

Chapter 2

Electric Fields and Field Lines

First Edition

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science

AGH University of Krakow

Physics 2

Physics for Technical and Engineering Students -
Volume II

Electricity, Magnetism, Waves, Optics, Relativity,
Quantum Physics, and Nuclear Physics

1 September 2026

Contents

Preface	iii
Conventions and notation	iv
2 Electric Fields and Field Lines	1
2.1 Motivation and roadmap	1
2.2 From action-at-a-distance force to field language	2
2.3 Dimensions, units and physical scale	4
2.4 Electric field of a point charge	5
2.5 Superposition of electric fields	7
2.6 Electric field lines as a visualization tool	11
2.7 Electric dipoles	12
2.7.1 Field on the dipole axis	13
2.7.2 Field on the perpendicular bisector	13
2.7.3 When is the dipole approximation accurate?	14
2.8 A point charge moving in an electric field	15
2.8.1 From exit displacement to a detector reading	16
2.9 Continuous charge distributions	17
2.9.1 Uniformly charged ring on its axis	18
2.9.2 Uniform finite line charge at a perpendicular point	20
2.9.3 A finite rod away from its perpendicular bisector	21
2.9.4 Field of a uniformly charged finite rod on its axis	22
2.9.5 Uniformly charged arc at the centre of curvature	23
2.9.6 Uniformly charged disk on its axis	24
2.9.7 What the disk formula does and does not describe	25

2.10	Electric dipoles in uniform external fields	26
2.11	Field probes, measurement and the small-test-charge idea	28
2.11.1	A force-balance measurement	28
2.11.2	Testing the nonperturbing-probe assumption	29
2.12	Engineering applications: drift and particle collection	30
2.13	Laboratory and computational connection: field mapping on a grid	31
2.14	Core synthesis and self-check	32
2.15	Advanced Topics for Technical and Engineering Students	34
2.15.1	From a localized distribution to its far field	34
2.15.2	Force on a permanent dipole in a nonuniform field	36
2.15.3	Equilibrium is a local question: two stability examples	37
2.15.4	Thermal motion and the mean orientation of a dipole	38
2.15.5	A nonuniform rod: density, components and complete integration	39
2.15.6	Finite plates: a calculable model of field uniformity	40
2.15.7	Numerical quadrature as a controlled source approximation	42
2.15.8	Tracing field lines and validating field maps	43
2.15.9	A reproducible field calculation with an analytic benchmark	45
2.15.10	Inferring a source from field data	47
2.16	Practice problems	48
	Conceptual reasoning	48
	Analytical and geometrical problems	48
	Numerical and computational problems	49
	Engineering, laboratory and data problems	50
	Advanced challenges	50
2.17	Selected detailed solutions	51
2.18	Bridge to Chapter 3	56
	Compact formula sheet and validity conditions	57

Preface

Chapter 2 introduces the electric field, the first genuine field concept in this volume. Chapter 1 described electrostatic interaction through Coulomb forces between charged particles. That language is useful and measurable, but it is not the most economical way to describe the electromagnetic influence of many charges, continuous charge distributions, devices, sensors, beams, capacitors, dielectrics, waves and radiation. The electric field is the next conceptual step: it describes what the source charges do to space, before any particular test charge is chosen.

The transition from force to field is a major intellectual transition. In mechanics, students often ask which object pushes which other object. In electromagnetism, we instead learn to ask what vector field exists at a point in space and how a charge placed there responds. This is not just a change of notation from $\mathbf{F}$ to $\mathbf{E}$. It is a change in modelling viewpoint. It prepares the way for electric flux, Gauss's law, electric potential, capacitance, current flow, electromagnetic waves and ultimately Maxwell's equations.

The chapter is written for Technical Physics, engineering, computer-science, space-science and applied-science students. For that reason, field diagrams, numerical field mapping, electrostatic deflection, dipoles, field probes, grid calculations and engineering interpretation are integrated with the mathematical derivations. The field of a point charge is developed from Coulomb's law; superposition is applied to several point charges; field lines are treated as a visualization tool rather than as physical strings; and continuous charge distributions are introduced using transparent one-dimensional and axial geometries.

The Guideline

When solving an electric-field problem, separate three roles before writing equations. The *source charge distribution* creates the field. The *field point* is the point in space where the field is evaluated. A *test charge*, if present, responds to the field. Mixing these roles is the most common source of conceptual and sign errors in this chapter.

Pedagogical boundary

This chapter studies electrostatic fields produced by stationary charge distributions. Electric flux, Gauss's law, electric potential, boundary-value methods, dielectric materials in detail and electromagnetic waves are reserved for later chapters. Rotational potential energy of a rigid dipole is derived here directly from mechanical work. Those topics need the field language developed here, but they also require additional ideas that will be introduced in later chapters.

Conventions and notation

Vectors are written using the macro $\mathbf{a}$, $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{r}$. Cartesian unit vectors are written as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$. The electric field is denoted by $\mathbf{E}$, with SI unit newton per coulomb, N C^{-1} , equivalently volt per metre, V m^{-1} . The equivalence of these units will become especially meaningful after electric potential is introduced.

A point in space where the field is evaluated is called the *field point*. Its position vector is usually written $\mathbf{r}$. A source element is located at $\mathbf{r}'$. The vector from the source element to the field point is

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \quad R = |\mathbf{R}|, \quad \hat{\mathbf{R}} = \frac{\mathbf{R}}{R}.$$

This convention is used throughout the chapter. The unit vector $\hat{\mathbf{R}}$ points from the source element toward the field point. This direction matters: it is the direction of the electric field contribution from a positive source element dq .

The Coulomb constant is

$$k_e = \frac{1}{4\pi\epsilon_0},$$

where ϵ_0 is the vacuum permittivity. The symbol q_0 denotes a small test charge used to define or measure the field. The symbol q or Q denotes a source charge unless otherwise stated. For continuous distributions, we use

$$\lambda = \frac{dq}{d\ell}, \quad \sigma = \frac{dq}{dA}, \quad \rho = \frac{dq}{dV},$$

for linear, surface and volume charge densities, respectively.

Symbol	Meaning	SI unit
$\mathbf{E}$	electric field vector	$\text{N C}^{-1} = \text{V m}^{-1}$
$\mathbf{F}$	electric force on a charge	N
q_0	test charge used to define or measure the field	C
q, Q	source charge or total charge	C
$\mathbf{r}$	field-point position vector	m
$\mathbf{r}'$	source-point position vector	m
$\mathbf{R} = \mathbf{r} - \mathbf{r}'$	displacement from source element to field point	m
λ	linear charge density	C m^{-1}
σ	surface charge density	C m^{-2}
ρ	volume charge density	C m^{-3}
$\mathbf{p}$	electric dipole moment, from negative to positive charge	C m
τ	torque on a dipole	N m
U	potential energy of a dipole in a uniform field	J

Chapter 2

Electric Fields and Field Lines

2.1 Motivation and roadmap

Chapter 1 used Coulomb's law to compute the force between charged particles. That force law is already powerful, but it has a practical limitation. Suppose an engineer wants to know how a small charged dust grain, an electron beam, an ion in a detector, or a probe charge will behave near a complicated charged object. It would be inefficient to repeat the full pairwise force calculation for every possible charge that might be placed at every possible location. A better strategy is to describe the electrical influence of the source distribution throughout space. That description is the electric field.

The electric field is a vector field: to every point in space it assigns a vector $\mathbf{E}$. The direction of $\mathbf{E}$ is the direction of the force on a positive test charge placed at that point. The magnitude tells how much force per coulomb such a charge would experience. Once the field is known, the force on any point charge q is simply

$$\mathbf{F} = q\mathbf{E}. \quad (2.1)$$

The source calculation and the response calculation are therefore separated. This separation is the conceptual bridge from Coulomb's law to Maxwell's theory.

The progression to later chapters is summarized in Figure 2.1.

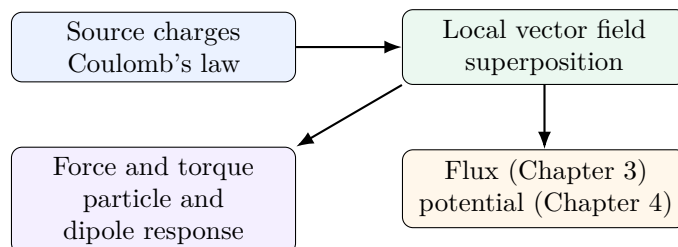


Figure 2.1: The field separates the source calculation from the response of a particle or dipole. The same local vectors also prepare the flux and potential descriptions developed in subsequent chapters.

Learning goals

After studying this chapter, the student should be able to:

- define the electric field as force per unit positive test charge and explain the test-charge limit;
- distinguish source charge, field point and test charge;
- derive the electric field of a point charge from Coulomb's law;
- calculate the net electric field from several point charges by vector superposition;
- interpret field-line diagrams correctly, including their direction and density conventions;
- explain why field lines never cross in electrostatics;
- calculate the field of an electric dipole on its axis and on its perpendicular bisector in the far-field approximation;
- calculate the force and acceleration of a charged particle in a uniform electric field;
- set up continuous charge-distribution integrals using $dq = \lambda d\ell$, $dq = \sigma dA$ or $dq = \rho dV$;
- derive transparent electric-field expressions for simple line, ring and disk geometries;
- compute the torque and potential energy of an electric dipole in a uniform external field;
- connect field mapping to numerical grids, electrostatic probes, deflection devices and engineering modelling.

Key message

The electric field is a local description of electrostatic influence. It tells what force a positive unit test charge would experience at a point, but it is not itself a charge and it is not a material thread connecting charges.

2.2 From action-at-a-distance force to field language

Coulomb's law for two point charges can be used directly to calculate the force on one charge due to another. If a charge q_0 at a field point experiences the force $\mathbf{F}$ due to a set of source charges, the ratio $\mathbf{F}/q_0$ is independent of q_0 as long as q_0 is small enough not to disturb the source distribution. This observation motivates the definition of the electric field.

The word “field” should be taken seriously. A scalar field assigns one number to each point in space, as temperature in a room approximately does. A vector field assigns a vector to each point, as the velocity field in flowing water does. The electric field assigns an electric vector to each point in space. The field may vary from point to point in magnitude and direction. It can be represented by arrows on a grid, by field lines, by formulas, by numerical arrays, by equipotential maps after potential is introduced, or by finite-element simulations in engineering software.

Definition 2.1: Electric field

The electric field at a point is defined by

$$\mathbf{E} = \lim_{q_0 \rightarrow 0} \frac{\mathbf{F}}{q_0}, \quad (2.2)$$

where $\mathbf{F}$ is the electric force exerted by the source charges on a positive test charge q_0 placed at that point. The limiting statement means that the test charge is small enough that it does not measurably rearrange the source charges.

Several details in Definition 2.1 are worth making explicit. First, the test charge is conventionally

positive. Therefore $\mathbf{E}$ points in the direction a positive test charge would be pushed. A negative charge placed at the same point would experience a force opposite to $\mathbf{E}$. Second, the field exists whether or not the test charge is actually present. A test charge is a measurement concept, not a necessary ingredient of the field. Third, the limiting process does not claim that charge can be divided continuously in nature. It expresses the idealized requirement that the measuring charge should not perturb the system being measured.

Caution

Do not rewrite the definition as $\mathbf{E} = q_0 \mathbf{F}$. The correct relation is $\mathbf{E} = \mathbf{F}/q_0$. Equivalently, once the field is known, the force on a charge q is $\mathbf{F} = q\mathbf{E}$. These two equations answer different questions.

Guided checks

1. A field point is empty: no particle is placed there. Can the electric field at that point be nonzero?
2. A negative charge is placed at a point where $\mathbf{E}$ points to the right. What is the direction of the force on the negative charge?
3. If a larger test charge is used and it polarizes a nearby conductor, why would the measured force per charge no longer represent the unperturbed field?

Method

To translate between force and field, use this two-step pattern.

1. Source calculation: determine $\mathbf{E}(\mathbf{r})$ from the source charges.
2. Response calculation: determine $\mathbf{F} = q\mathbf{E}$ for the particular charge placed at the point.

This separation is especially useful when many different test charges or particle trajectories must be studied in the same apparatus.

Which field enters the force law?

For a responding point charge, $q\mathbf{E}$ means the force due to the *external* field evaluated at the particle's location. Adding the particle's own singular Coulomb field at that same point is not a legitimate way to calculate its force. In a system of prescribed point charges, the force on particle i is therefore

$$\mathbf{F}_i = k_e q_i \sum_{j \neq i} q_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}. \quad (2.3)$$

The source-response separation can be repeated with a different particle designated as the responder. It does not imply that forces are one-way: in a dynamical system all particles may move, and their positions must be updated consistently. In this chapter the source is fixed or sufficiently heavy and supported that its rearrangement is negligible.

For an extended charged object, $Q\mathbf{E}$ at one arbitrarily chosen point is generally insufficient. Divide the object into elements and sum their forces:

$$\mathbf{F}_{\text{ext}} = \int \rho_{\text{body}}(\mathbf{r}) \mathbf{E}_{\text{ext}}(\mathbf{r}) d^3r. \quad (2.4)$$

If the external field is uniform across the object, it factors out and the result reduces to $Q_{\text{body}}\mathbf{E}_{\text{ext}}$. If it varies, a neutral body can have nonzero net force, as the dipole-gradient example later demonstrates. Likewise, an extended neutral body can have zero net force but nonzero torque.

Example 2.1: A neutral two-charge body tests the point-response approximation

A rigid object carries charges $+q$ and $-q$. Its left end is in a rightward field of magnitude E_0 , and its right end is in a rightward field of magnitude $1.10E_0$. The positive charge is at the right end. Find the net electric force.

Solution. The positive end feels $+1.10qE_0\hat{i}$ and the negative end feels $-qE_0\hat{i}$. Thus

$$\mathbf{F}_{\text{net}} = 0.10qE_0\hat{i}. \quad (2.5)$$

Its total charge is zero, but the fields sampled by its ends differ. Multiplying total charge by the centre field would lose that spatial information. The setup describes the field along the body's axis; it does not specify the field everywhere in surrounding space. If the two end fields were equal, the net force would vanish exactly.

2.3 Dimensions, units and physical scale

Because $\mathbf{E} = \mathbf{F}/q_0$, the SI unit of electric field is

$$[E] = \frac{\text{N}}{\text{C}}. \quad (2.6)$$

Later, after electric potential is introduced, the same unit will also be written as V m^{-1} . The unit N C^{-1} emphasizes force per charge, while V m^{-1} emphasizes potential difference per distance. Both describe the same physical quantity.

Electric fields can be weak or extremely strong depending on the source and distance. A small laboratory field between charged plates might be of order 10^3 to 10^5 N C^{-1} . Near microscopic charges at atomic distances, the field can be much larger. In air, sufficiently large electric fields can ionize molecules and produce electrical breakdown, but breakdown depends on pressure, geometry, humidity, impurities and electrode shape. This chapter uses the ideal electrostatic field concept and does not attempt a quantitative safety standard.

A useful scale estimate comes from the field of a single elementary charge at a nanometre distance:

$$E \sim k_e \frac{e}{(1 \text{ nm})^2} \approx 1.4 \times 10^9 \text{ N C}^{-1}. \quad (2.7)$$

This large value does not mean that macroscopic objects usually have huge net electric fields. Ordinary matter is nearly neutral, and nearby positive and negative charges cancel strongly at distances large compared with atomic separations.

Note

Electric fields can be large even when forces on individual charges are small. The force on an electron in a field $E = 10^5 \text{ N C}^{-1}$ has magnitude $eE \approx 1.6 \times 10^{-14} \text{ N}$. That is tiny on a human scale but large enough to cause substantial acceleration of an electron because the electron mass is very small.

Example 2.2: Field, force and acceleration in a uniform field

An electron enters a region where the electric field is uniform and has magnitude $E = 2.0 \times 10^4 \text{ N C}^{-1}$ directed upward. Find the force and acceleration of the electron, neglecting gravity.

Solution. The charge of the electron is $q = -e = -1.602 \times 10^{-19} \text{ C}$. The electric force is

$$\mathbf{F} = q\mathbf{E}. \quad (2.8)$$

Because $q < 0$, the force is opposite to the electric field, so it is downward. Its magnitude is

$$F = eE = (1.602 \times 10^{-19} \text{ C})(2.0 \times 10^4 \text{ N C}^{-1}) = 3.2 \times 10^{-15} \text{ N}. \quad (2.9)$$

Using $m_e = 9.11 \times 10^{-31} \text{ kg}$,

$$a = \frac{F}{m_e} \approx \frac{3.2 \times 10^{-15}}{9.11 \times 10^{-31}} \text{ m s}^{-2} \approx 3.5 \times 10^{15} \text{ m s}^{-2}, \quad (2.10)$$

directed downward. The acceleration is enormous compared with g , so neglecting gravity is justified for the electron motion in this field.

2.4 Electric field of a point charge

The electric field of a point charge follows directly from Coulomb's law. Place a source charge q at the origin and a positive test charge q_0 at position $\mathbf{r}$. Coulomb's law gives the force on the test charge:

$$\mathbf{F} = k_e \frac{qq_0}{r^2} \hat{\mathbf{r}}, \quad (2.11)$$

where $\hat{\mathbf{r}}$ points from the source charge to the field point. Dividing by q_0 gives

$$\mathbf{E}(\mathbf{r}) = k_e \frac{q}{r^2} \hat{\mathbf{r}} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}. \quad (2.12)$$

If $q > 0$, the field points radially outward. If $q < 0$, the field points radially inward. The magnitude is

$$E = k_e \frac{|q|}{r^2}. \quad (2.13)$$

For a point charge q located at position $\mathbf{r}'$, the field at $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = k_e q \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \quad (2.14)$$

This coordinate form is often the safest form for numerical work, because the sign of q and the direction from source to field point are both included.

Figure 2.2 compares the radial directions for both signs.

Caution

The formula $E = k_e|q|/r^2$ gives only the magnitude. It does not by itself tell the vector direction. The vector direction follows from geometry and the sign of the source charge. When in doubt, use Eq. (2.14).

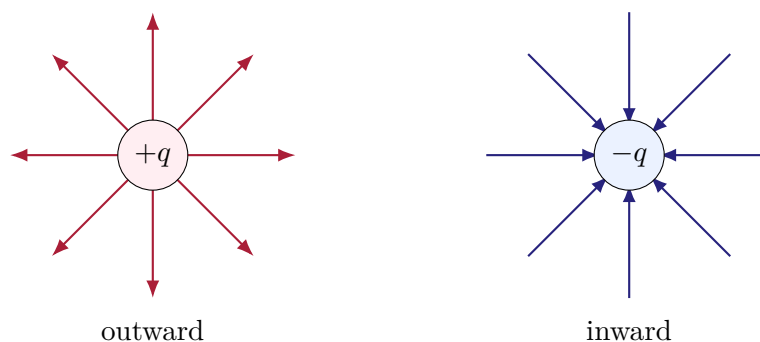


Figure 2.2: Radial field lines in a plane through an isolated point charge. Arrows indicate direction along the lines, not displacement from the charge to a field point or a calibrated field magnitude. The three-dimensional field is spherically symmetric.

Example 2.3: Electric field at a point due to one point charge

A source charge $q = +4.0 \text{ nC}$ is fixed at the origin. Find the electric field at the point P with coordinates $(0.30 \text{ m}, 0)$.

Solution. The distance from the charge to the field point is $r = 0.30 \text{ m}$. Since the charge is positive, the field points away from the origin, along $+\hat{i}$. The magnitude is

$$E = k_e \frac{|q|}{r^2} = (8.99 \times 10^9) \frac{4.0 \times 10^{-9}}{(0.30)^2} \text{ N C}^{-1} \approx 4.0 \times 10^2 \text{ N C}^{-1}. \quad (2.15)$$

Therefore

$$\mathbf{E}(P) \approx (4.0 \times 10^2 \text{ N C}^{-1})\hat{i}. \quad (2.16)$$

A positive test charge at P would be pushed to the right; a negative test charge would be pushed to the left.

Example 2.4: An off-axis field using a source-to-point vector

A charge $q = -2.0 \text{ nC}$ is at $(0.10, 0.00) \text{ m}$. Find the field at $P = (0.40, 0.40) \text{ m}$.

