

Chapter 17

Special Relativity

First Edition

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science

AGH University of Krakow

Physics 2

Physics for Technical and Engineering Students -
Volume II

Electricity, Magnetism, Waves, Optics, Relativity,
Quantum Physics, and Nuclear Physics

1 September 2026

Contents

Preface	ii
Conventions and notation	iii
17 Special Relativity	1
17.1 Motivation and roadmap	1
17.2 From Galilean relativity to a new kinematics	2
17.3 Events, reference frames and clock synchronization	4
17.4 Einstein’s postulates	5
17.5 Relativity of simultaneity	6
17.6 Time dilation and proper time	8
17.7 Length contraction and proper length	11
17.8 The Lorentz transformation	13
17.9 Velocity transformations and the speed limit	16
17.10 Relativistic Doppler effect for light	18
17.11 Relativistic momentum	19
17.12 Relativistic energy and the energy–momentum relation	20
17.13 Conservation laws, reactions and mass–energy accounting	23
17.14 Engineering and applied-science connections	24
17.15 Laboratory and computational connection: testing relativistic kinematics	27
17.16 Choosing the measurement before the formula	32
17.17 Main chapter summary	33
17.18 Advanced Topics for Technical and Engineering Students	35
17.19 Integrated applications and case studies	57
17.20 Original practice problems	61
17.21 Selected solutions to representative problems	64
17.22 Bridge to quantum and modern physics	67
A Notation dictionary	69
B Compact formula sheet	70

Preface

Special relativity is the point at which the familiar concepts of space and time must be reorganized. In Newtonian mechanics, two observers moving uniformly with respect to one another may disagree about velocities, but they share a common time coordinate and can speak unambiguously about events occurring “at the same time.” Electromagnetism does not fit naturally into that structure. Maxwell’s equations predict a distinguished propagation speed for electromagnetic waves, and experiment shows that the vacuum speed of light is the same for all inertial observers. The resolution is not to modify the speed of light by ordinary velocity addition, but to modify the transformation rules that connect space and time measurements made in different inertial frames.

This chapter develops that new kinematics carefully and then rebuilds momentum and energy so that the conservation laws remain valid. The emphasis is operational: an *event* is something that happens at a definite place and time; a reference frame is equipped with synchronized clocks and spatial coordinates; and every formula is tied to a stated measurement procedure. This is especially important for time dilation and length contraction. Those effects are not optical distortions and they are not caused by signal-travel delays. They describe how different inertial frames assign coordinates and intervals to the same physical events.

The chapter is written for Technical Physics, engineering, computer-science, space-science and applied-science students. Accordingly, the development repeatedly connects relativity to particle beams, timing systems, radiation, spectroscopy, detectors, accelerator design and numerical modeling. The main body uses the speed of light c explicitly rather than setting $c = 1$. A later enrichment section introduces spacetime intervals, four-vectors, rapidity, invariant mass, threshold kinematics and selected links to electromagnetism for students who want a deeper mathematical structure.

The Guideline

Before using any relativity formula, identify the **events** and the **frame**. For time dilation, ask whether the two events occur at the same place in some inertial frame; that frame measures the proper time. For length contraction, ask which frame contains the object at rest; that frame measures the proper length. Most sign and interpretation errors disappear when these questions are answered first.

Pedagogical boundary

The main chapter treats special relativity in inertial frames and uses acceleration only in limited applications such as comparing worldlines. General relativity, curved spacetime and a full relativistic treatment of gravity are outside the main scope. When GPS is discussed, the special-relativistic contribution is separated explicitly from the additional gravitational correction required by general relativity.

Conventions and notation

The vacuum speed of light is denoted by c ,

$$c = 299\,792\,458 \text{ m s}^{-1},$$

which is exact in the SI definition of the metre. For numerical estimates we often use $c \simeq 3.00 \times 10^8 \text{ m s}^{-1}$.

Two inertial frames in *standard configuration* are denoted by S and S' . Their Cartesian axes are parallel, the origins coincide at $t = t' = 0$, and S' moves with constant velocity $+v\hat{x}$ relative to S . The dimensionless speed parameter and Lorentz factor are

$$\beta \equiv \frac{v}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}.$$

Unless stated otherwise, $0 \leq v < c$ and therefore $\gamma \geq 1$.

An event has coordinates (t, x, y, z) in S and (t', x', y', z') in S' . Coordinate differences between two events A and B are written

$$\Delta t = t_B - t_A, \quad \Delta x = x_B - x_A,$$

with analogous notation in S' . The symbol $\Delta\tau$ denotes elapsed time along a specified ideal clock's worldline. For an inertial clock, the two clock events occur at the same spatial coordinate in its rest frame. For an accelerated clock there is generally no single global rest frame, and its proper time is obtained by integrating along its path. Many introductory texts use Δt_0 for the inertial-clock interval; $\Delta\tau$ is used consistently here.

The proper length of an object is denoted by L_0 and is measured in the object's rest frame. A length L measured in a frame in which the object moves parallel to its length is obtained from simultaneous endpoint positions in that measuring frame.

Particle momentum is $\mathbf{p}$, total relativistic energy is E , kinetic energy is $K = E - mc^2$, and m denotes invariant (rest) mass. We do not use the outdated “relativistic mass” convention. For four-vectors in the enrichment section, the metric convention is

$$s^2 = c^2 t^2 - x^2 - y^2 - z^2,$$

corresponding to signature $(+, -, -, -)$.

Note

A statement such as “the moving clock runs slow” is incomplete unless the comparison procedure and frame are specified. Each inertial observer can consistently describe the other frame's identically constructed clocks as running slow when comparing spatially separated synchronized clocks in their own frame. There is no contradiction because simultaneity itself is frame-dependent.

Appendix A provides the complete notation dictionary and units; Appendix B collects the working equations with their conditions.

Chapter 17

Special Relativity

17.1 Motivation and roadmap

Relativity is not a correction that is switched on only for exotic astronomical objects. It is the kinematic framework required whenever measurements made in different inertial frames are compared accurately, and its effects become quantitatively large when relative speeds are an appreciable fraction of c . Modern accelerators routinely operate in the regime $\gamma \gg 1$; unstable particles created high in the atmosphere survive to detectors because their decay times are frame-dependent; high-resolution spectroscopy uses relativistic Doppler shifts; and precision timing systems must account for relativistic clock effects.

The conceptual route of this chapter is shown schematically in Fig. 17.1. We begin with Galilean relativity and the meaning of an inertial frame. We then make measurement operational by defining events, arrays of clocks and synchronization. Einstein's two postulates replace Galilean transformations by Lorentz transformations. From those transformations follow the relativity of simultaneity, time dilation, length contraction and relativistic velocity addition. Momentum and energy are then reconstructed in a form compatible with the new kinematics and conservation laws.

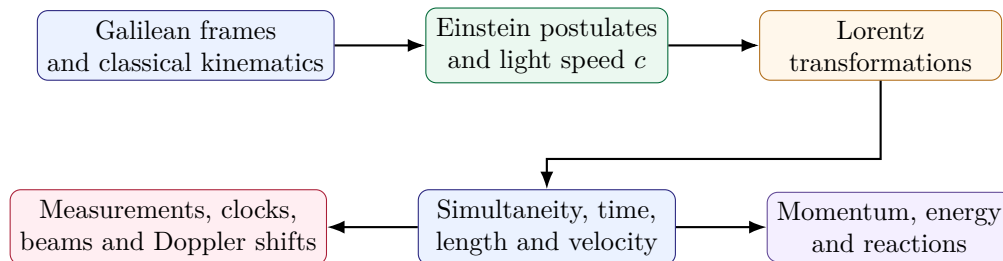


Figure 17.1: Roadmap for Chapter 17. Special relativity modifies the transformation rules connecting inertial frames while preserving the form of physical laws and the invariant speed c .

Learning goals

After studying the main part, the student should be able to:

- state the two postulates and define events, inertial frames and synchronized clocks;
- transform event coordinates and explain the relativity of simultaneity;
- identify proper time and proper length, and apply their measurement conditions;
- transform velocities, preserve the light-speed limit and calculate Doppler shifts;
- use relativistic momentum, energy and conservation laws, including reaction Q ;
- recover low-speed limits and judge the error of a classical approximation;
- connect lifetimes, beam measurements and precision timing to testable predictions.

How to use the chapter

The core route runs from events and synchronization through energy conservation and the computational laboratory. Work through the examples in that order before choosing advanced topics. The enrichment section develops three connected strands: spacetime geometry, collision invariants, and measurement applications. Its tensor notation, accelerated worldlines and detector models are optional extensions. The later case studies can be used selectively in problem-solving classes; the exercises identify where advanced material is needed.

Required prior knowledge

Students should be comfortable with one-dimensional and vector kinematics, relative velocity in Newtonian mechanics, inertial frames, conservation of momentum and energy, waves and the vacuum speed of electromagnetic radiation. Algebraic manipulation, square roots, binomial approximations and basic graph interpretation are used throughout. Calculus is needed only in selected enrichment topics.

Guided checks

1. In Newtonian mechanics, which quantity is assumed to be the same for every inertial observer: position, velocity, or time?
2. If two observers disagree about the coordinates of an event, does that mean they disagree that the event occurred?
3. Why is an array of synchronized clocks more useful than a single clock for assigning times to distant events?
4. At what qualitative speed scale should one expect γ to differ appreciably from unity?

17.2 From Galilean relativity to a new kinematics**17.2.1 Inertial frames and the classical transformation**

An inertial frame is a frame in which a free particle moves with constant velocity. If S is inertial, any frame S' moving at constant velocity relative to S is also inertial. In Newtonian mechanics, the transformation between frames in standard configuration is

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t. \quad (17.1)$$

The last equality expresses absolute time: the same time coordinate is assigned to an event by every inertial frame, apart from a possible constant offset chosen when clocks are set.

Differentiating Eq. (17.1) gives the Galilean velocity transformation,

$$u'_x = u_x - v, \quad (17.2)$$

with $u'_y = u_y$ and $u'_z = u_z$ for the simple parallel-axis configuration. This rule works extremely well for ordinary mechanical speeds. A passenger walking forward at 2 m s^{-1} in a train moving at 30 m s^{-1} relative to the ground moves at approximately 32 m s^{-1} relative to the ground.

The difficulty appears when the transported signal is light. If one simply used Galilean velocity addition, light emitted forward from a source moving at speed v would have ground-frame speed $c + v$, whereas light emitted backward would have speed $c - v$. That prediction is incompatible with the structure of Maxwell's equations and with the experimentally established invariance of the vacuum light speed.

Key message

Galilean relativity is not “wrong” in its domain. It is the low-speed approximation to special relativity. The question is not whether classical mechanics should be discarded, but which transformation law remains accurate when v/c is no longer negligible.

17.2.2 Why electromagnetism exposes the problem

Maxwell’s theory combines electric and magnetic fields into equations that predict electromagnetic waves with propagation speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \quad (17.3)$$

in vacuum. The expression contains properties of the electromagnetic field and no source velocity. Historically, this raised a profound question: relative to what frame is this speed measured? Attempts to interpret light as a mechanical wave propagating through a material “ether” did not produce a consistent measurable preferred frame. The modern conclusion is stronger and simpler: no inertial frame is preferred, and the vacuum speed of light is the same for every inertial observer.

A detailed historical reconstruction is not needed to use special relativity, but one lesson is important for scientific method. The new theory was not introduced because classical mechanics was numerically inaccurate for everyday motion. It was introduced because the transformation structure of Newtonian spacetime was incompatible with the observed symmetry of electromagnetism.

Caution

Do not say that special relativity was created merely because “light travels very fast.” A large numerical value of c is not the conceptual issue. The key fact is that the same value of c is obtained by every inertial observer, independent of the motion of source or detector.

17.2.3 The correspondence principle and the small- β limit

A successful new theory must reproduce the older theory where the older theory has been tested. Let

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c}. \quad (17.4)$$

For $\beta \ll 1$, the binomial expansion gives

$$\gamma = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots \quad (17.5)$$

Thus $\gamma - 1 \approx v^2/(2c^2)$, which is tiny at ordinary speeds.

Example 17.1: How relativistic is a high-speed aircraft?

Consider a hypothetical aircraft moving at $v = 1000 \text{ m s}^{-1}$. Estimate $\gamma - 1$.

Solution. The dimensionless speed is

$$\beta = \frac{1000}{2.998 \times 10^8} \simeq 3.34 \times 10^{-6}.$$

Using Eq. (17.5),

$$\gamma - 1 \simeq \frac{1}{2}\beta^2 \simeq 5.6 \times 10^{-12}.$$

For most engineering calculations involving such a vehicle, Newtonian kinematics is therefore more than adequate. Precision clocks, however, can detect effects far smaller than those relevant to ordinary mechanical design.

Guided checks

1. If $v = 0.01c$, estimate $\gamma - 1$ without a calculator.
2. Which term in Eq. (17.5) gives the first relativistic correction?
3. Why can relativity be essential for precision timing even when it is negligible for a vehicle's ordinary trajectory calculation?

17.3 Events, reference frames and clock synchronization

Special relativity is easiest to understand when measurements are defined before formulas are introduced. An *event* is an occurrence localized in space and time: a detector click, a spark, a particle decay, the emission of a laser pulse or the crossing of two worldlines. A frame assigns four coordinates to each event,

$$(t, x, y, z). \quad (17.6)$$

Different inertial frames generally assign different coordinate values to the same event, but the event itself is common to all descriptions.

17.3.1 A frame as a measuring network

Imagine a three-dimensional lattice of rulers and identical clocks at rest in frame S . Each clock remains at a fixed spatial coordinate and all clocks are synchronized according to a specified convention. When an event occurs next to one clock, its location is read from the local ruler coordinates and its time is read from that nearby clock. This procedure avoids confusing the coordinate time of the event with the later time at which a distant observer receives a signal reporting it.

Definition 17.1: Event

An *event* is a physical occurrence to which an inertial frame assigns one time coordinate and three spatial coordinates. A single event is not “moving”; a sequence of events can form the worldline of an object or signal.

17.3.2 Einstein synchronization

To synchronize two clocks A and B at rest in the same inertial frame, one may exchange light signals. Suppose clock A emits a pulse at $t_A^{(1)}$, clock B reflects it immediately, and A receives the return pulse at $t_A^{(2)}$. The synchronization convention assigns the reflection event at B the time

$$t_B = \frac{t_A^{(1)} + t_A^{(2)}}{2}. \quad (17.7)$$

The procedure relies on the isotropy of c in that inertial frame. Once synchronized, the distributed clocks provide a coordinate time for events throughout the frame.

Method

When reading a relativity problem, write a small event table before doing algebra. Label events A, B, C , list their coordinates in the frame that is given most naturally, and only then transform. This prevents the common error of combining a distance measured in one frame with a time interval measured in another.

Example 17.2: Signal travel time is not the event time

A flash occurs 900 m from the origin of a laboratory frame at laboratory coordinate time $t = 2.0 \mu\text{s}$. When does a detector at the origin receive the light signal, assuming straight-line propagation in vacuum?

Solution. The event itself has coordinate time $2.0 \mu\text{s}$. The signal then requires

$$\Delta t_{\text{signal}} = \frac{900 \text{ m}}{c} \simeq 3.0 \mu\text{s}$$

to reach the origin. The reception occurs near $5.0 \mu\text{s}$, but the laboratory frame does not assign $5.0 \mu\text{s}$ to the original flash. A synchronized local clock next to the flash reads $2.0 \mu\text{s}$ when the flash occurs.

Interpretation. Coordinate assignments can be made after correcting for propagation delay, but special-relativistic effects themselves are not caused by that delay.

Caution

“What I see now” and “what my frame assigns as happening now at a distant location” are different statements. Visual observation contains light-travel delay. Relativity of simultaneity remains after such delays are corrected using synchronized clocks and known light propagation.

17.4 Einstein’s postulates

Special relativity can be organized around two statements.

Definition 17.2: Postulates of special relativity

- 1. Principle of relativity.** The laws of physics have the same form in all inertial frames. No inertial frame is physically preferred.
- 2. Invariance of the vacuum speed of light.** Light in vacuum propagates with the same speed c in every inertial frame, independent of the uniform motion of the source or observer.

The first postulate extends the familiar Galilean relativity of mechanics to all physical laws, including electromagnetism. The second states that the transformation between inertial frames must preserve

the light-cone condition. If a pulse leaves the common origin at $t = t' = 0$, then in one spatial dimension its worldline satisfies

$$x = \pm ct \quad (17.8)$$

in S and

$$x' = \pm ct' \quad (17.9)$$

in S' .

The postulates force space and time coordinates to mix. The transformation cannot keep $t' = t$ and simultaneously preserve c for nonzero relative speed v . This is the origin of the relativity of simultaneity and the modified relations for durations and lengths.

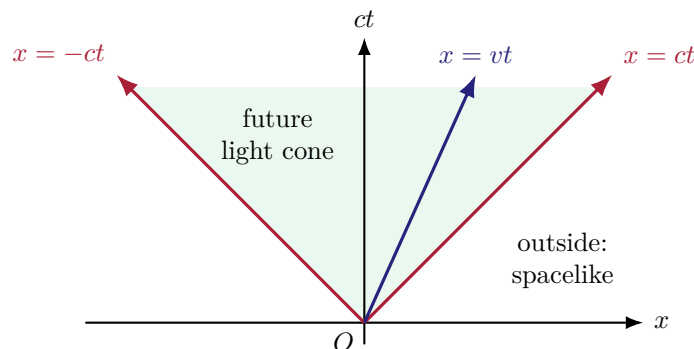


Figure 17.2: Future light cone of event O , with equal scales on x and ct . Light follows its two boundaries. A massive observer starting at O follows a timelike worldline inside the cone; the example shown has $v = 0.45c$.

17.4.1 Causality and the speed limit

The invariance of c is not merely a property of light as an electromagnetic phenomenon. In relativistic spacetime, c sets the invariant causal speed. Massive particles can approach but cannot reach c through finite energy input, while electromagnetic radiation in vacuum propagates at c . In the standard relativistic framework, no controllable signal carrying information can propagate faster than c without creating frame-dependent causal-order problems.

Key message

Special relativity does not say that “everything is relative.” It identifies both frame-dependent quantities and invariant structure. Event coordinates, simultaneity, time intervals and longitudinal lengths can depend on frame. The laws of physics, the vacuum light speed and later the spacetime interval are invariant.

Guided checks

1. Does the principle of relativity claim that all observers measure the same particle velocity?
2. If a laser source moves toward you, does special relativity predict a light speed larger than c or a change in frequency and wavelength?
3. Which postulate rules out using an internal experiment in a sealed inertial laboratory to determine an “absolute” uniform velocity?

17.5 Relativity of simultaneity

The most conceptually important consequence of the postulates appears before time dilation or length contraction: spatially separated events that are simultaneous in one inertial frame are generally not simultaneous in another.

Consider flashes at the front and rear of an inertially moving train, simultaneous in the ground frame. At that ground-frame instant, let a ground observer and the train's midpoint passenger coincide midway between the flash positions (Fig. 17.3). The ground observer receives both pulses together; the passenger moves toward the front pulse and away from the rear pulse and receives the front pulse first. This difference is not because the passenger assumes different light speeds. On the contrary, the passenger also measures both pulses moving at c . Therefore the passenger's frame assigns different emission times to the two strikes.

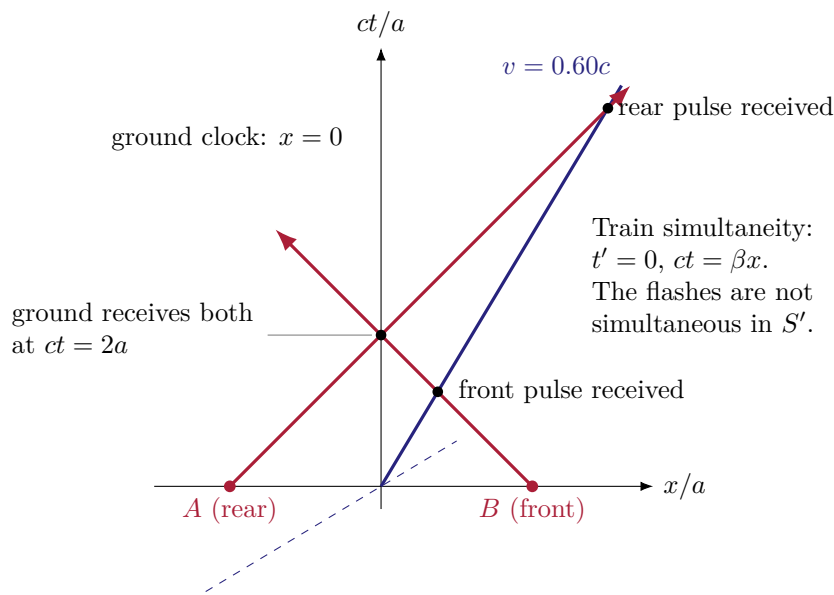


Figure 17.3: Two flashes at $x = \pm 2a$, simultaneous at $t = 0$ in the ground frame. The ground observer and passenger coincide at their midpoint then. The passenger meets the front pulse at $(x, ct) = (0.75a, 1.25a)$ and the rear pulse at $(3a, 5a)$; equal light speeds hold throughout. The dashed line is train-frame simultaneity through the origin.

The Lorentz transformation developed later gives the quantitative result directly. For two events,

$$\Delta t' = \gamma \left(\Delta t - \frac{v \Delta x}{c^2} \right). \quad (17.10)$$

If the events are simultaneous in S , so $\Delta t = 0$, then

$$\Delta t' = -\gamma \frac{v \Delta x}{c^2}. \quad (17.11)$$

Unless $v = 0$ or $\Delta x = 0$, the time separation in S' is nonzero.

Example 17.3: How non-simultaneous are simultaneous flashes?

Two flashes occur simultaneously in a laboratory frame at positions separated by $\Delta x = 600$ m. A spacecraft frame S' moves in the $+x$ direction at $v = 0.60c$. Find the time difference between the flashes in S' .

Solution. Here $\gamma = 1/\sqrt{1 - 0.60^2} = 1.25$ and $\Delta t = 0$. Equation (17.11) gives

$$\Delta t' = -1.25 \frac{(0.60c)(600 \text{ m})}{c^2} = -1.25 \frac{360 \text{ m}}{c} \simeq -1.50 \text{ } \mu\text{s}.$$

Thus the event at the larger x coordinate occurs $1.50 \text{ } \mu\text{s}$ earlier in S' if we define $\Delta t' = t'_B - t'_A$ with B at the larger x .

