometers are highly sensitive to refractive index and thickness.

Guided checks

If the geometric path difference is zero but one beam passes through glass and the other through air, can the phase difference be nonzero? Which quantity in Eq. (15.13) detects that change?

15.4 Coherence: when a stable pattern can exist

The existence of two waves at the same point is not sufficient to guarantee a visible, stationary interference pattern. The relative phase must remain sufficiently stable during the detector's averaging time. Two waves satisfying this condition are said to be coherent over the relevant time and path difference.

A useful idealization is *complete coherence*: the phase difference $\Delta\phi$ is constant. Young's experiment naturally approaches this condition because the two slits are illuminated by the same parent wave. The two emerging waves are not independent sources; they are two samples of one optical field. A common source is helpful, but it does not guarantee high contrast: the slit separation must lie within the spatial coherence range and the path delay within the temporal coherence range. Figure 15.2 shows explicitly how averaging different relative phases removes the modulation.

By contrast, two separate ordinary lamps are usually poor sources for a stable two-source interference pattern. Their microscopic emitters radiate with rapidly fluctuating phases. The cross term in Eq. (15.8) changes sign quickly, and the detector averages it toward zero. The result becomes approximately

$$\langle I \rangle \approx I_1 + I_2, \quad (15.16)$$

with no stable bright-dark fringe pattern.

Coherence has both temporal and spatial aspects. Temporal coherence concerns phase correlation at different times and therefore limits the optical-path mismatch over which fringes retain contrast. Spatial coherence concerns the phase relation across different transverse points of a wavefront and determines whether separated apertures can act as coherent secondary sources. The main chapter uses these ideas qualitatively; quantitative coherence functions are developed in the advanced section.

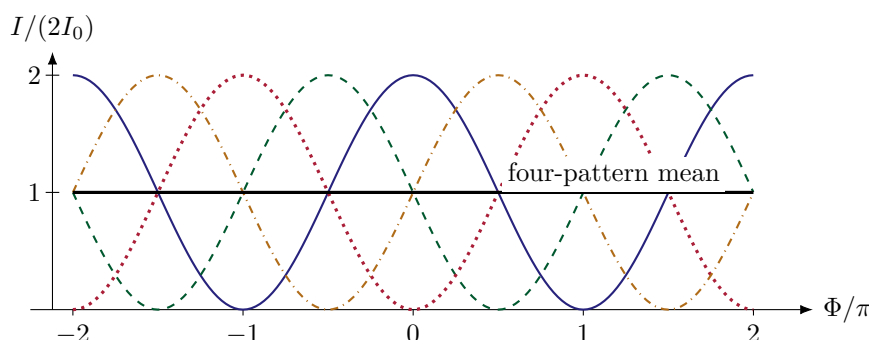


Figure 15.2: Four equal-beam patterns with phase offsets $0, \pi/2, \pi, 3\pi/2$. Each pattern separately has full visibility, while their equal-weight average is the constant $2I_0$. The same cancellation occurs for a uniformly sampled random phase during a long exposure.

Key message

Coherence does not mean that a wave has to be perfectly monochromatic or infinitely long. It means that the phase correlation is sufficiently stable over the path difference, time interval and aperture separation used in the experiment.

Example 15.2: Will a Michelson pattern survive?

A laser has a quoted coherence length of approximately 20 cm. A Michelson interferometer is adjusted so that one round-trip optical path is 6 cm longer than the other. Should high-contrast fringes be plausible?

The relevant comparison is the optical-path difference, not the one-way arm-length difference. Here the stated round-trip mismatch is 6 cm, which is well below the quoted coherence length. High-contrast fringes are therefore plausible, provided alignment and polarization are also adequate. If the path mismatch were many times larger than 20 cm, the fringe contrast would be expected to fall strongly.

Caution

A laser is not “perfectly coherent.” Real lasers have finite linewidth, finite spatial mode quality, technical noise and finite coherence length. Conversely, nonlaser sources can produce interference if the geometry selects sufficiently correlated portions of the field.

15.5 General two-wave interference and fringe visibility

Equation (15.8) immediately shows how unequal amplitudes affect an interference pattern. The maximum and minimum possible intensities are

$$I_{\max} = I_1 + I_2 + 2\sqrt{I_1 I_2} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2, \quad (15.17)$$

$$I_{\min} = I_1 + I_2 - 2\sqrt{I_1 I_2} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2. \quad (15.18)$$

Perfect darkness requires equal field amplitudes. If one beam is much stronger than the other, the weak beam can modulate the total intensity but cannot cancel the strong beam completely.

A standard measure of contrast is the fringe visibility

$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}. \quad (15.19)$$

For two perfectly coherent beams of common polarization,

$$\boxed{\mathcal{V} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}}. \quad (15.20)$$

The visibility equals one only when $I_1 = I_2$.

Example 15.3: Unequal beams and nonzero minima

Two coherent beams have intensities $I_1 = 9.0 \text{ W m}^{-2}$ and $I_2 = 4.0 \text{ W m}^{-2}$. Find the maximum intensity, minimum intensity and visibility.

Using square-root amplitudes,

$$I_{\max} = (3 + 2)^2 = 25 \text{ W m}^{-2},$$

$$I_{\min} = (3 - 2)^2 = 1 \text{ W m}^{-2}.$$

Then

$$\mathcal{V} = \frac{25 - 1}{25 + 1} = \frac{24}{26} \approx 0.923.$$

The minima are dim but not zero because the two field amplitudes are unequal.

Guided checks

Suppose $I_2/I_1 \rightarrow 0$. What happens to $I_{\max}$, $I_{\min}$ and $\mathcal{V}$? Explain physically why a single dominant beam cannot produce strong interference contrast with an arbitrarily weak second beam.

15.6 Young's double-slit experiment: geometry and path difference

Young's double-slit experiment provides the cleanest geometry for connecting phase to a visible spatial pattern. A monochromatic wave illuminates two narrow slits separated by a center-to-center distance d . The slits act as coherent secondary sources because they are driven by the same incident wave. At a distant observation point P , the two waves travel slightly different distances and therefore arrive with a phase difference.

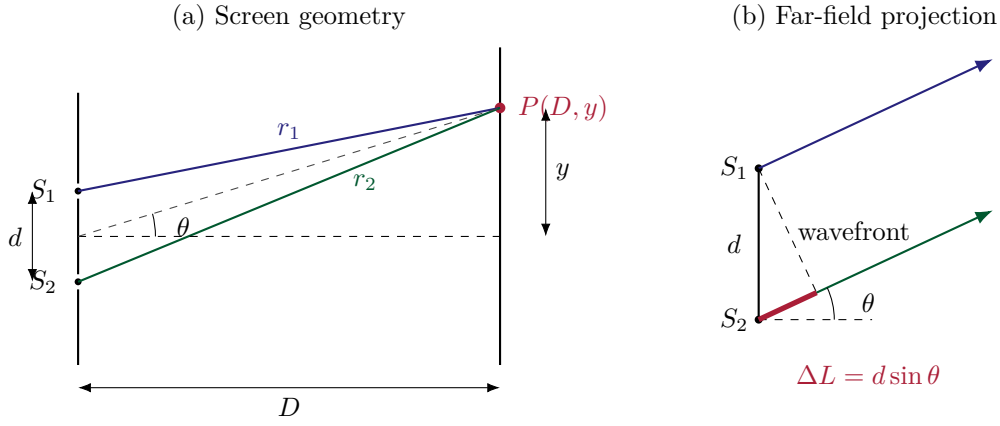


Figure 15.3: Young geometry with the upper slit labelled S_1 and positive y upward. The lower path is longer at the displayed point. In the parallel-ray approximation, projecting the slit separation onto the propagation direction gives $r_2 - r_1 \simeq d \sin \theta$. The drawing exaggerates the separation; panel (b) isolates the projection.

In Fig. 15.3, place S_1 at $y = +d/2$ and S_2 at $y = -d/2$, and define $\Delta L = r_2 - r_1$. At a point (D, y) ,

$$r_1 = \sqrt{D^2 + (y - d/2)^2}, \quad r_2 = \sqrt{D^2 + (y + d/2)^2}, \quad (15.21)$$

$$\Delta L = \frac{r_2^2 - r_1^2}{r_2 + r_1} = \frac{2yd}{r_2 + r_1}. \quad (15.22)$$

This exact point-source geometry also fixes the sign: for $y > 0$, the lower slit's path is longer. When $d \ll \sqrt{D^2 + y^2}$, the denominator is approximately $2\sqrt{D^2 + y^2}$. Since $\sin \theta = y/\sqrt{D^2 + y^2}$, the path difference is then

$$\boxed{\Delta L \approx d \sin \theta.} \quad (15.23)$$

If both paths lie in the same medium and no additional phase shift is introduced, the phase difference is

$$\Delta \phi = \frac{2\pi}{\lambda} d \sin \theta. \quad (15.24)$$

Constructive interference occurs when

$$\boxed{d \sin \theta_m = m\lambda, \quad m = 0, \pm 1, \pm 2, \dots} \quad (15.25)$$

and destructive interference occurs when

$$\boxed{d \sin \theta_m^{(\text{dark})} = \left(m + \frac{1}{2}\right) \lambda.} \quad (15.26)$$

The central bright fringe corresponds to $m = 0$ and lies on the perpendicular bisector when the two slit paths are symmetric.

There are two separate approximations here. Nearly parallel rays justify $\Delta L \simeq d \sin \theta$; a small observation angle is needed only for the later screen formula. For phase-sensitive work, require the error in path difference to be small compared with λ , not merely with D . For an extended aperture of total transverse span W , $D \gg W^2/\lambda$ is a useful sufficient Fraunhofer criterion in free propagation. A lens can instead form the far-field pattern in its focal plane. The complete aperture treatment is deferred to Chapter 16.

Method

Order is a phase count. The integer m tells how many whole wavelengths of path difference separate the two waves at a bright fringe. It is not the number of the slit, not the number of oscillations emitted, and not the number of fringes between the center and the observation point unless the counting convention is stated carefully.

Example 15.4: Angular position of a bright fringe

A double slit has separation $d = 0.250 \text{ mm}$ and is illuminated with $\lambda = 600 \text{ nm}$ light in air. Find the angle of the third-order bright fringe without a small-angle approximation.

For $m = 3$,

$$\sin \theta_3 = \frac{3\lambda}{d} = \frac{3(600 \times 10^{-9})}{0.250 \times 10^{-3}} = 7.20 \times 10^{-3}.$$

Therefore

$$\theta_3 = \sin^{-1}(7.20 \times 10^{-3}) \approx 0.412^\circ.$$

Here $m\lambda/d \ll 1$ justifies the small-angle limit. The independent far-field approximation still requires a sufficiently large observation distance; a small angle alone does not establish that condition.

Example 15.5: Highest possible interference order

A slit separation is $d = 5.0\lambda$. What is the highest possible bright order?

Equation (15.25) requires $|\sin \theta| \leq 1$, hence

$$|m| \leq \frac{d}{\lambda} = 5.0.$$

Thus the largest possible magnitude is $|m| = 5$, occurring formally at $|\theta| = 90^\circ$. In a real slit geometry, finite aperture effects and limited screen extent may make such extreme-angle orders impractical to observe, but the order bound follows directly from geometry.

Guided checks

1. If $d < \lambda$, can any noncentral bright order exist in Eq. (15.25)?
2. Why are the $+m$ and $-m$ fringes symmetric in the ideal setup?
3. Does increasing d move a given nonzero angular order toward or away from the central axis?

15.7 Fringe positions on a screen and the small-angle approximation

Let the screen be a perpendicular distance D from the slit plane and let y be the transverse coordinate measured from the central axis. Geometry gives

$$\tan \theta = \frac{y}{D}. \quad (15.27)$$

For small angles in radians,

$$\sin \theta \approx \tan \theta \approx \theta \approx \frac{y}{D}. \quad (15.28)$$

The bright-fringe positions then become

$$y_m \approx \frac{m\lambda D}{d} \quad (15.29)$$

and adjacent bright fringes are separated by

$$\Delta y \approx \frac{\lambda D}{d}. \quad (15.30)$$

Likewise, dark fringes occur at

$$y_m^{(\text{dark})} \approx \left(m + \frac{1}{2}\right) \frac{\lambda D}{d}. \quad (15.31)$$

Equation (15.30) makes the design dependencies transparent. Larger wavelength or larger screen distance spreads the pattern out. Larger slit separation compresses it. These trends are as important as the numerical formula because they provide immediate reasonableness checks.

Keeping the trigonometry, but still using the far-field path approximation, gives

$$y_m = D \tan \left[\arcsin \left(\frac{m\lambda}{d} \right) \right] = \frac{Dm\lambda/d}{\sqrt{1 - (m\lambda/d)^2}}. \quad (15.32)$$

Thus a useful estimate of the fractional small-angle error is $(m\lambda/d)^2/2$. For $y/D = 0.150$, replacing $\sin \theta$ by y/D overestimates the sine by $\sqrt{1 + 0.150^2} - 1 \simeq 1.12\%$. This is small for a demonstration but significant for a precise wavelength measurement. Equation (15.32) is exact in trigonometry, not an exact finite-distance diffraction solution.

Example 15.6: Fringe spacing in a teaching laboratory

A red laser with wavelength 650 nm illuminates a double slit with separation 0.300 mm. The screen is 2.00 m away. Estimate the separation of adjacent bright fringes.

Using Eq. (15.30),

$$\begin{aligned} \Delta y &= \frac{(650 \times 10^{-9} \text{ m})(2.00 \text{ m})}{0.300 \times 10^{-3} \text{ m}} \\ &= 4.33 \times 10^{-3} \text{ m} = 4.33 \text{ mm}. \end{aligned}$$

A few-millimetre spacing is easily resolved by eye or by an ordinary camera, so the result is experimentally reasonable.

Example 15.7: Measuring an unknown wavelength

A double slit of known separation $d = 0.200$ mm is placed $D = 1.50$ m from a screen. The distance from the central bright fringe to the tenth bright fringe on one side is 47.0 mm. Estimate the wavelength.

From Eq. (15.29),

$$\lambda = \frac{y_m d}{m D} = \frac{(47.0 \times 10^{-3})(0.200 \times 10^{-3})}{(10)(1.50)} = 6.27 \times 10^{-7} \text{ m}.$$

Thus

$$\lambda \approx 627 \text{ nm}.$$

Using several well-resolved orders reduces random position uncertainty. The angular approximation and the diffraction envelope must still be checked; increasing the order indefinitely does not guarantee a better measurement.

Caution

The formula $y_m = m\lambda D/d$ is a small-angle approximation. When y is not small compared with D , use the angular condition $d \sin \theta = m\lambda$ together with the exact geometry $\tan \theta = y/D$.

Guided checks

A student doubles both D and d while keeping the wavelength fixed. Predict the new fringe spacing without calculation. Then verify your reasoning with Eq. (15.30).

15.7.1 A plate over one slit: fringe displacement and its sign

A thin, normally traversed plate covering the upper slit S_1 adds optical path $\epsilon = (n_p - 1)t_p$ in air. Neglect its amplitude change and assume the added phase is approximately uniform over the observed angular range. With the sign convention of Fig. 15.3,

$$\Delta\phi = \frac{2\pi}{\lambda_0} \left(\frac{dy}{D} - \epsilon \right), \quad y_m \simeq \frac{D}{d}(m\lambda_0 + \epsilon). \quad (15.33)$$

The zero-order fringe moves toward the covered slit: its extra material delay is compensated by a shorter geometrical path from that slit. The spacing remains $\lambda_0 D/d$ because a constant phase offset does not change the phase gradient across the screen.

Example 15.8: Tracking a shifted central fringe

A 532 nm laser illuminates slits with $d = 0.250$ mm and $D = 1.50$ m. A plate over S_1 adds $\epsilon = 0.80$ μm . Determine the displacement of the original central fringe.

Strategy. Track the same interference order while adding the plate continuously. The original $m = 0$ fringe must regain zero relative phase. Equation (15.33) gives

$$\delta y = \frac{D\epsilon}{d} = 4.80 \text{ mm}, \quad \Delta y = \frac{\lambda_0 D}{d} = 3.192 \text{ mm}.$$

The tracked shift is $\epsilon/\lambda_0 = 1.504$ fringe periods toward S_1 . The phase increment of beam 1 is $2\pi(1.504)$; beam 2 minus beam 1 changes by its negative at a fixed pixel. A final photograph alone identifies the shift only modulo one fringe period, here 0.504 periods. Calling the tracked displacement simply “half a fringe” would lose one complete cycle. The final coordinate satisfies $|y|/D \ll 1$, as assumed.

15.8 Intensity distribution across the ideal double-slit pattern

The bright- and dark-fringe conditions give special locations, but a detector measures intensity at every position. For two equal narrow slits, let the intensity contribution of each slit at the observation point be I_0 . Combining Eqs. (15.9) and (15.24) gives

$$I(\theta) = 4I_0 \cos^2 \left(\frac{\pi d \sin \theta}{\lambda} \right). \quad (15.34)$$

If we define $I_{\max} = 4I_0$, then

$$I(\theta) = I_{\max} \cos^2 \left(\frac{\pi d \sin \theta}{\lambda} \right). \quad (15.35)$$

Under the small-angle approximation,

$$I(y) \approx I_{\max} \cos^2 \left(\frac{\pi dy}{\lambda D} \right). \quad (15.36)$$

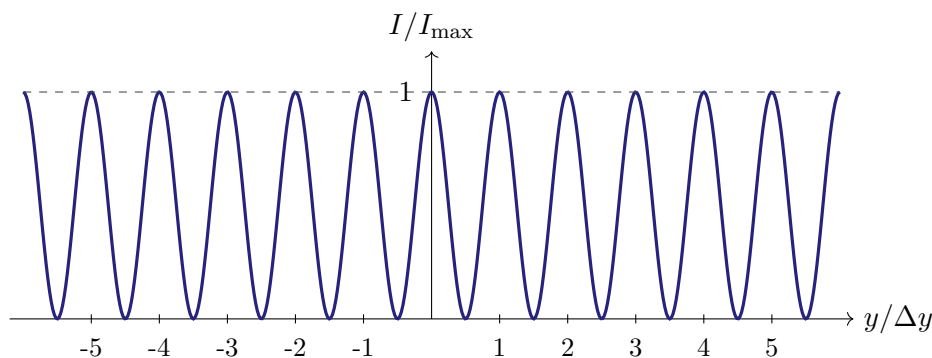


Figure 15.4: Normalized ideal double-slit intensity as a function of screen coordinate measured in units of the fringe spacing. Finite slit width is neglected here.

Figure 15.4 shows the ideal result, which predicts equally high maxima. A real slit has finite width, so each slit diffracts with an angle-dependent amplitude. The interference fringes are therefore multiplied by a slowly varying diffraction envelope. The next section explains this point without yet developing the full diffraction theory.

Example 15.9: Intensity at a specified phase

At some screen point the phase difference between the waves from two equal slits is $\Delta\phi = \pi/3$. Each slit alone produces intensity $I_0 = 2.0 \text{ W m}^{-2}$ at that point. Find the combined intensity. Equation (15.9) gives

$$I = 4I_0 \cos^2\left(\frac{\pi}{6}\right) = 8.0 \left(\frac{\sqrt{3}}{2}\right)^2 = 6.0 \text{ W m}^{-2}.$$

The incoherent sum would be only 4.0 W m^{-2} , showing directly that coherent superposition redistributes the intensity.

Guided checks

At which phase differences does the slope of $I(\Delta\phi)$ have the largest magnitude? Why are such operating points useful in sensors that convert a small phase change into an intensity change?

15.9 Real double slits: finite width and the diffraction envelope

The ideal formula assumes that each slit is so narrow that its emitted amplitude is effectively constant over the angular region of interest. Real slits have a finite width a . Different points across one slit then contribute slightly different phases at an off-axis observation point. Their self-interference produces a single-slit diffraction envelope.

For two identical rectangular slits of width a and separation d , the Fraunhofer pattern has the form

$$I(\theta) = I_{\text{scale}} \left(\frac{\sin \beta}{\beta}\right)^2 \cos^2 \alpha, \quad \beta = \frac{\pi a \sin \theta}{\lambda}, \quad \alpha = \frac{\pi d \sin \theta}{\lambda}. \quad (15.37)$$

This formula is included here only to identify the two physical scales.