Strategy. Subtract the source coordinates from the observation coordinates. Keep the signed charge in the vector formula; no additional sign is needed to make the field point toward a negative source.

$$\mathbf{R} = (0.30\hat{i} + 0.40\hat{j}) \text{ m}, \quad R = 0.50 \text{ m}, \quad (2.17)$$

$$\hat{\mathbf{R}} = 0.60\hat{i} + 0.80\hat{j}, \quad E = k_e \frac{|q|}{R^2} = 71.9 \text{ N C}^{-1}. \quad (2.18)$$

Consequently,

$$\mathbf{E}(P) = -(43\hat{i} + 58\hat{j}) \text{ N C}^{-1}. \quad (2.19)$$

Interpretation and check. Both components are negative: from P , the negative charge is to the left and below. The component ratio $E_y/E_x = 4/3$ matches the geometry, and $\sqrt{E_x^2 + E_y^2} \simeq 72 \text{ N C}^{-1}$. The units of $k_e q \mathbf{R}/R^3$ are $(\text{N m}^2/\text{C}^2)(\text{C})(\text{m})/(\text{m}^3) = \text{N/C}$. Using the position of P itself as $\mathbf{R}$ would give both a wrong direction and a wrong distance.

Guided checks

1. If a source charge is doubled while the field point is unchanged, what happens to the field magnitude?
2. If the distance from a point charge is tripled, what happens to the field magnitude?
3. A negative source charge lies to the left of a field point. Does the electric field at the point point left or right?

2.5 Superposition of electric fields

The electric field obeys the same superposition principle as the Coulomb force. If several source charges are present, the net electric field at a point is the vector sum of the electric fields that each source charge would produce separately:

$$\mathbf{E}_{\text{net}} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots + \mathbf{E}_N. \quad (2.20)$$

For point charges q_i at positions $\mathbf{r}_i$, the field at $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = k_e \sum_{i=1}^N q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \quad (2.21)$$

This equation is also the natural starting point for numerical field maps.

Method

For a field-superposition problem:

1. mark the field point clearly;
2. for each source charge, draw the direction of its field at the field point;
3. compute each field magnitude using distance from source to field point;
4. resolve fields into components;
5. add components and reconstruct the net vector;
6. only after the field is known, multiply by q if the force on a particular charge is requested.

The component cancellation is illustrated in Figure 2.3.

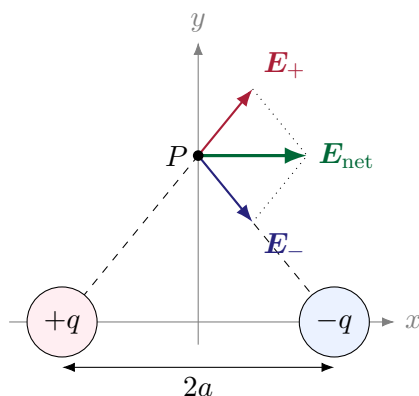


Figure 2.3: Vector addition on the perpendicular bisector of opposite equal charges. Both component arrows are parallel to their exact source-to-point directions, with the negative-source contribution reversed. The horizontal resultant equals their vector sum.

Example 2.5: Field at the midpoint between two unequal charges

Two point charges lie on the x axis. Charge $q_1 = +3.0 \text{ nC}$ is at $x = 0$ and charge $q_2 = -1.0 \text{ nC}$ is at $x = 0.40 \text{ m}$. Find the electric field at the midpoint.

Solution. The midpoint is at $x = 0.20 \text{ m}$. The field due to q_1 points away from the positive charge, so it points in the $+x$ direction. The field due to q_2 points toward the negative charge, also in the $+x$ direction. Therefore the magnitudes add:

$$E_1 = k_e \frac{3.0 \times 10^{-9}}{(0.20)^2} = 674 \text{ N C}^{-1}, \quad (2.22)$$

$$E_2 = k_e \frac{1.0 \times 10^{-9}}{(0.20)^2} = 225 \text{ N C}^{-1}. \quad (2.23)$$

Thus

$$\mathbf{E}_{\text{net}} = 9.0 \times 10^2 \text{ N C}^{-1} \hat{\mathbf{i}}. \quad (2.24)$$

The opposite signs of the charges do not make the fields cancel at the midpoint; in this geometry they make the fields point in the same direction.

Example 2.6: Field at a point above two equal positive charges

Two identical charges $+q$ are fixed at $x = \pm a$ on the x axis. Find the electric field at the point $P = (0, y)$.

Solution. By symmetry, the horizontal components cancel. The distance from either charge to P is

$$R = \sqrt{a^2 + y^2}. \quad (2.25)$$

The field magnitude from either charge is $k_e q / R^2$. The vertical component from one charge is

$$E_y^{(1)} = \frac{k_e q}{R^2} \frac{y}{R} = k_e q \frac{y}{R^3}. \quad (2.26)$$

For $y > 0$, both vertical components point upward, so

$$\mathbf{E}(0, y) = 2k_e q \frac{y}{(a^2 + y^2)^{3/2}} \hat{\mathbf{j}}. \quad (2.27)$$

A useful check is the far-field limit $y \gg a$. Then $(a^2 + y^2)^{3/2} \approx y^3$, and $E \approx 2k_e q / y^2$, the field of total charge $2q$ far away.

Example 2.7: Finding a point where the electric field is zero

A charge $+4q$ is fixed at $x = 0$ and a charge $+q$ is fixed at $x = L$. Find the point on the x axis between the charges where the electric field is zero.

Solution. Between two positive charges, the field from $+4q$ points to the right and the field from $+q$ points to the left. Let the zero-field point be at x with $0 < x < L$. Equality of magnitudes gives

$$k_e \frac{4q}{x^2} = k_e \frac{q}{(L-x)^2}. \quad (2.28)$$

Canceling common factors,

$$\frac{2}{x} = \frac{1}{L-x}, \quad (2.29)$$

where positive distances have been used. Hence $2(L-x) = x$, so

$$x = \frac{2L}{3}. \quad (2.30)$$

The zero-field point is closer to the smaller charge, as expected.

Caution

A point where $\mathbf{E} = 0$ is not necessarily a point where the electric potential is zero. Electric field and electric potential are different quantities. Potential is introduced in a later chapter.

The next example is a useful check that the symmetry argument and the trigonometry refer to the same centre. Figure 2.4 shows the centroid-to-vertex vectors explicitly.

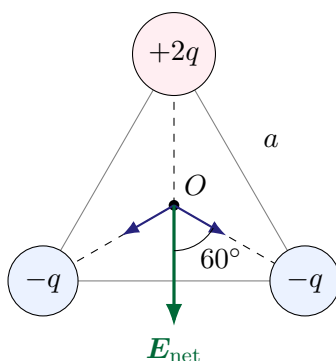


Figure 2.4: Charges on an equilateral triangle. At the centroid, each field from a lower negative charge is at 60° to the downward vertical. Their vertical components and the field from the upper positive charge all point downward. Arrow lengths indicate directions schematically.

Example 2.8: field at the centre of an equilateral triangle

Three charges are placed at the vertices of an equilateral triangle of side a . The top vertex carries charge $+2q$, and the two bottom vertices each carry charge $-q$. Find the direction of the electric field at the centre of the triangle.

Solution. The centre is equidistant from all three vertices. Let the top vertex lie on the positive y axis and the bottom side be horizontal. The two bottom charges are equal and symmetrically placed, so their horizontal field components cancel at the centre. Because the bottom charges are negative, each field contribution at the centre points toward the corresponding bottom charge. Their vertical components both point downward.

The top charge is positive, so its field at the centre points away from the top charge, also downward. Therefore all surviving components point downward. The net electric field is along the negative y direction.

The magnitude provides a useful independent geometry check. The bottom vertices are at $y = -a/(2\sqrt{3})$, so their vertical projection factor is $1/2$. The distance from the centre to each vertex is $r = a/\sqrt{3}$. The field from the top charge has magnitude

$$E_{\text{top}} = k_e \frac{2q}{r^2} = 6k_e \frac{q}{a^2}. \quad (2.31)$$

Each bottom charge contributes magnitude $E_b = k_e q/r^2 = 3k_e q/a^2$. The angle between each bottom contribution and the downward vertical is 60° , so the two vertical components together are

$$2E_b \cos 60^\circ = 6k_e \frac{q}{a^2} \frac{1}{2} = 3k_e \frac{q}{a^2}. \quad (2.32)$$

Thus

$$\mathbf{E}_{\text{centre}} = -9k_e \frac{q}{a^2} \hat{\mathbf{j}}. \quad (2.33)$$

The direction conclusion was available before the magnitude calculation.

Example 2.9: zero field for unequal opposite charges

A charge $+q$ is fixed at $x = 0$ and a charge $-9q$ is fixed at $x = L$. Determine where on the x axis the electric field can be zero.

Solution. Between opposite charges, the field from the positive charge points to the right and the field from the negative charge also points to the right, because the field points toward the negative charge. Therefore there is no zero-field point between the charges.

To the right of $-9q$, the field from $+q$ points right and the field from $-9q$ points left. The negative charge is both larger and closer than the positive charge for points just to the right of L , and far away the larger charge still dominates. No cancellation occurs on that side.

To the left of $+q$, the field from $+q$ points left and the field from $-9q$ points right. Cancellation is possible because the smaller charge $+q$ can be closer. Let the point be $x = -s$ with $s > 0$. The distances are s and $L + s$. Equating magnitudes,

$$k_e \frac{q}{s^2} = k_e \frac{9q}{(L + s)^2}. \quad (2.34)$$

Taking positive square roots gives

$$\frac{1}{s} = \frac{3}{L + s}. \quad (2.35)$$

Thus $L + s = 3s$, so $s = L/2$. The zero-field point is

$$x = -\frac{L}{2}. \quad (2.36)$$

This result is on the side of the smaller-magnitude charge, as expected.

2.6 Electric field lines as a visualization tool

Electric field lines are a graphical representation of the electric field. They are drawn so that the tangent to a field line at any point gives the direction of $\mathbf{E}$ at that point. The density of lines gives a qualitative indication of field strength: regions where lines are closer together correspond to larger field magnitude. Field lines begin on positive charge and end on negative charge, or extend to or from infinity in idealized diagrams.

Field lines are not physical threads, not paths carved into space and not necessarily particle trajectories. A charged particle released from rest initially accelerates along the local field direction if it is positive and opposite to the field if it is negative. But once the particle has velocity, its trajectory depends on its previous motion, inertia and forces over time. Field lines are a map of force directions for positive test charges, not a movie of particle paths.

Figure 2.5 compares numerically traced two-charge fields.

Theoretical point 2.1: Why field lines do not cross

At any ordinary point where $\mathbf{E} \neq 0$, the electric field has one vector value and one direction. If two field lines crossed at a point, the tangent to one line would give one direction for $\mathbf{E}$ while the tangent to the other line would give a different direction. That would assign two different directions to the same vector at the same point, which is impossible. Therefore distinct field lines cannot cross at a regular point with nonzero field. At a null, $\mathbf{E} = 0$, the tangent direction is undefined; limiting curves may approach the null, but a tracer must not assign a unique continuation through it.

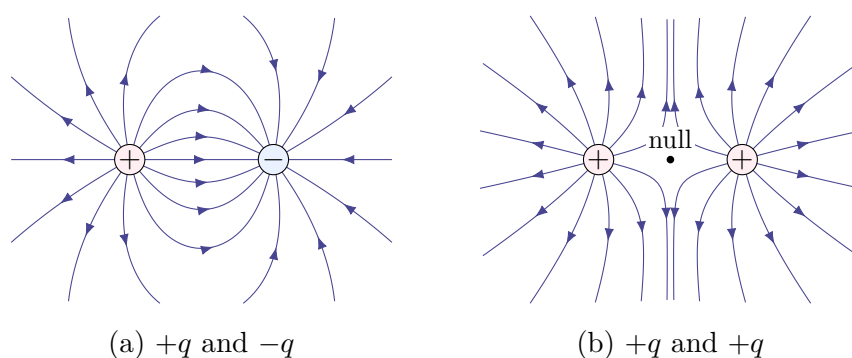


Figure 2.5: Field lines obtained by integrating the normalized Coulomb field of two point charges in the plane of the page. Curves stop near the sources or at the panel boundary. In (a), arrows run from positive to negative charge; in (b), lines leave both positive charges and the midpoint is a null. Seed choices are illustrative and do not calibrate line density as field strength.

Caution

Line density is a qualitative convention. Twice as many drawn lines on a printed figure does not automatically mean twice the physical field unless the drawing has been constructed with a precise rule. In introductory diagrams, line density should be read as “stronger here, weaker there,” not as a calibrated measurement.

Guided checks

1. Why must field lines start on positive charges and end on negative charges in electrostatic diagrams?
2. Why can two field lines never cross at a point where the field is nonzero?
3. If a positive charge is released from rest, is its initial acceleration along or opposite to the local field line?
4. If an electron is released from rest, is its initial acceleration along or opposite to the local field line?

2.7 Electric dipoles

An electric dipole consists of two charges of equal magnitude and opposite sign separated by a small distance. If the negative charge is at one end and the positive charge at the other, the electric dipole moment is defined as

$$\mathbf{p} = q\mathbf{d}, \quad (2.37)$$

where q is the positive charge magnitude and $\mathbf{d}$ points from the negative charge to the positive charge. The SI unit of dipole moment is C m.

Dipoles matter because many neutral systems are not electrically featureless. A molecule may have a permanent dipole moment if its positive and negative charge distributions have different centres. A neutral atom or molecule may also acquire an induced dipole moment in an external electric field. Dipoles are the first step in understanding how neutral matter responds to electric fields, an idea that later becomes important for dielectrics, polarization, sensors and molecular physics.

The source and observation coordinates are shown in Figure 2.6.

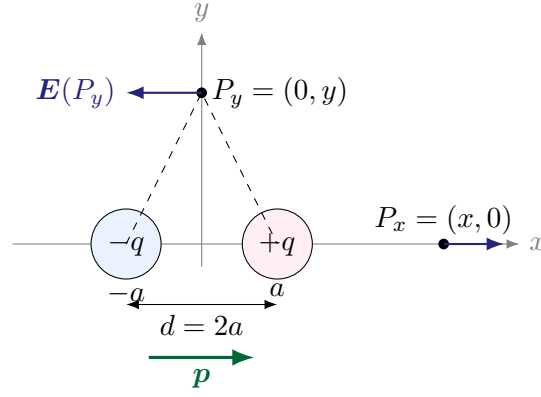


Figure 2.6: Coordinates for the exact axial and perpendicular-bisector fields of a dipole. The field on the bisector is opposite to $\mathbf{p}$; on the exterior positive axis it is parallel to $\mathbf{p}$.

2.7.1 Field on the dipole axis

Place charges $-q$ at $x = -a$ and $+q$ at $x = +a$, so the separation is $d = 2a$ and $\mathbf{p} = 2aq\hat{\mathbf{i}}$. At a point on the positive x axis with $x > a$, the field from the positive charge points in $+\hat{\mathbf{i}}$ and the field from the negative charge points in $-\hat{\mathbf{i}}$. The exact axial field for $x > a$ is

$$E_x = k_e q \left[\frac{1}{(x-a)^2} - \frac{1}{(x+a)^2} \right]. \quad (2.38)$$

Combining fractions gives

$$E_x = k_e q \frac{(x+a)^2 - (x-a)^2}{(x^2 - a^2)^2} = k_e q \frac{4ax}{(x^2 - a^2)^2}. \quad (2.39)$$

For $x \gg a$, this becomes

$$\mathbf{E}_{\text{axial}} \approx k_e \frac{2p}{x^3} \hat{\mathbf{i}}. \quad (2.40)$$

The important feature is the $1/x^3$ decrease. A dipole field falls faster than the field of a net charge because the total charge is zero and the leading monopole contribution cancels.

2.7.2 Field on the perpendicular bisector

Keep the same charges and coordinates as on the axis. At $P = (0, y)$, the displacement vectors from $+q$ and $-q$ are $-a\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $a\hat{\mathbf{i}} + y\hat{\mathbf{j}}$, respectively. Both distances are $\sqrt{a^2 + y^2}$. Superposition therefore gives

$$\mathbf{E}(0, y) = \frac{k_e q}{(a^2 + y^2)^{3/2}} [(-a\hat{\mathbf{i}} + y\hat{\mathbf{j}}) - (a\hat{\mathbf{i}} + y\hat{\mathbf{j}})] \quad (2.41)$$

$$= -\frac{2k_e qa}{(a^2 + y^2)^{3/2}} \hat{\mathbf{i}} = -\frac{k_e \mathbf{p}}{(a^2 + y^2)^{3/2}}. \quad (2.42)$$

This expression is exact for the two-point-charge model, including $y = 0$. In particular, the centre is not a null: $\mathbf{E}(0, 0) = -2k_e q\hat{\mathbf{i}}/a^2$. A neutral source need not have zero field either at its centre or outside it. The expression is even in y , so the field has the same direction above and below the midpoint.

At a point on the y axis, the two charges are at equal distances. The vertical components of the two fields cancel, while the horizontal components point from the positive charge toward the negative charge, opposite to $\mathbf{p}$. For $|y| \gg a$, the result is

$$\mathbf{E}_{\text{bisector}} \approx -k_e \frac{\mathbf{p}}{|y|^3}. \quad (2.43)$$

The axial field has magnitude approximately twice the bisector field at the same large distance, but the directions are different.

Example 2.10: Dipole field on the axis

An electric dipole consists of charges $\pm 5.0 \text{ nC}$ separated by 4.0 mm . Estimate the electric field on the dipole axis at a point 0.20 m from the dipole centre, on the positive side of the dipole moment.

Solution. The dipole moment magnitude is

$$p = qd = (5.0 \times 10^{-9} \text{ C})(4.0 \times 10^{-3} \text{ m}) = 2.0 \times 10^{-11} \text{ C m}. \quad (2.44)$$

Since the observation distance is much larger than the separation, the far-field axial expression is appropriate:

$$E_{\text{axial}} \approx k_e \frac{2p}{x^3} = (8.99 \times 10^9) \frac{2(2.0 \times 10^{-11})}{(0.20)^3} \approx 45 \text{ N C}^{-1}. \quad (2.45)$$

The field points along the dipole moment on the positive axial side.

Example 2.11: Dipole field on the perpendicular bisector

For the same dipole as in Example 2.10, estimate the electric field at a point 0.20 m from the centre on the perpendicular bisector.

Solution. The far-field bisector magnitude is

$$E_{\text{bisector}} \approx k_e \frac{p}{y^3} = (8.99 \times 10^9) \frac{2.0 \times 10^{-11}}{(0.20)^3} \approx 22 \text{ N C}^{-1}. \quad (2.46)$$

The direction is opposite to the dipole moment. The magnitude is half of the axial far-field value at the same distance, a useful check on the result.

Key message

A neutral object can still produce an external electric field if its positive and negative charge distributions do not coincide. The dipole is the simplest example. Its far field decreases as $1/r^3$, faster than the $1/r^2$ field of a net point charge.

2.7.3 When is the dipole approximation accurate?

The phrase “far away” should refer to a desired accuracy, not merely to a sketch. For $x > a$ the exact axial expression can be written as

$$E_{x,\text{exact}} = \frac{2k_e p}{x^3} \left(1 - \frac{a^2}{x^2} \right)^{-2}. \quad (2.47)$$

The leading approximation is smaller than the exact field. Defining fractional error relative to the exact result gives

$$\delta_{\text{ax}} \equiv \frac{E_{x,\text{exact}} - E_{x,\text{dip}}}{E_{x,\text{exact}}} = 1 - \left(1 - \frac{a^2}{x^2} \right)^2 = 2\frac{a^2}{x^2} - \frac{a^4}{x^4}. \quad (2.48)$$

Thus $x = 4a$ gives $\delta_{\text{ax}} = 31/256 = 12.1\%$, whereas $x = 10a$ gives 1.99% . The numerical dipole example above has $a/x = 0.01$, so its error is only 0.020% , well below the precision of the stated data.

On the perpendicular bisector, the far-field magnitude $k_e p/|y|^3$ instead *overestimates* the exact result. Its error relative to the exact field is

$$\delta_{\perp} = \left(1 + \frac{a^2}{y^2}\right)^{3/2} - 1 \simeq \frac{3a^2}{2y^2}. \quad (2.49)$$

The different signs of the errors are a reminder that an approximation must be checked in its actual geometry. The axial approximation is not valid between the charges; at $x = 0$ it is singular even though the exact field is finite. At large distances on *either* side of the dipole axis, the field points along $\mathbf{p}$.

