

Chapter 20

# Atomic Physics and Lasers

First Edition

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## Physics 2

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Electricity, Magnetism, Waves, Optics, Relativity,  
Quantum Physics, and Nuclear Physics

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# Preface

Atomic physics brings the ideas of introductory quantum mechanics into a working description of matter and instruments. Photons, matter waves and hydrogenic bound states provide the starting point. Real atoms require electron spin, magnetic moments, exclusion and many-electron energy ordering. These ideas explain periodic structure, characteristic x rays, magnetic resonance and laser operation.

Three themes organize the chapter. First, *states and measurements*: angular-momentum quantum numbers describe a state, while a chosen-axis measurement has discrete outcomes. Stern–Gerlach deflection makes this distinction experimentally visible; spin is intrinsic angular momentum, not a rotating charged sphere. Second, *state occupancy*: Pauli exclusion and atomic energy ordering together determine shell filling and the broad structure of the periodic table. Third, *controlled radiative transitions*: absorption, spontaneous emission and stimulated emission connect atomic states to optical fields. In the conventional lasers studied here, pumping, inversion, resonant feedback and losses determine whether coherent oscillation develops.

## The Guideline

For each problem, identify **(i)** the allowed states and their labels, **(ii)** the interaction or measurement, and **(iii)** the observable: wavelength, deflection, resonance frequency or population. Select the model and its assumptions before inserting numbers. Use units, signs and limiting cases to check the result.

## Pedagogical boundary

The core uses nonrelativistic quantum numbers and calculus-based undergraduate physics. It does not derive hydrogen wave functions, full  $LS$  or  $jj$  coupling theory, many-electron structure from first principles, or spontaneous emission from quantum electrodynamics. Where these theories matter, their physical meaning and validity limits are stated. Advanced enrichment is optional and is identified separately from the core.

# Conventions and notation

Quantum numbers are written without vector arrows:  $n$  is the principal quantum number,  $\ell$  the orbital angular-momentum quantum number,  $m_\ell$  the orbital magnetic quantum number,  $s$  the spin quantum number and  $m_s$  its magnetic component quantum number. Angular momenta and magnetic dipole moments are vectors and are written  $\mathbf{L}$ ,  $\mathbf{S}$ ,  $\mathbf{J}$  and  $\boldsymbol{\mu}$ . A selected laboratory measurement axis is denoted by  $z$ ; it is often defined physically by an applied magnetic field  $\mathbf{B} = B\hat{\mathbf{z}}$ .

We use  $h$  for Planck's constant,  $e > 0$  for the magnitude of electron charge, and ordinary frequency  $f$  or  $\nu$  rather than angular frequency  $\omega = 2\pi f$ :

$$\hbar = \frac{h}{2\pi}, \quad (0.1)$$

$$E_\gamma = hf = \frac{hc}{\lambda}, \quad (0.2)$$

$$1 \text{ eV} = 1.602\,176\,634 \times 10^{-19} \text{ J}. \quad (0.3)$$

The electronvolt conversion is exact in the SI; numerical results use the significant figures justified by the inputs.

The Bohr magneton is

$$\mu_B = \frac{e\hbar}{2m_e} \approx 9.274 \times 10^{-24} \text{ J T}^{-1}, \quad (0.4)$$

where  $e > 0$  is the magnitude of the electron charge and  $m_e$  is the electron mass. Because the electron charge is  $-e$ , its orbital and spin magnetic moments are oriented oppositely to the associated angular-momentum vectors in the simplest one-electron picture.

For a magnetic dipole in an externally imposed magnetic field,

$$U = -\boldsymbol{\mu} \cdot \mathbf{B}. \quad (0.5)$$

If the relevant magnetic-moment component is constant while the field varies in the  $z$  direction, the corresponding force component is

$$F_z = -\frac{\partial U}{\partial z} = \mu_z \frac{\partial B_z}{\partial z}. \quad (0.6)$$

Equation (0.6) is the elementary force model used for Stern–Gerlach deflection.

Throughout,  $N_i$  denotes a number of atoms and  $n_i$  a number density. The quantum number  $n$  is distinct from the refractive index  $n_r$ . We use  $L$  for cavity length and  $\ell_g$  for active-medium length when they differ. Frequency widths are full widths at half maximum (FWHM) unless explicitly described as order-of-magnitude estimates. Appendix A provides the full notation dictionary.

## Note

A magnetic  $g$ -factor differs from a level degeneracy  $g_i$ ; subscripts and context distinguish them. Spin arrows  $\uparrow$  and  $\downarrow$  label outcomes relative to the chosen axis, not literal rotation of an electron.

# Chapter 20

## Atomic Physics and Lasers

### 20.1 Motivation and roadmap

Atomic physics connects the abstract rules of quantum mechanics to observable structure. A spectrometer records narrow wavelengths rather than a continuous smear; a Stern–Gerlach magnet splits an atomic beam into discrete components; the periodic table repeats chemical patterns; an x-ray tube produces element-specific lines; an NMR spectrometer responds at sharply defined frequencies; and a laser produces radiation with unusually high directionality and coherence. These are different experiments, but they are linked by one idea: atomic systems possess discrete quantum states, and interactions select transitions or measurements between them.

The sequence of this chapter is shown schematically in Fig. 20.1. We begin with the spectroscopic and magnetic properties of atoms. We then introduce orbital angular momentum and its magnetic moment, electron spin, and the Stern–Gerlach experiment. Next comes the Pauli exclusion principle and the filling of many-electron states, which gives a quantum-mechanical basis for the broad architecture of the periodic table. Characteristic x rays provide another spectroscopic probe of inner-shell structure. The final part of the main chapter turns from diagnosis to control: absorption, spontaneous emission and stimulated emission are assembled into the working principles of lasers.

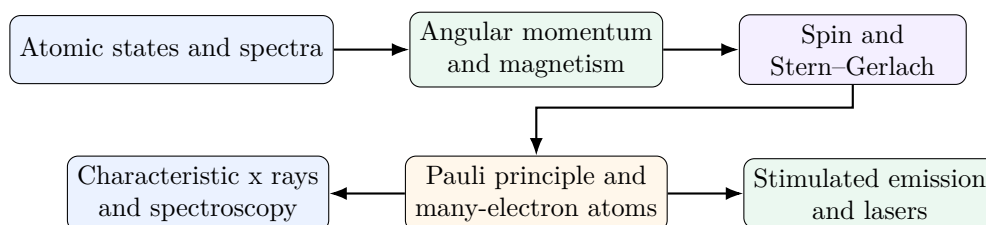


**Figure 20.1:** Roadmap of Chapter 20. The same discrete-state framework underlies spectroscopy, atomic magnetism, electron-state filling, characteristic x rays and laser operation.

#### Learning goals

After studying this chapter, the student should be able to:

- explain why discrete atomic spectra imply discrete internal energy differences;
- use  $n$ ,  $\ell$ ,  $m_\ell$  and  $m_s$  consistently and identify their allowed ranges;
- calculate the magnitude and selected-axis projection of orbital angular momentum;
- relate orbital angular momentum to the electron’s orbital magnetic moment and use the Bohr magneton;
- explain why electron spin is intrinsic and calculate the two allowed values of  $S_z$ ;
- describe the Stern–Gerlach experiment and calculate the force in a known magnetic-field gradient;
- explain qualitatively how magnetic resonance follows from quantized magnetic-energy levels;
- apply the Pauli exclusion principle to shell and subshell occupancy;

- connect electron configurations to broad periodic-table structure without treating filling mnemonics as exact many-body theory;
- calculate photon frequencies and wavelengths for atomic transitions and characteristic x rays;
- distinguish absorption, spontaneous emission and stimulated emission;
- explain metastable states, pumping, population inversion and optical feedback in a laser;
- perform simple laser photon-flux, cavity-mode and rate-equation calculations;
- interpret laboratory spectra and numerical population models with appropriate units and uncertainty awareness.

### Key message

Atomic physics is not a list of disconnected quantum rules. The rules constrain *which states exist, how many electrons can occupy them, and which transitions can occur*. Spectroscopy, magnetism, periodicity and lasers are different experimental consequences of this same structure.

## 20.2 From atomic spectra to quantum states

A low-pressure gas excited by an electrical discharge emits light at a set of discrete wavelengths characteristic of the atomic species. This observation differs fundamentally from the continuous thermal spectrum of a hot dense object. For isolated bound states, discrete internal energy differences produce discrete resonances. Continuum transitions, molecular bands and collisional broadening require additional interpretation. The presence of sharp lines therefore indicates that the emitting system can change energy only by particular amounts.

Suppose an atom makes a transition from an initial state of energy  $E_i$  to a lower state  $E_f$ . Energy conservation gives

$$E_i - E_f = hf = \frac{hc}{\lambda}, \quad E_i > E_f, \quad (20.1)$$

for photon emission, neglecting the small recoil of the atom and referring to its rest-frame transition. Moving atoms show Doppler shifts. For absorption, a photon can raise the atom from a lower state to a higher one only when its energy matches the allowed energy difference to within the physical linewidth of the transition:

$$hf = E_f - E_i, \quad E_f > E_i. \quad (20.2)$$

The same energy difference therefore appears in both emission and absorption spectroscopy, as shown in Fig. 20.2.

The word *state* is essential. In elementary diagrams we often draw horizontal energy levels, but the horizontal line is not a classical orbit. It represents an eigenstate or, in a simplified many-electron description, a configuration with a definite energy to the accuracy of the model. The vertical position in an energy-level diagram encodes energy; horizontal spacing is usually only a graphical convenience unless explicitly defined otherwise.

### Definition 20.1: Atomic transition

An *atomic transition* is a change between quantum states of an atom. If electromagnetic radiation supplies or removes the energy difference, the photon energy satisfies  $hf = |\Delta E|$ . The relation fixes the photon energy but does not, by itself, state how probable or how fast the transition is.

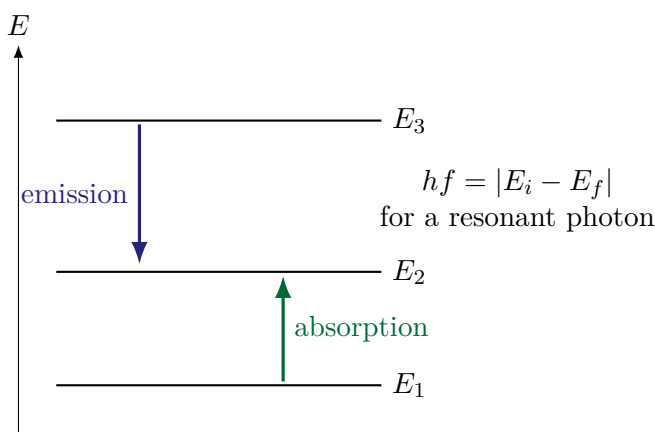


**Figure 20.2:** Energy-level representation of emission and absorption. The line positions indicate state energies; the arrows indicate transitions, not classical trajectories through space.

### Example 20.1: Photon wavelength from an atomic energy difference

An excited atom has a radiative transition with energy difference  $\Delta E = 2.10 \text{ eV}$ . Find the vacuum wavelength of the emitted photon and identify the approximate region of the visible spectrum.

**Solution.** Using  $\lambda = hc/\Delta E$  and the convenient product  $hc \approx 1240 \text{ eV nm}$ ,

$$\lambda \approx \frac{1240 \text{ eV nm}}{2.10 \text{ eV}} \approx 590 \text{ nm}. \quad (20.3)$$

A wavelength near 590 nm lies in the yellow–orange part of the visible spectrum. The important physical point is that a measured line wavelength directly determines an energy difference between quantum states.

**Check.** Visible photons have energies of order a few electronvolts, so a wavelength of several hundred nanometres is reasonable.

### Method

For a spectroscopy calculation, write the energy difference before inserting numbers. Decide whether the process is emission or absorption, determine  $|\Delta E|$ , then use  $hf = |\Delta E|$  and  $c = f\lambda$ . Keep energy units consistent: either convert electronvolts to joules before using SI  $h$ , or use a correctly converted constant such as  $hc \approx 1240 \text{ eV nm}$ .

### Guided checks

1. Why does a line spectrum contain information about energy *differences* rather than absolute atomic energies?
2. If two different transitions have the same  $\Delta E$ , must they have the same wavelength in vacuum?
3. Does the equation  $hf = \Delta E$  tell you whether a transition is strong, weak or forbidden by a selection rule?
4. Why is a horizontal line in an energy-level diagram not a picture of an electron orbit?

**Example 20.2: Reconstructing a simple three-level scheme**

A source is known to have a ground state at  $E_0 = 0$  and two excited states. It emits lines at 400 nm, 600 nm and 1200 nm. Test whether these can arise from all three pairwise transitions among three levels.

The photon energies are approximately

$$E_{400} = \frac{1240}{400} = 3.10 \text{ eV}, \quad E_{600} = 2.07 \text{ eV}, \quad E_{1200} = 1.03 \text{ eV}.$$

Within rounding,

$$2.07 + 1.03 = 3.10 \text{ eV}.$$

Therefore a consistent level scheme is

$$E_0 = 0, \quad E_1 \simeq 1.03 \text{ eV}, \quad E_2 \simeq 3.10 \text{ eV},$$

with transitions  $E_2 \rightarrow E_0$  at 400 nm,  $E_2 \rightarrow E_1$  at 600 nm and  $E_1 \rightarrow E_0$  at 1200 nm. The approximate closure of the energy differences is the key consistency test.

## 20.3 Quantum numbers and orbital angular momentum

The hydrogen atom provides the cleanest introduction to atomic quantum numbers. For hydrogenic Coulomb bound states,  $n$  labels the radial-shell structure, while  $\ell, m_\ell$  label angular momentum. Many-electron central-field orbitals inherit these labels approximately; arbitrary central potentials need not have hydrogenic degeneracies. Electron spin requires two additional labels,  $s$  and  $m_s$ . For an electron,

$$n = 1, 2, 3, \dots, \quad (20.4)$$

$$\ell = 0, 1, \dots, n-1, \quad (20.5)$$

$$m_\ell = -\ell, -\ell+1, \dots, +\ell, \quad (20.6)$$

$$s = \frac{1}{2}, \quad (20.7)$$

$$m_s = \pm \frac{1}{2}. \quad (20.8)$$

The letters  $s, p, d, f, \dots$  used for subshells correspond to  $\ell = 0, 1, 2, 3, \dots$

For a simultaneous eigenstate of  $L^2$  and  $L_z$ , the conventional angular-momentum magnitude denotes  $\sqrt{\langle L^2 \rangle}$ . It is not the magnitude of a classical vector with three definite components. For a state with orbital quantum number  $\ell$ , quantum mechanics gives

$$L = \sqrt{\ell(\ell+1)} \hbar. \quad (20.9)$$

A measurement of the component along a chosen  $z$  axis gives

$$L_z = m_\ell \hbar, \quad m_\ell = -\ell, \dots, +\ell. \quad (20.10)$$

Thus the magnitude and one component are simultaneously specified, but a classical vector picture with all three Cartesian components simultaneously definite is not valid.

The familiar “vector-cone” picture can be useful if treated carefully. For fixed  $L$  and  $L_z$ , one may define a semiclassical angle  $\theta$  through

$$\cos \theta = \frac{L_z}{L} = \frac{m_\ell}{\sqrt{\ell(\ell+1)}}. \quad (20.11)$$

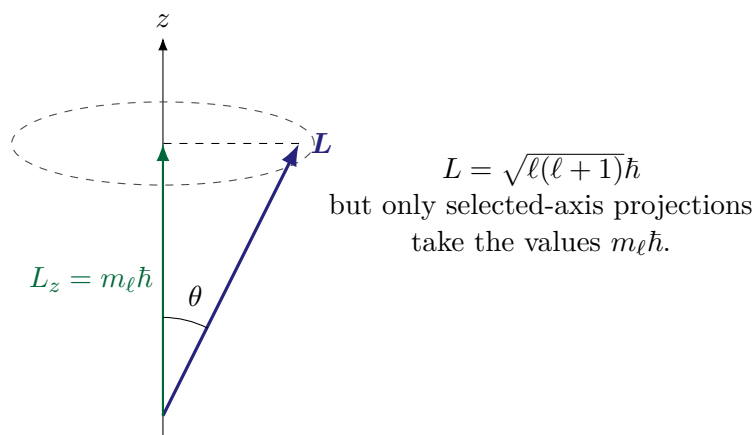


**Figure 20.3:** Semiclassical visualization of orbital angular-momentum quantization. The cone is a visualization of a fixed magnitude and fixed  $z$  projection, not a classical trajectory.

The construction in Fig. 20.3 visualizes the discrete projections but should not be interpreted as a literal orbit plane precessing with a simultaneously known azimuthal direction.

### Example 20.3: Allowed orbital angular-momentum values for a $d$ state

For an electron in a state with  $\ell = 2$ , determine the magnitude  $L$ , list all allowed  $L_z$  values, and calculate the semiclassical angle for  $m_\ell = +2$ .