Here $I_{\text{scale}} = I(0)$ for equal, in-phase illumination and $\sin \beta / \beta$ is defined by its limit 1 at $\beta = 0$. Figure 15.5 compares the fringe factor and its envelope. The product's local maxima can move slightly away from the maxima of the cosine factor when the envelope varies appreciably; “missing

order” refers to a nominal cosine maximum coincident with an envelope zero. The factor $\cos^2 \alpha$ gives the closely spaced two-slit fringes. The factor $(\sin \beta / \beta)^2$ gives the broad diffraction envelope. A full derivation belongs to the diffraction chapter.

If an interference maximum occurs at the same angle as a single-slit minimum, that interference order is suppressed. Such “missing orders” provide a useful reminder that the simple condition $d \sin \theta = m\lambda$ locates where the two slit-center contributions would be in phase; it does not guarantee that each slit emits appreciable amplitude at that angle.

Pedagogical boundary

Equation (15.37) is a preview, not a substitute for a diffraction derivation. In the present chapter, use it to interpret realistic fringe envelopes and missing orders. The physical origin and detailed mathematics of the single-slit factor are developed later.

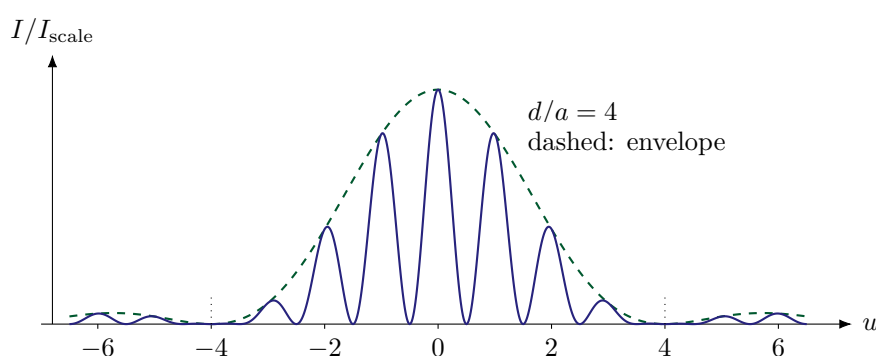


Figure 15.5: Finite-width double-slit pattern with $u = d \sin \theta / \lambda$ and $d/a = 4$. The curve is $[\sin(\pi u/4)/(\pi u/4)]^2 \cos^2(\pi u)$; the dashed curve is its envelope. Nominal orders $m = \pm 4$ coincide with the first envelope zeros.

Example 15.10: A missing interference order

Two identical slits have width $a = 0.050$ mm and separation $d = 0.200$ mm. Which two-slit bright orders coincide with single-slit minima?

A two-slit bright fringe satisfies

$$d \sin \theta = m\lambda.$$

A single-slit minimum satisfies

$$a \sin \theta = p\lambda, \quad p = 1, 2, 3, \dots$$

At coincidence,

$$\frac{m}{d} = \frac{p}{a} \Rightarrow m = p \frac{d}{a} = 4p.$$

Therefore the fourth, eighth, twelfth, ... interference orders are missing in the ideal rectangular-slit model.

15.10 Interference in a refractive medium

Immersing a double-slit apparatus in a uniform medium changes the wavelength but not the optical frequency. With refractive index n ,

$$\lambda = \frac{\lambda_0}{n}. \quad (15.38)$$

The small-angle fringe spacing becomes

$$\Delta y_n \approx \frac{\lambda_0 D}{nd} = \frac{1}{n} \Delta y_{\text{air}} \quad (15.39)$$

if air is approximated by $n \approx 1$. Thus a medium with $n > 1$ compresses the fringe pattern.

This result can also be derived directly from optical path: the same geometric path difference corresponds to a larger optical-path difference by a factor n , hence to a larger phase difference. A smaller angular displacement is therefore needed to accumulate the phase change from one fringe to the next.

The screen and both slits must share the same medium for this simple result. If a plane interface separates the slits from an external screen, refraction changes the ray geometry and must also be included.

Example 15.11: Double slit immersed in water

A double-slit pattern has fringe spacing 3.60 mm in air. The entire optical region between the slits and screen is then filled with water of refractive index 1.333, with all geometry unchanged. Estimate the new spacing.

$$\Delta y_{\text{water}} = \frac{\Delta y_{\text{air}}}{n} = \frac{3.60 \text{ mm}}{1.333} \approx 2.70 \text{ mm}.$$

The fringes move closer together because the wavelength in water is shorter.

Caution

Do not change the frequency when light enters a stationary transparent medium. The source sets the frequency. The phase velocity and wavelength change according to $v = c/n$ and $\lambda = \lambda_0/n$.

15.11 Thin-film interference: why reflection phases matter

The geometry in Fig. 15.6 produces interference because an incident wave is partially reflected at the first interface and partially transmitted into the film. The transmitted part can reflect at the second interface and return to the first medium, where it overlaps the first reflected wave. The two reflected waves have followed different optical histories.

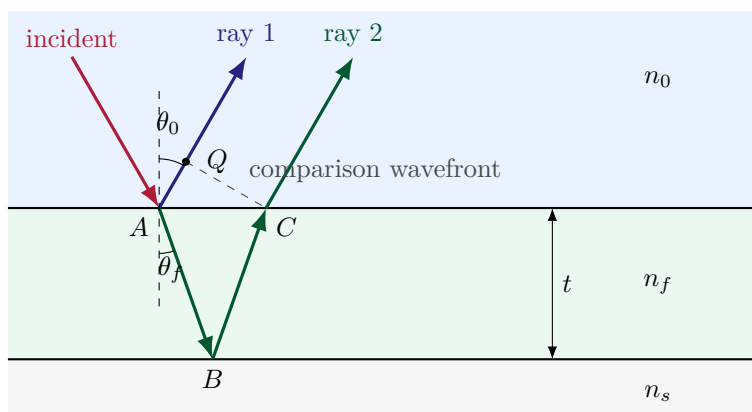


Figure 15.6: Two principal reflected rays at a film. A and C are the exit points; Q and C lie on a common outgoing wavefront. The drawn angles obey Snell's law for $n_0 = 1$ and $n_f = 1.5$ with $\theta_0 = 30^\circ$. These exaggerated oblique rays clarify the geometry; the core thickness rules take the near-normal limit.

Near normal incidence, the second ray travels an additional geometric distance approximately $2t$ inside the film. Its propagation phase relative to the first ray is therefore

$$\Delta\phi_{\text{prop}} = \frac{2\pi}{\lambda_0}(2n_ft) = \frac{4\pi n_ft}{\lambda_0}. \quad (15.40)$$

At oblique incidence the corresponding optical-path contribution is $2n_ft \cos \theta_f$ for the standard parallel-film geometry, where θ_f is the angle inside the film. The main chapter uses near-normal incidence so that $\cos \theta_f \approx 1$.

Propagation is only part of the phase ledger. Reflection can add a phase shift of π .

15.12 Phase changes on reflection

For a wave reflected at normal incidence from an interface between two ordinary, nonabsorbing dielectric media:

- reflection from a lower refractive index toward a higher refractive index produces a phase reversal of π in the reflected electric field;
- reflection from a higher refractive index toward a lower refractive index produces no π phase reversal.

A useful mechanical analogy is a pulse reflecting from a fixed end versus a free end, but the optical result should ultimately be understood from electromagnetic boundary conditions.

Reflection condition	Reflected-field phase change	Practical interpretation
$n_{\text{after}} > n_{\text{before}}$	π	one half-cycle reversal
$n_{\text{after}} < n_{\text{before}}$	0	no half-cycle reversal

For the pair of reflected rays from a film, only the *difference* between their reflection phase shifts matters. There are therefore two important classes:

1. **one net phase reversal:** one ray receives a π shift and the other does not;
2. **zero net phase reversal:** both rays receive the same reflection shift (both 0 or both π).

Method

Thin-film phase table. Make a two-row table for the two reflected rays. Record (i) the interface where each reflection occurs, (ii) whether that reflection is low-to-high or high-to-low index, and (iii) the corresponding 0 or π phase shift. Only after this table is complete should you write a thickness condition.

Caution

Do not say that “reflection always adds half a wavelength.” The π phase reversal occurs only for the reflected electric field when reflection is from lower to higher refractive index under the assumptions stated here. Transmission does not acquire the same universal half-cycle reversal.

Guided checks

For an air–oil–water system with $1 < n_{\text{oil}} < n_{\text{water}}$, determine the reflection phase shift of the ray reflected from the air–oil interface and of the ray reflected from the oil–water interface. Is there a net reflection-phase difference of 0 or π ?

15.13 Thin-film interference at near-normal incidence

Combining propagation and reflection phases gives a compact set of rules. For reflected light from a film of refractive index n_f and thickness t at near-normal incidence:

15.13.1 Case A: exactly one net π reflection reversal

The total phase difference may be written

$$\Delta\phi = \frac{4\pi n_f t}{\lambda_0} + \pi \pmod{2\pi}. \quad (15.41)$$

Constructive reflected interference requires $\Delta\phi = 2\pi m$, which gives

$$2n_f t = \left(m + \frac{1}{2}\right) \lambda_0, \quad \text{one net reflection reversal.} \quad (15.42)$$

Destructive reflected interference requires

$$2n_f t = m\lambda_0, \quad \text{one net reflection reversal.} \quad (15.43)$$

The order is relabelled so that $m = 0, 1, 2, \dots$ gives nonnegative thickness. For $m = 0$, the minimum condition includes $t = 0$. This identifies an out-of-phase relation, not necessarily zero reflected power. A vanishing film reduces to the direct interface between the two exterior media; its reflectance is zero only if those media are index matched. The complete sum of internal reflections recovers this limit exactly in the advanced treatment.

15.13.2 Case B: zero net π reflection reversal

Now the reflection contributions cancel in the relative phase, so

$$\Delta\phi = \frac{4\pi n_f t}{\lambda_0} \pmod{2\pi}. \quad (15.44)$$

Constructive reflected interference occurs for

$$2n_f t = m\lambda_0, \quad \text{zero net reflection reversal,} \quad (15.45)$$

and destructive reflected interference occurs for

$$2n_f t = \left(m + \frac{1}{2}\right) \lambda_0, \quad \text{zero net reflection reversal.} \quad (15.46)$$

Key message

The thin-film formulas are not a list to memorize independently. They are all consequences of one equation: propagation phase plus reflection phase equals the phase required for a maximum or minimum.

Example 15.12: Minimum thickness for strong reflected green light

A film of refractive index $n_f = 1.25$ lies between air ($n \approx 1$) and a substrate of refractive index 1.50. What is the smallest nonzero film thickness that gives constructive reflection for vacuum wavelength $\lambda_0 = 540$ nm at near-normal incidence?

At the air–film interface the reflection is low-to-high, so the first reflected ray gains π . At the film–substrate interface the reflection is also low-to-high, so the second reflected ray also gains π . The two reflection shifts are the same, giving zero net reflection-phase difference. We therefore use Eq. (15.45):

$$2n_ft = m\lambda_0.$$

The smallest nonzero thickness corresponds to $m = 1$:

$$t = \frac{\lambda_0}{2n_f} = \frac{540 \text{ nm}}{2(1.25)} = 216 \text{ nm}.$$

Example 15.13: A coating with two reflection reversals

A transparent film of index $n_f = 1.38$ is deposited on glass of index 1.52 and viewed from air. Find the smallest nonzero thickness that gives a reflected minimum at $\lambda_0 = 550$ nm.

Air-to-film reflection gives a π shift, and film-to-glass reflection also gives a π shift. Thus there is zero net reflection reversal. A reflected minimum therefore satisfies Eq. (15.46):

$$2n_ft = \left(m + \frac{1}{2}\right) \lambda_0.$$

With $m = 0$,

$$t = \frac{\lambda_0}{4n_f} = \frac{550 \text{ nm}}{4(1.38)} \approx 99.6 \text{ nm}.$$

This is the familiar quarter-wave optical-thickness condition.

Guided checks

Why can two films with the same n_f and t give opposite bright/dark behavior when the substrate refractive index is changed? Identify the term in the phase ledger that changes.

15.14 Anti-reflection coatings and amplitude matching

A single-layer anti-reflection coating is designed so that the principal reflected waves interfere destructively at a chosen wavelength. Suppose light is incident from medium n_0 onto a coating n_1 on a substrate n_2 , with

$$n_0 < n_1 < n_2. \quad (15.47)$$

Both principal reflected rays then undergo a π phase reversal, so their relative reflection phase is zero. Destructive reflection requires

$$2n_1t = \left(m + \frac{1}{2}\right) \lambda_0. \quad (15.48)$$

The thinnest useful coating is therefore the quarter-wave thickness

$$t = \frac{\lambda_0}{4n_1}. \quad (15.49)$$

Phase cancellation alone does not guarantee zero reflected intensity. The amplitudes must also satisfy a matching condition. For a lossless single layer at normal incidence, the complete sum of all internally reflected waves gives exact zero reflection at the design wavelength when

$$n_1 = \sqrt{n_0 n_2}. \quad (15.50)$$

This is an exact result for the stated single-layer model, not a first-two-ray approximation. The first internally reflected emerging wave also carries transmission factors, so matching just its bare interface coefficient to the first reflection is not a complete derivation. The advanced section derives the full reflection amplitude. Figure 15.7 shows how index mismatch leaves a finite residual reflectance even at a quarter-wave thickness. Material dispersion, oblique incidence and broadband requirements motivate more elaborate coatings.

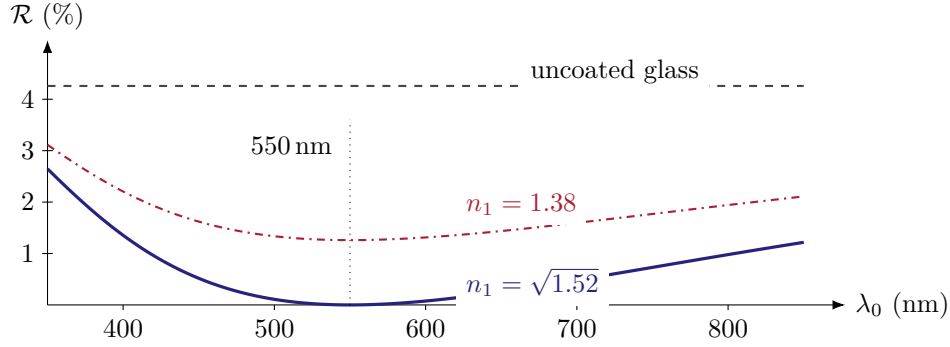


Figure 15.7: Calculated all-orders reflectance of a single layer on $n_2 = 1.52$ glass, viewed from air. Each coating has thickness $550 \text{ nm}/(4n_1)$. The matched layer reaches zero at the design wavelength; $n_1 = 1.38$ leaves 1.26%. Indices are held constant to isolate interference rather than material dispersion.

Example 15.14: Design wavelength for a coating

A coating has refractive index 1.30 and thickness 105 nm. For the simple quarter-wave design, what vacuum wavelength is most strongly targeted for reflection suppression?

From Eq. (15.49),

$$\lambda_0 = 4n_1 t = 4(1.30)(105 \text{ nm}) = 546 \text{ nm}.$$

The design wavelength lies in the green region. The minimum reflectance also depends on the substrate index and amplitude matching.

Caution

A quarter-wave thickness is a phase condition. It does not by itself guarantee perfect anti-reflection performance over all wavelengths and angles. Broadband optical coatings are an engineering optimization problem involving multiple layers, Fresnel amplitudes, material dispersion and manufacturing tolerances.

15.15 Thin-film colours, wedge films and qualitative diagnostics

Soap bubbles, oil films and oxide layers often display colours even when the material itself is nearly transparent. White light contains many wavelengths. At a given film thickness and viewing angle, some wavelengths are preferentially reinforced in reflection while others are suppressed. If the film thickness varies across the surface, the favored wavelengths change with position and a colour pattern appears.

This interpretation immediately explains several observations. First, the colour can change with viewing angle because the propagation phase at oblique incidence contains a $\cos \theta_f$ factor. Second,

very thin regions can appear dark for some interface configurations because the reflection phase relation alone may favor destructive interference. Third, the pattern can move when the film flows because thickness is changing in time.

A wedge-shaped film converts thickness variation into approximately straight fringes (Fig. 15.8). If the wedge angle is small, $t(x) \approx \alpha x$, and successive conditions differing by $\Delta(2nt) = \lambda_0$ occur at nearly equal spatial intervals. Wedge fringes are therefore useful for measuring small angles, testing surface flatness and estimating thickness changes.

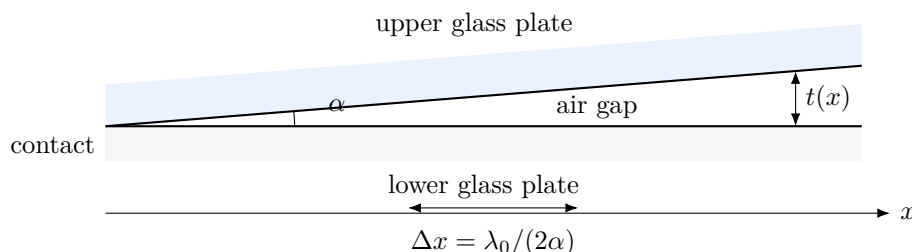


Figure 15.8: An air wedge converts a thickness gradient into equal-thickness reflected fringes. The gap and wedge angle are exaggerated. For one net reflection reversal, successive dark orders differ by $\Delta t = \lambda_0/2$; with $t \simeq \alpha x$, their spacing is $\lambda_0/(2\alpha)$.

Example 15.15: Wedge-film fringe spacing

A thin air wedge lies between two glass plates separated by a small angle $\alpha = 0.50 \text{ mrad}$. Monochromatic light of wavelength 600 nm is observed in reflection near normal incidence. Estimate the spacing between successive dark fringes, assuming the air film has one net reflection phase reversal.

For reflected minima with one net reversal,

$$2t = m\lambda_0.$$

Successive minima differ in thickness by

$$\Delta t = \frac{\lambda_0}{2}.$$

Since $t \approx \alpha x$,

$$\Delta x = \frac{\Delta t}{\alpha} = \frac{600 \times 10^{-9}/2}{0.50 \times 10^{-3}} = 6.0 \times 10^{-4} \text{ m} = 0.60 \text{ mm}.$$

15.16 Newton's rings

Newton's rings are concentric interference fringes produced by the thin film between a gently curved transparent surface and a flat plate. A common arrangement places a plano-convex lens of radius of curvature R on a flat glass plate. Near the contact point, an air film of thickness t increases with radial distance r .

In Fig. 15.9, the lower lens surface touches the plate at its centre. If the centre of curvature is a height R above the contact, circle geometry gives

$$r^2 + (R - t)^2 = R^2, \quad t = R - \sqrt{R^2 - r^2} = \frac{r^2}{2R} + O\left(\frac{r^4}{R^3}\right).$$

For $r \ll R$, this gives

$$t \approx \frac{r^2}{2R}. \quad (15.51)$$

For an air film viewed in reflection, the reflection at the lens–air interface is high-to-low index and has no π reversal, while the reflection at the air–plate interface is low-to-high and has a π reversal. There is therefore one net reflection reversal. Reflected dark rings satisfy

$$2t = m\lambda_0. \quad (15.52)$$

Using Eq. (15.51),

$$\boxed{r_m^2 = m\lambda_0 R, \quad m = 0, 1, 2, \dots} \quad (15.53)$$

for an air film. The $m = 0$ result predicts a dark central spot in the ideal contact geometry.

If the gap contains a transparent liquid of refractive index n , the propagation condition becomes $2nt = m\lambda_0$ and

$$r_m^2 = \frac{m\lambda_0 R}{n}. \quad (15.54)$$

Thus the ring pattern can be used to infer a refractive index or radius of curvature.

The liquid formula assumes that the gap index remains below the indices of both bounding glass surfaces, so that the same one-reversal phase pattern applies. Reflected bright rings instead satisfy

$$r_{m,\text{bright}}^2 = \left(m + \frac{1}{2}\right) \frac{\lambda_0 R}{n}, \quad m = 0, 1, 2, \dots$$

Equal increments in r^2 , rather than in r , explain why successive rings become closer together farther from the centre. In a transparent, lossless arrangement the corresponding transmitted pattern has complementary extrema once the appropriate output powers are included.