Check. Reversing the spacecraft velocity reverses the sign of the ordering. The magnitude scales with both separation and relative speed.

Caution

Two events at the same location in a frame cannot be simultaneous unless they are the same event. Relativity of simultaneity concerns spatially separated events. The phrase “at the same time” is therefore incomplete unless the reference frame is stated.

Guided checks

1. If two events satisfy $\Delta x = 0$ in S , can they be simultaneous in S and non-simultaneous in S' ? Explain using Eq. (17.10).
2. Two flashes are simultaneous in S with $\Delta x > 0$. What sign does $\Delta t'$ have for $v > 0$?
3. Why does correcting visual observations for light-travel time not remove relativity of simultaneity?

17.6 Time dilation and proper time

Time dilation is a statement about the interval between two events. The central task is to identify the frame in which the events occur at the same spatial position.

Definition 17.3: Proper time

The proper time $\Delta\tau$ along a specified timelike worldline is the elapsed time recorded by an ideal clock following that path. If the clock moves inertially between events A and B , its rest frame assigns both events the same spatial position. A clock taking a different, accelerated path between the same endpoints can record a different proper time.

For two ticks on the same inertial clock, let $\Delta\tau$ be the clock’s reading difference. An inertial frame in which that clock moves at constant speed v assigns

$$\Delta t = \gamma \Delta\tau \quad (17.12)$$

with $\gamma \geq 1$. Therefore $\Delta t \geq \Delta\tau$.

17.6.1 Light-clock derivation

A light clock consists of two mirrors separated by a transverse distance D . Define a full tick by emission at the lower mirror (event A) and return to that same mirror (event B), as in Fig. 17.4. The lower clock records $\Delta\tau$ between these events. In its rest frame, the coordinate time for one outward half-trip is

$$\frac{\Delta\tau}{2} = \frac{D}{c}. \quad (17.13)$$

In a laboratory frame where the clock moves horizontally at speed v , the light follows a diagonal path. During one half-tick the clock moves a horizontal distance $v\Delta t/2$, while light travels distance $c\Delta t/2$. The right triangle gives

$$\left(c\frac{\Delta t}{2}\right)^2 = D^2 + \left(v\frac{\Delta t}{2}\right)^2. \quad (17.14)$$

Using $D = c\Delta\tau/2$ from Eq. (17.13),

$$c^2\Delta t^2 = c^2\Delta\tau^2 + v^2\Delta t^2, \quad (17.15)$$

$$\Delta t^2 \left(1 - \frac{v^2}{c^2}\right) = \Delta\tau^2, \quad (17.16)$$

which yields Eq. (17.12).

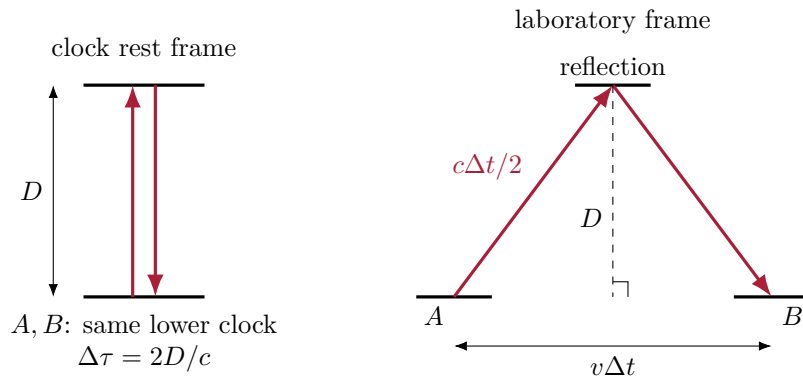


Figure 17.4: One full tick: emission A , reflection, and return B . The same lower clock records A and B in its rest frame. In the laboratory, the outward and return rays each form a right triangle; the drawn geometry corresponds to $v = 0.60c$. The mirrors are shown at the relevant interaction times.

Which interval is the proper time?

In Fig. 17.4, $\Delta\tau$ refers to emission and return at the *same lower clock*. Emission and reflection are connected by a light ray, so their spacetime interval is null; D/c is a coordinate travel time in the clock frame, not a photon's proper time. The full tick is used to compare a physical clock with the laboratory clock network. The unchanged transverse separation follows from $y' = y$; it is not a claim that the two frames agree on every distance.

17.6.2 Interpreting “moving clocks run slow”

Suppose a spacecraft clock ticks twice, defining events A and B at the same location in the spacecraft. The spacecraft clock directly records $\Delta\tau$. In the laboratory, the two events occur at different positions because the spacecraft has moved. The laboratory's synchronized clock network assigns a longer interval $\Delta t = \gamma\Delta\tau$.

The symmetry of inertial motion does not create a logical contradiction. In the spacecraft frame, a particular laboratory clock is not present at both events A and B ; rather, different laboratory clocks pass the spacecraft at those events. Comparing a remote clock network requires a simultaneity convention, and the two frames do not share the same surfaces of simultaneity.

Method

For two ticks of an inertial clock at constant relative speed:

1. Name the two events.
2. Find the frame in which they occur at the same position.
3. The time measured there is the proper time $\Delta\tau$.
4. Use $\Delta t = \gamma\Delta\tau$ only for the interval measured in another inertial frame.
5. Check that the non-proper interval is the larger one.

Example 17.4: Spacecraft travel time in two frames

A spacecraft travels from Earth to a station that is at rest relative to Earth and is 12.0 light-years away in the Earth frame. The spacecraft speed is $0.800c$. Neglect acceleration phases. Find the travel time in the Earth frame and on the spacecraft clock.

Earth frame. The departure and arrival occur at different Earth-frame positions, so the Earth-frame interval is not the spacecraft proper time. Using speed equals distance over time,

$$\Delta t = \frac{12.0 \text{ ly}}{0.800c} = 15.0 \text{ y},$$

where a light-year is the distance light travels in one year, so $c = 1 \text{ ly/y}$ when the same year convention is used in both units. The light-year is a unit of distance.

Spacecraft frame. The departure and arrival events occur at the spacecraft, so its clock measures the proper time. With

$$\gamma = \frac{1}{\sqrt{1 - 0.800^2}} = \frac{5}{3},$$

we obtain

$$\Delta\tau = \frac{\Delta t}{\gamma} = 9.00 \text{ y}.$$

Check. The proper time is smaller, as required. In the spacecraft frame the Earth–station separation is length-contracted to $12.0/(5/3) = 7.20 \text{ ly}$, and $7.20 \text{ ly}/(0.800c) = 9.00 \text{ y}$ gives the same result.

Example 17.5: Cosmic-ray muons and survival to the ground

A population of muons is produced high in the atmosphere. Take a representative speed $v = 0.998c$ and proper mean lifetime $\tau_0 = 2.20 \mu\text{s}$. Estimate the mean decay length in the Earth frame and compare it with the classical estimate.

Solution. The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - 0.998^2}} \simeq 15.8.$$

The Earth-frame mean lifetime is therefore

$$\tau = \gamma\tau_0 \simeq 34.8 \mu\text{s}.$$

The corresponding mean travel distance is approximately

$$\ell = v\tau \simeq (0.998c)(34.8 \times 10^{-6} \text{ s}) \simeq 1.04 \times 10^4 \text{ m}.$$

Without time dilation, the estimate would be

$$\ell_{\text{classical}} \simeq (0.998c)(2.20 \times 10^{-6} \text{ s}) \simeq 6.58 \times 10^2 \text{ m}.$$

Thus relativistic time dilation changes the characteristic travel scale from less than a kilometre to roughly ten kilometres for these chosen parameters.

Physical meaning. In the muon rest frame the lifetime is still $2.20 \mu\text{s}$; the atmospheric distance is instead length-contracted. Both descriptions predict the same set of decay and detection events.

Guided checks

1. A particle is created and later decays. Which frame measures its proper lifetime?
2. Two clocks at rest at different positions in a laboratory record departure and arrival of a spacecraft. Does either one alone measure the proper travel time?
3. If $\gamma = 10$, how does the laboratory lifetime of a moving unstable particle compare with its proper lifetime?
4. Can one infer time dilation from a single event? Why not?

17.7 Length contraction and proper length

A length is obtained by locating two endpoints at the same time in the measuring frame, as in Fig. 17.5. This simultaneous measurement condition is essential because the endpoints of a moving object change position with time.

Definition 17.4: Proper length

The proper length L_0 of an object is the distance between its endpoints measured in the inertial frame in which the object is at rest. For an object moving parallel to its length at speed v , another inertial frame measures

$$L = \frac{L_0}{\gamma} = L_0 \sqrt{1 - \frac{v^2}{c^2}}. \quad (17.17)$$

The contraction occurs only along the direction of relative motion.

17.7.1 Why simultaneity enters a length measurement

Suppose a rod moves in the $+x$ direction. To measure its length in the laboratory, record the position of the rear endpoint and the front endpoint at the same laboratory time. If the rear position is recorded earlier than the front position, the rod moves between the two readings and the difference is not the rod's instantaneous laboratory length. Because simultaneity differs between frames, the pair of endpoint events used by the laboratory is not simultaneous in the rod frame. Length contraction is therefore inseparable from relativity of simultaneity.

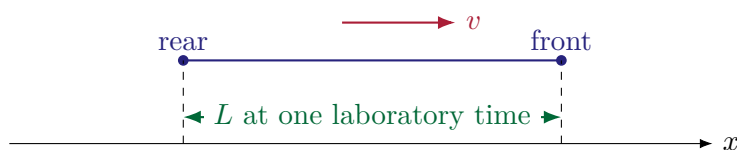


Figure 17.5: A laboratory length measurement of a moving rod requires simultaneous laboratory positions of both endpoints. The rod's proper length is measured in its own rest frame.

17.7.2 Deriving contraction from Lorentz transformation

Let the rod be at rest in S' with endpoints at fixed coordinates x'_1 and x'_2 . Its proper length is

$$L_0 = x'_2 - x'_1. \quad (17.18)$$

The laboratory S measures endpoint positions x_1 and x_2 simultaneously, so $\Delta t = 0$. The spatial Lorentz transformation for separations is

$$\Delta x' = \gamma(\Delta x - v\Delta t) = \gamma\Delta x. \quad (17.19)$$

Thus $L_0 = \gamma L$, giving Eq. (17.17).

Example 17.6: A moving probe: simultaneous positions and a single timing gate

A stress-free probe has proper length $L_0 = 50.0$ m and moves at $v = 0.800c$. Find its laboratory length and the time it takes to pass a fixed timing gate. Identify which clock measures a proper time.

Strategy and assumptions. The probe coasts at constant speed. An array of synchronized laboratory detectors can measure simultaneous endpoint positions. A single gate instead records two different events: the front passes the gate, then the rear passes the same gate.

Calculation. With $\gamma = 5/3$,

$$L = \frac{L_0}{\gamma} = 30.0 \text{ m}, \quad \Delta t_{\text{gate}} = \frac{L}{v} \simeq 125 \text{ ns}.$$

Both passage events occur at the gate's fixed laboratory position, so the gate clock measures the proper time of this event pair. In the probe frame they occur at different endpoints. The coordinate interval there is

$$\Delta t' = \gamma \Delta t_{\text{gate}} \simeq 208 \text{ ns} = \frac{L_0}{v}.$$

The transverse dimensions of the probe are unchanged.

Interpretation and check. The moving probe does not automatically supply the proper clock for every problem involving it. The event pair decides. A time has units L/v , and both the contracted-length and transformed-time methods give the same gate crossing history.

Caution

Length contraction is not a statement that a moving object has been physically compressed by stress. An ideal inertially moving rod can be stress-free in its own frame while having a smaller longitudinal coordinate length in another frame. Real acceleration protocols can introduce stresses, but that is a separate dynamical question.

Guided checks

1. Why must the endpoint positions of a moving rod be recorded simultaneously in the measuring frame?
2. Which is always larger for inertial longitudinal measurements, L_0 or L ?
3. Does a vertical rod moving horizontally become shorter vertically according to Eq. (17.17)? Explain.
4. A spacecraft says the Earth–star distance is contracted. What does the Earth–star frame say about the spacecraft's length?

17.8 The Lorentz transformation

The Lorentz transformation is the coordinate map that replaces the Galilean transformation while satisfying Einstein's postulates. In standard configuration, with S' moving at speed $+v$ along the common x direction relative to S , the forward transformation is

$$x' = \gamma(x - vt), \quad (17.20)$$

$$y' = y, \quad (17.21)$$

$$z' = z, \quad (17.22)$$

$$t' = \gamma \left(t - \frac{vx}{c^2} \right). \quad (17.23)$$

The inverse transformation is obtained by replacing v with $-v$ and interchanging primed and unprimed coordinates:

$$x = \gamma(x' + vt'), \quad (17.24)$$

$$y = y', \quad (17.25)$$

$$z = z', \quad (17.26)$$

$$t = \gamma \left(t' + \frac{vx'}{c^2} \right). \quad (17.27)$$

The transformation of event separations is often more useful than the transformation of individual coordinates:

$$\Delta x' = \gamma(\Delta x - v\Delta t), \quad (17.28)$$

$$\Delta t' = \gamma \left(\Delta t - \frac{v\Delta x}{c^2} \right). \quad (17.29)$$

These equations immediately contain simultaneity, time dilation and length contraction as special measurement arrangements.

17.8.1 Deriving the Lorentz transformation from symmetry and light-speed invariance

It is useful to see why the Lorentz transformation has its particular form. Consider two inertial frames in standard configuration. Homogeneity of space and time implies that a transformation between coordinates should be linear: translating an experiment in space or time must not change the transformation law. Write

$$x' = A(v)(x - vt), \quad (17.30)$$

where the origin of S' follows $x = vt$.

For a light pulse moving in the $+x$ direction, $x = ct$ in S and $x' = ct'$ in S' . For a pulse moving in the $-x$ direction, $x = -ct$ and $x' = -ct'$. To derive the time transformation rather than guess it, write $t' = Bx + Ct$. For the two light rays $x = \pm ct$, the requirements $x' = \pm ct'$ imply

$$A(c - v) = Bc^2 + Cc, \quad A(c + v) = -Bc^2 + Cc.$$

Adding and subtracting give $C = A$ and $B = -Av/c^2$. Thus

$$t' = A(v) \left(t - \frac{vx}{c^2} \right). \quad (17.31)$$

The remaining factor follows from reciprocity: the inverse transformation must have the same form with $v \rightarrow -v$. Applying the inverse to Eq. (17.30) gives

$$x = A(-v)(x' + vt').$$

Isotropy implies $A(-v) = A(v)$. Substituting x' and t' and demanding identity for arbitrary x, t yields

$$A^2(1 - v^2/c^2) = 1,$$

so the physically continuous solution with $A(0) = 1$ is

$$A(v) = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \quad (17.32)$$

Thus the Lorentz transformation follows from a small set of symmetry assumptions plus invariant light speed.

Theoretical point 17.1: Why linearity matters

The usual derivation assumes inertial frames, homogeneous space and time, spatial isotropy and a continuous transformation connected to the identity at $v = 0$. These assumptions are physical. They rule out arbitrary nonlinear coordinate maps that would turn uniform motion into nonuniform motion or make the laws depend on where and when an experiment is performed.

Guided checks

If c were formally allowed to approach infinity while v remained finite, what happens to γ , x' and t' ? Explain how the Galilean transformation is recovered.

17.8.2 Consistency checks

A transformation should be tested before it is trusted. First, if $v = 0$, then $\gamma = 1$ and $x' = x$, $t' = t$. Second, if $v \ll c$, the term vx/c^2 is usually small and $\gamma \simeq 1$, so the transformation approaches Galilean kinematics. Third, the worldline of the S' origin is $x' = 0$. Equation (17.20) then gives $x = vt$, exactly as required.

Most importantly, a light ray moving in the $+x$ direction satisfies $x = ct$. Then

$$x' = \gamma(c - v)t, \quad (17.33)$$

$$ct' = c\gamma\left(t - \frac{vct}{c^2}\right) = \gamma(c - v)t, \quad (17.34)$$

so $x' = ct'$. The light speed is c in both frames.

Theoretical point 17.2: Light-cone preservation

The Lorentz transformation mixes x and ct in such a way that $x = \pm ct$ implies $x' = \pm ct'$. This is the coordinate expression of the invariant causal structure demanded by the second postulate.

17.8.3 Spacetime axes and worldlines

A spacetime diagram helps organize event geometry; Fig. 17.2 already displayed the two future-directed light rays from a single event. With ct vertical and x horizontal, an object at rest in S has a vertical worldline. The origin of S' satisfies $x = vt$, so its time axis ct' is tilted toward the light line as v increases. The line $t' = 0$ follows from Eq. (17.23),

$$ct = \beta x, \quad (17.35)$$

which is the x' axis in the diagram. The two axes tilt symmetrically toward the light cone when equal x and ct scales are used (Fig. 17.6). They are not ordinary perpendicular Cartesian axes on

the page. A line parallel to the x' axis has constant t' , and a line parallel to the ct' axis has constant x' . The invariant hyperbolae in the enrichment section will calibrate their units.

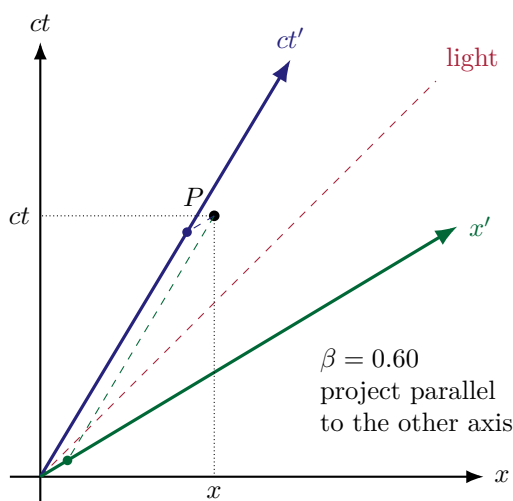


Figure 17.6: Lorentz axes for a definite boost $\beta = 0.60$, with equal laboratory scales. Lines of constant ct' are parallel to x' , and lines of constant x' are parallel to ct' . Primed coordinates require these oblique projections and calibrated primed ticks; ordinary perpendicular projection is incorrect.

Method

For coordinate-transformation problems, first confirm the sign convention. If S' moves in the $+x$ direction relative to S , use Eqs. (17.20)–(17.23). If instead S' moves in the $-x$ direction, change $v \rightarrow -v$. Never guess the sign from memory without drawing the relative motion.

Example 17.7: Transforming a single event

Frame S' moves at $v = 0.60c$ relative to S . An event occurs in S at

$$x = 900 \text{ m}, \quad t = 5.00 \text{ } \mu\text{s}, \quad y = z = 0.$$

Find x' and t' using $c \simeq 3.00 \times 10^8 \text{ m s}^{-1}$.

Solution. With $\gamma = 1.25$,

$$vt = (0.60)(3.00 \times 10^8)(5.00 \times 10^{-6}) \text{ m} = 900 \text{ m}.$$

Therefore

$$x' = 1.25(900 - 900) \text{ m} = 0.$$

For time,

$$\frac{vx}{c^2} = \frac{(0.60c)(900 \text{ m})}{c^2} = \frac{540 \text{ m}}{c} \simeq 1.80 \text{ } \mu\text{s},$$

so

$$t' = 1.25(5.00 - 1.80) \text{ } \mu\text{s} = 4.00 \text{ } \mu\text{s}.$$

Interpretation. The event lies on the S' origin worldline, so $x' = 0$ is exactly what the geometry predicts.

Example 17.8: Transforming a pair of events

Two detector events in S satisfy

$$\Delta x = 1200 \text{ m}, \quad \Delta t = 6.00 \text{ } \mu\text{s}.$$

Find their separation in a frame moving at $0.80c$ along $+x$.

Solution. Here $\gamma = 5/3$. The spatial separation becomes

$$\Delta x' = \frac{5}{3} \left[1200 - (0.80c)(6.00 \times 10^{-6} \text{ s}) \right].$$

Using $c \simeq 3.00 \times 10^8 \text{ m s}^{-1}$,

$$(0.80c)(6.00 \text{ } \mu\text{s}) = 1440 \text{ m},$$

so

$$\Delta x' = -400 \text{ m}.$$

Similarly,

$$\Delta t' = \frac{5}{3} \left[6.00 \text{ } \mu\text{s} - \frac{(0.80c)(1200 \text{ m})}{c^2} \right] = \frac{5}{3} (6.00 - 3.20) \text{ } \mu\text{s} \simeq 4.67 \text{ } \mu\text{s}.$$

The negative $\Delta x'$ means that event B lies at smaller x' than event A , even though $\Delta x > 0$ in S .

Guided checks

1. Show directly that the inverse Lorentz transformation recovers x and t from x' and t' .
2. If an event occurs at the origin of S' at time t' , what relation must x and t satisfy in S ?
3. If two events satisfy $\Delta x = c\Delta t$ in S , what relation do they satisfy in S' ?
4. Why is it dangerous to transform x but leave t unchanged?

17.9 Velocity transformations and the speed limit

Classical velocity addition follows from differentiating Galilean transformations. For a particle trajectory, differentiate the Lorentz transformations instead:

$$dx' = \gamma(dx - vdt), \quad dt' = \gamma \left(dt - \frac{vdx}{c^2} \right).$$

Dividing the first equation by the second, then using $u_x = dx/dt$, gives

$$u'_x = \frac{u_x - v}{1 - u_x v / c^2}. \quad (17.36)$$

Solving for u_x gives the inverse relation

$$u_x = \frac{u'_x + v}{1 + u'_x v / c^2}. \quad (17.37)$$

For transverse components,

$$u'_y = \frac{u_y}{\gamma(1 - u_x v / c^2)}, \quad (17.38)$$

$$u'_z = \frac{u_z}{\gamma(1 - u_x v / c^2)}. \quad (17.39)$$

The one-dimensional formula is sufficient for most introductory applications, but Eqs. (17.39) show that a boost along x also changes transverse velocity components because time itself transforms.

17.9.1 Light remains at c

Set $u_x = c$ in Eq. (17.36):

$$u'_x = \frac{c - v}{1 - v/c} = c. \quad (17.40)$$

Thus no inertial boost changes a light ray from speed c to $c - v$ or $c + v$.

For two subluminal positive velocities $u'_x < c$ and $v < c$, Eq. (17.37) produces $u_x < c$. The denominator is the crucial relativistic correction. For signed one-dimensional velocities, writing $a = u'_x/c$ gives the useful identity

$$1 - \frac{u_x^2}{c^2} = \frac{(1 - a^2)(1 - \beta^2)}{(1 + a\beta)^2} > 0$$

whenever $|a| < 1$ and $|\beta| < 1$. This proves the subluminal result without relying on positive directions only. At low speeds, $u'_x v/c^2 \ll 1$, so the familiar approximation $u_x \simeq u'_x + v$ returns.

