

Chapter 18

Photons, Matter Waves, and the Birth of Quantum Physics

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Preface

Classical electromagnetism, optics, and special relativity remain essential tools: fields propagate, waves interfere, and energy and momentum are conserved. Yet thermal radiation, photoemission, Compton scattering, and electron diffraction require a new description of the exchange of energy and the outcomes of measurements. This chapter builds that description from experiments before introducing its mathematical structure.

Radiation exchanges energy in quanta $E = hf$ and carries momentum $p = h/\lambda$. A material particle with momentum p has de Broglie wavelength $\lambda = h/p$. These relations connect quantities that can be measured; the slogan “wave–particle duality” alone does not predict an outcome. Probability amplitudes, the Born rule, and the Schrödinger equation provide the more precise language needed for interference, reflection, and tunneling.

The chapter is written for technical-physics, engineering, computer-science, space-science, and applied-science students. It assumes calculus, practical complex numbers, elementary differential equations, and energy–momentum conservation. Each new mathematical tool is introduced where it acquires a physical purpose. Photodetectors, electron diffraction, tunneling instruments, and numerical wave packets connect the foundations to measurement and design.

How to study the chapter. Read the main discussion in order, attempt the guided checks before continuing, and use the worked examples to identify a model before substituting numbers. The optional enrichment develops the mathematical details and engineering applications; the final problem groups identify where that additional material is needed. Selected solutions support independent study, and the formula sheet records the conditions attached to the equations.

The Guideline

Before calculating, identify the physical preparation, the quantity to be predicted or measured, and the model and approximations. A formula such as $\lambda = h/p$ or $K_{\max} = hf - \phi$ becomes useful only after these choices are clear.

Pedagogical boundary

The main discussion introduces nonrelativistic quantum states while retaining relativistic photon kinematics. Enrichment covers packets, current, scattering, coherence, counting, and computation. Detailed bound-state spectra belong to Chapter 19; spin, atoms, lasers, bands, nuclei, and particles are developed later in the volume.

The supplied university texts informed the coverage, teaching sequence, and visual comparisons. Explanations and drawings have been independently developed and revised; the final source note identifies the references and their scope.

Conventions and notation

We use SI units, with electronvolts for microscopic energies. The positive elementary charge magnitude is $e = 1.602176634 \times 10^{-19}$ C, and

$$h = 6.62607015 \times 10^{-34} \text{ J s}, \quad \hbar = \frac{h}{2\pi}, \quad 1 \text{ eV} = 1.602176634 \times 10^{-19} \text{ J}.$$

The stated values of h and e are exact SI definitions. Frequency and angular frequency obey $\omega = 2\pi f$; wave number is $k = 2\pi/\lambda$. Photon wavelengths are vacuum wavelengths unless stated otherwise.

For massive particles, $p = mv$ and $K = p^2/(2m)$ require $v \ll c$. When relativistic kinematics are needed, $E_{\text{tot}}^2 = p^2c^2 + m^2c^4$ and $K = E_{\text{tot}} - mc^2$. In the nonrelativistic Schrödinger equation, E denotes mechanical energy excluding rest energy.

The complete wave function is $\Psi(x, t)$. A stationary state has $\Psi = \psi(x)e^{-iEt/\hbar}$. Its probability density and interval probability are

$$\rho_P = |\Psi|^2, \quad P(a < x < b) = \int_a^b |\Psi|^2 dx, \quad \int_{-\infty}^{\infty} |\Psi|^2 dx = 1.$$

The last equality applies to normalized localized states. Ideal stationary scattering waves instead use incident-flux normalization. In one dimension, Ψ has units $\text{m}^{-1/2}$ and ρ_P has units m^{-1} .

Note

$V(x)$ denotes potential *energy*; V_s and V_{acc} denote voltages. For an electron, the electrostatic contribution is $V(x) = -e\varphi_{\text{el}}(x)$. Uncertainties Δx and Δp_x are standard deviations for identically prepared states, with $\Delta x \Delta p_x \geq \hbar/2$. They are distinct from instrument resolution. Order-of-magnitude estimates use $\sim$ or $\gtrsim$ explicitly.

Symbol	Meaning	Typical unit
$h, \hbar$	Planck constant, reduced Planck constant	J s
f, ω	frequency, angular frequency	Hz, rad s^{-1}
λ, k	wavelength, wave number	m, m^{-1}
E_γ, p_γ	photon energy and momentum	J, kg m s^{-1}
ϕ	work function	eV or J
V_s, V_{acc}	stopping magnitude, accelerating voltage	V
$\lambda_C = h/(m_e c)$	electron Compton wavelength	m
Ψ, ψ	complete and spatial wave functions	$\text{m}^{-1/2}$ in 1D
$V(x)$	potential energy	J or eV
T, R	transmitted and reflected flux fractions	dimensionless

Chapter 18

Photons, Matter Waves, and the Birth of Quantum Physics

18.1 Motivation and roadmap

By the end of classical optics, light has acquired an impressive wave description. Interference, diffraction, polarization, reflection, and refraction can all be organized through electromagnetic waves. It would therefore be natural to expect that increasingly careful experiments should merely refine that picture. Instead, microscopic experiments reveal a more subtle structure. The energy transferred by light to matter does not always behave as a continuously divisible wave quantity, while particles such as electrons can produce diffraction patterns that are impossible to explain by classical point-particle trajectories alone.

The central task is therefore not to choose once and for all between a classical wave picture and a classical particle picture. Quantum theory replaces both with a new framework. Classical wave concepts such as wavelength, phase, superposition, and interference survive, but their interpretation changes. Classical particle concepts such as energy, momentum, and localized detection events also survive, but trajectories are no longer the fundamental description between preparation and measurement.

The route through the chapter is experimental: photon energy and momentum lead to matter-wave diffraction, probability amplitudes, and the Schrödinger equation. Steps and barriers then connect that framework to instruments. The optional enrichment develops the derivations and measurement models in greater depth.

Learning goals

After studying this chapter, the student should be able to:

- explain why blackbody radiation, the photoelectric effect, Compton scattering, and electron diffraction motivate quantum physics;
- use $E = hf$ and $p = h/\lambda = E/c$ for photons and state the domain of validity of each relation;
- solve photoelectric-effect problems involving threshold frequency, work function, maximum kinetic energy, and stopping potential;
- derive and use the Compton wavelength-shift formula for scattering from an approximately free electron initially at rest;
- calculate de Broglie wavelengths for nonrelativistic and, where needed, relativistic particles;
- distinguish a quantum wave function Ψ from a classical mechanical or electromagnetic wave;
- interpret $|\Psi|^2$ as a probability density and normalize simple one-dimensional wave functions;
- state the time-dependent and time-independent Schrödinger equations for one nonrelativistic particle in one dimension;

- explain the position–momentum uncertainty relation in terms of state spreads rather than poor apparatus;
- describe quantum reflection at a potential step and tunneling through a finite barrier;
- connect these ideas to photodetectors, electron microscopy, scanning tunneling microscopy, tunnel devices, and numerical quantum simulations.

Key message

Quantum physics is not a patchwork rule saying “use waves here and particles there.” It is a predictive framework in which states carry probability amplitudes, observables have probability distributions, and experimental detections occur as localized outcomes with quantized energy and momentum transfer.

18.1.1 What prior knowledge is needed?

Students should be comfortable with conservation of energy and momentum, sinusoidal waves, wavelength and frequency, interference and diffraction, work and potential energy, exponential functions, and elementary calculus. The photon relations also connect directly to the special-relativistic energy–momentum relation from the preceding modern-physics material. In particular, setting the rest mass to zero in

$$E^2 = p^2 c^2 + m^2 c^4 \quad (18.1)$$

gives $E = pc$ for a photon in vacuum.

Guided checks

1. If light behaves as a wave in a double-slit experiment, does that logically require the energy delivered to a detector to be continuously divisible? Explain.
2. If an electron produces a localized spot on a screen, does that logically imply that it followed one classical trajectory through the apparatus? Explain why the conclusion is stronger than the observation.
3. Which conservation laws from classical mechanics remain essential in Compton scattering?

18.2 Why classical physics needed an extension

Classical physics is extraordinarily successful when action variables are large compared with h and when coherence at microscopic scales is unimportant. A satellite orbit, a bridge vibration, a radio antenna, and geometrical-optics imaging can all be treated without quantum mechanics to excellent accuracy. The failure that motivated quantum theory was therefore not a general collapse of classical reasoning. It was a mismatch between specific microscopic observations and specific classical predictions.

Three tensions are especially useful for this chapter. First, classical thermal-radiation theory assigns too much high-frequency energy to the modes of electromagnetic radiation in equilibrium. Second, a purely classical wave model of light predicts that energy transfer to an electron should depend primarily on incident intensity and exposure time, whereas the photoelectric effect exhibits a threshold frequency and an electron energy controlled by frequency. Third, a purely classical particle model of electrons predicts localized impacts but not diffraction and interference patterns.

The successful quantum response was to introduce discreteness in energy exchange and a probability-amplitude description of matter. Planck’s energy element hf and Einstein’s photon picture solve

the first two difficulties at a conceptual level. De Broglie's relation $\lambda = h/p$ extends wave concepts to matter. Schrödinger's equation then gives a dynamical rule for nonrelativistic wave functions.

Caution

Do not infer that classical physics is “wrong.” Classical mechanics and classical electromagnetism are limiting theories with enormous domains of validity. Quantum mechanics must reproduce classical behavior in appropriate macroscopic or short-wavelength limits. A good modern theory explains both the new phenomena and why the older theory worked so well where it did.

18.2.1 A useful scale argument involving Planck's constant

Because h is extremely small in SI units, de Broglie wavelengths of macroscopic objects are usually minute. Consider a 0.15 kg ball moving at 20 m s⁻¹. Its momentum is 3.0 kg m s⁻¹, giving

$$\lambda = \frac{h}{p} \approx 2.2 \times 10^{-34} \text{ m}. \quad (18.2)$$

No realistic slit system could resolve interference on this scale. For an electron with momentum around 10⁻²⁴ kg m s⁻¹, however, wavelengths are near atomic length scales and diffraction is experimentally accessible.

Example 18.1: Macroscopic and microscopic de Broglie wavelengths

A dust particle of mass $m_d = 1.0 \times 10^{-12}$ kg moves at $v_d = 1.0$ m s⁻¹. An electron moves nonrelativistically at $v_e = 1.0 \times 10^6$ m s⁻¹. Compare their de Broglie wavelengths. For the dust particle,

$$\lambda_d = \frac{h}{m_d v_d} = \frac{6.626 \times 10^{-34}}{10^{-12} \times 1.0} \approx 6.6 \times 10^{-22} \text{ m}.$$

For the electron,

$$\lambda_e = \frac{h}{m_e v_e} \approx \frac{6.626 \times 10^{-34}}{(9.109 \times 10^{-31})(10^6)} \approx 7.27 \times 10^{-10} \text{ m}.$$

The electron wavelength is comparable with interatomic dimensions, whereas the dust-particle wavelength is roughly six orders of magnitude smaller than a typical nuclear length of 10⁻¹⁵ m. The calculation does not say that the dust particle has no quantum state; it says that wave effects associated with its centre-of-mass motion are extraordinarily difficult to resolve in this preparation.

18.3 Thermal radiation and Planck's quantum hypothesis

Any body at nonzero thermodynamic temperature exchanges electromagnetic radiation with its surroundings. An ideal blackbody absorbs all incident radiation and, in thermal equilibrium, emits a spectrum determined only by its temperature. Experimentally, the spectral distribution is continuous in wavelength, but its shape changes systematically with temperature: hotter bodies radiate more total power and the peak shifts toward shorter wavelengths.

Classical electromagnetic theory correctly provides the mode structure of a cavity, but classical equipartition assigns an average energy of order $k_B T$ to each mode. Because the number of electromagnetic modes rises rapidly with frequency, this classical procedure predicts an unbounded high-frequency energy density. The contradiction is not a small numerical discrepancy; it reveals

that the classical statistical assumption cannot be applied without modification to electromagnetic modes.

Planck's successful hypothesis was that the energy exchanged by an oscillator of frequency f occurs in discrete amounts built from

$$\varepsilon = hf. \quad (18.3)$$

In modern language, the equilibrium mode energy follows from quantum statistics. For the present chapter, the key conceptual point is that the energy scale associated with a frequency is proportional to f , not independent of it.

Figure 18.1 compares calculated spectra and the classical long-wavelength approximation.

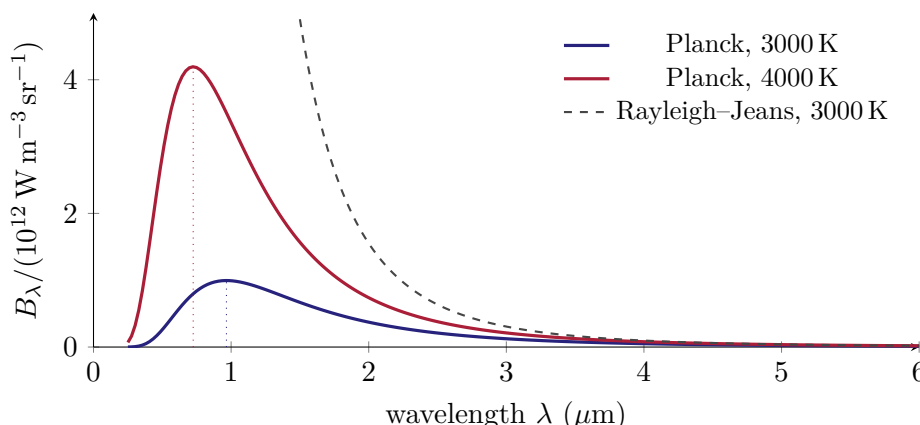


Figure 18.1: Calculated blackbody spectral radiance per unit wavelength, using Planck's law. The hotter spectrum lies above the cooler one at every wavelength and peaks at a shorter wavelength. The classical curve approaches the quantum result only at long wavelength and diverges toward short wavelength. Its out-of-range part is omitted.

A useful empirical relation is Wien's displacement law,

$$\lambda_{\max} T = b, \quad b \approx 2.898 \times 10^{-3} \text{ m K}, \quad (18.4)$$

which makes the peak shift quantitative. We will derive neither Wien's law nor the full Planck spectrum in the main chapter; the advanced section returns to the spectral formula and limiting behavior.

Example 18.2: Temperature from a thermal-radiation peak

A heated surface has a spectral maximum near $\lambda_{\max} = 1.00 \mu\text{m}$. Estimate its absolute temperature using Wien's law.

$$T = \frac{b}{\lambda_{\max}} = \frac{2.898 \times 10^{-3} \text{ m K}}{1.00 \times 10^{-6} \text{ m}} \approx 2.90 \times 10^3 \text{ K}.$$

This is an order-of-magnitude surface-temperature estimate. Real materials are not perfect blackbodies and may have wavelength-dependent emissivity, so precision thermometry requires calibration and a more detailed radiation model.

Guided checks

1. Why does an infinite number of high-frequency modes cause difficulty if every mode receives an energy of order $k_B T$?
2. Does Planck quantization imply that the wavelength spectrum itself becomes a set of

- discrete lines? Explain why not.
3. A hotter blackbody has a shorter $\lambda_{\max}$. Does that mean it emits no long-wavelength radiation? Explain.

18.3.1 A spectrum is a density, not a single photon energy

A spectral curve answers a different question from $E_\gamma = hc/\lambda$: how much radiant power is present in each wavelength interval? Define the spectral radiance B_λ so that the power emitted from projected area $dA \cos \vartheta$ into solid angle $d\Omega$ and wavelength interval $d\lambda$ is

$$dP = B_\lambda(T) \cos \vartheta dA d\Omega d\lambda. \quad (18.5)$$

Here ϑ is measured from the surface normal. For an ideal blackbody,

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp[hc/(\lambda k_B T)] - 1}, \quad [B_\lambda] = \text{W m}^{-2} \text{sr}^{-1} \text{m}^{-1}. \quad (18.6)$$

The final inverse metre refers to wavelength bandwidth. A graph quoted per micrometre has numerical values 10^{-6} times those quoted per metre. Integrating radiance over an emitting hemisphere gives spectral exitance $M_\lambda = \pi B_\lambda$ for this isotropic emitter; integrating again over wavelength gives $M = \sigma_{\text{SB}} T^4$. This connects the quantum spectrum with the thermal-radiation law from Physics I.

Planck's formula has two useful physical checks. At every fixed wavelength the radiance rises with temperature: ideal spectra at different temperatures cannot cross. At sufficiently short wavelength, the exponential outweighs the factor λ^{-5} and suppresses the spectrum. A hotter source still emits at long wavelengths; its maximum is only the location where power per unit wavelength is greatest.

Example 18.3: Net radiative exchange near room temperature

An idealized black surface of area $A = 0.020 \text{ m}^2$ is maintained at $T = 310 \text{ K}$ inside a much larger black enclosure at $T_0 = 293 \text{ K}$. Estimate its emitted power and its net radiative heat loss. Neglect convection and conduction.

Strategy. Emission depends on the surface temperature, while net loss subtracts radiation absorbed from the enclosure. With $\sigma_{\text{SB}} \simeq 5.67 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4}$,

$$P_{\text{emit}} = A\sigma_{\text{SB}}T^4 \simeq 10.47 \text{ W},$$

$$P_{\text{net}} = A\sigma_{\text{SB}}(T^4 - T_0^4) \simeq 2.12 \text{ W}.$$

Wien's law gives $\lambda_{\max} \simeq 9.35 \mu\text{m}$, in the infrared. The net heat loss is much smaller than the emitted power because the surroundings radiate back. It vanishes when $T = T_0$. A real surface requires emissivity and viewing geometry; the calculation is a radiation-only model, not a complete heat budget.

Caution

The maximum of a spectrum per unit wavelength is not obtained by taking c divided by the maximum frequency of a spectrum per unit frequency. Densities transform with a Jacobian: $B_f = B_\lambda |d\lambda/df| = B_\lambda c/f^2$. The advanced section derives both peak conditions. Wien's constant $2.898 \times 10^{-3} \text{ m K}$ applies to B_λ .

18.4 Photons: quantized energy and momentum

Einstein extended the quantization idea from oscillators to electromagnetic radiation itself. In processes where light exchanges energy with matter, it is useful to regard the radiation as consisting of photons. Each photon of frequency f carries

$$E_\gamma = hf = \hbar\omega. \quad (18.7)$$

For light in vacuum, $c = f\lambda$, and therefore

$$E_\gamma = \frac{hc}{\lambda}. \quad (18.8)$$

Because a photon has zero rest mass, relativistic energy–momentum conservation gives

$$p_\gamma = \frac{E_\gamma}{c} = \frac{h}{\lambda}. \quad (18.9)$$

The phrase “photon energy” refers to the energy transferred in an individual quantum event. The intensity of a nearly monochromatic beam depends on both the energy per photon and the photon arrival rate. Increasing intensity at fixed frequency usually means increasing the average number of photons crossing an area per unit time, not increasing the energy of each photon.

Definition 18.1: Photon

A photon is a quantum of electromagnetic radiation. In vacuum it has energy $E_\gamma = hf$, momentum magnitude $p_\gamma = E_\gamma/c = h/\lambda$, zero electric charge, and zero rest mass. A photon should not be pictured as a classical charged ball or as a tiny packet that follows a Newtonian trajectory between measurements.

Example 18.4: Photon energy and photon rate of a laser beam

A monochromatic laser of wavelength $\lambda = 532 \text{ nm}$ delivers an average power $P = 2.00 \text{ mW}$. Estimate the photon energy and the average number of photons emitted per second. The photon energy is

$$\begin{aligned} E_\gamma &= \frac{hc}{\lambda} \\ &\approx \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{532 \times 10^{-9}} \text{ J} \\ &\approx 3.73 \times 10^{-19} \text{ J} \approx 2.33 \text{ eV}. \end{aligned}$$

If $\dot{N}$ is the average photon rate, $P = \dot{N}E_\gamma$, so

$$\dot{N} = \frac{P}{E_\gamma} \approx \frac{2.00 \times 10^{-3}}{3.73 \times 10^{-19}} \approx 5.36 \times 10^{15} \text{ s}^{-1}.$$

The large number explains why ordinary optical beams often appear continuous even though individual detection events are quantized.

Method

For photon-conversion problems, the relation

$$E_\gamma(\text{eV}) \approx \frac{1240 \text{ eV nm}}{\lambda(\text{nm})}$$

is a useful check. Derive it once from hc rather than treating 1240 as an unexplained constant. For formal derivations and uncertainty work, use SI quantities consistently and convert to electronvolts only at the end.

18.4.1 Photon momentum and radiation pressure

Photon momentum is experimentally important even when the energy transfer is small. If a beam is completely absorbed, the momentum delivered per photon is $p_\gamma = E_\gamma/c$. For specular reflection at normal incidence, the normal momentum change is approximately twice as large. In classical electromagnetism the same momentum transport is described by the electromagnetic stress and by radiation pressure. Quantum and classical descriptions agree in the regime of many photons.

For a beam of intensity I incident normally on an ideal absorbing surface,

$$P_{\text{rad}} = \frac{I}{c}, \quad (18.10)$$

while an ideal perfectly reflecting surface has $P_{\text{rad}} = 2I/c$. These relations are not new quantum laws; they are the macroscopic consequence of momentum transfer and can also be derived classically.

Caution

Never use $p = mv$ for a photon. A photon has no rest frame and no nonzero rest mass to insert into that nonrelativistic formula. Use $p = E/c = h/\lambda$ in vacuum.