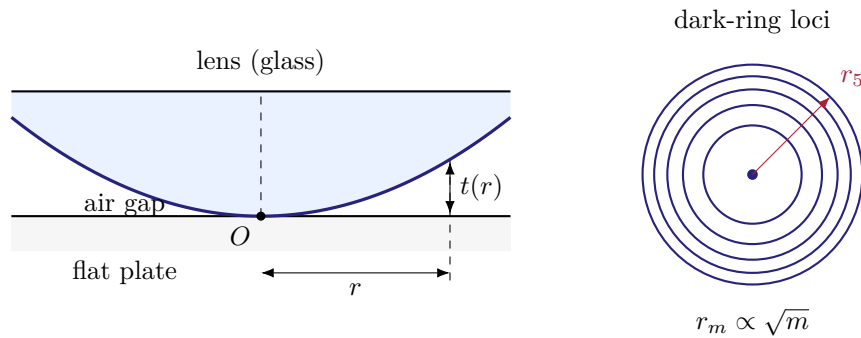


Figure 15.9: Newton's-rings geometry: the convex lower lens surface contacts the plate at O , and the gap grows outward. The vertical gap scale is exaggerated. The right panel shows loci of reflected minima, with radii proportional to $\sqrt{m}$, rather than a photograph or an intensity image.

Example 15.16: Radius of curvature from Newton's rings

In reflected light of wavelength 589 nm, the fifth dark Newton ring has radius 2.10 mm. Assuming an air film and ideal contact, estimate the radius of curvature R of the lens.

Using Eq. (15.53),

$$R = \frac{r_m^2}{m\lambda_0} = \frac{(2.10 \times 10^{-3})^2}{5(589 \times 10^{-9})} \approx 1.50 \text{ m}.$$

A radius of curvature much larger than the millimetre-scale ring radius is consistent with the shallow-surface approximation used in the derivation.

Guided checks

If a liquid with $n > 1$ is introduced into the lens–plate gap while the wavelength in vacuum is unchanged, do the dark-ring radii increase or decrease? Use Eq. (15.54) to justify the answer.

15.16.1 Imperfect contact and fitting ring diameters

A small residual central gap t_0 changes the geometry to

$$t(r) \simeq t_0 + \frac{r^2}{2R}.$$

For the same one-reversal reflected minima,

$$r_m^2 = \frac{m\lambda_0 R}{n} - 2Rt_0, \quad D_m^2 = \frac{4m\lambda_0 R}{n} - 8Rt_0, \quad (15.55)$$

where $D_m = 2r_m$ denotes a ring diameter, distinct from the screen distance D . Only orders with $r_m^2 \geq 0$ are present. The centre need not be dark when $t_0 \neq 0$.

Measuring opposite points on each ring reduces dependence on a guessed centre. Differences between orders eliminate the constant contact offset:

$$D_{m+p}^2 - D_m^2 = \frac{4p\lambda_0 R}{n}. \quad (15.56)$$

A fit to squared diameter versus order determines the slope without requiring perfect contact. Absolute t_0 additionally requires the correct order assignment. If the diameter uncertainty is u_D , first-order propagation gives $u_{D^2} \simeq 2D_m u_D$; equal diameter errors therefore do not imply equal errors in squared diameter. A weighted fit should reflect this difference. A displaced ring centre or a distorted lens can generate nonconstant errors, so not every imperfect dataset is repaired by adding an intercept.

15.17 The Michelson interferometer

Figure 15.10 shows a Michelson interferometer, which separates one input beam into two arms, reflects the two beams from mirrors and recombines them at a beam splitter. Because the same source feeds both arms, the recombined beams can maintain a stable relative phase when the optical-path difference lies within the coherence range and the instrument is properly aligned.

Let the one-way geometric arm lengths be L_1 and L_2 in the same medium. Each beam travels to its mirror and back, so the propagation optical-path difference is approximately

$$\Delta_{\text{OPL}} = 2n(L_2 - L_1) \quad (15.57)$$

apart from fixed beam-splitter and reflection phases that are normally absorbed into a constant reference phase.

The arm lengths alone therefore do not label a particular output port as bright or dark: the fixed beam-splitter phase also matters. The measured signal can be written $I = A + B \cos[2k_0 n(L_2 - L_1) + \phi_{\text{fixed}}]$. Parallel, perfectly overlapping beams can give a spatially uniform signal that varies with mirror position. A small relative tilt produces straight spatial fringes; a suitable divergent-beam geometry produces rings. Mirror travel can be inferred from the temporal cycles at one detector point in either case.

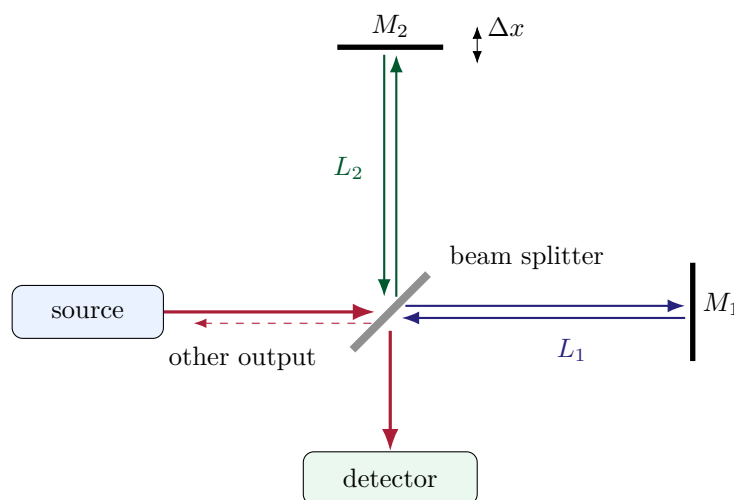


Figure 15.10: Michelson paths with a beam-splitter interface oriented to reflect the incoming beam upward. The beams return along their arms and combine at the detector port; a complementary output returns toward the source. Small transverse offsets separate outgoing and returning arrows for legibility. Moving M_2 changes the double-pass arm-2 path.

15.17.1 Mirror displacement and fringe count

If one mirror moves by a small distance Δx along the beam direction, the round-trip optical path changes by

$$\Delta(\text{OPL}) = 2n\Delta x. \quad (15.58)$$

In air, $n \approx 1$. Each change of optical path by one vacuum wavelength shifts the interference condition by one full fringe, so

$$\boxed{N = \frac{2\Delta x}{\lambda_0}} \iff \boxed{\Delta x = \frac{N\lambda_0}{2}}. \quad (15.59)$$

Example 15.17: Mirror motion from counted fringes

A Michelson interferometer uses 632.8 nm light. During a slow mirror motion, 240 fringes pass a fixed reference point. Find the mirror displacement.

$$\Delta x = \frac{N\lambda_0}{2} = \frac{240(632.8 \text{ nm})}{2} = 75.9 \mu\text{m}.$$

A submillimetre displacement has been converted into a large, easily counted number of optical cycles.

15.17.2 Inserting a transparent plate or gas cell

Suppose a plate of thickness t and index n replaces the same thickness of air in one Michelson arm. Because the beam passes through the plate twice, the optical-path increase is

$$\Delta(\text{OPL}) = 2t(n - 1). \quad (15.60)$$

This expression assumes normal traversal, unchanged geometry, a fixed wavelength, and negligible parasitic reflections. More generally, a cell whose index changes from n_a to n_b contributes $2t(n_b - n_a)$, while its unchanged windows cancel in the difference. The number of fringes shifted is therefore

$$\boxed{N = \frac{2t(n - 1)}{\lambda_0}}. \quad (15.61)$$

This can be inverted to determine n or t .

Example 15.18: Refractive index from a Michelson fringe shift

A cell of internal length $t = 5.00$ cm is placed in one arm of a Michelson interferometer using 632.8 nm light. The cell initially contains vacuum. When it is filled with a gas, 43 fringes pass the reference point. Estimate the gas refractive index, assuming the beam traverses the cell twice.

From Eq. (15.61),

$$n - 1 = \frac{N\lambda_0}{2t} = \frac{43(632.8 \times 10^{-9})}{2(5.00 \times 10^{-2})} = 2.72 \times 10^{-4}.$$

Thus

$$n \approx 1.000272.$$

Interferometry is well suited to such small refractive-index differences because phase accumulates over the full optical path.

Method

In a Michelson interferometer, always ask whether a physical element is traversed once or twice. A mirror displacement changes a round-trip path, and a plate in an arm is normally crossed on the outward and return journeys. Missing this factor of two is one of the most common calculation errors.

Guided checks

A mirror is moved by $\lambda_0/4$. By how much does the round-trip optical path change? Does a fixed detector point switch from bright to dark or return to the same interference state?

15.18 Interferometry as a measurement technology

Interference is especially valuable because phase is sensitive to several physical quantities at once. If an optical path changes by

$$\delta(\text{OPL}) = n \delta L + L \delta n \quad (15.62)$$

for a uniform segment, then an interferometer can respond to displacement δL , refractive-index change δn , or both. Temperature, pressure, strain and composition can influence one or both of these quantities.

15.18.1 Displacement and vibration sensing

A mirror attached to a vibrating structure converts mechanical displacement into optical phase. Near a quadrature operating point, where the slope of the cosine fringe is largest, a small phase variation produces an approximately linear intensity variation. This principle is used in precision positioning, vibration diagnostics and calibration systems.

15.18.2 Refractometry and environmental sensing

A gas or liquid cell inserted in one arm converts refractive index into fringe shift. Since refractive index can depend on density, pressure, temperature and composition, interferometric refractometers can become sensitive environmental or process sensors. Good practice requires independent temperature and pressure control because an interferometer measures optical path, not “composition” directly.

15.18.3 Surface and thickness metrology

When a reference wave reflects from a known surface and a measurement wave reflects from a test surface, local height differences become phase differences. Contour-like fringe patterns can reveal flatness errors or deformation. Thin-film thickness can similarly be inferred from spectral or angular interference if the refractive index is known.

15.18.4 Low-coherence and medical imaging connection

If a broadband source has a short coherence length, interference occurs only when the reference and sample optical paths match within a narrow range. This provides depth selectivity. Optical coherence tomography uses this principle in a much more sophisticated form to obtain depth-resolved images in scattering tissue. The present chapter provides the underlying ideas of optical-path difference and coherence, but not the full imaging theory.

Key message

An interferometer measures optical phase. It measures phase or fringe position, which depends on optical path. Converting that result into distance, refractive index, strain or temperature requires a physical model and calibration.

15.19 Laboratory connection: measuring wavelength with a double slit

A straightforward laboratory experiment is to determine an unknown laser wavelength from the slope of fringe position versus fringe order. Rather than measuring one fringe spacing, record the positions y_m of several bright fringes and fit

$$y_m = b + sm, \quad (15.63)$$

where the ideal small-angle model predicts

$$s = \frac{\lambda D}{d}. \quad (15.64)$$

The wavelength estimate is then

$$\lambda = \frac{sd}{D}. \quad (15.65)$$

The intercept b accounts for imperfect choice of coordinate origin. Fitting several orders reduces sensitivity to a single pixel or ruler reading and makes residuals available as a diagnostic of the model.

15.19.1 Suggested experimental procedure

1. Measure the slit-to-screen distance D and record its uncertainty.
2. Record the slit separation d from the manufacturer or microscope calibration, including uncertainty.
3. Photograph the pattern with the camera sensor approximately parallel to the screen.
4. Identify the central fringe and at least five positive and five negative orders if possible.
5. Calibrate image pixels into millimetres using a ruler or known target in the same plane.
6. Fit y_m against m using linear regression.

7. Convert the fitted slope to λ with Eq. (15.65).
8. Inspect residuals. Curvature can indicate failure of the small-angle approximation, perspective distortion or a nonuniform phase offset. A constant origin error changes the fitted intercept, not the residual curvature.

If s , d and D are independent, a first-order relative uncertainty estimate is

$$\left(\frac{\sigma_\lambda}{\lambda}\right)^2 \approx \left(\frac{\sigma_s}{s}\right)^2 + \left(\frac{\sigma_d}{d}\right)^2 + \left(\frac{\sigma_D}{D}\right)^2. \quad (15.66)$$

This formula immediately shows where improved instrumentation is most useful.

Example 15.19: Uncertainty budget for a wavelength measurement

A fitted fringe slope is $s = (3.25 \pm 0.03)$ mm/order, the slit separation is $d = (0.250 \pm 0.002)$ mm and the screen distance is $D = (1.300 \pm 0.005)$ m. Estimate the wavelength and its standard uncertainty.

The central value is

$$\lambda = \frac{sd}{D} = \frac{(3.25 \times 10^{-3})(0.250 \times 10^{-3})}{1.300} = 6.25 \times 10^{-7} \text{ m} = 625 \text{ nm}.$$

The relative uncertainty is

$$\begin{aligned} \frac{\sigma_\lambda}{\lambda} &\approx \sqrt{(0.03/3.25)^2 + (0.002/0.250)^2 + (0.005/1.300)^2} \\ &\approx 0.0128. \end{aligned}$$

Thus

$$\sigma_\lambda \approx (0.0128)(625 \text{ nm}) \approx 8 \text{ nm},$$

so a reasonable report is $\lambda = (625 \pm 8) \text{ nm}$ under the stated uncertainty model.

15.20 Computational connection: simulate the pattern and destroy coherence

A simple numerical model reinforces the distinction between deterministic phase and random phase. For an ideal double slit, choose a grid of screen coordinates y_i and evaluate

$$I(y_i) = I_{\max} \cos^2\left(\frac{\pi d y_i}{\lambda D}\right). \quad (15.67)$$

To include a finite slit width, multiply by the diffraction envelope from Eq. (15.37). Parameter scans over d/λ or D reveal directly how the fringe scale changes.

To model phase instability, replace the deterministic phase difference by

$$\Delta\phi_i(t) = \Delta\phi_i^{(0)} + \eta(t), \quad (15.68)$$

where $\eta(t)$ is a random phase that changes from one simulated exposure to the next. Average the intensity over many realizations. If η becomes uniformly distributed over 0 to 2π , then

$$\langle \cos(\Delta\phi_i^{(0)} + \eta) \rangle \rightarrow 0, \quad (15.69)$$

and the interference term disappears.

Method

A useful computational exercise is to compare three curves on the same coordinate grid: (i) ideal coherent interference, (ii) finite-width interference with a diffraction envelope, and (iii) an ensemble average with random phase noise. Each modification changes a different physical aspect of the pattern, so the comparison is more instructive than plotting only the final result.

Guided checks

In the random-phase simulation, does each individual realization necessarily have no fringes? Explain why averaging many realizations can erase a pattern that is present instantaneously.

15.21 Diagnosing an interference calculation

Before choosing a formula, identify the two fields, the wavelength convention, the relative propagation and reflection phases, and the detector's averaging conditions. A wrong wavelength or a missed double pass usually changes the phase scale; unequal amplitudes, background, polarization mismatch and phase fluctuations usually change contrast. These are distinct experimental signatures.

Caution

A path difference has units of length, whereas a phase difference is dimensionless. Use either nL with λ_0 or L with λ_0/n , never both index corrections together. In a Michelson instrument, count complete bright-to-bright cycles: a bright-to-dark change is half a cycle. At oblique incidence, angles are measured from the normal.

Guided checks

1. Why can a half-wavelength propagation difference give a reflected maximum in a film but a minimum in an unmodified Young experiment?
2. A uniform phase plate moves the fringe pattern. Which parameter would change its spacing instead?
3. Does a smaller measured visibility determine a unique coherence length? Identify two other possible causes.
4. A fixed mirror and changing gas pressure produce moving fringes. What is being measured directly?

15.22 Main chapter summary

Summary

- Interference follows from superposition of electromagnetic fields. The detector measures a time-averaged intensity after field addition.
- For two coherent, equally polarized beams,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \Delta\phi.$$

- Propagation phase is determined by optical path:

$$\Delta\phi_{\text{prop}} = \frac{2\pi}{\lambda_0} \Delta\text{OPL}.$$

Additional reflection and component phases must be added explicitly.

- Young's double-slit experiment gives $d \sin \theta = m\lambda$ for bright fringes and $d \sin \theta = (m + \frac{1}{2})\lambda$ for dark fringes under the ideal assumptions.
- For small angles, $\Delta y \approx \lambda D/d$.
- Finite slit width multiplies the two-slit fringe structure by a diffraction envelope.
- Thin-film interference requires both the extra propagation phase $4\pi n_f t/\lambda_0$ and reflection-phase accounting.
- A low-to-high-index reflection introduces a π phase reversal of the reflected electric field; a high-to-low reflection does not under the main-chapter assumptions.
- A quarter-wave coating suppresses reflection at a design wavelength when phase and amplitude conditions are both favorable.
- Newton's rings convert a radially varying film thickness into concentric fringes, with $r_m^2 = m\lambda_0 R$ for reflected dark rings in an air film.
- A Michelson interferometer converts optical-path changes into fringe counts; in air $N = 2\Delta x/\lambda_0$ for mirror motion.
- Coherence controls whether the interference cross term survives detector averaging.

The compact formula sheet in Appendix B collects the results with their assumptions.

15.23 Advanced Topics for Technical and Engineering Students

Pedagogical boundary

The main chapter deliberately used the smallest mathematical apparatus needed to solve standard two-beam, thin-film, Newton-ring and Michelson problems. The present section develops a more powerful language based on complex amplitudes, coherence functions, Fourier ideas, multiple-beam interference and quantitative measurement. These topics are intended for strong students and for readers interested in photonics, optical engineering, metrology, sensing and experimental physics. Diffraction as a full boundary-value problem remains the subject of the next chapter; here it appears only where it is inseparable from realistic interference measurements.

15.23.1 Complex amplitudes and phasor geometry

For sinusoidal waves, the trigonometric description is physically transparent but can become cumbersome when many phase shifts accumulate. A convenient alternative is to represent a real electric field as the real part of a complex quantity,

$$E(z, t) = \Re\{\tilde{E}(z) e^{-i\omega t}\}, \quad (15.70)$$

where the complex amplitude may be written

$$\tilde{E}(z) = E_0 e^{i(kz + \phi_0)}. \quad (15.71)$$

The measurable field is still real. Complex notation is only a calculation tool that stores amplitude and phase in one symbol.

Suppose two waves of the same frequency and polarization reach a detector:

$$\tilde{E}_1 = E_{01} e^{i\phi_1}, \quad \tilde{E}_2 = E_{02} e^{i\phi_2}. \quad (15.72)$$

Linearity gives

$$\tilde{E} = \tilde{E}_1 + \tilde{E}_2. \quad (15.73)$$

Because time-averaged intensity is proportional to $|\tilde{E}|^2$,

$$|\tilde{E}|^2 = (\tilde{E}_1 + \tilde{E}_2)(\tilde{E}_1^* + \tilde{E}_2^*) \quad (15.74)$$

$$= E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos(\phi_2 - \phi_1), \quad (15.75)$$

which immediately reproduces the two-beam interference law.

Method

When an optical system contains several propagation segments, reflections, plates or phase shifters, use a *phase ledger*. Multiply the complex amplitude by one phase factor for each element:

$$\tilde{E}_{\text{out}} = \tilde{E}_{\text{in}} \exp\left(i \frac{2\pi}{\lambda_0} \text{OPL}\right) \exp(i\phi_{\text{refl}}) \exp(i\phi_{\text{comp}}).$$

Only the relative phase between interfering contributions affects the fringe pattern.

Figure 15.11 makes the complex sum geometric: its squared length contains the dot-product term responsible for interference.

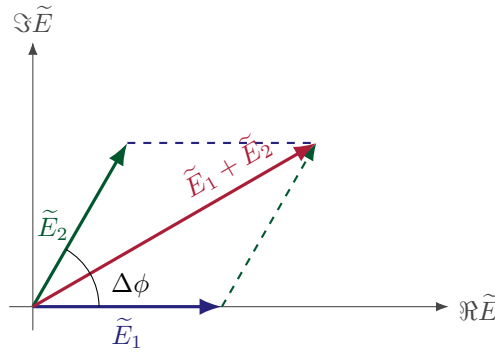


Figure 15.11: Equal-length phasors separated by 60° . Dashed translated copies complete the parallelogram. The resultant squared magnitude is $E_{01}^2 + E_{02}^2 + 2E_{01}E_{02} \cos \Delta\phi$. Both axes represent field amplitude, not spatial coordinates.

15.23.2 Polarization overlap and Jones vectors

The scalar formula assumes that the two electric fields point along the same polarization direction. More generally, let the complex vector amplitudes be $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{E}}_2$. The time-averaged intensity contains the cross term

$$2\Re(\tilde{\mathbf{E}}_1 \cdot \tilde{\mathbf{E}}_2^*). \quad (15.76)$$

If two linearly polarized waves have polarization unit vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ separated by an angle α , then

$$\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_2 = \cos \alpha. \quad (15.77)$$

For otherwise identical coherent beams,

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \alpha \cos \Delta\phi. \quad (15.78)$$

Thus parallel polarizations give the largest modulation, while orthogonal polarizations make the scalar interference term vanish in an ordinary polarization-insensitive detector.