**Solution.** The magnitude is

$$L = \sqrt{2(2+1)}\hbar = \sqrt{6}\hbar. \quad (20.12)$$

The allowed projections are

$$L_z = -2\hbar, -\hbar, 0, +\hbar, +2\hbar. \quad (20.13)$$

For  $m_\ell = 2$ ,

$$\cos \theta = \frac{2}{\sqrt{6}}, \quad \theta \approx 35.3^\circ. \quad (20.14)$$

**Interpretation.** There are  $2\ell + 1 = 5$  allowed orbital projections for the chosen axis. The angle is a visualization of the projection relation, not evidence for a classical orbit with a rigidly fixed orientation.

### Key message

The factor  $\sqrt{\ell(\ell+1)}$  is a genuinely quantum result. Replacing it by  $\ell$  changes the magnitude and destroys the relation between the angular-momentum spectrum and the allowed projections.

For a state with definite  $\ell, m_\ell$ , the remaining components fluctuate. Since  $L^2 = L_x^2 + L_y^2 + L_z^2$ ,

$$\langle L_x^2 + L_y^2 \rangle = [\ell(\ell+1) - m_\ell^2]\hbar^2. \quad (20.15)$$

Thus even  $|m_\ell| = \ell$  leaves nonzero transverse fluctuations when  $\ell > 0$ . The cone is a memory aid for the projection relation, not a distribution of hidden classical directions. The angle formula is undefined for  $\ell = 0$ , where both  $L$  and  $L_z$  vanish.

## 20.4 Orbital magnetic moments and the Bohr magneton

A classical current loop possesses both angular momentum and magnetic dipole moment. Quantum mechanics retains a precise relation between the electron's orbital angular momentum and its orbital

magnetic moment. For an electron,

$$\boldsymbol{\mu}_L = -\frac{e}{2m_e}\mathbf{L} = -\mu_B \frac{\mathbf{L}}{\hbar}. \quad (20.16)$$

The minus sign follows from the negative electron charge. The magnitude is

$$\mu_L = \mu_B \sqrt{\ell(\ell+1)}, \quad (20.17)$$

and the selected-axis projection is

$$\mu_{L,z} = -m_\ell \mu_B. \quad (20.18)$$

The Bohr magneton therefore sets the natural magnetic-moment scale of atomic electrons.

If a magnetic field  $\mathbf{B} = B\hat{\mathbf{z}}$  is applied and spin is temporarily neglected, the interaction energy is

$$U = -\mu_{L,z}B = m_\ell \mu_B B. \quad (20.19)$$

Different  $m_\ell$  values can then acquire different energies. This is the basic magnetic splitting mechanism behind the normal Zeeman effect, although real atomic spectra often require spin and total angular momentum for an accurate description.

#### 20.4.1 Origin of the orbital magnetic-moment coefficient

For a classical circular path of radius  $r$  and speed  $v$ , the revolution period is  $T = 2\pi r/v$ . The signed loop current is  $q/T$ , giving  $\mu = (q/T)\pi r^2 = qvr/2$ , while the mechanical angular momentum is  $L = m_e vr$ . Hence  $\boldsymbol{\mu} = (q/2m_e)\mathbf{L}$ . Quantum mechanics carries this coefficient into the orbital moment operator, even though an atomic orbital is not a classical path. The derivation explains the sign and dimensions without making the orbit picture literal. For spin, the coefficient is different and must be supplied by the spin magnetic moment. For an  $s$  orbital,  $\ell = 0$  gives zero orbital angular momentum and zero orbital magnetic moment, yet the electron still has spin. Zero orbital angular momentum therefore does not mean a zero total atomic magnetic response.

##### Example 20.4: Orbital magnetic-moment projection

An electron is in an orbital state with  $\ell = 3$  and  $m_\ell = -2$ . Determine  $\mu_{L,z}$  and the corresponding orbital magnetic interaction energy in a field  $B = 0.60$  T.

**Solution.** The magnetic-moment projection is

$$\mu_{L,z} = -m_\ell \mu_B = +2\mu_B. \quad (20.20)$$

The interaction energy is

$$U = -\mu_{L,z}B = -2\mu_B B \approx -2(9.274 \times 10^{-24})(0.60) \text{ J} \approx -1.11 \times 10^{-23} \text{ J}. \quad (20.21)$$

In electronvolts this is about  $-6.9 \times 10^{-5}$  eV.

**Check.** Magnetic atomic energy shifts in laboratory-scale fields are usually much smaller than ordinary optical excitation energies of order electronvolts.

##### Caution

The relation  $\boldsymbol{\mu}_L = -(e/2m_e)\mathbf{L}$  is for the electron's *orbital* magnetic moment. Spin carries a different magnetic-moment coefficient. Do not use the orbital factor blindly for electron spin.

**Guided checks**

1. Why is  $\mu_L$  antiparallel to  $\mathbf{L}$  for an electron?
2. For  $m_\ell = 0$ , is the orbital angular momentum itself zero?
3. If  $B$  doubles, what happens to the simple orbital magnetic energy splitting in Eq. (20.19)?
4. Why does a weak-field magnetic splitting reveal quantum numbers even when its energy is much smaller than the gross atomic binding energy?

**20.5 Electron spin: intrinsic angular momentum**

Electron spin is one of the places where classical intuition must be used with particular care. The electron has intrinsic angular momentum even when there is no orbital motion to which that angular momentum can be attributed. The name “spin” is historical. Treating the electron as a small charged sphere rotating about an axis leads to incorrect physical conclusions and is not the quantum theory.

For an electron,

$$s = \frac{1}{2}, \quad S = \sqrt{s(s+1)}\hbar = \frac{\sqrt{3}}{2}\hbar. \quad (20.22)$$

For a chosen quantization axis,

$$S_z = m_s \hbar, \quad m_s = \pm \frac{1}{2}. \quad (20.23)$$

Only two outcomes are possible for the component measurement.

The electron also has an intrinsic spin magnetic moment. To a very good introductory approximation,

$$\mu_S = -g_s \mu_B \frac{\mathbf{S}}{\hbar}, \quad g_s \approx 2.0023, \quad (20.24)$$

so that

$$\mu_{S,z} = -g_s m_s \mu_B. \quad (20.25)$$

Taking  $g_s \approx 2$  gives the two component values approximately  $\mp \mu_B$ .

For a spin-only model, with orbital coupling and hyperfine interactions neglected, in a field  $B\hat{\mathbf{z}}$  the spin magnetic energy is

$$U_s = -\mu_{S,z} B = g_s m_s \mu_B B. \quad (20.26)$$

The energy separation of the two electron-spin states is therefore

$$\Delta E = g_s \mu_B B \approx 2\mu_B B. \quad (20.27)$$

This splitting is central to electron-spin resonance and contributes to atomic Zeeman structure.

**Theoretical point 20.1: What “spin up” means**

“Spin up” means that a measurement of the selected spin component gives  $m_s = +1/2$ . Because the electron’s magnetic moment is antiparallel to its spin angular momentum, the corresponding magnetic-moment projection has the opposite sign. Careful work therefore distinguishes the labels  $m_s$ ,  $S_z$  and  $\mu_{S,z}$  rather than treating the arrows as interchangeable.

**Example 20.5: Electron-spin splitting and resonance frequency**

Estimate the energy separation and photon frequency associated with electron spin states in a uniform magnetic field of  $B = 0.80$  T. Use  $g_s = 2.00$  for this introductory calculation.

**Solution.** From Eq. (20.27),

$$\Delta E = 2\mu_B B = 2(9.274 \times 10^{-24})(0.80) \approx 1.48 \times 10^{-23} \text{ J.} \quad (20.28)$$

The resonant frequency is

$$f = \frac{\Delta E}{h} \approx \frac{1.48 \times 10^{-23}}{6.626 \times 10^{-34}} \approx 2.24 \times 10^{10} \text{ Hz} = 22.4 \text{ GHz.} \quad (20.29)$$

This lies in the microwave region.

**Check.** Electron-spin resonance frequencies are typically tens of gigahertz per tesla, so the order of magnitude is appropriate.

### Caution

A spin-1/2 particle does not have  $S = \hbar/2$ . The component magnitude is  $|S_z| = \hbar/2$ , whereas the total spin magnitude is  $S = \sqrt{3}\hbar/2$ .

## 20.6 The Stern–Gerlach experiment

The Stern–Gerlach experiment converts quantization of magnetic moment into a visible spatial separation (Fig. 20.4). A beam of neutral atoms is collimated and sent through a strongly nonuniform magnetic field. The use of neutral atoms is important: it suppresses the ordinary Lorentz deflection  $q\mathbf{v} \times \mathbf{B}$  that would dominate for charged particles. The relevant force acts on the atomic magnetic dipole because the magnetic field changes with position.

Along a trajectory where the moment adiabatically follows a nonzero local quantization field, the force is  $F_z = -\partial E_m(B)/\partial z$ . In the simple approximation of an almost  $z$ -directed field and fixed  $\mu_z$ ,

$$F_z \approx \mu_z \frac{dB_z}{dz}. \quad (20.30)$$

Classically, a randomly oriented set of magnetic dipoles would have a continuous range of  $\mu_z$  values and would be expected to smear the beam into a continuous distribution. In the quantum experiment with atoms having an effective spin-1/2 magnetic moment and an unpolarized incident ensemble, the beam separates into two components. The result is evidence that the measured magnetic-moment component has discrete values.

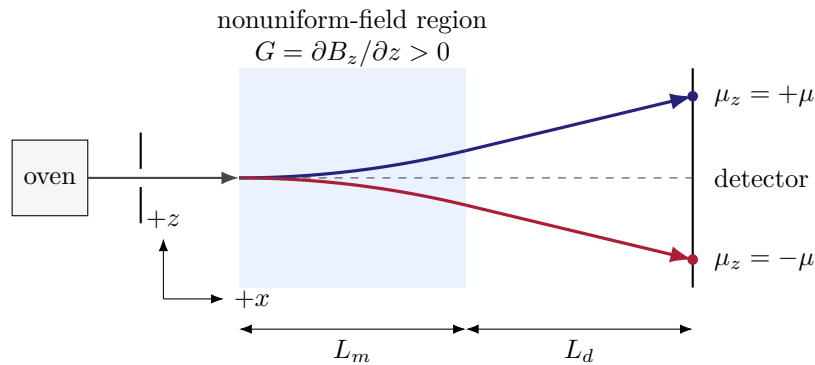


**Figure 20.4:** Stern–Gerlach kinematics in the constant-gradient approximation. The neutral beam curves while the force acts and follows straight paths during the drift. For positive gradient, positive magnetic-moment projection deflects toward  $+z$ ; the electron spin projection has the opposite sign. Other field components required by Maxwell’s equations are omitted from this trajectory schematic.

The historical silver-atom experiment is especially useful because the closed inner shells largely cancel their angular momenta and magnetic moments, while the relevant outer electron leaves an effective two-valued magnetic moment. Modern discussions describe the observed splitting in terms of the valence electron: the ground electronic configuration has  $L = 0$  and  $J = 1/2$ . The two-component treatment neglects nuclear hyperfine structure and assumes a regime in which the electronic moment gives the relevant deflection. A full atom may have more than two components when other angular momenta participate. The experiment also teaches a broader lesson: quantum measurement does not reveal a continuously pre-existing classical orientation. It returns one of the eigenvalues allowed for the selected component.

### Example 20.6: Beam deflection in a Stern–Gerlach field gradient

A neutral silver atom travels horizontally at  $v = 500 \text{ m s}^{-1}$  through a magnet of length  $L = 3.0 \text{ cm}$ . The effective magnetic-moment component has magnitude  $\mu_B$ , and the field gradient is  $\text{d}B_z/\text{d}z = 900 \text{ T m}^{-1}$ . Estimate the separation between the two beam components at the magnet exit. Use a silver atomic mass of  $107.9 u$  and neglect the small change in horizontal speed.

**Solution.** The force magnitude is

$$F = \mu_B \frac{\text{d}B_z}{\text{d}z} \approx (9.274 \times 10^{-24})(900) \approx 8.35 \times 10^{-21} \text{ N}. \quad (20.31)$$

The atomic mass is

$$M \approx 107.9(1.6605 \times 10^{-27}) \approx 1.79 \times 10^{-25} \text{ kg}, \quad (20.32)$$

so the vertical acceleration magnitude is

$$a_z = \frac{F}{M} \approx 4.66 \times 10^4 \text{ m s}^{-2}. \quad (20.33)$$

The transit time is

$$t = \frac{L}{v} = \frac{0.030}{500} = 6.0 \times 10^{-5} \text{ s}. \quad (20.34)$$

Each component is displaced by

$$y = \frac{1}{2}a_z t^2 \approx \frac{1}{2}(4.66 \times 10^4)(6.0 \times 10^{-5})^2 \approx 8.4 \times 10^{-5} \text{ m}. \quad (20.35)$$

The separation of the two oppositely deflected components is approximately

$$\Delta y \approx 2y \approx 1.7 \times 10^{-4} \text{ m} = 0.17 \text{ mm}. \quad (20.36)$$

**Check.** A sub-millimetre separation is experimentally observable with a sufficiently narrow and well-collimated beam.

### Method

In Stern–Gerlach problems, distinguish three separate ingredients: the quantized value of  $\mu_z$ , the spatial field gradient  $\text{d}B/\text{d}z$ , and the ordinary Newtonian motion after the force is known. Quantization determines the discrete force values; classical kinematics then converts those forces into beam positions.

**Guided checks**

1. Why is a uniform magnetic field not enough to produce a net Stern–Gerlach force on a fixed magnetic dipole?
2. Why are neutral atoms advantageous in the experiment?
3. What classical beam pattern would be expected from continuously distributed magnetic orientations?
4. Which part of the calculation is quantum and which part can be treated by ordinary mechanics?

**20.6.1 From force to detector pattern**

A complete Stern–Gerlach calculation has three stages: magnetic force inside the magnet, kinematic deflection while the force acts, and free drift after the atom leaves the field-gradient region.

Let the longitudinal beam direction be  $x$ , the splitting direction be  $z$ , magnet length be  $L_m$ , drift length be  $L_d$ , and longitudinal speed be  $v$ . For a magnetic projection  $\mu_z$ , the force inside the magnet is

$$F_z = \mu_z G, \quad G = \frac{\partial B_z}{\partial z}. \quad (20.37)$$

Assuming  $G$  and  $v$  are approximately constant, the transit time through the magnet is

$$t_m = \frac{L_m}{v}. \quad (20.38)$$

The vertical acceleration is  $a_z = F_z/M$ , so the displacement accumulated in the magnet is

$$z_m = \frac{1}{2} a_z t_m^2, \quad (20.39)$$

and the exit vertical velocity is

$$v_{z,\text{exit}} = a_z t_m. \quad (20.40)$$

During the field-free drift,

$$t_d = \frac{L_d}{v}, \quad z_d = v_{z,\text{exit}} t_d. \quad (20.41)$$

Therefore one component reaches

$$z = z_m + z_d = \frac{\mu_z G}{M} \left[ \frac{1}{2} \left( \frac{L_m}{v} \right)^2 + \frac{L_m L_d}{v^2} \right]. \quad (20.42)$$

For two symmetric components  $\mu_z = \pm\mu$ , the peak separation is  $2|z|$ .

**Example 20.7: Including the drift distance**

Take  $M = 108u$ ,  $v = 500$  m/s,  $G = 700$  T/m,  $L_m = 3.0$  cm and  $L_d = 20$  cm, with  $|\mu_z| = \mu_B$ . The acceleration magnitude is

$$a_z = \frac{\mu_B G}{M} \simeq 3.62 \times 10^4 \text{ m/s}^2.$$

The magnet transit time is  $t_m = 6.0 \times 10^{-5}$  s, giving

$$z_m \simeq 6.5 \times 10^{-5} \text{ m}, \quad v_{z,\text{exit}} \simeq 2.17 \text{ m/s}.$$

The drift time is  $t_d = 4.0 \times 10^{-4}$  s, so

$$z_d \simeq 8.7 \times 10^{-4} \text{ m}.$$

The drift contribution is much larger than the in-magnet displacement. Each beam reaches roughly 0.94 mm from the center, so the two-peak separation is about 1.9 mm.

This example explains why a Stern–Gerlach apparatus benefits from a drift region: the magnet creates a transverse velocity difference, and free propagation converts that velocity difference into a larger position separation.

**Why real peaks have width.** A real oven beam has distributions of longitudinal velocity, source position and initial angle. The field gradient is not perfectly uniform. Collisions and finite slit widths broaden the pattern. The experiment is therefore resolved when the quantum-state separation exceeds the convolution of these classical widths. This is a useful reminder that discrete quantum outcomes can be observed through an apparatus with continuous instrumental resolution.