Guided checks

1. Two laser beams have equal power, but one is red and one is violet. Which beam has more photons per second, assuming both are monochromatic? Explain without calculation.
2. If the intensity of monochromatic light doubles while its frequency stays fixed, what happens to the energy of each photon?
3. Why can radiation pressure be described consistently by both Maxwell's theory and photon momentum?

18.5 The photoelectric effect

When light illuminates a suitable material, electrons may be emitted from its surface. A simple photoelectric apparatus contains an illuminated emitter, a collector, a variable potential difference, and a current meter. Classical wave intuition suggests that sufficiently intense light should eventually supply enough energy to eject electrons at any frequency. The observed behavior is different.

Three features are central. First, for a given material there is a threshold frequency f_0 below which the ideal one-photon model predicts no photoemission. This statement assumes modest intensity and negligible thermal or field-assisted emission. Second, above threshold, the maximum kinetic energy of the emitted electrons rises approximately linearly with frequency. Third, changing the intensity at fixed frequency mainly changes the number of emitted electrons and therefore the photocurrent; it does not set the maximum kinetic energy in the way a classical continuous-energy-transfer picture would suggest.

Einstein's photon model organizes the maximum emission energy through a one-photon balance:

$$hf = \phi + K_{\text{max}}. \quad (18.11)$$

where ϕ is the work function, the minimum energy required to remove an electron from the material to vacuum. Because emitted electrons may originate in different microscopic states and may undergo

energy-losing collisions before escape, their kinetic energies form a distribution. The largest value is

$$K_{\max} = hf - \phi. \quad (18.12)$$

The threshold condition $K_{\max} = 0$ gives

$$f_0 = \frac{\phi}{h}, \quad \lambda_0 = \frac{hc}{\phi}. \quad (18.13)$$

The polarity and series current measurement are shown in Fig. 18.2.

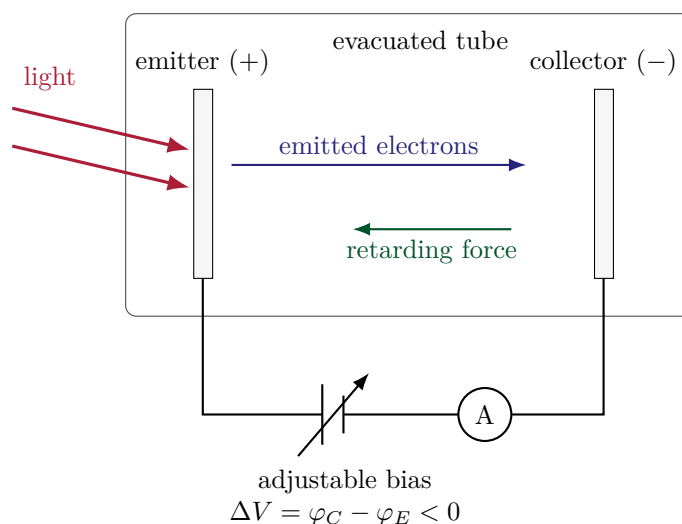


Figure 18.2: Photoelectric measurement in the retarding configuration. The ammeter is connected in series, and the collector is negative relative to the emitter. The arrows indicate emission and retarding force, not classical electron trajectories. The stopping condition is $\Delta V = -V_s$.

To measure $K_{\max}$ electrically, the collector is made negative relative to the emitter so that electrons are decelerated. At the stopping-potential magnitude V_s , even the most energetic photoelectrons fail to reach the collector. Then

$$K_{\max} = eV_s, \quad (18.14)$$

where e is the positive magnitude of the electron charge. Combining with Eq. (18.12),

$$eV_s = hf - \phi. \quad (18.15)$$

Thus a graph of V_s versus f is linear with slope h/e and intercept $-\phi/e$ in the idealized model.

Example 18.5: Photoelectron energy, stopping voltage, and speed

Light of wavelength 250 nm illuminates a material with work function $\phi = 2.20$ eV. Find the maximum kinetic energy, the stopping-potential magnitude, and the maximum electron speed using a nonrelativistic approximation.

The photon energy is

$$E_\gamma \approx \frac{1240 \text{ eV nm}}{250 \text{ nm}} = 4.96 \text{ eV}.$$

Therefore

$$K_{\max} = 4.96 - 2.20 = 2.76 \text{ eV}.$$

Since $K_{\max} = eV_s$, the numerical value in electronvolts equals the voltage magnitude in volts:

$$V_s = 2.76 \text{ V}.$$

For the speed,

$$v_{\max} = \sqrt{\frac{2K_{\max}}{m_e}} = \sqrt{\frac{2(2.76)(1.602 \times 10^{-19})}{9.109 \times 10^{-31}}} \approx 9.85 \times 10^5 \text{ m s}^{-1}.$$

This is only about $0.0033c$, so the nonrelativistic speed formula is fully adequate.

Example 18.6: Work function and Planck constant from two data points

Suppose an idealized experiment gives $V_s = 0.40 \text{ V}$ at $f = 6.0 \times 10^{14} \text{ Hz}$ and $V_s = 1.23 \text{ V}$ at $f = 8.0 \times 10^{14} \text{ Hz}$. Estimate h and ϕ from the two points.

From $V_s = (h/e)f - \phi/e$, the slope is

$$\frac{\Delta V_s}{\Delta f} = \frac{0.83}{2.0 \times 10^{14}} = 4.15 \times 10^{-15} \text{ V s}.$$

Hence

$$h = e \frac{\Delta V_s}{\Delta f} \approx (1.602 \times 10^{-19})(4.15 \times 10^{-15}) \approx 6.65 \times 10^{-34} \text{ J s},$$

close to the accepted value. Using the first data point,

$$\frac{\phi}{e} = \frac{h}{e}f - V_s \approx (4.15 \times 10^{-15})(6.0 \times 10^{14}) - 0.40 \approx 2.09 \text{ V}.$$

Thus $\phi \approx 2.09 \text{ eV}$. Real experiments use many points and a linear regression with uncertainties rather than two-point arithmetic.

Key message

In the photoelectric effect, frequency controls the energy available per photon, whereas intensity at fixed frequency primarily controls the rate at which photons arrive. This distinction is the conceptual heart of the experiment.

Caution

A stopping potential is not the same as the accelerating potential that may have been used elsewhere to create an electron beam. In photoelectric measurements it is a retarding voltage chosen so that the photocurrent falls to zero. Use the magnitude $V_s > 0$ in $K_{\max} = eV_s$ unless a signed circuit convention is explicitly defined.

Guided checks

1. Light is below threshold but extremely intense. Why does the simple one-photon photoelectric model still predict no emitted electrons?
2. Light is above threshold and its intensity doubles at fixed frequency. What changes in an ideal photoelectric experiment: $K_{\max}$, V_s , or the saturation current?
3. Why is $K_{\max}$ used instead of claiming that every photoelectron emerges with the same kinetic energy?
4. On a graph of V_s versus f , what experimental quantities determine the slope and intercept?

18.5.1 Reading current–voltage curves

It helps to separate two measurements. At fixed optical frequency, varying the collector voltage tests the distribution of emitted electron energies. At fixed surface conditions, varying optical frequency tests the photon energy relation. Define the signed voltage

$$\Delta V = \varphi_{\text{collector}} - \varphi_{\text{emitter}}. \quad (18.16)$$

Because an electron has charge $-e$, its potential-energy change on reaching the collector is $\Delta U = -e\Delta V$. A negative ΔV raises its potential energy and retards it; at cutoff $\Delta V = -V_s$. Thus $\Delta K = -\Delta U$ gives $K_{\text{max}} = eV_s$, with no sign ambiguity.

At a sufficiently positive collection voltage, nearly all emitted electrons are collected and the current approaches saturation. Doubling intensity at fixed frequency approximately doubles that saturation current in the linear regime. The stopping voltage stays the same. Figure 18.3 displays this distinction and the frequency dependence of the cutoff. The rounded current curves are schematics: their detailed shapes depend on energy and angular distributions, geometry, contact potentials, and surface properties.

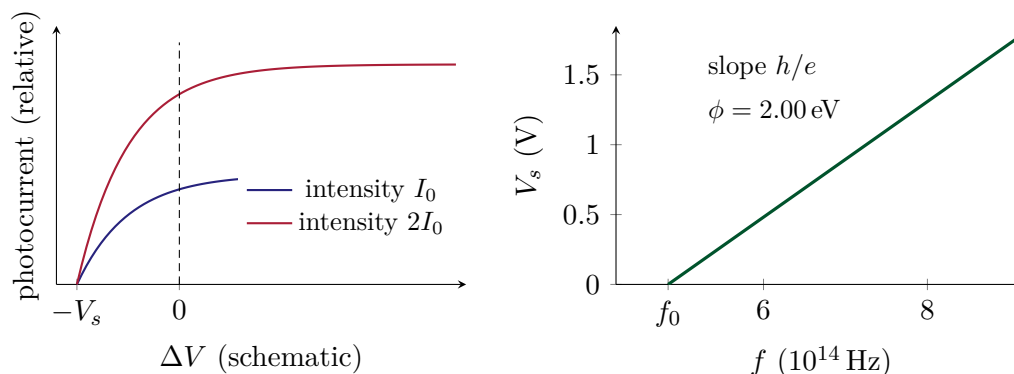


Figure 18.3: Left: schematic photocurrent curves for two intensities at the same frequency; the cutoff is shared. Right: calculated ideal stopping voltage for a 2.00 eV surface. No stopping potential is assigned below threshold, where this model emits no electrons.

Example 18.7: A threshold decision before calculating voltage

A surface with work function 2.80 eV is illuminated first at 500 nm and then at 400 nm. Which beam can produce one-photon photoemission?

The photon energies are 2.48 eV and 3.10 eV. The first is below threshold, so there is no emitted-electron stopping voltage to calculate in this model. The second gives $K_{\text{max}} = 3.10 - 2.80 = 0.30$ eV and $V_s = 0.30$ V. A formal negative result $2.48 - 2.80 = -0.32$ eV is a failed threshold test, not a negative kinetic energy.

Increasing the 500 nm intensity does not change that one-photon conclusion. At sufficiently strong fields, multiphoton and field-assisted processes require a different model; they are not part of the elementary threshold experiment.

18.6 Compton scattering: photons in a relativistic collision

The photoelectric effect establishes a frequency-dependent quantum of energy, but photon momentum is displayed especially cleanly in Compton scattering. Consider an incident photon of wavelength λ that scatters through an angle θ from an electron that is initially approximately free and at rest. The scattered photon has a longer wavelength λ' , while the electron recoils with momentum $\mathbf{p}_e$.

The two scattering angles are distinguished in Fig. 18.4.

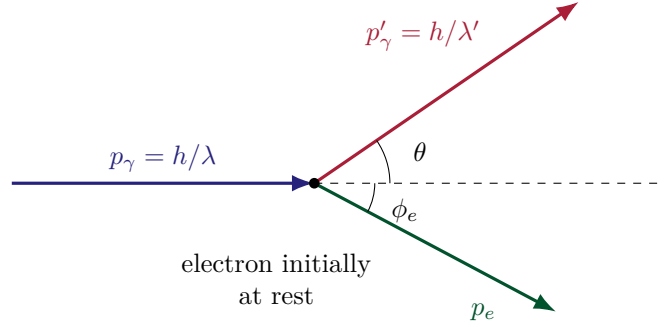


Figure 18.4: Compton geometry: θ is the photon deflection and ϕ_e the electron recoil angle below the incident axis. Arrows specify directions and are not drawn to momentum scale; the component equations determine their magnitudes.

Conservation of momentum gives

$$\frac{h}{\lambda} = \frac{h}{\lambda'} \cos \theta + p_e \cos \phi_e, \quad (18.17)$$

$$0 = \frac{h}{\lambda} \sin \theta - p_e \sin \phi_e. \quad (18.18)$$

Energy conservation requires the electron's relativistic total energy,

$$\frac{hc}{\lambda} + m_e c^2 = \frac{hc}{\lambda'} + E_e, \quad (18.19)$$

with

$$E_e^2 = p_e^2 c^2 + m_e^2 c^4. \quad (18.20)$$

To see the cancellation explicitly, write the incident and outgoing photon energies as $\mathcal{E}$ and $\mathcal{E}'$. Squaring the momentum-vector equation gives

$$p_e^2 c^2 = \mathcal{E}^2 + \mathcal{E}'^2 - 2\mathcal{E}\mathcal{E}' \cos \theta. \quad (18.21)$$

Meanwhile $E_e = m_e c^2 + \mathcal{E} - \mathcal{E}'$ implies

$$p_e^2 c^2 = (\mathcal{E} - \mathcal{E}')^2 + 2m_e c^2 (\mathcal{E} - \mathcal{E}'). \quad (18.22)$$

Equating the two expressions cancels the squared photon energies:

$$m_e c^2 (\mathcal{E} - \mathcal{E}') = \mathcal{E}\mathcal{E}' (1 - \cos \theta), \quad \frac{1}{\mathcal{E}'} - \frac{1}{\mathcal{E}} = \frac{1 - \cos \theta}{m_e c^2}. \quad (18.23)$$

Using $\lambda = hc/\mathcal{E}$ now turns an inverse-energy difference into a wavelength difference. The electron rest energy is essential in this step; a Newtonian recoil formula would not give the exact result.

Eliminating the recoil momentum yields the Compton shift

$$\boxed{\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)}. \quad (18.24)$$

The constant

$$\lambda_C = \frac{h}{m_e c} \approx 2.426 \times 10^{-12} \text{ m} \quad (18.25)$$

is the electron Compton wavelength. The shift is zero in forward scattering and reaches $2\lambda_C$ for backscattering through 180° .

Example 18.8: Compton wavelength shift at 60°

An x-ray photon scatters from an approximately free electron through $\theta = 60^\circ$. Find the wavelength increase.

$$\begin{aligned}\Delta\lambda &= \lambda_C(1 - \cos 60^\circ) \\ &= (2.426 \times 10^{-12})(1 - 0.5) \text{ m} \\ &\approx 1.21 \times 10^{-12} \text{ m} = 1.21 \text{ pm}.\end{aligned}$$

The result depends only on the scattering angle and electron mass, not on the initial photon wavelength. The fractional change $\Delta\lambda/\lambda$, however, is much easier to observe for x rays than for visible light because x-ray wavelengths are much shorter than optical wavelengths.

Example 18.9: Energy transferred in Compton backscattering

A photon of initial energy $E_\gamma = 100 \text{ keV}$ is backscattered from a free electron initially at rest. Find the scattered-photon energy. It is convenient to rewrite the Compton formula in energy form. Since $\lambda = hc/E$, one finds

$$E'_\gamma = \frac{E_\gamma}{1 + \frac{E_\gamma}{m_e c^2}(1 - \cos \theta)}.$$

For $\theta = 180^\circ$ and $m_e c^2 \approx 511 \text{ keV}$,

$$E'_\gamma = \frac{100}{1 + 2(100/511)} \text{ keV} \approx 71.8 \text{ keV}.$$

The recoil electron therefore receives kinetic energy $K_e \approx 28.2 \text{ keV}$. Energy conservation is explicit: tunneling or other quantum language is not involved here; this is an ordinary relativistic two-body conservation problem with a photon.

Method

A reliable Compton strategy is: (1) draw the momentum vectors, (2) use $p_\gamma = E_\gamma/c$, (3) conserve momentum in components, (4) conserve total relativistic energy, and (5) only then eliminate the electron recoil variables. Avoid inserting $K = p^2/(2m)$ unless you have verified that the recoil electron is nonrelativistic.

Guided checks

1. Why does the scattered photon's wavelength increase rather than decrease when the initially stationary electron gains kinetic energy?
2. At what scattering angle is the Compton shift largest?
3. Why is a relativistic energy–momentum relation required even if the electron starts at rest?
4. Why is the shift difficult to observe with visible photons scattering from electrons?

18.7 Wave–particle duality without classical pictures

The experiments discussed so far can tempt us into a misleading mental model: perhaps a photon is literally a tiny classical pellet that somehow also carries a classical sinusoidal wave. That picture

creates more problems than it solves. A photon is a quantum excitation whose preparation and detection statistics are described by quantum theory. Classical wave language remains extremely useful for propagation, coherence, and interference; localized detector clicks remain essential evidence of quantized exchange. Neither classical picture alone is complete.

A particularly important thought pattern is single-quantum interference. Suppose a source is attenuated so strongly that detection events are well separated in time. Individual detections remain localized, yet after many events the spatial distribution can reveal an interference pattern. The pattern is not produced by photons colliding with one another. It reflects the probability amplitudes associated with alternative paths through the apparatus.

Figure 18.5 separates the alternatives from the accumulated detection distribution.

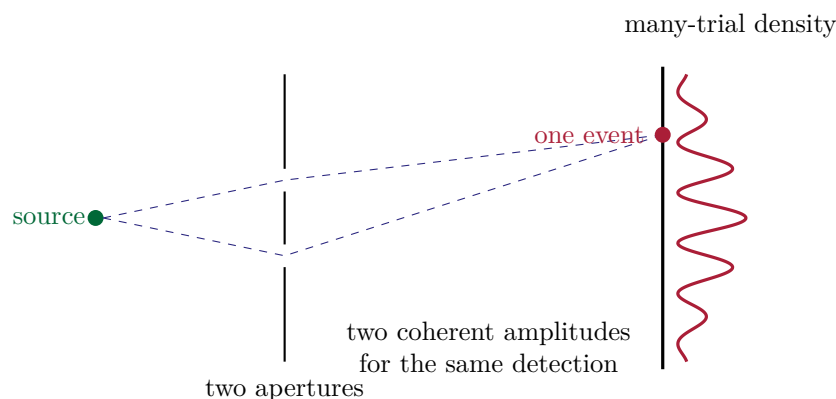


Figure 18.5: One localized event does not display a fringe. Repeated identically prepared trials reveal an interference distribution, sketched to the right. Dashed lines identify alternatives used in the amplitude calculation; they do not assert a measured trajectory.

Key message

Wave-particle duality is best understood as a warning against overextending classical categories. Quantum systems can produce wave-like interference probabilities and particle-like localized detection events. The predictive object is the quantum state, not a classical particle carrying a hidden mechanical ripple.

Caution

Do not say that an electron or photon “turns into a wave” while traveling and “turns back into a particle” when detected. That language suggests two classical substances. Quantum theory uses one state description and different measurement outcomes.

18.8 de Broglie matter waves

De Broglie’s proposal reverses the logic of photon momentum. If electromagnetic radiation with wavelength λ has momentum $p = h/\lambda$, perhaps a material particle with momentum p can be associated with a wavelength

$$\lambda_{\text{dB}} = \frac{h}{p}. \quad (18.26)$$

This relation is general when p is the relativistic momentum magnitude. In the nonrelativistic regime, $p \approx mv$, so

$$\lambda_{\text{dB}} \approx \frac{h}{mv}. \quad (18.27)$$

For an electron accelerated from rest through a potential difference of magnitude V and treated nonrelativistically,

$$eV = \frac{p^2}{2m_e}, \quad (18.28)$$

which gives

$$\lambda = \frac{h}{\sqrt{2m_e eV}}. \quad (18.29)$$

The formula is extremely useful at modest voltages, but the relativistic correction becomes important as eV is no longer small compared with $m_e c^2 \approx 511 \text{ keV}$.

Example 18.10: Electron wavelength after acceleration through 150 V

An electron starts approximately from rest and is accelerated through 150 V. Estimate its de Broglie wavelength.

$$\begin{aligned} \lambda &= \frac{h}{\sqrt{2m_e eV}} \\ &\approx \frac{6.626 \times 10^{-34}}{\sqrt{2(9.109 \times 10^{-31})(1.602 \times 10^{-19})(150)}} \text{ m} \\ &\approx 1.00 \times 10^{-10} \text{ m} = 0.100 \text{ nm}. \end{aligned}$$

This is comparable with atomic spacing in solids, so diffraction by a crystal lattice is plausible. The kinetic energy is only 150 eV, much smaller than 511 keV, validating the nonrelativistic approximation.

18.8.1 Relativistic correction for accelerated electrons

When the accelerating energy eV is not negligible compared with $m_e c^2$, use

$$E = m_e c^2 + eV, \quad (18.30)$$

with $E^2 = p^2 c^2 + m_e^2 c^4$. Therefore

$$pc = \sqrt{(eV)^2 + 2m_e c^2 (eV)}, \quad (18.31)$$

and

$$\lambda = \frac{hc}{\sqrt{(eV)^2 + 2m_e c^2 (eV)}}. \quad (18.32)$$

At 100 kV, the nonrelativistic formula overestimates the wavelength by several percent, which matters in electron-optical design.

Example 18.11: Relativistic de Broglie wavelength at 100 kV

For an electron accelerated through 100 kV, Eq. (18.32) gives

$$\lambda \approx 3.70 \text{ pm}.$$

The nonrelativistic expression gives about 3.88 pm. The difference is roughly 4.8%, no longer negligible for quantitative microscopy. This comparison illustrates an important modelling habit: quantum and relativistic corrections can become relevant at the same time.

Method

For a massive-particle de Broglie problem, do not begin with $\lambda = h/(mv)$ automatically. First decide how the momentum is known. If a speed is small compared with c , use $p \approx mv$. If kinetic energy or accelerating voltage is large enough that $K/(mc^2)$ is not tiny, use the relativistic energy–momentum relation.