This does not mean that the fields cease to superpose. It means that the time-averaged total intensity has no phase-dependent cross term between orthogonal components. A downstream polarizer can project both fields onto a common axis and thereby restore observable interference in the transmitted component.

Guided checks

Two perfectly coherent beams have equal intensities but mutually perpendicular linear polarizations. What does a polarization-insensitive camera record? What changes if a linear analyzer at 45° to both polarization axes is placed before the camera? Explain using projections, not by saying that the polarizer “creates coherence.”

Example 15.20: Visibility controlled by polarization angle

Two coherent beams have equal intensity I_0 and linear polarizations separated by 60° . From Eq. (15.78),

$$I = 2I_0 + 2I_0 \cos 60^\circ \cos \Delta\phi = 2I_0 + I_0 \cos \Delta\phi.$$

Therefore $I_{\max} = 3I_0$ and $I_{\min} = I_0$, giving

$$\mathcal{V} = \frac{3I_0 - I_0}{3I_0 + I_0} = \frac{1}{2}.$$

Polarization mismatch reduces the contrast even for fully coherent beams.

For fully polarized monochromatic light, polarization and phase can be combined in a two-component complex vector. In a chosen transverse basis,

$$\mathbf{J} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad (15.79)$$

where each component can carry its own phase. A linear polarizer with transmission axis at angle α can be represented by a projection matrix, while an ideal phase retarder changes the relative phase of two orthogonal components. The importance for interferometry is practical: an optical component can change not only the scalar phase but also the polarization state, so the two returning beams may no longer project onto the same detector polarization.

For example, if two arms return Jones vectors $\mathbf{J}_1$ and $\mathbf{J}_2$, the interference contribution is proportional to

$$2\Re(\mathbf{J}_1^\dagger \mathbf{J}_2). \quad (15.80)$$

This inner product simultaneously accounts for amplitude, relative phase and polarization overlap. If the vectors are orthogonal, the inner product vanishes even when each beam is individually coherent.

In laboratory interferometers this matters whenever mirrors are used at non-normal incidence, beam splitters have polarization-dependent coatings, fibers are birefringent, or stress changes the polarization state. One can often recover stable contrast by using polarizers, polarization-maintaining fibers or carefully chosen optical orientations, but the correct solution depends on the instrument rather than on a universal rule.

Example 15.21: A polarizer as a common projection

Let

$$\mathbf{J}_1 = E_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{J}_2 = E_0 e^{i\phi} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Without an analyzer, $\mathbf{J}_1^\dagger \mathbf{J}_2 = 0$ and no phase-dependent intensity term appears. Place a linear analyzer with unit axis

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The transmitted scalar amplitudes are both $E_0/\sqrt{2}$, with relative phase ϕ . They now interfere with full visibility in the transmitted channel. The analyzer has not made the original fields mutually nonorthogonal; it has selected a common measured component.

15.23.3 Partial coherence, linewidth and phase noise

Real optical fields need not have a perfectly fixed phase. A statistical description is therefore useful. Let $\mathcal{E}_j(t)$ denote the complex analytic field, whose real part is the physical electric field; it includes the optical time dependence. This distinguishes it from the real $E_j(t)$ used earlier. For stationary scalar fields with finite mean power, define

$$\Gamma_{12}(\tau) = \langle \mathcal{E}_1^*(t) \mathcal{E}_2(t + \tau) \rangle, \quad (15.81)$$

where the angle brackets denote an appropriate time or ensemble average. The normalized complex degree of coherence is

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}}. \quad (15.82)$$

For the superposition $\mathcal{E}_1(t) + e^{i\phi_{\text{controllable}}} \mathcal{E}_2(t + \tau)$, the detected intensity may be written

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \Re[\gamma_{12}(\tau) e^{i\phi_{\text{controllable}}}] . \quad (15.83)$$

The magnitude obeys $0 \leq |\gamma_{12}| \leq 1$.

Here τ is a signed time shift; a physical delay of the second waveform corresponds to $t - \tau_d$, or $\tau = -\tau_d$. The bound follows from the Cauchy–Schwarz inequality $|\langle \mathcal{E}_1^* \mathcal{E}_2 \rangle|^2 \leq \langle |\mathcal{E}_1|^2 \rangle \langle |\mathcal{E}_2|^2 \rangle$. The common detector proportionality factor cancels in γ_{12} . Perfect coherence corresponds to $|\gamma_{12}| = 1$; complete loss of mutual coherence corresponds to $|\gamma_{12}| = 0$ for the measurement under consideration.

If one scans a controllable phase through a full cycle while I_1 , I_2 and $|\gamma_{12}|$ remain fixed, then

$$\mathcal{V} = \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} |\gamma_{12}|. \quad (15.84)$$

For equal intensities this reduces to $\mathcal{V} = |\gamma_{12}|$. Visibility is therefore a practical probe of coherence, provided unequal powers, detector nonlinearity, polarization mismatch and background light have been controlled.

Caution

A measured visibility below unity does not uniquely determine a source’s coherence. Unequal beam powers, polarization mismatch, vibration, finite detector resolution, stray light and intensity noise can all reduce the observed contrast. Equation (15.84) isolates coherence only after these instrumental effects are treated appropriately.

Example 15.22: Estimating mutual coherence from calibrated visibility

An interferometer is adjusted so that the two beams reaching the detector have equal intensities and parallel polarizations. The measured fringe maximum and minimum are 12.0 and 4.0 arbitrary units after background subtraction. The visibility is

$$\mathcal{V} = \frac{12.0 - 4.0}{12.0 + 4.0} = 0.500.$$

Under the stated calibration conditions, Eq. (15.84) gives $|\gamma_{12}| \approx 0.500$ at that optical-path delay.

A perfectly monochromatic sinusoid would retain a predictable phase forever. Real sources have finite spectral width. The broader the optical spectrum, the more rapidly the phase correlation typically decays with delay. The characteristic time over which the field remains correlated is called a coherence time, and the corresponding propagation distance is called a coherence length.

For an order-of-magnitude estimate one often writes

$$\tau_c \sim \frac{1}{\Delta\nu}, \quad L_c \sim v_g \tau_c \approx \frac{c}{\Delta\nu} \quad (15.85)$$

in air or vacuum. Equivalently, for a narrow spectral width $\Delta\lambda \ll \lambda_0$,

$$\Delta\nu \approx \frac{c}{\lambda_0^2} \Delta\lambda, \quad L_c \sim \frac{\lambda_0^2}{\Delta\lambda}. \quad (15.86)$$

The proportionality factor depends on the spectral lineshape and on the precise convention used to define “coherence time” or “coherence length.” A full-width-at-half-maximum definition, a $1/e$ correlation definition and a first-zero definition need not give identical numerical factors.

For two copies of a stationary source with spectral density $S(\nu)$ and no additional spectral filtering, the normalized correlation is

$$\gamma(\tau) = \frac{\int S(\nu) e^{-i2\pi\nu\tau} d\nu}{\int S(\nu) d\nu}. \quad (15.87)$$

For a narrow line centred at ν_0 , a Lorentzian of frequency FWHM $\Delta\nu$ gives $|\gamma| = e^{-\pi\Delta\nu|\tau|}$, whereas a Gaussian with the same FWHM gives $|\gamma| = \exp[-\pi^2\Delta\nu^2\tau^2/(4\ln 2)]$. Their equal linewidths do not give equal visibility curves. In a dispersive arm, the delay is controlled by group optical path; the monochromatic phase still uses the phase index. Figure 15.12 compares these two precisely specified models.

Note

The inverse-bandwidth relation is more important pedagogically than a memorized numerical prefactor. Always state the linewidth convention and the coherence-length convention if a precision numerical result is required.

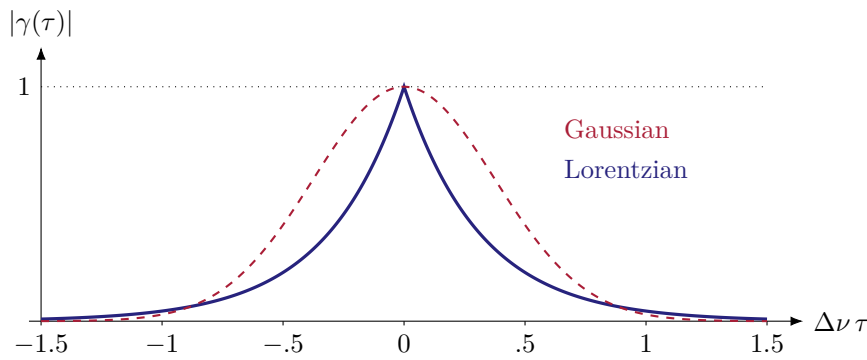


Figure 15.12: Normalized correlation magnitudes for narrow Lorentzian and Gaussian spectra with the same frequency FWHM $\Delta\nu$. The carrier phase is omitted. A linewidth needs a lineshape and a stated coherence-length convention before it gives a precise visibility at a specified delay.

Example 15.23: Coherence-length scale of a narrow-band source

A source centered at 650 nm has an optical linewidth of approximately 0.020 nm. Using the scale estimate in Eq. (15.86),

$$L_c \sim \frac{(650 \times 10^{-9})^2}{0.020 \times 10^{-9}} \approx 2.1 \times 10^{-2} \text{ m.}$$

Thus an optical-path mismatch of order centimetres can substantially reduce interference visibility. The precise visibility curve requires the actual spectral profile.

A Michelson interferometer can therefore be used as a Fourier-spectroscopy instrument: the intensity is recorded as a function of optical-path delay, producing an interferogram. Under suitable assumptions, the optical spectrum is related to the Fourier transform of the field-correlation information encoded in that interferogram. This connection is the foundation of Fourier-transform spectroscopy.

Suppose the relative phase is

$$\Delta\phi = \phi_0 + \eta, \quad (15.88)$$

where η is a zero-mean Gaussian random variable of variance σ_ϕ^2 that samples many statistically independent values during one measurement. For equal beam intensities,

$$I = 2I_0 [1 + \cos(\phi_0 + \eta)]. \quad (15.89)$$

A standard Gaussian average gives

$$\langle e^{i\eta} \rangle = e^{-\sigma_\phi^2/2}, \quad (15.90)$$

so

$$\langle I \rangle = 2I_0 \left[1 + e^{-\sigma_\phi^2/2} \cos \phi_0 \right]. \quad (15.91)$$

The visibility becomes

$$\mathcal{V} = e^{-\sigma_\phi^2/2}. \quad (15.92)$$

For $\sigma_\phi = 1$ rad, the visibility is about 0.607; for $\sigma_\phi = 2$ rad, it falls to about 0.135. This model shows quantitatively how rapid random phase fluctuations wash out fringes even though the instantaneous fields still interfere.

The result depends on the assumed Gaussian phase statistics and rapid averaging. Slowly varying noise instead causes the whole fringe pattern to wander between exposures rather than simply reducing each exposure to the same lower-visibility curve.

15.23.4 Spatial coherence and source angular extent

Young's experiment works most cleanly when the two slits sample portions of a field with a stable transverse phase relation. An extended incoherent source can be regarded, approximately, as many independent source points. Each point produces its own shifted fringe system; adding many such systems can wash out the contrast. Passing the source through a narrow entrance slit increases spatial coherence at the double slit by selecting a smaller angular range.

Spatial coherence is therefore not merely an abstract property of a laser. It is a statement about correlations between two points in a wavefront, and it depends both on the source and on the propagation geometry. In optical instrumentation, source size, aperture separation and propagation distance must be designed together.

Key message

Temporal coherence limits how far two optical paths may differ while retaining interference. Spatial coherence limits how widely separated two sampling points of a wavefront may be while retaining a stable phase relation. The two ideas answer different experimental questions.

A quantitative example can be obtained without a general Fourier-optics theorem. Consider a quasi-monochromatic, spatially incoherent line source of uniform brightness, with small angular coordinate ψ between $-\psi_s$ and $+\psi_s$ as seen from the two slits. Its independent source elements contribute intensities, not mutually coherent fields. For each element the two slits receive a relative incident phase approximately $k d \psi$. Their fringe systems are shifted relative to one another.

With equal slit transmissions, averaging these systems gives

$$I(y) = 2I_0 \left[1 + \frac{1}{2\psi_s} \int_{-\psi_s}^{\psi_s} \cos(\Phi(y) - k d \psi) d\psi \right] \quad (15.93)$$

$$= 2I_0 \left[1 + \frac{\sin(k d \psi_s)}{k d \psi_s} \cos \Phi(y) \right], \quad (15.94)$$

where $\Phi(y)$ is the propagation phase from slit to detector, and I_0 is the integrated contribution of either slit alone. Therefore

$$\mathcal{V} = \left| \frac{\sin(k d \psi_s)}{k d \psi_s} \right|, \quad d_{\text{first zero}} = \frac{\lambda}{2\psi_s}. \quad (15.95)$$

The zero-source-width limit gives visibility one. When the sinc factor becomes negative, the fringe phase reverses; visibility itself remains nonnegative. An extended source therefore need not produce a monotonic loss of contrast for every baseline, although the envelope decreases.

Example 15.24: How narrow must the entrance slit be?

A uniformly illuminated entrance slit of full width b is $Z = 1.00$ m from a double slit with separation $d = 0.300$ mm. At $\lambda = 600$ nm, find the entrance width giving the first visibility zero.

For $b \ll Z$, the angular half-width is $\psi_s \simeq b/(2Z)$. The first zero requires $kd\psi_s = \pi$, hence

$$b = \frac{\lambda Z}{d} = 2.00 \text{ mm}.$$

This is a cancellation of shifted fringe patterns, not a failure of either slit to transmit light. A narrower entrance slit improves the contrast at this baseline but reduces available power. The result assumes a spatially incoherent, uniform line source; a laser beam with the same geometric width need not obey it.

The more general connection between source angular brightness and mutual coherence is the van Cittert–Zernike relation. A circular source has a different visibility function, involving a Bessel function, so its numerical zero cannot be taken from this line-source example.

15.23.5 Oblique thin-film interference

For a film of refractive index n_f and internal propagation angle θ_f , the extra optical path associated with the down-and-up traversal contributes the standard phase thickness

$$\Delta\phi_{\text{prop}} = \frac{4\pi n_f t \cos \theta_f}{\lambda_0}. \quad (15.96)$$

The angle inside the film follows from Snell's law,

$$n_0 \sin \theta_0 = n_f \sin \theta_f. \quad (15.97)$$

The reflection phase must still be treated separately. To see why the propagation term contains a cosine, use the exit-plane construction in Fig. 15.6. The internal down-and-up optical path is $2n_f t / \cos \theta_f$, and the two emerging rays are laterally separated by $s = 2t \tan \theta_f$. Comparing phases on one wavefront subtracts the exterior contribution $n_0 s \sin \theta_0$. Consequently,

$$\Delta\phi_{\text{OPL,prop}} = \frac{2n_f t}{\cos \theta_f} - n_0 (2t \tan \theta_f) \sin \theta_0 \quad (15.98)$$

$$= \frac{2n_f t}{\cos \theta_f} - \frac{2n_f t \sin^2 \theta_f}{\cos \theta_f} = 2n_f t \cos \theta_f, \quad (15.99)$$

where Snell's law supplies $n_0 \sin \theta_0 = n_f \sin \theta_f$. Merely counting the internal ray distance misses the comparison wavefront.

At oblique incidence, polarization becomes important because the Fresnel reflection amplitudes differ for the two linear polarizations conventionally called s (TE) and p (TM). Close to Brewster's angle, the p -polarized reflected amplitude can become very small at a dielectric interface. Consequently, a thin-film design that performs one way at normal incidence may shift spectrally and behave differently for the two polarizations at oblique incidence.

Caution

Do not replace $2n_f t$ by $2n_f t / \cos \theta_f$. The phase difference between the two reflected beams includes the geometry of their exit points; the standard result is proportional to $2n_f t \cos \theta_f$. Deriving the geometry once is safer than relying on a remembered path sketch.

15.23.6 Fresnel amplitudes and the complete single-film sum

At normal incidence between two nonmagnetic, lossless dielectric media with indices n_i and n_j , the electric-field reflection amplitude is

$$r_{ij} = \frac{n_i - n_j}{n_i + n_j}. \quad (15.100)$$

The corresponding intensity reflectance is

$$R_{ij} = |r_{ij}|^2. \quad (15.101)$$

The sign of r_{ij} automatically contains the phase-reversal rule: if $n_j > n_i$, then $r_{ij} < 0$, corresponding to a phase shift of π relative to the incident electric field.

The transmitted electric-field amplitude coefficient at the same interface is $t_{ij} = 2n_i/(n_i + n_j)$. Its power transmittance is $(n_j/n_i)|t_{ij}|^2$; it is not simply $|t_{ij}|^2$ when the media differ. For real indices, $|r_{ij}|^2 + (n_j/n_i)|t_{ij}|^2 = 1$.

For a film of index n_1 between media n_0 and n_2 , let

$$\beta = \frac{2\pi n_1 t}{\lambda_0}$$

be the one-way phase. The first reflection has amplitude $r_{01}E_i$. The first wave returning from the lower interface has amplitude $t_{01}t_{10}r_{12}e^{2i\beta}E_i$. Each further internal round trip multiplies that amplitude by $r_{10}r_{12}e^{2i\beta}$. The geometric series therefore gives

$$r_{\text{eff}} = r_{01} + \frac{t_{01}t_{10}r_{12}e^{2i\beta}}{1 - r_{10}r_{12}e^{2i\beta}} \quad (15.102)$$

$$= \frac{r_{01} + r_{12}e^{2i\beta}}{1 + r_{01}r_{12}e^{2i\beta}}, \quad (15.103)$$

using $r_{10} = -r_{01}$ and $t_{01}t_{10} = 1 - r_{01}^2$. The observable reflectance is $\mathcal{R} = |r_{\text{eff}}|^2$. For a lossless film the transmitted power fraction satisfies $\mathcal{T} = 1 - \mathcal{R}$: suppressed reflection sends power into transmission.

Two limits test the result. At $t = 0$, it reduces to $(n_0 - n_2)/(n_0 + n_2)$, the direct exterior interface. At quarter-wave thickness, $e^{2i\beta} = -1$ and zero reflection requires $r_{01} = r_{12}$. Cross multiplying yields $n_1^2 = n_0 n_2$, proving the matching condition. The quarter-wave residual is

$$\mathcal{R}_{\text{QW}} = \left(\frac{n_0 n_2 - n_1^2}{n_0 n_2 + n_1^2} \right)^2. \quad (15.104)$$

These checks also show why the two-principal-ray picture is useful for phase intuition but insufficient for an exact cancellation or power budget.

Example 15.25: How much reflection remains after coating?

Consider air ($n_0 = 1.00$), a coating with $n_1 = 1.38$, and glass with $n_2 = 1.52$. Treat all indices as real and constant near the design wavelength 550 nm.

Strategy. First set the quarter-wave phase; then calculate the residual with the full film amplitude. The thickness and ideal matching index are

$$t = \frac{550}{4(1.38)} \text{ nm} = 99.6 \text{ nm}, \quad n_{1,\text{ideal}} = \sqrt{1.52} = 1.233.$$

The actual coating gives

$$\mathcal{R}_{\text{QW}} = \left(\frac{1.52 - 1.38^2}{1.52 + 1.38^2} \right)^2 = 0.0126 \simeq 1.26\%.$$

The bare air–glass interface reflects $[(1 - 1.52)/(1 + 1.52)]^2 = 0.0426$, or 4.26%. Thus the coating lowers reflection substantially but does not eliminate it. The indices and reflectance are dimensionless; the thickness has length units. Figure 15.7 compares the wavelength dependence with an ideally matched layer. Neither curve assumes material absorption or dispersion.

15.23.7 Transfer matrices with an explicit field convention

Multilayer coatings repeat the same propagation and boundary-matching operations. We retain the time convention $e^{-i\omega t}$ and take $+z$ from the incident medium toward the substrate. At normal incidence define the tangential-field state

$$\mathbf{f}(z) = \begin{pmatrix} \tilde{E}(z) \\ Z_0 \tilde{H}(z) \end{pmatrix}, \quad Z_0 = \sqrt{\mu_0/\epsilon_0}.$$

In layer j , a forward wave has $Z_0 \tilde{H} = +n_j \tilde{E}$, and a backward wave has $Z_0 \tilde{H} = -n_j \tilde{E}$. Both entries have electric-field units. Writing the fields as forward and backward exponentials and eliminating their amplitudes gives

$$\mathbf{f}(z_{j,\text{left}}) = M_j \mathbf{f}(z_{j,\text{right}}), \quad M_j = \begin{pmatrix} \cos \delta_j & -\frac{i}{n_j} \sin \delta_j \\ -in_j \sin \delta_j & \cos \delta_j \end{pmatrix}, \quad \delta_j = \frac{2\pi n_j t_j}{\lambda_0}. \quad (15.105)$$

The minus signs follow from propagating the state from the right face back to the left. A matrix mapping left to right has the opposite signs. Tangential E and H are continuous at an interface without surface sources, so no separate interface matrix is needed in this field basis.