### 20.6.2 Repeated measurements and a changed analysis axis

An ideal analyzer prepares a state as well as recording an outcome. Select the  $S_z = +\hbar/2$  output and analyze it again along  $z$ : every transmitted atom gives the same outcome. If the next analyzer instead measures  $S_x$ , the two outcomes occur with equal probability. Selecting its  $S_x = +\hbar/2$  branch and returning to a  $z$  analyzer again gives equal probabilities. The intermediate measurement has changed the prepared state. For a spin-1/2 state prepared along unit direction  $\hat{a}$  and measured along  $\hat{b}$  at angle  $\theta$ , quantum mechanics gives

$$P(+_{\hat{b}}|+_{\hat{a}}) = \cos^2(\theta/2), \quad P(-_{\hat{b}}|+_{\hat{a}}) = \sin^2(\theta/2). \quad (20.43)$$

The sum is one, and aligned, opposite and perpendicular analyzers give the expected limits 1, 0 and 1/2. These probabilities supplement the component eigenvalues; a list of allowed outcomes alone does not predict how often each appears.

#### Example 20.8: Changing the spin-analysis axis

An ideal first analyzer prepares  $+z$  atoms. A second analyzer is tilted by  $60^\circ$  from  $z$ . What fractions enter its two outputs? If 1000 prepared atoms are analyzed, what counts are expected?

**Strategy.** Use the angle between preparation and measurement axes, not a classical projection of angular momentum.

$$P_+ = \cos^2 30^\circ = 3/4, \quad P_- = 1/4.$$

The mean counts are 750 and 250. Individual counts fluctuate statistically; the allowed component values along the tilted axis remain exactly  $\pm\hbar/2$ . A changed probability does not change the eigenvalue spectrum.

## 20.7 Magnetic resonance: quantized energy meets radiofrequency radiation

A magnetic moment placed in an external magnetic field has orientation-dependent energy. Quantum mechanics restricts the allowed magnetic projections, so the field creates discrete magnetic-energy levels. If electromagnetic radiation supplies exactly the energy difference between two allowed levels, resonant transitions can occur. This is the core idea of magnetic resonance.

For a spin-1/2 magnetic moment with gyromagnetic ratio  $\gamma$ , it is common to write the angular resonance frequency as

$$\omega_0 = |\gamma|B, \quad f_0 = \frac{|\gamma|}{2\pi}B. \quad (20.44)$$

The same statement can be written as  $hf_0 = \Delta E$ . For protons,  $\gamma_p/(2\pi) \approx 42.58 \text{ MHz T}^{-1}$ . Thus a 1.5 T magnetic field corresponds to a proton resonance frequency near 63.9 MHz.

The RF or microwave magnetic field must have a component perpendicular to the static field to drive a spin flip in this model; matching the frequency alone is insufficient. The resonance frequency depends on the *local* magnetic field at the nucleus or electron. Small magnetic fields produced by surrounding electrons and neighbouring nuclei modify that local field. In nuclear magnetic resonance (NMR), these small shifts provide chemical information. In magnetic-resonance imaging (MRI), controlled magnetic-field gradients make the resonance frequency depend on position, allowing spatial information to be encoded into the detected signal.

### Example 20.9: Proton resonance frequency in a laboratory magnet

Estimate the proton resonance frequency in a field  $B = 3.0 \text{ T}$ .

**Solution.**

$$f_0 = (42.58 \text{ MHz T}^{-1})(3.0 \text{ T}) \approx 127.7 \text{ MHz.} \quad (20.45)$$

The resonant radiation is in the radiofrequency range.

**Interpretation.** MRI magnets can be extremely strong while the transition photon energy remains tiny compared with an optical photon because nuclear magnetic moments are much smaller than the energy scale of electronic optical transitions.

### Caution

MRI is not an x-ray technique. The static field and radiofrequency fields used for proton magnetic resonance are non-ionizing. A complete discussion of MRI safety is nevertheless more complex because strong static fields, field gradients, RF heating and projectile hazards require specialized engineering controls.

## 20.8 The Pauli exclusion principle

The quantum description of a single electron is not enough for real atoms. A neutral atom with atomic number  $Z$  contains  $Z$  electrons, and these electrons are indistinguishable fermions. Their many-electron wave function must have the exchange symmetry appropriate to spin-1/2 particles. The practical consequence introduced at this level is the Pauli exclusion principle.

### Definition 20.2: Pauli exclusion principle

No two electrons in the same atom can occupy the same one-electron quantum state. In the standard one-electron labeling, no two electrons can have the same complete set  $(n, \ell, m_\ell, m_s)$ .

For a given pair  $(n, \ell)$ , there are  $2\ell + 1$  possible values of  $m_\ell$  and two values of  $m_s$ . Therefore the maximum number of electrons in a subshell is

$$N_{n\ell} = 2(2\ell + 1). \quad (20.46)$$

Summing over  $\ell = 0, 1, \dots, n - 1$  gives the maximum number of electrons in a shell:

$$N_n = \sum_{\ell=0}^{n-1} 2(2\ell + 1) = 2n^2. \quad (20.47)$$

Thus  $s$ ,  $p$ ,  $d$  and  $f$  subshells can accommodate 2, 6, 10 and 14 electrons, respectively.

The state-counting result can be traced to exchange antisymmetry. In an independent-orbital approximation, two electrons occupying spin-orbitals  $a, b$  have the normalized state

$$\Psi(1, 2) = \frac{\phi_a(1)\phi_b(2) - \phi_b(1)\phi_a(2)}{\sqrt{2}}, \quad (20.48)$$

for orthonormal  $a \neq b$ . Here each coordinate includes position and spin. If  $a = b$ , the numerator vanishes, so that attempted occupation cannot represent a physical two-electron state. Two electrons can share a spatial orbital because opposite spin labels make their spin-orbitals distinct. Correlated atoms still obey antisymmetry, even when one configuration is insufficient to describe them.

### Example 20.10: Counting states in a shell and subshell

How many electron states are available in the  $n = 4$  shell? How many belong to the  $4d$  subshell?

**Solution.** The shell capacity is

$$N_4 = 2n^2 = 2(4)^2 = 32. \quad (20.49)$$

For a  $d$  subshell,  $\ell = 2$ , so

$$N_{4d} = 2(2\ell + 1) = 2(5) = 10. \quad (20.50)$$

The remaining 22 possible states in the  $n = 4$  shell belong to the  $4s$ ,  $4p$  and  $4f$  subshells.

The exclusion principle should not be confused with electrostatic repulsion. Electron–electron Coulomb repulsion changes the energy and spatial structure of many-electron atoms, but Pauli exclusion remains even in idealized models in which the direct interaction is neglected. Conversely, two electrons may occupy the same spatial orbital when their spin states are different; the familiar closed-shell pair therefore does not violate exclusion.

### Key message

The Pauli principle constrains the allowed many-electron state. Its consequences survive when direct Coulomb repulsion is neglected; Coulomb interaction and exclusion must both be considered in real atomic energies.

### Guided checks

1. Can two electrons in an atom have the same  $n$ ,  $\ell$  and  $m_\ell$  values? Under what condition?
2. Why can an  $s$  subshell hold only two electrons?
3. How many magnetic-orbital values occur in an  $f$  subshell?
4. What is wrong with the statement “electrons occupy different states because they repel each other”?

## 20.9 Many-electron atoms: shells, subshells and energy ordering

For hydrogen, the nonrelativistic energy depends primarily on the principal quantum number  $n$ . In a many-electron atom, the situation changes because each electron moves in the nuclear attraction and in the electric field produced by all the other electrons. The exact problem is therefore a correlated many-body quantum system. Introductory atomic physics replaces that difficult problem by an effective one-electron picture: each electron moves in an average central field and is assigned quantum numbers resembling those of hydrogen. This picture is approximate, but it organizes a remarkable amount of experimental information.

Two effects are especially useful for qualitative reasoning. First, inner electrons *shield* part of the

nuclear charge from outer electrons. Second, orbitals with different values of  $\ell$  penetrate the inner electron cloud by different amounts. Consequently, subshells with the same  $n$  need not have the same energy, as illustrated in Fig. 20.5. In neutral atoms the observed order of filling is therefore not simply  $1s, 2s, 2p, 3s, 3p, 3d, \dots$  in strict order of principal quantum number.

A commonly used mnemonic for the approximate ground-state filling order is

$$1s, 2s, 2p, 3s, 3p, 4s, 3d, 4p, 5s, 4d, 5p, 6s, 4f, 5d, 6p, \dots \quad (20.51)$$

with lower  $n + \ell$  usually filled first and, for equal  $n + \ell$ , lower  $n$  usually filled first. This is often called the Madelung or  $n + \ell$  rule. It is a mnemonic for many observed ground-state configurations, not a fundamental theorem. Transition metals and heavier elements contain important exceptions, and ionization can change the energetic ordering of subshells.

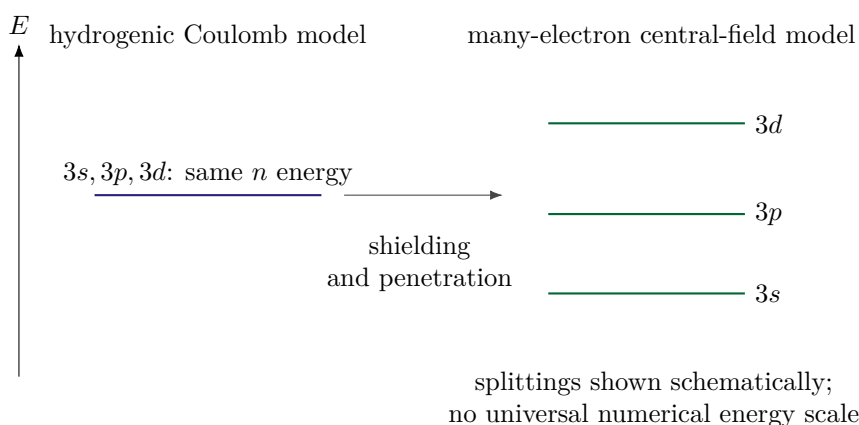


**Figure 20.5:** Why subshells with the same principal quantum number split in many-electron atoms. Different penetration and screening change the effective binding. This comparison illustrates a typical ordering within a shell, without asserting a universal fixed ordering of  $4s$  and  $3d$  across atoms and ions. Fine structure is omitted from the hydrogenic comparison.

Within a set of degenerate or nearly degenerate orbitals, Hund's first rule provides another useful qualitative guide: electrons tend to occupy distinct orbitals with parallel spin before pairing when doing so lowers the energy. This behavior reflects exchange symmetry and electron interactions. It should not be reduced to the misleading statement that electrons "prefer to stay far apart" in a classical sense.

### Example 20.11: Building simple ground-state configurations

Write the ground-state electron configurations of neutral sodium ( $Z = 11$ ), chlorine ( $Z = 17$ ) and calcium ( $Z = 20$ ) in the elementary filling picture. Identify the outermost occupied subshell.

**Solution.** For sodium, filling ten electrons closes the neon-like core and leaves one electron:

$$\text{Na} : 1s^2 2s^2 2p^6 3s^1. \quad (20.52)$$

The outermost occupied subshell is  $3s$ .

For chlorine,

$$\text{Cl} : 1s^2 2s^2 2p^6 3s^2 3p^5. \quad (20.53)$$

The outermost occupied subshell is  $3p$ .

For calcium,

$$\text{Ca} : 1s^2 2s^2 2p^6 3s^2 3p^6 4s^2. \quad (20.54)$$

The outermost occupied subshell is  $4s$ .

**Interpretation.** Sodium has a single electron outside a closed-shell core, chlorine is one electron short of a closed  $3p$  subshell, and calcium has a filled  $4s$  pair. These patterns help explain broad trends in chemical behavior.

### Caution

Electron configurations such as  $[\text{Ar}]4s^23d^6$  are useful labels, but the electrons in a real atom are not independent particles sitting in rigid hydrogen orbitals. Configuration notation is an approximate basis for describing a correlated many-electron quantum state.

### Example 20.12: Configuration reasoning for iron and an iron ion

Iron has  $Z = 26$ . The standard neutral configuration is

$$\text{Fe} : [\text{Ar}] 3d^6 4s^2.$$

For  $\text{Fe}^{2+}$ , two electrons are removed. In the usual spectroscopic description they are removed from  $4s$ , giving

$$\text{Fe}^{2+} : [\text{Ar}] 3d^6.$$

This does not contradict the pedagogical statement that  $4s$  is occupied before  $3d$  when building the neutral atom. The effective orbital energies have changed with occupancy and ionization.

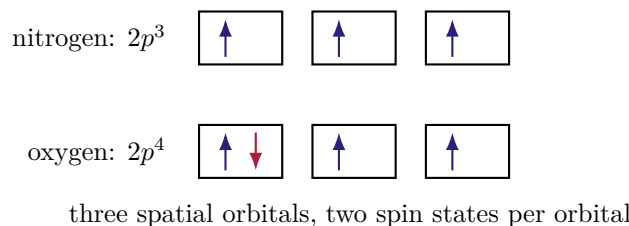


**Figure 20.6:** Orbital-box bookkeeping for the ground-state  $2p$  occupations of nitrogen and oxygen. The three boxes represent an orthonormal set of degenerate  $p$  orbitals; arrows label spin projections. Hund's first rule favors single occupation with parallel spins before pairing. The arrows are state labels, not rotating electrons.

Figure 20.6 distinguishes two rules: Pauli permits at most two electrons of opposite spin per spatial orbital, while Hund's rule addresses which permitted occupation has lower energy. Counting unpaired electrons is a useful first magnetic diagnostic; a quantitative moment requires angular-momentum coupling and the environment.

## 20.9.1 Closed shells, magnetic moments and chemical stability

A completely filled subshell contains every allowed  $m_\ell$  value with both spin projections. Its electronic ground configuration has total  $L = S = J = 0$  within this description. This excludes a permanent electronic moment from the closed shell, but does not exclude an induced diamagnetic response or a nuclear magnetic moment. In the simplest independent-electron picture, the vector contributions of orbital and spin angular momentum then cancel pairwise, giving no net angular momentum from that closed subshell. Closed shells are therefore especially symmetric and tend to have no permanent electronic magnetic moment from the closed set itself.

Noble-gas configurations are characterized by closed outer shells or subshells. Their high ionization energies and weak tendency to form ordinary chemical bonds reflect the energetic stability of

these filled structures. The word “stable” here does not mean that no excited states or chemical compounds are possible; it means that removing an electron or creating an open shell generally costs significant energy.

### Guided checks

1. Why does the  $n + \ell$  filling rule have to be presented as a mnemonic rather than as an exact law?
2. Why can a closed subshell have zero net magnetic moment even though every electron individually has spin magnetic moment?
3. Which is more fundamental for state occupancy: the Pauli principle or the order in which subshell energies happen to lie in a particular atom?
4. Why does ionization sometimes change the ordering that one memorizes for neutral atoms?

## 20.10 Periodic structure from quantum-state filling

The periodic table is ordered by atomic number  $Z$ , the number of protons in the nucleus. A neutral atom has  $Z$  electrons. As  $Z$  increases, electrons are added to available quantum states subject to the Pauli principle and the actual many-electron energy ordering. Similar outer-electron configurations recur, and these repeated valence structures produce recurring chemical behavior.

The broad block structure in Fig. 20.7 reflects the subshell being filled:

- the  $s$  block corresponds primarily to filling an outer  $s$  subshell;
- the  $p$  block corresponds primarily to filling an outer  $p$  subshell;
- transition metals involve  $d$ -subshell filling;
- lanthanides and actinides involve  $f$ -subshell filling.

This organization is one of the clearest macroscopic signatures of quantum mechanics. The sequence of elements is a sequence of nuclear charges, but the recurring chemical families arise because electron-state capacities repeat according to the angular-momentum quantum numbers.

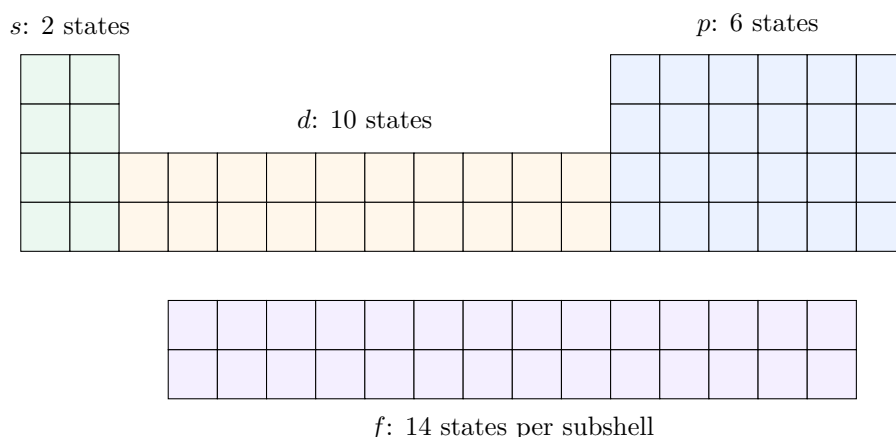


**Figure 20.7:** Schematic block widths connect the periodic table to subshell capacities  $2(2\ell + 1)$ . The drawing shows representative rows rather than a complete table. Helium has configuration  $1s^2$  and is placed with noble gases by chemistry; detailed  $d/f$  boundaries and exceptional configurations need the full table.