Example 17.9: A probe launched from a fast spacecraft

A spacecraft moves at $0.75c$ relative to Earth. It launches a probe forward at $0.80c$ relative to the spacecraft. What speed does Earth measure?

Solution.

$$\frac{u}{c} = \frac{0.80 + 0.75}{1 + (0.80)(0.75)} = \frac{1.55}{1.60} = 0.96875.$$

Therefore

$$u = 0.96875c.$$

Classical addition would give $1.55c$, which is not the relativistically correct transformation.

Example 17.10: A backward signal from a moving platform

A spacecraft moves at $+0.60c$ relative to Earth. In the spacecraft frame, a massive test particle is launched backward at $u' = -0.80c$. Find its Earth-frame velocity.

Solution.

$$\frac{u}{c} = \frac{-0.80 + 0.60}{1 + (-0.80)(0.60)} = \frac{-0.20}{0.52} \simeq -0.385.$$

The particle moves backward in the Earth frame as well, at about $0.385c$.

Interpretation. Direction is part of velocity. The signs of u' and v must be retained throughout the calculation.

Caution

Velocity addition is not a recipe for “subtracting Lorentz factors.” Transform velocities with Eq. (17.36) or transform event separations and form $u' = \Delta x'/\Delta t'$. The latter route is often safer when signs are complicated.

Guided checks

1. A laser pulse is emitted forward from a spacecraft moving at $0.9c$. What speed does Earth measure for the pulse?
2. Show that Eq. (17.37) reduces to Galilean addition when $|u'v| \ll c^2$.
3. If $u' = 0$, what does Eq. (17.37) give? What is the physical interpretation?

4. Can two individually subluminal collinear velocities combine to give exactly c ? Explain from the formula.

17.10 Relativistic Doppler effect for light

Relative motion changes the observed frequency of light. Figure 17.7 keeps the emission and reception event pairs separate. For sound in a material medium, source and observer motion enter asymmetrically because speeds are measured relative to the medium. Light in vacuum has no such preferred medium. For collinear motion, the Doppler ratio depends on their relative speed and whether they approach or recede. For an oblique ray, its propagation direction must also be specified in a stated frame.

For direct recession along the line of sight with relative speed $v > 0$, the received frequency is

$$f = f_0 \sqrt{\frac{1 - \beta}{1 + \beta}}, \quad \beta = \frac{v}{c}, \quad (17.41)$$

where f_0 is the source frequency in the source rest frame. The wavelength is correspondingly

$$\lambda = \lambda_0 \sqrt{\frac{1 + \beta}{1 - \beta}}. \quad (17.42)$$

For direct approach, replace $\beta \rightarrow -\beta$.

17.10.1 Derivation from emission intervals

Suppose a source recedes from an observer at speed v . In the source frame, successive wave crests are emitted one proper period $T_0 = 1/f_0$ apart. In the observer frame, the time between emission events is dilated to γT_0 . During that interval the source moves an additional distance $v\gamma T_0$ away. The second crest therefore has farther to travel by this amount, producing an arrival-time interval

$$T = \gamma T_0 \left(1 + \frac{v}{c}\right) = \gamma T_0 (1 + \beta). \quad (17.43)$$

Using $\gamma = (1 - \beta^2)^{-1/2}$,

$$\frac{T}{T_0} = \frac{1 + \beta}{\sqrt{1 - \beta^2}} = \sqrt{\frac{1 + \beta}{1 - \beta}}, \quad (17.44)$$

which yields Eq. (17.41) after inversion.

Example 17.11: A clean Doppler ratio

A monochromatic source recedes directly at $v = 0.60c$. What fraction of the emitted frequency is detected?

Solution.

$$\frac{f}{f_0} = \sqrt{\frac{1 - 0.60}{1 + 0.60}} = \sqrt{\frac{0.40}{1.60}} = \sqrt{0.25} = 0.50.$$

The observed frequency is half the source-rest-frame frequency. The observed wavelength is twice the wavelength measured in the source rest frame. A light wave itself has no rest frame or proper length.

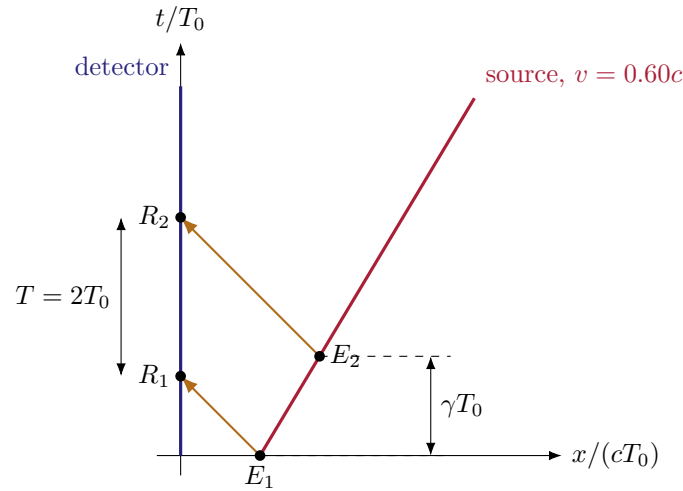


Figure 17.7: Doppler redshift as an event calculation. The source emits crests one proper period T_0 apart. In the detector frame the emissions are separated by γT_0 ; the extra travel distance further increases the reception interval to $T = \gamma(1 + \beta)T_0$. Photon lines have slope -1 in these normalized coordinates.

17.10.2 Low-speed limit

For $|\beta| \ll 1$,

$$\sqrt{\frac{1 - \beta}{1 + \beta}} \simeq 1 - \beta + \mathcal{O}(\beta^2), \quad (17.45)$$

so for recession

$$\frac{\Delta f}{f_0} \simeq -\frac{v}{c}, \quad \frac{\Delta \lambda}{\lambda_0} \simeq \frac{v}{c}. \quad (17.46)$$

These approximations are widely used when radial speeds are small compared with c .

Note

Cosmological redshift is not, in general, simply the special-relativistic Doppler shift of one inertial source through static spacetime. For nearby sources with small recession speeds the distinction can be negligible, but at cosmological distances the expansion of spacetime requires a general-relativistic description.

Guided checks

1. For recession, should f be larger or smaller than f_0 ? What about λ ?
2. If the observed frequency is exactly f_0 , what does the one-dimensional Doppler formula imply about radial relative speed?
3. Why is the relativistic light Doppler effect different in structure from the classical sound Doppler effect?

17.11 Relativistic momentum

A relativistic theory must preserve momentum conservation in isolated systems in every inertial frame. The Newtonian expression $\mathbf{p} = m\mathbf{v}$ is not compatible with Lorentz kinematics at high speed. The relativistically correct momentum of a particle of invariant mass m is

$$\boxed{\mathbf{p} = \gamma m \mathbf{v}.} \quad (17.47)$$

For $v \ll c$, $\gamma \simeq 1$ and Newtonian momentum is recovered.

The factor γ causes momentum to grow increasingly rapidly as v approaches c . This does not mean the particle's invariant mass changes. Instead, the relation between momentum and velocity is nonlinear.

17.11.1 Force and momentum change

The factor γ can also be understood from the derivative $m d\mathbf{r}/d\tau = \gamma m \mathbf{v}$: using the invariant clock parameter yields the spatial part of a Lorentz four-vector. This motivates the form; conservation of four-momentum is the dynamical law, and the low-speed limit fixes the normalization to m . The enrichment section develops the complete energy component.

A useful relativistic definition of three-force is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}. \quad (17.48)$$

Because γ depends on speed, $\mathbf{F}$ is generally not equal to $m\mathbf{a}$. For one-dimensional motion parallel to the force,

$$F_{\parallel} = \gamma^3 m a_{\parallel}, \quad (17.49)$$

whereas for an acceleration instantaneously perpendicular to the velocity,

$$F_{\perp} = \gamma m a_{\perp}. \quad (17.50)$$

Indeed, for fixed mass,

$$\frac{d\mathbf{p}}{dt} = m \left[\gamma \mathbf{a} + \gamma^3 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v} \right].$$

Resolving along and perpendicular to the instantaneous velocity gives the two expressions. In a magnetic field the acceleration is transverse and $\mathbf{F} \cdot \mathbf{v} = 0$, so speed and kinetic energy remain constant even as momentum direction changes. These relations are useful in accelerator physics, but the momentum law Eq. (17.48) is more fundamental and less likely to be misapplied.

Example 17.12: Momentum of a relativistic proton

A proton moves at $0.90c$. Express its momentum in terms of $m_p c$.

Solution. The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1 - 0.90^2}} \simeq 2.294.$$

Therefore

$$p = \gamma m_p v = (2.294)(0.90)m_p c \simeq 2.06 m_p c.$$

The Newtonian value would be only $0.90m_p c$, showing that the correction is already large at $0.90c$.

Key message

Use invariant mass m , not “relativistic mass.” The quantities that grow with speed are γ , momentum and total energy. Keeping m invariant makes the energy–momentum relation and conservation laws clearer.

17.12 Relativistic energy and the energy–momentum relation

For a particle of fixed invariant mass, the work–energy principle remains valid when momentum is relativistic. Starting from

$$dK = \mathbf{F} \cdot d\mathbf{r} = \frac{d\mathbf{p}}{dt} \cdot \mathbf{v} dt = \mathbf{v} \cdot d\mathbf{p}, \quad (17.51)$$

one can evaluate the work by choosing acceleration along a straight line. Since

$$\frac{d(\gamma v)}{dv} = \gamma^3, \quad \frac{d\gamma}{dv} = \frac{\gamma^3 v}{c^2},$$

we have

$$K = \int_0^v m\gamma(w)^3 w \, dw = mc^2 \int_1^{\gamma(v)} d\gamma.$$

Here w is an integration variable, and $K = 0$ at rest. Thus

$$\boxed{K = (\gamma - 1)mc^2.} \quad (17.52)$$

The work integral fixes changes of energy but cannot by itself determine an additive rest-energy constant. Compatibility of energy and momentum with Lorentz transformations and reaction conservation identifies that constant as mc^2 . The four-momentum construction below makes this relation explicit. The particle's total energy is

$$\boxed{E = K + mc^2 = \gamma mc^2.} \quad (17.53)$$

The rest energy is therefore

$$E_0 = mc^2. \quad (17.54)$$

Squaring Eqs. (17.47) and (17.53) and subtracting gives

$$E^2 - p^2 c^2 = \gamma^2 m^2 c^4 \left(1 - \frac{v^2}{c^2}\right) = m^2 c^4.$$

Thus eliminating the velocity yields

$$\boxed{E^2 = p^2 c^2 + m^2 c^4.} \quad (17.55)$$

Figure 17.8 displays this relation quantitatively. Substituting $E = K + mc^2$ also gives $(pc)^2 = K(K + 2mc^2)$, a convenient route from accelerating energy to beam momentum without first calculating a speed. This relation is one of the central equations of modern physics. For a particle at rest, $p = 0$ and $E = mc^2$. For a massless particle, $m = 0$ and the relation becomes $E = pc$.

17.12.1 Recovering classical kinetic energy

Using the expansion in Eq. (17.5),

$$K = (\gamma - 1)mc^2 \quad (17.56)$$

$$= \left(\frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots\right) mc^2 \quad (17.57)$$

$$= \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \dots \quad (17.58)$$

The first term is the Newtonian kinetic energy. This explicit limiting check shows how classical mechanics is embedded in relativity.

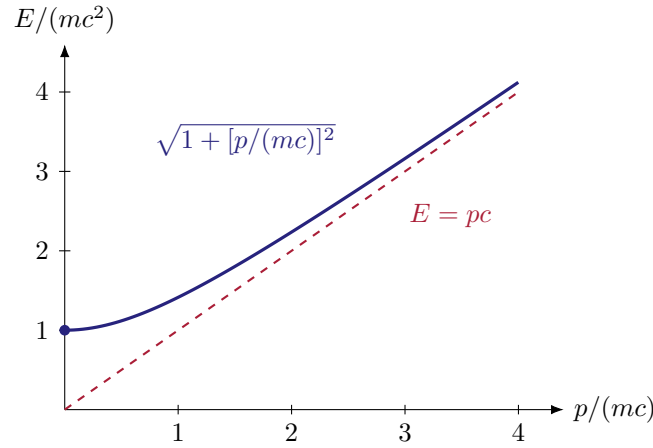


Figure 17.8: Calculated energy–momentum curve for a massive particle, with both axes normalized by its rest-energy scale. The dashed line is the massless relation on the same chosen scales. The massive curve begins at $E/(mc^2) = 1$ and approaches the line from above.

Example 17.13: Electron kinetic energy at $0.90c$

An electron moves at $v = 0.90c$. Its rest energy is approximately $m_e c^2 = 0.511$ MeV. Find its kinetic and total energies.

Solution. With $\gamma \simeq 2.294$,

$$E = \gamma m_e c^2 \simeq (2.294)(0.511 \text{ MeV}) \simeq 1.17 \text{ MeV}.$$

Thus

$$K = (\gamma - 1)m_e c^2 \simeq (1.294)(0.511 \text{ MeV}) \simeq 0.661 \text{ MeV}.$$

The check $E = K + m_e c^2$ gives 1.17 MeV within rounding.

Example 17.14: Speed from kinetic energy

A particle has kinetic energy equal to its rest energy, $K = mc^2$. What is its speed?

Solution. From $K = (\gamma - 1)mc^2$,

$$\gamma - 1 = 1 \quad \Rightarrow \quad \gamma = 2.$$

Hence

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2} \simeq 0.866.$$

Therefore $v \simeq 0.866c$.

17.12.2 Energy units in particle and nuclear physics

The joule is the SI energy unit, but electronvolts are convenient for microscopic systems:

$$1 \text{ eV} = 1.602176634 \times 10^{-19} \text{ J}. \quad (17.59)$$

Rest masses are often quoted as energy equivalents, for example in MeV/ c^2 or GeV/ c^2 . This notation does not set $c = 1$; it simply uses $E_0 = mc^2$ to express mass in energy-derived units.

Caution

The equation $E = mc^2$ is the rest-energy relation, not the complete energy formula for a moving massive particle. For motion, use $E = \gamma mc^2$, or equivalently $E^2 = p^2 c^2 + m^2 c^4$.

Guided checks

1. What happens to p and E as a massive particle approaches c ?
2. Show that $K \rightarrow \frac{1}{2}mv^2$ for $v \ll c$.
3. If $p = 0$, what does Eq. (17.55) reduce to?
4. For a photon, what relation connects energy and momentum?
5. Why does adding more energy to an already ultra-relativistic massive particle increase its momentum strongly but its speed only slightly?

17.13 Conservation laws, reactions and mass–energy accounting

Relativistic momentum and energy are constructed so that isolated systems obey conservation laws in every inertial frame. For a reaction or collision,

$$\sum_i \mathbf{p}_i = \sum_f \mathbf{p}_f, \quad (17.60)$$

$$\sum_i E_i = \sum_f E_f. \quad (17.61)$$

The rest energies of composite systems are part of this accounting. Chemical, nuclear and particle processes can convert rest energy into kinetic energy, radiation or excitation energy, and the reverse can occur as well.

For a reaction between specified asymptotic particles or bound objects, define $M_i = \sum_i m_i$ and $M_f = \sum_f m_f$ as the *sums of their individual rest masses*. These sums need not be equal. The reaction energy is

$$Q = (M_i - M_f)c^2, \quad (17.62)$$

where internal states and the choice of atomic or nuclear masses must be consistent. With all products, including radiation, counted, energy conservation gives $K_f - K_i = Q$, defining a massless particle's contribution here as its full energy. A positive Q makes energy available to product motion or radiation. A negative Q requires incident kinetic energy; the laboratory threshold can exceed $-Q$ because momentum must also be conserved.

These sums must not be confused with the invariant mass M_{sys} of the complete isolated system:

$$M_{\text{sys}}^2 c^4 = E_{\text{tot}}^2 - c^2 |\mathbf{p}_{\text{tot}}|^2.$$

Conservation of total four-momentum means M_{sys} is unchanged during an isolated reaction. In its CM frame this invariant includes internal kinetic energy and any retained radiation. A chosen subsystem can lose mass when energy escapes it, but then that subsystem is not isolated.

Example 17.15: Mass defect as an energy scale

The sum of the reactants' individual rest masses exceeds the sum of the products' individual rest masses by 1.0×10^{-30} kg, with specified internal states. Estimate the positive reaction Q -value.

Solution.

$$Q = \Delta Mc^2 = (1.0 \times 10^{-30})(2.998 \times 10^8)^2 \text{ J} \simeq 9.0 \times 10^{-14} \text{ J}.$$

In electronvolts,

$$Q \simeq \frac{9.0 \times 10^{-14}}{1.602 \times 10^{-19}} \text{ eV} \simeq 5.6 \times 10^5 \text{ eV} = 0.56 \text{ MeV}.$$

This illustrates why very small mass differences can correspond to substantial microscopic energies.

Note

A bound system can have an invariant mass smaller than the sum of the invariant masses of its separated constituents. The difference is related to binding energy. This does not violate mass-energy conservation; the energy and momentum of the complete system are the conserved quantities.

Example 17.16: A sticking collision increases the composite rest mass

Two identical particles of invariant mass m approach head-on in their CM frame, each at speed $0.600c$. They stick to form one composite object. Assume no energy or momentum leaves as radiation. Find the final object's mass and the energy stored internally.

Strategy. Momentum cancels in the chosen frame. The composite therefore rests there, and all initial energy becomes its rest energy, including excitation or heat.

$$Mc^2 = 2\gamma mc^2 = 2(1.25)mc^2, \quad M = 2.50m.$$

The initial kinetic energy is $2(\gamma - 1)mc^2 = 0.50mc^2$. It becomes internal energy of the composite. The sum of the initial individual rest masses is $2m$, while the final particle's rest mass is $2.50m$; for this specified excited final state, $Q = -0.50mc^2$.

Conservation check. The initial *complete system* already has invariant mass $2.50m$, because its total CM energy includes the two particles' kinetic energies. That system invariant does not change. If the composite later radiates, its own mass can decrease; the isolated system consisting of the composite plus all emitted radiation still has the original four-momentum. A lower-energy bound state and this excited composite are different final states and have different Q -values.

17.14 Engineering and applied-science connections

Special relativity becomes practical whenever timing, particle speed or energy reaches regimes where v/c cannot be neglected or where clocks are sufficiently precise to resolve small effects.

17.14.1 Particle accelerators and beam transport

In accelerator design, magnetic rigidity depends on momentum rather than simply on mv . A charged particle moving through a transverse magnetic field satisfies

$$r = \frac{p}{|q|B}, \quad (17.63)$$

for uniform circular motion when electric fields and radiation losses are neglected. At high energy one must use $p = \gamma mv$. Accelerator magnets, spectrometers and beam diagnostics are therefore calibrated in terms of relativistic momentum.

A useful high-energy relation follows from Eq. (17.55):

$$pc = \sqrt{E^2 - m^2 c^4}. \quad (17.64)$$

When $E \gg mc^2$, one has $E \simeq pc$, but a massive particle still remains below c .

17.14.2 Unstable particles and detector acceptance

Particle detectors often observe unstable particles that travel distances much longer than a naive calculation using the proper lifetime would suggest. The correct laboratory mean decay length is

$$\ell = \beta \gamma c \tau_0, \quad (17.65)$$

where τ_0 is the proper mean lifetime. The product $\beta \gamma = p/(mc)$ is especially useful experimentally because momentum may be measured directly from track curvature.

17.14.3 Precision timing and satellite systems

Satellite navigation requires relativistic clock modeling. In a common Earth-centered inertial coordinate system, satellite and ground clocks each have a kinematic rate depending on their speed. Their rate difference contains $-(v_{\text{sat}}^2 - v_{\text{ground}}^2)/(2c^2)$ to leading order. A difference in gravitational potential adds a separate general-relativistic contribution. Both must be modeled in a functioning precision timing system. Special relativity alone is therefore necessary but not sufficient for a complete GPS clock correction.

17.14.4 Spectroscopy and remote sensing

Doppler shifts connect measured wavelengths to relative radial motion. In laboratory spectroscopy, beam diagnostics and astronomical observations, the relativistic formula is needed whenever $|v|/c$ is not negligible at the required accuracy. At low speeds Eq. (17.46) provides a controlled approximation.

17.14.5 Acceleration by an electric potential difference

When a particle of charge q moves through an electrostatic potential difference and radiation losses are negligible, energy conservation gives

$$K_f - K_i = q(V_i - V_f). \quad (17.66)$$

If a particle starts from rest and gains kinetic energy $K = |q|\Delta V$, then

$$\gamma = 1 + \frac{|q|\Delta V}{mc^2}, \quad \beta = \sqrt{1 - \gamma^{-2}}. \quad (17.67)$$

For electrons, a potential difference of one volt supplies one electronvolt of kinetic energy in the ideal electrostatic picture. At hundreds of kilovolts, the relativistic correction is already substantial.

Example 17.17: Electron accelerated through 200 kV

An electron starts approximately from rest and gains $K = 200$ keV. With $m_e c^2 = 511$ keV,

$$\gamma = 1 + \frac{200}{511} = 1.391.$$

Hence

$$\beta = \sqrt{1 - \frac{1}{1.391^2}} \simeq 0.695.$$

The Newtonian estimate $v = \sqrt{2K/m}$ would be too large and becomes increasingly unreliable as the accelerating voltage rises.

17.14.6 Cyclotrons, synchrotrons and the breakdown of the nonrelativistic frequency

For a nonrelativistic particle in a uniform magnetic field, the cyclotron angular frequency is $\omega_c = |q|B/m$. Relativistically, transverse magnetic bending gives

$$\omega = \frac{|q|B}{\gamma m}. \quad (17.68)$$

As the particle gains energy, γ increases and the orbital frequency decreases if B remains fixed. This destroys the simple resonance condition of a classical cyclotron at sufficiently high energy.

A synchrotron avoids this limitation by varying the magnetic field and radio-frequency timing so that particles remain synchronized with the accelerating system while following an approximately fixed orbit. The distinction is a direct engineering consequence of relativistic momentum.

Guided checks

Why does a magnetic field change a charged particle's momentum direction without changing its kinetic energy? Use the magnetic Lorentz force and the power relation $dE/dt = \mathbf{F} \cdot \mathbf{v}$.