For layers numbered in their spatial order, $M = M_1 M_2 \cdots M_N$. Write its elements as M_{ab} and let n_s be the semi-infinite substrate index. A transmitted forward wave obeys $Z_0 H_s = n_s E_s$, giving the input optical admittance

$$Y = \frac{M_{21} + n_s M_{22}}{M_{11} + n_s M_{12}}. \quad (15.106)$$

At the incident face, $E = E_i + E_r$ and $Z_0 H = n_0(E_i - E_r)$. Therefore

$$r = \frac{n_0 - Y}{n_0 + Y}, \quad (15.107)$$

$$t_{\text{stack}} = \frac{2n_0}{n_0(M_{11} + n_s M_{12}) + M_{21} + n_s M_{22}}. \quad (15.108)$$

For real exterior indices, $\mathcal{R} = |r|^2$ and $\mathcal{T} = (n_s/n_0)|t_{\text{stack}}|^2$. A lossless calculation must give $\mathcal{R} + \mathcal{T} = 1$. Setting $N = 0$ gives the single interface; setting $N = 1$ reproduces Eq. (15.103). These are practical checks on matrix order, signs and normalization.

At the quarter-wave design wavelength, $\delta_H = \delta_L = \pi/2$ and one high–low pair reduces to

$$M_H M_L = \begin{pmatrix} -n_L/n_H & 0 \\ 0 & -n_H/n_L \end{pmatrix}.$$

For N_p identical pairs beginning with the high-index layer, $Y = n_s(n_H/n_L)^{2N_p}$. The resulting admittance mismatch produces a high-reflectance stack. Equal optical thickness nt , rather than equal physical thickness, controls the design phase.

Example 15.26: Optical thickness of a two-material quarter-wave pair

A high-index layer has $n_H = 2.10$ and a low-index layer has $n_L = 1.45$. For a design wavelength $\lambda_0 = 600$ nm, the quarter-wave physical thicknesses are

$$t_H = \frac{600 \text{ nm}}{4(2.10)} \approx 71.4 \text{ nm}, \quad t_L = \frac{600 \text{ nm}}{4(1.45)} \approx 103 \text{ nm}.$$

Although the physical thicknesses differ, both layers have the same optical thickness $nt = 150$ nm. That equality, not equal geometric thickness, sets their design phase.

15.23.8 Multiple-beam interference and Fabry–Pérot resonances

A pair of partially reflecting parallel mirrors forms a Fabry–Pérot cavity. In a symmetric lossless cavity, each emerging transmitted beam has made one additional round trip. Use power-normalized amplitudes in identical exterior media, and let each mirror have intensity reflectance R . The magnitude of the round-trip amplitude multiplier is R , while the product of the two transmission magnitudes is $1 - R$. Absorb the fixed mirror-reflection phases into a total round-trip phase δ . Then

$$\frac{E_T}{E_i} = (1 - R)e^{i\chi} \sum_{q=0}^{\infty} (Re^{i\delta})^q = \frac{(1 - R)e^{i\chi}}{1 - Re^{i\delta}}, \quad (15.109)$$

$$\mathcal{T}(\delta) = \frac{(1 - R)^2}{1 + R^2 - 2R \cos \delta} = \boxed{\frac{1}{1 + F \sin^2(\delta/2)}}, \quad F = \frac{4R}{(1 - R)^2}. \quad (15.110)$$

The phase χ of the first transmitted term does not affect transmitted power. Figure 15.13 shows that increasing R narrows the peaks while their ideal height remains unity. Internal loss or unequal mirrors removes this unit-transmission guarantee.

In the nondispersive model with the constant mirror phase chosen as the resonance reference,

$$\delta = \frac{4\pi nL\nu}{c} = \frac{4\pi nL}{\lambda_0}.$$

For real mirrors, add the two reflection phases; their frequency derivatives can also affect resonance spacing. Transmission maxima occur when

$$\delta = 2\pi m, \quad 2nL = m\lambda_0. \quad (15.111)$$

Adjacent resonances in frequency are separated approximately by the free spectral range

$$\Delta\nu_{\text{FSR}} = \frac{c}{2nL} \quad (15.112)$$

when dispersion and the frequency dependence of mirror phase are negligible. More generally, adjacent modes satisfy $(d\delta/d\nu)\Delta\nu_{\text{FSR}} \simeq 2\pi$, so a uniform dispersive intracavity medium gives $c/(2n_g L)$ when mirror delay is negligible. The same longitudinal resonance condition underlies a simple laser cavity; gain and mode selection are developed in Chapter 20.

The ideal-cavity finesse, defined as free spectral range divided by resonance FWHM, is

$$\mathcal{F} \approx \frac{\pi\sqrt{R}}{1 - R}, \quad (15.113)$$

which is the narrow-peak approximation. Solving $\mathcal{T} = 1/2$ gives $\sin^2(\delta/2) = 1/F$, hence

$$\Delta\delta_{\text{FWHM}} = 4 \arcsin(1/\sqrt{F}), \quad \mathcal{F}_{\text{exact}} = \frac{\pi}{2 \arcsin(1/\sqrt{F})}. \quad (15.114)$$

These FWHM expressions require $F \geq 1$; otherwise the transmission never falls to half its peak between resonances. For $F \gg 1$, $\arcsin(1/\sqrt{F}) \simeq 1/\sqrt{F}$, yielding Eq. (15.113). The coefficient F in the Airy function and the finesse $\mathcal{F}$ are different quantities. Absorption, scattering, surface errors and angular averaging can broaden measured resonances; poor mode matching can additionally alter contrast and the apparent line profile.

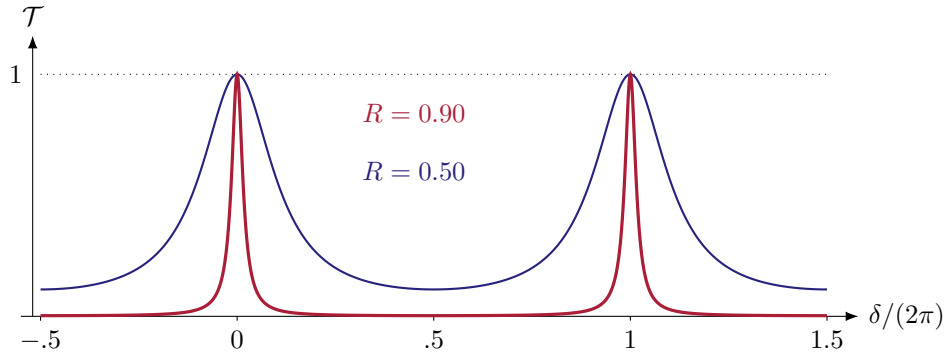


Figure 15.13: Airy transmission for identical lossless mirrors. The curves use $F = 8$ for $R = 0.50$ and $F = 360$ for $R = 0.90$. Successive peaks are one round-trip phase cycle apart; increased reflectance sharpens them without changing that phase spacing.

Example 15.27: Free spectral range and approximate finesse

An air-spaced Fabry–Pérot cavity has $L = 10.0$ cm and mirror reflectance $R = 0.95$. Its free spectral range is

$$\Delta\nu_{\text{FSR}} = \frac{3.00 \times 10^8}{2(0.100)} = 1.50 \times 10^9 \text{ Hz.}$$

The high-reflectivity finesse estimate is

$$\mathcal{F} \approx \frac{\pi\sqrt{0.95}}{1 - 0.95} \approx 61.$$

A corresponding resonance linewidth estimate is

$$\Delta\nu_{\text{linewidth}} \approx \frac{\Delta\nu_{\text{FSR}}}{\mathcal{F}} \approx 25 \text{ MHz.}$$

This example shows how multiple-beam interference converts moderate mirror reflectivity into narrow spectral features.

Key message

Young’s experiment is primarily a two-beam interferometer. A Fabry–Pérot cavity is a multiple-beam interferometer. The mathematical difference is not cosmetic: summing many coherent amplitudes produces much sharper resonances than the sinusoidal modulation of two equal beams.

15.23.9 Quadrature, statistical sensitivity and uncertainty

A sinusoidal fringe has maximum slope halfway between a maximum and a minimum. In a simple two-beam interferometer one may write locally

$$I(\phi) = I_{\text{av}} [1 + \mathcal{V} \cos \phi]. \quad (15.115)$$

Its phase sensitivity is

$$\left| \frac{dI}{d\phi} \right| = I_{\text{av}} \mathcal{V} |\sin \phi|, \quad (15.116)$$

which is largest near

$$\phi = \frac{\pi}{2}, \frac{3\pi}{2}, \dots \quad (15.117)$$

These points are called quadrature points. Operating near quadrature converts a small phase change approximately linearly into an intensity change,

$$\delta I \approx \left(\frac{dI}{d\phi} \right)_{\text{quad}} \delta \phi. \quad (15.118)$$

For a Michelson mirror displacement δx in air,

$$\delta \phi = \frac{4\pi}{\lambda_0} \delta x. \quad (15.119)$$

Thus interferometry converts sub-wavelength displacement into measurable phase.

Caution

High phase sensitivity does not automatically imply high absolute accuracy. Calibration of wavelength, refractive index, mechanical geometry, thermal expansion, detector response and environmental disturbances can dominate the uncertainty budget.

Any real interferometric measurement has noise. Important categories include detector read noise, laser intensity noise, electronic noise, vibration, acoustic coupling, air turbulence, thermal drift, polarization fluctuations and photon counting statistics.

If a counting interval has mean detected photon number $\bar{N}$ and the counts obey Poisson statistics, the standard deviation of the count is

$$\sigma_N = \sqrt{\bar{N}}. \quad (15.120)$$

The fractional counting fluctuation therefore scales as

$$\frac{\sigma_N}{\bar{N}} = \frac{1}{\sqrt{\bar{N}}}. \quad (15.121)$$

This square-root scaling is one reason more detected optical power, longer integration time or repeated averaging can improve statistical precision, until other noises dominate.

Near quadrature, a phase fluctuation $\delta \phi$ produces an intensity fluctuation approximately proportional to $\delta \phi$. In an ideal shot-noise-limited measurement, the phase uncertainty scales roughly as

$$\delta \phi \propto \frac{1}{\sqrt{\bar{N}}}, \quad (15.122)$$

with the precise coefficient depending on the interferometer and detection scheme.

Caution

“Resolution,” “sensitivity,” “precision” and “accuracy” are not synonyms. Resolution concerns the smallest distinguishable change; sensitivity is the response per change in the measurand; precision concerns repeatability or statistical spread; accuracy concerns closeness to the correct value after systematic effects are included.

Consider an ideal balanced two-beam measurement operated at quadrature with $N = 10^6$ detected photons during an integration interval. The Poisson count fluctuation scale is

$$\sqrt{N} = 10^3,$$

so the fractional counting fluctuation is 10^{-3} . An ideal phase estimator may therefore have a statistical phase scale of order

$$\delta\phi \sim 10^{-3} \text{ rad},$$

up to a coefficient set by the exact detection scheme.

For a Michelson interferometer at 633 nm, this phase corresponds to displacement scale

$$\delta x \sim \frac{\lambda_0}{4\pi} \delta\phi \sim 5 \times 10^{-11} \text{ m}.$$

This very small number should not be interpreted as a guaranteed experimental resolution. Mechanical vibration, laser frequency noise, detector electronics, imperfect visibility and environmental refractive-index noise can be much larger. The calculation is a statistical lower-scale estimate under idealized assumptions.

Suppose a wavelength is determined from

$$\lambda_0 = \frac{2\Delta x}{N} \quad (15.123)$$

using a Michelson mirror displacement Δx and a counted number of fringes N . Treating N as a measured quantity with uncertainty u_N and neglecting covariance gives

$$\left(\frac{u_\lambda}{\lambda_0}\right)^2 \approx \left(\frac{u_x}{\Delta x}\right)^2 + \left(\frac{u_N}{N}\right)^2. \quad (15.124)$$

If the displacement instrument has a scale-factor calibration uncertainty, that systematic contribution must also enter the budget.

For high-precision work, environmental refractive index cannot automatically be replaced by unity. The measured phase corresponds to optical path $n_{\text{air}}L$; variations of pressure, temperature, humidity and composition alter n_{air} . At ordinary undergraduate precision these effects may be negligible, but at metrology precision they are part of the measurand model.

Example 15.28: Comparing uncertainty contributions

A mirror is moved by $\Delta x = (1.5000 \pm 0.0005) \text{ mm}$ while $N = 4740 \pm 1$ fringes are counted. The fractional displacement uncertainty is

$$\frac{u_x}{\Delta x} = \frac{0.0005}{1.5000} = 3.33 \times 10^{-4},$$

while the fractional fringe-count uncertainty is

$$\frac{u_N}{N} = \frac{1}{4740} = 2.11 \times 10^{-4}.$$

Combining in quadrature gives

$$\frac{u_\lambda}{\lambda_0} \approx 3.94 \times 10^{-4}.$$

The displacement calibration contributes more strongly than the count uncertainty in this example, so improving the camera or fringe-count algorithm alone would not optimize the measurement.

15.23.10 Phase recovery, unwrapping and synthetic wavelength

Counting complete fringes is robust for large changes, but modern optical metrology often estimates phase directly. Suppose a controlled reference phase δ is added so that

$$I(\delta) = A + B \cos(\phi + \delta). \quad (15.125)$$

Record four images with phase steps

$$\delta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}. \quad (15.126)$$

Then

$$I_0 = A + B \cos \phi, \quad (15.127)$$

$$I_{\pi/2} = A - B \sin \phi, \quad (15.128)$$

$$I_\pi = A - B \cos \phi, \quad (15.129)$$

$$I_{3\pi/2} = A + B \sin \phi. \quad (15.130)$$

Taking differences removes the unknown background A :

$$I_0 - I_\pi = 2B \cos \phi, \quad I_{3\pi/2} - I_{\pi/2} = 2B \sin \phi. \quad (15.131)$$

Therefore a robust phase estimate is

$$\phi = \text{atan2}(I_{3\pi/2} - I_{\pi/2}, I_0 - I_\pi). \quad (15.132)$$

The two-argument arctangent is preferred because it returns the correct quadrant.

The measured phase is initially wrapped into an interval of length 2π . If phase varies smoothly across a surface and spans more than one cycle, numerical phase-unwrapping algorithms reconstruct a continuous phase map by adding suitable multiples of 2π . Poor signal, discontinuities or isolated phase singularities make unwrapping difficult, so data quality must be assessed before treating an unwrapped map as a physical surface.

Example 15.29: Surface displacement from a measured phase shift

A Michelson-based surface measurement at $\lambda_0 = 633 \text{ nm}$ finds a local phase increase of 0.80 rad . If the phase change is caused only by normal surface displacement,

$$\delta x = \frac{\lambda_0}{4\pi} \delta \phi = \frac{633 \times 10^{-9}}{4\pi} (0.80) \approx 40 \text{ nm}.$$

The factor $4\pi/\lambda_0$ rather than $2\pi/\lambda_0$ arises because the reflected beam travels the displacement twice.

A measured wrapped phase is usually confined to an interval such as $(-\pi, \pi]$. If the true phase changes smoothly by more than 2π , discontinuous jumps appear in the wrapped map. Phase unwrapping attempts to add integer multiples of 2π so that neighboring values form a physically plausible continuous surface.

In one dimension, a simple algorithm may add or subtract 2π whenever an adjacent difference exceeds π . In two dimensions, multiple paths through the image can disagree when noise or true discontinuities are present. More sophisticated algorithms use quality maps, branch cuts, least-squares minimization or regularization.

Unwrapping should not be treated as a purely cosmetic plotting operation. It embeds assumptions about continuity. Real surfaces can contain steps larger than half a fringe, disconnected regions or phase singularities. Low-intensity pixels also have poorly defined phase. A trustworthy reconstruction masks unreliable regions and validates the unwrapped result against independent geometric information when possible.

Two close wavelengths can create a slowly varying relative phase that extends the unambiguous range of an interferometric measurement. Let two vacuum wavelengths be λ_1 and λ_2 . Their wave-number difference is

$$\Delta k = 2\pi \left| \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right|. \quad (15.133)$$

Define the synthetic wavelength

$$\Lambda = \frac{2\pi}{\Delta k} = \frac{\lambda_1 \lambda_2}{|\lambda_1 - \lambda_2|}. \quad (15.134)$$

For $\lambda_1 = 632.8 \text{ nm}$ and $\lambda_2 = 633.0 \text{ nm}$,

$$\Lambda \approx \frac{(632.8)(633.0)}{0.2} \text{ nm} \approx 2.00 \text{ mm}.$$

Thus comparing phases at two very close optical wavelengths generates a millimetre-scale beat length, much larger than either optical wavelength. Multi-wavelength interferometry exploits this idea to enlarge the unambiguous measurement range while retaining fine optical phase sensitivity.

The phase difference repeats when the optical-path difference changes by Λ . For a double-pass mirror displacement in air, the corresponding displacement interval is $\Lambda/2$. Comparing two separately measured optical phases is sufficient; the detector need not follow an optical-frequency beat between the colours. This extension trades range against precision: for independent phase uncertainties u_{ϕ_1} and u_{ϕ_2} , the synthetic optical-path uncertainty is

$$u_{\text{OPL}} = \frac{\Lambda}{2\pi} \sqrt{u_{\phi_1}^2 + u_{\phi_2}^2}.$$

A large synthetic wavelength amplifies phase noise in the coarse range estimate.

Caution

A synthetic wavelength does not mean that the electromagnetic radiation has literally changed to a millimetre wavelength. It is a beat scale formed from the difference of two optical wave numbers.

15.23.11 Heterodyne and synchronous detection

In homodyne interferometry the two beams have the same optical frequency. In heterodyne interferometry one beam is shifted slightly, for example by an acousto-optic modulator. Let

$$E_1(t) = E_{01} \cos(\phi_1 - \omega t), \quad (15.135)$$

$$E_2(t) = E_{02} \cos[\phi_2 - (\omega + \Delta\omega)t]. \quad (15.136)$$

After averaging over the rapid optical oscillation, the detector retains a beat term at the difference frequency,

$$I(t) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos(\Delta\phi - \Delta\omega t). \quad (15.137)$$

The optical phase information is thereby translated to a much lower electronic frequency that can be measured accurately with digital signal processing.

The detector must resolve $|\Delta\omega|/(2\pi)$ while averaging over the optical carrier. Averaging over many beat periods without synchronous demodulation would erase the beat term as well. The sign in the cosine follows the common $e^{-i\omega t}$ convention; the electronic reference must use the same convention.

Heterodyne techniques are valuable when the sign and time dependence of displacement matter. Instead of merely observing a bright-to-dark change, electronics can track the phase of a radio-frequency beat. This connects optical interferometry directly to lock-in detection, phase-locked loops, digital demodulation and frequency metrology.

Interferometric signals are often small compared with slow detector drift or broadband noise. One strategy is to modulate a phase or optical path at a known reference frequency f_m and detect only the component of the detector signal synchronized with that modulation.

Suppose the measured voltage is

$$V(t) = V_s \cos(2\pi f_m t + \phi_s) + V_n(t). \quad (15.138)$$

Multiply by a reference $\cos(2\pi f_m t)$ and average over many cycles. Using

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)], \quad (15.139)$$

the average retains a DC term proportional to

$$\frac{V_s}{2} \cos \phi_s, \quad (15.140)$$

while noise components far from f_m average toward zero. Using $-\sin(2\pi f_m t)$ as the second reference yields $+(V_s/2) \sin \phi_s$; using $+\sin$ instead reverses that quadrature sign. Together the two channels recover amplitude and phase.

Lock-in detection does not magically eliminate all noise. Noise near the modulation frequency, reference jitter and systematic modulation artifacts can remain. Nevertheless, shifting a measurement away from low-frequency drift and using narrow-band synchronous detection is one of the most important practical techniques in precision experiments.

This topic connects interferometry to electronics and signal processing: the optical device encodes the measurand in phase, an intentional modulation translates it into a known temporal frequency, and digital or analog demodulation extracts the desired component.