### 20.10.1 Ionization energy as a periodic diagnostic

The first ionization energy is the energy required to remove the least tightly bound electron from a neutral atom in its ground state. Across a period, increasing nuclear charge tends to bind valence electrons more strongly, while the opening of a new shell tends to reduce the first ionization energy because the outer electron is farther from the nucleus and more strongly shielded. Real data contain substructure due to subshell energies, pairing and electron correlations, but the large-scale periodic pattern follows directly from quantum-shell structure.

An engineering student may meet ionization energies in plasma physics, gas-discharge devices, mass spectrometry and detector technology. In all these contexts, it is useful to remember that “ionization energy” is not one universal constant of an element; successive ionization energies differ, and the energy required depends on the charge state and electronic configuration.

#### Example 20.13: Relating valence configuration to a periodic-table trend

Consider two idealized neutral atoms with outer configurations  $ns^1$  and  $ns^2np^5$ , respectively, at comparable principal quantum number. Which one is qualitatively closer to a closed shell, and what broad chemical tendency follows from that statement?

**Solution.** The  $ns^2np^5$  atom is one electron short of a closed  $p$  subshell and therefore one electron short of a noble-gas-like outer configuration. The  $ns^1$  atom has one electron outside a closed inner-shell structure. Broadly, the former often tends to gain or share one electron, while the latter often tends to lose or share its single outer  $s$  electron.

**Boundary.** This argument explains broad families, not detailed reaction energetics. Bonding energies, molecular orbitals, electronegativity and condensed-matter environment require additional physics and chemistry.

## 20.11 Atomic transitions, line strengths and selection rules

The energy condition  $hf = |\Delta E|$  is necessary for an optical transition, but it is not sufficient to determine whether a transition is strong. A transition also requires a nonzero coupling matrix element between the initial and final states. At an introductory level, this fact is summarized by *selection rules*. For the dominant electric-dipole interaction in a one-electron central-field picture, the most familiar orbital rules are

$$\Delta\ell = \pm 1, \quad \Delta m_\ell = 0, \pm 1. \quad (20.55)$$

In a spin-independent electric-dipole transition one also has, approximately,

$$\Delta s = 0, \quad \Delta m_s = 0. \quad (20.56)$$

A parity change accompanies an electric-dipole transition. These rules are not statements that all other transitions are impossible. Magnetic-dipole, electric-quadrupole, spin-orbit mixing and other mechanisms can make nominally forbidden transitions weakly allowed. The word “forbidden” therefore usually means strongly suppressed relative to the dominant electric-dipole channel, not mathematically impossible in every physical environment.

Selection rules have important practical consequences. They determine which spectral lines are bright, which excited states are metastable, which pumping schemes are viable in lasers, and which transitions can serve as narrow frequency references. A long-lived metastable state is often valuable because its slow spontaneous decay permits population to accumulate.

**Note**

Selection rules depend on the interaction and on which quantum numbers are good labels in the physical regime. The rules in Eqs. (20.55)–(20.56) are the standard introductory electric-dipole rules, not universal transition rules for every atom and every field strength.

**Example 20.14: Testing an introductory electric-dipole transition**

In a simplified one-electron atom, consider a transition from a state with  $\ell = 2$ ,  $m_\ell = 1$  to a state with  $\ell = 1$ ,  $m_\ell = 0$ , with spin unchanged. Is the transition compatible with the elementary electric-dipole selection rules?

**Solution.** The changes are

$$\Delta\ell = 1 - 2 = -1, \quad \Delta m_\ell = 0 - 1 = -1. \quad (20.57)$$

Both satisfy Eq. (20.55), and the spin is unchanged. The transition is therefore electric-dipole allowed in this simplified classification.

**Caveat.** Being allowed does not specify the exact rate; the transition matrix element and detailed atomic structure still matter.

## 20.12 Characteristic x rays and inner-shell structure

An x-ray tube illustrates two physically distinct radiation mechanisms. Electrons accelerated through a high voltage strike a target. As the incident electrons are decelerated in the target, they can emit a continuous bremsstrahlung spectrum. Superimposed on this continuum are sharp characteristic x-ray lines determined by the target element.

If an electron crosses an accelerating potential difference  $V$  and loses essentially all of its kinetic energy in one photon-producing event, the maximum photon energy is  $eV$ . The corresponding minimum wavelength of the continuous spectrum is

$$\lambda_{\min} = \frac{hc}{eV}. \quad (20.58)$$

This cutoff depends on the accelerating voltage, not primarily on the target material.

Characteristic lines arise differently. An energetic incident electron can eject an inner-shell electron from a target atom, creating a vacancy. An electron from a higher shell then falls into the vacancy, and the atomic energy difference is emitted as an x-ray photon. These are differences of total atomic energies with different vacancy configurations; the one-electron shell diagram is an approximation to that many-electron process. A transition into the  $K$  shell ( $n = 1$ ) is labeled with  $K$  notation; for example, a  $K_\alpha$  line is associated primarily with a transition from the next shell into the  $K$  vacancy.

The inner-shell binding energies grow strongly with atomic number  $Z$ . Moseley's measurements showed that, for a given x-ray series, the square root of the characteristic frequency is approximately linear in an effective atomic number:

$$\sqrt{f} \approx C(Z - \sigma), \quad (20.59)$$

where  $C$  depends on the series and  $\sigma$  is an effective screening constant. This empirical relationship was historically important because it established atomic number, rather than atomic mass, as the natural ordering variable for the elements.

A simple hydrogenic estimate for a  $K_\alpha$  photon treats the remaining electrons as screening approximately one unit of nuclear charge. Using an effective charge  $Z_{\text{eff}} \approx Z - 1$  gives

$$E_{K_\alpha} \approx 13.6 \text{ eV} (Z - 1)^2 \left(1 - \frac{1}{4}\right) = 10.2 \text{ eV} (Z - 1)^2. \quad (20.60)$$

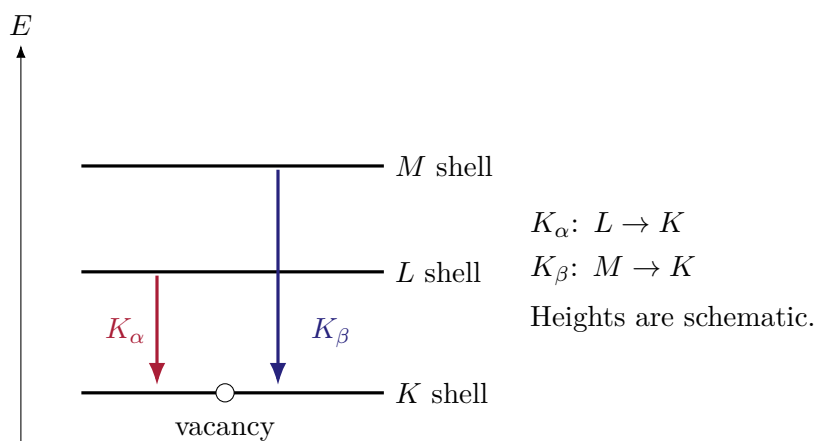


**Figure 20.8:** Schematic origin of characteristic x-ray lines. An inner-shell vacancy is filled by an electron from a higher shell, producing a photon whose energy is characteristic of the target atom.

This estimate captures the approximate  $Z^2$  scaling but is not a precision many-electron calculation.

An incident electron must have enough energy to create the vacancy. This threshold is the inner-shell binding energy, which exceeds the characteristic photon energy emitted when the vacancy is filled. Merely checking  $eV > E_{K_\alpha}$  is insufficient. For the same screened model, the approximate  $K$ -shell removal energy is  $E_K^{\text{bind}} \simeq 13.6(Z-1)^2 \text{ eV}$ . Competition with nonradiative Auger decay also matters: a vacancy can be filled while ejecting another electron rather than emitting an x ray. The fluorescence yield varies with element and shell.

#### Example 20.15: X-ray cutoff from tube voltage

An x-ray tube operates at  $V = 35 \text{ kV}$ . Estimate the minimum wavelength of its bremsstrahlung spectrum.

**Solution.** The maximum photon energy is 35 keV. Using  $hc \approx 1.240 \text{ keV nm}$ ,

$$\lambda_{\min} = \frac{1.240 \text{ keV nm}}{35 \text{ keV}} \approx 3.54 \times 10^{-2} \text{ nm} = 35.4 \text{ pm}. \quad (20.61)$$

**Interpretation.** The cutoff is fixed by the electron acceleration voltage. Changing the target changes the characteristic lines much more directly than it changes this ideal voltage-limited cutoff.

#### Example 20.16: Screened hydrogenic estimate of a copper $K_\alpha$ line

Use Eq. (20.60) to estimate the  $K_\alpha$  photon energy for copper,  $Z = 29$ , and convert it to wavelength.

**Solution.**

$$E_{K_\alpha} \approx 10.2 \text{ eV} (28)^2 \approx 7.997 \times 10^3 \text{ eV} \approx 8.00 \text{ keV}. \quad (20.62)$$

Then

$$\lambda \approx \frac{1.240 \text{ keV nm}}{8.00 \text{ keV}} \approx 0.155 \text{ nm}. \quad (20.63)$$

The estimate is close to the expected hard-x-ray scale of copper  $K$  radiation.

**Caveat.** Accurate characteristic energies require a better treatment of screening, relativistic effects and electron interactions.

**Example 20.17: Voltage, current and cutoff are different design variables**

An x-ray tube operates at 60 kV and 2.0 mA. The electrical input power is

$$P = VI = (60 \times 10^3)(2.0 \times 10^{-3}) = 120 \text{ W}.$$

The ideal cutoff wavelength is

$$\lambda_{\min} = \frac{1240 \text{ eV nm}}{60 \times 10^3 \text{ eV}} \simeq 0.0207 \text{ nm}.$$

If the current is doubled at the same voltage, the electrical power doubles, but the ideal cutoff wavelength is unchanged. The spectrum becomes more intense in a simplified picture, not harder at its endpoint.

Figure 20.9 separates endpoint energy from characteristic peak positions.

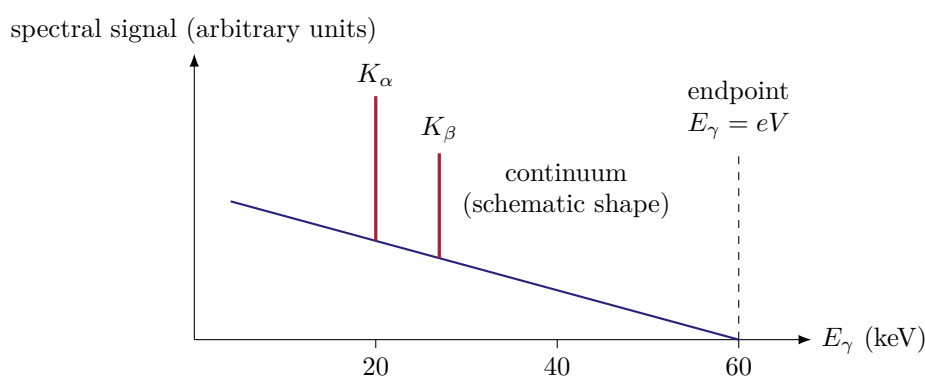


**Figure 20.9:** Schematic energy spectrum of a tube operating at 60 kV. The line positions are illustrative, not measured or assigned to an element. The endpoint is voltage limited; characteristic line positions remain approximately fixed when voltage changes above their vacancy thresholds. The continuum shape and relative line strengths depend on source and detector physics.

### 20.12.1 Why characteristic x rays matter technically

Characteristic x rays provide an elemental fingerprint because the inner-shell energies depend strongly on nuclear charge. This principle underlies x-ray fluorescence (XRF) analysis, electron-microscope energy-dispersive x-ray spectroscopy, materials identification and some planetary or space-instrument measurements. In these applications the measured spectrum is not interpreted by one isolated formula; detector response, background subtraction, line overlap, self-absorption and calibration matter. Nevertheless, the atomic-physics origin of the peaks is the same vacancy-filling process shown in Fig. 20.8.

#### Guided checks

1. Which feature of an x-ray tube spectrum is controlled mainly by accelerating voltage: the short-wavelength cutoff or the characteristic line positions?
2. Why do characteristic x-ray lines depend on the target element?
3. In the simple Moseley form, what does the screening parameter represent physically?
4. Why should Eq. (20.60) be called an estimate rather than an exact formula?

**Example 20.18: Identifying an element from an approximate  $K_\alpha$  energy**

A measured characteristic line is near 17.5 keV. Using Eq. (20.60),

$$(Z - 1)^2 \approx \frac{17.5 \times 10^3}{10.2} \simeq 1716,$$

so

$$Z - 1 \simeq 41.4, \quad Z \simeq 42.4.$$

The simple model therefore points to an element with atomic number near 42. The noninteger estimate is expected because the screening model is approximate. In an actual spectrometer, one would compare calibrated line energies with a database rather than rounding a one-parameter model blindly.

## 20.13 Three elementary light–matter processes

Figure 20.10 compares the three processes underlying laser physics. They involve two atomic levels  $E_1 < E_2$  and radiation of frequency

$$hf = E_2 - E_1. \quad (20.64)$$

The processes are absorption, spontaneous emission and stimulated emission.

### 20.13.1 Absorption

An atom in the lower state can absorb a resonant photon and move to the upper state:

$$E_1 + hf \longrightarrow E_2. \quad (20.65)$$

The probability per unit time is proportional to the spectral energy density of the radiation field at the transition frequency. In Einstein's notation the rate per atom is proportional to  $B_{12}\rho(f)$ .

### 20.13.2 Spontaneous emission

An atom in the upper state can decay without an externally incident resonant photon:

$$E_2 \longrightarrow E_1 + hf. \quad (20.66)$$

For one dominant radiative channel with no re-excitation or nonradiative loss, the population equation is

$$\frac{dN_2}{dt} = -A_{21}N_2, \quad (20.67)$$

with solution

$$N_2(t) = N_2(0)e^{-A_{21}t} = N_2(0)e^{-t/\tau}, \quad \tau = \frac{1}{A_{21}}. \quad (20.68)$$

The spontaneous photons are emitted with directions, phases and polarization properties governed by the transition and environment; there is no requirement that different atoms emit in phase with one another.

If several channels are open,  $1/\tau = \sum_f A_{2f} + W_{\text{nr}}$ , where  $W_{\text{nr}}$  is the nonradiative decay rate. A particular spectral line carries only its radiative branching fraction. A metastable state has a small total decay rate in the relevant environment; collisions can shorten its effective lifetime.

### 20.13.3 Stimulated emission

If an atom is already in the upper state and a resonant radiation field interacts with it, the radiation field can stimulate a transition to the lower state with emission of an additional photon. The added excitation enters the same optical mode, with the same frequency, propagation direction and polarization. The phase relation belongs to the field amplitude: individual photons do not carry an independently measurable classical phase. In a coherent field the stimulated contribution amplifies that field with the appropriate phase relation. This replication property is the microscopic basis of optical amplification.

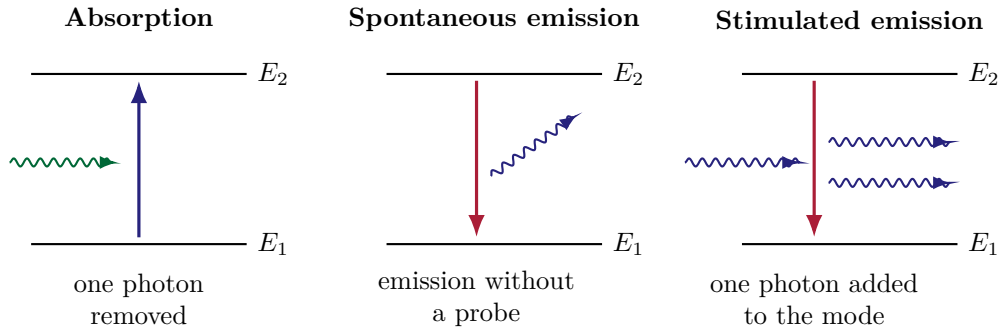


**Figure 20.10:** Three radiative processes for  $E_2 - E_1 = hu$ . Stimulated emission adds an excitation to the stimulating mode. The two outgoing wavy arrows are photon-number bookkeeping, not separate classical waves each carrying an individually measured photon phase.

#### Key message

Stimulated emission is not simply ordinary fluorescence made stronger. In stimulated emission the existing optical field drives the transition, and the emitted photon joins the stimulating mode. This mode-matching property is what makes coherent amplification possible.