Guided checks

1. An electron's momentum doubles. What happens to its de Broglie wavelength?
2. A proton and an electron have the same momentum. Which has the larger de Broglie wavelength?
3. A proton and an electron have the same nonrelativistic speed. Which has the larger wavelength?
4. Why can a very small de Broglie wavelength make a high-energy electron beam useful for microscopy?

18.9 Electron diffraction and the operational meaning of matter waves

The de Broglie relation would be an attractive analogy without experimental support. Its significance comes from diffraction experiments. Electrons scattered from a periodic crystal can produce intensity maxima corresponding to constructive interference, just as x rays do. Electron beams sent through slits or biprisms can also form interference fringes. At sufficiently low beam intensity, individual electrons arrive one at a time as localized detector events, yet the cumulative distribution approaches the interference pattern.

For a crystal plane spacing d , a simple Bragg condition is

$$2d \sin \theta = n\lambda, \quad (18.33)$$

where n is an integer diffraction order. In electron diffraction, λ is the de Broglie wavelength. The relation provides a direct route from measured diffraction angles to particle momentum.

Example 18.12: Electron diffraction from atomic planes

Electrons of de Broglie wavelength 0.100 nm encounter crystal planes separated by $d = 0.200$ nm. Estimate the first-order Bragg angle.

$$2d \sin \theta = \lambda \quad \Longrightarrow \quad \sin \theta = \frac{0.100}{0.400} = 0.250.$$

Thus

$$\theta \approx 14.5^\circ.$$

The example is deliberately idealized: real electron diffraction depends on three-dimensional crystal structure, multiple scattering, detector geometry, and finite coherence. The simple calculation nevertheless captures why atomic-scale wavelengths are necessary.

18.9.1 From a diffraction ring to a lattice spacing

In Bragg's law, θ is the angle between the incident beam and the diffracting planes. The deflection between the forward incident direction and the diffracted ray is 2θ . These definitions matter when

converting a measured ring radius into a wavelength. In a flat-screen transmission geometry with sample–screen distance D and ring radius r ,

$$2\theta = \arctan(r/D), \quad d = \frac{n\lambda}{2 \sin[\frac{1}{2} \arctan(r/D)]}. \quad (18.34)$$

For first order and $r \ll D$, this becomes $r \simeq D\lambda/d$. The angular geometry in Fig. 18.6 explains the factor of two; the diagram is a kinematic model, not a full calculation of crystal intensities.

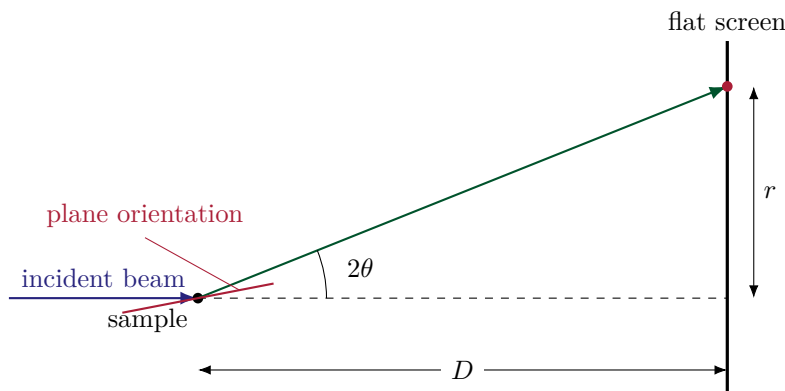


Figure 18.6: Transmission-diffraction geometry, not to scale. The beam is deflected through 2θ , twice the Bragg angle to the planes. A polycrystalline specimen supplies many plane orientations, producing diffraction rings.

Example 18.13: Lattice spacing from electron diffraction

Electrons accelerated through $V_{\text{acc}} = 2.00 \text{ kV}$ form a first-order ring of radius $r = 12.0 \text{ mm}$ on a flat screen $D = 100.0 \text{ mm}$ from a thin polycrystalline specimen. Estimate d and the effect of independent uncertainties $\sigma_r = 0.2 \text{ mm}$, $\sigma_D = 0.5 \text{ mm}$, and $\sigma_V = 10 \text{ V}$.

Model. Use elastic scattering and the Bragg condition, with a nonrelativistic wavelength. Its relative error at this voltage is about 0.10%, smaller than the stated measurement uncertainty. The wavelength is $\lambda \simeq 1.227/\sqrt{2000} = 0.0274 \text{ nm}$. The exact geometry gives $\theta = \frac{1}{2} \arctan(0.120) = 3.42^\circ$ and $d \simeq 0.230 \text{ nm}$. The small-angle value is $D\lambda/r \simeq 0.229 \text{ nm}$, lower by about 0.5%.

For a first uncertainty estimate using the small-angle relation,

$$\frac{\sigma_d}{d} \simeq \sqrt{\left(\frac{0.5}{100.0}\right)^2 + \left(\frac{0.2}{12.0}\right)^2 + \frac{1}{4} \left(\frac{10}{2000}\right)^2} \simeq 0.018.$$

Thus the inferred spacing is approximately $(0.230 \pm 0.004) \text{ nm}$ within this model. Ring-radius determination dominates. A curved screen or substantial multiple scattering requires a revised geometrical or scattering model, not simply more digits in the same formula.

18.9.2 What is actually waving?

For a classical string, the displacement of the string is a measurable wave variable. For an electromagnetic wave, electric and magnetic fields oscillate. For a nonrelativistic quantum particle, the wave function is a probability amplitude. It is generally complex and is not directly interpreted as a material displacement. Observable position probabilities are obtained from $|\Psi|^2$.

This point prevents a common misconception. A matter wave does not mean that an electron is physically smeared into a little sinusoidal charge-density rope in exactly the same sense as a classical continuous fluid. In a one-particle experiment, each detection is localized. The spatial wave function predicts the statistical distribution of detections over repeated identically prepared trials.

Caution

The phrase “probability wave” is useful only if it is completed by a mathematical statement: probabilities are calculated from the squared magnitude of a complex amplitude. Saying only that “the particle is a wave” hides the central quantum idea and encourages classical misinterpretation.

18.10 Probability amplitudes and the wave function

A quantum state contains more information than a probability distribution alone because amplitudes carry phase. Two amplitudes can interfere before their squared magnitude is taken. This is the mathematical origin of constructive and destructive interference in quantum experiments.

In one dimension, let the state be described by $\Psi(x, t)$. The probability density for position is

$$\rho_P(x, t) = |\Psi(x, t)|^2. \quad (18.35)$$

Therefore

$$P(a < x < b) = \int_a^b |\Psi(x, t)|^2 dx. \quad (18.36)$$

A physically acceptable state must be normalizable when it represents a particle localized somewhere on the full line:

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dx = 1. \quad (18.37)$$

Because probabilities are dimensionless, $|\Psi|^2$ has unit m^{-1} in one spatial dimension, so Ψ has unit $\text{m}^{-1/2}$.

Definition 18.2: Born probability rule

For a one-dimensional quantum state described by a normalized wave function $\Psi(x, t)$, the probability of a position measurement yielding a result between a and b is

$$P(a \leq x \leq b) = \int_a^b |\Psi(x, t)|^2 dx.$$

The complex function Ψ is a probability amplitude; $|\Psi|^2 = \Psi^* \Psi$ is a real, nonnegative probability density.

Example 18.14: Normalizing a simple exponential wave function

Consider the one-dimensional trial wave function

$$\psi(x) = Ae^{-|x|/a},$$

where $a > 0$. Determine A for a normalized state, choosing A real and positive. Normalization requires

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |A|^2 e^{-2|x|/a} dx \\ &= 2A^2 \int_0^{\infty} e^{-2x/a} dx \\ &= 2A^2 \left(\frac{a}{2} \right) = A^2 a. \end{aligned}$$

Hence

$$A = \frac{1}{\sqrt{a}}.$$

As a dimensional check, A indeed has units $\text{m}^{-1/2}$ in one dimension.

18.10.1 Why phases matter even when probabilities are positive

For a complex number $z = a + ib$, complex conjugation gives $z^* = a - ib$, so $|z|^2 = a^2 + b^2$. The identity $e^{i\varphi} = \cos \varphi + i \sin \varphi$ encodes phase without introducing negative probabilities. If two alternatives contribute amplitudes Ψ_1 and Ψ_2 to the same outcome, then

$$|\Psi_1 + \Psi_2|^2 = |\Psi_1|^2 + |\Psi_2|^2 + 2 \operatorname{Re}(\Psi_1^* \Psi_2). \quad (18.38)$$

The cross term can be positive, negative, or zero. Equal amplitudes with opposite phase cancel at a dark fringe. Adding probabilities first would erase this effect. By contrast, when alternatives are rendered distinguishable and the distinguishing record is ignored, the corresponding marginal probabilities add without that coherent cross term.

A common phase multiplying the entire state cancels from all expectation values. A phase that varies with position is different: it can change momentum and current even while leaving the position density unchanged. For example, $g(x)$ and $g(x)e^{ik_0x}$ have the same $|g(x)|^2$ but different mean momenta when g is real and suitably localized.

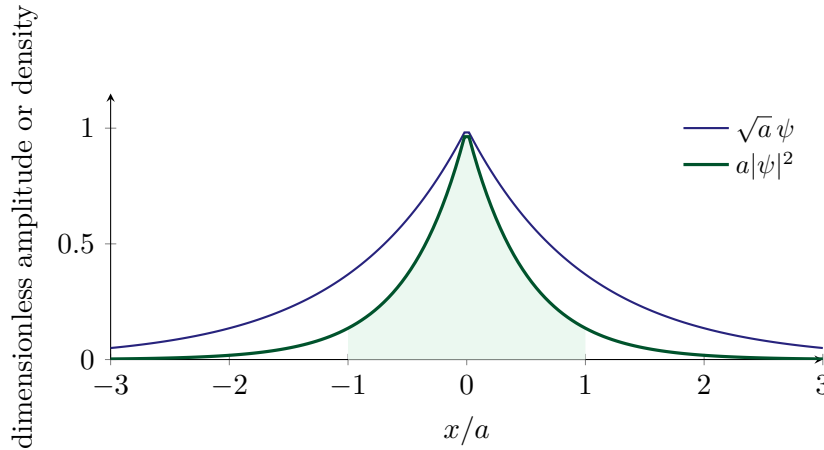


Figure 18.7: Normalized exponential trial state from Example 18.14. The axes use x/a and dimensionless vertical quantities. The shaded density area for $|x| < a$ is $1 - e^{-2} = 0.865$. Squaring the amplitude changes both its interpretation and its width.

Figure 18.7 turns normalization into a geometrical statement. For the exponential state, the probability in $|x| < a$ is

$$P(|x| < a) = \frac{2}{a} \int_0^a e^{-2x/a} dx = 1 - e^{-2}. \quad (18.39)$$

The cusp at the origin is acceptable for this normalization exercise. It is not a stationary solution of a smooth finite potential on the full line; such a stationary exponential requires a singular attractive interaction at the cusp. Normalizability alone does not establish that a proposed function solves a particular Hamiltonian.

18.10.2 Expectation values as statistical averages

For repeated measurements on identically prepared states, the expectation value of position is

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^*(x, t) x \Psi(x, t) dx. \quad (18.40)$$

The position spread is quantified by the standard deviation

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}. \quad (18.41)$$

We will not develop the full operator formalism in the main chapter, but this statistical language is necessary for a correct statement of uncertainty.

Guided checks

1. Can Ψ itself be interpreted as a probability density? Why not?
2. Why can a wave function be multiplied by a global phase $e^{i\alpha}$ without changing position probabilities?
3. If $|\Psi|^2$ is sharply localized, what qualitative conclusion can you draw about Δx ?
4. Why must the dimensions of a one-dimensional normalized wave function be $m^{-1/2}$ rather than dimensionless?

18.11 The Schrödinger equation: an introductory dynamical law

Newton's second law provides the time evolution of a classical particle once forces and initial conditions are specified. Maxwell's equations provide the evolution of electromagnetic fields. In nonrelativistic quantum mechanics, the corresponding state-evolution law is the Schrödinger equation.

For one particle of mass m moving in one dimension in a potential energy $V(x, t)$,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x, t) \Psi. \quad (18.42)$$

In operator notation,

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi, \quad (18.43)$$

where

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \quad (18.44)$$

is the Hamiltonian operator for this nonrelativistic one-particle model.

Equation (18.42) is not derived from Newton's laws; it is a fundamental postulate of nonrelativistic quantum mechanics whose consequences are tested experimentally. Its structure is constrained by the de Broglie and Planck relations and by the requirement that probabilities evolve consistently.

18.11.1 Time-independent potentials and stationary states

If V depends on position but not explicitly on time, one can seek separated solutions

$$\Psi(x, t) = \psi(x) e^{-iEt/\hbar}. \quad (18.45)$$

Substitution into Eq. (18.42) gives the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + V(x) \psi = E \psi. \quad (18.46)$$

A solution with definite E is called an energy eigenstate or stationary state. The word “stationary” does not mean that the particle is literally motionless. Instead,

$$|\Psi(x, t)|^2 = |\psi(x)|^2 \quad (18.47)$$

is time independent because the time factor has unit magnitude.

Theoretical point 18.1: Local wave number in a constant-potential region

In a region where $V(x) = V_0$ is constant and $E > V_0$, Eq. (18.46) becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0, \quad k = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

The corresponding local momentum magnitude is $p = \hbar k$, so

$$\lambda = \frac{2\pi}{k} = \frac{h}{p}.$$

Thus the de Broglie relation is built naturally into the Schrödinger wave equation.

18.11.2 Interpreting the equation before solving it

Every term in Eq. (18.42) has dimensions of energy times amplitude. For example, $\hbar\partial_t\Psi$ has units J times those of Ψ , and $\hbar^2\partial_x^2\Psi/m$ has the same units. The equation is linear: if two states solve the same equation, their linear combination also solves it. Probabilities, however, must be computed after the amplitudes have been added.

The curvature form of the stationary equation is especially revealing:

$$\psi''(x) = \frac{2m}{\hbar^2}[V(x) - E]\psi(x). \quad (18.48)$$

For a real spatial solution, $E > V$ makes curvature oppose the sign of ψ , producing oscillations. For $E < V$, curvature has the same sign, permitting exponential behavior. This is the mathematical basis of the allowed/forbidden-region distinction. The total mechanical energy E in this equation excludes the rest energy mc^2 .

A time-independent Hamiltonian does not imply that every state is stationary. If two energy eigenstates are combined,

$$\Psi = c_1\psi_1e^{-iE_1t/\hbar} + c_2\psi_2e^{-iE_2t/\hbar}, \quad (18.49)$$

the density generally contains a cross term oscillating at angular frequency $(E_2 - E_1)/\hbar$. Only a single-energy state, or special combinations with no observable cross term, has time-independent position density. A stationary scattering state can still carry a nonzero current.

18.11.3 Free-particle plane waves

For $V = 0$ and definite momentum $p = \hbar k$, a formal solution is

$$\Psi(x, t) = Ae^{i(kx - \omega t)}, \quad (18.50)$$

with

$$E = \hbar\omega = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}. \quad (18.51)$$

A single plane wave extends over all space and is not normalizable to one particle on the infinite line. Physical localized particles are represented by wave packets built from a range of k values. The advanced enrichment develops that idea more fully.

Caution

The nonrelativistic Schrödinger equation is not an equation for photons. Photons require a relativistic quantum-field description for a fundamental treatment. In this chapter photons are handled through experimentally established quantum relations and relativistic conservation laws, while Schrödinger's equation is used for nonrelativistic massive particles.

Guided checks

1. Why does the kinetic-energy operator contain a second spatial derivative?
2. In a stationary state, which quantity is time independent: Ψ itself or $|\Psi|^2$?
3. For a constant potential with $E > V_0$, what happens to the wavelength if V_0 increases while the total energy is fixed?
4. Why is a pure plane wave not a realistic model of a particle localized in a laboratory region?

18.12 The uncertainty principle

The uncertainty principle is often introduced with an unfortunate phrase such as “measuring position disturbs momentum.” Measurement back-action is real in quantum experiments, but it is not the complete meaning of the uncertainty relation. The stronger statement is about quantum states themselves. For an ensemble of identically prepared systems, the standard deviations of position and momentum satisfy

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}. \quad (18.52)$$

No improvement in instrument calibration can prepare a state that violates this bound.

The relation has a wave-theoretic origin. A state with a sharply defined momentum resembles a long nearly monochromatic wave with a narrow spread in wave number. To localize a wave packet strongly in space, one must superpose many wave numbers. Since $p = \hbar k$, a large spread in k means a large momentum spread.

Figure 18.8 shows the reciprocal widths of a normalized Gaussian pair.

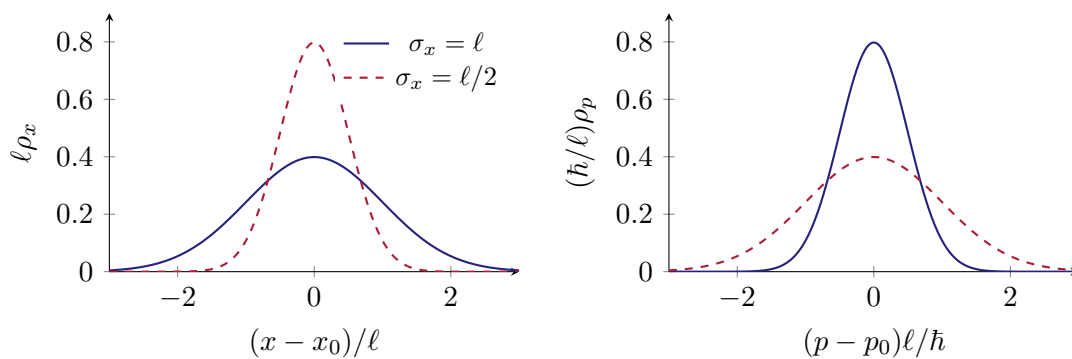


Figure 18.8: Calculated position and momentum probability densities for two unchirped Gaussian states. Each density integrates to one over its full dimensionless coordinate. Halving Δx doubles Δp ; each pair satisfies $\Delta x \Delta p = \hbar/2$. The arbitrary length ℓ only sets units.

Example 18.15: Localization, momentum spread, and kinetic energy

An electron is prepared with position uncertainty $\Delta x = 0.10 \text{ nm}$. Estimate the minimum momentum uncertainty from Eq. (18.52) and the corresponding nonrelativistic kinetic-energy

scale $(\Delta p)^2/(2m_e)$.

$$\Delta p_x \geq \frac{\hbar}{2\Delta x} = \frac{1.055 \times 10^{-34}}{2(1.0 \times 10^{-10})} \approx 5.27 \times 10^{-25} \text{ kg m s}^{-1}.$$

The associated kinetic-energy scale is

$$K \sim \frac{(\Delta p_x)^2}{2m_e} \approx 1.53 \times 10^{-19} \text{ J} \approx 0.95 \text{ eV}.$$

The estimate shows why microscopic confinement naturally creates non-negligible kinetic-energy scales. It should not be interpreted as a precise ground-state energy because the exact coefficient depends on the shape of the confining potential and the state.

18.12.1 From a spread bound to an energy bound

The uncertainty relation constrains standard deviations, not the mean momentum. For nonrelativistic motion in one dimension,

$$\langle K \rangle = \frac{\langle p_x^2 \rangle}{2m} = \frac{\langle p_x \rangle^2 + (\Delta p_x)^2}{2m} \geq \frac{\langle p_x \rangle^2}{2m} + \frac{\hbar^2}{8m(\Delta x)^2}. \quad (18.53)$$

This is a precise lower bound when the moments exist. A localized state with zero mean momentum still has kinetic energy associated with its momentum spread. The $1/(\Delta x)^2$ dependence explains the energy cost of localization without specifying a particular confining potential. Equality requires a minimum-uncertainty state; identifying Δx with a box width does not automatically satisfy that condition.

Instrumental blur and intrinsic spread must also be separated. For independent Gaussian preparation and detector distributions, $\sigma_{\text{obs}}^2 = \sigma_q^2 + \sigma_d^2$. A smaller observed width after better calibration need not violate uncertainty: it may simply remove detector blur. The uncertainty principle relates the intrinsic quantum distributions prepared in the same state.

18.12.2 Energy–time uncertainty: a careful remark

There is no universal rule obtained by replacing x and p_x by arbitrary measurement time and energy error. Time is the evolution parameter in the present theory. One precise relation is the Mandelstam–Tamm bound

$$\Delta E \tau_A \geq \frac{\hbar}{2}, \quad \tau_A = \frac{\Delta A}{|d\langle A \rangle/dt|}, \quad (18.54)$$

for a time-independent observable A when the denominator is nonzero. Here ΔE is the energy standard deviation and τ_A measures how rapidly an expectation changes compared with its spread. It is not simply the duration of every experiment.

A different statement concerns an ideal exponentially decaying unstable state: lifetime τ gives a Lorentzian spectral full width at half maximum $\Gamma = \hbar/\tau$. The Lorentzian standard deviation is not finite, so Γ must not be silently identified with ΔE in the preceding relation. Neither statement licenses temporary violations of energy conservation or explains tunneling by energy borrowing.

Caution

The uncertainty bound constrains the outcome distributions of a prepared quantum state. Detector resolution adds experimental broadening; it is a separate contribution and does not set the intrinsic quantum bound.