Guided checks

A dipole is observed at a distance ten times its *separation* $d = 2a$. Is the axial error approximately 2% or 0.5%? Use the ratio a/x , not d/x , in Eq. (2.48). The answer is approximately 0.5%.

2.8 A point charge moving in an electric field

Once the field is known, the motion of a charged particle follows from Newton's second law:

$$m\mathbf{a} = q\mathbf{E}. \quad (2.50)$$

In the nonrelativistic approximation, if $\mathbf{E}$ is constant in space and time and the particle remains in the field region, the acceleration is constant. Here the field is external: the particle's self-field is excluded, and magnetic forces, collisions, radiation losses and changes in the source are neglected. At relativistic speeds the general equation is $d\mathbf{p}_{\text{mech}}/dt = q\mathbf{E}$, so constant force does not imply constant acceleration. The problem becomes a kinematics problem from Physics 1, with acceleration determined by charge-to-mass ratio and field strength.

Figure 2.7 contrasts a positive-particle path with the straight field lines.

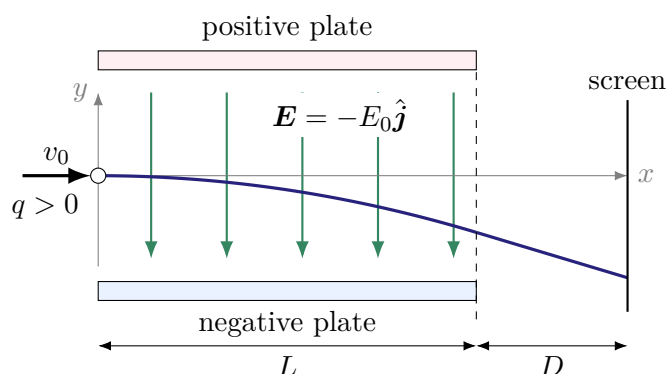


Figure 2.7: A positive particle in a downward uniform field: a parabolic path between the plates followed by straight field-free drift. The ideal trajectory clears the plates. A negative particle with the same entry conditions bends upward. Dimensions and curvature are schematic.

Choose $+y$ upward and write the field as $\mathbf{E} = E_y \hat{\mathbf{j}}$, where E_y is a *signed component*. For downward field of magnitude $E_0 > 0$, $E_y = -E_0$. A particle entering at $(0, y_0)$ with velocity $v_0 \hat{\mathbf{i}}$ obeys

$$a_y = \frac{qE_y}{m}, \quad x = v_0 t, \quad y = y_0 + \frac{qE_y}{2m} t^2. \quad (2.51)$$

Eliminating t gives

$$y(x) = y_0 + \frac{qE_y}{2mv_0^2} x^2. \quad (2.52)$$

This is a trajectory, not a field line. The horizontal velocity carries the particle across the straight field lines while the vertical force changes its vertical velocity. A negative particle curves upward in a downward field; a positive particle curves downward. The dimensions of $qE_y x^2/(mv_0^2)$ are length, since qE_y/m is acceleration and x/v_0 is time.

Constant acceleration does not always mean that speed increases. A particle initially moving opposite to its electric force slows down, stops, and reverses if it remains in the field. More generally, the power transferred by the field is $q\mathbf{E} \cdot \mathbf{v}$, whose sign determines whether kinetic energy increases or decreases.

Example 2.12: Electron-beam deflection between plates

An electron enters a uniform electric field of magnitude $1.5 \times 10^4 \text{ N C}^{-1}$ with horizontal speed $2.0 \times 10^7 \text{ m s}^{-1}$. The field region is 5.0 cm long. Estimate the vertical deflection at the exit, neglecting edge effects and gravity.

Solution. The time spent in the field is

$$t = \frac{L}{v_0} = \frac{0.050}{2.0 \times 10^7} \text{ s} = 2.5 \times 10^{-9} \text{ s}. \quad (2.53)$$

The electron acceleration magnitude is

$$a = \frac{eE}{m_e} = \frac{(1.602 \times 10^{-19})(1.5 \times 10^4)}{9.11 \times 10^{-31}} \approx 2.6 \times 10^{15} \text{ m s}^{-2}. \quad (2.54)$$

The vertical deflection magnitude is

$$|y| = \frac{1}{2}at^2 \approx \frac{1}{2}(2.6 \times 10^{15})(2.5 \times 10^{-9})^2 \approx 8.2 \times 10^{-3} \text{ m}. \quad (2.55)$$

The deflection is about 8 mm, opposite to the direction of the electric field because the electron has negative charge. Using unrounded values gives 8.24 mm, or 8.2 mm to two significant figures. The exit transverse speed is $6.6 \times 10^6 \text{ m s}^{-1}$, giving total speed about $0.070c$; the Newtonian result is adequate at the quoted precision. A beam entering at the midplane requires a gap larger than 16.5 mm in the ideal model to reach the exit without striking a plate.

2.8.1 From exit displacement to a detector reading

The field acts only during passage between the plates. At their exit, $t_L = L/v_0$, the particle has

$$y_L = \frac{qE_y L^2}{2mv_0^2}, \quad v_{y,L} = \frac{qE_y L}{mv_0}, \quad \tan \phi = \frac{qE_y L}{mv_0^2}, \quad (2.56)$$

for entry at $y_0 = 0$. In a subsequent field-free drift of length D , both velocity components remain constant. The extra displacement is $v_{y,L}D/v_0$. A screen therefore measures

$$y_{\text{screen}} = \frac{qE_y L}{mv_0^2} \left(D + \frac{L}{2} \right). \quad (2.57)$$

The factor $D + L/2$ matters: substituting the full distance $L + D$ into the parabolic formula would incorrectly apply a force in the drift region.

Example 2.13: Designing an ion deflector with a downstream screen

Singly charged positive ions of mass 4.0×10^{-26} kg enter horizontally at 8.0×10^4 m s⁻¹. The plates have length $L = 0.12$ m and gap $g = 8.0$ mm. A screen lies $D = 0.20$ m beyond the exit. What upward field component produces a screen displacement of 4.0 mm, and does the beam clear the plates?

Model. Treat each ion as a nonrelativistic particle, neglect fringing, beam width, space charge and other forces. The beam enters on the midplane. Equation (2.57) gives

$$E_y = \frac{m y_{\text{screen}} v_0^2}{e L (D + L/2)} = \frac{(4.0 \times 10^{-26})(0.0040)(8.0 \times 10^4)^2}{(1.602 \times 10^{-19})(0.12)(0.26)} \simeq 2.0 \times 10^2 \text{ N C}^{-1}. \quad (2.58)$$

The displacement at the plate exit follows without another force calculation:

$$y_L = y_{\text{screen}} \frac{L/2}{D + L/2} = 0.0040 \frac{0.060}{0.260} \text{ m} = 0.92 \text{ mm}. \quad (2.59)$$

It is smaller than $g/2 = 4.0$ mm. The exit angle is given by $\tan \phi = y_{\text{screen}}/(D + L/2) = 0.0154$, so $\phi \simeq 0.88^\circ$.

Interpretation. Most of the observed displacement accumulates during the drift. The ideal collision bound is $|E_y| < m g v_0^2 / (|q| L^2) \simeq 8.9 \times 10^2 \text{ N C}^{-1}$. This is a geometric bound for the stated particle model, not an engineering safety rating. A finite beam requires clearance for its radius, entry offset and velocity spread. Because displacement scales as v_0^{-2} , a 1% change in speed produces approximately a 2% change in deflection at fixed field.

2.9 Continuous charge distributions

Real objects usually contain many charges, and it is often useful to approximate their charge as continuously distributed. This is a macroscopic model. It does not deny charge quantization; it says that when a large number of elementary charges are spread over a length, area or volume, a smooth density can be a very accurate approximation for fields observed at distances large compared with microscopic separations.

The density describes a specified distribution. Uniform charge on an insulating object can be a useful imposed model; an isolated conducting rod or disk generally redistributes charge and need not have uniform density. The integration method calculates the field once the distribution is known. It does not determine an unknown conducting surface charge.

For a continuous distribution, the field is obtained by summing infinitesimal contributions:

$$d\mathbf{E} = k_e \frac{dq}{R^2} \hat{\mathbf{R}} = k_e \frac{\mathbf{R}}{R^3} dq, \quad (2.60)$$

so that

$$\mathbf{E}(\mathbf{r}) = k_e \int \frac{\mathbf{R}}{R^3} dq. \quad (2.61)$$

Depending on the geometry,

$$dq = \lambda d\ell, \quad dq = \sigma dA, \quad dq = \rho dV. \quad (2.62)$$

Method

To set up a continuous-field integral:

1. draw the source geometry and the field point;

2. choose a coordinate for the source element;
3. write dq in terms of density and the chosen coordinate;
4. write $\mathbf{R}$ from the source element to the field point;
5. identify components that cancel by symmetry;
6. integrate only the components that survive;
7. check limiting cases, such as far away from the distribution.

Caution

Do not integrate the scalar $k_e dq/R^2$ unless all contributions point in the same direction. In most continuous problems, directions vary across the source. Use components or vector notation.

2.9.1 Uniformly charged ring on its axis

A ring of radius a carrying total charge Q is a clean introduction because all source elements are the same distance from an axial field point. At a point on the symmetry axis a distance z from the centre, transverse components cancel and axial components add. Since $R = \sqrt{a^2 + z^2}$ and the axial projection factor is z/R ,

$$E_z = \int k_e \frac{dq}{R^2} \frac{z}{R} = k_e \frac{z}{(a^2 + z^2)^{3/2}} \int dq. \quad (2.63)$$

Therefore

$$\mathbf{E}_{\text{ring}}(z) = k_e \frac{Qz}{(a^2 + z^2)^{3/2}} \hat{\mathbf{z}}. \quad (2.64)$$

At $z = 0$, the field is zero by symmetry. For $z \gg a$, the field approaches $k_e Q/z^2$, the field of a point charge Q .

The cancellation of transverse components is shown in Figure 2.8.

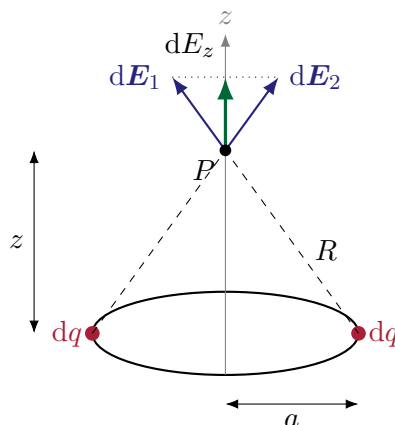


Figure 2.8: Opposite equal elements of a uniformly charged ring. The horizontal components cancel and the axial components add. The labelled dE_z is the axial component of one element; the pair contributes $2dE_z$. The ring is drawn in perspective.

Example 2.14: Field on the axis of a charged ring

A ring of radius $a = 0.050$ m carries total charge $Q = 20$ nC. Find the electric field on the axis at $z = 0.12$ m.

Solution. Using Eq. (2.64),

$$E_z = k_e \frac{Qz}{(a^2 + z^2)^{3/2}}. \quad (2.65)$$

Substitution gives

$$E_z = (8.99 \times 10^9) \frac{(20 \times 10^{-9})(0.12)}{(0.050^2 + 0.12^2)^{3/2}} \text{ N C}^{-1} \approx 9.8 \times 10^3 \text{ N C}^{-1}. \quad (2.66)$$

The field points in the $+z$ direction for positive Q and positive z . The result is smaller than $k_e Q/z^2$ because the charge is spread off the axis rather than concentrated at the centre.

Reading the whole axial field, not just one value

Introduce $u = z/a$. Then $E_z = (k_e Q/a^2)f(u)$ with

$$f(u) = \frac{u}{(1 + u^2)^{3/2}}. \quad (2.67)$$

For $Q > 0$, the field is negative below the ring and positive above it. Close to the centre, $f(u) \simeq u$, so $E_z \simeq k_e Qz/a^3$: cancellation is exact at the centre and becomes incomplete linearly with displacement. Far away, $f(u) \simeq u/|u|^3$, recovering the outward inverse-square field on both sides. These two limits explain the turnover in Figure 2.9.

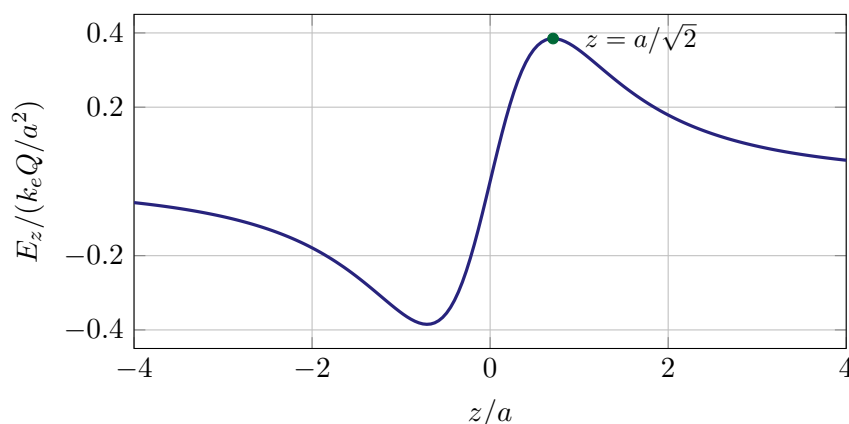


Figure 2.9: Signed axial field of a uniformly charged positive ring. The dimensionless plot uses the exact field; its maximum for $z > 0$ occurs at $a/\sqrt{2}$. The negative side follows by reflection and a sign change.

The natural field scale $k_e Q/a^2$ separates strength from shape. Doubling Q doubles the curve vertically; doubling a at fixed Q reduces the field at corresponding z/a by four. The ring is not well represented by a point charge near its centre, even though both models carry the same total charge.

Example 2.15: How far away is a ring effectively a point charge?

Find the minimum $z/a > 0$ for the point-charge approximation to overestimate the ring field by less than 1%, measured relative to the exact ring field.

Strategy. Form a dimensionless ratio so that charge and units cancel:

$$\frac{E_{\text{point}}}{E_{\text{ring}}} = \left(1 + \frac{a^2}{z^2}\right)^{3/2}. \quad (2.68)$$

The condition is $(1 + a^2/z^2)^{3/2} - 1 < 0.01$, hence

$$\frac{z}{a} > \left(1.01^{2/3} - 1\right)^{-1/2} \simeq 12.3. \quad (2.69)$$

At $z = 5a$, the overestimate is 6.06% relative to the exact value. The result shows why “several radii” may be sufficient for a rough estimate but insufficient for a 1% measurement. Stating which value normalizes a percentage error avoids ambiguous accuracy claims.

2.9.2 Uniform finite line charge at a perpendicular point

Consider a uniformly charged rod of length $2a$ lying along the x axis from $-a$ to $+a$, with total charge Q and linear charge density $\lambda = Q/(2a)$. We calculate the field at the point $(0, y)$ on the perpendicular bisector. A source element at position x has charge $dq = \lambda dx$ and distance $R = \sqrt{x^2 + y^2}$ from the field point. The horizontal components cancel by symmetry. The vertical component is

$$dE_y = k_e \frac{\lambda dx}{R^2} \frac{y}{R} = k_e \lambda y \frac{dx}{(x^2 + y^2)^{3/2}}. \quad (2.70)$$

Thus

$$E_y = k_e \lambda y \int_{-a}^a \frac{dx}{(x^2 + y^2)^{3/2}} \quad (2.71)$$

$$= k_e \lambda y \left[\frac{x}{y^2 \sqrt{x^2 + y^2}} \right]_{-a}^a \quad (2.72)$$

$$= \frac{2k_e \lambda a}{y \sqrt{a^2 + y^2}}. \quad (2.73)$$

The field points away from the rod for positive λ .

For a very long rod, $a \gg |y|$ at fixed λ , this approaches

$$E_y \approx \frac{2k_e \lambda}{y} = \frac{\lambda}{2\pi\epsilon_0 y}. \quad (2.74)$$

This limiting expression will later be derived more elegantly using Gauss’s law.

Figure 2.10 fixes the source coordinate and vector projection.

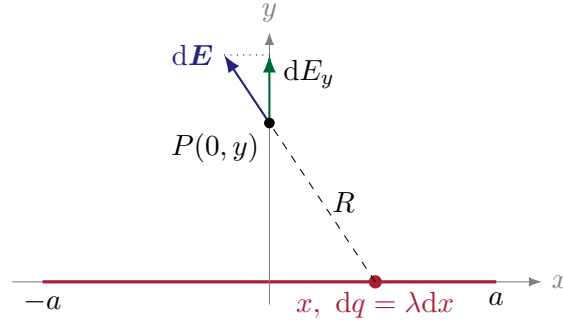


Figure 2.10: Field contribution from a positive element of a rod on its perpendicular bisector. Pairing the element at x with one at $-x$ cancels the horizontal component. The axial coordinate is a source variable, whereas y remains fixed in the integral.

Example 2.16: Field of a finite charged rod on its perpendicular bisector

A rod of length 0.60 m carries total charge $Q = 18 \text{ nC}$ uniformly. Find the field at a point 0.20 m from the centre on the perpendicular bisector.

Solution. Here $2a = 0.60 \text{ m}$, so $a = 0.30 \text{ m}$ and $\lambda = Q/(2a) = 3.0 \times 10^{-8} \text{ C m}^{-1}$. With $y = 0.20 \text{ m}$,

$$E_y = \frac{2k_e\lambda a}{y\sqrt{a^2 + y^2}} = \frac{2(8.99 \times 10^9)(3.0 \times 10^{-8})(0.30)}{(0.20)\sqrt{0.30^2 + 0.20^2}}. \quad (2.75)$$

This gives

$$E_y \approx 2.2 \times 10^3 \text{ N C}^{-1}. \quad (2.76)$$

The field points away from the positively charged rod, along the perpendicular bisector.

2.9.3 A finite rod away from its perpendicular bisector

The bisector formula is a special case: it relies on pairing equal charge at x and $-x$ about the perpendicular through the field point. If the observation point is not above the midpoint, a component along the rod remains. This more general geometry is useful for assessing edge effects and for checking field-sum programs.

Let a rod of constant density λ occupy $x_1 \leq x \leq x_2$ on the horizontal axis. Evaluate the field at $P = (x_0, y)$ with $y > 0$. Use a source coordinate $u = x - x_0$, so $u_1 = x_1 - x_0$ and $u_2 = x_2 - x_0$. Then

$$\mathbf{R} = -u\hat{\mathbf{i}} + y\hat{\mathbf{j}}, \quad d\mathbf{E} = k_e\lambda \frac{-u\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{(u^2 + y^2)^{3/2}} du. \quad (2.77)$$

The two components require different integrals:

$$E_x = -k_e\lambda \int_{u_1}^{u_2} \frac{u du}{(u^2 + y^2)^{3/2}} = k_e\lambda \left[\frac{1}{\sqrt{u^2 + y^2}} \right]_{u_1}^{u_2}, \quad (2.78)$$

$$E_y = k_e\lambda y \int_{u_1}^{u_2} \frac{du}{(u^2 + y^2)^{3/2}} = \frac{k_e\lambda}{y} \left[\frac{u}{\sqrt{u^2 + y^2}} \right]_{u_1}^{u_2}. \quad (2.79)$$

With $r_i = \sqrt{u_i^2 + y^2}$, these become

$$E_x = k_e\lambda \left(\frac{1}{r_2} - \frac{1}{r_1} \right), \quad E_y = \frac{k_e\lambda}{y} \left(\frac{u_2}{r_2} - \frac{u_1}{r_1} \right). \quad (2.80)$$

If the rod extends farther right than left of x_0 , then $r_2 > r_1$ and $E_x < 0$ for a positive rod. The vertical component is positive for $\lambda > 0$ because $u/\sqrt{u^2 + y^2}$ increases monotonically with u .

Figure 2.11 shows how signed endpoint angles encode the same geometry. Define $\theta_i = \tan^{-1}(u_i/y)$ measured from the downward perpendicular at P , positive toward the right. Since $u_i/r_i = \sin \theta_i$ and $y/r_i = \cos \theta_i$,

$$E_x = \frac{k_e \lambda}{y} (\cos \theta_2 - \cos \theta_1), \quad E_y = \frac{k_e \lambda}{y} (\sin \theta_2 - \sin \theta_1). \quad (2.81)$$

This endpoint-angle form is convenient when lengths are specified geometrically. The angles must be signed; replacing both by positive acute angles can erase the contribution from the left half of a rod.