15.23.12 Optical-path sensors in free space and waveguides

An interferometer responds to optical path length, not only geometric distance. A differential optical path

$$\text{OPL} = nL \quad (15.141)$$

changes according to

$$d(nL) = n dL + L dn. \quad (15.142)$$

A temperature change can influence both terms: thermal expansion changes L , while the thermo-optic effect changes n . Similarly, mechanical strain changes geometry and can also change refractive index through photoelasticity.

For a single-pass phase sensor,

$$d\phi = \frac{2\pi}{\lambda_0} d(nL). \quad (15.143)$$

A double-pass reflection geometry doubles the corresponding optical-path contribution. This is the basis of many fiber-optic and free-space sensors.

The same phase principles apply when light is guided rather than propagating through free space. In a fiber Mach–Zehnder interferometer, a coupler splits light into two guided paths and a second coupler recombines them. In a fiber Michelson configuration, reflections return light through the arms. Fiber Fabry–Pérot sensors use two partially reflecting surfaces to create a cavity.

The guided phase accumulated along a path is more accurately written in terms of an effective refractive index n_{eff} ,

$$\phi = \frac{2\pi}{\lambda_0} n_{\text{eff}} L. \quad (15.144)$$

Changes in strain, temperature, pressure or surrounding refractive index can alter n_{eff} or L , shifting the phase. Integrated-photonics interferometers use the same idea on planar waveguide chips, where path lengths, splitters and phase shifters can be fabricated lithographically.

This is an important engineering lesson: the physics of interference is platform-independent. What changes from a tabletop Michelson to a fiber sensor or photonic chip is the mode structure, coupling, loss, polarization behavior and environmental sensitivity, not the basic requirement that coherent amplitudes recombine with a controlled relative phase.

Consider a Mach–Zehnder sensor in which one arm contains a microfluidic channel of interaction length L . A change in analyte concentration changes the effective refractive index by Δn_{eff} , producing phase shift

$$\Delta\phi = \frac{2\pi L}{\lambda_0} \Delta n_{\text{eff}}. \quad (15.145)$$

If a phase-processing algorithm can resolve standard deviation u_ϕ , the corresponding idealized refractive-index resolution is

$$u_n \approx \frac{\lambda_0}{2\pi L} u_\phi. \quad (15.146)$$

Increasing interaction length improves this phase sensitivity, but also increases propagation loss, temperature sensitivity and footprint. Engineering design is therefore a compromise rather than a one-parameter maximization.

If the interferometer is biased at quadrature and intensity rather than phase is measured directly, laser-power fluctuations can mimic the signal. Balanced detection or a reference channel can suppress common intensity noise. If the sensing arm changes polarization, fringe visibility can also drift. A robust sensor design therefore separates the desired refractive-index response from nuisance responses to temperature, power, polarization and mechanics.

Example 15.30: Displacement plus refractive-index change in one Michelson arm

A Michelson interferometer operates at $\lambda_0 = 633 \text{ nm}$. During an experiment, mirror 2 moves away from the beam splitter by $4.00 \mu\text{m}$. Simultaneously, a 2.00 cm gas cell in that same arm changes refractive index by $+1.50 \times 10^{-5}$. The beam traverses the cell twice. Find the total phase change and the equivalent number of fringe cycles.

The mirror motion changes the round-trip geometric path by

$$\Delta\text{OPL}_x = 2\Delta x = 8.00 \mu\text{m}.$$

The gas-cell change contributes

$$\Delta\text{OPL}_n = 2L\Delta n = 2(0.0200)(1.50 \times 10^{-5}) = 6.00 \times 10^{-7} \text{ m} = 0.600 \mu\text{m}.$$

Therefore

$$\Delta\text{OPL}_{\text{total}} = 8.60 \mu\text{m}.$$

The number of phase cycles is

$$N_{\text{equiv}} = \frac{\Delta\text{OPL}_{\text{total}}}{\lambda_0} = \frac{8.60 \times 10^{-6}}{633 \times 10^{-9}} \approx 13.59.$$

The unwrapped phase change is

$$\Delta\phi = 2\pi N_{\text{equiv}} \approx 85.4 \text{ rad}.$$

If an experiment records only the final intensity rather than tracking the phase continuously, it cannot distinguish the integer number of full 2π cycles. Fringe counting or phase unwrapping is therefore essential for large displacements.

15.23.13 Dispersion and group delay

The main chapter often treated refractive index as a constant at the wavelength of interest. For broadband light, however, n depends on angular frequency. The phase accumulated through length L is

$$\phi(\omega) = \frac{\omega}{c} n(\omega) L. \quad (15.147)$$

Differentiating with respect to angular frequency gives the group delay,

$$\tau_g = \frac{d\phi}{d\omega} = \frac{L}{c} \left[n + \omega \frac{dn}{d\omega} \right] = \frac{n_g L}{c}, \quad (15.148)$$

where

$$n_g = n + \omega \frac{dn}{d\omega} \quad (15.149)$$

is the group index in this convention.

Two interferometer arms can therefore be matched in phase at one wavelength yet remain mismatched in group delay across a broad spectrum. White-light or ultrashort-pulse interferometry is especially sensitive to this distinction. Compensation plates are often introduced so that the two beams traverse similar thicknesses of dispersive material.

Expanding the phase difference about a central frequency,

$$\Delta\phi(\omega) \approx \Delta\phi_0 + \left. \frac{d\Delta\phi}{d\omega} \right|_{\omega_0} (\omega - \omega_0) + \frac{1}{2} \left. \frac{d^2\Delta\phi}{d\omega^2} \right|_{\omega_0} (\omega - \omega_0)^2 + \cdots, \quad (15.150)$$

shows a useful hierarchy. The constant term sets carrier phase, the first derivative sets relative group delay, and higher derivatives describe dispersion that can broaden or distort short pulses.

Caution

For monochromatic calculations, replacing $n(\omega)$ by a single value is often appropriate. For broadband interference, the phase index and group index play different roles. Do not use $v = c/n$ as a universal pulse velocity in a dispersive medium.

15.23.14 Low-coherence ranging and tomography

Broadband light has short temporal coherence. At first this seems disadvantageous, but it creates a powerful ranging method. When the optical-path mismatch between the two arms of an interferometer is scanned, high-contrast fringes appear only near the position where the paths are matched within the source coherence length. The result is a rapidly oscillating carrier fringe multiplied by a localized coherence envelope.

A simplified model is

$$I(\Delta_{\text{OPL}}) = I_1 + I_2 + 2\sqrt{I_1 I_2} |\gamma(\Delta_{\text{OPL}}/c)| \cos\left(\frac{2\pi}{\lambda_c} \Delta_{\text{OPL}} + \phi_0\right), \quad (15.151)$$

where λ_c is the vacuum centre wavelength and $|\gamma|$ provides the envelope. This form assumes a spectrum symmetric about its centre frequency and a nondispersive optical-path difference; otherwise the full complex spectral integral is required. A broad source spectrum yields a narrow correlation envelope and therefore good axial localization of reflecting surfaces.

This principle is used in optical profilometry and in low-coherence ranging. Because only surfaces whose path delay falls inside the coherence gate contribute strongly at a given reference position, reflections from different depths can be separated. The exact relationship between spectral bandwidth and axial point-spread function depends on the source spectrum and on whether bandwidth is specified in frequency or wavelength.

Example 15.31: Bandwidth and axial scale

Consider a source near $\lambda_0 = 850 \text{ nm}$ with spectral width $\Delta\lambda = 40 \text{ nm}$. The simple scale estimate gives

$$L_c \sim \frac{(850 \text{ nm})^2}{40 \text{ nm}} \approx 18 \mu\text{m}.$$

A detailed axial-resolution formula would include a lineshape-dependent numerical factor and the refractive index of the sample. Nevertheless, the estimate correctly shows why broadband sources can localize reflecting structures on micrometre-scale depths.

The quoted L_c is an optical-path scale, not a one-way sample depth. A reflection at depth z has a round-trip group delay approximately $2n_g z/c$, so the corresponding depth scale is $L_c/(2n_g)$ under the same convention.

Optical coherence tomography (OCT) is a low-coherence interferometric imaging method. A reference field interferes with light backscattered from structures inside a sample. Because the source has a short coherence length, strong interference occurs only when the reference-arm delay matches the delay of a particular sample depth within the coherence gate.

In time-domain OCT the reference delay is scanned mechanically and broadband-integrated interference is selected by the coherence envelope. In spectral-domain approaches, narrow spectral channels resolve the interference oscillations even when the broadband sum would cancel. A Fourier transform of data sampled uniformly in wave number then separates path delays and reconstructs depth. Thus the short coherence envelope sets axial resolution; it is not a requirement that every reflecting depth be within one coherence length of a fixed reference delay. Spectral resolution and sampling also limit the measurable depth range. The underlying chapter ideas remain recognizable:

- optical path, not merely geometric depth, sets phase;
- coherence controls which path differences can interfere;
- the measured signal is an interference cross term;
- Fourier analysis converts a family of phase oscillations into spatial information.

A full OCT treatment requires scattering theory, source spectra, detection bandwidth, Fourier reconstruction and biological-optics considerations. The present discussion is intended to show how a basic interference concept develops into a modern imaging technology.

15.23.15 Fourier-transform spectroscopy

For a source containing many frequencies, each spectral component produces a different phase as the Michelson path difference is scanned. The detector records the sum of all these oscillations. Let $S(\nu)$ denote an optical power spectrum. In a simplified ideal description, the variable part of the interferogram can be written

$$G(\tau) \propto \int_0^\infty S(\nu) \cos(2\pi\nu\tau) d\nu, \quad (15.152)$$

where τ is the relative time delay. Thus the interferogram and spectrum form a Fourier-transform pair, subject to the exact normalization and one-sided/two-sided conventions used.

The central burst near zero path difference occurs because all spectral components are approximately in phase there. Farther away, different frequencies dephase and the interferogram becomes smaller or more complicated. Recording a longer maximum optical-path difference improves spectral resolution because it allows more cycles of closely spaced frequencies to be distinguished.

This connects elementary interference with computational physics: the optical system encodes spectral information in a spatially scanned phase signal, while numerical Fourier transformation decodes it.

Method

In Fourier-transform spectroscopy, do not interpret every oscillation in the interferogram as a separate physical path. The oscillations arise because many optical frequencies accumulate different phases as one controlled path delay is scanned.

15.23.16 Spatial carriers, digital demodulation and holography

In the small-angle regime of a two-slit experiment,

$$\Delta\phi(y) \approx \frac{2\pi d}{\lambda D} y. \quad (15.153)$$

For equal beams,

$$I(y) = I_{\text{av}} [1 + \cos(2\pi f_y y)], \quad f_y = \frac{d}{\lambda D}. \quad (15.154)$$

The interference fringes are therefore a spatial carrier with period

$$P_y = \frac{1}{f_y} = \frac{\lambda D}{d}. \quad (15.155)$$

This language is especially useful when a camera records the pattern. For direct detection in the observation plane, a sensor with pixel pitch p must sample the fringes sufficiently finely. If an imaging system of transverse magnification M relays a screen to the sensor, the sensor period is $|M|P_y$. The formal Nyquist condition requires

$$|M|P_y > 2p \quad (\text{with } M = 1 \text{ for direct detection}), \quad (15.156)$$

but robust phase or visibility estimation typically benefits from several pixels per fringe, not merely two.

For a realistic finite slit, the recorded signal is a fast fringe carrier multiplied by a slowly varying diffraction envelope. In Fourier language, the slit separation sets a carrier spatial frequency while the slit width controls an envelope bandwidth. This viewpoint becomes central in coherent imaging and Fourier optics.

Suppose a measured one-dimensional fringe profile is approximately

$$I(y) = A(y) + B(y) \cos[2\pi f_c y + \phi(y)], \quad (15.157)$$

where $A(y)$ and $B(y)$ vary slowly compared with the carrier frequency f_c . A Fourier transform typically contains a low-frequency component from A and sidebands near $\pm f_c$ from the modulated cosine. A common digital strategy is:

1. Fourier transform the recorded fringe image or profile;
2. isolate one carrier sideband with a suitable filter;
3. shift that sideband toward zero spatial frequency;
4. inverse transform to obtain a complex analytic field;
5. calculate phase with $\arg$ of the complex field;
6. unwrap the phase if a continuous surface is expected.

The method is powerful because one camera exposure can encode a full spatial phase map when a known carrier is present. It also illustrates why sampling and image quality matter: aliasing, window edges, dust, saturation and strong amplitude variation contaminate the Fourier sidebands and therefore the reconstructed phase.

Example 15.32: Carrier frequency in a tilted-reference interferogram

A reference beam is tilted so that straight fringes with period $P = 0.50$ mm cross the camera. The spatial carrier frequency is

$$f_c = \frac{1}{P} = 2.0 \text{ mm}^{-1}.$$

If the pixel pitch is $5 \mu\text{m} = 0.005$ mm, the digital sampling frequency is 200 mm^{-1} , with Nyquist limit 100 mm^{-1} . The carrier is therefore very safely sampled. The practical design might intentionally use a higher carrier to separate the Fourier sideband more cleanly from the low-frequency background, while retaining sufficient pixels per fringe.

A hologram records interference between an object wave and a known reference wave. The recorded intensity has the form

$$I = |E_o + E_r|^2 = |E_o|^2 + |E_r|^2 + E_o E_r^* + E_o^* E_r. \quad (15.158)$$

The cross terms contain relative phase information that is absent from an ordinary intensity image of $|E_o|^2$ alone. When the hologram is illuminated appropriately, diffraction from the recorded pattern can reconstruct a wavefront related to the original object wave.

Digital holography replaces photographic recording with an electronic sensor and reconstructs the complex field numerically. Off-axis reference beams create a spatial carrier that allows the desired cross term to be separated in Fourier space, closely connecting holography with the carrier-demodulation method described above.

This is an important conceptual extension: interference can encode otherwise inaccessible phase information into an intensity pattern that a detector can record.

15.23.17 Beam-splitter phases and complementary outputs

The elementary Michelson diagram often labels a beam splitter as “50% reflected, 50% transmitted.” For phase-sensitive calculations, intensity percentages alone are insufficient. A lossless beam splitter must transform complex field amplitudes in a way that conserves energy and preserves the appropriate relative phases between output ports.

A convenient idealized symmetric representation is

$$\begin{pmatrix} E_{\text{out},1} \\ E_{\text{out},2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} E_{\text{in},1} \\ E_{\text{in},2} \end{pmatrix}. \quad (15.159)$$

The factors of i represent a particular phase convention for reflection. Other conventions are possible, but physical predictions are unchanged when the entire optical system is treated consistently.

If only input port 1 is illuminated, the two output amplitudes have equal magnitude, so each carries half the input intensity. If both input ports are illuminated coherently, the output powers depend on their relative phase. One output can become bright while the other becomes dark, consistent with energy conservation: interference redistributes energy between output modes rather than destroying it.

This point resolves a common conceptual discomfort. In a two-beam pattern, destructive interference at one location does not violate energy conservation. Energy appears elsewhere in the spatial field

or in the complementary output port of an interferometer. Interference changes where energy flows; it does not make energy disappear.

Example 15.33: Complementary powers from a lossless combiner

Let equal power-normalized amplitudes E and $Ee^{i\phi}$ enter the two input ports of Eq. (15.159). If either input alone has power $P_0 \propto |E|^2$, then

$$E_{\text{out},1} = \frac{E(1 + ie^{i\phi})}{\sqrt{2}}, \quad E_{\text{out},2} = \frac{E(i + e^{i\phi})}{\sqrt{2}}.$$

Taking absolute squares gives

$$P_1 = P_0(1 - \sin \phi), \quad P_2 = P_0(1 + \sin \phi), \quad P_1 + P_2 = 2P_0.$$

At $\phi = 0$ the powers are equal; at $\phi = \pi/2$ port 1 is dark and port 2 carries all the incident power. The chosen convention places extinction at $\pi/2$, not at zero phase. Propagation and both splitting events must be included before applying this to a complete interferometer. The sum provides an immediate energy-conservation check.

15.23.18 Stability and differential measurements

A highly sensitive interferometer responds to every phase perturbation, desired or undesired. A separated-arm Michelson can be sensitive to mirror vibration, air flow and temperature gradients because the two beams sample different environments. A common-path interferometer arranges the interfering fields to share much of the same physical route. Environmental disturbances then act more similarly on both contributions and partially cancel in the relative phase.

This idea is called common-mode rejection. If both arms acquire the same additional phase $\delta\phi_c$, then

$$(\phi_2 + \delta\phi_c) - (\phi_1 + \delta\phi_c) = \phi_2 - \phi_1. \quad (15.160)$$

Perfect cancellation is rarely achieved because the beams may differ in polarization, time of passage, spatial mode or exact path. Nevertheless, designing a measurement so that nuisance effects are common while the target signal is differential is a general engineering strategy extending far beyond optics.

A gravitational wave produces a differential change in spatial separations. Interferometric detectors exploit this by comparing optical phases accumulated along two nearly perpendicular arms. The natural dimensionless quantity is strain,

$$h \sim \frac{\Delta L}{L}, \quad (15.161)$$

where L is a characteristic arm length and ΔL is the differential change induced by the wave.

The conceptual link to the laboratory Michelson is direct: split a coherent laser beam, propagate the parts along different arms, reflect them and recombine them. However, modern gravitational-wave interferometers are far more sophisticated than the elementary diagram. Optical cavities increase the effective interaction with the arm-length change, recycling optics manage optical power and signal response, suspended mirrors reduce mechanical coupling, and elaborate feedback and data-analysis systems maintain operation near a sensitive working point.

Pedagogical boundary

The purpose here is not to derive the response of a gravitational-wave observatory. The relevant lesson is that the same phase-to-displacement principle introduced with the Michelson interferometer can be engineered into an instrument sensitive to extraordinarily small differential optical-path changes. The actual detector response requires general relativity, cavity optics, control theory and noise modelling beyond this chapter.

15.23.19 Fitting an image: what can actually be inferred?

A realistic camera profile can be modeled by

$$I(y) = B + A \left[\frac{\sin \beta(y)}{\beta(y)} \right]^2 [1 + \mathcal{V} \cos(2\alpha(y) + \phi_0)], \quad (15.162)$$

with

$$\alpha(y) = \frac{\pi d}{\lambda} \sin \theta(y), \quad \beta(y) = \frac{\pi a}{\lambda} \sin \theta(y), \quad \theta(y) = \arctan \frac{y - y_0}{D}. \quad (15.163)$$

Here B is a background offset, A is a scale, y_0 is the optical-axis location, $\mathcal{V}$ is the local visibility parameter and ϕ_0 is an offset phase. The model is a scalar Fraunhofer approximation with slowly varying illumination. Ratios such as d/λ and a/λ , not all lengths separately, determine the pattern. A wavelength fit therefore needs independently calibrated geometry: do not fit λ, d, a, D freely from one uncalibrated profile.

There is a second ambiguity. In the small-angle limit the phase contains $2\pi d(y - y_0)/(\lambda D) + \phi_0$, so a change of y_0 is indistinguishable from a compensating phase offset. Fix y_0 from independent geometry or fix ϕ_0 by a physical symmetry convention; fitting both without a constraint produces a degenerate or poorly conditioned estimate. The envelope may constrain its centre when it is resolved, but cannot be assumed to do so for a nearly flat envelope.

A good computational workflow is:

1. subtract or model the camera background;
2. inspect saturation and exclude saturated pixels;
3. estimate the fringe period using peak spacing or a Fourier spectrum;
4. initialize nonlinear parameters from the analytic small-angle formulas;
5. fit the full model, including the diffraction envelope if visible;
6. examine residuals rather than accepting a visually plausible curve;
7. estimate parameter covariance or use resampling when uncertainties are needed;
8. test sensitivity to the chosen fitting window and background model.

Caution

A nonlinear fit can return a very small formal statistical uncertainty while the model is systematically wrong. For example, ignoring the finite slit width, lens distortion, a tilted camera plane or wavelength spread can bias the fitted wavelength. Residual structure is evidence that the model or uncertainty assumptions are incomplete.

Example 15.34: Designing a camera-based Young experiment

A laboratory has a 532 nm laser, a camera with active width $W = 12$ mm and pixel pitch $p = 6.0 \mu\text{m}$. The sensor is placed directly in the observation plane at $D = 1.20$ m, without a camera lens. If a screen is imaged through a lens, its magnification must instead be included. Choose a slit separation that gives about ten fringe periods across the sensor width.