Guided checks

1. Can a state have $\Delta x = 0$ and finite Δp_x ? Use Eq. (18.52).
2. Does the uncertainty principle forbid a precise momentum measurement? What must then happen to position spread?
3. Why does narrowing a wave packet in position require adding more wave-number components?
4. Why should $\Delta E \Delta t \gtrsim \hbar/2$ be discussed more carefully than $\Delta x \Delta p_x \geq \hbar/2$?

18.13 Potential steps: quantum reflection and transmission

Potential-energy diagrams are useful in both classical and quantum mechanics. In classical mechanics, a particle with total energy E moving toward a step from V_1 to a higher constant value V_2 crosses the step if $E > V_2$ and is reflected if $E < V_2$. For an abrupt quantum step, the first statement changes: even when $E > V_2$, the wave can be partially reflected because the wavelength changes at the boundary.

Consider

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geq 0. \end{cases} \quad (18.55)$$

For $E > V_0$, the time-independent Schrödinger equation has oscillatory solutions in both regions:

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}, \quad k_1 = \frac{\sqrt{2mE}}{\hbar}, \quad (18.56)$$

$$\psi_{II}(x) = Ce^{ik_2x}, \quad k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}. \quad (18.57)$$

The terms proportional to A , B , and C represent incident, reflected, and transmitted traveling-wave components. Continuity of ψ and $d\psi/dx$ at a finite step determines the amplitudes. For equal mass on both sides, the resulting probabilities are

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2, \quad T = \frac{4k_1k_2}{(k_1 + k_2)^2}, \quad R + T = 1. \quad (18.58)$$

Here R and T are ratios of reflected and transmitted probability currents to the incident current. The factor k_2/k_1 in $T = (k_2/k_1)|C/A|^2$ accounts for the different speeds. The advanced section derives this flux relation and the boundary conditions.

Figure 18.9 separates the potential-energy profile from the directions of probability flow.

Example 18.16: Reflection at a potential step

A nonrelativistic electron of total energy $E = 10\text{ eV}$ encounters a sharp upward step $V_0 = 6\text{ eV}$. Find the reflection probability using Eq. (18.58).

Only the ratio k_2/k_1 is needed:

$$\frac{k_2}{k_1} = \sqrt{\frac{E - V_0}{E}} = \sqrt{\frac{4}{10}} \approx 0.632.$$

Thus

$$R = \left(\frac{1 - 0.632}{1 + 0.632} \right)^2 \approx 0.0508.$$

Reflection accounts for about 5.1% of the incident flux; $T \approx 94.9\%$. Each transmitted electron retains total energy 10 eV, with kinetic energy $E - V_0 = 4\text{ eV}$.

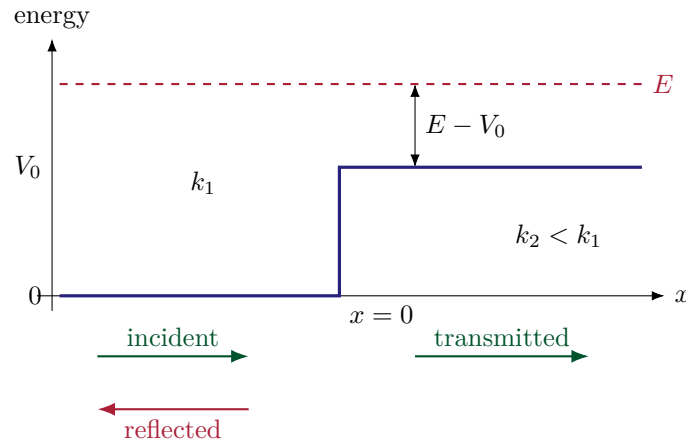


Figure 18.9: An upward potential-energy step with $E > V_0$. The energy zero agrees with the horizontal axis. Arrows below the profile indicate flux directions only; their heights are not energies. The transmitted wavelength is longer because its kinetic energy is smaller.

Key message

Quantum reflection does not require a classical turning point. A sudden change in the wave equation can produce a reflected amplitude even when the particle's total energy exceeds the potential step.

18.13.1 Penetration without transmitted flux

For a semi-infinite upward step with $0 < E < V_0$, the right-hand solution must decay as $\psi_{II} = Ce^{-\kappa x}$ with $\kappa = \sqrt{2m(V_0 - E)}/\hbar$. There is nonzero probability density near the interface, but no rightward outgoing traveling wave at infinity. The stationary transmitted current is zero and $R = 1$.

The amplitude drops by a factor e^{-1} over $1/\kappa$, while the density drops by that factor over $1/(2\kappa)$. These are different penetration lengths. A finite barrier changes the boundary conditions: there is another allowed region beyond it, and matching at the second boundary permits a nonzero outgoing flux. A nonzero tail and tunneling transmission are therefore related but distinct statements.

18.14 Tunneling through a finite barrier

Now consider a finite rectangular barrier of height V_0 and width L , with a particle energy $E < V_0$. Classical mechanics predicts certain reflection because the kinetic energy $K = E - V_0$ would be negative inside the barrier. Quantum mechanics does not interpret $E - V_0 < 0$ as an ordinary negative kinetic energy. Instead, the Schrödinger equation changes from oscillatory to exponential form in the barrier.

For $0 < x < L$,

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V_0\psi = E\psi, \quad (18.59)$$

so

$$\frac{d^2\psi}{dx^2} - \kappa^2\psi = 0, \quad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \quad (18.60)$$

The barrier-region wave function therefore contains decaying and growing exponential pieces. Matching the wave function and its derivative at both interfaces leads to a nonzero transmitted amplitude on the far side.

For a sufficiently high or wide barrier, a useful approximation is

$$T \propto e^{-2\kappa L}, \quad (18.61)$$

with a prefactor that depends on E/V_0 . The exponential dependence is the most important engineering feature: small changes in width or barrier height can change a tunneling current by orders of magnitude.

The exact stationary solution in Fig. 18.10 distinguishes potential energy, amplitude, and probability density.

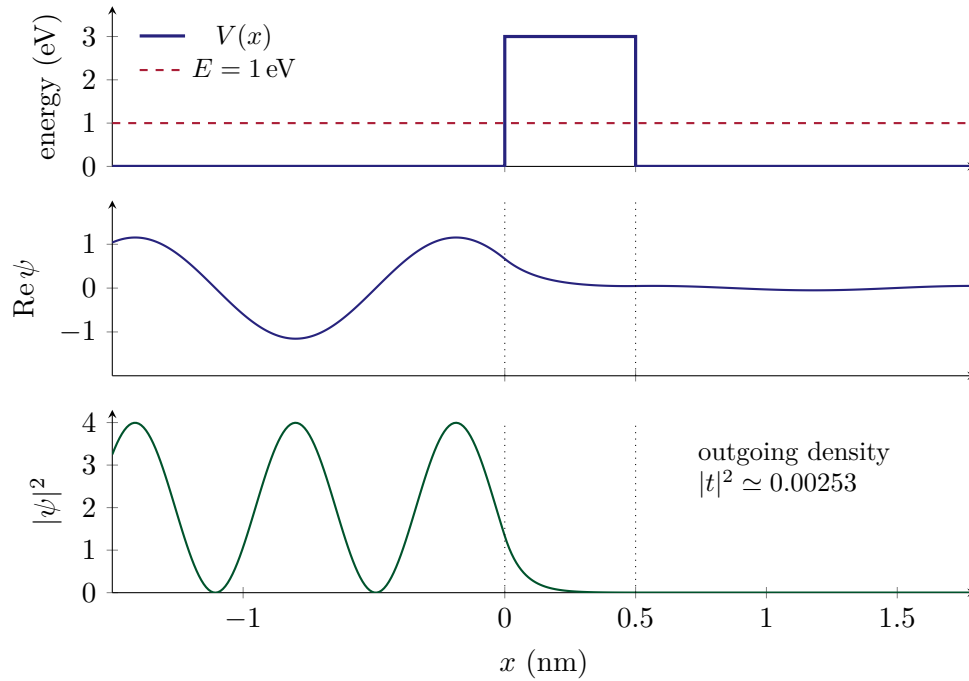


Figure 18.10: Calculated electron scattering state for $E = 1.0$ eV, $V_0 = 3.0$ eV, and $L = 0.50$ nm. The incident amplitude is set to one, so the lower panels are relative amplitudes and densities of a nonnormalizable stationary scattering state. The left-side modulation comes from incident–reflected interference. Both ψ and its derivative match at the interfaces; the small outgoing density is constant although $\text{Re } \psi$ oscillates.

Example 18.17: Sensitivity of tunneling to barrier width

Suppose an electron with $V_0 - E = 2.0$ eV tunnels through a rectangular barrier. Estimate the exponential factor $e^{-2\kappa L}$ for $L = 0.50$ nm and compare it with the factor for $L = 0.60$ nm. For an electron,

$$\kappa = \frac{\sqrt{2m_e(V_0 - E)}}{\hbar} \approx 7.25 \times 10^9 \text{ m}^{-1}.$$

Then

$$\begin{aligned} e^{-2\kappa(0.50 \text{ nm})} &\approx e^{-7.25} \approx 7.1 \times 10^{-4}, \\ e^{-2\kappa(0.60 \text{ nm})} &\approx e^{-8.70} \approx 1.7 \times 10^{-4}. \end{aligned}$$

Increasing the barrier width by only 0.10 nm reduces the approximate factor by more than a factor of four. This extraordinary sensitivity is the physical basis of scanning tunneling microscopy. The numbers are exponential factors, not full transmission probabilities. The energy-dependent prefactor can remain different from unity even for a very thick barrier; only the ratio at fixed energy becomes independent of that prefactor in this limit.

Caution

Tunneling does not mean that a particle temporarily “borrows” energy and violates energy conservation. For a time-independent barrier, the incident, reflected, and transmitted stationary-state components have the same total energy E . The classically forbidden region corresponds to an exponential solution of the wave equation, not to a measurable classical trajectory with negative kinetic energy.

Guided checks

1. If the barrier width increases, what happens to T qualitatively? Why is the change so strong?
2. If E approaches V_0 from below, what happens to κ ?
3. Does an electron emerge from a static barrier with less total energy simply because it tunneled?
4. Why does the exponential sensitivity make tunneling useful for measuring sub-nanometre distances?

18.15 Engineering and applied-science connections

Quantum mechanics is often introduced through experiments performed a century ago, but the same equations sit inside contemporary devices. This section emphasizes how the abstract quantities used above become design variables, calibration parameters, and measurable signals.

18.15.1 Photodetectors and photon counting

A photodiode converts incident optical power into an electrical response through semiconductor processes that will be studied later. Even before band theory, the photon relation $E = hf$ gives two essential design ideas. First, the material must have an energetically allowed absorption mechanism at the chosen wavelength. Second, optical power can be translated into an average photon rate $\dot{N} = P/E_\gamma$. In low-light imaging, astronomy, lidar, and quantum communication, fluctuations in the number of detected photons can be as important as the average signal.

A detector has a quantum efficiency $0 \leq \eta \leq 1$, defined in a simple model as the fraction of incident photons that produce registered detection events. The mean registered count rate is then

$$\dot{N}_{\text{det}} = \eta \frac{P}{hf}. \quad (18.62)$$

This relation is idealized: dark counts, dead time, afterpulsing, bandwidth, gain, and background radiation can all matter in real instrumentation.

Example 18.18: Photon-counting estimate for a weak optical signal

A 650 nm optical signal has power $P = 10$ pW and reaches a detector of quantum efficiency $\eta = 0.60$. Estimate the mean registered count rate.

The photon energy is

$$E_\gamma = \frac{hc}{\lambda} \approx 3.06 \times 10^{-19} \text{ J}.$$

The incident photon rate is

$$\dot{N} \approx \frac{10^{-11}}{3.06 \times 10^{-19}} \approx 3.27 \times 10^7 \text{ s}^{-1}.$$

Hence

$$\dot{N}_{\text{det}} \approx 0.60(3.27 \times 10^7) \approx 1.96 \times 10^7 \text{ s}^{-1}.$$

Whether a real detector can record this rate depends on its response time and electronics; quantum efficiency alone does not determine maximum count rate.

18.15.2 Electron microscopy

The resolving power of an imaging system is not determined by wavelength alone, but short wavelength is a major advantage. Electrons accelerated through tens or hundreds of kilovolts have de Broglie wavelengths of a few picometres, far shorter than visible optical wavelengths. Electron microscopes therefore have access to atomic-scale spatial information, although practical resolution is limited by lens aberrations, source coherence, specimen interactions, detector response, stability, and radiation damage.

The relation between accelerating voltage and wavelength also illustrates a recurring engineering lesson: at high voltage, relativistic corrections must be included even though the imaging calculation is otherwise quantum mechanical and nonrelativistic in spirit.

18.15.3 Scanning tunneling microscopy

A scanning tunneling microscope brings a conducting tip within roughly a nanometre of a conducting surface and applies a bias voltage. Electrons tunnel across the vacuum gap. Because the current depends approximately exponentially on the tip-sample separation,

$$I \sim I_0 e^{-2\kappa s}, \quad (18.63)$$

very small height changes produce measurable current changes. A feedback loop can maintain nearly constant current while scanning laterally, producing a height signal that contains both surface geometry and electronic-structure contrast. A more complete treatment requires the electronic density of states and tip geometry, but the elementary barrier model already explains the remarkable distance sensitivity.

18.15.4 Tunnel junctions and nanoscale electronics

Thin insulating barriers between conductors can support tunnel currents. In semiconductor and superconducting devices, quantum tunneling can be either a desired transport mechanism or a leakage path that limits scaling. The same exponential sensitivity that enables an STM also makes fabrication thickness and interface quality critical.

18.15.5 Quantum sensors and interferometry

Interferometers using atoms, neutrons, or photons convert phase shifts into measurable intensity or count-rate changes. External acceleration, rotation, electromagnetic fields, or gravity can change the phase accumulated along alternative paths. Modern quantum sensing exploits these phase relationships while carefully controlling decoherence and technical noise. The detailed device physics is advanced, but the core principle is already present in this chapter: amplitudes add, phases matter, and probabilities are obtained only after interference.

Key message

The engineering value of quantum mechanics often comes from a parameter dependence that classical intuition would miss: threshold behavior in photoemission, picometre matter wavelengths, phase-sensitive interference, and exponentially sensitive tunneling.

18.16 Laboratory and computational connection

Quantum experiments frequently produce data that are statistical rather than deterministic on an event-by-event basis. Computational work therefore has two complementary roles: solving wave equations and analysing measured distributions. Both should be present in an applied Physics 2 course.

18.16.1 Laboratory connection: estimating Planck's constant from photoelectric data

A conceptually transparent experiment measures stopping potential at several optical frequencies. The ideal model

$$V_s = \frac{h}{e}f - \frac{\phi}{e} \quad (18.64)$$

turns the experiment into a straight-line fit. A professional analysis should report the fitted slope and intercept, their uncertainties, the frequency calibration, voltage uncertainty, and evidence for whether the linear model is adequate over the selected range.

A practical workflow is:

1. measure or obtain calibrated optical frequencies f_i ;
2. determine $V_{s,i}$ from current–voltage curves rather than from a single visual threshold whenever possible;
3. fit $V_s = af + b$ by weighted least squares if uncertainties differ substantially;
4. infer $h = ea$ and $\phi = -eb$;
5. propagate the fit uncertainties to h and ϕ ;
6. inspect residuals for systematic trends.

Method

A straight line is not automatically evidence for the intended physical law. Check axis units, calibration, uncertainty, residual structure, and whether the fitted intercept has the physically expected sign. In experimental physics, the model is tested by the quality and consistency of the data, not by visual resemblance alone.

18.16.2 Computational connection: a one-dimensional wave packet

A minimal numerical project can represent a Gaussian wave packet on a one-dimensional grid,

$$\Psi(x, 0) = A \exp \left[-\frac{(x - x_0)^2}{4\sigma_x^2} \right] e^{ik_0x}, \quad (18.65)$$

normalize it numerically, and calculate $|\Psi(x, 0)|^2$. A more advanced version propagates the packet under Eq. (18.42). Several numerical schemes are possible: finite differences with Crank–Nicolson time stepping, split-operator Fourier propagation, or direct eigenstate expansion for time-independent potentials.

At the introductory level, the important computational checks are independent of algorithm:

- probability should remain normalized to numerical accuracy;
- grid spacing must resolve the shortest wavelength present;

- the time step must be consistent with the stability and accuracy requirements of the algorithm;
- boundaries should be far enough away or treated with absorbing conditions if reflections are unwanted;
- results should be checked against a solvable limit such as free propagation.

Example 18.19: Grid resolution for a matter-wave simulation

A simulation contains de Broglie wavelengths as short as $\lambda_{\min} = 0.20 \text{ nm}$. Suppose a conservative numerical rule uses at least 20 grid points per shortest wavelength. What is the largest acceptable grid spacing?

$$\Delta x \leq \frac{\lambda_{\min}}{20} = 0.010 \text{ nm} = 1.0 \times 10^{-11} \text{ m}.$$

A 100 nm domain would then require at least 10^4 spatial intervals. This simple estimate shows why quantum simulations can become computationally demanding when several length scales coexist.

18.16.3 Measurement connection: histograms are part of the physics

Suppose identical electrons are sent one by one through an interference apparatus. A single detection event does not reveal the probability distribution. A histogram accumulated over many events estimates the underlying detection probability. The expected statistical fluctuation in a count N is often of order $\sqrt{N}$ for independent Poisson-like counting, so relative counting noise scales approximately as $1/\sqrt{N}$. This is one reason long acquisition times are common in low-rate quantum experiments.

Guided checks

1. In a photoelectric fit, why is the slope more directly connected to h than the intercept?
2. Why should numerical probability normalization be monitored during time evolution?
3. If the number of detected events increases by a factor of 100, how does the approximate relative counting noise change in a Poisson model?
4. Why can a beautiful simulated wave packet still be physically wrong if the grid does not resolve its shortest wavelength?

18.17 A structured problem-solving strategy for introductory quantum physics

The formulas in this chapter are compact, but model selection is often harder than algebra. A useful strategy is to identify the experimental process first.

Method

Step 1: classify the process. Is this photon absorption, photoemission, photon scattering, matter-wave diffraction, probability normalization, uncertainty estimation, or a Schrödinger boundary problem?

Step 2: write the conserved or normalized quantity. Examples are energy, momentum, total probability, or probability current.

Step 3: select the correct kinematics. Photons require $E = pc$; slow massive particles may use $K = p^2/(2m)$; energetic electrons may need relativistic momentum.

Step 4: state the approximation. Examples: free electron initially at rest, one-photon photoelectric model, one-dimensional nonrelativistic potential, ideal rectangular barrier.

Step 5: perform a dimensional and limiting check. Does a threshold go to zero in the correct limit? Does T decrease as a barrier becomes wider? Does wavelength shrink as momentum grows?

A strong written solution should make the physics visible. For example, in a photoelectric problem, the line

$$hf = \phi + K_{\max}$$

is more informative than jumping directly to a numerical substitution. In a tunneling problem, defining κ before inserting values makes the exponential scale transparent.

18.18 Common mistakes and how to avoid them

Common mistake	Better reasoning
Treating a photon as a charged classical ball	A photon is electrically neutral and is described by quantum electrodynamics fundamentally; in this course use $E = hf$ and $p = E/c$ together with conservation laws.
Using $p = mv$ for light	Photons have zero rest mass. Use $p = h/\lambda = E/c$.
Saying intensity changes photon energy	At fixed frequency, photon energy is fixed. Intensity mainly changes photon flux.
Ignoring the work function in photoemission	The incident photon must supply the material-dependent removal energy ϕ before electron kinetic energy is available.
Giving all photoelectrons the energy $hf - \phi$	That relation gives the maximum kinetic energy in the simple model; emitted electrons generally have an energy distribution.
Using nonrelativistic recoil energy automatically in Compton scattering	Conserve relativistic energy and momentum unless a nonrelativistic recoil approximation is explicitly justified.
Writing $\lambda = h/(mv)$ at any speed	The fundamental relation is $\lambda = h/p$; use relativistic p when needed.
Interpreting Ψ as probability	Probability density is $ \Psi ^2$, not Ψ .
Forgetting normalization	A bound one-particle state must have total probability one.
Treating uncertainty as instrument error	Δx and Δp describe state distributions; detector resolution is an additional experimental effect.
Calling $E < V$ a region of negative classical kinetic energy	In quantum mechanics the Schrödinger equation gives exponential spatial behavior there.
Saying tunneling violates energy conservation	A static barrier does not change the conserved total energy of the stationary state.
Using a Schrödinger wave function for a photon	The nonrelativistic Schrödinger equation is for massive particles in the present treatment.
Calling a stationary state motionless	“Stationary” means the probability density is time independent, not necessarily that momentum or current vanish.
Assuming all exponentials are normalizable	Plane waves and growing exponentials over infinite domains are not normalizable as one-particle states; boundary conditions decide which solutions are physical.