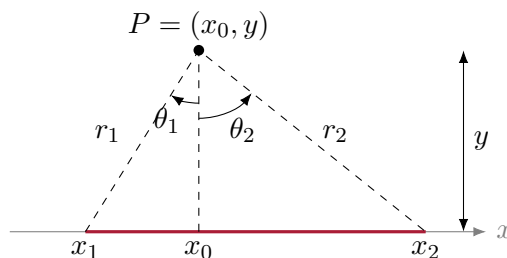


Figure 2.11: A general observation point above a uniform finite rod. Endpoint angles are signed relative to the downward perpendicular: the displayed θ_1 is negative and θ_2 positive. The two endpoint distances need not be equal.

Example 2.17: Field near the end of a finite charged strip model

A thin insulating rod extends from $x = 0$ to 0.40 m and has $\lambda = 10$ nC/m. Find the field at $(0.10, 0.10)$ m. Would the infinite-line approximation describe its direction?

Calculation. Here $u_1 = -0.10$ m, $u_2 = 0.30$ m, $r_1 = 0.1414$ m and $r_2 = 0.3162$ m. Thus

$$E_x = (89.9) \left(\frac{1}{0.3162} - \frac{1}{0.1414} \right) \text{ N/C} \simeq -351 \text{ N/C}, \quad (2.82)$$

$$E_y = \frac{89.9}{0.10} \left(\frac{0.30}{0.3162} + \frac{0.10}{0.1414} \right) \text{ N/C} \simeq 1489 \text{ N/C}. \quad (2.83)$$

The magnitude is 1.53×10^3 N/C and the vector is tilted 13.3° left of the upward vertical. The infinite-line field would be $2k_e \lambda / y = 1798$ N/C straight upward. It misses both the horizontal component and the finite-length reduction of the vertical component.

Interpretation. The nearer endpoint is only one perpendicular distance away, so the source cannot be treated as extending very far in both directions. A large total length alone is not sufficient: the field point's distance from *each* end matters.

At the bisector, $u_1 = -a$ and $u_2 = a$, so $r_1 = r_2$ and $E_x = 0$. If both endpoints tend to infinity in opposite directions at fixed λ , $E_y \rightarrow 2k_e \lambda / y$. If only one endpoint tends to infinity while the other lies at the foot of the perpendicular, both a horizontal and a vertical component remain. These limiting cases provide useful independent checks of the general expression.

2.9.4 Field of a uniformly charged finite rod on its axis

The field on the perpendicular bisector of a rod uses symmetry to cancel horizontal components. A different useful geometry is a field point on the axis of a finite rod but outside the rod. Let a uniformly charged rod extend from $x = 0$ to $x = L$ with linear charge density $\lambda > 0$. Find the field at a point $x = b$ with $b > L$. A source element at coordinate x has charge $dq = \lambda dx$ and lies a distance $b - x$ from the field point. Every contribution points in the $+x$ direction. Thus

$$E_x = k_e \int_0^L \frac{\lambda dx}{(b - x)^2}. \quad (2.84)$$

Using $u = b - x$, $du = -dx$, this becomes

$$E_x = k_e \lambda \left[\frac{1}{b-x} \right]_0^L = k_e \lambda \left(\frac{1}{b-L} - \frac{1}{b} \right). \quad (2.85)$$

Because $b > L$, the expression is positive. Far from the rod, $b \gg L$, expand

$$\frac{1}{b-L} = \frac{1}{b} \frac{1}{1-L/b} \approx \frac{1}{b} \left(1 + \frac{L}{b} \right). \quad (2.86)$$

Therefore

$$E_x \approx k_e \lambda \frac{L}{b^2} = k_e \frac{Q}{b^2}, \quad (2.87)$$

where $Q = \lambda L$. The far-field limit again becomes the point-charge field.

Example 2.18: Axial field of a short charged rod

A rod of length $L = 0.40$ m has linear charge density $\lambda = 25 \text{ nC m}^{-1}$. Find the field at a point on the rod axis 0.20 m beyond the right end.

Solution. The rod extends from 0 to $L = 0.40$ m. The field point is at $b = 0.60$ m. Equation (2.85) gives

$$E_x = k_e \lambda \left(\frac{1}{b-L} - \frac{1}{b} \right). \quad (2.88)$$

Substituting,

$$E_x = (8.99 \times 10^9)(25 \times 10^{-9}) \left(\frac{1}{0.20} - \frac{1}{0.60} \right) \approx 7.5 \times 10^2 \text{ N C}^{-1}. \quad (2.89)$$

The field points away from the positively charged rod, in the positive x direction.

2.9.5 Uniformly charged arc at the centre of curvature

Figure 2.12 makes the sign in the arc integral explicit: at the centre, the field of a positive source element is opposite to its outward radius vector. This is a case where all distances are equal but the contribution directions are not.

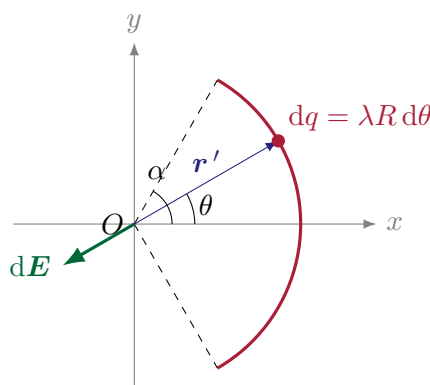


Figure 2.12: Uniform arc symmetric about the positive x axis. Pairing elements at θ and $-\theta$ cancels the vertical field components and leaves a negative horizontal field. The arc angle is measured in radians when writing its length $2\alpha R$.

Arc geometries are excellent tests of component thinking. Consider a uniformly charged circular arc of radius R , total charge Q and angle from $-\alpha$ to $+\alpha$ about the positive x axis. The field point is the centre of curvature. A source element at polar angle θ has charge

$$dq = \lambda R d\theta, \quad (2.90)$$

where $\lambda = Q/(2\alpha R)$. For positive charge, the field at the centre points from the charge element toward the centre, i.e. opposite to the radial unit vector of the element. Thus

$$d\mathbf{E} = -k_e \frac{dq}{R^2} (\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}). \quad (2.91)$$

The y components cancel by symmetry. The x component is

$$E_x = -k_e \frac{\lambda}{R} \int_{-\alpha}^{\alpha} \cos \theta d\theta = -k_e \frac{\lambda}{R} (2 \sin \alpha). \quad (2.92)$$

Therefore

$$\mathbf{E} = -\frac{2k_e \lambda \sin \alpha}{R} \hat{\mathbf{i}} = -\frac{k_e Q}{\alpha R^2} \sin \alpha \hat{\mathbf{i}}. \quad (2.93)$$

For a semicircle in the right half-plane, $\alpha = \pi/2$ and the field points to the left.

Example 2.19: Quarter-circle arc

A uniformly charged quarter-circle arc of radius R lies in the first quadrant from $\theta = 0$ to $\theta = \pi/2$, with total charge $Q > 0$. Find the field at the centre.

Solution. There is no symmetry about the x axis or y axis separately, but the two components can be integrated. The density is $\lambda = Q/[(\pi/2)R] = 2Q/(\pi R)$. The field contribution at the centre is

$$d\mathbf{E} = -k_e \frac{\lambda R d\theta}{R^2} (\cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{j}}). \quad (2.94)$$

Thus

$$E_x = -\frac{k_e \lambda}{R} \int_0^{\pi/2} \cos \theta d\theta = -\frac{k_e \lambda}{R}, \quad (2.95)$$

$$E_y = -\frac{k_e \lambda}{R} \int_0^{\pi/2} \sin \theta d\theta = -\frac{k_e \lambda}{R}. \quad (2.96)$$

Substituting $\lambda = 2Q/(\pi R)$ gives a field toward the third quadrant:

$$\mathbf{E} = -\frac{2k_e Q}{\pi R^2} (\hat{\mathbf{i}} + \hat{\mathbf{j}}). \quad (2.97)$$

2.9.6 Uniformly charged disk on its axis

A disk of radius R with uniform surface charge density σ can be built from concentric rings. A ring of radius s and thickness ds has area $dA = 2\pi s ds$ and charge $dq = \sigma 2\pi s ds$. The axial field contribution from this ring at height z is

$$dE_z = k_e \frac{z dq}{(s^2 + z^2)^{3/2}}. \quad (2.98)$$

Therefore

$$E_z = k_e 2\pi \sigma z \int_0^R \frac{s ds}{(s^2 + z^2)^{3/2}} \quad (2.99)$$

$$= \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right), \quad z > 0. \quad (2.100)$$

For $R \gg z$, the field approaches $\sigma/(2\epsilon_0)$, the field near a very large uniformly charged sheet. For $z \gg R$, the disk behaves like a point charge $Q = \sigma \pi R^2$.

Example 2.20: On-axis field of a uniformly charged disk

A disk of radius $R = 0.10$ m has surface charge density $\sigma = 3.0 \text{ nC m}^{-2}$. Estimate the field on its axis at $z = 0.020$ m above the centre.

Solution. Use Eq. (2.100):

$$E_z = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right). \quad (2.101)$$

The bracket is

$$1 - \frac{0.020}{\sqrt{0.020^2 + 0.10^2}} \approx 0.804. \quad (2.102)$$

Thus

$$E_z \approx \frac{3.0 \times 10^{-9}}{2(8.85 \times 10^{-12})} (0.804) \approx 1.36 \times 10^2 \text{ N C}^{-1}. \quad (2.103)$$

The field points away from the disk for positive surface charge.

Guided checks

1. Why is it legitimate to replace a disk by a set of rings for an axial field calculation?
2. In the charged-ring formula, why is the field zero at the centre?
3. For a very distant field point, why should a finite charged object resemble a point charge with the same total charge?
4. In a continuous distribution, why is dq not the same as q ?

2.9.7 What the disk formula does and does not describe

The disk calculation assumes a prescribed, uniform, infinitesimally thin surface charge. A real finite conducting disk generally has enhanced charge density near its edge; the uniform-disk formula is not automatically its solution. The derivation in Figure 2.13 uses rings of charge, not a uniform distribution of electric-field directions.

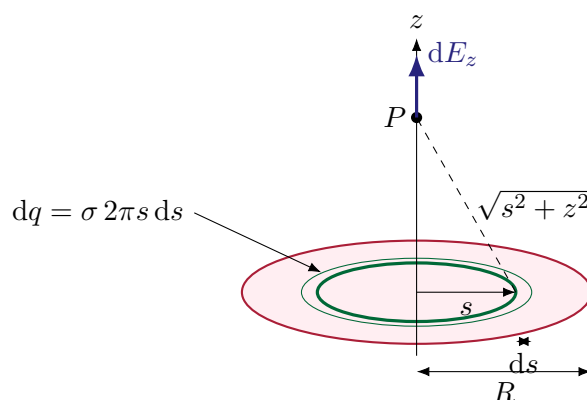


Figure 2.13: A disk decomposed into concentric annuli. Each annulus contributes the axial field of a ring. The radial width is exaggerated; the ellipse is a perspective view of a circular disk.

For $z < 0$, reflection symmetry reverses the field. Combining both sides gives

$$E_z(z) = \frac{\sigma}{2\epsilon_0} \left[\text{sgn}(z) - \frac{z}{\sqrt{z^2 + R^2}} \right], \quad z \neq 0. \quad (2.104)$$

The one-sided limits at the centre are $E_z(0^\pm) = \pm\sigma/(2\epsilon_0)$. There is no unique continuous value on the ideal surface itself. Averaging the two limiting values is a separate convention, not a field to substitute into the force law on an individual surface charge.

For $z \gg R$, expand $(1 + R^2/z^2)^{-1/2}$ to obtain

$$E_z = \frac{k_e Q}{z^2} \left[1 - \frac{3R^2}{4z^2} + \mathcal{O}\left(\frac{R^4}{z^4}\right) \right], \quad Q = \pi R^2 \sigma. \quad (2.105)$$

The field therefore tends to the point-charge result from below. Near the centre, at fixed σ with $z \ll R$, it approaches the infinite-sheet field. Holding Q fixed while increasing R instead dilutes σ ; that is a different limit.

Example 2.21: How large must a disk be to resemble a sheet?

At height $z > 0$, require the exact uniform-disk field to differ from the sheet value $E_{\text{sheet}} = \sigma/(2\epsilon_0)$ by less than 5% of E_{sheet} . Find a condition on R/z .

Solution. The omitted fraction is

$$\frac{E_{\text{sheet}} - E_{\text{disk}}}{E_{\text{sheet}}} = \frac{z}{\sqrt{z^2 + R^2}} < 0.05. \quad (2.106)$$

Thus $R/z > \sqrt{0.05^{-2} - 1} \simeq 20$. At $z = 2$ mm, a radius exceeding roughly 4 cm is needed under this criterion. A criterion normalized to the exact field would be slightly stricter.

Interpretation. The field of a finite plate is never distance independent everywhere. The sheet approximation is local to a broad central region. The criterion above is for the axis of a prescribed uniform disk; it does not quantify fringing at arbitrary points near conducting electrodes.

2.10 Electric dipoles in uniform external fields

A dipole in a uniform external electric field experiences no net force if the field is exactly uniform, but it generally experiences a torque. The positive charge is pushed along $\mathbf{E}$ and the negative charge is pushed opposite $\mathbf{E}$. The two forces have equal magnitudes and opposite directions, so their sum is zero, but their lines of action are different unless the dipole is parallel or antiparallel to the field.

Assume a rigid dipole with fixed separation $\mathbf{d}$ and moment $\mathbf{p} = q\mathbf{d}$, where $q > 0$. Evaluate the torque about the midpoint; the endpoint position vectors are $\pm\mathbf{d}/2$. The external forces are $\mathbf{F}_+ = q\mathbf{E}$ and $\mathbf{F}_- = -q\mathbf{E}$. Consequently,

$$\mathbf{F}_{\text{net}} = q\mathbf{E} - q\mathbf{E} = 0, \quad (2.107)$$

$$\boldsymbol{\tau} = \frac{\mathbf{d}}{2} \times q\mathbf{E} + \left(-\frac{\mathbf{d}}{2}\right) \times (-q\mathbf{E}) = q\mathbf{d} \times \mathbf{E}. \quad (2.108)$$

Because the net force vanishes, changing the origin used to compute this torque does not change the answer: the two forces form a couple. The field in these expressions excludes the dipole's own field.

The torque is

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}, \quad (2.109)$$

with magnitude

$$\tau = pE \sin \theta, \quad (2.110)$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{E}$. The potential energy is

$$U = -\mathbf{p} \cdot \mathbf{E} = -pE \cos \theta. \quad (2.111)$$

This energy follows from rotational work, without introducing electric potential. Choose a signed rotation coordinate θ measured from the field direction. The torque component tending to increase θ is $\tau_\theta = -pE \sin \theta$. The change in potential energy is minus the work by this torque:

$$U(\theta) - U(\pi/2) = - \int_{\pi/2}^{\theta} \tau_\theta d\theta' = pE \int_{\pi/2}^{\theta} \sin \theta' d\theta' = -pE \cos \theta. \quad (2.112)$$

Thus Eq. (2.111) chooses $U(\pi/2) = 0$. It describes the interaction with the external field, not the internal binding energy of the rigid dipole.

Both $\theta = 0$ and $\theta = \pi$ have zero torque. The first has minimum energy $-pE$ and is stable against a small rotation; the second has maximum energy $+pE$ and is unstable. A small departure from alignment gives $\tau_\theta \simeq -pE\theta$, a restoring torque. An isolated undamped dipole generally oscillates around alignment rather than settling there. Relaxation to the minimum requires dissipation, for example through collisions. Zero net force also does not mean zero translational velocity.

For a rotation from θ_i to θ_f , the field does work $W_{\text{field}} = -\Delta U$. Only in quasistatic rotation with negligible change in kinetic energy is the external agent's work $W_{\text{ext}} = \Delta U$.

Figure 2.14 shows why zero torque alone does not establish stability.

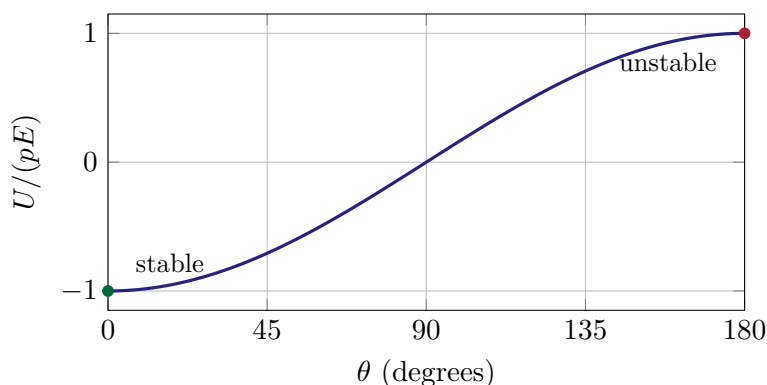


Figure 2.14: Interaction energy of a rigid permanent dipole in a uniform field. The curve is calculated from $U/(pE) = -\cos\theta$. Its stationary points have opposite stability.

The angle in Figure 2.15 is measured between the dipole moment and the external field.

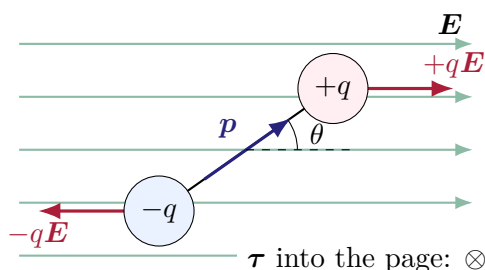


Figure 2.15: A rigid dipole in a horizontal external field. The angle θ is measured from $\mathbf{E}$ to $\mathbf{p}$. Equal and opposite forces give zero net force and a clockwise torque, into the page, tending to reduce θ .

Example 2.22: Torque and energy of a molecular dipole

A small molecule has dipole moment $p = 6.0 \times 10^{-30} \text{ C m}$. It is placed in a uniform electric field of magnitude $E = 4.0 \times 10^6 \text{ N C}^{-1}$ at an angle 60° to the field. Find the torque magnitude and the potential energy.

Solution. The torque magnitude is

$$\tau = pE \sin \theta = (6.0 \times 10^{-30})(4.0 \times 10^6) \sin 60^\circ \approx 2.1 \times 10^{-23} \text{ N m}. \quad (2.113)$$

The potential energy is

$$U = -pE \cos \theta = -(6.0 \times 10^{-30})(4.0 \times 10^6) \cos 60^\circ = -1.2 \times 10^{-23} \text{ J}. \quad (2.114)$$

The negative sign means the configuration has lower energy than the reference orientation with $\theta = 90^\circ$.

Key message

A uniform electric field can exert a torque on a rigid neutral dipole but gives it no net translational acceleration. A centre of mass already in motion continues at constant velocity when no other forces act. A nonuniform field can exert a net force on a dipole, because the forces at its two ends need not cancel.

2.11 Field probes, measurement and the small-test-charge idea

The definition $\mathbf{E} = \mathbf{F}/q_0$ involves a test charge, but real measurement devices are never perfectly passive. A probe has finite size, finite charge, finite conductivity, finite capacitance and mechanical support. If it is placed near a conductor, it may rearrange charges. If it draws current, it changes the system. If it is too large, it averages the field over a region instead of measuring the field at a point.

The ideal test charge is therefore a limiting model. It should be small enough that the source distribution is essentially unchanged, and its physical size should be small compared with the spatial scale over which the field varies. This is familiar from other parts of physics. A thermometer should not significantly heat or cool the object it measures. A voltmeter should not significantly change the circuit it measures. A field probe should not significantly change the field it measures.

Note

In practice, electric fields are often inferred indirectly: from forces, from particle deflection, from voltage differences, from charge distributions on conductors, from sensor calibration or from numerical solutions constrained by measured boundary conditions. Directly placing a literal isolated point test charge at every point is not how most engineering field maps are produced.

2.11.1 A force-balance measurement

A small charged bead suspended by an insulating thread provides a mechanical field probe. Suppose the bead has known mass m and positive charge q , and a horizontal external field deflects it by angle β from the vertical. In static equilibrium, resolving the tension T gives

$$T \cos \beta = mg, \quad T \sin \beta = qE, \quad E = \frac{mg \tan \beta}{q}. \quad (2.115)$$

Here the electric force is inferred from balance with gravity; gravity cannot be neglected simply because the object is charged. The field must be nearly uniform over the bead, charge leakage must be slow during the reading, and the bead's charge and support must not appreciably alter the source.

Figure 2.16 distinguishes the measured angle from the directions of the three forces.