Example 17.18: Magnetic rigidity of a relativistic particle

A singly charged particle has momentum $p = 1.00$ GeV/ c . Express the magnetic rigidity $B\rho = p/|q|$ numerically in T m using 1 eV = 1.602×10^{-19} J and $e = 1.602 \times 10^{-19}$ C.

Solution. Convert the momentum through $pc = 1.00$ GeV:

$$p = \frac{1.00 \times 10^9 (1.602 \times 10^{-19} \text{ J})}{2.998 \times 10^8 \text{ m s}^{-1}} \simeq 5.34 \times 10^{-19} \text{ kg m s}^{-1}.$$

Then

$$B\rho = \frac{p}{e} \simeq \frac{5.34 \times 10^{-19}}{1.602 \times 10^{-19}} \text{ T m} \simeq 3.34 \text{ T m}.$$

This conversion is widely useful in beam physics.

Take-home message

Relativity is an engineering correction whenever an instrument measures time, frequency, momentum or energy precisely enough that the Newtonian approximation exceeds the allowed error. The relevant question is not whether a system “looks relativistic,” but whether the required accuracy and value of v/c make relativistic terms measurable.

17.15 Laboratory and computational connection: testing relativistic kinematics

Although special relativity is conceptually demanding, many of its quantitative predictions are straightforward to explore numerically. Computation is particularly useful for answering a practical question: *at what speed does the classical approximation cease to be accurate for the precision required by a given experiment or engineering system?*

17.15.1 A numerical map of the Lorentz factor

The Lorentz factor

$$\gamma(\beta) = \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta = \frac{v}{c},$$

is nearly unity for small β but rises rapidly as $\beta \rightarrow 1$. Table 17.1 illustrates the transition.

Table 17.1: Representative Lorentz factors. Values are rounded for orientation; numerical work should retain adequate precision.

β	γ	$\gamma - 1$
0.01	1.00005	5.0×10^{-5}
0.10	1.00504	5.0×10^{-3}
0.30	1.04828	4.8×10^{-2}
0.50	1.15470	0.15470
0.80	1.66667	0.66667
0.90	2.29416	1.29416
0.99	7.08881	6.08881
0.999	22.3663	21.3663

The small-speed approximation follows from the binomial expansion

$$\gamma = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \mathcal{O}(\beta^6). \quad (17.69)$$

Thus the leading fractional departure of time dilation from the Newtonian expectation is approximately $\beta^2/2$.

Example 17.19: Speed at which a one-percent timing correction is needed

Suppose an instrument requires a time-dilation correction once $\gamma - 1 \geq 0.010$. Estimate the threshold speed and compare the approximation with the exact result.

Approximate solution. Using $\gamma - 1 \simeq \beta^2/2$ gives

$$\beta \simeq \sqrt{0.020} = 0.141.$$

Exact solution. Set $\gamma = 1.010$:

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{(1.010)^2}} \simeq 0.140.$$

The approximation is already very good at this level. The important lesson is that “relativistic” is an accuracy-dependent statement: a system operating at $0.05c$ may be safely classical for a ten-percent estimate but not for a precision timing experiment.

The quantitative curves in Fig. 17.9 show why one universal cutoff between “classical” and “relativistic” is inadequate. The Lorentz factor, kinetic-energy error and a clock experiment’s accumulated time offset answer different questions.

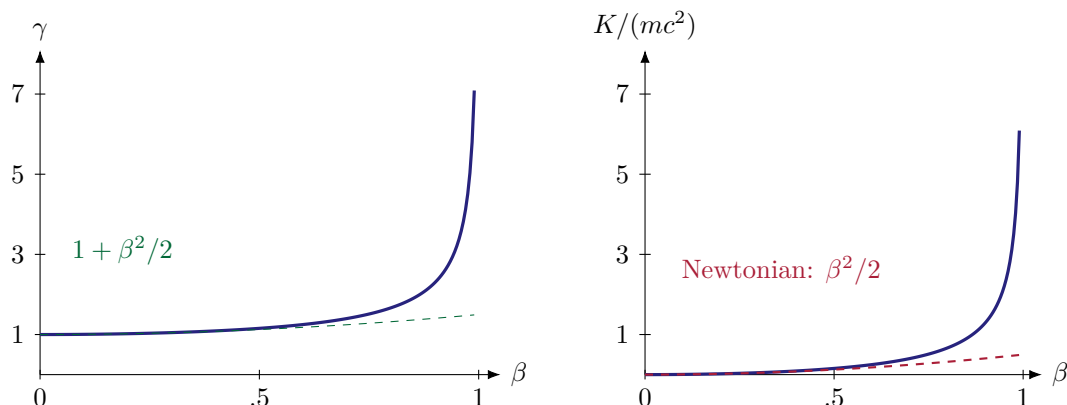


Figure 17.9: Calculated Lorentz factor (left) and normalized kinetic energy (right), ending at $\beta = 0.99$. Dashed curves are low-speed approximations. Both exact curves diverge as $\beta \rightarrow 1$; the endpoint shown is not the divergence. No experimental data are represented.

17.15.2 Computing the classical kinetic-energy error

For a particle of invariant mass m , compare

$$K_{\text{rel}} = (\gamma - 1)mc^2, \quad K_{\text{N}} = \frac{1}{2}mv^2.$$

A useful dimensionless error measure is

$$\epsilon_K(\beta) = \frac{K_{\text{rel}} - K_{\text{N}}}{K_{\text{rel}}}. \quad (17.70)$$

The mass cancels. Therefore the domain of validity of the Newtonian kinetic-energy expression can be explored without choosing any particular particle.

Method

For numerical studies of relativity, work with dimensionless quantities whenever possible. Use $\beta = v/c$, γ , $K/(mc^2)$ and $p/(mc)$ rather than carrying very large or very small SI numbers through every calculation. Restore units at the end. This reduces round-off mistakes and makes limiting behavior easier to inspect.

At very small β , directly subtracting 1 from a rounded γ loses precision. Rationalization gives

$$\gamma - 1 = \frac{\beta^2 \gamma^2}{\gamma + 1}, \quad \epsilon_K = \frac{(\gamma - 1)(2\gamma + 1)}{2\gamma^2}. \quad (17.71)$$

The second expression has the continuous value zero at $\beta = 0$. The following complete Python calculation uses only the standard library and prints dimensionless results:

```
from math import sqrt
for j in range(200):
    beta = 0.99*j/199
    gamma = 1/sqrt((1-beta)*(1+beta))
    km = beta*beta*gamma*gamma/(gamma+1)
    knew = 0.5*beta*beta
    error = km*(2*gamma+1)/(2*gamma*gamma)
    print(beta, gamma, km, knew, error)
```

Use the output to locate an error tolerance. For example, solving $\epsilon_K = 0.05$ gives $\gamma \simeq 1.0349$, hence $\beta \simeq 0.2575$. The Newtonian kinetic energy underestimates the relativistic value by five percent there. A numerical scan should approach this result when the grid is refined; changing particle mass cannot change the answer because the mass cancelled from the ratio.

17.15.3 Simulating Lorentz transformations of events

A second useful computational activity is to create a set of events (ct, x) in one frame and transform them to another. For one spatial dimension,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \quad (17.72)$$

Students can verify numerically that

$$(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t)^2 - (\Delta x)^2$$

to floating-point accuracy. This is a necessary diagnostic, not a sufficient one. A consistent boost with the opposite sign of v also preserves the interval. Test the moving origin $x = vt$, the two light directions, and composition with the inverse as well. Near a null separation, normalize numerical residuals to the input scale $c^2\Delta t^2 + \Delta x^2$, not to the nearly zero invariant itself.

Example 17.20: Numerical interval check

Two events in frame S have separation

$$\Delta x = 2.40 \times 10^8 \text{ m}, \quad \Delta t = 1.00 \text{ s}.$$

Transform to a frame moving at $v = 0.60c$ and verify the invariant interval.

Solution. Here $\gamma = 1.25$ and $v = 1.80 \times 10^8 \text{ m s}^{-1}$. Therefore

$$\Delta x' = \gamma(\Delta x - v\Delta t) = 1.25(2.40 - 1.80) \times 10^8 = 7.50 \times 10^7 \text{ m},$$

and

$$\Delta t' = \gamma \left(\Delta t - \frac{v\Delta x}{c^2} \right) = 1.25(1.00 - 0.48) = 0.650 \text{ s}.$$

Using $c = 3.00 \times 10^8 \text{ m s}^{-1}$ for this check,

$$s^2 = c^2(1.00)^2 - (2.40 \times 10^8)^2 = 3.24 \times 10^{16} \text{ m}^2,$$

while

$$s'^2 = c^2(0.650)^2 - (7.50 \times 10^7)^2 = 3.24 \times 10^{16} \text{ m}^2.$$

The agreement is not an additional assumption; it is a consequence of the Lorentz transformation.

17.15.4 A laboratory-style lifetime analysis

A realistic particle-lifetime analysis often measures decay positions rather than directly watching a microscopic clock. If an unstable particle beam has approximately fixed $\beta\gamma$, the surviving fraction after laboratory distance x is

$$N(x) = N_0 \exp\left(-\frac{x}{\beta\gamma c\tau_0}\right). \quad (17.73)$$

Taking logarithms gives

$$\ln N(x) = \ln N_0 - \frac{x}{\beta\gamma c\tau_0}. \quad (17.74)$$

The slope is $-1/\ell$, so a logarithmic plot is useful for diagnosing exponential attenuation. At large counts a suitably weighted log fit can estimate the decay length. For sparse data, zero counts, backgrounds or correlated survivor samples, fit the counts with an appropriate Poisson, binomial or joint likelihood instead; taking logarithms does not preserve simple counting uncertainties. An independent momentum and mass determine $\beta\gamma = p/(mc)$ and therefore the proper lifetime.

Note

Equation (17.73) assumes a narrow momentum distribution and an idealized acceptance. Real detectors require efficiency corrections, backgrounds and a momentum-dependent model. The simple form is nevertheless an excellent bridge between relativistic kinematics and experimental inference.

17.15.5 From measured counts to a proper lifetime

A decay-length measurement becomes a test of time dilation only when the boost is determined independently. Measuring one long attenuation length alone does not separate a longer rest-frame lifetime from a larger Lorentz factor. Momentum bins provide the separation: for a known mass, the prediction is $\ell(p) = p\tau_0/m$ and the inferred $\tau_0 = m\ell/p$ should be independent of the bin.

Example 17.21: An idealized lifetime measurement with counting uncertainty

Consider illustrative teaching counts from two independent runs with equal, known beam exposures and equal calibrated efficiencies. A narrow beam gives $N_0 = 4000$ counts at the entrance and $N_L = 1000$ surviving-particle counts at distance $L = 3.00$ m. Independent momentum measurements give $\beta\gamma = 4.00 \pm 0.08$. Estimate the proper mean lifetime and its uncertainty. Assume Poisson counts, negligible background, negligible energy loss and negligible uncertainty in L .

Model and inversion. The count ratio estimates $r = N_L/N_0 = e^{-L/\ell}$, so

$$\ell = \frac{L}{\ln(N_0/N_L)} = \frac{3.00}{\ln 4} \text{ m} = 2.164 \text{ m}.$$

The lifetime then follows from the independently measured boost:

$$\tau_0 = \frac{\ell}{(\beta\gamma)c} \simeq 1.805 \text{ ns}.$$

Uncertainty. For independent large Poisson counts, $\sigma_N \simeq \sqrt{N}$. With $a = \ln(N_0/N_L)$,

$$\sigma_a \simeq \sqrt{\frac{1}{N_0} + \frac{1}{N_L}} = 0.0354, \quad \sigma_\ell = \frac{L}{a^2} \sigma_a \simeq 0.055 \text{ m}.$$

Treating the momentum estimate as independent,

$$\frac{\sigma_{\tau_0}}{\tau_0} \simeq \sqrt{\left(\frac{0.055}{2.164}\right)^2 + \left(\frac{0.08}{4.00}\right)^2} = 0.0324.$$

A suitably rounded result is $\tau_0 = (1.80 \pm 0.06) \text{ ns}$ under this model.

Checks and limitations. The ratio is dimensionless and ℓ/c is a time. These counts illustrate the analysis and are not an experimental dataset. If the downstream efficiency is lower, ignoring that difference underestimates the survival and hence the lifetime. If the same particles contribute to nested upstream and downstream counts, the counts are correlated and a binomial or joint likelihood replaces the independent-Poisson propagation used here.

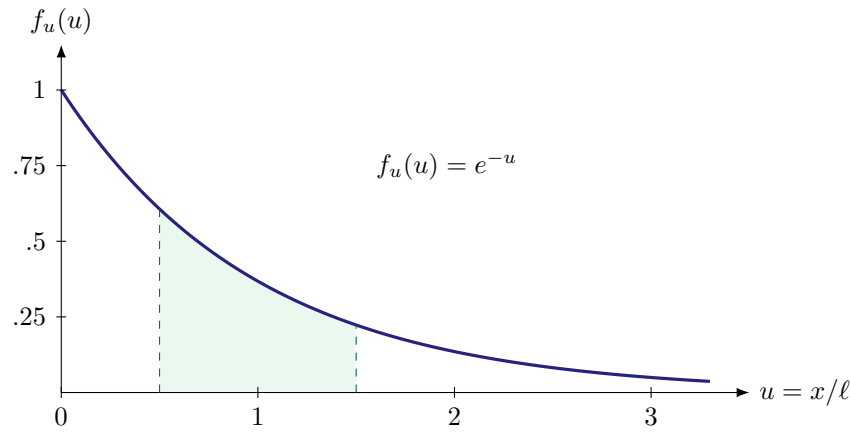


Figure 17.10: Calculated probability density of the normalized decay distance $u = x/\ell$ for a monoenergetic beam. The shaded area, from $u = 0.5$ to 1.5 , is the decay probability in that interval. It also equals the difference of survival probabilities $S(x_1) - S(x_2)$, with $S(x) = e^{-x/\ell}$. The density for the dimensional variable x is $e^{-x/\ell}/\ell$.

Figure 17.10 also connects survival with detector acceptance. For a constant-speed particle, the probability to decay in a path interval is

$$P(x_1 < x < x_2) = e^{-x_1/\ell} - e^{-x_2/\ell}. \quad (17.75)$$

For $\ell = 2.0$ m and a sensitive path interval from 0.50 to 1.50 m, the ideal probability is $e^{-0.25} - e^{-0.75} \simeq 0.307$. If efficiency depends on position, replace it by $\int_{x_1}^{x_2} \epsilon(x) e^{-x/\ell} dx/\ell$. A radial detector boundary must first be converted into path length for a nonradial track.

17.15.6 Uncertainty amplification near the speed of light

Since

$$\frac{d\gamma}{d\beta} = \beta\gamma^3, \quad (17.76)$$

a small uncertainty σ_β propagates approximately as

$$\sigma_\gamma \simeq \beta\gamma^3 \sigma_\beta. \quad (17.77)$$

At large γ , a direct speed measurement is therefore a poor way to determine the particle energy: tiny changes in β correspond to large changes in γ . High-energy experiments instead infer momentum or energy from magnetic curvature, calorimetry and other observables.

Example 17.22: Why accelerator physicists quote energy rather than speed

Compare electrons with $\gamma = 10$ and $\gamma = 100$. Their speed parameters are

$$\beta = \sqrt{1 - \gamma^{-2}}.$$

For $\gamma = 10$, $\beta \simeq 0.994987$; for $\gamma = 100$, $\beta \simeq 0.999950$. The total energy has increased by a factor of ten, yet the speed has changed by only about 0.5% of c . Energy and momentum remain highly informative while speed saturates close to c .

Guided checks

1. Why is $\gamma - 1$, rather than γ , a useful measure of the size of the time-dilation correction at low speed?
2. Why does the mass cancel from the comparison between relativistic and Newtonian kinetic energy?
3. What invariant can be used to debug a numerical Lorentz-transformation routine?
4. Why can a small uncertainty in β produce a large uncertainty in γ when β is very close to one?
5. In a lifetime experiment, which measured quantities are needed to infer a proper lifetime from a decay-length distribution?

17.16 Choosing the measurement before the formula

A correct equation can give a wrong answer when its inputs refer to different events. The following comparison collects the main distinctions in one place. It is a diagnostic tool to use after drawing the geometry, not a substitute for that drawing.

Table 17.2: Conditions that distinguish common relativistic measurements.

Question	Required condition	Frequent wrong substitution
Elapsed clock time	One specified clock path joins the two events; constant speed permits $\Delta t = \gamma \Delta \tau$.	Using a proper time from a different pair of events.
Moving length	Both endpoints are located at one time in the measuring frame.	Combining endpoint positions recorded at different times.
Velocity	Distance and coordinate time come from the same frame.	Dividing laboratory distance by onboard proper time and calling the result a laboratory speed.
Received frequency	Compare successive reception events at the detector.	Equating a Doppler period with the dilated emission period alone.
Reaction energy	Conserve total energy and momentum, including radiation.	Treating a sum of particle rest masses as the invariant mass of the whole system.

Method

State the frames and boost direction; name the events or endpoints; identify any proper quantity; then choose a transformation or invariant. Check dimensions, signs and a limiting case. If an answer seems paradoxical, compare the actual event pairs before interpreting the algebra.

17.17 Main chapter summary

Summary

- Special relativity describes flat spacetime; its standard transformations connect inertial frames and rest on two statements: the laws of physics have the same form in all inertial frames, and the vacuum speed of light c is invariant.
- A physical event has space and time coordinates. Different inertial frames assign different coordinates to the same event, but the assignments are connected systematically by Lorentz transformations.
- Simultaneity is not invariant for spatially separated events. This is central, not incidental: it underlies both time dilation and length contraction.
- Proper time $\Delta\tau$ is elapsed time along a specified clock path. For constant-speed inertial motion, a frame in which the clock moves assigns $\Delta t = \gamma\Delta\tau$. Accelerated paths require an integral.
- Proper length L_0 is measured in the rest frame of an object or pair of endpoints. A frame in which that length moves parallel to itself measures $L = L_0/\gamma$.
- The Lorentz transformation mixes space and time while preserving the light cone and the spacetime interval.
- Velocities do not add linearly at high speed. The relativistic law ensures that a light signal remains at speed c in every inertial frame.
- Relativistic Doppler shifts combine wave propagation with time dilation. At low speeds they reduce to the familiar first-order Doppler relation.
- Relativistic momentum is $\mathbf{p} = \gamma m\mathbf{v}$ and total energy is $E = \gamma mc^2$. Their invariant relation is $E^2 = p^2c^2 + m^2c^4$.
- Newtonian mechanics is recovered as a controlled low-speed approximation: $\gamma \simeq 1 + \beta^2/2$ and $K \simeq mv^2/2$.
- Practical relativistic effects appear in accelerator and detector design, unstable-particle lifetimes, precision timing, spectroscopy, remote sensing, and high-energy medical or industrial beams.

Formula summary

For standard configuration, with S' moving at $+v\hat{x}$ relative to S ,

$$\begin{aligned}\beta &= \frac{v}{c}, & \gamma &= \frac{1}{\sqrt{1-\beta^2}}, \\ \Delta t &= \gamma \Delta \tau, & L &= \frac{L_0}{\gamma}, \\ x' &= \gamma(x - vt), & t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ y' &= y, & z' &= z, \\ u'_x &= \frac{u_x - v}{1 - u_x v/c^2}, & u_x &= \frac{u'_x + v}{1 + u'_x v/c^2}, \\ f_{\text{rec}} &= f_0 \sqrt{\frac{1-\beta}{1+\beta}}, & \mathbf{p} &= \gamma m \mathbf{v}, \\ K &= (\gamma - 1)mc^2, & E &= \gamma mc^2, \\ E^2 &= p^2 c^2 + m^2 c^4, & \ell &= \beta \gamma c \tau_0.\end{aligned}$$

Here time dilation uses a constant-speed clock and its two tick events; length contraction uses simultaneous endpoint readings for an inertial rod; the Doppler formula is for collinear recession; $p = \gamma m v$ and $E = \gamma m c^2$ apply to a massive particle; and ℓ assumes a fixed boost and an exponential proper-lifetime distribution.

Take-home message

The deepest operational lesson of special relativity is not that “everything is relative.” Rather, coordinate descriptions depend on the inertial frame while physical law is common to all inertial frames. Relativity replaces separate absolute notions of space and time with a consistent spacetime structure whose invariants allow all observers to agree on the underlying physics.

17.18 Advanced Topics for Technical and Engineering Students

This section is intended for strong students who want a more structural view of special relativity and a closer connection with high-energy physics, electromagnetism, precision measurement and computational modeling. The material is self-contained enough to be read after the main chapter, but it deliberately goes beyond what is needed for a first examination. The central organizing idea is that Lorentz transformations are not a collection of separate corrections; they form a geometry and symmetry structure that unifies the results already derived.

Pedagogical boundary

The enrichment section introduces invariant intervals, four-vectors, rapidity, invariant mass, threshold kinematics and selected electromagnetic-field transformations. It does not develop tensor calculus, differential geometry, quantum field theory or the full general theory of relativity. Natural units $c = 1$ are mentioned only as a notation used in advanced physics; equations in this chapter continue to display c explicitly unless a short, clearly marked comparison is being made.

17.18.1 The spacetime interval and causal classification

For two events, define the interval in one spatial dimension by

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2, \quad (17.78)$$

and in three dimensions by

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2. \quad (17.79)$$

In one dimension, direct substitution gives

$$(c\Delta t')^2 - (\Delta x')^2 = \gamma^2 [(c\Delta t - \beta\Delta x)^2 - (\Delta x - \beta c\Delta t)^2].$$

The mixed terms cancel, leaving $\gamma^2(1 - \beta^2)[(c\Delta t)^2 - (\Delta x)^2] = \Delta s^2$. The unchanged transverse coordinates establish $\Delta s'^2 = \Delta s^2$ in three dimensions. This invariant replaces the separate Newtonian invariants of spatial distance and time interval.

The sign of Δs^2 divides event pairs into three classes:

Timelike: $\Delta s^2 > 0$. A subluminal object can travel from one event to the other. A frame exists in which the events occur at the same position.

Lightlike or null: $\Delta s^2 = 0$. A light signal can connect the events. All inertial frames agree that the interval is null.

Spacelike: $\Delta s^2 < 0$. No causal influence traveling at or below c can connect the events. A frame exists in which the events are simultaneous.

Example 17.23: Classifying two event pairs

For pair A, $\Delta t = 5.0 \mu\text{s}$ and $\Delta x = 900 \text{ m}$. For pair B, $\Delta t = 2.0 \mu\text{s}$ and $\Delta x = 900 \text{ m}$. Classify each pair.