Ten periods across W means

$$\Delta y \approx \frac{W}{10} = 1.2 \text{ mm}.$$

Using $\Delta y = \lambda D/d$,

$$d \approx \frac{\lambda D}{\Delta y} = \frac{532 \times 10^{-9} (1.20)}{1.2 \times 10^{-3}} \approx 5.32 \times 10^{-4} \text{ m} = 0.532 \text{ mm}.$$

Each fringe spans about

$$\frac{1.2 \text{ mm}}{6.0 \mu\text{m}} = 200$$

pixels, so sampling is excellent. A practical design must also choose slit width so that the diffraction envelope illuminates a useful number of these fringes, and must keep the camera below saturation.

15.23.20 Measuring a coherence envelope

A conceptually direct experiment measures fringe visibility as a function of Michelson path mismatch. The procedure is:

1. balance the two arm intensities and align their polarizations;
2. scan the mirror position over a range large compared with the expected coherence length;
3. at each delay, estimate local fringe maximum and minimum or demodulated fringe amplitude;
4. calculate visibility $\mathcal{V}(\Delta L)$;
5. fit the visibility envelope using a correlation model consistent with the expected spectral lineshape;
6. convert the fitted coherence scale to spectral linewidth using the corresponding Fourier relation.

The experiment teaches an important inverse-problem principle. The measured quantity is not linewidth directly; it is an interferometric correlation curve. A linewidth is inferred through a model. If the source spectrum is multimode or nonstationary, a single fitted linewidth may be a poor description.

15.23.21 Connection to quantum interference

Classical electromagnetic theory correctly predicts the average interference intensity for ordinary coherent light. When light is sufficiently weak that detections occur one photon at a time, individual events appear as localized detector counts, yet after many events the statistical distribution can build the same interference fringe pattern.

A quantum description associates probability amplitudes with indistinguishable alternatives and adds those amplitudes before taking the absolute square. This mathematical structure resembles the addition of complex classical field amplitudes but has a different interpretation. It is not correct to imagine each photon as a tiny classical pellet secretly choosing one slit while the other slit is

irrelevant. Conversely, it is also unnecessary in this chapter to replace classical optics by quantum electrodynamics to explain ordinary two-slit intensities.

The key conceptual bridge is

Classical optics: fields add, then intensity is formed.

whereas in elementary quantum language,

Quantum mechanics: probability amplitudes add, then probabilities are formed.

Which-path information can destroy interference because the alternatives become distinguishable. This prepares the reader for later quantum-mechanical discussions without confusing classical optical coherence with quantum entanglement.

Guided checks

Why is the statement “a dark fringe means photons avoid that region because they repel one another” incorrect? Explain first in classical field language and then, qualitatively, in single-photon probability-amplitude language.

15.23.22 Advanced synthesis

Summary

The advanced treatment replaces isolated bright/dark conditions by a unified complex-amplitude viewpoint. Polarization enters through vector projection; partial coherence enters through correlation functions; finite bandwidth limits useful path mismatch; camera fringes can be treated as spatial carriers; and realistic thin films require amplitude as well as phase. Multiple coherent reflections lead to Fabry–Pérot resonances, while phase-shifting and heterodyne methods convert optical phase into digitally recoverable signals. Interferometric sensing is therefore a synthesis of wave physics, materials, geometry, statistics, electronics and computation.

15.24 Original Practice Problems

The 50 problems progress from conceptual reasoning to measurement and modelling. Unless stated otherwise, take air as $n = 1$ and use vacuum wavelengths. Specify the approximation and the sign convention before calculating. Problems involving Jones vectors, Fresnel sums, phase recovery or cavity resonances use the advanced section. Numerical data are illustrative teaching inputs, not experimental datasets.

Conceptual questions

- C1.** Two identical monochromatic waves reach a detector with a phase difference of 7π . Is the point bright or dark for equal amplitudes? Reduce the phase modulo 2π and explain the result.
- C2.** A student says: “A path difference of $\lambda/2$ always means destructive interference.” Give two physically relevant reasons why this statement can fail when applied mechanically to an optical system.
- C3.** The entire Young apparatus is immersed in a transparent liquid of refractive index $n > 1$, with the geometric dimensions fixed and the source frequency unchanged. Explain what happens to the wavelength in the medium and to the fringe spacing.
- C4.** Why do two independent household lamps not normally produce a stationary interference pattern even if filters select nearly the same central wavelength from both lamps?
- C5.** In the scalar model, two equal, parallel-polarized field contributions cancel at an ideal dark fringe. Is either individual contribution zero there? Explain why cancellation of the detected sum neither blocks a slit nor destroys energy. Why should this scalar statement not be generalized to every component of $\mathbf{E}$ and $\mathbf{B}$ for arbitrary nonparallel waves?
- C6.** Two coherent beams have exactly equal intensities but mutually orthogonal linear polarizations. Why is the ordinary interference modulation absent on a polarization-insensitive camera? How can an analyzer make fringes visible again?
- C7.** For a film with one net π reflection reversal, explain why $t = 0$ satisfies a reflected-minimum phase condition. Does this ensure zero reflection if the exterior media have different indices? Identify the limiting physical interface.
- C8.** Two sources have the same quoted linewidth but different centre wavelengths. Compare the cases in which “same linewidth” means equal frequency width $\Delta\nu$ or equal wavelength width $\Delta\lambda$. State what must also be specified to predict the detailed coherence envelope.

Two-beam and double-slit calculations

- D1.** A laser of wavelength 632.8 nm illuminates two narrow slits separated by 0.250 mm. A screen is 2.00 m away. Calculate the small-angle fringe spacing and the position of the fourth bright fringe above the central maximum.
- D2.** In a Young experiment the third bright fringe is observed at $y = 7.20$ mm on a screen 1.50 m from slits separated by 0.300 mm. Determine the wavelength using the small-angle approximation and state whether the result lies in the visible range.

- D3.** The first-order bright fringe of a double slit occurs at $\theta = 12.0^\circ$ for $\lambda = 520 \text{ nm}$. Find the slit separation without making a small-angle approximation.
- D4.** Equal coherent beams each have intensity 4.0 W m^{-2} . Calculate the combined intensity for relative phase differences $0, \pi/3, \pi/2, 2\pi/3$ and π .
- D5.** Two parallel-polarized beams give individual intensities 16.0 and 9.0 W m^{-2} at a detector. A phase scan records $I_{\max} = 51.0$ and $I_{\min} = 3.0 \text{ W m}^{-2}$, including a separately measured incoherent background $B = 2.0 \text{ W m}^{-2}$. Find the raw visibility, the background-subtracted visibility and the inferred $|\gamma_{12}|$. Explain which contrast reduction is caused by unequal beam powers.
- D6.** A 589 nm source illuminates slits separated by 0.200 mm . At $D = 0.750 \text{ m}$, compare the fifth-order screen position from Eq. (15.32) with $m\lambda D/d$. Report the fractional difference and identify the approximation still shared by both formulas.
- D7.** For two slits of finite width $a = 0.050 \text{ mm}$ and separation $d = 0.300 \text{ mm}$, determine which double-slit bright orders coincide with single-slit diffraction minima using the conditions $d \sin \theta = m\lambda$ and $a \sin \theta = p\lambda$ with nonzero integer p .
- D8.** A double-slit pattern has fringe spacing 1.60 mm at $D = 1.20 \text{ m}$. The entire setup is then filled with water of refractive index 1.333 . Estimate the new fringe spacing, assuming the light frequency and geometry are unchanged.
- D9.** Two waves arrive with equal intensity I_0 but their relative phase fluctuates rapidly and uniformly over 0 to 2π during each detector exposure. Calculate the exposure-averaged intensity. Compare it with the maximum and minimum for fixed phase.

Films, coatings and Newton's rings

- F1.** A film of refractive index 1.40 is surrounded by air. Light of wavelength 560 nm is incident nearly normally. Because the upper reflection has a π shift and the lower one does not, find the smallest nonzero film thickness giving a reflected maximum. Also find the smallest positive thickness for a reflected minimum, distinguishing it from the zero-thickness limit.
- F2.** A film of index 1.50 lies on a substrate of index 1.80 , with air above. At both interfaces the reflected ray undergoes the same π phase change relative to the incident direction. Find the smallest positive thickness for a reflected minimum at 600 nm .
- F3.** Oil with $n = 1.25$ floats on water with $n = 1.33$. For nearly normal illumination from air, determine the minimum nonzero oil thickness that strongly reflects 530 nm light. Construct a two-interface reflection-phase table before using a formula.
- F4.** A magnesium-fluoride-like coating with $n_1 = 1.38$ is designed as a quarter-wave layer at 550 nm . Calculate the physical thickness. Then compute the ideal matching index for a glass substrate of $n_2 = 1.52$ in air and comment on whether exact zero reflection is expected from the actual n_1 .
- F5.** A wedge-shaped air film between glass plates increases in thickness at a rate of $1.00 \mu\text{m}$ per 5.00 cm along x . Using 600 nm light at near-normal incidence, determine the spacing between successive reflected dark fringes, assuming one net reflection reversal.

- F6.** A thin film has thickness 400 nm and refractive index 1.33. Under the one-net- π -reversal case, find the visible vacuum wavelengths between 400 and 700 nm that give reflected maxima at normal incidence.
- F7.** For a film of index 1.45 illuminated at 30° from air, calculate the internal angle using Snell's law. Then compute the propagation phase $4\pi nt \cos \theta_f / \lambda_0$ for $t = 250$ nm and $\lambda_0 = 550$ nm, before adding any reflection phase.
- F8.** In a Newton's-rings experiment using an air film, the radius of the 20th reflected dark ring is 4.50 mm for light of wavelength 589 nm. Determine the radius of curvature of the lens.
- F9.** A reflected Newton-ring pattern in air is recorded at $\lambda_0 = 589$ nm. Correctly assigned dark orders $m = 5$ and $m = 15$ have diameters $D_5 = 4.00$ mm and $D_{15} = 7.20$ mm. Assuming a spherical lens and a constant central gap, determine R using diameter differences, then infer t_0 . Explain why an unknown common shift of the order labels leaves R unchanged but changes the inferred t_0 .
- F10.** A single-layer coating is placed between air and glass. Using the normal-incidence amplitude coefficient $r_{ij} = (n_i - n_j)/(n_i + n_j)$, calculate r_{01} and r_{12} for $n_0 = 1.00$, $n_1 = 1.23$, $n_2 = 1.51$. Are the magnitudes close enough for a quarter-wave layer to yield strong cancellation?

Michelson interferometry and measurement

- M1.** A Michelson interferometer uses 632.8 nm light. How far must one mirror move to produce 250 complete fringe shifts at a fixed detector point?
- M2.** A mirror is translated by 0.250 mm and 790 fringes pass the reference point. Determine the wavelength.
- M3.** A gas cell of length 10.0 cm in one arm is evacuated and then filled. At 632.8 nm, 95 fringes pass. Determine $n - 1$ assuming a double pass.
- M4.** A Michelson instrument operates near quadrature at 633 nm. A small mirror displacement produces a phase change 0.020 rad. Find the displacement.
- M5.** Another pixel gives $I_0 = 90$, $I_{\pi/2} = 50$, $I_\pi = 30$, $I_{3\pi/2} = 70$. Calculate the phase using $\text{atan2}(I_{3\pi/2} - I_{\pi/2}, I_0 - I_\pi)$.
- M6.** A single-pass optical sensor has $n = 1.45$ and $L = 0.100$ m. Temperature changes the length by $\Delta L = +2.0$ μm and the refractive index by $\Delta n = -1.0 \times 10^{-5}$. Determine the first-order optical-path change $n\Delta L + L\Delta n$. Do the two effects reinforce or partially cancel?
- M7.** In a Michelson wavelength measurement, $\Delta x = (1.0000 \pm 0.0004)$ mm and $N = 3160 \pm 1$. Determine λ_0 and estimate its standard uncertainty using independent-error propagation.
- M8.** A heterodyne interferometer has frequency difference $\Delta f = 40.0$ kHz. Write the beat intensity term and determine the beat period. If the optical path slowly changes, what measurable feature of the electronic beat carries the displacement information?

Computation and data analysis

- N1.** Numerically compare the exact double-slit path difference in Eq. (15.22) with $d \sin \theta$ for $d = 0.50$ mm, $D = 0.20$ m and $|y| \leq 30$ mm. Plot the phase error in radians for $\lambda = 633$ nm.
- N2.** Generate a synthetic finite-width double-slit profile from Eq. (15.162) with known d, a, D, y_0 , fixed $\phi_0 = 0$ and a chosen wavelength. Label the data as simulated, add a constant background and Gaussian detector noise, and fit $\lambda, A, B, \mathcal{V}$. Compare a pure fringe model with the full envelope model. Inspect residuals and parameter correlations; explain why freely fitting all geometric lengths and the wavelength from one profile is not identifiable.
- N3.** Simulate independent Gaussian phase offsets of standard deviation $\sigma_\phi = 0, 0.2, 0.5, 1.0, 2.0$ rad. For each case estimate $\left| N_s^{-1} \sum_j e^{i\eta_j} \right|$ and compare with $e^{-\sigma_\phi^2/2}$. Verify from an averaged phase scan that this quantity is the visibility for equal beams. Repeat with a single fixed offset per exposure and explain why short individual exposures retain contrast.
- N4.** A camera has 2048 pixels of pitch $5.5 \mu\text{m}$. For $\lambda = 532$ nm and $D = 1.00$ m, determine a range of slit separations that gives between 8 and 30 fringe periods across the sensor width. State any small-angle assumption.
- N5.** Create a near-normal thin-film phase-accounting program with inputs $n_0, n_f, n_s, t, \lambda_0$. Use the signs of the two Fresnel reflection coefficients to obtain the relative reflection phase, and test air–oil–water and air–film–air. Handle an index-matched interface explicitly: a zero reflected amplitude has no defined reflection phase and cannot form a two-ray fringe by itself. Compare phase extrema with the complete reflection formula where both contributions are nonzero.
- N6.** Plot the ideal Fabry–Pérot transmission $\mathcal{T}(\delta)$ for $R = 0.2, 0.7, 0.9, 0.99$. Measure numerically how the full width at half maximum changes and compare the resulting finesse with Eq. (15.113) where appropriate.
- N7.** Simulate Newton-ring radii for a known R , nonzero central gap t_0 , and known order assignments. Add equal independent radius errors. Fit r_m^2 against m with a free intercept, using $u_{r^2} \simeq 2r_m u_r$ as weights. Compare with an unweighted fit and with a fit forced through the origin. Explain the distinct statistical and model errors.

Engineering and advanced challenges

- E1.** A displacement sensor must resolve 5 nm motion at 633 nm. Calculate the corresponding Michelson phase change. Explain why resolving this phase change is not equivalent to claiming 5 nm absolute accuracy.
- E2.** A quarter-wave coating is required at both 450 nm and 650 nm with one single homogeneous layer whose refractive index may be treated as constant over that interval. Explain quantitatively why one thickness cannot be the thinnest quarter-wave layer at both wavelengths. Suggest, conceptually, how multilayer coating design addresses broadband performance.
- E3.** A camera-based interferometer shows visibility decreasing from 0.90 at the center to 0.30 near the edge of the field. Propose at least four distinct physical or instrumental causes and one diagnostic test for each.
- E4.** A fiber interferometric temperature sensor has phase response $d\phi/dT$. Starting from $\phi =$

- $(2\pi/\lambda_0)n_{\text{eff}}L$, derive the first-order expression in terms of dn_{eff}/dT and the thermal expansion coefficient $\alpha = (1/L)dL/dT$.
- E5.** For an ideal symmetric Fabry–Pérot cavity, derive Eq. (15.110) by summing the transmitted complex-amplitude geometric series. State where the lossless and identical-mirror assumptions enter.
- E6.** A source has a Gaussian optical spectrum rather than a Lorentzian one. Explain why the detailed decay shape of fringe visibility with path delay differs even if the two sources have the same quoted FWHM linewidth. No full Fourier transform is required, but identify the underlying mathematical connection.
- E7.** A Michelson interferometer is placed in an enclosure. Both arms lengthen by the same amount due to thermal expansion, but the measured output phase barely changes. Explain common-mode rejection. Then give two reasons why perfect rejection may fail in a real instrument.
- E8.** Two neighbouring samples differ in true phase by 1.6π , but the wrapped difference is -0.4π . Show that both give the same complex phase factor. What does nearest-neighbour unwrapping choose if it assumes differences smaller than π ? Explain why continuity alone cannot recover the true jump and identify additional sampling or measurement information that could resolve it.

15.25 Selected Solutions to Representative Problems

These solutions show the model, reasoning, intermediate steps and checks. A numerical answer without its assumptions does not specify an interference calculation completely.

15.25.1 Solution to D1: fringe spacing and fringe position

For the requested order, $m\lambda/d = 4(632.8 \text{ nm})/(0.250 \text{ mm}) = 0.0101 \ll 1$, so the small-angle approximation is justified. The stated screen distance also makes the point-source far-field geometry appropriate. The fringe spacing is

$$\begin{aligned}\Delta y &\approx \frac{\lambda D}{d} \\ &= \frac{(632.8 \times 10^{-9} \text{ m})(2.00 \text{ m})}{0.250 \times 10^{-3} \text{ m}} \\ &= 5.06 \times 10^{-3} \text{ m} = 5.06 \text{ mm}.\end{aligned}$$

The fourth bright fringe above the center has $m = 4$:

$$y_4 \approx 4\Delta y = 20.2 \text{ mm}.$$

A quick proportionality check is useful: if d were doubled, both numbers would halve; if D were doubled, both would double.

15.25.2 Solution to D3: exact angular condition

For a bright fringe,

$$d \sin \theta = m\lambda.$$

The first order has $m = 1$, so

$$\begin{aligned}d &= \frac{\lambda}{\sin 12.0^\circ} \\ &= \frac{520 \times 10^{-9} \text{ m}}{0.2079} \\ &\approx 2.50 \times 10^{-6} \text{ m} = 2.50 \mu\text{m}.\end{aligned}$$

Using $\sin \theta \approx \theta$ would give $d \approx 2.48 \mu\text{m}$, already a visible difference at this angle. The exact trigonometric relation is therefore preferable.

15.25.3 Solution to D5: contrast, background and coherence

The raw extrema give

$$\mathcal{V}_{\text{raw}} = \frac{51 - 3}{51 + 3} = \frac{8}{9} \simeq 0.889.$$

Background subtraction must precede a coherence inference. The corrected extrema are 49 and 1 W m^{-2} , so

$$\mathcal{V}_{\text{corr}} = \frac{49 - 1}{49 + 1} = 0.960.$$

For the independently measured beam powers, the maximum possible contrast is

$$\frac{2\sqrt{I_1 I_2}}{I_1 + I_2} = \frac{2\sqrt{16 \times 9}}{25} = 0.960.$$

Thus $|\gamma_{12}| = \mathcal{V}_{\text{corr}}/0.960 = 1$ within the ideal model. Unequal amplitudes leave a corrected minimum $(4 - 3)^2 = 1 \text{ W m}^{-2}$ even at complete coherence. The additional reduction from 0.960 to 0.889 is due to incoherent background. All these inferences assume a linear detector and negligible polarization or spatial-overlap loss.

15.25.4 Solution to D6: trigonometric versus small-angle position

The fifth bright fringe satisfies

$$\sin \theta_5 = \frac{5\lambda}{d} = \frac{5(589 \times 10^{-9})}{0.200 \times 10^{-3}} = 0.014725.$$

Therefore

$$\theta_5 = \arcsin(0.014725) \approx 0.0147255 \text{ rad.}$$

Retaining exact screen trigonometry within the far-field model gives

$$y_{\text{exact}} = D \tan \theta_5 \approx (0.750)(0.0147266) = 0.011045 \text{ m.}$$

The small-angle result is

$$y_{\text{small}} = \frac{5\lambda D}{d} = (0.750)(0.014725) = 0.011044 \text{ m.}$$

The fractional difference is $[1 - (5\lambda/d)^2]^{-1/2} - 1 = 1.08 \times 10^{-4}$, about 0.011%. Both expressions still use $r_2 - r_1 \simeq d \sin \theta$; Eq. (15.22) is available if the finite-distance path error must also be assessed.

15.25.5 Solution to D7: missing orders in a finite-width double slit

Double-slit interference maxima obey

$$d \sin \theta = m\lambda,$$

whereas single-slit minima obey

$$a \sin \theta = p\lambda, \quad p = \pm 1, \pm 2, \dots$$

A nominal interference maximum disappears when both conditions hold at the same angle. Dividing them gives

$$\frac{m}{p} = \frac{d}{a} = \frac{0.300}{0.050} = 6.$$

Hence

$$m = 6p.$$

The $m = \pm 6, \pm 12, \dots$ interference orders coincide with diffraction zeros whenever those orders are geometrically allowed. This is a good example of why the two-slit cosine factor cannot be interpreted independently of the single-slit envelope.