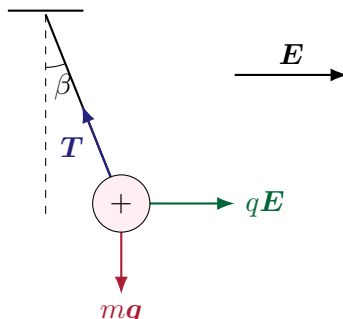


Figure 2.16: Inferring a horizontal field from the equilibrium of a positive charged bead. The force diagram is schematic; arrow lengths are not a calibrated force scale. The angle is measured from the vertical.

Example 2.23: Field strength and uncertainty from a suspended bead

A bead has $m = (1.00 \pm 0.01) \text{ mg}$ and $q = (2.00 \pm 0.10) \text{ nC}$. Its equilibrium angle is $\beta = (12.0 \pm 0.5)^\circ$. Estimate the field and the uncertainty due to these three measurements. Treat the quoted uncertainties as independent standard uncertainties and take $g = 9.81 \text{ m s}^{-2}$.

Calculation. Converting mass to 10^{-6} kg gives

$$E = \frac{(1.00 \times 10^{-6})(9.81) \tan 12.0^\circ}{2.00 \times 10^{-9}} = 1.04 \times 10^3 \text{ N C}^{-1}. \quad (2.116)$$

For independent small uncertainties, differentiate the logarithm of Eq. (2.115):

$$\left(\frac{u_E}{E}\right)^2 \simeq \left(\frac{u_m}{m}\right)^2 + \left(\frac{u_q}{q}\right)^2 + \left(\frac{u_\beta}{\sin \beta \cos \beta}\right)^2. \quad (2.117)$$

The angular uncertainty must be in radians: $u_\beta = 0.5\pi/180 = 0.00873$. The relative contributions are 0.010, 0.050 and 0.0429, so $u_E/E = 0.0667$. Thus

$$E = (1.04 \pm 0.07) \times 10^3 \text{ N C}^{-1}. \quad (2.118)$$

Interpretation. Charge calibration and angular reading dominate. This uncertainty budget does not include changes in the field caused by the bead, nonuniform fields, charge leakage or air currents. Repeating the measurement tests repeatability; it does not remove a common calibration bias. Reversing the bead charge should reverse the deflection if the external field and other conditions remain fixed.

2.11.2 Testing the nonperturbing-probe assumption

A small ratio $|q_0/Q|$ is neither necessary nor sufficient by itself. Two mechanically fixed ideal point charges preserve their positions regardless of that ratio. In contrast, a weakly supported source or a nearby conductor may respond strongly even to a much smaller probe. Compare the disturbance with the source's mechanical or electrical response scale.

For example, suppose a source charge Q is attached to a spring of stiffness κ_s and a probe q_0 is a

distance r away. If displacement is small, the probe exerts a force of magnitude $k_e|Qq_0|/r^2$, causing

$$|\Delta r| \simeq \frac{k_e|Qq_0|}{\kappa_s r^2}, \quad \frac{|\Delta E|}{E} \simeq \frac{2|\Delta r|}{r}. \quad (2.119)$$

For $Q = 1.0 \text{ nC}$, $q_0 = 0.10 \text{ nC}$ and $r = 0.10 \text{ m}$, the force is $9.0 \times 10^{-8} \text{ N}$. With $\kappa_s = 10^{-3} \text{ N m}^{-1}$, the source shifts by $90 \text{ }\mu\text{m}$ and changes the field by about 0.18% . A spring a hundred times softer would invalidate the small-disturbance assumption at the same charge ratio. The estimate concerns source motion; conductor polarization needs a different response model.

Potential mapping on conducting paper offers another laboratory route, developed in Chapter 4. It measures voltages in a steady conduction problem and uses a mathematical analogy with charge-free electrostatics for homogeneous media and matching boundaries. Such a map is not a direct photograph of electric field lines, and a two-dimensional paper model should not be mistaken for the three-dimensional field of point charges.

2.12 Engineering applications: drift and particle collection

A particle in vacuum can accelerate continually while it remains in a field. In a gas or liquid, collisions oppose its motion. A simple linear-drag model for the velocity component v along a uniform field is

$$m \frac{dv}{dt} = qE - bv, \quad b > 0. \quad (2.120)$$

If the particle starts from rest, solving this first-order equation gives

$$v(t) = v_d(1 - e^{-t/\tau}), \quad v_d = \frac{qE}{b}, \quad \tau = \frac{m}{b}. \quad (2.121)$$

The early-time limit is $v \simeq (qE/m)t$, recovering the vacuum acceleration before drag develops. After several relaxation times, the particle has almost constant drift velocity, even though a nonzero electric force still acts. The force is then balanced by drag. For a negative charge, the signed drift velocity is opposite to the field.

Example 2.24: Collection of a charged particle from a gas stream

A model spherical particle has radius $r_p = 1.0 \mu\text{m}$, mass density $\rho_p = 2.0 \times 10^3 \text{ kg m}^{-3}$, and positive charge $100e$. The gas has dynamic viscosity $\eta = 1.8 \times 10^{-5} \text{ Pa s}$. An electric field $E = 5.0 \times 10^4 \text{ N C}^{-1}$ acts transverse to a gas stream. Estimate the time to drift $w = 5.0 \text{ mm}$ to a collecting plate.

Assumptions and strategy. Use Stokes drag $b = 6\pi\eta r_p$ for slow motion of an isolated sphere; neglect gravity along the collection direction, diffusion, slip corrections, turbulence and changing charge. These are supplied model parameters, not measured device performance.

$$b = 3.39 \times 10^{-10} \text{ kg s}^{-1}, \quad m = \frac{4}{3}\pi r_p^3 \rho_p = 8.38 \times 10^{-15} \text{ kg}, \quad (2.122)$$

$$qE = 8.01 \times 10^{-13} \text{ N}, \quad v_d = 2.36 \times 10^{-3} \text{ m s}^{-1}, \quad (2.123)$$

$$\tau = 2.47 \times 10^{-5} \text{ s}, \quad t_{\text{collect}} \simeq \frac{w}{v_d} = 2.1 \text{ s}. \quad (2.124)$$

The collection time is much longer than τ , justifying the terminal-speed approximation. In a gas stream of longitudinal speed 0.10 m s^{-1} , a collection region would need a length of order 0.21 m under this ideal model.

Checks and limits. The units $qE/b = \text{N}/(\text{kg/s})$ are velocity. For gas density 1.2 kg m^{-3} , the transverse Reynolds number $2\rho_{\text{gas}} r_p v_d / \eta \simeq 3.1 \times 10^{-4}$ is small, consistent with viscous drag. Slip at micrometre scales and Brownian diffusion can require corrections; their neglect limits the estimate. The vacuum formula $y = qEt^2/(2m)$ would greatly overestimate collection in this collision-dominated regime.

Ionization detectors use the same separation of electric force and transport response. Electrons and positive ions experience opposite electric forces, but their drift speeds depend on the medium. One should not infer equal speeds from equal charge magnitudes. Dielectric polarization, microscopic mobility and space-charge effects require the material physics of later chapters.

2.13 Laboratory and computational connection: field mapping on a grid

A simple computational experiment can make the field concept concrete. Choose several source charges, place a grid of field points in the plane and compute

$$\mathbf{E}(x, y) = k_e \sum_i q_i \frac{(x - x_i)\hat{\mathbf{i}} + (y - y_i)\hat{\mathbf{j}}}{[(x - x_i)^2 + (y - y_i)^2]^{3/2}}. \quad (2.125)$$

At each grid point, draw an arrow proportional to the field vector or a normalized arrow showing only direction. The result is an arrow plot. One can also trace approximate field lines by stepping along the local field direction.

Figure 2.17 is a direction-only sample of this calculation.

Method

A minimal field-map algorithm is:

1. store the source charges as triples (q_i, x_i, y_i) ;
2. loop over grid points (x_j, y_k) ;
3. initialize $E_x = 0$ and $E_y = 0$;
4. for each source charge, compute $\Delta x = x_j - x_i$, $\Delta y = y_k - y_i$, $R = (\Delta x^2 + \Delta y^2)^{1/2}$;

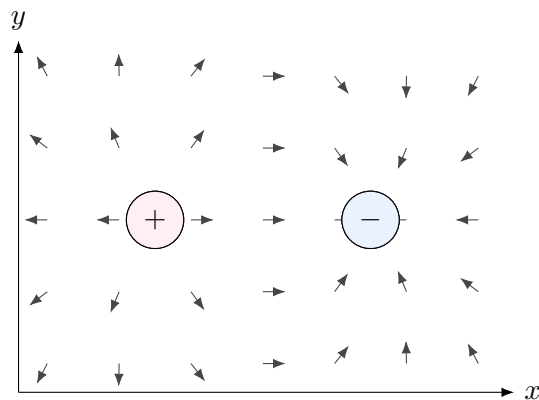


Figure 2.17: A normalized arrow plot computed from Coulomb superposition for a dipole pair of point charges. Each arrow shows the direction of the computed field at a grid point. Arrow lengths are equal by construction and carry no magnitude information. Coordinates are in arbitrary length units.

5. add $k_e q_i \Delta x / R^3$ to E_x and $k_e q_i \Delta y / R^3$ to E_y ;
6. mask points inside a stated exclusion radius around each ideal point charge;
7. plot arrows, streamlines or contours of field magnitude.

Example 2.25: Numerical field from three point charges

Three source charges are located in the plane: $q_1 = +2.0 \text{ nC}$ at $(0, 0)$, $q_2 = -2.0 \text{ nC}$ at $(0.20 \text{ m}, 0)$ and $q_3 = +1.0 \text{ nC}$ at $(0, 0.20 \text{ m})$. Find the electric field at $P = (0.20 \text{ m}, 0.20 \text{ m})$.

Solution. Use the coordinate formula. From q_1 , $\mathbf{R}_1 = 0.20\hat{\mathbf{i}} + 0.20\hat{\mathbf{j}}$ and $R_1 = 0.283 \text{ m}$. From q_2 , $\mathbf{R}_2 = 0.20\hat{\mathbf{j}}$ and $R_2 = 0.20 \text{ m}$. From q_3 , $\mathbf{R}_3 = 0.20\hat{\mathbf{i}}$ and $R_3 = 0.20 \text{ m}$.

The three contributions are

$$\mathbf{E}_1 = k_e(2.0 \times 10^{-9}) \frac{0.20\hat{\mathbf{i}} + 0.20\hat{\mathbf{j}}}{(0.283)^3} \approx (159\hat{\mathbf{i}} + 159\hat{\mathbf{j}}) \text{ N C}^{-1}, \quad (2.126)$$

$$\mathbf{E}_2 = k_e(-2.0 \times 10^{-9}) \frac{0.20\hat{\mathbf{j}}}{(0.20)^3} \approx -449\hat{\mathbf{j}} \text{ N C}^{-1}, \quad (2.127)$$

$$\mathbf{E}_3 = k_e(1.0 \times 10^{-9}) \frac{0.20\hat{\mathbf{i}}}{(0.20)^3} \approx 225\hat{\mathbf{i}} \text{ N C}^{-1}. \quad (2.128)$$

Adding components gives

$$\mathbf{E}(P) \approx (384\hat{\mathbf{i}} - 290\hat{\mathbf{j}}) \text{ N C}^{-1}. \quad (2.129)$$

The magnitude is about $4.8 \times 10^2 \text{ N C}^{-1}$, directed below the positive x axis.

2.14 Core synthesis and self-check

Summary

The field calculation specifies the source distribution and the observation point. The response calculation then uses the external field to obtain force, torque or motion. Superposition is vector addition; symmetry removes components only when the complete signed source distribution has the corresponding symmetry. A continuous density replaces a many-particle model with an integral, but does not change the underlying Coulomb law. Approximations such as a point

source, a dipole, an infinite line or a sheet are justified by scale ratios and a stated accuracy.

Question	Reliable reasoning	Error to avoid
What is the field direction?	Force direction for a positive test charge	Using the force direction on an electron
How do contributions combine?	Add signed Cartesian components	Adding magnitudes at different angles
What makes a source continuous?	A coarse-grained density with a defined geometry	Assuming a conductor has uniform charge
What does a field line represent?	Local tangent direction where $\mathbf{E} \neq 0$	Confusing a line with a particle path
Is a dipole in equilibrium?	Check both force and torque, then stability	Inferring stability from zero torque alone
Is a field map trustworthy?	Check units, symmetry, convergence and analytic limits	Treating normalized arrows as magnitude data

Guided checks

1. A negative charge feels a leftward force. Which way is the field? *Rightward*, because dividing by a negative charge reverses direction.
2. Two equal positive sources give zero field at their midpoint. Is the neighbourhood field-free? *No*; cancellation holds at one point. Stability requires examining nearby positions.
3. Why is the ring field zero at its centre but the dipole field nonzero there? Opposite ring elements carry equal signs and cancel; opposite dipole elements carry opposite signs and their centre fields add.
4. Can a neutral dipole move in a uniform field? It can retain an initial velocity, but the field gives it no net translational acceleration in the rigid two-charge model.
5. A particle follows a curve through a uniform field. What is curved? Its *trajectory*; the field lines remain straight.

The core reading can now proceed to the practice problems. The following enrichment develops the same ideas with Taylor expansions, stability analysis and numerical methods. It is optional for a first reading and does not replace the later chapters on flux or potential.

2.15 Advanced Topics for Technical and Engineering Students

This section asks what the field reveals about a source, how a neutral object responds to spatial variation, and how numerical models can be checked. The additional tools are first-order Taylor expansions, elementary differential equations and numerical quadrature. Every result remains tied to the electrostatic source and response distinction developed in the core.

2.15.1 From a localized distribution to its far field

For a specified volume density, the vector integral is

$$\mathbf{E}(\mathbf{r}) = k_e \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'. \quad (2.130)$$

The integration variable is the source position $\mathbf{r}'$; the observation vector $\mathbf{r}$ is fixed. The same notation applies to line or surface densities after changing the integration measure. We use the formula away from singular point, line or surface sources. Boundary values on ideal charge sheets require specified one-sided limits.

Suppose all source points lie within a characteristic size ℓ_s about the origin and $r \gg \ell_s$. Define $\hat{\mathbf{n}} = \mathbf{r}/r$. Then

$$|\mathbf{r} - \mathbf{r}'|^{-3} = r^{-3} \left(1 - \frac{2\hat{\mathbf{n}} \cdot \mathbf{r}'}{r} + \frac{r'^2}{r^2} \right)^{-3/2}. \quad (2.131)$$

Using $(1 + u)^{-3/2} = 1 - 3u/2 + \mathcal{O}(u^2)$ and retaining first order in ℓ_s/r gives

$$\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{\hat{\mathbf{n}}}{r^2} + \frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{r}') - \mathbf{r}'}{r^3} + \mathcal{O}\left(\frac{r'^2}{r^4}\right). \quad (2.132)$$

Integrating term by term introduces two source moments,

$$Q_{\text{tot}} = \int dq, \quad \mathbf{p} = \int \mathbf{r}' dq, \quad (2.133)$$

and yields

$$\mathbf{E}(\mathbf{r}) = k_e \frac{Q_{\text{tot}}}{r^2} \hat{\mathbf{n}} + \frac{k_e}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} - \mathbf{p}] + \mathcal{O}\left(\frac{k_e Q_{\text{abs}} \ell_s^2}{r^4}\right), \quad (2.134)$$

where $Q_{\text{abs}} = \int |dq|$ is a scale for the total amount of charge of either sign. The remainder is an absolute estimate, not a guaranteed small *fraction* of a leading term that may itself cancel. This is the beginning of a multipole expansion derived directly from Coulomb's field, without a potential expansion.

The first term depends only on net charge. The second retains information about the separation of positive and negative charge. For a neutral two-charge source it reduces to $q(\mathbf{r}_+ - \mathbf{r}_-) = q\mathbf{d}$. With polar angle θ measured from $\mathbf{p}$, its radial and angular components are

$$E_r = \frac{2k_e p \cos \theta}{r^3}, \quad E_\theta = \frac{k_e p \sin \theta}{r^3}, \quad |\mathbf{E}| = \frac{k_e p}{r^3} \sqrt{1 + 3 \cos^2 \theta}. \quad (2.135)$$

At $\theta = 0$ or π , the vector is parallel to $\mathbf{p}$. At $\theta = \pi/2$, it is opposite to $\mathbf{p}$. These statements reproduce both core special cases and determine all intermediate directions. The factor-of-two axial/equatorial magnitude ratio is a geometric result, not a change in the dipole strength.

Example 2.26: A dipole observed at an oblique angle

A compact neutral source has dipole moment $\mathbf{p} = p\hat{\mathbf{z}}$. Find the field direction at $\theta = 60^\circ$ in the far zone, expressing it in the local radial and angular basis.

Solution. Equation (2.135) gives

$$\mathbf{E} = \frac{k_e p}{r^3} \left(\hat{\mathbf{r}} + \frac{\sqrt{3}}{2} \hat{\boldsymbol{\theta}} \right). \quad (2.136)$$

The field has magnitude $\sqrt{7} k_e p / (2r^3)$ and is tilted from the outward radial direction toward increasing θ by

$$\psi = \tan^{-1} \left(\frac{E_\theta}{E_r} \right) = \tan^{-1}(\sqrt{3}/2) \simeq 40.9^\circ. \quad (2.137)$$

The dipole field is generally neither radial nor parallel to its axis. Both components scale as r^{-3} , so the direction at fixed θ becomes independent of distance in this leading approximation.

Origins, nearly neutral sources and cancellation

If the origin is shifted by $\mathbf{b}$, the new source coordinates are $\mathbf{r}'' = \mathbf{r}' - \mathbf{b}$, so

$$\mathbf{p}' = \mathbf{p} - Q_{\text{tot}} \mathbf{b}. \quad (2.138)$$

A neutral source has an origin-independent dipole moment. A charged source does not; its complete field is nevertheless independent of the coordinate origin. Changing origins rearranges the terms of the expansion consistently.

A nonzero net charge eventually dominates at sufficiently large distance. If $|Q_{\text{tot}}|$ is very small, however, the condition $r \gg \ell_s$ alone does not ensure monopole dominance. Comparing scales gives $k_e |Q_{\text{tot}}| / r^2$ versus $k_e p / r^3$, so the crossover is of order $r \sim p / |Q_{\text{tot}}|$, with angular factors. A nearly neutral object may look dipolar over a wide intermediate range. Conversely, if both net charge and dipole moment vanish, a quadrupolar term often leads and the field typically falls as r^{-4} ; particular observation directions can have still stronger cancellations.

Example 2.27: Far-field comparison of a charge and a dipole

A point charge $Q = 1.0 \text{ nC}$ and a dipole with $p = 1.0 \times 10^{-11} \text{ C m}$ are observed at $r = 1.0 \text{ m}$. Compare the order of magnitude of their electric fields.

Solution. The point-charge field is

$$E_Q \sim k_e \frac{Q}{r^2} \approx (9.0 \times 10^9)(1.0 \times 10^{-9}) = 9 \text{ N C}^{-1}. \quad (2.139)$$

The dipole field is of order

$$E_p \sim k_e \frac{p}{r^3} \approx (9.0 \times 10^9)(1.0 \times 10^{-11}) = 9.0 \times 10^{-2} \text{ N C}^{-1}, \quad (2.140)$$

up to angular factors of order unity. Even though both sources involve small charges, the neutral dipole field is much weaker far away because of cancellation.

Guided checks

For charges $+q$ at $x = \pm a$ and $-2q$ at the origin, compute Q_{tot} and $\mathbf{p}$. Both vanish. On the positive axis with $x \gg a$, expansion of the three fields gives $E_x \simeq 6k_e q a^2 / x^4$. The absence of an r^{-2} or r^{-3} term is a source-property check on the calculation.

2.15.2 Force on a permanent dipole in a nonuniform field

Uniform-field cancellation fails when the two charges sample different external fields. Put the centre at $\mathbf{r}$ and the endpoints at $\mathbf{r} \pm \mathbf{d}/2$. The exact net external force in the two-charge model is

$$\mathbf{F} = q [\mathbf{E}(\mathbf{r} + \mathbf{d}/2) - \mathbf{E}(\mathbf{r} - \mathbf{d}/2)]. \quad (2.141)$$

Expand each field component about the centre. For a sufficiently short dipole,

$$F_i \simeq \sum_j p_j \frac{\partial E_i}{\partial x_j}, \quad \mathbf{F} \simeq (\mathbf{p} \cdot \nabla) \mathbf{E}. \quad (2.142)$$

Here i, j label Cartesian components. The derivative measures how the field changes when one moves along the dipole axis. In the symmetric endpoint model, the next force term contains third spatial derivatives and is of order qd^3 times a typical third derivative. The approximation requires the field variation length to exceed d , and the field must be smooth across the dipole.