Solution. The light-travel distance corresponding to $5.0 \mu\text{s}$ is $c\Delta t \simeq 1500 \text{ m}$, larger than 900 m , so pair A is timelike. For pair B, $c\Delta t \simeq 600 \text{ m} < 900 \text{ m}$, so pair B is spacelike. The classification is invariant: no boost can change a timelike pair into a spacelike pair.

Frame with zero spatial separation

For timelike-separated events, set $\Delta x' = 0$:

$$0 = \gamma(\Delta x - v\Delta t) \Rightarrow v = \frac{\Delta x}{\Delta t},$$

which is less than c precisely when $|\Delta x| < c|\Delta t|$.

Frame with zero temporal separation

For spacelike-separated events, set $\Delta t' = 0$:

$$0 = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \Rightarrow v = \frac{c^2\Delta t}{\Delta x},$$

which is less than c precisely when $c|\Delta t| < |\Delta x|$.

Key message

Causal order is preserved for timelike and lightlike separated events. Different inertial frames may reverse the time order of spacelike-separated events, but those events cannot be causally connected by any signal traveling at or below c . This is why relativity of simultaneity does not create causal paradoxes.

Example 17.24: Explicit reversal of order for spacelike events

In frame S , event A occurs at $(t_A, x_A) = (0, 0)$ and event B at $(1.00 \mu\text{s}, 600 \text{ m})$. Show that the events are spacelike separated and find their order in a frame moving at $0.80c$. The light-travel distance during $1.00 \mu\text{s}$ is about 300 m , which is smaller than the 600 m spatial separation. Thus the pair is spacelike.

For $\beta = 0.80$, $\gamma = 5/3$:

$$\Delta t' = \gamma\left(1.00 \mu\text{s} - \frac{(0.80c)(600 \text{ m})}{c^2}\right).$$

Since $600/c \simeq 2.00 \mu\text{s}$,

$$\Delta t' = \frac{5}{3}(1.00 - 1.60) \mu\text{s} = -1.00 \mu\text{s}.$$

So B, which follows A in S , precedes A in S' .

Causality check. Because $600 \text{ m} > c(1.00 \mu\text{s})$, no signal at or below c can travel from A to B. Reversing their coordinate-time order therefore does not reverse a causal relation.

17.18.2 Proper time as spacetime length

For a timelike infinitesimal displacement,

$$c^2 d\tau^2 = c^2 dt^2 - d\mathbf{r}^2. \quad (17.80)$$

Since $d\mathbf{r}^2 = v^2 dt^2$,

$$d\tau = dt \sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma}. \quad (17.81)$$

An ideal clock obeys this local relation even when its speed changes. This clock hypothesis neglects acceleration-induced disturbances of the physical clock mechanism. Integrating along the specified trajectory in flat spacetime gives

$$\Delta\tau = \int_{t_1}^{t_2} \sqrt{1 - \frac{v(t)^2}{c^2}} dt. \quad (17.82)$$

This is the natural generalization of time dilation beyond constant relative speed.

Example 17.25: Piecewise journey and elapsed proper time

A spacecraft is described in an Earth inertial frame as traveling for 4.0 y at $0.60c$, then for 3.0 y at $0.80c$. Ignore the short acceleration intervals. How much proper time elapses on the spacecraft clock?

Solution. For $0.60c$, $1/\gamma = 0.80$; for $0.80c$, $1/\gamma = 0.60$. Therefore

$$\Delta\tau = (4.0 \text{ y})(0.80) + (3.0 \text{ y})(0.60) = 5.0 \text{ y}.$$

Earth clocks record 7.0 y over the two cruising segments.

17.18.3 The twin scenario as a proper-time comparison

The famous twin scenario is often presented as a paradox, but the proper-time integral removes the apparent contradiction. One traveler remains approximately in one inertial frame while the other changes velocity during the journey. Their spacetime paths are different, so the proper times along them need not agree.

Consider an idealized outbound and inbound journey at constant speed v , with instantaneous turnaround. If the stay-at-home clock records total time T , the traveler's elapsed time is

$$\tau = \frac{T}{\gamma}. \quad (17.83)$$

The instantaneous-turnaround approximation is not physically realistic, but the age difference persists when the turnaround is smoothed over finite acceleration.

Example 17.26: A symmetric out-and-back journey

A traveler goes to a star 4.0 light-years away in the Earth frame at $0.80c$ and immediately returns at the same speed. Ignore acceleration time. Compare elapsed times.

Solution. Each Earth-frame leg lasts

$$\frac{4.0 \text{ ly}}{0.80c} = 5.0 \text{ y},$$

so Earth records 10.0 y. With $\gamma = 5/3$, the traveler records

$$\tau = \frac{10.0 \text{ y}}{5/3} = 6.0 \text{ y}.$$

The difference is 4.0 y. There is no symmetry between the complete worldlines because the traveler changes inertial frames at turnaround.

The traveler's change of simultaneity

For the numerical journey just calculated, the turnaround is at Earth coordinates $(t_T, x_T) = (5.0 \text{ y}, 4.0 \text{ ly})$. An outbound simultaneity line through it obeys

$$t - t_T = \frac{v(x - x_T)}{c^2}.$$

At Earth ($x = 0$), this selects $t_- = 5.0 - 0.80(4.0) = 1.8 \text{ y}$. The incoming inertial frame has velocity $-v$, so its simultaneity line through the same event selects $t_+ = 5.0 + 0.80(4.0) = 8.2 \text{ y}$. Figure 17.11 displays both choices.

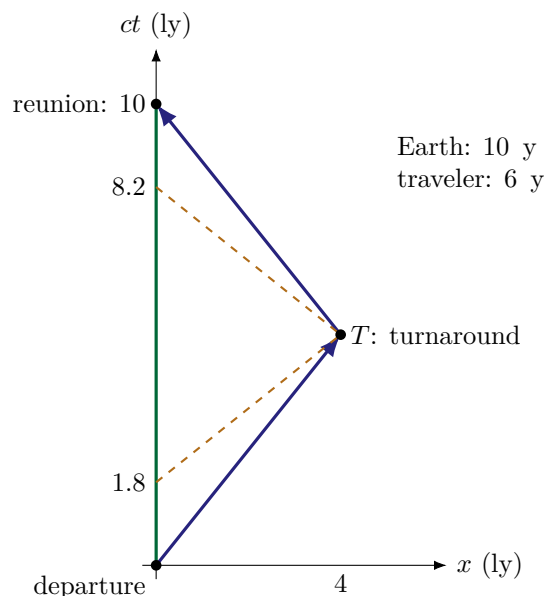


Figure 17.11: Two paths between the same departure and reunion, with equal spatial and temporal length scales. Dashed lines are outbound and inbound simultaneity through the turnaround, not light rays. The Earth's worldline remains continuous.

The change $t_+ - t_- = 6.4$ y is a reassignment of which *distant* Earth event is called simultaneous when the traveler changes frames. Earth does not suddenly age, and the traveler does not see a discontinuity in received light. With a finite turnaround the frame change is continuous. The two proper times compared at reunion are local clock readings and are independent of this bookkeeping choice.

Between fixed timelike endpoints, the straight inertial path has the largest proper time in flat spacetime. In the rest frame of that path the endpoints have the same position, and any competing path satisfies

$$\Delta\tau_{\text{path}} = \int_{t_A}^{t_B} \sqrt{1 - v(t)^2/c^2} dt \leq t_B - t_A.$$

Equality requires $v(t) = 0$ throughout apart from sets of zero duration. This compares *different clock paths*. By contrast, elementary time dilation compares coordinate intervals for one inertial clock path; confusing these comparisons creates the false claim that the same proper time must be both minimum and maximum.

What role does acceleration play?

Acceleration marks the asymmetry and permits the traveler to reverse direction, but the accumulated proper time is determined by the entire worldline. One can compare two clocks that reunite after following different timelike paths; the proper-time integral decides the result. General relativity is not required to analyze the flat-spacetime version.

17.18.4 Minkowski diagrams and hyperbolic geometry

A Minkowski diagram plots ct vertically and x horizontally. Light rays follow $ct = \pm x$. A subluminal inertial worldline through the origin lies inside the light cone. The ct' axis is the worldline $x = vt$ of the moving origin. The x' axis consists of events satisfying $t' = 0$ and therefore obeys $ct = \beta x$.

Figure 17.12 shows actual level curves of the invariant. For $c^2t^2 - x^2 = a^2 > 0$, the future branch is $ct = \sqrt{a^2 + x^2}$, opening upward. A Lorentz boost moves the coordinates along this hyperbola; it cannot take a timelike event outside the cone. The analogy with an ordinary rotation is suggestive

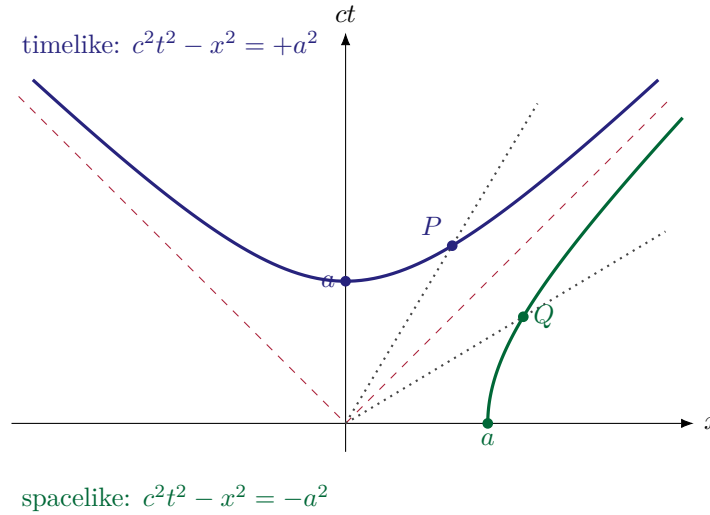


Figure 17.12: Calculated invariant hyperbolae, drawn with $a = 2$ in the displayed coordinate units. For a $\beta = 0.60$ boost, the primed coordinate points $(ct', x') = (a, 0)$ and $(0, a)$ lie at $P = (ct, x) = (2.5, 1.5)$ and $Q = (1.5, 2.5)$. Their hyperbolae calibrate equal primed units; Euclidean distances on tilted axes do not.

but incomplete. A Euclidean rotation preserves $x^2 + y^2$ and uses trigonometric functions. A Lorentz boost preserves $c^2t^2 - x^2$ and is naturally described by hyperbolic functions.

17.18.5 Rapidity: turning velocity composition into addition

Define the rapidity η by

$$\beta = \tanh \eta, \quad \gamma = \cosh \eta, \quad \beta\gamma = \sinh \eta. \quad (17.84)$$

The Lorentz transformation becomes

$$ct' = ct \cosh \eta - x \sinh \eta, \quad (17.85)$$

$$x' = x \cosh \eta - ct \sinh \eta. \quad (17.86)$$

If two collinear boosts have rapidities η_1 and η_2 , the combined rapidity is simply

$$\eta_{\text{total}} = \eta_1 + \eta_2. \quad (17.87)$$

The nonlinear velocity-addition formula is therefore a consequence of the relation $\beta = \tanh \eta$.

Example 17.27: Two equal boosts using rapidity

Two successive frames each move at $0.60c$ relative to the preceding one in the same direction. Find the first-to-third relative speed.

Solution. Directly,

$$\beta = \frac{0.60 + 0.60}{1 + 0.60^2} = \frac{1.20}{1.36} = 0.882.$$

In rapidity language, $\eta_0 = \text{artanh}(0.60)$ and the combined rapidity is $2\eta_0$; using $\tanh(2\eta_0) = 2 \tanh \eta_0 / (1 + \tanh^2 \eta_0)$ gives the same result immediately.

17.18.6 Lorentz matrices and invariant scalar products

Write the boost in (ct, x) coordinates as

$$\Lambda(\beta) = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}. \quad (17.88)$$

Introduce the Minkowski metric matrix

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (17.89)$$

Lorentz invariance is expressed compactly as

$$\Lambda^T G \Lambda = G. \quad (17.90)$$

Students familiar with linear algebra can verify this directly. The equation is the spacetime analogue of the Euclidean rotation condition $R^T R = I$, with G replacing the identity matrix.

Example 17.28: Determinant of a boost

For the 2×2 boost matrix,

$$\det \Lambda = \gamma^2 - (\beta\gamma)^2 = \gamma^2(1 - \beta^2) = 1.$$

The boost is therefore invertible and continuously connected to the identity for $|\beta| < 1$.

17.18.7 Four-position and four-velocity

Introduce the four-position

$$X^\mu = (ct, \mathbf{r}). \quad (17.91)$$

The notation $\mu = 0, 1, 2, 3$ labels one time and three spatial components. The invariant interval is the Minkowski norm of ΔX^μ .

Differentiating with respect to proper time gives the four-velocity

$$U^\mu = \frac{dX^\mu}{d\tau} = \gamma(c, \mathbf{v}). \quad (17.92)$$

For any four-vector $A^\mu = (A^0, A^1, A^2, A^3)$, lowering the index in our convention gives $A_\mu = (A^0, -A^1, -A^2, -A^3)$; a repeated index is summed. Thus $A^\mu B_\mu = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}$. Applying this rule, the four-velocity's invariant magnitude is

$$U^\mu U_\mu = c^2. \quad (17.93)$$

Although the ordinary three-velocity is bounded by c , the spatial part $\gamma \mathbf{v}$ of the four-velocity can grow without bound. This quantity is sometimes called the proper velocity.

Note

Index notation is used here only to reveal structure. A complete treatment of covariant and contravariant components, metrics and tensors belongs to a later course. For this chapter, it is enough to recognize four-vectors as objects whose components mix under Lorentz transformations while their Minkowski scalar products remain invariant.

17.18.8 Four-momentum and invariant mass

For a massive particle, multiplying four-velocity by invariant mass gives the four-momentum

$$P^\mu = mU^\mu = \left(\frac{E}{c}, \mathbf{p} \right). \quad (17.94)$$

Its invariant norm is

$$P^\mu P_\mu = \frac{E^2}{c^2} - p^2 = m^2 c^2, \quad (17.95)$$

which is exactly the energy–momentum relation.

Its energy and longitudinal momentum transform as

$$E' = \gamma(E - vp_x), \quad (17.96)$$

$$p'_x = \gamma\left(p_x - \frac{vE}{c^2}\right), \quad p'_y = p_y, \quad p'_z = p_z. \quad (17.97)$$

Here γ belongs to the *frame boost*, not necessarily to the particle's velocity. The combination vp_x has energy units and vE/c^2 has momentum units. Because these transformations are linear, a sum of conserved four-momenta in one frame remains conserved in every other inertial frame.

For a many-particle system whose asymptotic constituents are noninteracting, total four-momentum is the sum

$$P_{\text{tot}}^\mu = \sum_i P_i^\mu. \quad (17.98)$$

The invariant mass M of the complete system is defined by

$$M^2 c^4 = E_{\text{tot}}^2 - p_{\text{tot}}^2 c^2. \quad (17.99)$$

The invariant M includes internal kinetic energy. For a bound or interacting system, field and interaction contributions must also be included in the total energy–momentum; summing free constituent rest energies alone is insufficient. The sum $\sum_i m_i$ generally differs from M .

Example 17.29: Invariant mass of two photons

Two photons each have energy E_γ and travel in exactly opposite directions. What is the invariant mass of the two-photon system?

Solution. The total momentum is zero because the photon momenta cancel, while the total energy is $2E_\gamma$. Hence

$$M^2 c^4 = (2E_\gamma)^2, \quad M = \frac{2E_\gamma}{c^2}.$$

Each photon individually has zero invariant mass, but the two-photon system has nonzero invariant mass because it has nonzero energy in its center-of-momentum frame.

Key message

Invariant mass is a property of the complete energy–momentum content of a system, not merely a count of constituent rest masses. This viewpoint is essential in nuclear, particle and collider physics.

17.18.9 Massless particles and the lightlike limit

The formula $P^\mu = mU^\mu$ uses a massive particle's proper time and cannot be applied to a photon by multiplying $m = 0$ by an infinite Lorentz factor. Instead, use the independently defined four-momentum $(E/c, \mathbf{p})$ and its norm. Setting $m = 0$ in that norm gives

$$E = pc. \quad (17.100)$$

A massless particle cannot have a rest frame. If one attempted to transform to a frame moving at c with the particle, γ would diverge and the Lorentz transformation would cease to define an inertial frame.

The statement “photons experience no time” is therefore not a valid frame-based statement: there is no inertial photon rest frame with a proper clock. A null worldline has $d\tau = 0$, but this geometrical fact should not be rephrased as observations made by a photon.

Caution

Never ask “what does a photon see in its own rest frame?” Special relativity provides no inertial rest frame for light. Questions about measurements must be formulated in physically realizable timelike frames.

17.18.10 Center-of-momentum frame

For a system with timelike total four-momentum ($M > 0$), the center-of-momentum (CM) frame is an inertial frame in which

$$\mathbf{p}_{\text{tot}}^* = 0. \quad (17.101)$$

In this frame, the total energy is purely the system’s invariant mass energy:

$$E_{\text{CM}} = Mc^2. \quad (17.102)$$

The CM frame simplifies collision and reaction calculations because momentum conservation often becomes geometrically transparent.

For two particles in one dimension with total laboratory energy E_{tot} and total momentum p_{tot} , the velocity of the CM frame is

$$V_{\text{CM}} = \frac{p_{\text{tot}}c^2}{E_{\text{tot}}}. \quad (17.103)$$

It follows by setting the boosted total longitudinal momentum to zero in Eq. (17.97); in three dimensions, $\mathbf{V}_{\text{CM}} = c^2\mathbf{p}_{\text{tot}}/E_{\text{tot}}$. A single photon, or a group of exactly parallel photons, has null total four-momentum and would require $|\mathbf{V}_{\text{CM}}| = c$. Such a system has no inertial CM frame.

Example 17.30: Center-of-momentum speed for two photons

Two photons travel in opposite directions with energies $3E_0$ and E_0 . Find the speed and direction of their center-of-momentum frame.

Solution. Take the higher-energy photon along $+x$. Then

$$E_{\text{tot}} = 4E_0, \quad p_{\text{tot}} = \frac{3E_0}{c} - \frac{E_0}{c} = \frac{2E_0}{c}.$$

Therefore

$$V_{\text{CM}} = \frac{p_{\text{tot}}c^2}{E_{\text{tot}}} = \frac{(2E_0/c)c^2}{4E_0} = \frac{c}{2},$$

along $+x$.

17.18.11 Threshold reactions and invariant energy

A beam’s laboratory energy is not all available to create rest mass. If the initial system has nonzero total momentum, the final system must carry that momentum as well. The center-of-momentum energy removes this compulsory bulk motion from the calculation.

For projectile a and target b , define

$$s \equiv (E_a + E_b)^2 - c^2|\mathbf{p}_a + \mathbf{p}_b|^2. \quad (17.104)$$

Here s has dimensions of energy squared; it is distinct from the spacetime interval Δs^2 , which has dimensions of length squared. The energy $\sqrt{s} = Mc^2$ is the total CM energy. For a target at rest, $E_b = m_b c^2$ and $\mathbf{p}_b = 0$, so expansion and use of $E_a^2 - p_a^2 c^2 = m_a^2 c^4$ give

$$s = m_a^2 c^4 + m_b^2 c^4 + 2m_b c^2 E_a. \quad (17.105)$$

The projectile quantity E_a is its *total* energy.

For an allowed reaction with massive final particles of specified rest masses, the minimum CM energy occurs when their relative momenta tend to zero. Thus

$$s_{\text{thr}} = \left(\sum_f m_f c^2 \right)^2. \quad (17.106)$$

This is a kinematic threshold, not a guarantee of a nonzero rate at threshold. Charge and other conservation laws must also permit the reaction. A final massless particle cannot be at rest; if its energy can become arbitrarily small, the corresponding expression describes a limiting threshold rather than a state with a stationary photon.

Example 17.31: Producing an extra particle in a fixed-target collision

A projectile and stationary target both have mass m . The final state consists of the two original particles and a new particle of mass $\mu > 0$. Assume the reaction is dynamically allowed. Determine the minimum projectile kinetic energy.

Strategy. Use the same total four-momentum invariant in the laboratory and CM frames. At threshold the three final particles share one laboratory velocity but have no relative motion in their CM frame.

$$2m^2 c^4 + 2mc^2 E_{a,\text{thr}} = (2m + \mu)^2 c^4.$$

Subtracting the projectile rest energy gives

$$K_{\text{thr}} = E_{a,\text{thr}} - mc^2 = \left(2\mu + \frac{\mu^2}{2m} \right) c^2. \quad (17.107)$$

For hypothetical masses with $mc^2 = 1.00 \text{ GeV}$ and $\mu c^2 = 0.140 \text{ GeV}$,

$$K_{\text{thr}} = 0.280 \text{ GeV} + \frac{(0.140 \text{ GeV})^2}{2(1.00 \text{ GeV})} = 0.2898 \text{ GeV} \simeq 0.290 \text{ GeV}.$$

The ratio has energy units. The negative reaction Q is $-\mu c^2$, but the laboratory threshold exceeds $-Q$: momentum conservation prevents all projectile kinetic energy from becoming additional rest energy.

Example 17.32: Collider and fixed target: comparing specified beam energies

Compare two geometries for hypothetical particles of rest energy $mc^2 = 1.00 \text{ GeV}$. In the first, one particle of total energy $E_b = 100 \text{ GeV}$ strikes an identical stationary target. In the second, two beams each have that total energy and collide head-on.

Fixed target.

$$E_{\text{CM}} = \sqrt{2m^2 c^4 + 2mc^2 E_b} = \sqrt{202} \text{ GeV} \simeq 14.2 \text{ GeV}.$$

Collider. Equal and opposite momenta cancel, giving $E_{\text{CM}} = 2E_b = 200 \text{ GeV}$.

Interpretation. This comparison holds the energy *per moving beam* fixed; the collider uses two accelerated beams. More generally, at $E_b \gg mc^2$, the fixed-target CM energy grows as $\sqrt{2mc^2 E_b}$, whereas the symmetric collider energy grows linearly with E_b . The advantage concerns available collision energy; it does not determine luminosity, cost or event rate.

17.18.12 Invariant mass reconstruction from detector measurements

Measured daughter particles can reveal a parent without observing the parent directly. If all daughter four-momenta are known, conservation gives

$$M_{\text{cand}}^2 c^4 = \left(\sum_i E_i \right)^2 - c^2 \left| \sum_i \mathbf{p}_i \right|^2. \quad (17.108)$$

The energies must correspond to the assigned daughter masses. Missing particles or incorrect assignments generally change the reconstructed mass. A collection of correct candidates may cluster around a parent mass, with spread from the decay width and measurement resolution.