18.19 Main chapter summary

Summary

- Quantum physics becomes necessary when classical wave and particle descriptions fail to account for microscopic experiments.
- The photon energy and momentum in vacuum are $E_\gamma = hf = hc/\lambda$ and $p_\gamma = E_\gamma/c = h/\lambda$.
- The photoelectric equation is $K_{\max} = hf - \phi = eV_s$. Frequency sets energy per photon; intensity at fixed frequency mainly controls photon flux.
- Compton scattering combines photon momentum with relativistic energy–momentum conservation and gives $\Delta\lambda = (h/m_e c)(1 - \cos\theta)$.
- A massive particle with momentum p has de Broglie wavelength $\lambda = h/p$. The relation is general; $p = mv$ is only a low-speed approximation.
- Quantum states are described by probability amplitudes. In one dimension $|\Psi(x, t)|^2$ is a position probability density and must be normalized for localized one-particle states.
- The nonrelativistic time-dependent Schrödinger equation is $i\hbar\partial_t\Psi = \hat{H}\Psi$. For time-independent potentials, stationary states satisfy the time-independent Schrödinger equation.
- The uncertainty relation $\Delta x\Delta p_x \geq \hbar/2$ describes intrinsic spreads of quantum observables for a state, not merely poor measurement equipment.
- Abrupt potential changes can produce quantum reflection. Finite barriers permit tunneling, with transmission often dominated by an exponential factor $e^{-2\kappa L}$.
- The same physics underlies photodetection, electron microscopy, scanning tunneling microscopy, tunnel devices, interferometric sensors, and numerical quantum modelling.

Take-home message

The birth of quantum physics is not the abandonment of conservation laws, waves, or energy. It is a change in what those concepts describe. Energy and momentum are exchanged in quantum events, amplitudes rather than classical trajectories determine interference, and the Schrödinger equation evolves probabilities through a complex wave function.

18.20 Advanced Topics for Technical and Engineering Students

The following optional modules deepen the main chapter through derivations, measurement models, and reproducible computation. They assume the core discussion and introduce each additional mathematical tool locally. The one-particle, nonrelativistic scattering model remains the default; relativistic photon kinematics are used only where explicitly stated.

18.21 The Planck spectrum beyond the qualitative picture

The spectral radiance of an ideal blackbody may be written in wavelength form as

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp\left(\frac{hc}{\lambda k_B T}\right) - 1}. \quad (18.66)$$

Different texts use related quantities—spectral energy density, spectral exitance, or radiance—that differ by geometric factors. Equation (18.66) is a radiance per unit wavelength and solid angle. The distinction matters experimentally because a detector measures power through a finite area, angle, and bandwidth rather than an abstract spectrum.

The factor

$$x = \frac{hc}{\lambda k_B T} \quad (18.67)$$

controls the crossover between long- and short-wavelength behavior. When $x \ll 1$, the exponential can be expanded as $e^x - 1 \approx x$, giving the Rayleigh–Jeans limit

$$B_\lambda(T) \approx \frac{2ck_B T}{\lambda^4}. \quad (18.68)$$

This classical form works at sufficiently long wavelengths. At short wavelength, $x \gg 1$ and

$$B_\lambda(T) \approx \frac{2hc^2}{\lambda^5} e^{-hc/(\lambda k_B T)}, \quad (18.69)$$

so the spectrum is exponentially suppressed rather than diverging.

18.21.1 Mode energy and the classical limit

For an oscillator with excitation energies $E_n = nhf$, the Boltzmann weights are proportional to q^n with $q = \exp[-hf/(k_B T)]$. Summing the geometric series gives

$$Z = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \quad \langle E \rangle = hf \frac{q}{1-q} = \frac{hf}{\exp[hf/(k_B T)] - 1}. \quad (18.70)$$

These are the thermal excitation energies relevant for the spectrum; the oscillator zero-point energy is not counted as thermally emitted radiation. The electromagnetic mode density, including two polarizations, is $8\pi f^2/c^3$ per unit volume per unit frequency. Thus

$$u_f = \frac{8\pi hf^3}{c^3} \frac{1}{e^{hf/(k_B T)} - 1}, \quad B_f = \frac{c}{4\pi} u_f. \quad (18.71)$$

The relation $B_\lambda = B_f |df/d\lambda|$ with $|df/d\lambda| = c/\lambda^2$ then yields Eq. (18.66). At $hf \ll k_B T$, the mean excitation energy approaches $k_B T$, recovering classical equipartition. At $hf \gg k_B T$, exciting even one quantum is thermally unlikely.

18.21.2 Peak location and integrated power

At fixed temperature, substitute $x = hc/(\lambda k_B T)$. Apart from constants, $B_\lambda \propto x^5/(e^x - 1)$, so its nonzero maximum satisfies

$$5(1 - e^{-x}) = x, \quad x = 4.9651 \dots, \quad \lambda_{\max} T = \frac{hc}{4.9651 k_B}. \quad (18.72)$$

For B_f , the power of x is three, giving $3(1 - e^{-x}) = x$ and $hf_{\max}/(k_B T) = 2.8214 \dots$. The two maxima describe different spectral densities; both represent the same radiation when integrated over corresponding intervals.

Integrating $M_\lambda = \pi B_\lambda$ over wavelength gives

$$M = \frac{2\pi k_B^4 T^4}{h^3 c^2} \int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{2\pi^5 k_B^4}{15 h^3 c^2} T^4 = \sigma_{\text{SB}} T^4. \quad (18.73)$$

The integral equals $\pi^4/15$. This derivation explains the fourth power of temperature rather than introducing it as an unrelated empirical rule.

Example 18.20: Dimensionless parameter for thermal radiation

For a blackbody at $T = 3000 \text{ K}$ and wavelength $\lambda = 1.0 \mu\text{m}$,

$$x = \frac{hc}{\lambda k_B T} \approx \frac{(6.626 \times 10^{-34})(2.998 \times 10^8)}{(10^{-6})(1.381 \times 10^{-23})(3000)} \approx 4.80.$$

The wavelength is therefore not in the Rayleigh–Jeans limit $x \ll 1$. A classical long-wavelength approximation would be quantitatively poor near this region of the spectrum.

Guided checks

1. Why does the Rayleigh–Jeans approximation improve as λ becomes large at fixed T ?
2. Why is it unsafe to compare a theoretical B_λ curve directly with detector counts before including the detector bandwidth and spectral responsivity?
3. If the temperature doubles, does the peak wavelength double or halve? What happens qualitatively to the total emitted power?

18.22 Photoelectric data as a parameter-estimation problem

The photoelectric equation can be rewritten as a linear model,

$$V_s = \frac{h}{e} f - \frac{\phi}{e}. \quad (18.74)$$

This form is valuable experimentally because it separates two parameters. The slope is h/e , whereas the vertical intercept is $-\phi/e$. A set of stopping-potential measurements at several frequencies therefore provides more information than a single threshold measurement. It also makes uncertainty analysis natural.

Suppose measured values are (f_i, V_i) with standard uncertainties σ_{V_i} . If frequency uncertainties are negligible, a weighted least-squares fit minimizes

$$\chi^2(a, b) = \sum_i \frac{[V_i - (af_i + b)]^2}{\sigma_{V_i}^2}, \quad (18.75)$$

with $a = h/e$ and $b = -\phi/e$. If both frequency and voltage uncertainties are relevant, ordinary least squares is no longer the complete statistical model; an errors-in-variables treatment or a likelihood including both uncertainties is preferable.

Example 18.21: Interpreting a fitted photoelectric slope

A linear fit to idealized stopping-potential data gives

$$a = (4.14 \pm 0.05) \times 10^{-15} \text{ V s}, \quad b = -(2.30 \pm 0.04) \text{ V}.$$

Because $h = ea$,

$$h = (1.602 \times 10^{-19} \text{ C})(4.14 \times 10^{-15} \text{ V s}) = 6.63 \times 10^{-34} \text{ J s}.$$

The work function is

$$\phi = -eb = 2.30 \text{ eV}.$$

The uncertainty in h from the fitted slope alone is approximately

$$\sigma_h = e\sigma_a \approx 8.0 \times 10^{-36} \text{ J s}.$$

The example illustrates an important laboratory habit: identify which fitted parameter corresponds to which physical constant before inserting numbers.

A real photoelectric experiment contains additional systematics: contact potentials, finite bandwidth of the light source, surface contamination, nonuniform work function, finite instrument resolution, and the operational definition of the stopping potential. These effects do not invalidate Einstein's relation; they determine how accurately its parameters can be extracted from a particular apparatus.

Method

For a laboratory fit of Eq. (18.74):

1. plot the raw data before fitting;
2. fit V_s versus f , not the inverse relation, if the dominant measurement uncertainty is in voltage;
3. inspect residuals for curvature or an outlying wavelength;
4. convert the fitted slope to h only after checking units;
5. quote statistical and identifiable systematic uncertainties separately when possible.

18.23 Wave packets, dispersion, phase velocity, and group velocity

A single plane wave

$$\Psi(x, t) = Ae^{i(kx - \omega t)} \quad (18.76)$$

has a perfectly definite wave number and momentum $p = \hbar k$, but its magnitude $|\Psi|^2 = |A|^2$ is uniform over all space. It therefore cannot represent a localized particle. Localization requires a superposition of wave numbers,

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\Psi}(k) e^{i[kx - \omega(k)t]} dk. \quad (18.77)$$

The envelope of this superposition is a wave packet.

For a free nonrelativistic particle,

$$E = \frac{p^2}{2m}, \quad E = \hbar\omega, \quad p = \hbar k, \quad (18.78)$$

so

$$\omega(k) = \frac{\hbar k^2}{2m}. \quad (18.79)$$

The phase velocity of a single Fourier component is

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{\hbar k}{2m} = \frac{v}{2}, \quad (18.80)$$

whereas the group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = v. \quad (18.81)$$

Thus the wave-packet envelope propagates at the classical particle speed in the nonrelativistic free-particle limit.

Because $\omega(k)$ is nonlinear, different Fourier components have different group velocities. A free wave packet generally spreads. This dispersion is not friction and does not represent loss of total probability. It is a consequence of superposing components with a range of momenta.

Caution

Do not identify the phase velocity of a matter-wave Fourier component with the speed of the particle. For a localized quantum state, the motion of the wave-packet envelope and the probability distribution is the physically useful comparison. In the nonrelativistic free-particle case, the group velocity equals p/m .

Example 18.22: Characteristic spreading scale

A minimum-uncertainty Gaussian electron packet has initial position spread $\Delta x = 0.20 \text{ nm}$. Its minimum momentum spread is

$$\Delta p = \frac{\hbar}{2\Delta x} \approx 2.64 \times 10^{-25} \text{ kg m s}^{-1}.$$

The kinetic-energy scale associated with this spread is

$$\frac{(\Delta p)^2}{2m_e} \approx 0.238 \text{ eV}.$$

This estimate shows why highly localized electron packets inevitably contain a broad range of momenta.

18.23.1 Gaussian packets and the minimum-uncertainty state

A normalized Gaussian packet at $t = 0$ can be written

$$\Psi(x, 0) = \frac{1}{(2\pi\sigma_x^2)^{1/4}} \exp\left[-\frac{(x - x_0)^2}{4\sigma_x^2}\right] \exp(ik_0x). \quad (18.82)$$

Its probability density is a Gaussian centered at x_0 with standard deviation σ_x ,

$$|\Psi(x, 0)|^2 = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{(x - x_0)^2}{2\sigma_x^2}\right]. \quad (18.83)$$

The Fourier transform is also Gaussian. Its wave-number standard deviation is

$$\Delta k = \frac{1}{2\sigma_x}. \quad (18.84)$$

Since $p = \hbar k$,

$$\Delta p = \hbar \Delta k = \frac{\hbar}{2\sigma_x}, \quad (18.85)$$

and therefore

$$\boxed{\Delta x \Delta p = \frac{\hbar}{2}}. \quad (18.86)$$

This unchirped Gaussian saturates the product bound at $t = 0$. A Gaussian with an additional quadratic phase generally has nonzero position–momentum covariance and a larger $\Delta x \Delta p$. Free evolution also creates such covariance, even though the state remains Gaussian.

Theoretical point 18.2: Fourier structure behind uncertainty

The position–momentum uncertainty relation is closely tied to Fourier analysis. A narrow function in position requires a broad spectrum of spatial frequencies k , while a narrow spectrum in k produces a broad spatial function. Quantum mechanics connects k to momentum by $p = \hbar k$, turning a general wave property into the position–momentum uncertainty relation.

18.23.2 Free-packet spreading

For an initially minimum-uncertainty Gaussian packet of position width σ_0 , free evolution gives a position variance

$$\sigma_x^2(t) = \sigma_0^2 \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2 \right]. \quad (18.87)$$

The spreading law can be derived without evaluating the full Fourier integral. For a free particle, the Heisenberg operator obeys $x(t) = x(0) + p(0)t/m$. Subtract the mean, square, and take the expectation to obtain

$$(\Delta x(t))^2 = (\Delta x(0))^2 + \frac{2t}{m} C_{xp}(0) + \frac{t^2}{m^2} (\Delta p(0))^2, \quad (18.88)$$

where $C_{xp} = \frac{1}{2} \langle (x - \langle x \rangle)(p - \langle p \rangle) + (p - \langle p \rangle)(x - \langle x \rangle) \rangle$ is the symmetrized covariance. For the unchirped Gaussian, $C_{xp}(0) = 0$ and $\Delta p(0) = \hbar/(2\sigma_0)$, reproducing Eq. (18.87). A correlated packet can initially narrow, so monotonic spreading is not a claim about every possible initial state.

A characteristic spreading time is therefore

$$t_s = \frac{2m\sigma_0^2}{\hbar}. \quad (18.89)$$

For $t \ll t_s$, the width changes little; for $t \gg t_s$, it grows approximately linearly with time.

Example 18.23: Electron versus heavy-particle spreading

For an electron with $\sigma_0 = 1.0 \text{ nm}$,

$$t_s = \frac{2m_e \sigma_0^2}{\hbar} \approx 1.73 \times 10^{-14} \text{ s}.$$

For a particle a million times heavier with the same initial width, the spreading time is a million times larger. Mass therefore suppresses free wave-packet spreading strongly, another route toward classical-looking dynamics for heavy objects.

18.23.3 Operator viewpoint and the origin of the uncertainty bound

The main chapter introduced uncertainty through standard deviations. A deeper formulation associates observables with operators. Position and momentum in one dimension are represented by

$$\hat{x} = x, \quad \hat{p}_x = -i\hbar \frac{\partial}{\partial x}. \quad (18.90)$$

Their commutator is

$$[\hat{x}, \hat{p}_x] \equiv \hat{x}\hat{p}_x - \hat{p}_x\hat{x} = i\hbar. \quad (18.91)$$

For any two Hermitian observables A and B , the Robertson relation gives

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|. \quad (18.92)$$

Applying Eq. (18.91),

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}. \quad (18.93)$$

The result is therefore connected to the mathematical incompatibility of the position and momentum operators, not to a particular microscope design. Measurement disturbance can be physically important in real experiments, but it is not the fundamental derivation of the standard state-preparation uncertainty relation.

Pedagogical boundary

A rigorous proof of the Robertson inequality uses the Cauchy–Schwarz inequality in Hilbert space and distinguishes state preparation from measurement-error relations. Here the operator result is included to show the structural origin of $\hbar/2$, not to develop a complete mathematical formalism of observables.

18.23.4 Ehrenfest’s theorem and the classical correspondence

Quantum mechanics does not discard Newtonian mechanics; classical equations reappear under appropriate conditions. For a particle with Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}), \quad (18.94)$$

Ehrenfest’s theorem gives

$$\frac{d\langle x \rangle}{dt} = \frac{\langle p \rangle}{m}, \quad (18.95)$$

$$\frac{d\langle p \rangle}{dt} = - \left\langle \frac{dV}{dx} \right\rangle. \quad (18.96)$$

If a wave packet is narrow enough that the force does not vary substantially across it, then

$$\left\langle \frac{dV}{dx} \right\rangle \approx \left. \frac{dV}{dx} \right|_{x=\langle x \rangle}, \quad (18.97)$$

and the packet center approximately follows Newton’s second law.

Key message

The classical limit is not obtained by declaring quantum physics “turned off.” It emerges when relevant actions are large compared with $\hbar$, phases average rapidly, wave packets remain sufficiently localized for the observable of interest, and environmental decoherence suppresses interference between macroscopically distinct alternatives.

18.24 Probability current and local conservation of probability

Normalization states that total probability is conserved, but a stronger local statement is available. For the one-dimensional Schrödinger equation with a real potential, define the probability density

$$\rho(x, t) = |\Psi(x, t)|^2 \quad (18.98)$$

and probability current

$$j(x, t) = \frac{\hbar}{m} \text{Im} \left(\Psi^* \frac{\partial \Psi}{\partial x} \right). \quad (18.99)$$

Subtracting the Schrödinger equation times Ψ^* from its conjugate times Ψ , with signs arranged to isolate the time derivative, gives

$$\partial_t |\Psi|^2 = \frac{i\hbar}{2m} (\Psi^* \partial_x^2 \Psi - \Psi \partial_x^2 \Psi^*) \quad (18.100)$$

$$= -\partial_x \left[\frac{\hbar}{2mi} (\Psi^* \partial_x \Psi - \Psi \partial_x \Psi^*) \right]. \quad (18.101)$$

The real-potential terms cancel. The expression in brackets is exactly j from Eq. (18.99). Integrating over $a < x < b$ gives $dP_{[a,b]}/dt = j(a) - j(b)$: probability in the interval changes by inflow minus outflow. With one-dimensional normalization, $[j] = \text{s}^{-1}$.

The result is the continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0.} \quad (18.102)$$

This has the same mathematical form as local conservation laws in fluid mechanics, charge conservation, and heat transport, although the physical meaning of the transported quantity is different.

For a plane wave $Ae^{i(kx - \omega t)}$,

$$j = \frac{\hbar k}{m} |A|^2. \quad (18.103)$$

The sign of k determines the direction of probability flow. This is why transmission coefficients in a potential-step problem must be calculated as ratios of currents, not merely ratios of squared amplitudes when the wave numbers differ.

Guided checks

1. If ρ decreases in a small interval, what must Eq. (18.102) imply about the net probability current through its boundaries?
2. Why can j be negative while ρ cannot?
3. Why is current, rather than wave amplitude alone, the correct quantity for defining transmission probability across regions with different wave numbers?

18.24.1 Exact scattering from a one-dimensional potential step

Consider

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geq 0, \end{cases} \quad (18.104)$$

and a particle incident from the left with $E > V_0$. In the two regions,

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x}, \quad k_1 = \frac{\sqrt{2mE}}{\hbar}, \quad (18.105)$$

$$\psi_2(x) = Ce^{ik_2x}, \quad k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}. \quad (18.106)$$

The boundary conditions follow from the differential equation, rather than from a choice to make the graph attractive. Integrating across $-\epsilon < x < \epsilon$ gives

$$-\frac{\hbar^2}{2m}[\psi'(\epsilon) - \psi'(-\epsilon)] + \int_{-\epsilon}^{\epsilon} (V - E)\psi \, dx = 0. \quad (18.107)$$

For a bounded potential and finite amplitude, the integral vanishes as $\epsilon \rightarrow 0$, so ψ' is continuous. A jump in ψ would itself create singular derivatives absent from this finite-potential equation, so ψ is also continuous. These conditions assume constant mass; effective-mass interfaces in solids require their own Hamiltonian and matching conditions.

Continuity of ψ and $d\psi/dx$ at $x = 0$ gives

$$A + B = C, \quad k_1(A - B) = k_2C. \quad (18.108)$$

Hence

$$r \equiv \frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}, \quad t \equiv \frac{C}{A} = \frac{2k_1}{k_1 + k_2}. \quad (18.109)$$

The current ratios are

$$R = |r|^2, \quad T = \frac{k_2}{k_1}|t|^2 = \frac{4k_1k_2}{(k_1 + k_2)^2}, \quad (18.110)$$

and $R + T = 1$.

For $E < V_0$, the second-region wave number is imaginary. Writing $k_2 = i\kappa$ gives an evanescent wave

$$\psi_2(x) = Ce^{-\kappa x}, \quad \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \quad (18.111)$$

For a semi-infinite step this decaying tail carries no transmitted probability current to $x \rightarrow +\infty$, and $R = 1$. The nonzero tail nevertheless becomes crucial when the forbidden region has finite width, because it can connect to a propagating wave on the far side of a barrier.

18.25 Exact rectangular-barrier transmission

For a rectangular barrier

$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & 0 \leq x \leq L, \\ 0, & x > L, \end{cases} \quad (18.112)$$

with $0 < E < V_0$ and the same mass and potential on the two exterior sides, choose incident amplitude one:

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx}, & x < 0, \\ Ae^{\kappa x} + Be^{-\kappa x}, & 0 < x < L, \\ te^{ikx}, & x > L, \end{cases} \quad k = \frac{\sqrt{2mE}}{\hbar}. \quad (18.113)$$

An efficient route propagates the pair (ψ, ψ') across the barrier:

$$\begin{pmatrix} \psi(L) \\ \psi'(L) \end{pmatrix} = \begin{pmatrix} \cosh \kappa L & \sinh(\kappa L)/\kappa \\ \kappa \sinh \kappa L & \cosh \kappa L \end{pmatrix} \begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix}. \quad (18.114)$$

Insert $\psi(0) = 1 + r$, $\psi'(0) = ik(1 - r)$, $\psi(L) = te^{ikL}$, and $\psi'(L) = ikte^{ikL}$. Eliminating r gives

$$t = \frac{e^{-ikL}}{\cosh(\kappa L) + i\frac{\kappa^2 - k^2}{2k\kappa} \sinh(\kappa L)}. \quad (18.115)$$

Since exterior wave numbers are equal, $T = |t|^2$. Using $\cosh^2 z = 1 + \sinh^2 z$ and $(k^2 + \kappa^2)^2 / (4k^2\kappa^2) = V_0^2 / [4E(V_0 - E)]$ produces the expression below. For unequal exterior potentials, the current-velocity factor must be restored.