For a dipole aligned along x on a symmetry line, only the sampled axial component may be needed:

$$F_x \simeq p \frac{dE_x}{dx}. \quad (2.143)$$

A prescribed permanent dipole aligned with an increasing positive field is pulled toward larger field strength. An antiparallel dipole has the opposite force unless it rotates. Thus “neutral objects are attracted to stronger fields” is not a universal statement about all fixed dipoles. Orientation, field gradients and the object’s polarization law matter.

A spatially varying axial field is usually a restriction of a three-dimensional field to a symmetry axis. It should not automatically be interpreted as a field $E_x(x)\hat{\mathbf{i}}$ everywhere in a source-free three-dimensional region. Transverse components away from the axis may be required.

Example 2.28: Estimate of dipole force in a field gradient

A polarizable particle has a local induced dipole moment approximately $p = 2.0 \times 10^{-24} \text{ C m}$ aligned with a nonuniform field. The field changes by $1.0 \times 10^5 \text{ N C}^{-1}$ over 1.0 mm. Estimate the force.

Solution. The field gradient is approximately

$$\frac{dE}{dx} \sim \frac{1.0 \times 10^5}{1.0 \times 10^{-3}} = 1.0 \times 10^8 \text{ N C}^{-1} \text{ m}^{-1}. \quad (2.144)$$

Then

$$F \sim p \frac{dE}{dx} = (2.0 \times 10^{-24})(1.0 \times 10^8) = 2.0 \times 10^{-16} \text{ N}. \quad (2.145)$$

This is tiny macroscopically but may be relevant for microscopic particles in fluids, where masses and drag forces are also small.

Induced dipoles and the factor of one half

An isotropic linearly polarizable object has $\mathbf{p} = \alpha \mathbf{E}$ with positive constant polarizability α . Its dipole moment is not held fixed as the applied field changes. The internal work required to build the moment can be modelled by $U_{\text{int}} = p^2/(2\alpha)$, while its coupling to the external field is $-\mathbf{p} \cdot \mathbf{E}$. Minimizing

$$U(p) = \frac{p^2}{2\alpha} - pE \quad (2.146)$$

for an aligned moment gives $p = \alpha E$ and

$$U_{\text{eff}} = -\frac{1}{2}\alpha E^2, \quad \mathbf{F} = \frac{1}{2}\alpha \nabla(E^2). \quad (2.147)$$

The gradient notation means the vector of spatial derivatives. This result assumes quasistatic adjustment, a weak enough field for linear response, and an object small compared with the field variation scale. It is the simplest vacuum electrostatic model of dielectrophoretic attraction. Effective response in a surrounding fluid or an alternating field can be more complicated and need not have the same sign.

The force also follows from substituting the *local* induced moment into Eq. (2.142) and using the curl-free character of electrostatics. It does not follow by differentiating $-\alpha E^2$ as though that expression were the complete induced-dipole energy; doing so counts twice the force. The internal polarization cost supplies the missing half.

Example 2.29: A polarizable particle near a fixed point charge

A small isotropic particle of polarizability $\alpha > 0$ is a distance r from a fixed point charge Q . Ignore the particle's influence on the source and use its leading induced-dipole response. Find its radial force.

Solution. The external field is $\mathbf{E} = k_e Q \hat{\mathbf{r}}/r^2$, so $E^2 = k_e^2 Q^2/r^4$. Therefore

$$\mathbf{F} = \frac{\alpha}{2} \frac{d}{dr} \left(\frac{k_e^2 Q^2}{r^4} \right) \hat{\mathbf{r}} = -\frac{2\alpha k_e^2 Q^2}{r^5} \hat{\mathbf{r}}. \quad (2.148)$$

The force is attractive for either sign of Q . Its r^{-5} decrease combines the r^{-2} induced moment with the r^{-3} gradient of the external field. The estimate fails when the particle's size becomes comparable with r , or when polarization is nonlinear. Its energy is $-\alpha k_e^2 Q^2/(2r^4)$, whose negative radial derivative verifies the sign and factor.

2.15.3 Equilibrium is a local question: two stability examples

A point where $\mathbf{E} = 0$ gives zero electric force on a probe at that instant, but stability depends on the first changes in the force after a displacement. This distinction connects electrostatics directly to the small-oscillation methods of mechanics.

For two positive sources $+q$ at $x = \pm a$, the field on the line between them is

$$E_x(x) = k_e q \left[\frac{1}{(a+x)^2} - \frac{1}{(a-x)^2} \right] \simeq -\frac{4k_e q}{a^3} x, \quad |x| \ll a. \quad (2.149)$$

A positive test charge q_t constrained to this line has a restoring force and oscillation frequency

$$\omega_x = \sqrt{\frac{4k_e q q_t}{ma^3}}. \quad (2.150)$$

However, on the perpendicular bisector,

$$E_y(0, y) = \frac{2k_e q y}{(a^2 + y^2)^{3/2}} \simeq \frac{2k_e q}{a^3} y. \quad (2.151)$$

The same positive probe is pushed away from the centre after a transverse displacement. A negative probe reverses both responses. Neither sign has stable confinement in all directions in this arrangement. A constraint that holds a bead on a line is part of the physics, not an irrelevant mathematical convenience.

For a positive ring, the near-centre axial field is $E_z \simeq k_e Q z/a^3$. A negative charge $q_t = -|q_t|$ constrained to the axis obeys

$$m\ddot{z} = -\frac{k_e Q |q_t|}{a^3} z, \quad \omega_z = \sqrt{\frac{k_e Q |q_t|}{ma^3}}. \quad (2.152)$$

The approximation requires an amplitude much smaller than a . It predicts axial oscillation, not a three-dimensional electrostatic trap.

Example 2.30: Checking the missing transverse stability of a ring

Show directly from Coulomb's integral that a small radial displacement s in the ring plane produces $E_s \simeq -k_e Q s / (2a^3)$ for a uniform positive ring.

Strategy. Let the observation point be $(s, 0, 0)$ and a source element be $a(\cos \varphi, \sin \varphi, 0)$. The x component is

$$E_x = \frac{k_e Q}{2\pi} \int_0^{2\pi} \frac{s - a \cos \varphi}{(a^2 + s^2 - 2as \cos \varphi)^{3/2}} d\varphi. \quad (2.153)$$

To first order in s/a , the integrand is

$$\frac{s - a \cos \varphi}{a^3} \left(1 + \frac{3s}{a} \cos \varphi \right) = -\frac{\cos \varphi}{a^2} + \frac{s}{a^3} (1 - 3 \cos^2 \varphi) + \mathcal{O}(s^2/a^4). \quad (2.154)$$

The averages are $\langle \cos \varphi \rangle = 0$ and $\langle \cos^2 \varphi \rangle = 1/2$, giving

$$E_s \simeq -\frac{k_e Q}{2a^3} s. \quad (2.155)$$

A negative test charge therefore experiences an *outward* radial force. It is stable along the axis but unstable sideways. Axial stability alone cannot establish trapping. This explicit example anticipates the general restrictions on static electrostatic confinement discussed in more advanced electromagnetism.

2.15.4 Thermal motion and the mean orientation of a dipole

A torque gives a preferred direction but does not eliminate thermal motion. For independent rigid permanent dipoles in classical thermal equilibrium, an orientation has Boltzmann weight proportional to $\exp[-U/(k_B T)]$. With $U = -pE \cos \theta$, define the dimensionless field parameter $\xi = pE/(k_B T)$. The solid-angle measure is $\sin \theta d\theta d\varphi$; orientations near the equator are more numerous than orientations near a pole.

Integrating over angles gives the orientational partition factor

$$Z = 2\pi \int_0^\pi e^{\xi \cos \theta} \sin \theta d\theta = 4\pi \frac{\sinh \xi}{\xi}. \quad (2.156)$$

The average alignment follows either by doing the weighted integral of $\cos \theta$ or differentiating $\ln Z$:

$$\langle \cos \theta \rangle = \frac{\partial \ln Z}{\partial \xi} = \coth \xi - \frac{1}{\xi} \equiv \mathcal{L}(\xi). \quad (2.157)$$

Here $\mathcal{L}$ is the classical Langevin function. Its apparent singularity at $\xi = 0$ cancels. For weak fields,

$$\mathcal{L}(\xi) = \frac{\xi}{3} - \frac{\xi^3}{45} + \cdots, \quad \langle p_{\parallel} \rangle \simeq \frac{p^2 E}{3k_B T}. \quad (2.158)$$

For $\xi \gg 1$, $\mathcal{L}(\xi) \simeq 1 - 1/\xi$: alignment approaches unity gradually. The factor $1/3$ in the weak-field limit comes from averaging over three-dimensional orientations. Dropping the $\sin \theta$ measure gives a different, inappropriate planar model.

Example 2.31: Dipole energy compared with thermal energy

A molecule has dipole moment $p = 6.0 \times 10^{-30}$ C m in a field $E = 1.0 \times 10^6$ N C⁻¹. Compare pE with $k_B T$ at $T = 300$ K, using $k_B = 1.38 \times 10^{-23}$ J K⁻¹.

Solution. The dipole energy scale is

$$pE = (6.0 \times 10^{-30})(1.0 \times 10^6) = 6.0 \times 10^{-24} \text{ J.} \quad (2.159)$$

The thermal energy scale is

$$k_B T = (1.38 \times 10^{-23})(300) = 4.14 \times 10^{-21} \text{ J.} \quad (2.160)$$

Thus

$$\frac{pE}{k_B T} \approx 1.4 \times 10^{-3}. \quad (2.161)$$

The electric field gives a preferred orientation, but thermal agitation is much larger in this estimate. Strong alignment would require stronger fields, larger dipoles, lower temperature or cooperative material effects.

For those numerical parameters, Eq. (2.157) predicts $\langle \cos \theta \rangle \simeq 4.8 \times 10^{-4}$. Each molecule can have a substantial permanent dipole while the ensemble has only a weak preferred orientation. This derivation assumes equilibrium, negligible dipole-dipole interactions and a classical rotor. Dense liquids, collective order and quantum rotational spectra can invalidate those assumptions; the model should not be used as an unqualified description of every dielectric.

2.15.5 A nonuniform rod: density, components and complete integration

Uniform density permits symmetry arguments that a varying density may destroy. Consider a rod from $x = 0$ to L with $\lambda(x) = \lambda_0 x/L$, where $\lambda_0 > 0$. At the point $(0, y)$ with $y > 0$,

$$\mathbf{R} = -x\hat{\mathbf{i}} + y\hat{\mathbf{j}}, \quad dq = \frac{\lambda_0 x}{L} dx. \quad (2.162)$$

Every element contributes leftward and upward. The geometry fixes the signs before any integration:

$$E_x = -\frac{k_e \lambda_0}{L} \int_0^L \frac{x^2 dx}{(x^2 + y^2)^{3/2}}, \quad (2.163)$$

$$E_y = \frac{k_e \lambda_0 y}{L} \int_0^L \frac{x dx}{(x^2 + y^2)^{3/2}}. \quad (2.164)$$

For the first integral write $x^2 = (x^2 + y^2) - y^2$; for the second use $u = x^2 + y^2$. The antiderivatives are

$$\int \frac{x^2 dx}{(x^2 + y^2)^{3/2}} = \text{arsinh}(x/y) - \frac{x}{\sqrt{x^2 + y^2}}, \quad (2.165)$$

$$\int \frac{x dx}{(x^2 + y^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + y^2}}. \quad (2.166)$$

Thus the complete field is

$$E_x = -\frac{k_e \lambda_0}{L} \left[\text{arsinh}(L/y) - \frac{L}{\sqrt{L^2 + y^2}} \right], \quad (2.167)$$

$$E_y = \frac{k_e \lambda_0}{L} \left[1 - \frac{y}{\sqrt{L^2 + y^2}} \right]. \quad (2.168)$$

The inverse hyperbolic function may be written as $\operatorname{arsinh}(L/y) = \ln[(L + \sqrt{L^2 + y^2})/y]$, with a dimensionless logarithm argument. The bracket in E_x is positive, so the signs agree with the geometric prediction.

Example 2.32: Far-field audit of the nonuniform rod

Use the source moments to check the leading terms for $y \gg L$.

Solution. The total charge and dipole moment about the origin are

$$Q = \int_0^L \frac{\lambda_0 x}{L} dx = \frac{\lambda_0 L}{2}, \quad \mathbf{p} = \int_0^L x \frac{\lambda_0 x}{L} dx \hat{\mathbf{i}} = \frac{\lambda_0 L^2}{3} \hat{\mathbf{i}}. \quad (2.169)$$

With $\hat{\mathbf{n}} = \hat{\mathbf{j}}$, Eq. (2.134) predicts

$$E_y \simeq \frac{k_e \lambda_0 L}{2y^2}, \quad E_x \simeq -\frac{k_e \lambda_0 L^2}{3y^3}. \quad (2.170)$$

Expanding the exact expressions gives precisely these terms. The weaker horizontal component records that the charge lies to the right of the observation axis. Replacing the rod by a point charge at the origin retains only the leading vertical field and loses that directional information.

2.15.6 Finite plates: a calculable model of field uniformity

The infinite-sheet approximation predicts a uniform field, but a finite source must ultimately produce a decaying field. A pair of prescribed uniform disks provides an exact on-axis model that connects these regimes without solving a conducting-electrode boundary problem.

Let two coaxial disks of radius R lie in the planes $z = -g/2$ and $z = +g/2$. The lower disk carries surface density $+\sigma$ and the upper disk $-\sigma$, so the field between them points upward. Inside the gap, define positive distances $d_+ = z + g/2$ from the positive disk and $d_- = g/2 - z$ from the negative disk. Superposition of the two single-disk fields gives

$$E_z(z) = \frac{\sigma}{2\epsilon_0} \left[2 - \frac{z + g/2}{\sqrt{R^2 + (z + g/2)^2}} - \frac{g/2 - z}{\sqrt{R^2 + (g/2 - z)^2}} \right], \quad |z| < g/2. \quad (2.171)$$

Both contributions point in $+z$. Subtracting their magnitudes because their charges have opposite signs would be a direction error. The total source is neutral; its far field outside the apparatus is dipolar, while the interior can be almost uniform.

At the midplane,

$$E_z(0) = \frac{\sigma}{\epsilon_0} \left[1 - \frac{g/2}{\sqrt{R^2 + (g/2)^2}} \right]. \quad (2.172)$$

The infinite-plane value is $E_\infty = \sigma/\epsilon_0$. The fractional centre-field deficit is

$$\frac{E_\infty - E_z(0)}{E_\infty} = \frac{g/2}{\sqrt{R^2 + (g/2)^2}} \simeq \frac{g}{2R} \quad (g \ll R). \quad (2.173)$$

This quantifies one finite-size correction. It does not yet measure how much the field varies *within* the gap.

Write $h = g/2$ and $f(s) = s/\sqrt{R^2 + s^2}$. Since the bracket in Eq. (2.171) contains $f(h+z) + f(h-z)$, the linear term in z cancels. With

$$f''(s) = -\frac{3R^2 s}{(R^2 + s^2)^{5/2}}, \quad (2.174)$$

the expansion about the midplane is

$$E_z(z) = E_z(0) + \frac{3\sigma R^2 h}{2\epsilon_0(R^2 + h^2)^{5/2}} z^2 + \mathcal{O}(z^4). \quad (2.175)$$

Thus in this prescribed-source model the centre is a shallow axial minimum. When g/R is small, the absolute reduction from the infinite-plane value can be much larger than the variation across the central part of the gap. “Uniform” and “equal to the infinite-plane value” are different accuracy questions.

Figure 2.18 shows this distinction in two representative aspect ratios. Neither curve describes the edge region away from the axis.

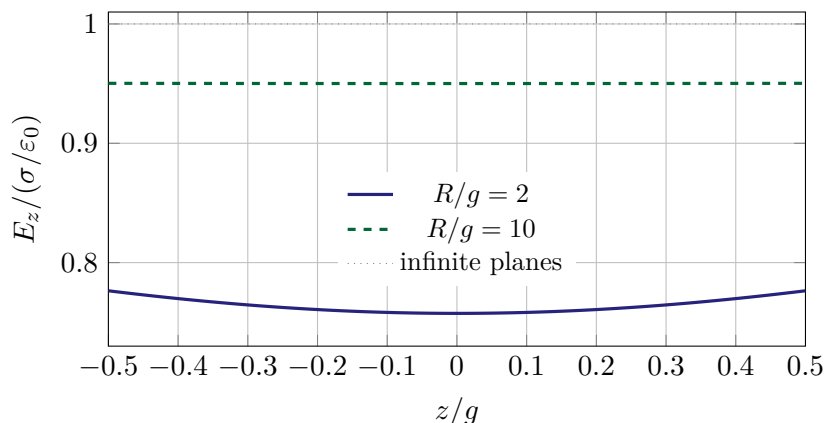


Figure 2.18: Exact on-axis field between opposite uniform disks. The endpoints show limits taken from inside the gap; ideal charged surfaces have discontinuous normal field. Increasing R/g reduces the finite-size deficit and flattens the profile.

Example 2.33: A uniform field can still have the wrong assumed magnitude

Use the model with $R = 0.10$ m and $g = 0.010$ m. Compare the centre field with the infinite-plane value and with the axial field immediately inside either disk.

Solution. At the centre,

$$\frac{E_z(0)}{E_\infty} = 1 - \frac{0.005}{\sqrt{0.10^2 + 0.005^2}} = 0.95006. \quad (2.176)$$

At $z \rightarrow g/2^-$, the upper-disk contribution has its limiting surface value and the lower disk is a distance g away:

$$\frac{E_z(g/2^-)}{E_\infty} = 1 - \frac{1}{2} \frac{g}{\sqrt{R^2 + g^2}} = 0.95025. \quad (2.177)$$

The centre is about 5.0% below the infinite-plane prediction, while the on-axis change across a half-gap is only 0.020% of the centre field. A nearly uniform field need not have the magnitude inferred from an inaccurate source idealization.

Model boundary. These are fixed uniform charge sheets, not finite metallic electrodes held at a specified voltage. Conductors redistribute surface charge. The two-disk model isolates a geometric effect and gives an analytic check; it cannot replace the correct boundary-value calculation for a particular device.

What does a beam measure in a varying field?

The transverse-deflection method also has a useful generalization. Suppose a beam samples a weakly deflecting field $E_y(x)$ over $0 < x < L$, with $v_x \simeq v_0$ and initial $v_y = 0$. Evaluate the field along the unperturbed path; this is a first-order, small-deflection approximation for a general electrostatic geometry. Then

$$\frac{dv_y}{dx} = \frac{qE_y(x)}{mv_0}, \quad v_{y,L} = \frac{q}{mv_0} \int_0^L E_y(x) dx. \quad (2.178)$$

Integrating the transverse displacement a second time gives

$$y_L = \frac{q}{mv_0^2} \int_0^L (L - x) E_y(x) dx, \quad (2.179)$$

and after a field-free drift D ,

$$y_{\text{screen}} = \frac{q}{mv_0^2} \int_0^L (L + D - x) E_y(x) dx. \quad (2.180)$$

An early kick has more flight time to produce displacement than a late kick of the same strength. Exit angle measures an unweighted field integral, whereas screen position measures a spatially weighted integral. They cannot generally be replaced by one unspecified “average field.”

For the illustrative on-path profile $E_y(x) = E_0(1 + \eta x/L)$, integration yields

$$y_{\text{screen}} = \frac{qE_0}{mv_0^2} \left[LD + \frac{L^2}{2} + \eta \left(\frac{LD}{2} + \frac{L^2}{6} \right) \right]. \quad (2.181)$$

A constant-field inference would return

$$E_{\text{inferred}} = E_0 \left[1 + \eta \frac{D/2 + L/6}{D + L/2} \right]. \quad (2.182)$$

For $D = 0$ the gradient enters with factor $1/3$, whereas for a long drift it approaches $1/2$. These weights help interpret fringe-field corrections and explain why two instruments sampling the same nonuniform field can report different effective values.