For two massless daughters, let θ be the opening angle between their momentum vectors in one frame. Since $p_i = E_i/c$,

$$c^2 |\mathbf{p}_1 + \mathbf{p}_2|^2 = E_1^2 + E_2^2 + 2E_1 E_2 \cos \theta.$$

Subtracting this from $(E_1 + E_2)^2$ yields

$$M^2 c^4 = 2E_1 E_2 (1 - \cos \theta). \quad (17.109)$$

Parallel photons have zero combined invariant mass; oppositely directed photons have $Mc^2 = 2\sqrt{E_1 E_2}$. An energy sum alone cannot determine the mass because the opening angle controls the total momentum.

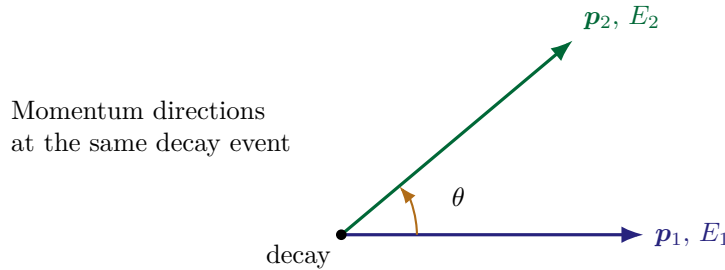


Figure 17.13: Opening-angle geometry for reconstructing a two-body invariant mass. Arrow lengths indicate directions only. Both daughter energies and the angle must be evaluated in the same frame.

Example 17.33: Reconstructing a moving parent

Two effectively massless daughters have measured energies 5.0 and 8.0 GeV and opening angle 40° , as in Fig. 17.13. Infer the parent mass, momentum magnitude and speed, assuming no unobserved daughter.

Calculation.

$$M^2 c^4 = 80(1 - \cos 40^\circ) \text{ GeV}^2 \simeq 18.7 \text{ GeV}^2, \quad Mc^2 \simeq 4.33 \text{ GeV}.$$

Keeping extra digits internally,

$$E_{\text{tot}} = 13.0 \text{ GeV}, \quad p_{\text{tot}} c = \sqrt{169 - 18.7} \text{ GeV} \simeq 12.3 \text{ GeV}.$$

Therefore $\beta_{\text{parent}} = p_{\text{tot}} c / E_{\text{tot}} \simeq 0.943$ and $\gamma_{\text{parent}} = E_{\text{tot}} / (Mc^2) \simeq 3.00$. These are illustrative values; the input precision supports about two significant figures.

Check. The invariant mass energy is below the laboratory energy because the parent moves. The energy-momentum relation reproduces the inferred mass; $\beta < 1$ as required.

At fixed daughter energies, differentiating Eq. (17.109) gives

$$\frac{\delta M}{M} \simeq \frac{1}{2} \cot\left(\frac{\theta}{2}\right) \delta\theta,$$

with the small angular error in radians. Nearly parallel daughters therefore require particularly accurate angular reconstruction. This is a geometric source of experimental sensitivity, not a change of the parent's invariant mass.

17.18.13 Relativistic acceleration and constant proper acceleration

A traveler can carry an accelerometer. Its reading is called the proper acceleration α . For one-dimensional motion,

$$\alpha = \gamma^3 a, \quad (17.110)$$

where $a = dv/dt$ is the coordinate acceleration measured in the laboratory frame.

For a positive constant proper acceleration from rest, rapidity provides a derivation. Since $v/c = \tanh \eta$, differentiation gives $\alpha = c d\eta/d\tau$. Hence $\eta = \alpha\tau/c$ with $\eta(0) = 0$. Integrating $dt/d\tau = \cosh \eta$ and $dx/d\tau = c \sinh \eta$, with $x(0) = t(0) = 0$, gives

$$ct(\tau) = \frac{c^2}{\alpha} \sinh \left(\frac{\alpha\tau}{c} \right), \quad (17.111)$$

$$x(\tau) = \frac{c^2}{\alpha} \left[\cosh \left(\frac{\alpha\tau}{c} \right) - 1 \right], \quad (17.112)$$

$$\frac{v(\tau)}{c} = \tanh \left(\frac{\alpha\tau}{c} \right). \quad (17.113)$$

The speed approaches c asymptotically but never reaches it at finite proper time. At small $\alpha\tau/c$, these expressions reduce to $t \simeq \tau$, $v \simeq \alpha t$ and $x \simeq \alpha t^2/2$. The ratio c^2/α is a length. For fixed mass in one dimension, a constant laboratory force also has $F = m\alpha$, so $p = Ft$ and $v = Ft/[m\sqrt{1 + (Ft/mc)^2}]$; the coordinate acceleration falls even while momentum increases uniformly.

Note

These equations describe accelerated motion in flat spacetime. Calling the traveler “non-inertial” does not by itself require general relativity. General relativity becomes necessary when gravitation and curved spacetime are essential to the problem.

17.18.14 Acceleration protocols and the Bell-spaceship lesson

Two spacecraft start at rest, separated by D in a laboratory, with a thread of unstressed rest length D between them. Give both spacecraft identical acceleration histories at the same laboratory times. Their laboratory separation remains D . After acceleration ends, suppose both coast at speed v . Their common final rest frame assigns the spacecraft separation

$$D_{\text{final rest}} = \gamma D.$$

The thread must span a rest-frame distance larger than its original natural length. A sufficiently fragile thread therefore breaks. Equivalently, an unstressed coasting thread of natural length D would occupy only D/γ in the laboratory, too little to connect anchors separated by D .

This argument compares the initial and final inertial stages and avoids assuming that one global inertial frame is attached to the accelerating pair. During acceleration the stress depends on elastic signal propagation and on how forces are applied. Maintaining Born-rigid longitudinal acceleration, meaning fixed local rest-frame distances, requires different proper accelerations at different positions. Identical laboratory acceleration histories do not achieve it.

Caution

Length contraction is a relation between measurements, not a mechanical squeezing force. Stress arises from the acceleration protocol and material response. A uniformly moving rod can be stress-free in its own rest frame.

17.18.15 Oblique Doppler shift, transverse geometry and aberration

A transverse Doppler formula is ambiguous until the frame defining the angle is specified. Let a source rest in S' and move at $+v\hat{x}$ in laboratory S . The source-rest frequency is f_0 . Let θ denote the angle between the photon propagation vector and $+x$ in S , and θ' the corresponding angle in S' .

The phase of a plane wave describes the same crest in either frame. Its invariance makes $K^\mu = (\omega/c, \mathbf{k})$ a four-vector, with $k = \omega/c$ in vacuum. A boost gives

$$\omega' = \gamma(\omega - vk_x) = \gamma\omega(1 - \beta \cos \theta).$$

Since $\omega' = 2\pi f_0$, the laboratory frequency is

$$f = \frac{f_0}{\gamma(1 - \beta \cos \theta)} = \gamma f_0(1 + \beta \cos \theta'). \quad (17.114)$$

The first equality uses a laboratory angle; the second uses a source-rest angle. Transforming k_x and dividing by $k' = \omega'/c$ gives

$$\cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta}, \quad \cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}. \quad (17.115)$$

These expressions recover the longitudinal redshift for photons emitted backward toward a detector behind a receding source ($\theta = \pi$), and the blueshift for forward emission ($\theta = 0$).

Example 17.34: Two different meanings of transverse emission

A source moves at $0.60c$ and has $f_0 = 500$ THz. Compare photons transverse in the laboratory with photons transverse in the source frame.

Laboratory-transverse ray. For $\theta = 90^\circ$,

$$f = \frac{f_0}{\gamma} = 400 \text{ THz}, \quad \cos \theta' = -0.60. \quad (17.116)$$

The ray is emitted backward of transverse in the source frame: $\theta' \simeq 126.9^\circ$.

Source-transverse ray. For $\theta' = 90^\circ$,

$$f = \gamma f_0 = 625 \text{ THz}, \quad \cos \theta = 0.60,$$

so $\theta \simeq 53.1^\circ$. These are different photon directions, not contradictory predictions for one measurement.

Experimental interpretation. The relevant line of sight connects the emission event to reception. A source's apparent direction on a photograph, or its position simultaneous with reception, must not be substituted without allowing for light travel time.

Aberration also explains forward beaming. A source-rest transverse ray obeys $\cos \theta = \beta$, hence $\sin \theta = 1/\gamma$. At large γ this angle is approximately $1/\gamma$. Radiation in the forward hemisphere of the source frame is mapped inside this cone. Its intensity requires additional frequency, rate and solid-angle transformations; aberration alone specifies directions. A ray's propagation direction $\mathbf{k}$ points toward the detector; the direction on the sky *toward the source* is the opposite direction.

17.18.16 Relativistic Doppler radar and lidar

A reflected electromagnetic signal undergoes two frequency transformations. Assume an ideal mirror moving at constant speed, with negligible recoil and elastic reflection in its instantaneous rest frame. For low target speed, the familiar monostatic radar result contains approximately twice the one-way fractional shift because the wave is Doppler shifted on the way to the moving target and again on reflection. At larger speeds a fully relativistic treatment is cleaner.

For an ideal mirror receding directly at speed v , the returned-to-emitted frequency ratio is

$$\frac{f_{\text{return}}}{f_{\text{emit}}} = \frac{1 - \beta}{1 + \beta}. \quad (17.117)$$

The incoming photon is redshifted into the receding mirror frame by $D = \sqrt{(1 - \beta)/(1 + \beta)}$. Reflection reverses its direction there without changing its frequency. Transformation back to the laboratory gives another factor D , so the return ratio is D^2 . The energy lost by the light is transferred to the mirror or the external apparatus that maintains its speed.

Example 17.35: Relativistic reflected-frequency shift

A mirror recedes at $0.20c$. Then

$$\frac{f_{\text{return}}}{f_{\text{emit}}} = \frac{0.80}{1.20} = 0.667.$$

The returned frequency is two-thirds of the emitted frequency. At ordinary automotive speeds, expanding Eq. (17.117) gives the usual first-order result $\Delta f/f \simeq -2v/c$.

17.18.17 Electric and magnetic fields under a Lorentz boost

Special relativity explains why electric and magnetic fields are not independent entities but different components of one electromagnetic field. For a boost with velocity $\mathbf{v}$ from S to S' , components parallel to $\mathbf{v}$ satisfy

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad (17.118)$$

while the perpendicular components are

$$\mathbf{E}'_{\perp} = \gamma (\mathbf{E} + \mathbf{v} \times \mathbf{B})_{\perp}, \quad (17.119)$$

$$\mathbf{B}'_{\perp} = \gamma \left(\mathbf{B} - \frac{\mathbf{v} \times \mathbf{E}}{c^2} \right)_{\perp}. \quad (17.120)$$

The signs correspond to the same convention used throughout this chapter: S' moves with velocity $+\mathbf{v}$ relative to S .

Example 17.36: A magnetic field appears in a moving frame

Suppose S contains a uniform electric field $\mathbf{E} = E_0 \hat{\mathbf{y}}$ and no magnetic field. Frame S' moves at $+v\hat{\mathbf{x}}$. Then

$$\mathbf{E}' = \gamma E_0 \hat{\mathbf{y}},$$

and

$$\mathbf{B}' = \gamma \left(-\frac{\mathbf{v} \times \mathbf{E}}{c^2} \right) = -\gamma \frac{vE_0}{c^2} \hat{\mathbf{z}}.$$

An observer who sees a purely electric field in one frame can therefore correspond to an observer who sees both electric and magnetic fields in another.

Key message

Magnetism is not an optional correction added to electrostatics. The Lorentz transformation mixes electric and magnetic field components. This is one of the deepest links between special relativity and Maxwell's theory.

17.18.18 Electromagnetic field invariants

Two combinations of the electric and magnetic fields are invariant under Lorentz transformations:

$$I_1 = E^2 - c^2 B^2, \quad I_2 = \mathbf{E} \cdot \mathbf{B}. \quad (17.121)$$

Their numerical values are the same in all inertial frames. These invariants can answer qualitative questions without explicitly transforming every component.

For example, if $\mathbf{E} \cdot \mathbf{B} \neq 0$ in one frame, no inertial boost can make either field vanish completely, because a frame with $\mathbf{E}' = 0$ or $\mathbf{B}' = 0$ would have $I_2' = 0$.

For a vacuum plane wave, $E = cB$ and $\mathbf{E} \perp \mathbf{B}$, so

$$I_1 = 0, \quad I_2 = 0. \quad (17.122)$$

The null character of electromagnetic radiation is therefore visible in the field invariants as well as in the wave four-vector.

Example 17.37: Can a purely magnetic frame exist?

Suppose perpendicular fields at one event satisfy $E < cB$. Choose $\mathbf{v} = \mathbf{E} \times \mathbf{B}/B^2$. Its speed is $E/B < c$, and $\mathbf{v} \times \mathbf{B} = -\mathbf{E}$, so the transformation gives $\mathbf{E}' = 0$. The remaining magnitude is $B' = \sqrt{B^2 - E^2/c^2}$. For $E > cB$, choosing $\mathbf{v} = c^2 \mathbf{E} \times \mathbf{B}/E^2$ instead gives $\mathbf{B}' = 0$. For a nonzero null field with $E = cB$ and $\mathbf{E} \perp \mathbf{B}$, either construction would require $v = c$ and is not an inertial frame. These are local statements; a single boost need not eliminate a field throughout a nonuniform region.

Example 17.38: Field transformation and force consistency

In frame S , a uniform electric field is $\mathbf{E} = E_0 \hat{\mathbf{y}}$ and $\mathbf{B} = 0$. A charged particle is instantaneously at rest in S . Frame S' moves at $+v\hat{\mathbf{x}}$. Determine the transformed fields, the particle velocity and the transverse force in S' .

The fields are

$$\mathbf{E}' = \gamma E_0 \hat{\mathbf{y}}, \quad \mathbf{B}' = -\gamma \frac{v E_0}{c^2} \hat{\mathbf{z}}.$$

The particle, at rest in S , moves at $-v\hat{\mathbf{x}}$ in S' .

The Lorentz force in S' is

$$\mathbf{F}' = q(\mathbf{E}' + \mathbf{u}' \times \mathbf{B}').$$

Now

$$(-v\hat{\mathbf{x}}) \times \left(-\gamma \frac{v E_0}{c^2} \hat{\mathbf{z}} \right) = -\gamma \frac{v^2 E_0}{c^2} \hat{\mathbf{y}},$$

so

$$\mathbf{E}' + \mathbf{u}' \times \mathbf{B}' = \gamma E_0 (1 - \beta^2) \hat{\mathbf{y}} = \frac{E_0}{\gamma} \hat{\mathbf{y}}.$$

The force components transform nontrivially, but the description is self-consistent. What was a purely electric situation in one frame involves both electric and magnetic fields in another.

17.18.19 Four-force and orthogonality to four-velocity

For a particle of fixed mass, the four-force is defined as

$$F^\mu = \frac{dP^\mu}{d\tau}. \quad (17.123)$$

Using $dE/dt = \mathbf{F} \cdot \mathbf{v}$ and $dt/d\tau = \gamma$, its components are

$$F^\mu = \gamma \left(\frac{\mathbf{F} \cdot \mathbf{v}}{c}, \mathbf{F} \right).$$

Because $P^\mu P_\mu = m^2 c^2$ is constant for a particle of fixed invariant mass,

$$\frac{d}{d\tau}(P^\mu P_\mu) = 2P_\mu F^\mu = 0. \quad (17.124)$$

Thus

$$P_\mu F^\mu = 0, \quad (17.125)$$

and since $P^\mu = mU^\mu$,

$$U_\mu F^\mu = 0. \quad (17.126)$$

The four-force is Minkowski-orthogonal to the four-velocity. This compact result encodes the familiar fact that acceleration changes the direction of a timelike four-velocity while its invariant magnitude remains c .

Pedagogical boundary

This four-force discussion is included to show the structural power of four-vectors. Detailed relativistic continuum mechanics and covariant electromagnetism are beyond the scope of this chapter.

17.18.20 GPS timing as a mixed special- and general-relativistic problem

Satellite navigation illustrates the importance of separating contributions. A moving satellite clock experiences special-relativistic time dilation relative to an idealized Earth-centered inertial frame. At the same time, its different gravitational potential changes its clock rate according to general relativity. Earth's rotation also matters when converting between inertial and Earth-fixed coordinates, and signal propagation must be modeled accurately.

Relative to the coordinate time of an ideal Earth-centered inertial frame, the kinematic rate of a satellite clock of speed $v \ll c$ is

$$\frac{\Delta\tau}{\Delta t} \simeq 1 - \frac{v^2}{2c^2}. \quad (17.127)$$

Even though v/c is tiny, modern timing precision makes the accumulated correction operationally important.

Example 17.39: Order-of-magnitude satellite timing estimate

Take a constant orbital speed $v = 4.0 \times 10^3 \text{ m s}^{-1}$ in an ideal Earth-centered inertial frame. Ignore gravity and compare the satellite proper time with that frame's coordinate time, not with a rotating surface clock. Estimate the kinematic clock-rate deficit.

Solution.

$$\frac{v^2}{2c^2} \simeq \frac{(4.0 \times 10^3)^2}{2(3.0 \times 10^8)^2} \simeq 8.9 \times 10^{-11}.$$

Over one day, $8.64 \times 10^4 \text{ s}$, this corresponds to a magnitude of order

$$(8.9 \times 10^{-11})(8.64 \times 10^4 \text{ s}) \simeq 7.7 \text{ } \mu\text{s}.$$

This is only the special-relativistic part; the gravitational contribution must also be included in an actual navigation system.

17.18.21 Ultra-relativistic expansions

At $\gamma \gg 1$,

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} \simeq 1 - \frac{1}{2\gamma^2}, \quad (17.128)$$

so

$$c - v \simeq \frac{c}{2\gamma^2}. \quad (17.129)$$

This formula explains why massive particles with enormously different energies can all have speeds extremely close to c .

For $pc \gg mc^2$,

$$E = pc \sqrt{1 + \frac{m^2 c^2}{p^2}} \simeq pc + \frac{m^2 c^3}{2p} + \dots \quad (17.130)$$

Example 17.40: How close to c at $\gamma = 1000$?

Using Eq. (17.129),

$$c - v \simeq \frac{c}{2 \times 10^6} \simeq 150 \text{ m s}^{-1}.$$

A particle with $\gamma = 1000$ is slower than light by only about 150 m s^{-1} , even though increasing its energy substantially can still increase γ by large factors.

17.18.22 Relativistic time of flight

Time-of-flight systems measure the arrival time over a known path length L . For speed $v = \beta c$,

$$t = \frac{L}{\beta c}. \quad (17.131)$$

Relative to a light signal,

$$\Delta t = \frac{L}{c} \left(\frac{1}{\beta} - 1 \right). \quad (17.132)$$

At high momentum, using $\beta = pc/E$, particles of different mass but equal momentum have slightly different arrival times. This is one basis of particle identification.

17.18.23 Time-of-flight particle identification

Suppose two particle species have the same measured momentum p but different masses. Their speeds are

$$\beta_i = \frac{pc}{\sqrt{p^2c^2 + m_i^2c^4}}. \quad (17.133)$$

Over path length L , the time difference is

$$\Delta t = \frac{L}{c} \left(\frac{1}{\beta_1} - \frac{1}{\beta_2} \right). \quad (17.134)$$

At high momentum, both β_i approach one and the separation becomes very small.

Using the high-momentum expansion

$$\frac{1}{\beta} \simeq 1 + \frac{m^2c^2}{2p^2},$$

one obtains

$$\Delta t \simeq \frac{Lc}{2p^2}(m_1^2 - m_2^2), \quad (17.135)$$

with consistent units understood. This shows explicitly why timing-based mass separation becomes harder as momentum increases.

Method

A detector often combines momentum from track curvature with speed from time of flight. Once p and β are known, the mass follows from

$$m^2c^2 = p^2 \left(\frac{1}{\beta^2} - 1 \right).$$

This is an excellent example of using two independent observables to infer a particle property.

Example 17.41: Same momentum, different time of flight

Two hypothetical particles have the same momentum $p = 1.00 \text{ GeV}/c$. Their rest energies are $m_1c^2 = 0.140 \text{ GeV}$ and $m_2c^2 = 0.494 \text{ GeV}$. Compare their times over $L = 5.00 \text{ m}$. For each species,

$$E_i = \sqrt{(pc)^2 + (m_ic^2)^2}, \quad \beta_i = \frac{pc}{E_i}.$$

Thus

$$E_1 \simeq \sqrt{1.000 + 0.0196} = 1.0098 \text{ GeV}, \quad \beta_1 \simeq 0.9903,$$

and

$$E_2 \simeq \sqrt{1.000 + 0.244} = 1.115 \text{ GeV}, \quad \beta_2 \simeq 0.897.$$

The flight times are

$$t_i = \frac{L}{\beta_i c}.$$

Using $L/c \simeq 16.68 \text{ ns}$,

$$t_1 \simeq 16.84 \text{ ns}, \quad t_2 \simeq 18.60 \text{ ns}.$$

The difference is about 1.76 ns , large enough in principle for a fast timing detector to distinguish the species statistically.

17.18.24 Cherenkov-threshold connection

A charged particle can emit Cherenkov radiation in a material medium when its speed exceeds the phase speed of light in that medium, c/n . This does *not* violate relativity because the invariant limiting speed is the vacuum speed c , not c/n .

For a transparent approximately isotropic medium at rest in the laboratory, with refractive index $n(\omega) > 1$, the threshold at a selected frequency is

$$\beta > \frac{1}{n}. \quad (17.136)$$

When emission occurs, the idealized Cherenkov angle satisfies

$$\cos \theta_C = \frac{1}{n\beta}. \quad (17.137)$$

Thus a measured angle can provide information about β and, when combined with momentum, about particle identity.

Example 17.42: Cherenkov threshold

For a medium with refractive index $n = 1.50$, the threshold is $\beta_{\text{thr}} = 1/n = 0.667$. The corresponding Lorentz factor is

$$\gamma_{\text{thr}} = \frac{1}{\sqrt{1 - 1/n^2}} \simeq 1.34.$$

A particle can therefore move faster than light propagates in this medium while remaining safely below c .