15.25.6 Solution to F3: oil on water

The index sequence is

$$1.00 \rightarrow 1.25 \rightarrow 1.33.$$

The reflection at the air–oil interface is low-to-high, so the reflected field receives π . The reflection at the oil–water interface is also low-to-high, so it also receives π . The relative reflection phase is therefore zero: the two reflected rays have the same reflection reversal.

For zero net reflection-phase difference, a reflected maximum at near-normal incidence requires

$$2nt = m\lambda_0.$$

The smallest nonzero thickness corresponds to $m = 1$:

$$\begin{aligned} t &= \frac{\lambda_0}{2n} \\ &= \frac{530 \text{ nm}}{2(1.25)} \\ &= 212 \text{ nm.} \end{aligned}$$

The phase table is essential: blindly using the “one reversal” quarter-wave result would give the wrong thickness.

15.25.7 Solution to F6: visible reflected maxima of a film

For one net π reflection reversal, reflected maxima satisfy

$$2nt = \left(m + \frac{1}{2}\right) \lambda_0.$$

Here

$$2nt = 2(1.33)(400 \text{ nm}) = 1064 \text{ nm}.$$

Hence

$$\lambda_0 = \frac{1064 \text{ nm}}{m + 1/2}.$$

We require $400 \leq \lambda_0 \leq 700 \text{ nm}$. Testing nonnegative integers:

$$\begin{aligned} m = 1 : \lambda_0 &= \frac{1064}{1.5} \text{ nm} \approx 709 \text{ nm}, \\ m = 2 : \lambda_0 &= \frac{1064}{2.5} \text{ nm} \approx 426 \text{ nm}, \\ m = 3 : \lambda_0 &= \frac{1064}{3.5} \text{ nm} \approx 304 \text{ nm}. \end{aligned}$$

Within the stated 400–700 nm interval, the only maximum is therefore approximately 426 nm. The $m = 1$ result lies just beyond the specified upper boundary.

15.25.8 Solution to F8: lens curvature from Newton's rings

For reflected dark rings in an air film,

$$r_m^2 = m\lambda_0 R.$$

Therefore

$$\begin{aligned} R &= \frac{r_m^2}{m\lambda_0} \\ &= \frac{(4.50 \times 10^{-3} \text{ m})^2}{20(589 \times 10^{-9} \text{ m})} \\ &\approx 1.72 \text{ m}. \end{aligned}$$

The result is plausible for a weakly curved optical surface: the ring radii are millimetric while the lens radius of curvature is macroscopic.

15.25.9 Solution to M3: refractive index of a gas

The cell has physical length $L = 0.100 \text{ m}$ and is traversed twice. Filling it changes the optical path by

$$\Delta\text{OPL} = 2L(n - 1).$$

Each fringe shift corresponds to one wavelength of optical-path change:

$$N = \frac{2L(n - 1)}{\lambda_0}.$$

Thus

$$\begin{aligned} n - 1 &= \frac{N\lambda_0}{2L} \\ &= \frac{95(632.8 \times 10^{-9})}{2(0.100)} \\ &\approx 3.01 \times 10^{-4}. \end{aligned}$$

Therefore

$$n \approx 1.000301.$$

The small value of $n - 1$ is physically typical of gases at ordinary conditions, providing a useful reasonableness check.

15.25.10 Solution to M5: four-step phase with both quadratures

Now

$$Y = I_{3\pi/2} - I_{\pi/2} = 70 - 50 = 20,$$

while

$$X = I_0 - I_{\pi} = 90 - 30 = 60.$$

Thus

$$\phi = \text{atan2}(20, 60) = \arctan\left(\frac{1}{3}\right) \approx 0.322 \text{ rad}.$$

Both X and Y are positive, so the phase lies in the first quadrant. Using a two-argument arctangent is important when signs differ.

15.25.11 Solution to M6: competing thermo-optic and expansion effects

To first order,

$$\Delta(nL) \approx n\Delta L + L\Delta n.$$

The geometric term is

$$n\Delta L = (1.45)(2.0 \times 10^{-6}) = 2.90 \times 10^{-6} \text{ m}.$$

The refractive-index term is

$$L\Delta n = (0.100)(-1.0 \times 10^{-5}) = -1.00 \times 10^{-6} \text{ m}.$$

Therefore

$$\Delta\text{OPL} = 1.90 \times 10^{-6} \text{ m}.$$

The effects partially cancel because thermal expansion increases the optical path while the refractive-index change decreases it.

15.25.12 Solution to M7: wavelength and propagated uncertainty

The wavelength model is

$$\lambda_0 = \frac{2\Delta x}{N}.$$

With $\Delta x = 1.0000 \text{ mm}$ and $N = 3160$,

$$\begin{aligned} \lambda_0 &= \frac{2(1.0000 \times 10^{-3})}{3160} \\ &= 6.329 \times 10^{-7} \text{ m} = 632.9 \text{ nm}. \end{aligned}$$

The independent fractional uncertainty is

$$\begin{aligned} \left(\frac{u_\lambda}{\lambda_0}\right)^2 &= \left(\frac{0.0004}{1.0000}\right)^2 + \left(\frac{1}{3160}\right)^2 \\ &= (4.00 \times 10^{-4})^2 + (3.16 \times 10^{-4})^2. \end{aligned}$$

Hence

$$\frac{u_\lambda}{\lambda_0} \approx 5.10 \times 10^{-4}$$

and

$$u_\lambda \approx (632.9 \text{ nm})(5.10 \times 10^{-4}) \approx 0.32 \text{ nm}.$$

A suitable result is therefore about

$$\lambda_0 = (632.9 \pm 0.3) \text{ nm}$$

for the stated uncertainty model. More digits would not be justified unless systematic calibration effects are smaller than this budget.

15.25.13 Solution to E4: temperature sensitivity of a fiber interferometer

Start from

$$\phi(T) = \frac{2\pi}{\lambda_0} n_{\text{eff}}(T) L(T),$$

assuming the vacuum wavelength is externally fixed. Differentiate:

$$\frac{d\phi}{dT} = \frac{2\pi}{\lambda_0} \left(L \frac{dn_{\text{eff}}}{dT} + n_{\text{eff}} \frac{dL}{dT} \right).$$

With thermal expansion coefficient

$$\alpha = \frac{1}{L} \frac{dL}{dT},$$

we obtain

$$\frac{d\phi}{dT} = \frac{2\pi L}{\lambda_0} \left(\frac{dn_{\text{eff}}}{dT} + n_{\text{eff}} \alpha \right).$$

The two terms are the thermo-optic and geometric-expansion contributions. Their relative signs determine whether they reinforce or partially cancel.

15.25.14 Solution to F9: ring curvature without perfect contact

Subtract Eq. (15.55) for the two orders. The unknown central gap cancels:

$$R = \frac{D_{15}^2 - D_5^2}{4(15 - 5)\lambda_0} = \frac{(7.20^2 - 4.00^2) \times 10^{-6} \text{ m}^2}{40(589 \times 10^{-9} \text{ m})} = 1.52 \text{ m}.$$

Using the unrounded curvature in the fifth-order equation gives

$$t_0 = \frac{1}{2} \left(5\lambda_0 - \frac{D_5^2}{4R} \right) \simeq 1.58 \times 10^{-7} \text{ m} = 0.158 \mu\text{m}.$$

Both the millimetre radii and the submicrometre gap are small compared with R , as required. Shifting every order label by integer q leaves the difference 10 unchanged, so the curvature is unchanged. It changes t_0 by $q\lambda_0/2$. Absolute gap inference therefore requires the correct order assignment, whereas the curvature from differences does not.

15.25.15 Solution to N3: Gaussian phase averaging

For a zero-mean Gaussian phase error, the characteristic function gives

$$\langle e^{i\eta} \rangle = e^{-\sigma_\phi^2/2}, \quad \langle I \rangle = 2I_0[1 + e^{-\sigma_\phi^2/2} \cos \phi_0].$$

The visibility is therefore $\mathcal{V} = e^{-\sigma_\phi^2/2}$:

σ_ϕ (rad)	0	0.2	0.5	1.0	2.0
$\mathcal{V}$	1.000	0.980	0.882	0.607	0.135

Monte Carlo estimates fluctuate around these values. A phase fixed throughout one exposure shifts its fringes without reducing visibility; the loss requires averaging different offsets within an exposure or across unregistered images. Appendix C gives reproducible code.

Chapter A

Notation Dictionary for Interference of Light

This appendix gathers the notation used in the chapter. Symbols in optics are not universal across textbooks, so the definitions below take precedence within this textbook chapter.

Symbol	Meaning in this chapter	SI unit
$E, \tilde{E}$	real electric-field component and its complex amplitude	V m^{-1}
E_0	electric-field amplitude	V m^{-1}
I	time-averaged optical intensity	W m^{-2}
I_1, I_2	intensities of two interfering beams	W m^{-2}
$I_{\max}, I_{\min}$	maximum and minimum fringe intensities	W m^{-2}
$\mathcal{V}$	fringe visibility	dimensionless
f, ν	ordinary frequency	Hz
ω	angular frequency, $2\pi\nu$	rad s^{-1}
λ_0	wavelength in vacuum; air approximation when stated	m
λ	wavelength in a specified material	m
k_0	vacuum wave number, $2\pi/\lambda_0$	rad m^{-1}
k	material wave number, nk_0 in a simple dielectric	rad m^{-1}
ϕ	optical phase	rad
$\Delta\phi$	relative phase between two contributions	rad
L	geometric path length; context may also use cavity or cell length	m
OPL	optical path length $\int n \, ds$	m
ΔL	geometric path difference; optical path is denoted ΔOPL	m
n	refractive index	dimensionless
n_g	group index	dimensionless
d	double-slit center-to-center separation	m
a	individual slit width	m
D	distance from slit plane to screen/camera	m
y	transverse coordinate on screen/camera	m
θ	observation or incidence angle, according to context	rad
m	integer interference order	integer
t	thin-film or plate thickness	m
R	lens radius of curvature in Newton's rings; mirror reflectance when explicitly defined in cavity context	m or dimensionless
r_m	radius of the m th Newton ring	m

Symbol	Meaning in this chapter	SI unit
r_{ij}	complex/amplitude reflection coefficient from medium i to j	dimensionless
β	one-way film phase; also the single-slit argument where defined	rad
δ	round-trip phase in a Fabry–Pérot cavity	rad
F	Airy coefficient $4R/(1 - R)^2$	dimensionless
$\mathcal{F}$	cavity finesse	dimensionless
$\Delta\nu_{\text{FSR}}$	free spectral range	Hz
Γ_{12}	mutual coherence function	field-correlation units
γ_{12}	normalized complex degree of coherence	dimensionless
τ_c	coherence-time scale	s
L_c	coherence-length scale	m
$\Delta\nu$	optical frequency linewidth/bandwidth	Hz
$\Delta\lambda$	wavelength linewidth/bandwidth	m
N	fringe cycles or layer count as stated; $\bar{N}$ is a mean photon count	dimensionless
u_x	standard uncertainty associated with quantity x	unit of x

Note

The symbol R is traditionally overloaded in optics: it may denote a radius of curvature or an intensity reflectance. The surrounding equation makes the meaning clear in this chapter. In formal technical writing, a different symbol such as $\mathcal{R}$ for reflectance can be introduced if both quantities appear repeatedly in the same derivation.

Chapter B

Compact Formula Sheet

Two-beam interference

These scalar formulas assume common detection medium, common polarization, stable relative phase and paraxial overlap; I_0 is the intensity from one beam alone.

$$\begin{aligned} I &= I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \Delta\phi, \\ I_{\max} &= (\sqrt{I_1} + \sqrt{I_2})^2, \\ I_{\min} &= (\sqrt{I_1} - \sqrt{I_2})^2, \\ \mathcal{V} &= \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}, \\ I &= 4I_0 \cos^2(\Delta\phi/2) \quad \text{for equal beams.} \end{aligned}$$

Phase and optical path

$$\begin{aligned} \text{OPL} &= \int n \, ds, \\ \Delta\phi_{\text{prop}} &= \frac{2\pi}{\lambda_0} \Delta\text{OPL}, \\ \lambda &= \frac{\lambda_0}{n} \quad \text{at the stated frequency, using the phase index.} \end{aligned}$$

A low-to-high-index reflection gives a π electric-field phase reversal at normal incidence for the elementary lossless dielectric case; a high-to-low reflection gives no such reversal.

Young double slit

$$\begin{aligned} \Delta L &\approx d \sin \theta, \\ d \sin \theta_m &= m\lambda && \text{bright,} \\ d \sin \theta_m &= \left(m + \frac{1}{2}\right) \lambda && \text{dark,} \\ y_m &\approx \frac{m\lambda D}{d} && \text{small angle,} \\ \Delta y &\approx \frac{\lambda D}{d}, \\ I(\theta) &= I_{\max} \cos^2 \left(\frac{\pi d \sin \theta}{\lambda} \right) && \text{equal narrow slits.} \end{aligned}$$

For finite slit width a ,

$$I(\theta) = I_{\text{scale}} \left(\frac{\sin \beta}{\beta} \right)^2 \cos^2 \alpha, \quad \alpha = \frac{\pi d \sin \theta}{\lambda}, \quad \beta = \frac{\pi a \sin \theta}{\lambda}.$$

Thin films at near-normal incidence

Propagation phase between the first two reflected rays:

$$\Delta\phi_{\text{prop}} = \frac{4\pi n_f t}{\lambda_0}.$$

With exactly one net π reflection reversal:

$$\begin{aligned} 2n_f t &= \left(m + \frac{1}{2}\right) \lambda_0 && \text{reflected maximum,} \\ 2n_f t &= m \lambda_0 && \text{reflected minimum.} \end{aligned}$$

With zero net π reflection reversal:

$$\begin{aligned} 2n_f t &= m \lambda_0 && \text{reflected maximum,} \\ 2n_f t &= \left(m + \frac{1}{2}\right) \lambda_0 && \text{reflected minimum.} \end{aligned}$$

For oblique incidence inside the film:

$$\Delta\phi_{\text{prop}} = \frac{4\pi n_f t \cos \theta_f}{\lambda_0}.$$

Quarter-wave design:

$$t = \frac{\lambda_0}{4n_1}, \quad n_1 = \sqrt{n_0 n_2}$$

for exact ideal single-layer amplitude matching at normal incidence.

Newton's rings

For the standard reflected dark air-film rings with ideal contact and $r \ll R$:

$$r_m^2 = m \lambda_0 R.$$

If the gap has refractive index n , the corresponding ideal relation becomes

$$r_m^2 = \frac{m \lambda_0 R}{n}$$

under the same phase-reversal pattern. A central gap changes the squared-radius intercept as in Eq. (15.55).

Michelson interferometer

Mirror displacement in air:

$$N = \frac{2\Delta x}{\lambda_0}, \quad \Delta\phi = \frac{4\pi\Delta x}{\lambda_0}.$$

Double-pass cell or plate replacing air:

$$N = \frac{2t(n-1)}{\lambda_0}.$$

Advanced coherence and cavities

Visibility assumes common polarization and corrected background. The Airy expression assumes identical lossless mirrors; the stated FSR neglects dispersion and mirror delay, and the finesse formula is the narrow-peak approximation.

$$\begin{aligned}
 I &= I_1 + I_2 + 2\sqrt{I_1 I_2} \Re[\gamma_{12} e^{i\phi}], \\
 \mathcal{V} &= \frac{2\sqrt{I_1 I_2}}{I_1 + I_2} |\gamma_{12}|, \\
 \tau_c &\sim \frac{1}{\Delta\nu}, \\
 L_c &\sim \frac{c}{\Delta\nu} \sim \frac{\lambda_0^2}{\Delta\lambda} \quad (\text{scale relations; convention-dependent factors}), \\
 \mathcal{T}(\delta) &= \frac{1}{1 + F \sin^2(\delta/2)}, \\
 F &= \frac{4R}{(1 - R)^2}, \\
 \Delta\nu_{\text{FSR}} &\approx \frac{c}{2nL}, \\
 \mathcal{F} &\approx \frac{\pi\sqrt{R}}{1 - R}, \\
 \Lambda &= \frac{\lambda_1 \lambda_2}{|\lambda_1 - \lambda_2|}.
 \end{aligned}$$

Caution

A formula sheet is a map, not a substitute for assumptions. Before using any relation, identify the geometry, wavelength convention, coherence condition, reflection phases and approximation regime.

Chapter C

Computational and Laboratory Reference

Recommended computational workflow for Young fringes

A compact numerical model can be organized as follows:

1. define physical inputs λ, d, a, D and a screen coordinate array y ;
2. compute $\theta = \arctan(y/D)$ rather than assuming $\theta = y/D$ automatically;
3. calculate $\alpha = \pi d \sin \theta / \lambda$;
4. if finite width is included, calculate $\beta = \pi a \sin \theta / \lambda$;
5. evaluate the intensity and normalize it;
6. sample the continuous model at camera-pixel locations;
7. optionally add background, noise and finite exposure effects;
8. fit the simulated data with the same method planned for laboratory data.

The following standalone Python calculation uses only the standard library. It compares Monte Carlo phase averaging with the exact Gaussian prediction and evaluates the finite-width pattern without any ambiguity about the definition of sinc. All inputs are illustrative, and no measured data are implied.

```
from math import sin, cos, pi, exp
from random import Random
rng = Random(15)
ns = 20000
for sigma in [0.0, 0.2, 0.5, 1.0, 2.0]:
    phases = [rng.gauss(0.0, sigma) for _ in range(ns)]
    cbar = sum(cos(q) for q in phases) / ns
    sbar = sum(sin(q) for q in phases) / ns
    visibility = (cbar*cbar + sbar*sbar)**0.5
    print(sigma, visibility, exp(-sigma*sigma/2))
```

Continue in the same Python session to evaluate the finite-width pattern:

```
wavelength, d, a, D = 633e-9, 0.20e-3, 0.05e-3, 1.5
def normalized_intensity(y):
    sin_theta = y / (D*D + y*y)**0.5
    alpha = pi*d*sin_theta/wavelength
    beta = pi*a*sin_theta/wavelength
    sinc = 1.0 if abs(beta) < 1e-12 else sin(beta)/beta
    return sinc*sinc*cos(alpha)**2
print("centre:", normalized_intensity(0.0))
```

The centre returns 1 by construction. The visibility output should approach the values in the selected solution to N3; changing the seed or sample size changes the finite-sample scatter. To model an exposure, the phase samples must represent fluctuations during that exposure. Treating every image as one fixed phase and then averaging their unregistered intensities is a different acquisition procedure, although its ensemble mean can agree.

Recommended data columns for a double-slit laboratory

A useful data table contains

Order m	Pixel coordinate	Physical coordinate y_m	Note
-3	measured	calibrated	left-side maximum
-2	measured	calibrated	left-side maximum
-1	measured	calibrated	left-side maximum
0	measured	calibrated	central maximum
+1	measured	calibrated	right-side maximum
+2	measured	calibrated	right-side maximum
+3	measured	calibrated	right-side maximum

A linear regression of y_m against m then estimates the fringe spacing and central-axis intercept simultaneously.

Recommended metadata to record

For reproducibility, record more than the final fringe coordinates:

- source type and nominal wavelength;
- slit separation and slit width, including tolerance if known;
- screen/camera distance and how it was measured;
- camera pixel pitch, image size and any lens magnification;
- exposure time, gain and whether pixels were saturated;
- ambient-light conditions;
- approximate beam polarization if relevant;
- date, setup changes and raw image filenames;
- fitting model and software version if numerical fitting is used.

These records allow a future reader to reproduce the inference rather than only the final number.

A minimal uncertainty model for wavelength from fringe spacing

For

$$\lambda = \frac{d\Delta y}{D},$$

independent input uncertainties give

$$\left(\frac{u_\lambda}{\lambda}\right)^2 \approx \left(\frac{u_d}{d}\right)^2 + \left(\frac{u_{\Delta y}}{\Delta y}\right)^2 + \left(\frac{u_D}{D}\right)^2.$$

If Δy is obtained from the slope of a many-fringe linear fit, its uncertainty should come from that regression rather than from one ruler reading. If d comes from manufacturer specification rather than an in-lab calibration, its tolerance can dominate the final uncertainty.

Laboratory safety and good practice

Laser demonstrations should use the lowest practical optical power, a beam height and layout that avoid eye exposure, stable mounts, beam stops and local institutional laser-safety procedures. Never look into a laser beam or its specular reflection. Reflective jewelry and uncontrolled mirror-like objects should be kept away from the beam path. The optical physics of interference does not remove the ordinary responsibilities of laboratory safety.

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