2.15.7 Numerical quadrature as a controlled source approximation

A numerical source sum is an approximation with a resolution scale. For a straight rod with prescribed $\lambda(x)$, divide its length into N panels of width $h = L/N$. Put a representative charge $\Delta q_j = \lambda(x_j)h$ at each midpoint x_j . Then

$$\mathbf{E}_N(\mathbf{r}) = k_e \sum_{j=1}^N \Delta q_j \frac{\mathbf{r} - x_j \hat{\mathbf{i}}}{|\mathbf{r} - x_j \hat{\mathbf{i}}|^3}. \quad (2.183)$$

This is the midpoint quadrature rule applied to the vector integral. For a smooth integrand at a fixed observation point away from the source, its error is generally proportional to h^2 . If h is not small compared with the observation distance, nearby panels can dominate and apparent convergence may be poor.

Example 2.34: A convergence table for the finite rod

For a rod extending from $-a$ to a , evaluate the bisector field at $y = a$ with midpoint quadrature. Normalize by the exact field $E_{\text{exact}} = \sqrt{2} k_e \lambda / a$. The dimensionless sum is

$$\frac{E_N}{k_e \lambda / a} = \frac{2}{N} \sum_{j=1}^N \frac{1}{[1 + u_j^2]^{3/2}}, \quad u_j = -1 + \left(j - \frac{1}{2}\right) \frac{2}{N}. \quad (2.184)$$

N	E_N/E_{exact}	Relative error (%)
4	1.00768040	0.76804
8	1.00194841	0.19484
16	1.00048800	0.04880
32	1.00012205	0.01221
64	1.00003052	0.00305

The entries are calculated from the displayed sum. After reaching the smooth-error regime, doubling N reduces the error by approximately four. The same observation point and total charge must be used at every resolution; changing either one is not a convergence test.

Without an exact solution, compare three resolutions. If $E_N = E + C/N^2 + \dots$, then

$$E \simeq \frac{4E_{2N} - E_N}{3}, \quad |E_{2N} - E| \simeq \frac{|E_{2N} - E_N|}{3}. \quad (2.185)$$

Apply this component by component, and verify that the difference ratio is near four. Near a component zero, an absolute error or a physically chosen scale is more informative than division by that nearly zero component.

Singular points, masking and softened sources

At the position of an ideal point charge the field diverges. A program should mask that point and clearly mark any excluded neighbourhood. Clipping long arrows for readability changes only the drawing, whereas replacing the field law changes the source model.

For example, the softened kernel

$$\mathbf{E}_s(\mathbf{R}) = k_e q \frac{\mathbf{R}}{(R^2 + a_s^2)^{3/2}} \quad (2.186)$$

is finite and zero at its centre. It represents a spread-out charge model, not the exact point-charge field. Away from the origin its magnitude relative to the point field is

$$\frac{E_s}{E_{\text{point}}} = \left(1 + \frac{a_s^2}{R^2}\right)^{-3/2} = 1 - \frac{3}{2} \frac{a_s^2}{R^2} + \dots \quad (2.187)$$

For a 1% relative tolerance, this already suggests $R \gtrsim 12a_s$. Decreasing grid spacing at fixed a_s converges to the softened model, not to a point charge. Discretization error and model error must be checked separately.

2.15.8 Tracing field lines and validating field maps

A regular field line satisfies

$$\frac{d\mathbf{r}}{ds} = \frac{\mathbf{E}(\mathbf{r})}{|\mathbf{E}(\mathbf{r})|}, \quad |\mathbf{E}| \neq 0, \quad (2.188)$$

where s is arc length in the exact differential equation. A normalized tracer spends comparable steps on equal path lengths, instead of taking enormous steps near a charge. It contains no particle mass and has no time variable, so it cannot be a particle trajectory integrator.

The explicit Euler update is

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h \frac{\mathbf{E}(\mathbf{r}_n)}{|\mathbf{E}(\mathbf{r}_n)|}. \quad (2.189)$$

Its straight steps can overshoot a curved line. A second-order midpoint step first forms $\mathbf{u}_n = \mathbf{E}(\mathbf{r}_n)/|\mathbf{E}(\mathbf{r}_n)|$, evaluates the unit direction at $\mathbf{r}_n + h\mathbf{u}_n/2$, and uses that midpoint direction for the full step. A step-halving comparison tests whether visible curves have stabilized. Higher-order adaptive methods are useful when the field changes rapidly.

Example 2.35: One step of a field-line tracing algorithm

At a grid point, a computed field is $\mathbf{E} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}}) \text{ N C}^{-1}$. A field line is traced with step size $h = 0.010 \text{ m}$. What is the next step direction and displacement?

Solution. The field magnitude is $|\mathbf{E}| = 5 \text{ N C}^{-1}$. The unit tangent direction is

$$\frac{\mathbf{E}}{|\mathbf{E}|} = 0.6\hat{\mathbf{i}} + 0.8\hat{\mathbf{j}}. \quad (2.190)$$

Thus the displacement for one step is

$$\Delta\mathbf{r} = h \frac{\mathbf{E}}{|\mathbf{E}|} = (0.006\hat{\mathbf{i}} + 0.008\hat{\mathbf{j}}) \text{ m}. \quad (2.191)$$

This step follows the field direction but does not encode field magnitude in the distance moved because the direction field has been normalized.

Three stopping conditions are essential: entry into an excluded source neighbourhood, exit from the plotting domain, and approach to a null where normalization becomes ill-conditioned. A small absolute threshold must be chosen relative to a declared field scale. It must not silently classify every weak distant field as a physical zero. Seeding more curves in one region changes their visual density even though the underlying field is unchanged.

Example 2.36: Code validation using a symmetric pair

A program computes the field of two equal positive charges located at $x = \pm a$. What analytic checks should be applied before trusting its plots?

Solution. Several checks are available.

1. At the origin, the net field must be zero by symmetry.
2. On the y axis, the x component must be zero and the y component must be $2k_e qy/(a^2 + y^2)^{3/2}$.
3. Far from the pair, the field should approach that of a point charge $2q$ at the origin.
4. The field magnitude should become large near either charge, unless the code deliberately regularizes the singularity.
5. Reflecting the plot across the y axis should not change the field pattern.

A code that fails any of these checks is not ready for physical interpretation.

Under reflection across the y axis in that example, the *components* obey $E_x(-x, y) = -E_x(x, y)$ and $E_y(-x, y) = E_y(x, y)$. The complete vector pattern is symmetric; the x component is not an even function. Similar component-aware checks are much stronger than judging whether a colour map “looks symmetric.”

Method

A useful field atlas contains a point charge, equal like charges, a neutral dipole and a source with vanishing net charge and dipole moment. For each source, report coordinates and units, an unnormalized magnitude map, a direction plot, the source exclusion radius, at least one analytic comparison, and a resolution study. If displaying a logarithm, plot $\log_{10}(|\mathbf{E}|/E_{\text{ref}})$ for a specified reference field; mask true zeros separately. A logarithm requires a dimensionless argument.

2.15.9 A reproducible field calculation with an analytic benchmark

The following compact Python example answers a specific question: how finely must a finite rod be divided before its calculated bisector field agrees with the exact result? It uses arrays for source coordinates, returns the physical vector before any plotting normalization, and refuses an observation point inside an excluded source neighbourhood. The source positions are three-dimensional even when the geometry happens to lie in a plane.

Method

Keep the physical field routine separate from plotting. Store lengths in metres and charge in coulombs. A display routine may normalize vectors or choose logarithmic colours, but those transformations must never change the values used for force or trajectory calculations.

```
import numpy as np

KE = 8.99e9                      # N m^2 / C^2

def electric_field(point, positions, charges, rmin=1e-8):
    # point: (3,), positions: (N, 3), charges: (N,)
    # rmin is a masking radius in metres, not softening.
    point = np.asarray(point, dtype=float)
    positions = np.asarray(positions, dtype=float)
    charges = np.asarray(charges, dtype=float)
    displacement = point - positions
    distance2 = np.sum(displacement**2, axis=1)
    if np.any(distance2 <= rmin**2):
        raise ValueError("Observation point is too near a source")
    terms = charges[:, None] * displacement
    terms /= distance2[:, None]**1.5
    return KE * np.sum(terms, axis=0)

a = 0.30                        # rod half-length, m
lam = 3.0e-8                    # C/m
y = 0.20                        # observation height, m
point = np.array([0.0, y, 0.0])
exact_y = 2 * KE * lam * a / (y * np.sqrt(a*a + y*y))

for count in (8, 16, 32, 64, 128):
    width = 2*a / count
    x = -a + (np.arange(count) + 0.5)*width
    positions = np.column_stack((x, 0*x, 0*x))
    charges = np.full(count, lam*width)
    calculated = electric_field(point, positions, charges)
    relative_error = abs(calculated[1] / exact_y - 1)
    print(count, calculated[1], relative_error)
    # Geometrical cancellation is checked independently.
    assert abs(calculated[0]) < 1e-10 * abs(exact_y)
```

```

assert abs(calculated[2]) < 1e-10 * abs(exact_y)

def one_step_midpoint(point, step, positions, charges):
    # step is a geometrical path length in metres.
    def direction(p):
        field = electric_field(p, positions, charges)
        norm = np.linalg.norm(field)
        if norm < 1e-10:      # N/C, chosen for this example scale
            raise ValueError("Direction unresolved near a null")
        return field / norm
    midpoint = point + 0.5*step*direction(point)
    return point + step*direction(midpoint)

```

The rod benchmark is the same physical geometry as the core finite-rod example. Representative output, rounded for presentation, is:

Panels	E_y (N/C)	Relative error (%)
8	2247.729	0.16442
16	2244.971	0.04148
32	2244.273	0.01039
64	2244.098	0.00260
128	2244.054	0.00065

The exact field is 2244.0 N/C to the displayed precision. Both nonvertical components cancel to rounding precision in a symmetric discretization. The normalization used inside the optional midpoint routine affects only the tracing direction, never the physical field returned by `electric_field`.

To turn the step routine into a complete tracer, add a domain boundary, a maximum path length and a source-approach stopping rule. A finite step can cross an excluded region even if its endpoint lies outside, so monitor the step segment or reduce the step when approaching a source. The fixed null threshold shown is suitable only as a stated numerical choice for this benchmark's units and scale; a general application needs an appropriate absolute and relative criterion.

The core dipole field-line figure was obtained by integrating the same normalized Coulomb direction field with an adaptive numerical method. The displayed curves are embedded in this chapter's TikZ source, so compiling the chapter does not require Python. The code here is an independent reproducibility exercise, not a hidden compilation dependency.

Avoiding cancellation in a distant disk field

A mathematically exact expression can be numerically awkward. In the disk field at $z \gg R$, direct evaluation of $1 - z/\sqrt{z^2 + R^2}$ subtracts two numbers very close to one. Their small difference can lose significant digits. Rationalization gives the identical expression

$$1 - \frac{z}{\sqrt{z^2 + R^2}} = \frac{R^2}{\sqrt{z^2 + R^2} [\sqrt{z^2 + R^2} + z]}, \quad z > 0. \quad (2.192)$$

The right-hand side avoids that subtraction and makes the far-field scaling $R^2/(2z^2)$ explicit. This is numerical stability, distinct from physical model accuracy and quadrature convergence. A smaller grid spacing cannot fix cancellation in a badly evaluated analytic formula.

A similar stable axial dipole expression is $4k_e q a x / (x^2 - a^2)^2$, rather than subtracting $k_e q / (x - a)^2$ and $k_e q / (x + a)^2$ at extremely large x/a . In either case, the algebraic reformulation preserves the original model; it does not introduce softening or alter the source.

2.15.10 Inferring a source from field data

For known source positions, Coulomb superposition is linear in the unknown charge values. At measurement point j , one measured component can be written

$$E_{x,j} = \sum_i A_{ji} q_i, \quad A_{ji} = k_e \frac{x_j - x_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}. \quad (2.193)$$

The matrix A is fixed by geometry. More independent measurements can constrain more unknown charges, but measurements at nearly equivalent points may add little information. Noise, uncertain source locations and probe disturbance affect the inferred charges.

Example 2.37: What can two perpendicular-bisector measurements determine?

Unknown charges q_L and q_R lie at $x = -a$ and $x = +a$. At $P = (0, y)$ with $a, y > 0$, suppose both field components are measured. Determine the two charges.

Solution. With $R = \sqrt{a^2 + y^2}$, superposition gives

$$E_x = \frac{k_e a}{R^3} (q_L - q_R), \quad E_y = \frac{k_e y}{R^3} (q_L + q_R). \quad (2.194)$$

Adding and subtracting these combinations yields

$$q_L = \frac{R^3}{2k_e} \left(\frac{E_y}{y} + \frac{E_x}{a} \right), \quad (2.195)$$

$$q_R = \frac{R^3}{2k_e} \left(\frac{E_y}{y} - \frac{E_x}{a} \right). \quad (2.196)$$

For $a = 0.050$ m, $y = 0.10$ m, $E_x = 965$ N/C and $E_y = 643$ N/C, the inferred charges are approximately $q_L = 2.0$ nC and $q_R = -1.0$ nC.

Identifiability. Measuring only E_y determines the sum, not the two charges separately. At $y = 0$, the vertical component vanishes for every pair of charge values and cannot determine their sum. Far away, the difference signal in E_x decreases as y^{-3} , while the sum signal decreases as y^{-2} . A distant measurement can therefore constrain net charge more readily than internal charge separation. This is the measurement counterpart of the multipole hierarchy.

The inverse problem also clarifies the limits of simulation. Summing fields of *specified* charges is a forward calculation. Finding unknown surface charge on electrodes held at given voltages is a boundary-value problem. It needs additional constraints and the potential formalism of Chapter 4; merely assigning arbitrary charges to mesh panels does not solve it.

2.16 Practice problems

The following 45 problems progress from core concepts to analytical, computational, measurement and enrichment tasks. Use $k_e = 8.99 \times 10^9 \text{ N m}^2 \text{ C}^{-2}$, $e = 1.602 \times 10^{-19} \text{ C}$, $m_e = 9.11 \times 10^{-31} \text{ kg}$ and $m_p = 1.67 \times 10^{-27} \text{ kg}$ where needed. Unless stated otherwise, sources are fixed in vacuum and responding particles do not alter their distribution. In every quantitative answer give a direction or signed components as well as units. Selected detailed solutions follow the complete set.

Conceptual reasoning

- C1.** A positive and a negative probe are placed one at a time at the same empty point. Explain which of field, force and acceleration remain the same, assuming both probes leave the source unchanged. What extra information is needed to compare accelerations?
- C2.** An electron experiences an upward force in a device. Determine the field direction. Does the field itself depend on the electron mass?
- C3.** A field vanishes at the midpoint between equal positive charges. Explain why this implies neither absence of sources nor stable three-dimensional equilibrium for a charged probe.
- C4.** A sensor gives different values of force per charge when its charge is increased. Give two plausible physical causes and distinguish them from a numerical error in calculating F/q .
- C5.** A normalized arrow plot has equal-length arrows everywhere. Can it show a nonuniform field? Can it distinguish a null from a weak nonzero field without an additional plotting rule?
- C6.** A student draws field lines for $+2q$ and $-q$ with six lines leaving the positive charge and six entering the negative charge, and two regular curves crossing at a point where $\mathbf{E} \neq 0$. Diagnose both errors if line count is intended to be proportional to charge magnitude.
- C7.** A neutral dipole in a uniform field has no net force. Explain why it can nevertheless change orientation, and why it need not remain stationary or settle into alignment.
- C8.** Compare the meaning of $\mathbf{E} = 0$ at the centre of a uniform ring with the nonzero centre field of an equal-and-opposite charge pair. Identify the cancelling or adding components.
- C9.** Why is a uniformly charged insulating disk a different model from a finite conducting disk in electrostatic equilibrium? Which part of the field calculation is still valid for both?
- C10.** In the ring and disk far-field limits, what is held fixed? Explain why a disk of fixed total charge spread over an increasingly large area does not have a nonzero fixed sheet field.

Analytical and geometrical problems

- A1.** A source -8.0 nC is at $(0.10, -0.20) \text{ m}$. Find the field at $(0.40, 0.20) \text{ m}$. Then find the force on a $+2.0 \text{ nC}$ test charge there.
- A2.** Two equal charges $+q$ are at $x = \pm a$. Derive $\mathbf{E}(0, y)$ for positive and negative y . What changes if both source signs reverse?
- A3.** A charge $+9q$ is at $x = 0$ and $+q$ is at $x = L$, with $q, L > 0$. Find all finite zero-field points on the x axis. Explain why a root obtained by squaring an equation must still satisfy the direction conditions.
- A4.** A charge $+q$ is at $x = 0$ and $-4q$ at $x = L$. Find all finite zero-field points on the axis, considering the three intervals separately.

- A5.** Charges $+q, +q, -q$ occupy $(0, 0)$, $(a, 0)$ and $(0, a)$, respectively. Find the field at (a, a) and its angle relative to the positive x axis.
- A6.** An equilateral triangle of side a carries $+2q$ at the top and $-q$ at each lower vertex. Derive the field at the centroid using explicit coordinates. Explain why using 30° rather than 60° relative to the downward vertical gives an incorrect result.
- A7.** A dipole has $-q$ at $-a$ and $+q$ at a . Derive its exact field on the perpendicular bisector, including at the centre, and compare the direction with $\mathbf{p}$.
- A8.** For the same dipole, determine the minimum exterior axial distance x/a for the far-field approximation to have error below 1% relative to the exact field. Give the condition also in units of the separation $d = 2a$.
- A9.** Differentiate the axial ring field to find the positive- z maximum and its value. Explain the behaviour for $z < 0$ without repeating the derivative.
- A10.** A uniform positive rod extends from $-L$ to 0 and carries charge Q . Find the field at $x = b > 0$. Compare it with a point charge at the rod centre in the limit $b \gg L$.
- A11.** A positive arc of radius R and total charge Q spans $-\alpha \leq \theta \leq \alpha$ about the positive x axis, with $0 < \alpha \leq \pi$. Derive the centre field. Check the limits $\alpha \rightarrow 0$ at fixed Q and $\alpha = \pi$.
- A12.** A disk of radius R has uniform positive density σ . Derive its axial field by integrating annuli. Obtain the field below the plane and the two limits as $z \rightarrow 0^\pm$.
- A13.** A rigid dipole of moment p rotates from 120° to 30° in a uniform field of magnitude E . Find ΔU and the work by the field. What is the external work for a quasistatic rotation? What additional term is needed if rotational kinetic energy changes?

Numerical and computational problems

- N1.** Write a component-based field-sum algorithm with inputs (q_i, x_i, y_i) and output (E_x, E_y) . Include exclusion of source neighbourhoods. State the units of every input and give a sign test using one negative charge.
- N2.** Charges $+1.0 \text{ nC}$ and -1.0 nC are at $(0, 0)$ and $(0.10, 0) \text{ m}$. Compute the field at $(0.05, 0.05) \text{ m}$ and $(0.05, 0.20) \text{ m}$. Explain why the direction is the same while the ratio of magnitudes is not exactly 4^3 .
- N3.** Plot the normalized ring field for $-4 \leq z/a \leq 4$. Mark its zero and extrema. Compare the exact field with the point-charge model at $z/a = 2, 5, 10, 20$, defining the relative error explicitly.
- N4.** Approximate a uniform rod by $N = 8, 16, 32, 64$ midpoint charges. At $y = a$ and then at $y = 0.05a$, compare with the exact bisector field. Explain why the near-source calculation requires more panels.
- N5.** For the softened point-charge kernel in the enrichment section, derive an exact condition on R/a_s that keeps its magnitude error below 1% relative to the true point field. Explain why arrow normalization cannot detect this error for a single source.
- N6.** Trace one dipole field line with Euler and midpoint methods at two step sizes. Give a quantitative way to compare curves, state stopping conditions, and explain why step count is not physical elapsed time.
- N7.** Two calculations give visually identical normalized dipole maps, but one uses charges a thousand times larger. Propose checks that distinguish them. Include a magnitude comparison and a unit

conversion check.