17.18.25 Uncertainty propagation for relativistic observables

Suppose a measured momentum p has uncertainty σ_p and a particle mass hypothesis m is treated as exact. Since

$$E(p) = \sqrt{p^2 c^2 + m^2 c^4}, \quad (17.138)$$

first-order uncertainty propagation gives

$$\sigma_E \simeq \left| \frac{\partial E}{\partial p} \right| \sigma_p = \frac{pc^2}{E} \sigma_p = v \sigma_p. \quad (17.139)$$

The derivative $\partial E / \partial p = v$ is itself a useful physical relation.

For a Doppler measurement with receding-source ratio

$$R \equiv \frac{f}{f_0} = \sqrt{\frac{1 - \beta}{1 + \beta}},$$

one can solve

$$\beta = \frac{1 - R^2}{1 + R^2}. \quad (17.140)$$

This form is often more convenient for extracting velocity from data.

Example 17.43: Velocity inferred from a relativistic Doppler ratio

A receding source is measured at $f/f_0 = 0.80$. Then

$$\beta = \frac{1 - 0.80^2}{1 + 0.80^2} = \frac{0.36}{1.64} \simeq 0.220.$$

A purely first-order estimate, $\Delta f/f_0 \simeq -\beta$, would give $\beta \simeq 0.20$ and would already be noticeably less accurate.

Example 17.44: Uncertainty near the light-speed limit

A speed measurement gives $\beta = 0.9900 \pm 0.0005$. Estimate the uncertainty in γ by first-order propagation.

At $\beta = 0.9900$,

$$\gamma = \frac{1}{\sqrt{1 - 0.9900^2}} \simeq 7.09.$$

Using $d\gamma/d\beta = \beta\gamma^3$,

$$\sigma_\gamma \simeq (0.990)(7.09)^3(0.0005) \simeq 0.177.$$

Thus $\gamma \simeq 7.09 \pm 0.18$ from this linear estimate. A tiny absolute uncertainty in β produces a few-percent uncertainty in γ .

17.18.26 Spectroscopy and propagated uncertainty

Suppose a laboratory transition has known proper frequency f_0 , and a moving source produces a line at measured frequency f . A robust analysis proceeds as follows:

1. determine whether the geometry is longitudinal, transverse or oblique;
2. correct instrumental calibration and identify the rest-frame line;
3. use the appropriate relativistic Doppler relation;
4. infer β and propagate the frequency uncertainty;
5. check whether the low-speed approximation would have been adequate.

For pure longitudinal recession,

$$R = \frac{f}{f_0}, \quad \beta = \frac{1 - R^2}{1 + R^2}.$$

The derivative needed for uncertainty propagation is

$$\frac{d\beta}{dR} = -\frac{4R}{(1 + R^2)^2}. \quad (17.141)$$

Hence $\sigma_\beta \simeq |d\beta/dR|\sigma_R$ for a small uncertainty.

Example 17.45: Doppler uncertainty estimate

If $R = 0.900 \pm 0.001$, then

$$\beta = \frac{1 - 0.81}{1 + 0.81} = 0.10497.$$

The derivative magnitude is

$$\frac{4(0.900)}{(1 + 0.81)^2} \simeq 1.10,$$

so $\sigma_\beta \simeq 0.00110$ if the uncertainty in R dominates.

17.18.27 Momentum spread and the decay distribution

A beam is rarely perfectly monoenergetic. Real particles occupy distributions of momentum and energy. For a narrow distribution, the mean decay length can be approximated using a representative $\beta\gamma$. For a broad distribution, the survival probability should instead be averaged over the momentum spectrum:

$$P(x) = \int dp f(p) \exp\left[-\frac{x}{\beta(p)\gamma(p)c\tau_0}\right], \quad (17.142)$$

where $p > 0$ is the momentum magnitude, $f(p)$ is normalized to unity, and energy loss and acceptance effects have been neglected.

This equation illustrates an important modeling principle: nonlinear relativistic functions should not generally be evaluated only at the mean momentum when the distribution is broad. A full treatment of relativistic statistical mechanics is outside the present scope.

17.18.28 Monte Carlo model for an unstable relativistic beam

A simple Monte Carlo simulation can generate decay distances. Draw a uniform random number r in $(0, 1)$ and use the inverse exponential distribution

$$x = -\ell \ln r, \quad \ell = \beta\gamma c\tau_0. \quad (17.143)$$

Repeated samples form an exponential histogram whose fitted scale should approach ℓ .

A more realistic exercise can draw momenta from a chosen distribution $f(p)$, calculate a different $\ell(p)$ for each particle, and then generate x . Students can compare the resulting distribution with a single exponential and observe how momentum spread distorts the simple model.

Method

When validating a Monte Carlo model, compare generated sample means and survival fractions with analytic expectations in a limit where the answer is known. Random-looking output is not evidence of a correct simulation.

17.18.29 Unstable beams in a storage ring

A circulating unstable-particle beam may survive for many laboratory revolutions because its proper lifetime is dilated. If the ring circumference is C and the particle speed is βc , one revolution takes

$$T_{\text{rev}} = \frac{C}{\beta c}. \quad (17.144)$$

Assuming constant speed and the clock hypothesis for the circulating particle, the mean path length divided by the circumference (allowing a final fractional turn) is

$$N_{\text{rev}} = \frac{\gamma\tau_0}{T_{\text{rev}}} = \frac{\beta\gamma c\tau_0}{C} = \frac{\ell}{C}. \quad (17.145)$$

The same decay length that governs a straight detector therefore controls beam storage in a ring.

Example 17.46: Mean revolutions of an unstable beam

Take $C = 300$ m and laboratory mean decay length $\ell = 12$ km. Then

$$N_{\text{rev}} \simeq \frac{12\,000}{300} = 40.$$

The particle can circle the machine dozens of times on average even if its proper lifetime is microscopically short.

17.18.30 A computational worldline integrator

For sampled motion $v(t_i)$ over small laboratory intervals Δt_i , the proper time can be approximated by

$$\Delta\tau \simeq \sum_i \Delta t_i \sqrt{1 - \frac{v_i^2}{c^2}}. \quad (17.146)$$

A simple algorithm is

```
tau = 0
for i in range(N-1):
    dt = t[i+1] - t[i]
    v = velocity_at_midpoint(i)
    tau += dt*sqrt(1 - (v/c)**2)
```

Students can compare different velocity histories that start and finish at the same spacetime events. This makes the geometric character of differential aging concrete without requiring advanced mathematics.

Numerical convergence check

Repeat the calculation with twice as many time steps. If the trajectory interpolation is well behaved, the estimated $\Delta\tau$ should converge. Reporting only a numerical answer without a step-size test is poor computational practice.

17.18.31 Advanced mini-project: Lorentz-transform an event cloud

Construct a synthetic dataset of events in one spatial dimension. Include:

1. a stationary clock worldline $x = 0$;
2. a moving clock worldline $x = 0.7ct$;
3. two simultaneous events separated by a known Δx ;
4. a right-moving and left-moving light ray;
5. a timelike decay chain with three ordered events.

Transform the full event set to frames with $\beta = -0.8, -0.4, 0, 0.4, 0.8$. For each transformed set:

- verify interval invariance for selected event pairs;
- identify which pairs change simultaneity order;
- verify that transformed light-ray velocities dx'/dt' remain $\pm c$;

- compute the proper time on the straight inertial segment connecting a timelike pair, distinguishing it from proper time on a different path;
- plot ct versus x with equal axis scales.

A short technical report should separate numerical output from interpretation and should state the sign convention for every boost.

17.18.32 Advanced synthesis: one geometry, many formulas

The main formulas of special relativity can be organized around three related structures:

1. **Spacetime interval:** governs simultaneity, time dilation, length contraction and causal classification.
2. **Four-momentum norm:** governs relativistic energy, momentum, invariant mass and reaction thresholds.
3. **Lorentz transformations of physical quantities:** four-velocity gives velocity composition, the wave four-vector gives Doppler shift and aberration, and the electromagnetic field tensor mixes electric and magnetic components. The fields themselves are not a single four-vector.

This organization is more powerful than memorizing separate correction formulas because it provides a strategy for unfamiliar problems.

Table 17.3: Useful choices of relativistic tools.

Problem type	Efficient first tool
Two event coordinates	Lorentz transformation or interval
Elapsed clock time	Proper-time condition and $\Delta\tau$
Moving object length	Simultaneous endpoint events in measurement frame
High-speed composition	Velocity transformation or rapidity
Light frequency/direction	Wave four-vector, Doppler and aberration
Particle energy/momentum	$E^2 - p^2c^2 = m^2c^4$
Reaction threshold	Total four-momentum invariant s
Detector decay distance	$\ell = \beta\gamma c\tau_0$
Track curvature	$p = q Br$
Changing velocity history	Proper-time integral

17.18.33 Advanced guided checks

Guided checks

1. Why can the temporal order of spacelike-separated events reverse without violating causality?
2. What property of the Lorentz transformation is analogous to a Euclidean rotation, and what important sign difference changes the geometry?
3. Why does rapidity add linearly while ordinary velocity does not?
4. How can a system of massless particles have nonzero invariant mass?
5. Why is the center-of-momentum energy the maximum energy available for creating rest mass and internal excitation?

6. In a fixed-target collision, where does most of the very large beam energy go from the CM point of view?
7. Why is $\partial E/\partial p = v$ physically reasonable in both nonrelativistic and ultra-relativistic limits?
8. How do electric and magnetic fields demonstrate that relativity is built into electromagnetism?
9. Why does a charged particle exceeding c/n in a medium not violate the relativistic speed limit?
10. Which invariant checks would you implement first in a relativity simulation?

Take-home message

At an advanced level, special relativity is best understood as Lorentz symmetry of spacetime. Time dilation, length contraction, Doppler shift, relativistic momentum, threshold energies and electromagnetic field transformations are different projections of one common structure. Invariants are the bridge between observers: coordinates change, but spacetime intervals and four-vector scalar products do not.

17.19 Integrated applications and case studies

These case studies combine measurement procedures and conservation laws. The pole-and-barn problem develops the core event method; the remaining topics use the advanced material where indicated.

17.19.1 The pole-and-barn scenario as an event calculation

A moving pole can fit inside a shorter barn at one barn-frame instant, although the barn is even shorter when measured in the pole frame. To resolve this, compare the door events rather than arguing about which object's length is “really” contracted. Figure 17.14 specifies a geometry with enough clearance to avoid contact.

Example 17.47: Door closure, clearance and simultaneity

A pole of proper length 10.0 m coasts at $0.800c$ through an 8.0 m barn. Let the entrance be $x = 0$, the exit $x = 8.0$ m, and at $t = 0$ let the rear and front of the pole lie at $x = 1.0$ and 7.0 m. The doors close briefly and simultaneously at $t = 0$ in the barn frame and reopen before contact. Compare the descriptions.

Barn-frame geometry. With $\gamma = 5/3$, the moving pole has length $10.0/\gamma = 6.0$ m. The stated positions leave 1.0 m clearance at each end. The front reaches the exit after $1.0/(0.800c) \simeq 4.17$ ns, so a sufficiently brief closure is possible in this idealized model. No instantaneous stopping of the pole is assumed.

Transform the same two door events. Write $A = (t, x) = (0, 0)$ and $B = (0, 8.0 \text{ m})$. In the pole frame,

$$\Delta t' = -\gamma \frac{v(8.0 \text{ m})}{c^2} \simeq -35.6 \text{ ns}, \quad \Delta x' = \gamma(8.0 \text{ m}) = 13.3 \text{ m}.$$

The exit closes earlier than the entrance. The value 13.3 m is the spatial separation of these *non-simultaneous closure events*, not the barn's length. A simultaneous pole-frame measurement gives the barn length $8.0/\gamma = 4.8$ m.

Resolution. Both frames agree on all local passages and any contact. They disagree about whether spatially separated door events are simultaneous. The pole is never permanently trapped without additional dynamics; finite elastic signals would matter if it were stopped.

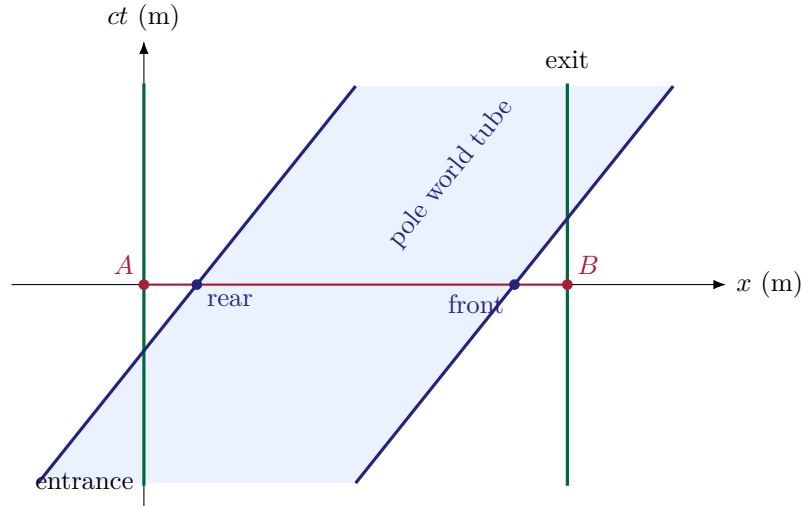


Figure 17.14: Barn-frame spacetime geometry. The blue boundaries are the pole endpoints $x_{\text{rear}} = 1 + 0.8ct$ and $x_{\text{front}} = 7 + 0.8ct$, with coordinates in metres. The red segment joins the two simultaneous door-closure events. It intersects the pole over a length of 6.0 m.

17.19.2 Tutorial 4: Fizeau-type light propagation in a moving medium

Special relativity also clarifies why light propagation in a moving transparent medium does not obey naive Galilean addition. In the idealized collinear, nondispersive model, let the light propagation speed in the medium rest frame be

$$u' = \frac{c}{n},$$

and let the medium move at speed v along the light direction relative to the laboratory. The laboratory speed is

$$u = \frac{c/n + v}{1 + v/(nc)}. \quad (17.147)$$

For $v \ll c$, expand to first order:

$$u \simeq \left(\frac{c}{n} + v \right) \left(1 - \frac{v}{nc} \right) \quad (17.148)$$

$$\simeq \frac{c}{n} + v \left(1 - \frac{1}{n^2} \right). \quad (17.149)$$

The coefficient $1 - 1/n^2$ is the nondispersive first-order drag factor. In a dispersive medium, specify phase versus group speed and account for the frequency dependence of n ; the simple fixed- n expression is then not a complete experimental model. Relativistic velocity addition therefore reproduces an important optical result in the low-speed limit.

Example 17.48: Moving water estimate

Take $n = 1.33$ and water moving at $v = 10 \text{ m s}^{-1}$. The first-order change in laboratory light speed is

$$\Delta u \simeq v \left(1 - \frac{1}{1.33^2} \right) \simeq 4.35 \text{ m s}^{-1}.$$

The effect is tiny relative to c/n but measurable interferometrically with sufficient precision.

17.19.3 Apparent superluminal motion without superluminal transport

Apparent transverse motion inferred from angular displacement and arrival times can exceed c ; relativistic jets provide a physical setting for this geometry. This does not mean matter or information travels faster than light. The effect arises because the emitting material moves nearly toward the observer, so light emitted later has a shorter distance to travel.

In flat spacetime, let a compact source move at constant speed $v = \beta c$ at angle θ to the direction toward a distant observer. Approximate the photon paths as parallel and neglect cosmological expansion over the modelled region. In laboratory time Δt , its transverse displacement is

$$\Delta x_{\perp} = v \Delta t \sin \theta.$$

The arrival-time separation of the corresponding photons is reduced to

$$\Delta t_{\text{arr}} = \Delta t (1 - \beta \cos \theta).$$

Thus the apparent transverse speed is

$$v_{\text{app}} = \frac{\beta c \sin \theta}{1 - \beta \cos \theta}. \quad (17.150)$$

This quantity can exceed c because it is not a local signal speed; it is a ratio inferred from angular displacement and photon-arrival timing.

Example 17.49: An apparent speed greater than c

Take $\beta = 0.99$ and $\theta = 10^\circ$. Then

$$\frac{v_{\text{app}}}{c} = \frac{0.99 \sin 10^\circ}{1 - 0.99 \cos 10^\circ} \simeq 6.87.$$

The jet material itself still moves at $0.99c$.

17.19.4 Why one photon cannot create a massive pair in empty space

A single photon has four-momentum satisfying

$$P_{\gamma}^2 = 0.$$

Suppose it could convert in empty space into an electron–positron pair. The final two-particle system would have invariant mass at least $2m_e$, so

$$P_{\text{final}}^2 c^2 \geq (2m_e c^2)^2 > 0.$$

Four-momentum conservation would require $P_{\gamma} = P_{\text{final}}$ and therefore equal invariants, which is impossible.

Pair production can occur in the electromagnetic field of a nucleus, which can recoil and absorb momentum, or through collisions of two photons when the combined photon system has sufficient invariant mass. The argument is purely relativistic kinematics; the probability and microscopic mechanism are quantum questions.

Key message

Energy conservation alone is not enough in relativistic reactions. Momentum conservation and invariant mass can forbid a process even when a naive energy budget appears sufficient.

17.19.5 Electron–positron annihilation at rest

Consider an electron and positron initially at rest in their center-of-momentum frame. The initial total momentum is zero and total energy is $2m_e c^2$. A two-photon final state can conserve both quantities if the photons travel back-to-back with equal energies:

$$E_{\gamma 1} = E_{\gamma 2} = m_e c^2. \quad (17.151)$$

With $m_e c^2 \simeq 0.511$ MeV, each photon has energy about 0.511 MeV.

If the pair has kinetic energy before annihilation, the photon energies and angles depend on the chosen frame, but total four-momentum remains conserved. This is a clean bridge from relativistic energy accounting to photon physics.

17.19.6 Relativistic beam power and particle rate

A beam containing N particles per second, each with kinetic energy K , carries kinetic beam power

$$P_{\text{beam}} = NK. \quad (17.152)$$

This is not the total rest-energy flow unless the application specifically requires that quantity. For accelerator engineering, deposited power and radiation load often depend on kinetic or total particle energy and on how much of it is absorbed.

Example 17.50: Beam power from particle energy

A beam delivers 2.0×10^{12} electrons per second, each with kinetic energy 10 MeV. Since

$$10 \text{ MeV} = 10^7 (1.602 \times 10^{-19}) \text{ J} = 1.602 \times 10^{-12} \text{ J},$$

the kinetic beam power is

$$P = (2.0 \times 10^{12})(1.602 \times 10^{-12}) \text{ W} \simeq 3.20 \text{ W}.$$

The particles are individually highly relativistic, yet the average beam power can still be only a few watts if the particle rate is modest.

17.19.7 Invariant mass and missing energy

Suppose a reaction begins with known initial four-momentum P_{init} but only some final particles are measured. Define the missing four-momentum

$$P_{\text{miss}} = P_{\text{init}} - \sum_{\text{measured}} P_i. \quad (17.153)$$

The missing invariant mass satisfies

$$M_{\text{miss}}^2 c^2 = P_{\text{miss}}^2. \quad (17.154)$$

This is a powerful general method in particle and nuclear physics. It can reveal an unobserved neutral particle or test whether an event is kinematically consistent with a hypothesized final state.

The method is only as reliable as the initial-state knowledge and detector calibration. Measurement errors can produce unphysical reconstructed values such as slightly negative M_{miss}^2 ; these must be interpreted statistically rather than by forcing every event to satisfy an exact idealization.

17.20 Original practice problems

Unless stated otherwise, use $c = 2.998 \times 10^8 \text{ m s}^{-1}$. Label all frames and identify the relevant events before calculating. The final group uses the enrichment section; the numerical invariance exercise also draws on it. Values explicitly described as teaching data are illustrative rather than historical measurements.

Conceptual questions

1. **Inertial laboratories.** Two sealed laboratories drift past one another at constant relative velocity. A student claims that a sufficiently accurate internal mechanical experiment can determine which laboratory is “really moving.” Identify the principle that rules this out and explain what the experiment can determine instead.
2. **Events versus observations.** A flash occurs 900 m from an observer. Distinguish the time at which the flash is assigned to occur in the observer’s synchronized reference frame from the later time at which light from the flash arrives at the observer.
3. **Simultaneity.** Two spatially separated lamps flash simultaneously in frame S . Frame S' moves in the $+x$ direction. Without calculation, explain why S' generally assigns different times to the flashes. Which flash occurs earlier in S' if the second lamp has the larger x coordinate?
4. **Proper time.** A spacecraft passes Earth and later passes a distant beacon. Earth and the beacon are at rest with respect to each other. The spacecraft carries one clock; Earth and the beacon each carry synchronized clocks. Which clock arrangement measures the proper time between the two passing events?
5. **Proper length.** A train has rest length L_0 . Explain why measuring the moving train’s front at noon and its rear one second later does not determine its length in the station frame.
6. **Reciprocal time dilation.** Observer A says B’s moving clock runs slow. B says A’s moving clock runs slow. Explain why the statements are not contradictory for inertial motion.
7. **Transverse dimensions.** A cube moves rapidly along one of its edges. Which dimensions are Lorentz contracted in the laboratory frame? What happens to its transverse dimensions?
8. **Doppler shift.** A source recedes from a detector. Is the detected frequency larger or smaller than the proper emission frequency? How does your answer change for an approaching source?
9. **Energy saturation misconception.** A massive particle’s speed approaches c . Does its kinetic energy approach a finite maximum? Explain using γ .
10. **Rest energy.** Explain the difference among rest energy, kinetic energy and total energy. For a particle at rest, which of these are zero?

Analytical and derivation problems

11. **Lorentz-factor inversion.** Starting from $\gamma = (1 - \beta^2)^{-1/2}$, derive $\beta = \sqrt{1 - \gamma^{-2}}$. Find the speed parameter for $\gamma = 3$.