## 20.14 Thermal populations and why inversion is special

At positive-temperature thermal equilibrium, the population per sublevel decreases with energy. The total population of a highly degenerate upper level can exceed that of a less degenerate lower level without producing gain. Including level degeneracies, the Boltzmann relation is

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp \left[ -\frac{E_2 - E_1}{k_B T} \right]. \quad (20.69)$$

For equal degeneracies,  $N_2 < N_1$  at every positive temperature. If resonant light passes through such a medium, absorption from the lower level competes with stimulated emission from the upper level, and the larger lower-state population means that ordinary equilibrium matter is not a net laser amplifier.

A laser therefore requires a nonequilibrium condition. For equal degeneracies the elementary population-inversion criterion is

$$N_2 > N_1. \quad (20.70)$$

More generally, when the degeneracies differ, the condition for positive small-signal gain is expressed in terms of population per sublevel:

$$\frac{N_2}{g_2} > \frac{N_1}{g_1}. \quad (20.71)$$

Population inversion cannot be created by simply heating an ideal two-level system to an ordinary positive temperature. Energy must be supplied in a way that drives the medium out of thermal equilibrium; this is called *pumping*.

**Example 20.19: Thermal population ratio for an optical transition**

Two nondegenerate atomic levels are separated by  $\Delta E = 1.80$  eV. Estimate  $N_2/N_1$  at  $T = 300$  K in thermal equilibrium.

**Solution.** At room temperature,  $k_B T \approx 0.025852$  eV. Therefore

$$\frac{N_2}{N_1} = \exp\left(-\frac{1.80}{0.025852}\right) \approx e^{-69.63} \approx 5.8 \times 10^{-31}. \quad (20.72)$$

The upper optical level is essentially unpopulated thermally.

**Interpretation.** Optical population inversion is profoundly nonequilibrium. It must be created by pumping and by exploiting level lifetimes and decay pathways.

**Caution**

The phrase “negative temperature” is sometimes used in advanced discussions of inverted finite-level systems. It does not mean “colder than absolute zero.” It is a specialized statistical-mechanics description and is not needed to operate the basic laser model developed here.

## 20.15 How a laser works

The acronym LASER stands for *light amplification by stimulated emission of radiation*. A practical laser requires more than stimulated emission alone. Four ingredients are conceptually distinct:

1. an **active medium** with suitable quantum states;
2. a **pump** that supplies energy and creates a nonequilibrium population;
3. a mechanism that produces **population inversion** on the desired transition;
4. an **optical resonator** or another feedback mechanism that allows selected optical modes to make repeated passes through the gain medium while a controlled fraction leaves as useful output.

A two-level system is generally unsuitable for steady population inversion using only resonant optical pumping because the same radiation that excites atoms also drives stimulated emission back down. Practical lasers therefore use at least three effective levels, and many common systems are better represented by four-level schemes.

### 20.15.1 Three-level scheme

In an idealized three-level laser, atoms are pumped from a ground level 1 to a high level 3. They rapidly decay, often nonradiatively, to a relatively long-lived metastable level 2. Population accumulates in level 2, and the laser transition occurs from 2 to the ground level 1. Because the lower laser level is the ground state, more than half the relevant atoms must often be transferred out of the ground state before inversion is obtained. This makes three-level lasers comparatively demanding to pump.

### 20.15.2 Four-level scheme

In a four-level laser, pumping raises atoms from level 1 to 4, followed by rapid decay to the upper laser level 3. The laser transition is  $3 \rightarrow 2$ , and level 2 rapidly empties into the ground state 1. Because the lower laser level remains sparsely populated, achieving  $N_3 > N_2$  is much easier than

achieving inversion against a heavily populated ground state. Figure 20.11 compares the population pathways.

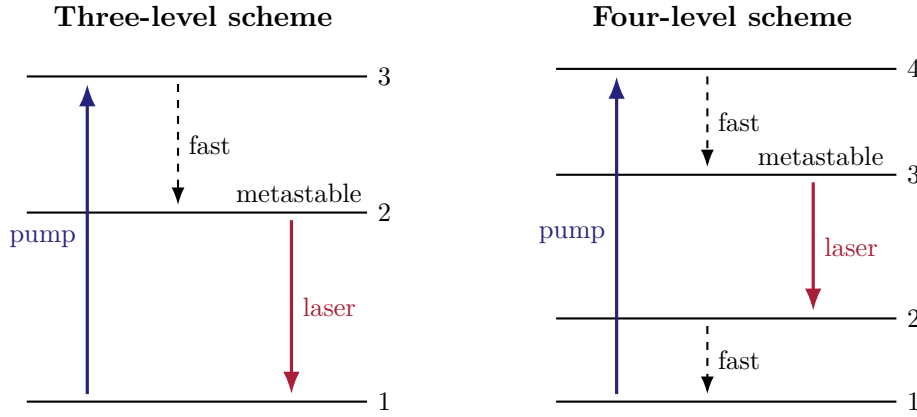


**Figure 20.11:** Multilevel routes to inversion. The upper laser level accumulates population, while rapid relaxation bypasses a reversible two-level pump. In the four-level scheme, fast emptying of the lower laser level reduces reabsorption. Heights are schematic; dashed arrows indicate rapid non-lasing relaxation.

### Definition 20.3: Population inversion

For the conventional bound-level gain model used here, a pair of laser levels is population inverted when the upper level is more strongly populated, per available sublevel, than the lower level. For equal degeneracy this reduces to  $N_{\text{upper}} > N_{\text{lower}}$ . Population inversion is a nonequilibrium condition required for net stimulated-emission gain.

### 20.15.3 Optical feedback and output coupling

A simple laser resonator uses two facing mirrors surrounding the gain medium (Fig. 20.12). One mirror is highly reflecting, while the output coupler is partially transmitting. Light traveling near the resonator axis passes repeatedly through the active medium. Frequencies satisfying the longitudinal resonance condition are preferentially reinforced. In the simplest cavity of optical length  $n_r L$ ,

$$\nu_q \approx q \frac{c}{2n_r L}, \quad q = 1, 2, 3, \dots, \quad (20.73)$$

for a uniform, nondispersive medium, neglecting mirror phase and transverse Gouy phase. Adjacent longitudinal modes are separated by the free spectral range

$$\Delta\nu_{\text{FSR}} = \frac{c}{2n_r L}. \quad (20.74)$$

A mode must lie in the gain spectrum and have sufficient modal gain to exceed its losses. Gain bandwidth alone does not predict how many modes lase. In a dispersive medium, the FSR uses group delay,  $\Delta\nu_{\text{FSR}} \simeq c/(2n_g L)$  for a uniform cavity, rather than simply the phase index.

### Example 20.20: Longitudinal mode spacing

A linear air-filled laser cavity has mirror separation  $L = 30$  cm. Estimate the spacing between adjacent longitudinal modes.

**Solution.** Taking  $n_r \approx 1$ ,

$$\Delta\nu_{\text{FSR}} = \frac{c}{2L} = \frac{3.00 \times 10^8}{0.60} \approx 5.00 \times 10^8 \text{ Hz} = 500 \text{ MHz}. \quad (20.75)$$

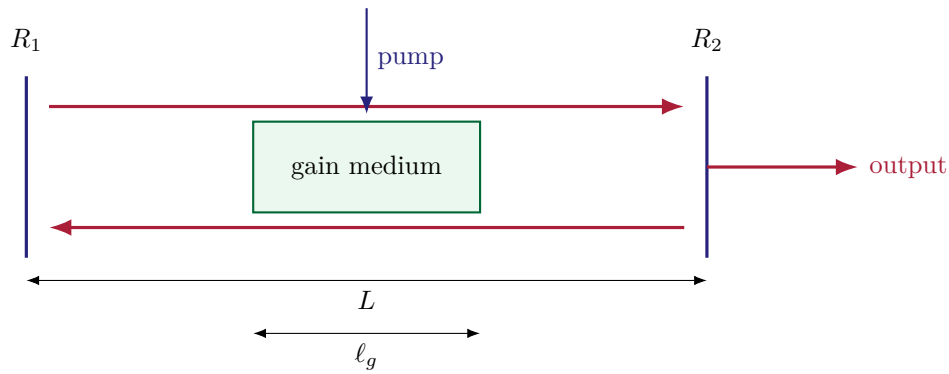


**Figure 20.12:** A linear resonator with a partially transmitting output mirror. The cavity length  $L$  fixes round-trip delay, while the active length  $\ell_g$  controls gain per pass. They need not be equal. Arrows show repeated propagation; the drawing separates the two lengths used in the generalized threshold condition.

**Interpretation.** The optical carrier frequency is hundreds of terahertz, so many cavity modes may fit under a sufficiently broad gain curve unless mode-selection mechanisms suppress most of them.

#### Guided checks

1. Why is stimulated emission alone not enough to define a working laser?
2. Why is a four-level system generally easier to invert than a three-level system?
3. What role does the metastable level play?
4. What happens to the cavity mode spacing if the mirror separation doubles?
5. Why can only a subset of cavity modes lase even though the cavity has many resonances?

#### 20.15.4 Cavity length as a design variable

Cavity length influences longitudinal mode spacing, package size, mechanical sensitivity and thermal stability. Equations (20.73) and (20.74) show that doubling optical length halves the free spectral range. A longer cavity therefore places more resonances within a fixed gain interval; a short cavity can simplify longitudinal mode selection.

Suppose a gain medium has useful gain bandwidth  $\Delta\nu_g$ . A rough count of cavity longitudinal modes within that interval is

$$N_{\text{modes}} \sim \frac{\Delta\nu_g}{\Delta\nu_{\text{FSR}}} = \frac{2n_r L \Delta\nu_g}{c}. \quad (20.76)$$

The number need not be an integer in this estimate, and actual lasing depends on threshold gain for each mode.

#### Example 20.21: Mode-count design estimate

A gain feature is 6.0 GHz wide. Compare cavities of length  $L = 2.5$  cm and  $L = 25$  cm in air. For 2.5 cm,

$$\Delta\nu_{\text{FSR}} = \frac{3.00 \times 10^8}{2(0.025)} \simeq 6.0 \text{ GHz}.$$

Only about one longitudinal-mode spacing fits inside the gain width. For 25 cm,

$$\Delta\nu_{\text{FSR}} \simeq 0.60 \text{ GHz},$$

so roughly ten mode spacings fit inside the same gain interval. The longer cavity therefore has denser longitudinal modes.

**But short is not automatically better.** A shorter cavity increases FSR but may worsen thermal handling, mode-volume requirements or mirror geometry. A longer cavity can provide useful access for intracavity optics and may support lower beam divergence for a chosen resonator design. Real cavity design is an optimization problem with optical, mechanical and thermal constraints.

**Length noise becomes frequency noise.** For a fixed longitudinal index,

$$\nu_q \propto \frac{1}{L}.$$

A small length change gives

$$\frac{\delta\nu}{\nu} \simeq -\frac{\delta L}{L}. \quad (20.77)$$

This simple relation explains why precision lasers use mechanically and thermally stable reference cavities: nanometre-scale or smaller length fluctuations can matter when the optical frequency is hundreds of terahertz.

## 20.16 Laser gain, threshold and saturation: a first quantitative model

Suppose an optical intensity  $I$  propagates through an inverted medium. In a simplified small-signal model the stimulated-emission gain can be written

$$\frac{dI}{dz} = gI, \quad (20.78)$$

where  $g$  is the intensity gain coefficient. If  $g$  is constant,

$$I(z) = I(0)e^{gz}. \quad (20.79)$$

Real media also have distributed losses. If  $\alpha$  is an intensity-loss coefficient, a convenient net model is

$$\frac{dI}{dz} = (g - \alpha)I. \quad (20.80)$$

A laser oscillates when the small-signal round-trip gain reaches the round-trip loss for a resonant mode. If the gain fills a uniform cavity of length  $L$ , with distributed intensity-loss coefficient  $\alpha$ , the round-trip intensity factor is

$$R_1 R_2 \exp[2(g - \alpha)L]. \quad (20.81)$$

Unity defines threshold; a value above unity makes a weak resonant field grow until saturation changes the gain. Thus

$$g_{\text{th}} = \alpha + \frac{1}{2L} \ln \left( \frac{1}{R_1 R_2} \right). \quad (20.82)$$

The reflectivities are power ratios, so no amplitude square root belongs in this intensity balance. Both  $g$  and  $\alpha$  have units  $\text{m}^{-1}$ . If the active medium occupies only length  $\ell_g$  of a cavity of total length  $L$ , while losses act throughout the cavity, the corresponding balance is

$$R_1 R_2 e^{2g\ell_g - 2\alpha L} = 1, \quad g_{\text{th}} = \frac{\alpha L + \frac{1}{2} \ln[1/(R_1 R_2)]}{\ell_g}. \quad (20.83)$$

For spatially varying gain and loss, replace  $g\ell_g$  and  $\alpha L$  by one-pass integrals along the mode. A longitudinal resonance is necessary as well as gain; an off-resonant wave does not become an oscillating cavity mode merely because the intensity factor is large.

Above threshold, the intracavity field depletes the inversion through stimulated emission. The gain therefore cannot grow indefinitely with pump power. This reduction of gain as intensity increases is called *gain saturation*. Saturation is one of the mechanisms that stabilizes laser output and must be included in realistic rate-equation models.

### Example 20.22: Threshold gain in a simple cavity

A uniform cavity completely filled by gain medium has length  $L = 0.20$  m and mirrors with power reflectivities  $R_1 = 0.995$  and  $R_2 = 0.95$ . Neglect distributed losses. Estimate the threshold gain coefficient.

**Solution.**

$$g_{\text{th}} = \frac{1}{2L} \ln \left( \frac{1}{R_1 R_2} \right) = \frac{1}{0.40} \ln \left( \frac{1}{0.94525} \right). \quad (20.84)$$

Because  $\ln(1/0.94525) \approx 0.0563$ ,

$$g_{\text{th}} \approx 0.141 \text{ m}^{-1}. \quad (20.85)$$

A smaller output-coupler reflectivity increases the loss and therefore raises the gain needed for threshold.

### Caution

The exponential law  $I = I_0 e^{gz}$  is a small-signal approximation. Once the optical field becomes strong enough to change the level populations appreciably,  $g$  depends on intensity and the simple constant- $g$  exponential no longer describes the full dynamics.

## 20.17 Photon rate, optical power and quantum efficiency

Laser output power is a macroscopic energy flow, while laser action occurs photon by photon. If a monochromatic beam of power  $P$  has photon energy  $hf$ , the photon emission rate is

$$\dot{N}_\gamma = \frac{P}{hf} = \frac{P\lambda}{hc}. \quad (20.86)$$

Even a milliwatt laser therefore contains an enormous photon flux.

### Example 20.23: Photon rate from a red laboratory laser

A red laser emits  $P = 2.0$  mW at  $\lambda = 633$  nm. Estimate the number of photons emitted per second.

**Solution.** The energy per photon is

$$E_\gamma = \frac{hc}{\lambda} \approx \frac{(6.626 \times 10^{-34})(3.00 \times 10^8)}{633 \times 10^{-9}} \approx 3.14 \times 10^{-19} \text{ J}. \quad (20.87)$$

Thus

$$\dot{N}_\gamma = \frac{2.0 \times 10^{-3}}{3.14 \times 10^{-19}} \approx 6.4 \times 10^{15} \text{ s}^{-1}. \quad (20.88)$$

**Interpretation.** A low-power beam can contain quadrillions of photons per second. Photon energy is microscopic, while optical power measures the collective energy transfer rate.

In pumped lasers it is useful to distinguish several efficiencies: absorbed pump efficiency, quantum efficiency, slope efficiency and wall-plug efficiency. They are not identical. For example, even if every absorbed pump photon ultimately contributes to one laser photon, energy conservation limits the optical conversion efficiency when the laser photon has less energy than the pump photon. Real devices have additional nonradiative, optical and electrical losses.

## 20.18 What makes laser light distinctive?

Laser light is often described as monochromatic, coherent, directional and intense. Each word needs qualification.

**Spectral concentration.** A laser can have a much narrower optical spectrum than a thermal lamp, but no real laser is perfectly monochromatic. Finite cavity lifetime, spontaneous emission, technical noise, Doppler effects and other processes produce a finite linewidth.

**Temporal coherence.** If the optical field maintains a predictable phase relation over a long time, interference remains observable for a large path delay. A rough coherence time is related to linewidth by

$$\tau_c \sim \frac{1}{\Delta\nu}, \quad (20.89)$$

up to convention-dependent numerical factors. A corresponding coherence length is of order  $L_c \sim c\tau_c$  in vacuum.

**Spatial coherence.** A laser operating in a well-defined transverse mode can exhibit a stable phase relation across the beam profile, enabling tight focusing and clean interference patterns.

**Directionality.** Resonator geometry favors propagation close to the cavity axis. Nevertheless, diffraction prevents an ideal beam from remaining perfectly parallel. The advanced section develops the Gaussian-beam estimate of divergence.

**Brightness and intensity.** Lasers can concentrate substantial power into a small area and small solid angle, but “laser” does not automatically mean high power. A low-power alignment laser is still a laser because of the underlying gain and mode physics.