Engineering, laboratory and data problems

- E1.** Compare the accelerations of a proton, an electron and an alpha particle in a field of 5.0×10^4 N/C. Use alpha charge $2e$ and mass $4m_p$. Specify directions and check the approximation of neglecting gravity.
- E2.** An electron enters horizontally at 1.5×10^7 m/s. It traverses plates of length $L = 0.040$ m and gap $g = 0.010$ m, then drifts $D = 0.10$ m to a screen. Its upward screen displacement is 0.012 m. Infer the field magnitude and direction, the plate-exit deflection and the exit angle. Check plate clearance and nonrelativistic motion.
- E3.** A bead of mass 2.0 mg and charge -3.0 nC hangs at 8.0° to the left of vertical. Find the horizontal electric field. Draw the force directions before assigning signs.
- E4.** A horizontal-field bead measurement uses $E = mg \tan \beta / q$ with $m = 1.00$ mg, $q = 2.00$ nC and $\beta = 12.0^\circ$. Its independent relative mass and charge uncertainties are 1% and 5%, and $u_\beta = 0.5^\circ$. If only one uncertainty could be halved, which change would most reduce u_E/E ? Quantify all three options.
- E5.** A charged dust particle obeys $m\dot{v} = qE - bv$ and starts from rest. Integrate its velocity to obtain the transverse displacement $y(t)$. Identify conditions under which $y \simeq qEt/b$ is accurate and compare the early-time limit with constant acceleration.
- E6.** A circular uniform disk is used as a model of a broad charged surface. At $z = 3.0$ mm, find the minimum radius for the sheet approximation to differ by less than 2% of the sheet field. Explain why the answer does not directly establish a tolerance for an arbitrary conducting plate.
- E7.** A finite probe is modelled as the average of the undisturbed field component along a segment of length ℓ , centred at x_0 . Taylor-expand this average through order ℓ^2 . Does a linear field gradient cause a bias for a symmetric probe? Distinguish averaging from physical disturbance of the source.

Advanced challenges

- H1.** Charges $+q$ are at $x = \pm a$ and $-2q$ is at the origin. Show that net charge and dipole moment vanish. Expand the exact field for $x > a$ to obtain its leading far-field term.
- H2.** For two positive sources $+q$ at $x = \pm a$, linearize the field in both the axial and transverse directions near the origin. Determine the stability of positive and negative test charges, with and without a line constraint.
- H3.** Complete the nonuniform-rod calculation for $\lambda(x) = \lambda_0(1 + x/L)$ on $0 \leq x \leq L$, at $(0, y)$ with $y > 0$. Use the uniform and linearly increasing results by superposition, and check the far field against $Q = 3\lambda_0 L/2$.
- H4.** For a permanent dipole aligned along a varying external field on a symmetry axis, expand the exact two-charge force through the first correction beyond pE'_x . State how the approximation changes when the leading gradient vanishes.
- H5.** Starting from $U = p^2/(2\alpha) - pE$, minimize with respect to p and derive the force on an isotropic linearly polarizable particle. Explain the factor-of-two error obtained by differentiating only $-pE$ after setting $p = \alpha E$.

- H6.** Charges q_L, q_R lie at $x = \mp a$. At $(0, y)$ both field components have independent standard uncertainties u_x, u_y , while positions are known exactly. Derive the variances and covariance of the inferred charges. Explain what happens to the reconstruction as $y \rightarrow 0$.
- H7.** Derive the weak-field mean orientation $\langle \cos \theta \rangle \simeq pE/(3k_B T)$ using a first-order expansion of the Boltzmann factor and the three-dimensional solid-angle measure.
- H8.** A negative particle is constrained to the axis of a positive ring. Derive its small-amplitude angular frequency. Then remove the constraint and use the in-plane linear field to decide whether the particle is trapped in three dimensions.

2.17 Selected detailed solutions

The thirteen solutions below emphasize different reasoning methods. They do not replace the modelling and direction checks requested in the remaining problems.

Example 2.38: Solution C6: auditing a field-line diagram

At a regular point with nonzero field, the field has a single direction, so two distinct field-line tangents cannot cross. A numerical or hand-drawn crossing there is an error, not a physical feature.

If six lines represent $2q$, three lines represent q . Thus three of the six lines leaving $+2q$ should end on $-q$, while three should continue to infinity. Alternatively, retain six lines entering $-q$ and draw twelve leaving $+2q$. The absolute number is a drawing convention; the ratio and net outward excess express the signed source charges. A flat drawing is not a literal measurement of three-dimensional field magnitude.

Example 2.39: Solution A4: cancellation for opposite unequal charges

For $0 < x < L$, both fields point right, so there is no root. For $x > L$, the negative source is four times larger and closer, so its leftward contribution exceeds the positive source's rightward contribution. A root can exist only for $x < 0$.

Write $x = -s$ with $s > 0$. The positive source gives a leftward field of magnitude $k_e q/s^2$, whereas the negative source gives a rightward field $4k_e q/(L+s)^2$. Equality requires

$$\frac{1}{s^2} = \frac{4}{(L+s)^2} \implies L+s = 2s \implies x = -L. \quad (2.197)$$

At this point the distances are L and $2L$, so the factor of four in charge is cancelled by the factor of four in squared distance. The direction analysis is essential: equating unsigned magnitudes in the middle interval could produce a spurious answer.

Example 2.40: Solution A6: the equilateral-triangle geometry

Place the centroid at the origin. The vertices are $(0, a/\sqrt{3})$ and $(\pm a/2, -a/(2\sqrt{3}))$. Every vertex is at distance $r = a/\sqrt{3}$.

The top charge contributes $-6k_e q \hat{\mathbf{j}}/a^2$. Each bottom negative charge produces a field toward its own vertex, with magnitude $3k_e q/a^2$. Its vertical unit-vector component is

$$\frac{-a/(2\sqrt{3})}{a/\sqrt{3}} = -\frac{1}{2}. \quad (2.198)$$

The bottom horizontal components cancel and their vertical sum is $-3k_e q \hat{\mathbf{j}}/a^2$. Therefore

$$\mathbf{E} = -\frac{9k_e q}{a^2} \hat{\mathbf{j}}. \quad (2.199)$$

The bottom vectors are 60° from the downward vertical, not 30° . The latter is their angle from the horizontal. The field has the required inverse-square dimension and points toward the negative side of the source.

Example 2.41: Solution A8: an accuracy-based dipole distance

Set $u = a^2/x^2$, where $0 < u < 1$ on the exterior positive axis. The fractional underestimate is $2u - u^2$. The 1% condition is

$$2u - u^2 < 0.01 \quad \Longleftrightarrow \quad (1 - u)^2 > 0.99. \quad (2.200)$$

Since $1 - u > 0$, this gives $u < 1 - \sqrt{0.99}$ and

$$\frac{x}{a} > \frac{1}{\sqrt{1 - \sqrt{0.99}}} \simeq 14.1, \quad \frac{x}{d} > 7.06. \quad (2.201)$$

The small- u estimate $2u < 0.01$ gives $x/a > \sqrt{200} = 14.14$, already close. A separation-based statement must include the factor $d = 2a$; confusing half-separation and separation changes the estimated error by a factor of four at leading order.

Example 2.42: Solution A10: rod field and the correct observation distance

For the rod from $-L$ to 0 , $\lambda = Q/L$ and every contribution at $b > 0$ points right. Hence

$$E_x = k_e \lambda \int_{-L}^0 \frac{dx}{(b-x)^2} = k_e \lambda \left[\frac{1}{b-x} \right]_{-L}^0 \quad (2.202)$$

$$= k_e \lambda \left(\frac{1}{b} - \frac{1}{b+L} \right) = \frac{k_e Q}{b(b+L)}. \quad (2.203)$$

The distance from the rod centre is $r_c = b + L/2$, so the exact expression is

$$E_x = \frac{k_e Q}{r_c^2 - (L/2)^2} = \frac{k_e Q}{r_c^2} \left[1 + \frac{L^2}{4r_c^2} + \dots \right]. \quad (2.204)$$

The point-charge approximation at the centre underestimates the exact axial field. There is no first-order correction in L/r_c because the centred source is symmetric. Writing a point charge at an endpoint instead introduces a coordinate-dependent first-order distance correction.

Example 2.43: Solution A13: rotation, work and energy bookkeeping

Using $U = -pE \cos \theta$,

$$\Delta U = -pE \cos 30^\circ + pE \cos 120^\circ = -\frac{1+\sqrt{3}}{2} pE. \quad (2.205)$$

The field does positive work

$$W_{\text{field}} = -\Delta U = \frac{1+\sqrt{3}}{2} pE. \quad (2.206)$$

If an external agent rotates the dipole quasistatically, the change in kinetic energy is negligible and $W_{\text{ext}} = \Delta U < 0$: the agent removes energy while restraining the field-driven motion. For arbitrary motion, the work–energy theorem instead gives

$$W_{\text{ext}} + W_{\text{field}} = \Delta K_{\text{rot}}, \quad W_{\text{ext}} = \Delta K_{\text{rot}} + \Delta U. \quad (2.207)$$

The quasistatic statement is a special case, not a general identity for an accelerating rotor.

Example 2.44: Solution N2: two samples of a dipole field

Let $a = 0.050$ m be half the separation and measure y above the midpoint. The positive source lies to the left and the negative source to the right, so vertical components cancel and horizontal components add rightward:

$$E_x = \frac{2k_e qa}{(a^2 + y^2)^{3/2}}, \quad E_y = 0. \quad (2.208)$$

At $y = 0.050$ m, the result is $E_x = 2.54 \times 10^3$ N/C. At $y = 0.20$ m, it is 1.03×10^2 N/C. Their ratio is

$$\left(\frac{0.050^2 + 0.20^2}{0.050^2 + 0.050^2} \right)^{3/2} = 8.5^{3/2} \simeq 24.8. \quad (2.209)$$

It is not 64, because the nearer point has $y = a$ and is outside the far-dipole regime. Applying a power-law scaling to both points without checking the source size would be unjustified.

Example 2.45: Solution N5: a quantified softening error

The softened magnitude divided by the point-field magnitude is $(1 + a_s^2/R^2)^{-3/2}$, which is smaller than one. The condition for less than 1% error is

$$\left(1 + \frac{a_s^2}{R^2} \right)^{-3/2} > 0.99. \quad (2.210)$$

Solving without a series expansion,

$$\frac{R}{a_s} > \left(0.99^{-2/3} - 1 \right)^{-1/2} \simeq 12.2. \quad (2.211)$$

For a single charge, both kernels are exactly radial away from their centre, so a normalized arrow map is identical wherever both are defined and nonzero. The model discrepancy is in magnitude. A convergence study that examines directions alone will miss it.

Example 2.46: Solution E2: inferring a field from a screen displacement

Take upward as $+y$. The electron has $q = -e$ and $y_{\text{screen}} > 0$, so the field component must be negative. Using the magnitude form of Eq. (2.57),

$$E_0 = \frac{m_e y_{\text{screen}} v_0^2}{eL(D + L/2)} = \frac{(9.11 \times 10^{-31})(0.012)(1.5 \times 10^7)^2}{(1.602 \times 10^{-19})(0.040)(0.12)} \simeq 3.2 \times 10^3 \text{ N/C}. \quad (2.212)$$

Thus $\mathbf{E} = -E_0 \hat{\mathbf{j}}$. The plate-exit displacement is

$$y_L = y_{\text{screen}} \frac{L/2}{D + L/2} = 2.0 \text{ mm}, \quad (2.213)$$

smaller than the half-gap 5.0 mm. The exit slope is $\tan \phi = y_{\text{screen}}/(D + L/2) = 0.100$, so $\phi = 5.71^\circ$. The transverse speed is 1.5×10^6 m/s and the total exit speed about 1.51×10^7 m/s $\simeq 0.050c$, consistent with the nonrelativistic calculation at the stated precision.

Using 12 mm as the deflection *inside* the plates would wrongly predict a collision. The screen and exit are different observation points.

Example 2.47: Solution E4: choosing a measurement improvement

The three independent relative uncertainty contributions are 0.010, 0.050, and

$$\frac{0.5\pi/180}{\sin 12^\circ \cos 12^\circ} = 0.0429. \quad (2.214)$$

The current combined relative uncertainty is 6.67%. Halving only the mass uncertainty gives $\sqrt{0.005^2 + 0.050^2 + 0.0429^2} = 6.61\%$. Halving only the charge uncertainty gives $\sqrt{0.010^2 + 0.025^2 + 0.0429^2} = 5.07\%$. Halving only the angle uncertainty gives $\sqrt{0.010^2 + 0.050^2 + 0.02145^2} = 5.53\%$.

Improving the charge calibration gives the largest reduction under the stated equal-cost choice. The result concerns the listed independent measurement uncertainties only. If probe-induced field distortion dominates, improving any of these readings would leave that model bias unresolved.

Example 2.48: Solution E7: spatial averaging by a symmetric probe

The model reading is

$$\bar{E} = \frac{1}{\ell} \int_{-\ell/2}^{\ell/2} E(x_0 + u) du. \quad (2.215)$$

Expand the smooth field about the probe centre:

$$E(x_0 + u) = E(x_0) + uE'(x_0) + \frac{u^2}{2}E''(x_0) + \cdots. \quad (2.216)$$

Odd powers integrate to zero on the symmetric interval, while $\ell^{-1} \int_{-\ell/2}^{\ell/2} u^2 du = \ell^2/12$. Therefore

$$\bar{E} = E(x_0) + \frac{\ell^2}{24}E''(x_0) + \mathcal{O}(\ell^4). \quad (2.217)$$

A purely linear gradient gives no bias in this ideal symmetric average; curvature gives the leading error. An offset centre or asymmetric sensitivity can reintroduce a first-derivative term. This calculation averages an undisturbed field. A real conductive or charged probe may also change the field, which is a separate physical effect.

Example 2.49: Solution H2: axial stability and transverse instability

Near the midpoint, direct expansion gives

$$E_x \simeq -\frac{4k_e q}{a^3} x, \quad E_y \simeq \frac{2k_e q}{a^3} y, \quad (2.218)$$

and by rotational symmetry about the source axis, $E_z \simeq 2k_e qz/a^3$. For $q_t > 0$, the axial equation is

$$\ddot{x} + \frac{4k_e q q_t}{ma^3} x = 0, \quad (2.219)$$

so a line-constrained particle oscillates. Transversely,

$$\ddot{y} - \frac{2k_e q q_t}{ma^3} y = 0, \quad (2.220)$$

which has an exponentially growing solution. The field null is therefore a saddle-type equilibrium for the probe. For $q_t < 0$, transverse directions become restoring but the axial direction becomes unstable. Reversing the probe sign does not produce stability in all directions.

Example 2.50: Solution H6: uncertainty and covariance in charge reconstruction

Define $A = R^3/(2k_e y)$ and $B = R^3/(2k_e a)$ with $R = \sqrt{a^2 + y^2}$. The inferred charges are

$$q_L = AE_y + BE_x, \quad q_R = AE_y - BE_x. \quad (2.221)$$

For independent component errors of variances u_y^2 and u_x^2 , linear propagation gives

$$\text{Var}(q_L) = \text{Var}(q_R) = A^2 u_y^2 + B^2 u_x^2, \quad (2.222)$$

$$\text{Cov}(q_L, q_R) = A^2 u_y^2 - B^2 u_x^2. \quad (2.223)$$

The shared E_y error moves both inferred charges in the same direction, whereas the shared E_x error moves them in opposite directions. Thus independent measurement errors do not imply independent inferred charges.

As $y \rightarrow 0$ with finite absolute u_y , the coefficient A diverges. The charge sum becomes unidentifiable because its field component tends to zero. At exactly $y = 0$, E_y contains no information about the sum, so the inversion formula must not be used. Extra measurements away from that symmetry line are needed.

2.18 Bridge to Chapter 3

The electric field assigns a vector to each observation point. The next step is to describe its relation to an entire surface. Chapter 3 introduces electric flux and Gauss's law, connecting the flux through a closed surface to the net enclosed charge. The field itself remains the same local quantity developed here.

The symmetry arguments used for rings, rods and planes will become especially valuable. Gauss's law is always valid, but it becomes a direct method for determining field magnitude only when enough symmetry is available. The law does not make arbitrary finite-electrode geometries uniform. Chapter 4 then introduces electric potential as a scalar description of electrostatic work; Chapter 5 develops capacitance and dielectric response.

Take-home message

Before choosing a formula, identify the source, the observation point, the geometry and the approximation. After calculating, check components, units, limiting behaviour and the quantity actually measured. Those habits carry directly into flux, potential and the rest of electromagnetism.

Compact formula sheet and validity conditions**Formula summary****Definition and prescribed electrostatic sources in vacuum**

$$\begin{aligned}
 \mathbf{E} &= \lim_{q_0 \rightarrow 0^+} \frac{\mathbf{F}_{\text{ext}}}{q_0}, & \mathbf{F}_{\text{ext}} &= q \mathbf{E}_{\text{ext}}, \\
 \mathbf{E}(\mathbf{r}) &= k_e \sum_i q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}, & k_e &= \frac{1}{4\pi\epsilon_0}, \\
 \mathbf{E}(\mathbf{r}) &= k_e \int \frac{\mathbf{R}}{R^3} dq, & \mathbf{R} &= \mathbf{r} - \mathbf{r}', \\
 dq &= \lambda d\ell, \quad \sigma dA, \quad \rho dV, & [E] &= \text{N/C} = \text{V/m}.
 \end{aligned}$$

The source-to-point displacement and the signed source charge determine direction. Exclude a point particle's self-field from its response law. The integral requires a specified charge distribution; it does not find unknown conducting surface charges.

Formula summary**Uniform prescribed sources, away from the source itself**

$$\begin{aligned}
 E_{z,\text{ring}} &= \frac{k_e Q z}{(a^2 + z^2)^{3/2}}, & \text{ring radius } a, \\
 E_{y,\text{rod}} &= \frac{2k_e \lambda a}{y \sqrt{a^2 + y^2}}, & -a \leq x \leq a, \quad y \neq 0, \\
 E_{x,\text{rod}} &= k_e \lambda \left(\frac{1}{b-L} - \frac{1}{b} \right), & 0 \leq x \leq L, \quad b > L, \\
 E_{z,\text{disk}} &= \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right), & \text{disk radius } R, \quad z > 0, \\
 E_{r,\text{line}} &= \frac{\lambda}{2\pi\epsilon_0 r}, & \text{infinite line, } r > 0.
 \end{aligned}$$

The ring expression is valid on both sides; the disk field reverses across its plane. The long-line limit holds at fixed λ . A local sheet limit of the disk requires $z/R \ll 1$ at fixed σ , with $E_z \simeq \sigma/(2\epsilon_0)$ above the sheet.

Formula summary**Dipoles and response to external fields**

$$\begin{aligned}
\mathbf{p} &= q\mathbf{d} & (\mathbf{d}: -q \text{ to } +q), \\
\mathbf{E}_{\text{dip}} &\simeq \frac{k_e}{r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] & (r \gg d), \\
\boldsymbol{\tau} &= \mathbf{p} \times \mathbf{E}, & U = -\mathbf{p} \cdot \mathbf{E}, \\
\mathbf{F}_{\text{perm}} &\simeq (\mathbf{p} \cdot \nabla)\mathbf{E}, & \mathbf{F}_{\text{ind}} = \frac{\alpha}{2}\nabla(E^2).
\end{aligned}$$

Torque and U above apply to a rigid permanent dipole; $U(\pi/2) = 0$. The force-gradient approximation requires a dipole small compared with the field variation scale. The induced formula assumes a quasistatic isotropic linear response $\mathbf{p} = \alpha\mathbf{E}$, for which $U_{\text{eff}} = -\alpha E^2/2$.

Formula summary**Uniform transverse deflection, nonrelativistic particle**

$$\begin{aligned}
a_y &= \frac{qE_y}{m}, & t_L &= \frac{L}{v_0}, \\
y_L &= \frac{qE_y L^2}{2mv_0^2}, & v_{y,L} &= \frac{qE_y L}{mv_0}, \\
y_{\text{screen}} &= \frac{qE_y L}{mv_0^2} \left(D + \frac{L}{2} \right).
\end{aligned}$$

Entry is at the midplane with zero transverse velocity; the field is uniform over length L and zero over the subsequent drift D . The field component E_y is signed. Plate clearance requires $|y_L| < g/2$ for an ideal zero-width beam, and the whole path must remain nonrelativistic. Collisions require a transport model such as $m\dot{v} = qE - bv$ instead.

Bibliography

- [1] D. Halliday, R. Resnick and J. Walker, *Fundamentals of Physics*, 12th edition, Volume 2, Wiley, Chapter 22, “Electric Fields.”
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th edition, Cengage Learning (2014), Sections 23.4–23.7.
- [3] S. J. Ling, J. Sanny and W. Moebs, *University Physics*, Volume 2, OpenStax (2016; revised edition 2018), Sections 5.4–5.7. <https://openstax.org/details/books/university-physics-volume-2>.