Matching ψ and ψ' at both interfaces gives

$$T = \left[1 + \frac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E)} \right]^{-1}, \quad (18.116)$$

where

$$\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}. \quad (18.117)$$

When $\kappa L \gg 1$,

$$T \approx 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0} \right) e^{-2\kappa L}. \quad (18.118)$$

The frequently quoted $e^{-2\kappa L}$ expression captures the dominant exponential sensitivity but omits the prefactor.

18.25.1 Limits, above-barrier interference, and energy filtering

The exact formula passes several checks. At fixed $E > 0$, $L \rightarrow 0$ gives $T \rightarrow 1$. At fixed nonzero width, $E \rightarrow 0^+$ gives $T \rightarrow 0$. At the barrier top the apparent denominator $V_0 - E$ is removable because $\sinh(\kappa L) \simeq \kappa L$:

$$\lim_{E \rightarrow V_0} T = \left(1 + \frac{mV_0 L^2}{2\hbar^2} \right)^{-1}. \quad (18.119)$$

Thus $\kappa \rightarrow 0$ does not mean unit transmission through a barrier of fixed width. The thick-barrier approximation fails precisely in this limit.

For $E > V_0$, write $q = \sqrt{2m(E - V_0)}/\hbar$; continuation from hyperbolic to trigonometric functions gives

$$T = \left[1 + \frac{V_0^2 \sin^2(qL)}{4E(E - V_0)} \right]^{-1}. \quad (18.120)$$

When $qL = n\pi$ with positive integer n , reflections from the two interfaces cancel and $T = 1$. Such resonances are ordinary amplitude interference in a quantum scattering problem. Multiple barriers and finite wells extend this idea, but detailed heterostructures belong to later chapters.

For an incident energy distribution $F(E)$, a static barrier filters the outgoing distribution as $F_{\text{out}}(E) \propto T(E)F(E)$. A finite-width packet therefore has

$$T_{\text{packet}} = \int_0^\infty |a(k)|^2 T(k) dk, \quad (18.121)$$

if its positive-momentum incident spectrum $a(k)$ is normalized and the outgoing pieces are asymptotically separated. Comparing a numerical packet directly with T evaluated only at the mean energy is justified only for a sufficiently narrow spectrum.

Figure 18.11 separates two experimental questions: changing the incident energy at fixed geometry and changing the width at fixed energy. Reflection can persist above the barrier, while interference can also give perfect transmission at specific energies. In the thick regime, a semilogarithmic width plot approaches a straight line with slope -2κ .

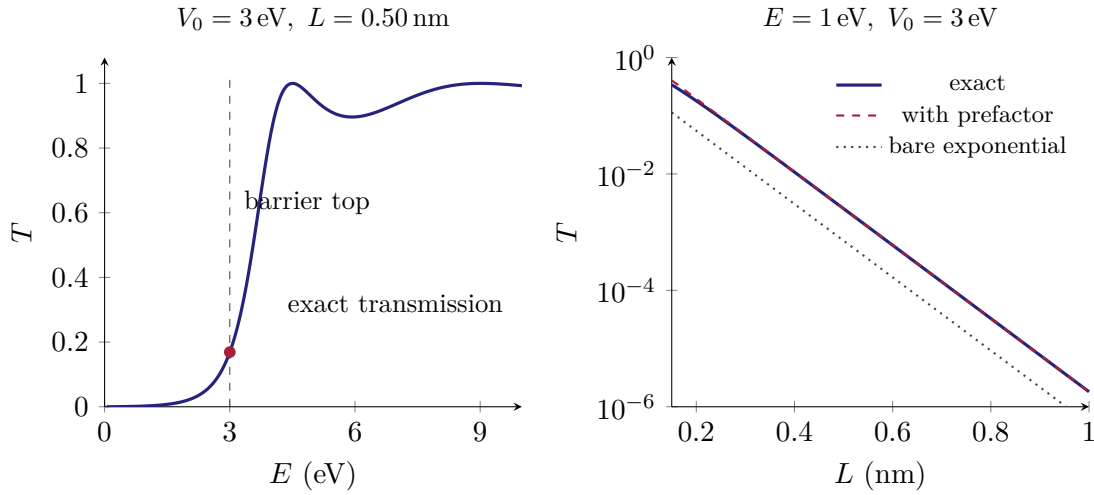


Figure 18.11: Calculated electron transmission through a rectangular barrier. Left: the exact curve is continuous at $E = V_0$, and its first unit-transmission resonance is near 4.50 eV. Right: the thick-barrier approximation approaches the exact curve; the bare exponential retains a multiplicative error. The vertical scale in the right panel is logarithmic.

Example 18.24: A barrier selects an incident energy distribution

An incident ensemble has a normalized energy density

$$F(E) = C^{-1} \exp\left[-\frac{(E - E_0)^2}{2\sigma_E^2}\right], \quad E > 0, \quad E_0 = 1.00 \text{ eV}, \quad \sigma_E = 0.20 \text{ eV},$$

and meets a barrier with $V_0 = 3.00 \text{ eV}$ and $L = 0.50 \text{ nm}$. What fraction transmits, and what is the mean energy of the transmitted subset?

Strategy. Average the probability $T(E)$ over the prepared distribution. Then normalize the selected distribution before computing its mean. The integral $C = \int_0^\infty \exp[-(E - E_0)^2/(2\sigma_E^2)] dE$ has energy units, so F has inverse-energy units. Its numerical value is $C \simeq 0.501326 \text{ eV}$.

$$P_T = \int_0^\infty F(E)T(E)dE \simeq 2.773 \times 10^{-3},$$

$$\langle E \rangle_T = \frac{\int_0^\infty EF(E)T(E)dE}{P_T} \simeq 1.091 \text{ eV}.$$

These values follow by one-dimensional numerical quadrature of the exact transmission function. Using only $T(E_0) = 2.534 \times 10^{-3}$ would underestimate the average: the actual average is about 9.4% larger. Higher-energy members transmit more readily in this energy range.

The transmitted mean exceeds the incident mean because the barrier selects different members of the ensemble. Each transmitted member conserves its total energy in the static potential. The same spectral weighting describes a suitably prepared incident packet after its reflected and transmitted parts separate; it does not imply an energy gain during tunneling.

Example 18.25: Exact and asymptotic tunneling estimates

An electron has $E = 1.0 \text{ eV}$ and encounters a $V_0 = 3.0 \text{ eV}$ barrier of width $L = 0.50 \text{ nm}$. The decay constant is

$$\kappa \approx 7.25 \text{ nm}^{-1}, \quad \kappa L \approx 3.62.$$

Equation (18.116) gives

$$T \approx 2.53 \times 10^{-3}.$$

The thick-barrier approximation including its prefactor gives 2.54×10^{-3} , already very close. The bare exponential $e^{-2\kappa L}$ is 7.14×10^{-4} and therefore captures the scale but not the numerical coefficient.

Caution

Tunneling does not mean that a particle temporarily acquires classical kinetic energy $E - V_0 < 0$. The classically forbidden region is described by an evanescent quantum amplitude. The particle is detected on the transmitted side with the same conserved total energy E for a static barrier.

18.25.2 Semiclassical tunneling and the WKB exponent

The rectangular-barrier formula is exact for a very special potential. Many physical barriers vary smoothly. A useful semiclassical approximation is the Wentzel–Kramers–Brillouin (WKB) method. In a classically forbidden region where $V(x) > E$, define

$$\kappa(x) = \frac{\sqrt{2m[V(x) - E]}}{\hbar}. \quad (18.122)$$

Between classical turning points x_1 and x_2 , the leading tunneling dependence is

$$T \sim \exp\left[-2 \int_{x_1}^{x_2} \kappa(x) dx\right]. \quad (18.123)$$

For a rectangular barrier, κ is constant and Eq. (18.123) reduces to $\exp(-2\kappa L)$.

The WKB expression shows that the relevant “barrier size” is an action-like integral, not width alone. A barrier that is wider but only slightly above the particle energy may be more transparent than a narrower but much taller barrier.

Caution

WKB is not uniformly accurate near turning points where $E = V(x)$, and it can fail for abrupt structure on the scale of the local wavelength. It is an approximation whose validity must be checked, not a universal replacement for solving the Schrödinger equation.

18.25.3 Field emission as barrier shaping

A strong electric field near a metal surface can tilt the electron potential-energy barrier. In a simplified one-dimensional picture, the vacuum-side potential contains a term approximately linear in distance,

$$V(x) \approx \Phi - e\mathcal{E}x, \quad (18.124)$$

where Φ is an effective work-function scale and $\mathcal{E}$ is the electric-field magnitude. The barrier becomes triangular rather than rectangular. Increasing $\mathcal{E}$ narrows the forbidden region and can increase tunneling emission dramatically.

For an electron whose energy is the chosen zero, the turning point is $x_2 = \Phi/(e\mathcal{E})$. The leading exponent is found by direct integration:

$$2 \int_0^{x_2} \kappa(x) dx = \frac{2\sqrt{2m_e}}{\hbar} \int_0^{\Phi/(e\mathcal{E})} \sqrt{\Phi - e\mathcal{E}x} dx \quad (18.125)$$

$$= \frac{4\sqrt{2m_e} \Phi^{3/2}}{3\hbar e\mathcal{E}}. \quad (18.126)$$

Hence T has a leading dependence $\exp[-4\sqrt{2m_e}\Phi^{3/2}/(3\hbar e\mathcal{E})]$. This is the origin of the strong field dependence in elementary field-emission models. It is not a complete current law: occupation factors, supply of electrons, and prefactors remain necessary.

Example 18.26: How changing barrier shape changes the exponent

Compare a triangular and a rectangular forbidden region with the same entrance deficit Φ and width $L = \Phi/(e\mathcal{E})$. Write $\kappa_0 = \sqrt{2m_e}\Phi/\hbar$. The rectangular exponent is $2\kappa_0 L$; the triangular exponent is $(4/3)\kappa_0 L$. If $\kappa_0 L = 6$, their exponential factors are e^{-12} and e^{-8} , a ratio $e^4 \simeq 55$. Both remain small, and unknown prefactors prevent treating 55 as an exact ratio of physical currents. The comparison isolates why reshaping the forbidden region matters, even without changing its width.

A quantitative treatment requires image-charge corrections, material parameters, surface geometry, and a careful definition of local electric field, but the underlying quantum mechanism is already visible: an external field reshapes the potential and changes the tunneling action.

This connection is important in electron sources, vacuum nanoelectronics, high-field surfaces, and breakdown studies. It also illustrates why sharp tips can matter: electrostatic field concentration changes the local barrier profile.

18.26 Scanning tunneling microscopy as exponential metrology

A scanning tunneling microscope (STM) brings a conducting tip within roughly a nanometre of a conducting surface. Electrons tunnel through the vacuum gap. A simplified model writes the current as

$$I(s) \propto e^{-2\kappa s}, \quad \kappa \approx \frac{\sqrt{2m_e}\Phi}{\hbar}, \quad (18.127)$$

where s is the tip-sample separation and Φ is an effective barrier height related to the work functions and applied bias. Real STM theory is richer because the density of electronic states and tip geometry also enter, but Eq. (18.127) explains the extraordinary vertical sensitivity.

Example 18.27: STM current sensitivity

Take an effective barrier height $\Phi = 4.0 \text{ eV}$. Then

$$\kappa \approx 10.25 \text{ nm}^{-1}.$$

Increasing the gap by only 0.10 nm changes the simplified current by the factor

$$\frac{I(s + 0.10 \text{ nm})}{I(s)} = \exp[-2\kappa(0.10 \text{ nm})] \approx 0.129.$$

Thus a 0.10 nm increase can reduce the current by nearly a factor of eight. This exponential response is both the power and the experimental challenge of STM: vibration isolation, feedback stability, and tip condition become critical.

STM commonly operates in two modes. In constant-height mode, the tip follows a nominally fixed vertical coordinate and current variations are recorded. In constant-current mode, feedback adjusts the tip height to maintain a target current, and the required vertical displacement forms the image signal. The recorded topography generally reflects both geometric height and electronic structure; interpreting every contrast feature as literal surface height would be an oversimplification.

18.26.1 Displacement sensitivity and calibration

Example 18.28: Sensitivity, dynamic range, and calibration

A conceptual displacement sensor, treated in a fixed-bias barrier model, uses a tunnel current

$$I(s) = I_0 e^{-2\kappa s}$$

with effective $\kappa = 9.0 \text{ nm}^{-1}$. At the operating point s_0 , the measured current is $I = 10 \text{ nA}$.

Local sensitivity. Differentiating,

$$\frac{dI}{ds} = -2\kappa I.$$

At the operating point,

$$\frac{dI}{ds} = -18 \text{ nm}^{-1} (10 \text{ nA}) = -180 \text{ nA nm}^{-1}.$$

A displacement of $1 \text{ pm} = 10^{-3} \text{ nm}$ therefore produces a local current change of about 0.18 nA in the linearized approximation.

Nonlinearity. For a larger displacement $\Delta s = 0.05 \text{ nm}$, linearization is poor because

$$\frac{I(s_0 + \Delta s)}{I(s_0)} = e^{-0.9} \approx 0.407.$$

The sensor response is strongly nonlinear over such a range.

Calibration lesson. Inferring s from I requires knowledge of I_0 and κ , both of which can drift with tip condition, barrier chemistry, bias, and temperature. High intrinsic sensitivity does not remove the need for calibration and feedback.

18.26.2 Feedback and electronic contrast

The exponential current–distance relation makes STM naturally connected to control engineering. In constant-current operation, the measured current is compared with a set point I_{set} . A feedback controller adjusts the vertical actuator to reduce the error

$$e_I(t) = I_{\text{set}} - I(t). \quad (18.128)$$

The recorded actuator displacement is then used to build an image.

A simplified local relation is

$$\frac{\delta I}{I} \approx -2\kappa \delta s. \quad (18.129)$$

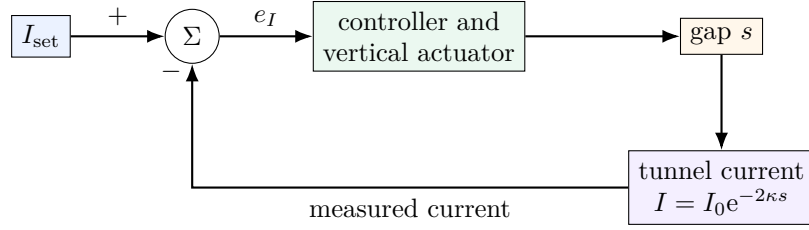
For $\kappa \sim 10 \text{ nm}^{-1}$, a 1 pm displacement gives a relative current change of about 2%. This provides excellent sensitivity, but it also means that vibration and electronic noise are strongly coupled into the signal.

The feedback bandwidth must be high enough to follow intended surface variations but low enough to reject excessive noise and avoid instability. A realistic image is therefore a product of quantum tunneling, mechanical dynamics, piezoelectric actuation, electronic amplification, and feedback design.

Figure 18.12 shows the closed feedback path and the required correction sign.

18.27 Quantitative double-slit interference for matter waves

The qualitative statement that single particles build an interference pattern can be sharpened. Consider two narrow openings separated by distance d , observed on a distant screen at coordinate



For $I < I_{\text{set}}$, the controller reduces s to increase I : the correction opposes the error.

Figure 18.12: Constant-current STM feedback. The sign of the actuator response must compensate the negative derivative dI/ds . Current contrast also depends on the electronic structure, so the recorded height signal is not automatically a purely geometric surface profile.

y . In the paraxial limit, the path difference is approximately

$$\delta \approx d \sin \theta \approx d \frac{y}{D}, \quad (18.130)$$

where D is the slit–screen distance. The relative phase is

$$\Delta\varphi = \frac{2\pi}{\lambda} \delta \approx \frac{2\pi d}{\lambda D} y. \quad (18.131)$$

If the two alternatives have equal amplitudes and remain coherent, the probability density has the familiar interference factor

$$P(y) \propto 1 + \cos \Delta\varphi = 2 \cos^2 \left(\frac{\Delta\varphi}{2} \right). \quad (18.132)$$

For matter waves, $\lambda = h/p$. The bright-fringe spacing is therefore approximately

$$\Delta y = \frac{\lambda D}{d} = \frac{hD}{pd}. \quad (18.133)$$

The distances entering the fringe-spacing formula are defined in Fig. 18.13.

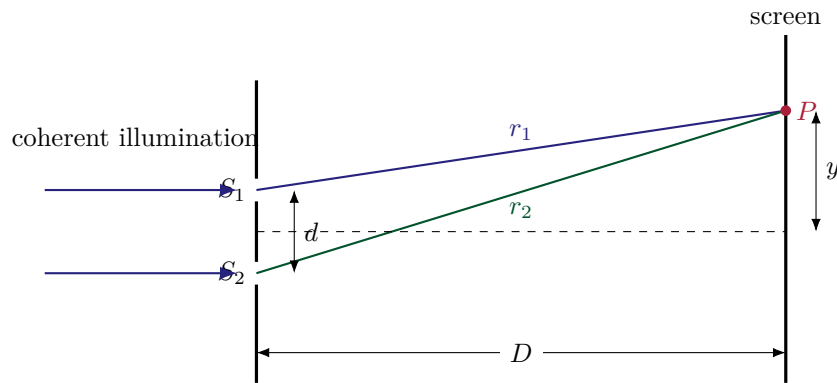


Figure 18.13: Geometry for matter-wave interference, not to scale. In the far field and at small angles, $r_2 - r_1 \simeq dy/D$. The fringe spacing follows from changing this path difference by one de Broglie wavelength. Finite slit width supplies a diffraction envelope.

Example 18.29: Fringe spacing for electrons

Electrons with wavelength $\lambda = 0.050 \text{ nm}$ pass through two effective openings separated by

$d = 2.0 \mu\text{m}$, and a detector is $D = 1.0 \text{ m}$ away. In the far-field approximation,

$$\Delta y = \frac{(5.0 \times 10^{-11} \text{ m})(1.0 \text{ m})}{2.0 \times 10^{-6} \text{ m}} = 2.5 \times 10^{-5} \text{ m} = 25 \mu\text{m}.$$

The individual electron impacts are localized, but the statistical distribution develops fringes separated by about $25 \mu\text{m}$.

The interference formula also clarifies what destroys visible fringes. If environmental interactions or path information make the two alternatives distinguishable, the observable cross term can be suppressed. Uncontrolled phase averaging is one mechanism; correlation with distinguishable path markers is another, even without a literal random phase for each run. This is one operational route toward the classical limit: not because h changes, but because coherent phase information becomes inaccessible.

18.27.1 Coherence, which-path information, and visibility

For two contributions with intensities I_1 and I_2 , a scalar two-beam model with matched polarization can be written as

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} |\gamma| \cos(\Delta\varphi + \varphi_0), \quad (18.134)$$

where $0 \leq |\gamma| \leq 1$ is a degree of coherence. In quantum language, the same mathematical structure appears in detection probabilities when two probability amplitudes remain coherent.

The fringe visibility is

$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}. \quad (18.135)$$

More generally, $\mathcal{V} = 2\sqrt{I_1 I_2} |\gamma| / (I_1 + I_2)$; for equal intensities it reduces to $\mathcal{V} = |\gamma|$. Reduced visibility can arise from finite source coherence, path fluctuations, detector averaging, unequal amplitudes, or entanglement with uncontrolled environmental degrees of freedom.

Caution

Loss of visible interference does not automatically prove that a microscopic object has become a classical particle following a definite unobserved trajectory. The experimental statement is narrower: the measured statistics no longer contain the phase-sensitive cross term with appreciable visibility under the given preparation and detection conditions.

Guided checks

1. If two slits transmit very unequal amplitudes, can the pattern still be coherent but have reduced visibility?
2. Why can finite detector pixel size wash out fringes even when the state reaching the detector is coherent?
3. What physical information must be controlled to compare measured visibility with a theoretical idealization?

18.27.2 A Mach–Zehnder viewpoint on single-quantum interference

A two-path interferometer provides a clean conceptual model because path amplitudes are separated and recombined deliberately. After the first beam splitter, a single quantum can be represented

schematically by the superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|A\rangle + e^{i\varphi} |B\rangle \right). \quad (18.136)$$

The notation here is introduced only as compact bookkeeping; a full bra-ket formalism is not required. A relative phase accumulated between the arms changes the probabilities at the two outputs after recombination.

Both arms reach the recombining beam splitter in Fig. 18.14.

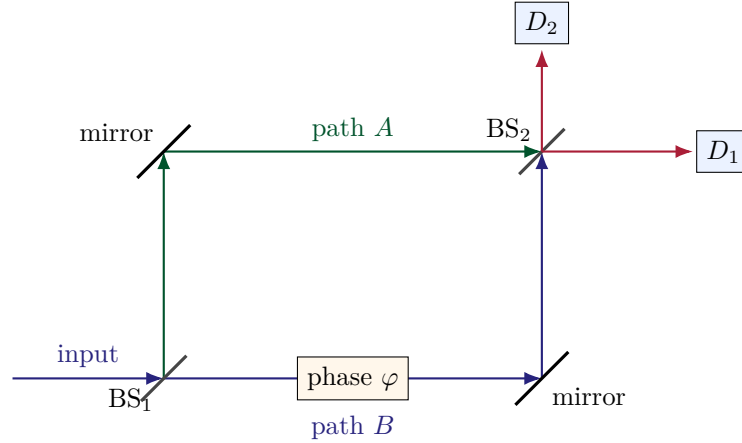


Figure 18.14: A balanced Mach-Zehnder interferometer with both paths connected to BS₂. The phase element lies in path B. Beam-splitter and mirror phases are included in the defined total relative phase φ ; relabeling the ports can exchange the sine and cosine probabilities.