### Key message

A laser is defined by its physical mechanism, not by one visual property. Narrow linewidth, coherence and directionality are consequences of stimulated emission plus resonant mode selection; high optical power is optional rather than defining.

## 20.19 Representative laser systems and applications

Different laser technologies implement the same conceptual architecture using very different active media and pumps. The examples below are intended to connect the state-transition picture to engineering systems rather than to provide a catalogue of commercial devices.

### 20.19.1 Helium–neon laser

A helium–neon laser uses an electrical discharge in a gas mixture. Excited helium atoms efficiently transfer energy to selected excited neon states through collisions. Neon provides the lasing transitions, while helium helps create the required population distribution. The familiar red line near 633 nm is one possible output. The system is a good teaching example because the pump, energy transfer, metastable behavior and optical cavity can be discussed separately.

### 20.19.2 Solid-state lasers

In solid-state lasers, active ions are embedded in a crystalline or glass host. Optical pumping excites the active ions, and the host determines thermal, mechanical and spectroscopic properties. Solid-state systems can generate continuous or pulsed output and are widely used in manufacturing, measurement and scientific instrumentation.

### 20.19.3 Semiconductor lasers

A semiconductor laser uses electron and hole populations in a forward-biased junction or quantum-well structure. Recombination across the band structure provides photons, and optical feedback is supplied by cleaved facets or engineered resonators. A full semiconductor-laser treatment belongs with solid-state physics, but the same basic requirement remains: optical gain must exceed loss for selected cavity modes. Semiconductor lasers are central to optical communication, barcode readers, data storage and integrated photonics.

### 20.19.4 Fiber lasers

Rare-earth-doped optical fibers can act simultaneously as gain medium and waveguide. Long interaction lengths, efficient pump coupling and excellent heat handling make fiber lasers important in telecommunications and high-power industrial systems.

### 20.19.5 Ultrashort-pulse lasers

A laser can also operate with many longitudinal modes phase-locked to one another. Their interference produces a train of short pulses. This mode-locking concept connects frequency-domain mode structure to time-domain pulses and leads to femtosecond-scale sources used in spectroscopy, microscopy and precision frequency metrology. Detailed pulse formation and nonlinear optics are beyond the main chapter, but the connection illustrates how cavity modes become engineering resources.

| Laser family  | Typical active medium                 | Main technical connection                              |
|---------------|---------------------------------------|--------------------------------------------------------|
| Gas           | atoms or molecules in discharge       | alignment, metrology, spectroscopy, education          |
| Solid state   | dopant ions in crystal/-glass         | machining, medical systems, pulsed science             |
| Semiconductor | electron-hole states in semiconductor | communications, storage, sensing, integrated photonics |
| Fiber         | doped optical fiber                   | telecommunications, amplification, industrial power    |
| Dye / tunable | organic molecules in solution         | tunable spectroscopy and specialized sources           |

### 20.19.6 Spectroscopy and chemical sensing

Atomic and molecular spectroscopy identifies materials through transition frequencies and line shapes. Narrowband lasers can probe one transition at a time, measure Doppler shifts, detect trace gases and characterize plasmas. In remote sensing and atmospheric science, wavelength selection allows a laser to target an absorption line of a species of interest.

### 20.19.7 Interferometry and precision metrology

Long coherence length and stable optical frequency make lasers natural sources for interferometers. Length changes much smaller than an optical wavelength can be inferred from phase shifts. Frequency-

stabilized lasers also provide references for dimensional metrology and precision timing.

### 20.19.8 Communications and computing

Semiconductor lasers couple efficiently into optical fibers and can be modulated at high rates. Photodetectors convert the received optical signal back into electrical information. Modern data centers and long-distance networks therefore rely directly on controlled electronic and photonic transitions.

### 20.19.9 Medicine and materials processing

Laser energy can be localized in space, time and wavelength. Depending on wavelength and pulse duration, laser interaction with tissue or materials may involve absorption, heating, ablation, photochemical change or nonlinear processes. Safe application requires understanding both optical dose and material response; the word “laser” alone does not determine the biological or engineering effect.

## 20.20 Laboratory connection: observing atomic spectra

A compact atomic-spectroscopy laboratory can be built around a discharge lamp, a diffraction grating and a calibrated detector or camera. The experiment links three chapters of physics at once: atomic transitions determine the wavelengths, diffraction separates the wavelengths spatially, and detector electronics converts optical power into a recorded signal.

A basic workflow is:

1. record a spectrum from a calibration source with known lines;
2. map detector position or grating angle to wavelength;
3. record the unknown or teaching-source spectrum;
4. identify peaks after subtracting background;
5. estimate each peak wavelength and uncertainty;
6. convert wavelengths to photon energies using  $E = hc/\lambda$ ;
7. compare energy spacings with an atomic-level model.

The calibration step matters because a camera pixel is not itself a wavelength. If the instrument relation is approximately linear over a narrow interval,

$$\lambda(x) = a + bx, \quad (20.90)$$

where  $x$  is a pixel coordinate or angular coordinate. For wider ranges a polynomial or physically derived grating relation may be more appropriate.

### 20.20.1 Uncertainty propagation for transition energy

From  $E = hc/\lambda$ , a small wavelength uncertainty  $\sigma_\lambda$  gives approximately

$$\sigma_E \approx \left| \frac{\partial E}{\partial \lambda} \right| \sigma_\lambda = \frac{hc}{\lambda^2} \sigma_\lambda, \quad (20.91)$$

so the fractional uncertainties satisfy

$$\frac{\sigma_E}{E} \approx \frac{\sigma_\lambda}{\lambda}. \quad (20.92)$$

This simple result is useful whenever spectral peak positions are converted to energy differences.

**Example 20.24: Energy uncertainty from a measured line wavelength**

A spectral line is measured at  $\lambda = (589.0 \pm 0.2) \text{ nm}$ . Estimate the photon energy and its uncertainty.

**Solution.** Using  $hc \approx 1239.842 \text{ eV nm}$ ,

$$E \approx \frac{1239.842}{589.0} \text{ eV} \approx 2.105 \text{ eV}. \quad (20.93)$$

The fractional wavelength uncertainty is

$$\frac{\sigma_\lambda}{\lambda} \approx \frac{0.2}{589.0} \approx 3.40 \times 10^{-4}. \quad (20.94)$$

Thus

$$\sigma_E \approx E \frac{\sigma_\lambda}{\lambda} \approx (2.105)(3.40 \times 10^{-4}) \text{ eV} \approx 7.2 \times 10^{-4} \text{ eV}. \quad (20.95)$$

A suitable result is therefore  $E = (2.1050 \pm 0.0007) \text{ eV}$ , subject to systematic calibration errors not included in the quoted wavelength uncertainty.

**Method**

When analyzing a measured spectrum, separate *resolution* from *accuracy*. A narrow instrumental peak does not guarantee that the wavelength scale is correctly calibrated, and an accurately calibrated spectrometer may still be unable to resolve two lines that are closer than its instrumental resolution.

**20.20.2 From pixels to calibrated transition energies**

A modern spectrum often begins as an array of detector counts indexed by pixel number rather than as a table of wavelengths. Turning that array into physics requires a chain of calibration and inference steps. A professional workflow can be organized as follows.

**Step 1: dark and background correction.** Record detector response with no intended signal and subtract dark current, ambient light and electronic offset according to the instrument protocol. If the background varies with time, propagate that uncertainty instead of treating one subtraction as exact.

**Step 2: wavelength calibration.** Measure a reference source with known spectral lines. Fit a calibration relation

$$\lambda(x) = a_0 + a_1x + a_2x^2 + \dots, \quad (20.96)$$

where  $x$  is pixel coordinate. Use the lowest polynomial order justified by residuals and instrument geometry. An unnecessarily high order can overfit calibration noise.

**Step 3: line model and fitting window.** Choose a Gaussian, Lorentzian, Voigt or instrument-specific line-spread model and include a local baseline. The model must be motivated by the physics and detector response, not selected solely because it gives a visually smooth curve.

**Step 4: parameter covariance.** Peak position, width, amplitude and baseline may be correlated. A fit should return a covariance matrix or equivalent uncertainty estimate. Reporting a peak wavelength with many decimal places while ignoring calibration error is not precision metrology.

**Step 5: transform to physical quantities.** For  $E = hc/\lambda$ , uncertainty propagation gives, for small independent wavelength uncertainty,

$$\frac{\sigma_E}{E} \simeq \frac{\sigma_\lambda}{\lambda}. \quad (20.97)$$

The sign of  $dE/d\lambda$  is negative, but standard uncertainty uses its magnitude.

**Step 6: residual analysis.** Plot residuals. Structured residuals can reveal unresolved lines, saturation, asymmetry, calibration error or an inappropriate profile. A small reduced chi-squared value by itself is not proof that the physical model is correct.

### Method

For any measured spectrum, keep three uncertainty classes conceptually separate: **statistical** uncertainty from finite counts/noise, **calibration** uncertainty from mapping detector coordinate to wavelength or frequency, and **model** uncertainty from the chosen peak/background description. Combining them requires knowing which contributions are independent and which are correlated.

### Example 20.25: Combining an uncertainty budget

Suppose the standard uncertainties in wavelength are

| Contribution               | standard uncertainty |
|----------------------------|----------------------|
| repeatability of line fit  | 0.0015 nm            |
| reference calibration      | 0.0020 nm            |
| calibration-model residual | 0.0012 nm            |
| instrument drift           | 0.0010 nm            |

If these contributions can reasonably be treated as independent, the combined standard uncertainty is

$$\sigma_\lambda = \sqrt{0.0015^2 + 0.0020^2 + 0.0012^2 + 0.0010^2} \simeq 0.0030 \text{ nm}.$$

The calculation justifies the quoted 0.003 nm scale. If calibration and drift share a common temperature dependence, however, adding them independently would be questionable.

## 20.21 Computational connection: rate equations for a pumped laser

The physical question is whether pumping can accumulate population faster than the upper laser state loses it. Consider a three-level scheme whose pump level empties rapidly. Eliminate that short-lived level and denote the remaining reservoir population by  $N_1 = N - N_2$ , with an effective one-way pump rate  $W_p$  from the reservoir to the metastable state. This transfer occurs through a separate pump transition and rapid relaxation; it is not direct resonant pumping of the laser transition. For equal degeneracies, let a weak laser-frequency field drive absorption and stimulated emission at rate  $W_s$ . Then

$$\frac{dN_2}{dt} = W_p(N - N_2) - \frac{N_2}{\tau} - W_s(2N_2 - N), \quad (20.98)$$

$$N_1 = N - N_2. \quad (20.99)$$

The rates  $W_p, W_s, 1/\tau$  have units  $\text{s}^{-1}$ . Pump transfer increases  $N_2$ , decay decreases it, and the resonant field reduces the population difference. Setting  $W_s = 0$  gives

$$\frac{N_{2,\infty}}{N} = \frac{W_p\tau}{1 + W_p\tau}, \quad N_2(t) = N_{2,\infty} + [N_2(0) - N_{2,\infty}]e^{-(W_p+1/\tau)t}. \quad (20.100)$$

For this effective three-level model, inversion requires  $N_2 > N/2$ , hence  $W_p\tau > 1$ . In a four-level laser, a rapidly emptied lower laser state allows gain before half the total atoms are transferred. These are different population bookkeeping models.

### Example 20.26: Pump strength and approach to an inverted state

In the effective three-level model, choose  $\tau = 1.0$  ms and seek an upper-state fraction 0.80 before stimulated depletion matters. Find  $W_p$  and the approach timescale. **Strategy.** First solve the steady-state fraction, then read the exponential rate.

$$0.80 = \frac{x}{1+x}, \quad x = W_p\tau = 4, \quad W_p = 4.0 \times 10^3 \text{ s}^{-1}.$$

The approach time is  $(W_p + 1/\tau)^{-1} = 0.20$  ms. From an initially empty upper state, after 0.60 ms the fraction is  $0.80(1 - e^{-3}) \simeq 0.760$ . The lifetime alone is not the approach time because pumping also changes the population.

#### 20.21.1 An exactly checkable numerical activity

Let  $u = N_2/N$ ,  $x = W_p\tau$  and  $\theta = t/\tau$ . With  $W_s = 0$ ,  $du/d\theta = x - (1+x)u$ . The following code uses only Python's standard library and prints the numerical and exact fractions.

```
from math import exp
x, h, steps = 4.0, 0.01, 60
u = 0.0
for k in range(steps):
    u += h * (x - (1 + x) * u)
theta = steps * h
exact = x / (1 + x) * (1 - exp(-(1 + x) * theta))
print(theta, u, exact, abs(u - exact))
```

Repeat with  $h = 0.005$  and twice as many steps, keeping the final time fixed. A smaller error should result. For explicit Euler integration, linear stability requires  $0 < h < 2/(1+x)$ , while  $h \leq 1/(1+x)$  preserves a monotonic, positive approach from  $u = 0$ . Accuracy generally requires a smaller step. This illustrates why a stable computation can still be inaccurate. The coupled photon-population model in Section 20.24.19 is needed to describe laser turn-on and gain clamping.

## 20.22 Laser safety and responsible laboratory practice

Laser beams can present eye, skin, fire and secondary hazards because optical power may be concentrated into a small area and because the eye can focus visible and near-infrared radiation onto the retina. Safety therefore depends on wavelength, power or pulse energy, exposure duration, beam diameter, divergence and the optical path. Reflections from mirrors, lenses, polished metal or glass can be hazardous even when the direct beam is not intentionally viewed.

IEC 60825-1 classifies laser products by accessible emission and associated optical hazard [5]. The commonly encountered IEC-style classes include Class 1, 1M, 1C, 2, 2M, 3R, 3B and 4; Class 1C addresses certain contact applications with engineered controls. Classification limits depend on wavelength and exposure conditions, so a class label must not be reconstructed from optical power alone without the relevant standard. In teaching laboratories, the practical hierarchy is simpler:

use the lowest-power source that accomplishes the experiment, keep beams below eye level when possible, terminate beams with appropriate stops, remove unnecessary specular reflectors, avoid direct viewing into a beam, use wavelength-appropriate protective eyewear when required by the local risk assessment, and follow the laboratory's written procedures.

### Caution

Never infer that a beam is eye-safe because it looks dim. Near-infrared radiation may be poorly visible or invisible while still reaching the retina, and pulsed lasers can have high peak power even when their average power seems modest. Safety decisions must follow the source classification and the laboratory's formal risk assessment.

## 20.23 Main chapter summary

### Summary

Atomic spectra reveal discrete energy differences, with photon energy  $hf = |\Delta E|$ . Orbital angular momentum is quantized according to  $L = \sqrt{\ell(\ell + 1)}\hbar$  and  $L_z = m_\ell\hbar$ . The associated electron orbital magnetic moment is measured in units of the Bohr magneton. Electron spin is intrinsic angular momentum with  $s = 1/2$  and two possible selected-axis projections. A Stern–Gerlach field gradient converts quantized magnetic projections into spatially separated atomic beams. Magnetic resonance drives transitions between magnetic-energy levels at a sharply defined frequency.

In many-electron atoms, the Pauli exclusion principle limits each one-electron state to one electron and therefore gives subshell capacities  $2(2\ell + 1)$  and shell capacities  $2n^2$ . Together with the actual energy ordering of many-electron orbitals, this produces electron configurations and the broad periodic structure of the elements. Characteristic x rays arise when inner-shell vacancies are filled and reveal element-dependent atomic energies.

Laser physics combines absorption, spontaneous emission and stimulated emission. Net optical gain requires population inversion, usually produced through multi-level pumping schemes. An optical resonator supplies feedback and mode selection; laser oscillation begins when gain overcomes loss. Real lasers have finite linewidth, finite divergence, saturation and efficiency limits. Their technological importance follows from the ability to control optical frequency, phase, direction and power with extraordinary precision.

## 20.24 Advanced Topics for Technical and Engineering Students

This section develops a deeper layer of atomic and laser physics for students who want to connect the introductory picture to modern spectroscopy, photonics, magnetic-resonance instrumentation and computational modelling. The aim is not to reproduce a full quantum-mechanics or laser-physics course. Instead, we extend the central ideas of the chapter far enough to show how the elementary formulas fit into a more systematic theoretical framework and how they become quantitative tools in real instruments.

A recurring theme will be the distinction between an *exact quantum statement* and a *semiclassical visualization*. Angular-momentum quantum numbers, selection rules and transition rates are quantum statements. Vector cones, orbit-like pictures and ray diagrams are often useful pedagogical representations, but they must not be interpreted as literal microscopic trajectories.

### Pedagogical boundary

The enrichment material uses angular-momentum coupling, phenomenological linewidths, Einstein coefficients, Gaussian beams, rate equations, Bloch-equation ideas and uncertainty propagation. We do not derive atomic structure from the many-electron Schrödinger equation, quantum electrodynamics, or density-matrix dynamics. Whenever a phenomenological formula is introduced, its assumptions and range of validity are stated explicitly.