12. **Low-speed expansion.** Use the binomial theorem to derive $\gamma \simeq 1 + \beta^2/2 + 3\beta^4/8$ through fourth order. Use it to derive the first relativistic correction to Newtonian kinetic energy.
13. **Light-clock derivation.** A light clock has mirror separation D perpendicular to its direction of motion. Derive the time-dilation relation by comparing the light path in the rest frame of the clock and in a frame where the clock moves at speed v .
14. **Length contraction from Lorentz transformations.** Consider a rod at rest in S' with endpoints $x'_1 = 0$ and $x'_2 = L_0$. Use simultaneous endpoint measurements in S to derive $L = L_0/\gamma$.
15. **Relativity of simultaneity.** Two events have $\Delta t = 0$ and separation $\Delta x = L$ in S . Derive their time separation in S' moving at $+v$.
16. **Frame of simultaneity.** Two events have separations Δx and Δt with $|c\Delta t| < |\Delta x|$. Derive the velocity of the inertial frame in which they are simultaneous and show that its magnitude is less than c .
17. **Frame of co-location.** For timelike-separated events, derive the velocity of the frame in which the two events occur at the same position.
18. **Velocity transformation.** Differentiate the Lorentz transformations to derive $u'_x = (u_x - v)/(1 - u_x v/c^2)$.
19. **Transverse velocity.** Starting from $y' = y$ and the Lorentz time transformation, derive $u'_y = u_y/[\gamma(1 - u_x v/c^2)]$.
20. **Doppler inversion.** For a receding source, use $R = f/f_0 = \sqrt{(1 - \beta)/(1 + \beta)}$ to derive $\beta = (1 - R^2)/(1 + R^2)$.
21. **Energy from work.** Starting with $\mathbf{p} = \gamma m \mathbf{v}$ and $dK = \mathbf{v} \cdot d\mathbf{p}$, show that $K = (\gamma - 1)mc^2$.
22. **Energy–momentum invariant.** Use $E = \gamma mc^2$ and $p = \gamma mv$ to derive $E^2 - p^2 c^2 = m^2 c^4$.

Numerical, graphical and computational problems

23. **Reading quantitative plots.** Using Fig. 17.9, estimate γ and $K/(mc^2)$ at $\beta = 0.8$. Calculate the exact values and explain why their difference is one. Sketch γ as a function of β beyond the plotted endpoint without drawing a finite value at $\beta = 1$.
24. **Newtonian kinetic-energy error.** Compute the fractional underestimate $(K_{\text{rel}} - K_{\text{N}})/K_{\text{rel}}$ for $\beta = 0.1, 0.2, 0.3, 0.5, 0.8$. At what listed speed does the Newtonian formula first differ by more than 5% from the relativistic result?
25. **Muon flight.** A muon has proper mean lifetime $2.20 \mu\text{s}$ and speed $0.995c$. Compute its laboratory mean lifetime and mean decay length.
26. **Spacecraft clocks.** A spacecraft travels 18.0 light-years in the Earth frame at $0.90c$. Compute the trip duration according to Earth clocks and spacecraft clocks. What distance does the spacecraft crew assign to the route?
27. **Two boosts.** Frame S' moves at $0.70c$ relative to S , and an object moves at $0.80c$ relative to S' in the same direction. Find its speed in S . Repeat for motion opposite the S' velocity.
28. **Relativistic redshift.** A spectral line has proper wavelength 500 nm. It is emitted by a source receding at $0.40c$. Find the detected wavelength.

29. **Electron energy.** Using $m_e c^2 = 0.511$ MeV, find total energy, kinetic energy and momentum magnitude in MeV/ c for an electron at $0.95c$.
30. **Event-cloud code.** Write a program that generates 20 events along $x = 0.50ct$ for $0 \leq t \leq 10$ ns and transforms them to frames with $\beta = 0.2, 0.5, 0.8$. Verify numerically that the transformed points remain on a straight inertial worldline.
31. **Testing a boost numerically.** Generate 100 random event separations and transform them at $\beta = 0.75$. Add the test separation $(c\Delta t, \Delta x) = (1000, 1000)$ m. Report the interval residual normalized to $(c\Delta t)^2 + \Delta x^2$ for each nonzero input, and test that the inverse boost returns the original components. Why is dividing by $|\Delta s^2|$ unreliable near a null interval? Why can a transformation with the wrong sign of v pass the interval test?

Engineering and measurement problems

32. **Beam rigidity.** A singly charged particle has momentum 4.00 GeV/ c . Find the magnetic rigidity $B\rho$ in Tm. What uniform magnetic field is required for a bending radius of 2.50 m?
33. **Detector decay acceptance.** An unstable particle has $\beta\gamma = 8.0$ and proper lifetime 0.50 ns. What fraction survives to a detector layer 0.80 m from the production point?
34. **Timing resolution and its convention.** Two particles are emitted simultaneously and traverse 4.0 m with speeds $0.92c$ and $0.97c$. Find their arrival-time separation. If each arrival has independent Gaussian error with the same standard deviation σ_t , require a separation of at least $3\sigma_{\text{difference}}$. Find the largest allowed σ_t . Compare it with a criterion that instead uses $3\sigma_t$. Neglect path-length and start-time uncertainty.
35. **Satellite timing.** Estimate the special-relativistic time loss accumulated over one day relative to ideal Earth-centered inertial coordinate time, for a satellite moving at 3.9 km s^{-1} . Use the low-speed approximation and clearly state that gravitational effects are excluded.
36. **Does one lifetime describe two momentum bins?** In an idealized teaching experiment, independent narrow beam samples have $\beta\gamma = 3.00$ and 6.00 , treated as exact. Their efficiency-corrected mean decay lengths are (1.78 ± 0.09) m and (3.62 ± 0.12) m. Infer both proper lifetimes and uncertainties. Quantify the difference in units of its combined standard deviation. Explain why agreement tests the boost dependence more strongly than a single long decay length.
37. **Cherenkov threshold.** A transparent medium has refractive index $n = 1.33$. Find the threshold β and γ for Cherenkov emission. Explain why this does not imply a particle exceeds the vacuum speed of light.
38. **Electron-beam accelerator.** An electron beam has kinetic energy 5.00 MeV. Using $m_e c^2 = 0.511$ MeV, find γ , β and momentum in MeV/ c .
39. **Precision clock transport.** A clock moves at 250 ms^{-1} for 12 h. Estimate the special-relativistic time difference relative to an ideal stationary clock using the leading low-speed term. Is the effect in seconds, microseconds, nanoseconds or picoseconds?

Advanced challenges

40. **Invariant two-photon mass.** Two photons have energies E_1 and E_2 and meet at angle θ . Show that the invariant mass of the pair satisfies

$$M^2 c^4 = 2E_1 E_2 (1 - \cos \theta).$$

Discuss the limits $\theta = 0$ and $\theta = \pi$.

41. **Fixed-target threshold.** A projectile of mass m strikes an identical target at rest and produces an additional particle of mass μ while both original particles remain. Derive the projectile threshold total energy using invariant s .
42. **Worldline extremum.** Two events are timelike separated. Compare a direct inertial path between them with a piecewise inertial path that changes velocity at an intermediate event. Use the proper-time integral or a numerical example to show that the direct inertial path has the larger proper time.
43. **Simultaneity reversal.** Construct explicit numerical values for two spacelike-separated events such that event A occurs before B in S but after B in S' . Verify that no signal at or below c can connect them.
44. **Field transformation.** In frame S , $\mathbf{B} = B_0 \hat{z}$ and $\mathbf{E} = 0$. Frame S' moves at $+v \hat{x}$. Determine $\mathbf{E}'$ and $\mathbf{B}'$. Check dimensions and directions, and verify $E'^2 - c^2 B'^2 = -c^2 B_0^2$.
45. **Constant proper acceleration.** Starting from the parametrization in Eq. (17.113), show that $c^2 t^2 - (x + c^2/\alpha)^2 = -c^4/\alpha^2$. Interpret the resulting hyperbola in a spacetime diagram.
46. **High- γ time of flight.** Derive the approximation

$$\Delta t \simeq \frac{L}{2c\gamma^2}$$

for the delay of an ultra-relativistic massive particle relative to light over distance L . State the assumptions.

47. **CM energy of unequal photon beams.** Two counter-propagating photon beams have photon energies E_1 and E_2 . For $E_1, E_2 > 0$, find the center-of-momentum speed relative to the laboratory and the invariant mass energy per photon pair.
48. **Design synthesis.** You must distinguish two unstable charged particle species with the same momentum but different masses. Propose a conceptual detector strategy using at least two of the following: track curvature, time of flight, decay length, Cherenkov angle, calorimetric energy. Explain what each observable measures and which relativistic equations connect the measurements.

17.21 Selected solutions to representative problems

These solutions emphasize why the chosen method applies. Attempt each problem first. Numerical results below use adequate intermediate precision and are rounded at the end.

Solution to Problem 11: Lorentz-factor inversion

Starting from

$$\gamma^{-2} = 1 - \beta^2,$$

we obtain

$$\beta^2 = 1 - \gamma^{-2}, \quad \beta = \sqrt{1 - \gamma^{-2}}$$

for a positive speed. For $\gamma = 3$,

$$\beta = \sqrt{1 - \frac{1}{9}} = \sqrt{\frac{8}{9}} = 0.943.$$

Solution to Problem 15: relativity of simultaneity

For simultaneous events in S , $\Delta t = 0$. The time-separation transformation gives

$$\Delta t' = \gamma \left(0 - \frac{vL}{c^2} \right) = -\gamma \frac{vL}{c^2}.$$

If $v > 0$ and the second event has larger x , then $\Delta t' < 0$: the larger- x event is earlier in S' under the adopted convention.

Solution to Problem 24: Newtonian kinetic-energy error

For each β compute

$$k_{\text{rel}} \equiv \frac{K_{\text{rel}}}{mc^2} = \gamma - 1, \quad k_{\text{N}} \equiv \frac{K_{\text{N}}}{mc^2} = \frac{1}{2}\beta^2,$$

and then

$$\epsilon = \frac{k_{\text{rel}} - k_{\text{N}}}{k_{\text{rel}}}.$$

Representative values are:

β	k_{rel}	k_{N}	ϵ
0.10	0.00504	0.00500	0.75%
0.20	0.02062	0.02000	3.0%
0.30	0.04828	0.04500	6.8%
0.50	0.15470	0.12500	19.2%
0.80	0.66667	0.32000	52.0%

The first listed speed exceeding a five-percent error is $0.30c$.

Solution to Problem 27: two boosts

For forward motion,

$$\frac{u}{c} = \frac{0.80 + 0.70}{1 + (0.80)(0.70)} = \frac{1.50}{1.56} = 0.962.$$

For the object moving opposite the S' velocity, take $u' = -0.80c$:

$$\frac{u}{c} = \frac{-0.80 + 0.70}{1 - (0.80)(0.70)} = \frac{-0.10}{0.44} = -0.227.$$

The sign shows that the object still moves in the negative direction in S , but much more slowly than in S' .

Solution to Problem 33: detector decay acceptance

The mean decay length is

$$\ell = \beta\gamma c\tau_0 = 8.0(3.00 \times 10^8)(0.50 \times 10^{-9}) \text{ m} = 1.20 \text{ m}.$$

The survival fraction at $x = 0.80 \text{ m}$ is

$$P = \exp(-x/\ell) = \exp(-0.80/1.20) = 0.513.$$

Solution to Problem 34: timing resolution

The slower particle arrives later by

$$\Delta t = \frac{4.0 \text{ m}}{c} \left(\frac{1}{0.92} - \frac{1}{0.97} \right) \simeq 0.748 \text{ ns.}$$

The difference of two independent arrival times has variance $2\sigma_t^2$. Requiring $\Delta t \geq 3\sqrt{2}\sigma_t$ gives

$$\sigma_t \leq \frac{0.748}{3\sqrt{2}} \text{ ns} \simeq 0.176 \text{ ns.}$$

A convention based on separation divided by the single-arrival width would instead give $\sigma_t \leq 0.249 \text{ ns}$. Neither criterion is meaningful until its width convention is specified. Shared start-time errors can partly cancel in a difference; they do not necessarily contribute as independent errors.

Solution to Problem 35: satellite timing

At low speed,

$$\Delta\tau \simeq \Delta t \left(1 - \frac{v^2}{2c^2} \right),$$

so the moving clock loses approximately

$$\delta t \simeq \Delta t \frac{v^2}{2c^2}.$$

With $v = 3.9 \times 10^3 \text{ m s}^{-1}$ and $\Delta t = 86400 \text{ s}$,

$$\delta t \simeq 86400 \frac{(3.9 \times 10^3)^2}{2(2.998 \times 10^8)^2} \simeq 7.31 \text{ } \mu\text{s}.$$

This is not the net GPS correction because gravitational effects have intentionally been omitted.

Solution to Problem 36: two momentum bins

For each bin use $\tau_0 = \ell/[(\beta\gamma)c]$, with $\sigma_{\tau_0} = \sigma_\ell/[(\beta\gamma)c]$ under the stated assumptions:

$$\tau_{01} = (1.98 \pm 0.10) \text{ ns}, \quad \tau_{02} = (2.01 \pm 0.07) \text{ ns}.$$

Using unrounded values, the difference is 0.0334 ns and its standard deviation is

$$\sqrt{(0.100)^2 + (0.0667)^2} \text{ ns} \simeq 0.120 \text{ ns}.$$

The discrepancy is only 0.28 standard deviations. The doubling of the boost accompanies an approximately doubled decay length, while the inferred proper lifetime is unchanged within this model's uncertainties. Momentum calibration, backgrounds and acceptance would need independent control in a real test.

Solution to Problem 37: Cherenkov threshold

The threshold is

$$\beta_{\text{thr}} = \frac{1}{n} = \frac{1}{1.33} = 0.752.$$

Then

$$\gamma_{\text{thr}} = \frac{1}{\sqrt{1 - \beta_{\text{thr}}^2}} \simeq 1.52.$$

The particle exceeds the light phase speed c/n in the medium, not the vacuum invariant speed c .

Solution to Problem 40: two-photon invariant mass

Choose photon 1 along a reference direction and let photon 2 make angle θ . The total energy is $E_1 + E_2$. Since $p_i = E_i/c$,

$$|\mathbf{p}_1 + \mathbf{p}_2|^2 = \frac{E_1^2 + E_2^2 + 2E_1E_2 \cos \theta}{c^2}.$$

Therefore

$$\begin{aligned} M^2 c^4 &= (E_1 + E_2)^2 - c^2 |\mathbf{p}_1 + \mathbf{p}_2|^2 \\ &= E_1^2 + E_2^2 + 2E_1E_2 - E_1^2 - E_2^2 - 2E_1E_2 \cos \theta \\ &= 2E_1E_2(1 - \cos \theta). \end{aligned}$$

For parallel photons $\theta = 0$ and $M = 0$; for opposite photons $\theta = \pi$, $Mc^2 = 2\sqrt{E_1E_2}$.

Solution to Problem 42: maximum proper time

Choose the rest frame of the inertial worldline that joins the two timelike endpoints. Its elapsed time is T and its position is constant. For a competing path that leaves and returns to that position,

$$\tau = \int_0^T \sqrt{1 - v(t)^2/c^2} dt \leq T.$$

A path with nonzero speed for a nonzero duration gives strict inequality. For example, a symmetric out-and-back path at $0.60c$ for a total coordinate duration 10 y has $\tau = 8$ y; the straight path has 10 y. A smooth turnaround changes the integral continuously. This comparison concerns two different clocks that reunite, not two coordinate descriptions of one clock's inertial ticks.

Solution to Problem 46: ultra-relativistic time-of-flight delay

From

$$t = \frac{L}{\beta c}, \quad t_\gamma = \frac{L}{c},$$

the delay is

$$\Delta t = \frac{L}{c} \left(\frac{1}{\beta} - 1 \right).$$

For $\gamma \gg 1$, $\beta \simeq 1 - 1/(2\gamma^2)$. To leading order,

$$\frac{1}{\beta} \simeq 1 + \frac{1}{2\gamma^2},$$

so

$$\Delta t \simeq \frac{L}{2c\gamma^2}.$$

The approximation assumes a highly relativistic massive particle and neglects terms of order γ^{-4} .

17.22 Bridge to quantum and modern physics

Special relativity changes the kinematics of particles; quantum physics changes the framework used to describe microscopic states and measurements. The two theories meet immediately in modern physics. A photon satisfies $E = pc$ and, in quantum theory, $E = hf$ and $p = h/\lambda$. Relativistic energy and momentum are therefore already present when photons are introduced.

The next modern-physics topics also depend on relativistic ideas in less obvious ways. Particle creation requires enough invariant energy; atomic spectra observed from moving sources are Doppler shifted; unstable-particle lifetimes are frame-dependent while proper lifetime is invariant; and high-energy scattering is naturally formulated with four-momentum conservation.

At the same time, this chapter has deliberately avoided quantum concepts in deriving Lorentz transformations. Special relativity is a classical spacetime theory. Quantum mechanics and quantum field theory add new principles; they do not replace Lorentz symmetry in their relativistic domains.

Key message

The conceptual bridge is therefore two-way: relativity supplies the kinematic and symmetry structure for modern high-energy physics, while quantum theory supplies the microscopic description of particles, radiation and measurement. Their combination is essential for particle physics but begins with the careful event and energy–momentum language developed here.

Appendix A

Notation dictionary

Symbol	Meaning	Typical SI unit
c	speed of light in vacuum	m s^{-1}
v	relative speed of reference frames	m s^{-1}
β	speed parameter v/c	dimensionless
γ	Lorentz factor $(1 - \beta^2)^{-1/2}$	dimensionless
S, S'	inertial reference frames	—
t, t'	event time coordinates	s
x, y, z	event spatial coordinates	m
$\Delta\tau$	proper-time interval	s
L_0	proper or rest length	m
L	longitudinal moving-frame length	m
u, u'	object velocities in different frames	m s^{-1}
f_0	proper/source frequency where specified	Hz
f	detected frequency	Hz
λ_0	source/rest-frame wavelength where specified	m
$\mathbf{p}$	relativistic three-momentum	kg m s^{-1}
m	invariant mass of a particle	kg
E_0	rest energy mc^2	J or eV
K	kinetic energy	J or eV
E	total relativistic energy	J or eV
Δs^2	invariant spacetime interval squared	m^2
X^μ	four-position $(ct, \mathbf{r})$	m
U^μ	four-velocity	m s^{-1}
P^μ	four-momentum $(E/c, \mathbf{p})$	kg m s^{-1}
M	invariant mass of a complete system	kg
η	rapidity	dimensionless
τ_0	proper mean lifetime of unstable particle	s
ℓ	laboratory mean decay length	m
$B\rho$	magnetic rigidity	T m
α	proper acceleration	m s^{-2}

Appendix B

Compact formula sheet

This sheet collects equations with their measurement conditions. The symbol γ always refers to the velocity in the equation being used: a frame boost, a particle velocity, or a specified clock motion.

Kinematics

For constant-speed clock motion and a coasting rod parallel to the boost; the moving length uses simultaneous endpoints:

$$\begin{aligned}\beta &= v/c, & \gamma &= (1 - \beta^2)^{-1/2}, \\ \Delta t &= \gamma \Delta \tau, & L &= L_0/\gamma.\end{aligned}$$

Lorentz transformation: S' moves at $+v\hat{x}$

$$\begin{aligned}x' &= \gamma(x - vt), & t' &= \gamma\left(t - \frac{vx}{c^2}\right), \\ y' &= y, & z' &= z, \\ x &= \gamma(x' + vt'), & t &= \gamma\left(t' + \frac{vx'}{c^2}\right).\end{aligned}$$

Velocity transformation

$$\begin{aligned}u'_x &= \frac{u_x - v}{1 - u_x v/c^2}, \\ u'_y &= \frac{u_y}{\gamma(1 - u_x v/c^2)}, \\ u'_z &= \frac{u_z}{\gamma(1 - u_x v/c^2)}.\end{aligned}$$

Inverse formulas follow by $v \rightarrow -v$ and interchange of primed and unprimed quantities.

Relativistic Doppler effect for collinear recession

$$\begin{aligned}f &= f_0 \sqrt{\frac{1 - \beta}{1 + \beta}}, \\ \lambda &= \lambda_0 \sqrt{\frac{1 + \beta}{1 - \beta}}.\end{aligned}$$

For approach, reverse the signs of β in the ratios.

Momentum and energy

For a free massive particle of fixed invariant mass; the last relation also applies to a massless particle, for which $E = pc$:

$$\begin{aligned}\mathbf{p} &= \gamma m \mathbf{v}, \\ E_0 &= mc^2, \\ K &= (\gamma - 1)mc^2, \\ E &= \gamma mc^2 = K + mc^2, \\ E^2 &= p^2 c^2 + m^2 c^4.\end{aligned}$$

Spacetime and four-vectors

$$\begin{aligned}\Delta s^2 &= c^2 \Delta t^2 - |\Delta \mathbf{r}|^2, \\ d\tau &= dt \sqrt{1 - v^2/c^2}, \\ U^\mu &= \gamma(c, \mathbf{v}), \\ P^\mu &= (E/c, \mathbf{p}), \\ M^2 c^4 &= E_{\text{tot}}^2 - p_{\text{tot}}^2 c^2.\end{aligned}$$

Useful applications

The lifetime formulas use a fixed boost and exponential decay. The bending formula assumes momentum perpendicular to a uniform magnetic field and negligible losses. A CM frame requires timelike total four-momentum. The Cherenkov formula uses the medium rest frame and its phase index at the measured frequency.

$$\begin{aligned}\ell &= \beta \gamma c \tau_0, \\ P(x) &= \exp(-x/\ell), \\ p &= |q| B r, \\ V_{\text{CM}} &= \frac{p_{\text{tot}} c^2}{E_{\text{tot}}}, \\ \cos \theta_C &= \frac{1}{n\beta} \quad (n\beta > 1).\end{aligned}$$

Low- and high-speed expansions

$$\begin{aligned}\gamma &\simeq 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \dots \quad (\beta \ll 1), \\ K &\simeq \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \dots, \\ \beta &\simeq 1 - \frac{1}{2\gamma^2} \quad (\gamma \gg 1).\end{aligned}$$

Bibliography

- [1] D. Halliday, R. Resnick and J. Walker, *Fundamentals of Physics*, Vol. 2, 12th ed., Chapter 37, especially Sections 37.1–37.6: simultaneity, the full light-clock tick, Lorentz transformations, Doppler shift, momentum and energy.
- [2] R. A. Serway and J. W. Jewett, *Physics for Scientists and Engineers with Modern Physics*, supplied edition, Chapter 39, especially Sections 39.4–39.8: event-based time dilation, the pole-and-barn construction, velocity transformations and relativistic dynamics.
- [3] S. J. Ling, J. Sanny and W. Moebs, *University Physics*, OpenStax. The supplied Volume 2 provides the electromagnetic background in Chapter 16; relativity itself is in Volume 3, Chapter 5. The online Sections 5.5 and 5.9 provide alternative derivations and applications:
[Section 5.5: The Lorentz Transformation](#); [Section 5.9: Relativistic Energy](#).
- [4] I. Neutelings, “Minkowski diagrams & Lorentz transformations,” TikZ.net, consulted for diagram implementation and coordinate-projection techniques. The figures here use independently specified geometry and equations:
https://tikz.net/relativity_minkowski_diagram/