For an ideal balanced interferometer, output probabilities can take the form

$$P_1 = \cos^2\left(\frac{\varphi}{2}\right), \quad P_2 = \sin^2\left(\frac{\varphi}{2}\right). \quad (18.137)$$

The sum is unity. Each detection is discrete, while the phase dependence emerges statistically after repeated trials. This is a useful model for optical and matter-wave interferometric sensors.

18.28 Photon counting, Poisson statistics, and shot noise

At low optical power, photodetection is naturally described as counting discrete events. For ideal detection of a coherent optical state over a fixed interval, the count distribution is Poissonian. It is also a useful approximation for independent rare arrivals with a stable rate. Thermal light and nonclassical states need not obey this model. For the Poisson case,

$$P(N; \mu) = \frac{\mu^N e^{-\mu}}{N!}, \quad (18.138)$$

with mean and variance

$$\langle N \rangle = \mu, \quad \text{Var}(N) = \mu. \quad (18.139)$$

The standard deviation is $\sqrt{\mu}$, so the ideal shot-noise-limited signal-to-noise ratio for a count proportional to N scales as

$$\text{SNR} \sim \frac{\mu}{\sqrt{\mu}} = \sqrt{\mu}. \quad (18.140)$$

Collecting four times as many photons therefore doubles, rather than quadruples, the shot-noise-limited SNR.

Example 18.30: Photon-number scale in a short measurement

A 1.0 nW optical signal at 850 nm is observed for 10 μ s. The incident photon number is approximately

$$N = \frac{P\Delta t}{hc/\lambda} \approx 4.28 \times 10^4.$$

For ideal Poisson statistics the count fluctuation is about

$$\sqrt{N} \approx 207,$$

corresponding to a relative shot-noise scale of $1/\sqrt{N} \approx 0.48\%$. Real detectors add quantum efficiency, dark counts, read noise, afterpulsing, dead time, and background.

Photon statistics connect the quantum picture to engineering measurement directly. A detector does not simply report an abstract intensity; it converts incident quanta into electrical signals with finite efficiency and noise.

18.28.1 Detecting a small intensity change

Consider a detector receiving a mean of N useful photons from one image element during an exposure. If shot noise dominates and counts are independent, the standard count fluctuation is $\sqrt{N}$. If a small fractional signal change δ changes the mean count by δN , then a rough single-pixel detectability criterion is

$$\delta N \gtrsim q\sqrt{N}, \quad (18.141)$$

where q is a desired significance factor. Thus

$$N \gtrsim \frac{q^2}{\delta^2}. \quad (18.142)$$

Example 18.31: Photon budget for a one-percent change

To detect a 1% fractional change with a shot-noise-only criterion of $q = 5$,

$$N \gtrsim \frac{25}{(0.01)^2} = 2.5 \times 10^5$$

useful detected photons are required when the reference mean is known with negligible uncertainty. Comparing two independent exposures of equal duration gives variance approximately $2N$, so the corresponding requirement doubles to 5.0×10^5 photons in each exposure. At 850 nm each photon has energy about 2.34×10^{-19} J, so the optical energy represented by these detected photons is only about 5.9×10^{-14} J. Real systems require more incident energy because quantum efficiency is below unity and background plus read noise contribute.

This calculation links the quantization of light directly to imaging-system design: photon discreteness becomes a measurable noise floor when the photon budget is small.

18.28.2 Measurement uncertainty versus quantum uncertainty

Two uses of the word uncertainty must be kept distinct.

1. **Quantum standard deviation.** Δx and Δp characterize the spreads predicted for repeated measurements on identically prepared quantum systems.

2. **Experimental uncertainty.** A reported quantity such as a fitted wavelength or stopping voltage has uncertainty because of finite calibration accuracy, resolution, statistical sampling, background, and model assumptions.

These concepts can coexist. For example, an experiment may estimate the position spread Δx of a prepared wave packet. The true quantum state has an intrinsic Δx , while the experimenter's estimate of that number has a confidence interval or standard uncertainty due to finite data and detector resolution.

A simple instrumental broadening model illustrates the distinction. If a true Gaussian position distribution of variance σ_q^2 is measured by a detector with an independent Gaussian resolution variance σ_d^2 , the observed variance is

$$\sigma_{\text{obs}}^2 = \sigma_q^2 + \sigma_d^2. \quad (18.143)$$

Recovering σ_q requires knowledge of the detector response. Calling σ_d “Heisenberg uncertainty” would be conceptually wrong.

18.29 Finite-difference Schrödinger equation

The one-dimensional time-independent Schrödinger equation is well suited to numerical discretization. On a uniform grid $x_j = x_0 + j\Delta x$, approximate

$$\left. \frac{d^2\psi}{dx^2} \right|_{x_j} \approx \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{(\Delta x)^2}. \quad (18.144)$$

Then

$$-\frac{\hbar^2}{2m} \frac{\psi_{j+1} - 2\psi_j + \psi_{j-1}}{(\Delta x)^2} + V_j\psi_j = E\psi_j. \quad (18.145)$$

Defining

$$\alpha = \frac{\hbar^2}{2m(\Delta x)^2}, \quad (18.146)$$

the discrete Hamiltonian is tridiagonal:

$$H_{jj} = 2\alpha + V_j, \quad H_{j,j\pm 1} = -\alpha. \quad (18.147)$$

With boundary conditions specified, the numerical problem becomes a matrix eigenvalue problem

$$\mathbf{H}\psi = E\psi. \quad (18.148)$$

The grid is a model choice. It must resolve the shortest physically important wavelength and the narrowest potential structure. Decreasing Δx improves discretization accuracy but increases the matrix dimension. Convergence should be demonstrated by repeating the calculation with a finer grid and verifying that observables change negligibly.

Method

A reliable finite-difference workflow is:

1. choose a physical interval large enough that the boundary conditions are appropriate;
2. choose Δx small compared with the shortest expected wavelength;
3. assemble the tridiagonal Hamiltonian with consistent SI or electronvolt-based units;
4. compute several low eigenpairs rather than the full spectrum if only low states are needed;
5. normalize each eigenvector using the discrete approximation $\sum_j |\psi_j|^2 \Delta x = 1$;
6. halve Δx and verify convergence of the reported energies.

18.29.1 Time propagation and the Crank–Nicolson idea

A direct forward-Euler discretization of the time-dependent Schrödinger equation is generally a poor choice because it does not preserve norm and can be unstable. A widely used second-order implicit scheme is Crank–Nicolson. For a time-independent Hamiltonian,

$$\left(\mathbf{I} + \frac{i\Delta t}{2\hbar}\mathbf{H}\right)\Psi^{n+1} = \left(\mathbf{I} - \frac{i\Delta t}{2\hbar}\mathbf{H}\right)\Psi^n. \quad (18.149)$$

The update is closely related to a unitary time-evolution operator and is norm-conserving up to numerical roundoff for a Hermitian discrete Hamiltonian solved consistently.

A wave-packet simulation can therefore visualize incidence on a step or barrier. The packet separates into reflected and transmitted pieces, making the meaning of R and T operational. The simulation also reveals finite packet width, dispersion, and interference that stationary plane-wave formulas suppress.

Note

A numerical animation is not evidence by itself that the numerical method is correct. Check norm conservation, grid convergence, time-step convergence, and agreement with an analytic limit. A visually smooth movie can still solve the wrong discretized equation.

18.29.2 Numerical stability, boundaries, and error budgets

Computational quantum mechanics combines physical approximation error and numerical error. These should be separated explicitly.

Spatial discretization

A second-order finite difference has a local truncation error of order $(\Delta x)^2$ for sufficiently smooth functions. Halving Δx should reduce a discretization-dominated error by roughly a factor of four in the asymptotic regime.

Time discretization

Crank–Nicolson is second order in Δt for a time-independent Hamiltonian, but a very large time step can still destroy phase accuracy even while the norm remains stable. Norm conservation is necessary, not sufficient.

Finite computational domain

Artificial boundaries can reflect outgoing waves. The simulation time must be short enough that unwanted reflections do not return to the region of interest, or an appropriate absorbing-boundary strategy must be used.

Floating-point scale

When amplitudes become extremely small, underflow or loss of relative precision can occur. Logarithmic representations are useful for transmission probabilities spanning many orders of magnitude.

Method

A defensible numerical error budget should state at least three separate checks: grid convergence, time-step convergence, and domain-size or boundary-condition sensitivity. Agreement among these checks is stronger evidence than a single high-resolution run.

18.29.3 A reproducible free-packet calculation

A useful first computational target is the freely evolving Gaussian, because its center, width, and norm are known independently. Choose a length scale ℓ , define $\xi = x/\ell$ and $\tau = \hbar t/(m\ell^2)$, and set $u = m\ell^2 V/\hbar^2$. The dimensionless equation is

$$i\partial_\tau \Psi = -\frac{1}{2}\partial_\xi^2 \Psi + u(\xi)\Psi. \quad (18.150)$$

The following Python/NumPy calculation uses Fourier propagation on a periodic interval. For a free particle, each discrete Fourier component is propagated by its exact phase; errors arise from the finite interval, sampling, and roundoff. The output is a physical test of motion and spreading, not just an animation.

```
import numpy as np
n, dx, tau = 4096, 0.05, 2.0
x = (np.arange(n) - n//2) * dx
x0, sigma, k0 = -15.0, 1.0, 2.0
psi = np.exp(-(x-x0)**2/(4*sigma**2) + 1j*k0*x)
psi /= np.sqrt(dx*np.sum(abs(psi)**2))
k = 2*np.pi*np.fft.fftfreq(n, d=dx)
psi_t = np.fft.ifft(np.fft.fft(psi)*np.exp(-0.5j*k*k*tau))
p = abs(psi_t)**2
norm = dx*np.sum(p)
mean = dx*np.sum(x*p)/norm
width = np.sqrt(dx*np.sum((x-mean)**2*p)/norm)
print(norm, mean, width)
```

The analytic values are 1, $x_0 + k_0\tau = -11$, and $\sqrt{1 + \tau^2/4} = \sqrt{2} = 1.41421\dots$. Run again with half the grid spacing at the same interval length, then with twice the interval length at fixed spacing. Agreement tests sampling and boundary effects separately. A packet that reaches an edge will wrap around in this periodic implementation; that is not transmission through a physical barrier.

For a barrier calculation, use the Crank–Nicolson update or a properly validated split-operator method, and compare with the spectral average in Eq. (18.121). At intermediate times, measure probability still near the barrier as well:

$$R(t) + T(t) + P_B(t) = 1 \quad (18.151)$$

for a lossless closed-domain calculation before unwanted boundary returns. If absorbing boundaries are introduced, record absorbed probability separately; the norm remaining on the grid then need not equal one.

18.30 Bridge to bound states in Chapter 19

A scattering problem specifies incoming and outgoing waves; a bound-state problem instead requires a normalizable state that remains localized. These boundary conditions select different allowed energies. The simplest preview is an infinite well on $0 < x < L$: $\psi(0) = \psi(L) = 0$ requires $kL = n\pi$, hence

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}, \quad n = 1, 2, \dots \quad (18.152)$$

The zero function corresponding to $n = 0$ cannot be normalized. For an electron and $L = 1.00$ nm, $E_1 \simeq 0.376$ eV. This is a boundary-condition result, not the energy obtained merely by inserting the full width L into a standard-deviation bound.

Chapter 19 develops the normalized states, position probabilities, finite wells, and atomic confinement systematically. Chapter 20 connects bound states to spectra and lasers; Chapter 21 develops the

electronic structure needed for semiconductor devices. Here the main conclusion is already clear: the same amplitude equation accounts for propagation, reflection, evanescence, and confinement, while the preparation and boundary conditions determine which problem is being solved.

18.31 Enrichment synthesis

Summary

The Planck spectrum connects quantum mode energies with thermal power. Gaussian packets make the relation between localization, momentum spread, and dynamics quantitative. Probability current supplies the correct definition of reflection and transmission, and boundary matching explains both tunneling and above-barrier interference. Coherence and counting statistics connect amplitudes to measured signals. Numerical propagation becomes reliable through comparisons with analytic limits and separate checks of sampling and boundary effects.

18.32 Practice problems

Use $h = 6.62607015 \times 10^{-34} \text{ J s}$, $c = 2.99792458 \times 10^8 \text{ m s}^{-1}$, $e = 1.602176634 \times 10^{-19} \text{ C}$, and $m_e \simeq 9.10938 \times 10^{-31} \text{ kg}$. Set $\hbar = h/(2\pi)$. Values of h , c , and e are exact SI definitions; the quoted mass is rounded. State the model, show units, and check a physical limit. The last two groups use the optional enrichment.

Conceptual questions

1. A surface is illuminated below its one-photon threshold. Explain what increasing intensity changes, and state the assumptions needed before claiming there will be no photoemission.
2. Two monochromatic beams have equal power, one red and one ultraviolet. Compare their photon rates and their total momentum delivered per second to an ideal absorbing target at normal incidence.
3. Classical electromagnetic waves already carry momentum. What additional prediction of Compton scattering supports the photon energy–momentum model?
4. Explain how single-electron detections can build an interference pattern without collisions between the electrons. What distinguishes one detection from the many-trial distribution?
5. A student claims that $|\Psi(x)|^2$ is the probability of detecting the particle at exactly x . Correct this statement, including the dimensions in one spatial dimension.
6. Compare the states $\psi(x)$, $-\psi(x)$, and $\psi(x)e^{ik_0x}$. Which position densities are identical? Must the momentum distributions also be identical?
7. Why can a plane wave describe a stationary incident flux but not a localized normalized particle on the infinite line?
8. Distinguish the intrinsic standard deviation Δx from the statistical uncertainty in an experimental estimate of Δx . Does better detector resolution invalidate the uncertainty relation?
9. Explain why a step can reflect a particle even when $E > V_0$, and why a decaying tail for a semi-infinite step at $E < V_0$ carries no transmitted current.
10. A particle crosses a static barrier by tunneling. Explain why neither temporary energy borrowing nor a classical negative kinetic-energy trajectory is needed.
11. Why does a 4 pm electron wavelength not guarantee 4 pm microscope resolution? Identify at least three relevant instrument or specimen effects.
12. A numerical wave packet preserves its norm to twelve digits. Is this sufficient evidence for correct dynamics? Give separate examples of phase error and boundary error.

Analytical and quantitative problems

13. Calculate photon energies at 450 nm, 632.8 nm, and 1550 nm. Rank their momenta and explain the ranking.
14. A surface has work function 2.40 eV. Find the threshold wavelength and decide whether 600 nm illumination gives one-photon photoemission.
15. Light at 300 nm illuminates a 2.10 eV surface. Find K_{max} , the magnitude V_s , and the signed collector–emitter voltage at cutoff.

16. Idealized stopping-voltage data are $(6.0 \times 10^{14} \text{ Hz}, 0.62 \text{ V})$ and $(8.0 \times 10^{14} \text{ Hz}, 1.45 \text{ V})$. Infer h and ϕ . Which result changes if both voltages have an unknown common offset?
17. An x ray of wavelength 0.080 nm scatters through 90° from a free electron initially at rest. Find the outgoing wavelength, fractional wavelength change, and recoil kinetic energy.
18. A 200 keV photon is Compton-backscattered from a free electron initially at rest. Find the outgoing photon energy and the recoil energy.
19. An electron and a proton each move at $2.0 \times 10^6 \text{ m s}^{-1}$. Calculate both wavelengths, using $m_p = 1.673 \times 10^{-27} \text{ kg}$. Explain how the mass dependence differs at equal kinetic energy.
20. Calculate the relativistic and nonrelativistic wavelengths for an electron accelerated through 200 kV . Quote the relative error of the nonrelativistic result with the relativistic value as denominator.
21. Electrons with wavelength 0.050 nm scatter from planes separated by 0.20 nm . Find the first-order Bragg angle and the full deflection angle. State which angle a flat-screen radius measures.
22. A state is $\psi(x) = Ax(L - x)$ for $0 < x < L$ and zero elsewhere. Normalize it, find $\langle x \rangle$ and Δx , and evaluate the probability in the central half of the interval. Treat it as a trial state, not as a stated energy eigenstate.
23. An initially unchirped Gaussian electron packet has $\Delta x = 0.050 \text{ nm}$ and zero mean momentum. Find Δp , $\Delta p/m_e$, and its mean kinetic energy in the nonrelativistic model.
24. On the full line, a normalized trial state is $\psi(x) = Axe^{-x/a}$ for $x \geq 0$ and zero for $x < 0$, with $a > 0$. Find A and $P(0 < x < a)$. Explain why this is a normalization exercise rather than an assertion about an unspecified stationary Hamiltonian.
25. A free electron plane wave has $k = 4.0 \times 10^{10} \text{ m}^{-1}$. Calculate its momentum, wavelength, and kinetic energy. Check the nonrelativistic approximation.
26. An electron with $E = 12 \text{ eV}$ reaches an upward step of $V_0 = 5 \text{ eV}$. Find R and T , using current ratios rather than amplitude ratios alone.
27. For an electron with $E = 2.0 \text{ eV}$ and a semi-infinite step $V_0 = 5.0 \text{ eV}$, calculate κ , the amplitude $1/e$ length, and the density $1/e$ length.
28. Use the exact rectangular-barrier formula for an electron with $E = 1.0 \text{ eV}$, $V_0 = 3.0 \text{ eV}$, and $L = 0.50 \text{ nm}$. Compare it with the bare exponential and the thick-barrier formula including its prefactor.

Engineering and measurement problems

29. A blackbody at 1200 K is viewed through a narrow infrared band centered on its wavelength-spectrum maximum. After optical losses, 20 pW reaches a detector with efficiency 0.40 . Estimate the peak wavelength and count rate. State the additional integral needed for a broad band.
30. A known reference intensity is to be tested for a 1% change at a shot-noise-only significance of five standard deviations. Estimate the required detected photon number. Repeat when the reference comes from an independent exposure of equal duration.
31. A gain-free photodiode receives 5.0 nW at 850 nm and has external quantum efficiency 0.75 . Find the collected current. State why internal gain would require a different relation.

32. A detector receives 10 fW at 650 nm for 1.0 ms, with efficiency 0.60 and dark-count rate 200 s^{-1} . Calculate the mean signal and dark counts and the standard deviation of the total. How does the variance change if an independent dark exposure of the same duration is subtracted?
33. A tunnel junction has effective barrier height 4.0 eV. At fixed energy, bias, and prefactor, find the current ratio for a gap increase from 0.70 to 0.75 nm. Find the gap increment that halves the current.
34. In first-order small-angle diffraction, $d \simeq D\lambda/r$ and $\lambda \propto V_{\text{acc}}^{-1/2}$. Derive the relative uncertainty for independent errors in D , r , and V_{acc} . Explain the scaling of ring radius when the voltage doubles.
35. For $\lambda = 0.050 \text{ nm}$, slit separation $d = 2.0 \mu\text{m}$, and screen distance $D = 1.0 \text{ m}$, calculate fringe spacing. Integrate a cosine fringe over a pixel of width w to show that the visibility is multiplied by $|\sin(\pi w/\Delta y)/(\pi w/\Delta y)|$. Evaluate it at $w = \Delta y$.
36. A small detector views a uniform blackbody through projected source area A and small accepted solid angle Ω . Derive a spectral integral for the mean detected count in time τ , with optical transmission $\mathcal{T}(\lambda)$ and quantum efficiency $\eta(\lambda)$. Check units and recover the narrow-band approximation.

Computational and data-analysis problems

37. Generate eight equally spaced frequencies between 5.0×10^{14} and $9.0 \times 10^{14} \text{ Hz}$ for a 2.0 eV photoelectric surface. Add independent Gaussian voltage noise of 0.03 V and fit slope and intercept. Repeat with a fixed recorded random seed over many trials. Compare the scatter of fitted h with fit uncertainties, then add a common voltage offset.
38. Plot $B_\lambda(T)$ for 1000, 3000, and 6000 K on physically appropriate wavelength ranges. Numerically locate each peak and verify Wien scaling. Transform to B_f with the Jacobian and show why the maxima do not transform by $f = c/\lambda$ alone.
39. Superpose a Gaussian spectrum of at least 100 complex plane-wave amplitudes. Plot the position density, then repeat after randomizing phases between trials. Explain the difference between coherent amplitude summation and averaging probabilities.
40. Run the dimensionless free-packet code in the enrichment. Verify the analytic center, width, and norm at $\tau = 2$. Repeat at two spatial resolutions and two interval lengths. Explain why preserving norm does not detect periodic wrap-around.
41. Propagate a Gaussian packet incident on a rectangular barrier. Compare the late transmitted probability both with T at the mean energy and with the spectrum-averaged expression. Monitor $R + T + P_B$ before the outgoing packets reach the computational boundaries.
42. Generate fixed-bias STM data with $\Phi = 4.0 \text{ eV}$ and 2% independent relative noise. Fit $\ln[I(s)/I_{\text{ref}}]$ against s to infer κ and Φ . Explain why an additive current-noise model should generally be fitted differently.
43. Generate independent Poisson counts for means 10, 100, 1000, and 10^4 . Compare sample means, variances, and relative standard deviations. Explain what this simulation tests and why it does not establish that every optical source has Poisson statistics.

Advanced challenges

44. Derive the Compton shift from total energy and two-dimensional momentum conservation. Then show that a free electron in vacuum cannot absorb a single real photon and remain a lone

on-shell electron; explain why a material can absorb photon energy.