### 20.24.1 Coupling orbital and spin angular momentum

For a single electron in a central potential, two intrinsic angular-momentum contributions are especially important: the orbital angular momentum  $\mathbf{L}$  and the spin angular momentum  $\mathbf{S}$ . When the interaction between them is weak compared with the central binding interaction but strong enough to be spectroscopically relevant, the useful total angular momentum is

$$\mathbf{J} = \mathbf{L} + \mathbf{S}. \quad (20.101)$$

Quantum mechanics does not permit an arbitrary magnitude for  $\mathbf{J}$ . If the orbital quantum number is  $\ell$  and the spin quantum number is  $s$ , then

$$j = |\ell - s|, |\ell - s| + 1, \dots, \ell + s, \quad (20.102)$$

and

$$J = \sqrt{j(j+1)}\hbar, \quad J_z = m_j\hbar, \quad m_j = -j, -j+1, \dots, j. \quad (20.103)$$

For an electron,  $s = 1/2$ . Therefore an orbital state with  $\ell > 0$  gives two possible values,

$$j = \ell + \frac{1}{2}, \quad j = \ell - \frac{1}{2}. \quad (20.104)$$

For an  $s$  state,  $\ell = 0$  and only  $j = 1/2$  occurs.

This coupling is already enough to explain the notation used for many spectroscopic levels. A one-electron state may be written in the form

$$n^{2S+1}L_J, \quad (20.105)$$

where the letter  $L = S, P, D, F, \dots$  denotes total orbital angular momentum  $L = 0, 1, 2, 3, \dots$ , the superscript  $2S + 1$  is the spin multiplicity, and the subscript gives  $J$ . For a one-electron  $p$  state, for example,  $L = 1$  and  $S = 1/2$ , so the two levels are conventionally labeled  $^2P_{1/2}$  and  $^2P_{3/2}$ .

**Caution**

The vector addition in Eq. (20.101) does not mean that  $\mathbf{L}$  and  $\mathbf{S}$  are classical arrows with simultaneously definite Cartesian components. The vector model is a compact way to encode allowed quantum numbers and expectation-value geometry. The exact quantum statement is expressed through angular-momentum operators and their eigenvalues.

**Example 20.27: Allowed  $j$  values for a  $d$  electron**

An electron occupies a state with  $\ell = 2$ . Since  $s = 1/2$ ,

$$j = 2 \pm \frac{1}{2} = \frac{5}{2}, \frac{3}{2}.$$

The  $j = 5/2$  multiplet has  $2j + 1 = 6$  possible  $m_j$  values, while the  $j = 3/2$  multiplet has  $2j + 1 = 4$ . Thus the two fine-structure multiplets together contain ten magnetic substates, consistent with the ten spin-orbitals of a  $d$  subshell.

**Reasonableness check.** The counting agrees with  $2(2\ell + 1) = 2(5) = 10$ , so no one-electron states have been lost or double counted.

**20.24.2 Landé factor and the weak-field Zeeman effect**

The orbital and spin magnetic moments do not have the same proportionality to angular momentum. In the usual one-electron approximation,

$$\boldsymbol{\mu}_L = -\mu_B \frac{\mathbf{L}}{\hbar}, \quad \boldsymbol{\mu}_S = -g_s \mu_B \frac{\mathbf{S}}{\hbar}, \quad (20.106)$$

with  $g_s \simeq 2.0023$ . When  $\mathbf{L}$  and  $\mathbf{S}$  are coupled to  $\mathbf{J}$ , their rapidly precessing transverse contributions average in such a way that the magnetic response along  $\mathbf{J}$  can be summarized by the Landé factor

$$g_J = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}, \quad (20.107)$$

where we have used  $g_s \approx 2$  for the compact textbook expression. In a weak external field  $B$  directed along  $z$ , the first-order Zeeman shift is then

$$\Delta E_Z = \mu_B g_J m_j B. \quad (20.108)$$

The sign convention in Eq. (20.108) is the standard one for the electron-energy shift after the electron's negative charge has been incorporated into the effective  $g_J$  relation.

**Example 20.28: Landé factor for a  $^2P_{3/2}$  state**

For  $\ell = 1$ ,  $s = 1/2$  and  $j = 3/2$ ,

$$\ell(\ell+1) = 2, \quad s(s+1) = \frac{3}{4}, \quad j(j+1) = \frac{15}{4}.$$

Equation (20.107) gives

$$g_J = 1 + \frac{15/4 + 3/4 - 2}{2(15/4)} = 1 + \frac{5/2}{15/2} = \frac{4}{3}.$$

Adjacent  $m_j$  levels therefore differ by

$$\Delta E_{m_j \rightarrow m_j+1} = \mu_B g_J B.$$

For  $B = 0.40$  T this is about

$$(9.274 \times 10^{-24} \text{ J/T}) \frac{4}{3} (0.40 \text{ T}) \simeq 4.95 \times 10^{-24} \text{ J},$$

corresponding to a frequency interval  $\Delta f = \Delta E/h \simeq 7.47$  GHz.

### Guided checks

1. A state has  $\ell = 0$ ,  $s = 1/2$  and  $j = 1/2$ . Show that Eq. (20.107) gives  $g_J = 2$  within the approximation  $g_s = 2$ .
2. Why does a weak magnetic field split a  $j$  multiplet into  $2j + 1$  components?
3. Which assumption fails when the applied field becomes so strong that it competes with spin-orbit coupling?

### Example 20.29: From Zeeman splitting to a measured wavelength shift

An optical transition near 500 nm has an upper-state magnetic shift relative to the lower state of  $\Delta E_Z = \mu_B B$  for one component. Estimate the wavelength shift at  $B = 1.0$  T.

The frequency shift is

$$\Delta\nu = \frac{\mu_B B}{h} \simeq 1.40 \times 10^{10} \text{ Hz}.$$

For small shifts,  $\nu = c/\lambda$  gives

$$|\Delta\lambda| \simeq \frac{\lambda^2}{c} |\Delta\nu|.$$

Thus

$$|\Delta\lambda| \simeq \frac{(500 \times 10^{-9})^2}{2.998 \times 10^8} (1.40 \times 10^{10}) \simeq 1.17 \times 10^{-11} \text{ m} = 0.0117 \text{ nm}.$$

Resolving such a shift near 500 nm requires resolving power of order

$$R \sim \frac{500}{0.0117} \simeq 4.3 \times 10^4,$$

illustrating why high-resolution spectroscopy is needed for modest Zeeman splittings in optical lines.

Figure 20.13 makes the signed level shifts visible. For an optical transition,  $h\delta\nu = \mu_B B(g_u m_u - g_l m_l)$ ; measuring an optical line separation cannot automatically be interpreted as  $g_J \mu_B B/h$  for a single multiplet.

### 20.24.3 Fine structure, spin-orbit interaction and field regimes

A nonrelativistic central-potential model is not the complete description of atomic energy levels. Even in hydrogen, smaller corrections split levels that would otherwise be degenerate. Collectively these corrections are called *fine structure*. Three ideas contribute in a relativistic quantum treatment: a relativistic correction to kinetic energy, spin-orbit coupling and, for  $s$  states, the Darwin term. At the level appropriate here, the most physically transparent piece is spin-orbit coupling.

In the electron's rest-frame intuition, motion through the nuclear electric field is associated with an effective magnetic field. The electron spin magnetic moment interacts with this effective field. A schematic Hamiltonian therefore has the form

$$H_{\text{SO}} = \xi(r) \mathbf{L} \cdot \mathbf{S}, \quad (20.109)$$

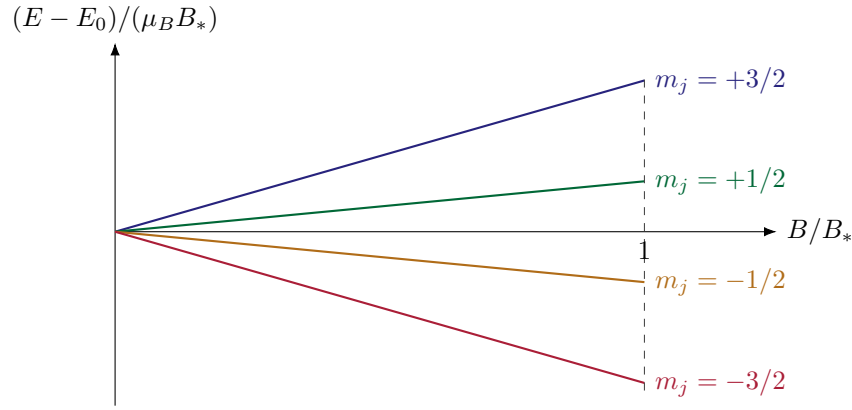


**Figure 20.13:** Linear Zeeman fan for a  $^2P_{3/2}$  level with  $g_J = 4/3$ . The scale  $B_*$  must be chosen within the weak-field regime. The four slopes are  $g_J m_j = \pm 2, \pm 2/3$ . These level separations differ from an optical line splitting, which depends on both upper and lower shifts.

where  $\xi(r)$  depends on the radial electric field and the atomic state. The scalar product can be rewritten using

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2} (J^2 - L^2 - S^2). \quad (20.110)$$

Thus levels with different  $j$  acquire different energies even when they share the same  $n$  and  $\ell$ .

For sufficiently weak external magnetic fields, spin–orbit coupling keeps  $\mathbf{L}$  and  $\mathbf{S}$  coupled, and  $j, m_j$  remain the natural quantum numbers. This is the weak-field Zeeman regime. In much stronger fields, the magnetic interaction can become larger than the spin–orbit interaction. Then  $\mathbf{L}$  and  $\mathbf{S}$  respond more independently to the field, and  $m_\ell, m_s$  become more natural labels. This limiting behavior is called the Paschen–Back regime.

#### Key message

A “good quantum number” is not merely a preferred notation. It identifies a quantity associated with a symmetry or an operator that approximately commutes with the Hamiltonian in the regime being considered. Changing the relative size of interactions can change which quantum numbers are useful.

#### 20.24.4 Selection rules in angular-momentum language

The elementary rule  $\Delta\ell = \pm 1$  is a useful first guide to electric-dipole transitions. A more complete angular-momentum description uses total  $J$ . For an electric-dipole (E1) transition in the usual weak-field coupling regime, commonly encountered rules include

$$\Delta J = 0, \pm 1, \quad J = 0 \not\leftrightarrow J' = 0, \quad (20.111)$$

$$\Delta m_J = 0, \pm 1, \quad \Delta S \simeq 0. \quad (20.112)$$

In addition, the spatial parity must change. The  $\Delta m_J$  value is connected to photon polarization relative to the chosen quantization axis:  $\Delta m_J = 0$  is conventionally called a  $\pi$  transition, while  $\Delta m_J = \pm 1$  correspond to  $\sigma^\pm$  components.

A transition that violates the leading E1 rules is not necessarily impossible. It may proceed through a weaker magnetic-dipole, electric-quadrupole or higher-order process. Such transitions can have much longer lifetimes, which is why some “forbidden” transitions are exceptionally useful in precision spectroscopy and atomic clocks.

**Method**

When analyzing a proposed atomic transition, use four separate filters:

1. **Energy:** is  $h\nu = |E_i - E_f|$  satisfied?
2. **Angular momentum:** are the relevant  $\Delta J$ ,  $\Delta m_J$  and spin conditions satisfied for the coupling mechanism?
3. **Parity:** does the required parity behavior hold?
4. **Timescale:** even if the leading process is suppressed, could a weaker multipole transition occur on an experimentally relevant timescale?

This prevents the common mistake of treating “forbidden” as synonymous with “mathematically impossible.”

**20.24.5 Hyperfine structure: coupling electrons to the nucleus**

Fine structure organizes the coupling of an electron’s orbital and spin angular momenta. At a still smaller energy scale, the nucleus can carry its own spin angular momentum  $\mathbf{I}$  and magnetic moment. The interaction of the electronic magnetic field with the nuclear magnetic moment produces *hyperfine structure*. A useful total angular momentum for an atomic level is then

$$\mathbf{F} = \mathbf{I} + \mathbf{J}, \quad (20.113)$$

with quantum numbers

$$F = |I - J|, |I - J| + 1, \dots, I + J, \quad m_F = -F, -F + 1, \dots, F. \quad (20.114)$$

The repeated pattern is worth noticing. Quantum angular momenta combine according to the same triangle rule, whether the ingredients are  $\mathbf{L} + \mathbf{S}$  or  $\mathbf{I} + \mathbf{J}$ .

For a simple magnetic-dipole hyperfine interaction, a schematic Hamiltonian is

$$H_{\text{hfs}} = A_{\text{hfs}} \frac{\mathbf{I} \cdot \mathbf{J}}{\hbar^2}, \quad (20.115)$$

where the energy-valued hyperfine constant  $A_{\text{hfs}}$  depends on the electronic wave function and nuclear properties. Using

$$\mathbf{I} \cdot \mathbf{J} = \frac{1}{2}(F^2 - I^2 - J^2), \quad (20.116)$$

we obtain the level shift

$$E_F = \frac{A_{\text{hfs}}}{2} [F(F + 1) - I(I + 1) - J(J + 1)] \quad (20.117)$$

in this explicitly stated convention. Spectroscopic tables often quote  $A_{\text{hfs}}/h$  in hertz. Electric-quadrupole hyperfine terms are omitted here; the magnetic-dipole term alone is not a universal model for nuclei with  $I \geq 1$ .

The ground-state hyperfine splitting of cesium-133 is the basis of the SI definition of the second discussed earlier. That fact makes hyperfine structure a remarkable bridge between nuclear magnetic properties, electron quantum states and macroscopic timekeeping.

**Example 20.30: Counting hyperfine levels**

Suppose an atom has nuclear spin  $I = 3/2$  and an electronic level with  $J = 1/2$ . The allowed total values are

$$F = |3/2 - 1/2|, \dots, 3/2 + 1/2 = 1, 2.$$

The  $F = 1$  level has  $2F + 1 = 3$  magnetic substates, while  $F = 2$  has five. The total is eight states. This equals

$$(2I + 1)(2J + 1) = 4 \times 2 = 8,$$

which is the expected number of product states before coupling. Angular-momentum coupling reorganizes the basis but preserves the total state count.

**Hyperfine Zeeman structure.** An external magnetic field can further split the  $m_F$  substates. At weak field,  $F$  remains a useful quantum number; at sufficiently strong field, electronic and nuclear angular momenta increasingly decouple. As before, the useful coupling scheme depends on the competition between internal and external interactions.

### 20.24.6 Einstein coefficients and the origin of stimulated emission

Consider two atomic levels, 1 and 2, separated by  $E_2 - E_1 = h\nu$ . Let a radiation field have spectral energy density  $\rho(\nu)$ , defined as electromagnetic energy per unit volume per unit frequency interval. Einstein described three transition rates:

$$R_{1 \rightarrow 2}^{\text{abs}} = N_1 B_{12} \rho(\nu), \quad (20.118)$$

$$R_{2 \rightarrow 1}^{\text{stim}} = N_2 B_{21} \rho(\nu), \quad (20.119)$$

$$R_{2 \rightarrow 1}^{\text{spont}} = N_2 A_{21}. \quad (20.120)$$

Here  $A_{21}$  has units of inverse time. The  $B$  coefficients depend on the convention used for spectral energy density, so quoted numerical values must always be accompanied by a definition of  $\rho$ .

At thermal equilibrium, detailed balance requires upward and downward rates to match:

$$N_1 B_{12} \rho = N_2 B_{21} \rho + N_2 A_{21}. \quad (20.121)$$

For degeneracies  $g_1$  and  $g_2$ , Boltzmann statistics give

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/k_B T}. \quad (20.122)$$

Requiring consistency with Planck's blackbody spectrum,

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1}, \quad (20.123)$$

leads, for this frequency-density convention, to

$$g_1 B_{12} = g_2 B_{21}, \quad \frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}. \quad (20.124)$$

These relations are historically important because stimulated emission is not an optional extra added to laser theory. It is required for a consistent equilibrium theory of radiation and matter.

#### Caution

Einstein  $B$  coefficients look different in different books because authors may use energy density per unit frequency, per unit angular frequency, per unit wavelength, or photon flux. Never transplant a  $B$  coefficient between conventions without transforming the spectral quantity as well.

**Example 20.31: Spontaneous lifetime and transition rate**

Suppose an excited state has radiative lifetime  $\tau = 25$  ns and one dominant decay channel. Then

$$A_{21} \simeq \frac{1}{\tau} = \frac{1}{25 \times 10^{-9} \text{ s}} = 4.0 \times 10^7 \text{ s}^{-1}.$$

For an ensemble containing  $N_2 = 3.0 \times 10^8$  atoms in that state, the initial spontaneous photon emission rate is approximately

$$R_\gamma(0) = N_2 A_{21} = 1.2 \times 10^{16} \text{ s}^{-1},$$

provided the emitted photons escape and reabsorption is negligible. The total rate subsequently decreases as the upper-state population decays.