45. Derive the probability continuity equation for a real potential, integrate it over a finite interval, and use the current to justify $T = (k_2/k_1)|t|^2$ at a step.
46. For the unchirped Gaussian, calculate Δx and Δp . Add a quadratic phase $\exp[ib(x - x_0)^2]$ with real b and determine how the momentum spread and position–momentum covariance change while the initial position density stays fixed.
47. For $V_0 = 3.0 \text{ eV}$ and $L = 0.50 \text{ nm}$, find the finite limit of transmission at $E = V_0$ and the first above-barrier unit-transmission energy. Plot $T(E)$ across the barrier top using the proper limiting value there.
48. Derive the leading WKB exponent for $V(x) - E = \Phi(1 - x/L)$ on $0 < x < L$. Compare it with a rectangular forbidden region of the same width and entrance deficit. Discuss endpoint matching and why the result alone is not an exact emission current.

18.33 Selected detailed solutions

Attempt the problems before consulting these solutions. The linked numbers refer to the final exercise set, and each solution makes its assumptions and checks explicit.

Problem 13: Photon energies and momenta

Use

$$E_\gamma = \frac{hc}{\lambda} \approx \frac{1240 \text{ eV nm}}{\lambda[\text{nm}]}.$$
 (18.153)

Thus

$$\begin{aligned} E(450 \text{ nm}) &\approx 2.76 \text{ eV}, \\ E(632.8 \text{ nm}) &\approx 1.96 \text{ eV}, \\ E(1550 \text{ nm}) &\approx 0.800 \text{ eV}. \end{aligned}$$

Because $p = E/c$, the momentum ranking is the same as the energy ranking: 450 nm is largest and 1550 nm smallest. As a check, shorter wavelength must correspond to larger momentum because $p = h/\lambda$.

Problem 14: Threshold wavelength

The threshold condition is $hc/\lambda_0 = \phi$. Hence

$$\lambda_0 = \frac{hc}{\phi} \approx \frac{1240 \text{ eV nm}}{2.40 \text{ eV}} \approx 517 \text{ nm}.$$
 (18.154)

Since $600 \text{ nm} > 517 \text{ nm}$, its photons have less energy than the work function and do not produce ideal photoemission.

Problem 16: Fitting the photoelectric relation

From

$$V_s = \frac{h}{e}f - \frac{\phi}{e},$$
 (18.155)

the slope of a two-point line is

$$a = \frac{V_2 - V_1}{f_2 - f_1}.$$
 (18.156)

Using

$$\begin{aligned} (f_1, V_1) &= (6.0 \times 10^{14} \text{ Hz}, 0.62 \text{ V}), \\ (f_2, V_2) &= (8.0 \times 10^{14} \text{ Hz}, 1.45 \text{ V}), \end{aligned}$$

we find

$$a = \frac{0.83 \text{ V}}{2.0 \times 10^{14} \text{ Hz}} = 4.15 \times 10^{-15} \text{ V s}.$$
 (18.157)

Hence

$$h = ea \approx 6.65 \times 10^{-34} \text{ J s}.$$
 (18.158)

The intercept is

$$b = V_1 - af_1 = 0.62 - (4.15 \times 10^{-15})(6.0 \times 10^{14}) \approx -1.87 \text{ V},$$
 (18.159)

so

$$\phi = -eb \approx 1.87 \text{ eV}.$$
 (18.160)

The value of h is close to the accepted value, as expected for deliberately clean data. Real laboratory data should use more than two points because a line through two points has no redundancy for detecting scatter or systematics.

A common added offset V_{off} leaves the slope unchanged but shifts the inferred work function by $-eV_{\text{off}}$. Without calibration of such offsets, the intercept determines an effective work function.

Problem 17: Compton wavelength and recoil

At 90° , $1 - \cos \theta = 1$. Therefore

$$\Delta\lambda = \lambda_C = 2.426 \text{ pm} = 0.002426 \text{ nm}. \quad (18.161)$$

For $\lambda = 0.080 \text{ nm}$,

$$\lambda' = 0.082426 \text{ nm}. \quad (18.162)$$

The fractional change is

$$\frac{\Delta\lambda}{\lambda} \approx \frac{0.002426}{0.080} \approx 3.03\%. \quad (18.163)$$

The shift is positive, as required for scattering from an initially stationary free electron. The recoil energy is the photon energy loss, not $hc/\Delta\lambda$:

$$K_e = hc \left(\frac{1}{\lambda} - \frac{1}{\lambda'} \right) \simeq 0.456 \text{ keV}.$$

The incident energy is about 15.50 keV, so the recoil is a small positive fraction of the available energy.

Problem 20: Relativistic wavelength

For an electron accelerated through $V = 200 \text{ kV}$, use Eq. (18.32):

$$\lambda = \frac{h}{\sqrt{2m_e eV \left(1 + \frac{eV}{2m_e c^2} \right)}}. \quad (18.164)$$

Numerically,

$$\lambda_{\text{rel}} \approx 2.51 \text{ pm}. \quad (18.165)$$

The nonrelativistic estimate is

$$\lambda_{\text{nr}} \approx 2.74 \text{ pm}. \quad (18.166)$$

The nonrelativistic value is about 9.3% larger. At this voltage, the correction is clearly significant for microscopy or diffraction calculations.

Problem 22: Polynomial normalization, variance, and probability

Suppose

$$\psi(x) = Ax(L - x), \quad 0 < x < L, \quad (18.167)$$

and zero elsewhere. Normalization requires

$$1 = |A|^2 \int_0^L x^2(L - x)^2 dx. \quad (18.168)$$

Expand the polynomial:

$$x^2(L - x)^2 = L^2 x^2 - 2Lx^3 + x^4. \quad (18.169)$$

Therefore

$$\int_0^L x^2(L - x)^2 dx = L^2 \frac{L^3}{3} - 2L \frac{L^4}{4} + \frac{L^5}{5} \quad (18.170)$$

$$= L^5 \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) \quad (18.171)$$

$$= \frac{L^5}{30}. \quad (18.172)$$

Thus

$$|A|^2 \frac{L^5}{30} = 1, \quad \boxed{A = \frac{\sqrt{30}}{L^{5/2}}} \quad (18.173)$$

for a real positive normalization choice. The dimensions $[A] = L^{-5/2}$ are necessary because $x(L-x)$ contributes L^2 while a one-dimensional wave function has dimensions $L^{-1/2}$.

Symmetry gives $\langle x \rangle = L/2$. Set $u = x/L$, so the normalized density in u is $30u^2(1-u)^2$. Then

$$\langle x^2 \rangle = 30L^2 \int_0^1 u^4(1-u)^2 du = \frac{2L^2}{7}, \quad (\Delta x)^2 = \frac{2L^2}{7} - \frac{L^2}{4} = \frac{L^2}{28}.$$

The probability in $L/4 < x < 3L/4$ is

$$P = \left[10u^3 - 15u^4 + 6u^5 \right]_{1/4}^{3/4} = \frac{203}{256} \simeq 0.793.$$

It exceeds one half because the density is concentrated near the center. The dimensions of $\Delta x = L/\sqrt{28}$ are length, while the probability is dimensionless.

Problem 23: Localization energy

The unchirped Gaussian saturates the uncertainty product. Thus

$$\begin{aligned} \Delta p &= \frac{\hbar}{2\Delta x} \simeq 1.05 \times 10^{-24} \text{ kg m s}^{-1}, \\ \frac{\Delta p}{m_e} &\simeq 1.16 \times 10^6 \text{ m s}^{-1}, \\ \langle K \rangle &= \frac{(\Delta p)^2}{2m_e} \simeq 6.10 \times 10^{-19} \text{ J} = 3.81 \text{ eV}. \end{aligned}$$

The mean momentum is zero, which is why no drift-energy term appears. The energy is far below $m_e c^2$, validating the nonrelativistic model. For a different state with the same Δx , this value is a lower bound rather than an exact energy prediction.

Problem 24: A continuous half-line trial state

Choose A real and positive. The integral $\int_0^\infty x^2 e^{-\beta x} dx = 2/\beta^3$ follows by two integrations by parts. With $\beta = 2/a$,

$$1 = A^2 \frac{a^3}{4}, \quad A = \frac{2}{a^{3/2}}.$$

The requested probability is

$$P = \frac{4}{a^3} \int_0^a x^2 e^{-2x/a} dx = \frac{1}{2} \int_0^2 u^2 e^{-u} du = 1 - 5e^{-2} \simeq 0.323.$$

The state is continuous at the origin and is square integrable. It has not been shown to satisfy a specified stationary equation; boundary behavior and the Hamiltonian must also be checked for that stronger claim. Normalization fixes probabilities independently of whether the state is stationary.

Problem 25: Momentum and kinetic energy

For $k = 4.0 \times 10^{10} \text{ m}^{-1}$,

$$p = \hbar k \approx 4.22 \times 10^{-24} \text{ kg m s}^{-1}. \quad (18.174)$$

The wavelength is

$$\lambda = \frac{2\pi}{k} \approx 1.57 \times 10^{-10} \text{ m} = 0.157 \text{ nm}. \quad (18.175)$$

If the particle is an electron and a nonrelativistic estimate is used,

$$K = \frac{p^2}{2m_e} \approx 61.0 \text{ eV}. \quad (18.176)$$

Because $61 \text{ eV} \ll 511 \text{ keV}$, the nonrelativistic kinetic-energy expression is self-consistent.

Problem 26: Step current ratios

For $E = 12 \text{ eV}$ and $V_0 = 5 \text{ eV}$,

$$\frac{k_2}{k_1} = \sqrt{\frac{E - V_0}{E}} = \sqrt{\frac{7}{12}} \approx 0.764. \quad (18.177)$$

Therefore

$$R = \left(\frac{1 - 0.764}{1 + 0.764} \right)^2 \approx 0.0179, \quad T \approx 0.982. \quad (18.178)$$

The coefficients add to unity within rounding.

Problem 27: Two penetration lengths

For $E = 2.0 \text{ eV}$ and $V_0 = 5.0 \text{ eV}$,

$$\kappa = \frac{\sqrt{2m_e(3.0 \text{ eV})}}{\hbar}. \quad (18.179)$$

The amplitude decay length is

$$\ell = \frac{1}{\kappa}. \quad (18.180)$$

Numerically, $\kappa \approx 8.87 \text{ nm}^{-1}$, so

$$\ell \approx 0.113 \text{ nm}. \quad (18.181)$$

The probability density falls as $e^{-2\kappa x}$ and therefore has an even shorter $1/e$ length $1/(2\kappa)$. Numerically, the density $1/e$ length is 0.0564 nm . Neither length is a classical travel distance inside the forbidden region.

Problem 28: Exact transmission and its approximation

The deficit is 2.0 eV , so $\kappa = 7.2453 \text{ nm}^{-1}$ and $\kappa L = 3.6226$. The three comparisons are

$$\begin{aligned} T_{\text{exact}} &= \left[1 + \frac{9 \sinh^2(3.6226)}{8} \right]^{-1} \simeq 2.534 \times 10^{-3}, \\ T_{\text{exp}} &= e^{-7.2453} \simeq 7.135 \times 10^{-4}, \\ T_{\text{thick}} &= 16 \frac{1}{3} \frac{2}{3} e^{-7.2453} \simeq 2.537 \times 10^{-3}. \end{aligned}$$

The thick-barrier result with its prefactor is within about 0.11% of the exact model. The bare exponential is lower by a factor close to $32/9$. Making the barrier even thicker improves the asymptotic expansion but does not force that prefactor toward one.

Problem 31: Gain-free photodiode current

The photon energy is

$$E_\gamma = \frac{hc}{850 \text{ nm}} \approx 2.34 \times 10^{-19} \text{ J}. \quad (18.182)$$

The incident photon rate is

$$\dot{N}_\gamma = \frac{5.0 \times 10^{-9} \text{ J s}^{-1}}{2.34 \times 10^{-19} \text{ J}} \approx 2.14 \times 10^{10} \text{ s}^{-1}. \quad (18.183)$$

With quantum efficiency $\eta = 0.75$, the collected-electron rate is $\eta\dot{N}_\gamma$. The current is

$$I = e\eta\dot{N}_\gamma \approx 2.57 \text{ nA}. \quad (18.184)$$

This estimate neglects gain, recombination, bandwidth limitations, and electronic noise.

Problem 32: Low counts and dark subtraction

At 650 nm, $E_\gamma \simeq 3.06 \times 10^{-19} \text{ J}$. With the power here in femtowatts,

$$\mu_s = \eta \frac{P\tau}{E_\gamma} \simeq 19.6, \quad \mu_d = D\tau = 0.200.$$

Independent Poisson counts have additive variances. Therefore the total standard deviation is $\sqrt{\mu_s + \mu_d} \simeq 4.45$ counts. Subtracting an independently measured equal-duration dark exposure removes the mean dark contribution but adds its variance:

$$\text{Var}(N_{\text{on}} - N_{\text{dark}}) = \mu_s + 2\mu_d \simeq 20.0.$$

The corrected standard deviation is about 4.47 counts. Subtracting a known mean background and subtracting a noisy background measurement are different statistical operations.

Problem 33: Tunnel-current ratio

With effective barrier height 4 eV,

$$\kappa \approx 10.25 \text{ nm}^{-1}. \quad (18.185)$$

For a gap increase $\Delta s = 0.05 \text{ nm}$,

$$\frac{I_2}{I_1} \approx e^{-2\kappa\Delta s} = e^{-1.025} \approx 0.359. \quad (18.186)$$

A five-hundredth of a nanometre change reduces the simplified current to roughly 36 percent of its initial value.

For half the current, $e^{-2\kappa\Delta s} = 1/2$, giving $\Delta s = \ln 2 / (2\kappa) \simeq 0.0338 \text{ nm} = 33.8 \text{ pm}$. Only a difference in separation is inferred without needing the prefactor, provided that prefactor is stable.

Problem 36: Broad-band detected photon number

Spectral power reaching the detector is $dP = A\Omega\mathcal{T}(\lambda)B_\lambda(T)d\lambda$ for the specified small, uniform viewing geometry. Divide by the photon energy and multiply by efficiency and exposure:

$$\mu = \tau A\Omega \int \eta(\lambda)\mathcal{T}(\lambda)B_\lambda(T) \frac{\lambda}{hc} d\lambda.$$

The product $A\Omega B_\lambda d\lambda$ has units of power; $\tau\lambda/(hc)$ has units seconds per joule, so the result is dimensionless. If all factors vary negligibly over a narrow band at λ_0 and its total received power is P , the result is $\mu \simeq \eta(\lambda_0)P\tau\lambda_0/(hc)$. A broad band generally cannot be represented by the peak wavelength alone.

Problem 40: Free-packet verification

In dimensionless units, the mean position is $\langle \xi \rangle = \xi_0 + k_0\tau = -15 + 2(2) = -11$. The initial Gaussian has $\sigma_\xi = 1$ and $\Delta k = 1/2$. Thus

$$\sigma_\xi(2) = \sqrt{1 + (2/2)^2} = \sqrt{2}, \quad \int |\Psi|^2 d\xi = 1.$$

The Fourier update multiplies every mode by a unit-modulus factor. It therefore preserves the discrete norm even if the periodic interval is too small and the packet wraps around. Separate interval-size and sampling checks are essential. The packet must also remain far from the boundaries for its computed linear-coordinate moments to be compared with the infinite-line formulas.

Problem 47: Barrier top and the first resonance

Use $\hbar^2/(2m_e) \simeq 0.03810 \text{ eV nm}^2$. The finite barrier-top value is

$$T(V_0) = \left[1 + \frac{V_0 L^2}{4[\hbar^2/(2m_e)]} \right]^{-1} = \left[1 + \frac{3.0(0.50)^2}{4(0.03810)} \right]^{-1} \simeq 0.169.$$

For $E > V_0$, the first nonzero resonance is $qL = \pi$:

$$E_{\text{res}} = V_0 + \frac{\pi^2 \hbar^2}{2m_e L^2} \simeq 3.0 + 1.50 = 4.50 \text{ eV}.$$

There $T = 1$ in the ideal stationary model. The barrier-top case $q = 0$ is not another unit-transmission resonance because the prefactor multiplying $\sin^2(qL)$ also has a singular limit. A finite spectral-width packet averages over this resonance and generally has a lower maximum transmitted fraction.

Chapter A

Compact formula sheet

Select the model before using these equations; the stated conditions are part of each result.

Formula summary

Thermal radiation and photons

$$\begin{aligned} B_\lambda(T) &= \frac{2hc^2}{\lambda^5 [\exp(hc/\lambda k_B T) - 1]}, & M &= \sigma_{\text{SB}} T^4, \\ \lambda_{\text{max}} T &\simeq 2.898 \times 10^{-3} \text{ m K}, & p_\gamma &= \frac{E_\gamma}{c}, \\ E_\gamma &= hf = \hbar\omega = \frac{hc}{\lambda}, & \dot{N}_{\text{det}} &= \eta \dot{N}, \\ \dot{N} &= \frac{P}{E_\gamma}, \end{aligned}$$

B_λ is radiance per unit wavelength; the temperature and total-exittance laws assume an ideal black-body. Photon wavelength formulas here use vacuum wavelength. The count-rate formula assumes monochromatic input and a linear detector without saturation.

Photoemission and Compton scattering

$$\begin{aligned} K_{\text{max}} &= hf - \phi = eV_s, & f_0 &= \phi/h, \\ \lambda' - \lambda &= \frac{h}{m_e c} (1 - \cos \theta), \\ E'_\gamma &= \frac{E_\gamma}{1 + [E_\gamma/(m_e c^2)](1 - \cos \theta)}. \end{aligned}$$

The one-photon photoelectric equation applies above threshold; V_s is a nonnegative magnitude. At cutoff the signed collector voltage is $-V_s$ relative to the emitter. Compton scattering assumes an initially stationary free electron.

Formula summary

Matter waves, diffraction, and uncertainty

$$\begin{aligned} \lambda &= \frac{h}{p}, & p_{\text{nr}} &= \sqrt{2mK}, \\ \lambda_e &= \frac{hc}{\sqrt{K(K + 2m_e c^2)}}, & K &= eV_{\text{acc}}, \\ 2d \sin \theta &= n\lambda, & r &= D \tan(2\theta), \\ \Delta y &\simeq \frac{\lambda D}{d}, & \Delta x \Delta p_x &\geq \frac{\hbar}{2}, \\ \langle K \rangle &\geq \frac{\langle p_x \rangle^2}{2m} + \frac{\hbar^2}{8m(\Delta x)^2}. \end{aligned}$$

The electron-voltage relation neglects losses and initial kinetic energy. The ring geometry is for a flat screen; θ is the Bragg angle to the planes. The two-slit spacing formula assumes far-field paraxial coherent illumination. The energy bound assumes one-dimensional nonrelativistic kinetic energy and finite moments.

Formula summary**Amplitudes, evolution, and probability current**

$$\begin{aligned}
P(a < x < b) &= \int_a^b |\Psi|^2 dx, & \int |\Psi|^2 dx &= 1, \\
i\hbar \partial_t \Psi &= -\frac{\hbar^2}{2m} \partial_x^2 \Psi + V \Psi, \\
-\frac{\hbar^2}{2m} \psi'' + V \psi &= E \psi, & \Psi &= \psi e^{-iEt/\hbar}, \\
j &= \frac{\hbar}{m} \text{Im}(\Psi^* \partial_x \Psi), & \partial_t |\Psi|^2 + \partial_x j &= 0.
\end{aligned}$$

These expressions describe a massive nonrelativistic particle with constant mass and a real scalar potential energy. The stationary ansatz requires time-independent V . Normalization to one applies to localized states; stationary scattering waves instead use incoming-flux normalization.

Step and rectangular barrier

$$\begin{aligned}
k_1 &= \frac{\sqrt{2mE}}{\hbar}, & k_2 &= \frac{\sqrt{2m(E - V_0)}}{\hbar}, \\
R &= \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2, & T &= \frac{4k_1 k_2}{(k_1 + k_2)^2}, \\
\kappa &= \frac{\sqrt{2m(V_0 - E)}}{\hbar}, \\
T_{\text{barrier}} &= \left[1 + \frac{V_0^2 \sinh^2(\kappa L)}{4E(V_0 - E)} \right]^{-1}, \\
T_{\text{thick}} &\simeq 16 \frac{E}{V_0} \left(1 - \frac{E}{V_0} \right) e^{-2\kappa L}.
\end{aligned}$$

Step coefficients require $E > V_0$ and equal mass. The barrier formulas require $0 < E < V_0$, exterior potential zero on both sides, and width L ; the last line also requires $\kappa L \gg 1$. A static, lossless scatterer obeys $R + T = 1$ and conserves total energy.

Formula summary**Enrichment: packets, counting, and tunneling metrology**

$$\begin{aligned}
v_g &= \frac{d\omega}{dk} = \frac{\hbar k_0}{m}, \\
\sigma_x^2(t) &= \sigma_0^2 \left[1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2 \right], \\
P(N; \mu) &= \frac{\mu^N e^{-\mu}}{N!}, & \text{Var } N &= \mu, \\
I(s) &\simeq I_0 e^{-2\kappa s}, & \frac{\delta I}{I} &\simeq -2\kappa \delta s.
\end{aligned}$$

The width law assumes a free, initially unchirped minimum-uncertainty Gaussian. Poisson counting requires the specified counting model, not merely low light. The STM relation assumes fixed bias and approximately stable barrier height and electronic prefactors; the differential relation is local.

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