**20.24.7 Natural, Doppler and collision broadening**

Real spectral lines have finite width. The observed profile contains information about lifetime, temperature, pressure, motion and the instrument itself. Three broadening mechanisms are especially useful to separate conceptually.

**Natural broadening.** Assume an upper-state population lifetime  $\tau$  and a stable lower state, with no additional dephasing. Population decays as  $e^{-t/\tau}$ , so the radiating amplitude decays as  $e^{-t/(2\tau)}$ . Its Fourier amplitude near  $\omega_0$  is proportional to

$$\int_0^\infty e^{-t/(2\tau)} e^{i(\omega - \omega_0)t} dt = \frac{1}{1/(2\tau) - i(\omega - \omega_0)}.$$

Squaring gives a Lorentzian. Half maximum occurs at  $|\omega - \omega_0| = 1/(2\tau)$ , hence the angular-frequency FWHM is  $1/\tau$  and

$$\Delta\nu_{\text{nat}} = \frac{1}{2\pi\tau}. \quad (20.125)$$

If both levels have finite lifetimes, their decay rates add in the natural transition width. With a general optical coherence time  $T_2$ , the Lorentzian FWHM is  $1/(\pi T_2)$ ; in the lifetime-limited case  $T_2 = 2\tau$ . The linewidth formula therefore follows from a stated decay model, not an arbitrary convention factor.

**Doppler broadening.** Atoms in a gas have a thermal velocity distribution. The line-of-sight velocity component Doppler shifts the observed resonance. For a gas with transition frequency  $\nu_0$ , atomic mass  $m$  and temperature  $T$ , the Gaussian Doppler FWHM is

$$\Delta\nu_D = \nu_0 \sqrt{\frac{8k_B T \ln 2}{mc^2}}. \quad (20.126)$$

The line-of-sight Maxwell distribution has variance  $k_B T/m$ . Using  $\delta\nu \simeq \nu_0 v_z/c$  gives  $\sigma_\nu = (\nu_0/c)\sqrt{k_B T/m}$ ; multiplying by the Gaussian FWHM factor  $2\sqrt{2 \ln 2}$  yields Eq. (20.126). The fractional Doppler width therefore scales as  $\sqrt{T/m}$ .

**Collision and pressure broadening.** Collisions interrupt the phase evolution of the atomic dipole and perturb the energy levels. In many regimes this produces an approximately Lorentzian contribution whose width grows with pressure. The detailed coefficient is species- and temperature-dependent and must be measured or calculated for the gas mixture under study.

When Gaussian Doppler broadening and Lorentzian lifetime/collision broadening are both important, their convolution is a *Voigt profile*. Fitting a measured line with a Voigt profile is a common laboratory task because it separates physically different broadening mechanisms.

**Example 20.32: Thermal Doppler width of a visible transition**

Estimate the Doppler FWHM for a  $\lambda_0 = 600$  nm transition of atoms with mass  $40u$  at  $T = 500$  K. The central frequency is

$$\nu_0 = \frac{c}{\lambda_0} \simeq 5.00 \times 10^{14} \text{ Hz},$$

and the atomic mass is

$$m = 40(1.6605 \times 10^{-27}) \text{ kg} \simeq 6.64 \times 10^{-26} \text{ kg}.$$

Then

$$\frac{\Delta\nu_D}{\nu_0} = \sqrt{\frac{8k_B T \ln 2}{mc^2}} \simeq 2.53 \times 10^{-6},$$

so

$$\Delta\nu_D \simeq 1.27 \times 10^9 \text{ Hz}.$$

A gigahertz-scale width is large compared with many natural linewidths. This explains why Doppler-free techniques are valuable for precision gas spectroscopy.

**Guided checks**

1. If the excited-state lifetime doubles, how does the natural linewidth scale?
2. If the gas temperature is quadrupled, by what factor does the Doppler width change?
3. Why can a measured line be wider than both the natural and Doppler widths predicted separately?
4. Why must instrument resolution be included when extracting an intrinsic linewidth from data?

**20.24.8 Coherent two-level driving and Rabi oscillations**

Rate equations describe populations when phase information can be ignored. If an atom is driven coherently by a nearly resonant monochromatic field, the quantum amplitudes of the two states evolve coherently and the population can oscillate back and forth. In the rotating-wave approximation for an ideal isolated two-level system, define the detuning

$$\Delta = \omega - \omega_0 \quad (20.127)$$

and the resonant Rabi frequency  $\Omega$ , which is proportional to the transition dipole moment and electric-field amplitude. If the atom begins in the lower state, the excited-state probability is

$$P_e(t) = \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2\left(\frac{1}{2}\sqrt{\Omega^2 + \Delta^2}t\right). \quad (20.128)$$

On exact resonance,  $\Delta = 0$ , so

$$P_e(t) = \sin^2\left(\frac{\Omega t}{2}\right). \quad (20.129)$$

A pulse satisfying  $\Omega t = \pi$  transfers an ideal atom from lower to upper state and is called a  $\pi$  pulse; a pulse with  $\Omega t = \pi/2$  creates an equal coherent superposition in the idealized model.

This behavior is fundamentally different from an incoherent exponential pumping law. The oscillation depends on phase coherence between atom and drive. Spontaneous emission, collisions, laser phase noise and inhomogeneous detuning damp the ideal oscillations in real ensembles.

**Example 20.33: Duration of an ideal resonant  $\pi$  pulse**

A coherently driven transition has resonant Rabi frequency  $\Omega/(2\pi) = 25$  kHz. Then

$$\Omega = 2\pi(25 \times 10^3) \text{ rad/s.}$$

The ideal  $\pi$ -pulse duration is

$$t_\pi = \frac{\pi}{\Omega} = \frac{1}{2(25 \times 10^3)} = 20 \text{ } \mu\text{s}.$$

If the drive is detuned, a pulse of this same duration no longer gives complete transfer because both the oscillation frequency and its amplitude are modified by Eq. (20.128).

**Caution**

Rabi oscillations require a coherent quantum treatment. They should not be inserted into an ordinary Einstein-rate model by simply replacing a transition rate with  $\Omega$ . Rate equations and coherent amplitude equations answer different dynamical questions.

**20.24.9 Saturation and power broadening in driven spectroscopy**

A weak probe measures an atomic resonance with minimal disturbance. A strong resonant field can substantially change the populations and coherence of the transition being measured. This is *saturation*. In a simple two-level picture, the excitation rate grows with intensity at low power but cannot increase without limit because the atom spends a substantial fraction of time in the excited state and stimulated emission competes with absorption.

A commonly used phenomenological saturation parameter is

$$s = \frac{I}{I_{\text{sat}}}, \quad (20.130)$$

where  $I_{\text{sat}}$  is a transition-dependent saturation intensity. For a homogeneously broadened two-level transition, the resonant scattering or excitation response often contains a factor of the form

$$\frac{s}{1+s}. \quad (20.131)$$

At  $s \ll 1$ , the response is linear in intensity. At  $s \gg 1$ , the response approaches a limit.

Strong driving also broadens the observed homogeneous resonance. A standard two-level result has the qualitative scaling

$$\Delta\nu_{\text{eff}} \propto \Delta\nu_0 \sqrt{1+s}, \quad (20.132)$$

with convention-dependent prefactors and a precise definition tied to the optical Bloch equations. This *power broadening* creates an experimental trade-off: more optical power improves signal strength up to a point but can degrade spectral resolution and shift the line through additional effects.

**Example 20.34: Recognizing saturation experimentally**

A student records a fluorescence signal while increasing laser power by factors of two. At low power the signal doubles each time. Above a certain power, doubling the laser increases fluorescence by only 20%, while the fitted linewidth grows. This behavior is inconsistent with a purely linear detector-limited response and is qualitatively consistent with atomic saturation

plus power broadening. A careful experiment would still test detector linearity separately before attributing the effect entirely to the atoms.

### Caution

A broadened line at high optical power is not automatically proof of textbook two-level power broadening. Laser-frequency noise, heating, detector saturation, unresolved light shifts and optical pumping can produce similar symptoms. Vary one control parameter at a time and use independent diagnostics.

## 20.24.10 AC Stark shifts and light as a perturbing field

A static electric field can shift atomic energies through the Stark effect. An oscillating optical field can also shift levels even when the light is not exactly resonant. This is the *AC Stark shift* or *light shift*. In a simplified two-level, far-detuned picture, the shift scales approximately as

$$\Delta E_g \simeq \frac{\hbar|\Omega|^2}{4\Delta}, \quad \Delta E_e \simeq -\frac{\hbar|\Omega|^2}{4\Delta}, \quad (20.133)$$

where  $\Omega$  is an angular Rabi frequency proportional to electric-field amplitude and  $\Delta = \omega - \omega_0$  is detuning. These level shifts use  $\Delta = \omega - \omega_0$ , the rotating-wave approximation, and  $|\Delta| \gg |\Omega|$  with decay negligible. The factor  $\hbar$  is essential for energy units. For red detuning ( $\Delta < 0$ ), the ground-state shift is negative; real multilevel atoms require contributions from all coupled levels.

This effect has two opposite roles. It is a source of systematic error in precision clocks and spectroscopy because the probe light can shift the very transition being measured. It is also a control tool: spatially varying light intensity creates spatially varying energy shifts, allowing optical dipole traps and optical lattices to confine neutral atoms.

**Measurement lesson.** If a measured line center changes when probe intensity changes, extrapolation toward zero probe intensity can help estimate an unperturbed frequency, provided other effects are controlled. Precision measurement is often the art of deliberately varying a nuisance parameter to expose its systematic influence.

## 20.24.11 Doppler-free saturated-absorption spectroscopy

Thermal Doppler broadening can hide narrow natural or hyperfine structure. One widely used way to recover narrow features is saturated-absorption spectroscopy. A strong pump beam and a weak probe beam propagate in opposite directions through the same vapor. For most laser frequencies, the pump and probe interact resonantly with different velocity classes because their Doppler shifts have opposite signs.

At the unshifted line center, however, atoms with nearly zero velocity component along the beams are resonant with both pump and probe. The strong pump reduces the population difference available to absorb the weak probe. The probe therefore sees a narrow reduction in absorption at line center, often called a *Lamb dip*, superimposed on the broad Doppler background.

The method does not cool the gas or eliminate thermal motion. It uses counterpropagating geometry and saturation to select a narrow velocity class. The resulting feature can approach a homogeneous linewidth scale and provides a much sharper frequency reference than the full Doppler-broadened profile.

If several hyperfine transitions lie within one Doppler envelope, additional crossover resonances can occur when one velocity class is resonant with different transitions for the pump and probe. These features are useful experimentally but can complicate line identification.

**Method**

When interpreting a Doppler-free spectrum, do not assume every narrow peak is a true atomic transition. Identify expected hyperfine components, possible crossover resonances, power broadening and residual magnetic-field shifts before assigning frequencies.

**20.24.12 Coherence time, coherence length, quality factor and finesse**

A perfectly monochromatic wave would require an infinite observation time. Real optical fields have finite spectral width  $\Delta\nu$ , and therefore finite temporal coherence. An order-of-magnitude connection is

$$\tau_c \sim \frac{1}{\Delta\nu}, \quad L_c \sim \frac{c}{n_r} \tau_c, \quad (20.134)$$

where factors of  $\pi$  depend on the chosen linewidth and correlation-function convention. The coherence length estimates the path-length difference over which stable interference can be maintained.

For a stationary Lorentzian power spectrum with frequency FWHM  $\Delta\nu$ , the normalized first-order field correlation is

$$|g^{(1)}(\delta t)| = e^{-\pi\Delta\nu|\delta t|}. \quad (20.135)$$

Define  $\tau_c$  as its  $1/e$  time:  $\tau_c = 1/(\pi\Delta\nu)$  and the vacuum coherence length is  $c\tau_c$ . In a balanced interferometer without polarization mismatch, fringe visibility equals  $|g^{(1)}|$  at the arm delay. For  $\Delta\nu = 200$  kHz, this convention gives  $\tau_c = 1.59$   $\mu\text{s}$  and  $L_c = 477$  m. The rough  $1/\Delta\nu$  scale differs by  $\pi$  and must not be interpreted as the same precisely defined time.

For a resonator centered at frequency  $\nu_0$  with resonance FWHM  $\delta\nu$ , the quality factor is

$$Q = \frac{\nu_0}{\delta\nu}. \quad (20.136)$$

If the resonator modes are separated by the free spectral range  $\Delta\nu_{\text{FSR}}$ , its finesse is

$$\mathcal{F} = \frac{\Delta\nu_{\text{FSR}}}{\delta\nu}. \quad (20.137)$$

The quantities  $Q$  and  $\mathcal{F}$  describe different aspects of a resonator. A high finesse means each resonance is narrow compared with the spacing to neighboring longitudinal modes, while  $Q$  compares the resonance frequency itself with its width.

**Example 20.35: Cavity finesse from measured peaks**

A scanning cavity shows resonances separated by 1.20 GHz, each with FWHM 12.0 MHz. Then

$$\mathcal{F} = \frac{1.20 \times 10^9}{12.0 \times 10^6} = 100.$$

If the light frequency is  $4.74 \times 10^{14}$  Hz, the quality factor associated with the measured linewidth is

$$Q = \frac{4.74 \times 10^{14}}{12.0 \times 10^6} \simeq 3.95 \times 10^7.$$

The two large but different dimensionless numbers should not be interchanged.

**20.24.13 Fabry–Pérot transmission and the Airy resonance**

The earlier cavity discussion emphasized resonance frequencies, FSR and finesse. The frequency-dependent transmission of an ideal symmetric lossless Fabry–Pérot cavity gives a more quantitative

picture. For mirror intensity reflectivity  $R$  and round-trip phase  $\delta$ , the normalized transmitted intensity can be written

$$\mathcal{T}(\delta) = \frac{1}{1 + \mathcal{F}_c \sin^2(\delta/2)}, \quad \mathcal{F}_c = \frac{4R}{(1 - R)^2}, \quad (20.138)$$

where  $\mathcal{F}_c$  is sometimes called the coefficient of finesse. It should not be confused with the cavity finesse  $\mathcal{F} = \Delta\nu_{\text{FSR}}/\delta\nu$ .

For a high-reflectivity symmetric cavity with negligible other losses,

$$\mathcal{F} \simeq \frac{\pi\sqrt{R}}{1 - R}. \quad (20.139)$$

As  $R$  approaches one, resonances become narrower and finesse increases. The physical reason is repeated interference: light makes many effective round trips before leaking out, so the cavity strongly discriminates phase and therefore frequency.

#### Example 20.36: Reflectivity needed for moderate finesse

For  $R = 0.95$ , the high-reflectivity estimate gives

$$\mathcal{F} \simeq \frac{\pi\sqrt{0.95}}{0.05} \simeq 61.$$

If the FSR is 1.0 GHz, the expected resonance width is of order

$$\delta\nu \simeq \frac{1.0 \text{ GHz}}{61} \simeq 16 \text{ MHz}.$$

Real mirror loss, mode mismatch and imperfect alignment can reduce the observed finesse below the ideal estimate.

A scanning Fabry–Pérot is therefore a practical diagnostic for laser longitudinal modes and relative linewidth. It does not usually give an absolute optical frequency by itself; that requires an independently calibrated frequency reference or known cavity length with sufficient accuracy.

#### 20.24.14 Laser frequency noise and the meaning of linewidth

A single quoted laser linewidth can hide different noise processes. Fundamental spontaneous-emission noise contributes phase diffusion, while technical sources include current noise, cavity vibration, temperature drift, acoustic coupling and electronic feedback noise. These processes act on different timescales.

If the optical phase is written

$$\phi(t) = 2\pi\nu_0 t + \delta\phi(t), \quad (20.140)$$

then instantaneous frequency fluctuations are related to the phase derivative,

$$\delta\nu(t) = \frac{1}{2\pi} \frac{d\delta\phi}{dt}. \quad (20.141)$$

Slow drift broadens a spectrum measured over long observation times even if the short-term line is narrow. Conversely, a fast spectrum analyzer may show a narrower feature than a long-term frequency counter would suggest.

For this reason, precision laser specifications often use frequency-noise spectral density, Allan deviation, integrated linewidth or linewidth measured with a stated observation time rather than one context-free number. The metrology lesson is general: a stability specification must include the timescale over which stability is defined.

**Guided checks**

1. What physical information is present in Rabi oscillations that is absent from a population-only rate equation?
2. Why do counterpropagating pump and probe beams isolate a narrow velocity class at line center?
3. What is the difference between the coefficient of finesse in the Airy formula and the experimentally defined cavity finesse?
4. Why can a laser have different apparent linewidths when measured over different observation times?

**20.24.15 Gaussian-beam optics and the diffraction limit of laser beams**

A real laser beam cannot remain perfectly parallel. Even an ideal fundamental transverse mode diffracts. For a paraxial Gaussian beam, the radius  $w(z)$  at which the intensity has fallen to  $1/e^2$  of its on-axis value is

$$w(z) = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2}, \quad z_R = \frac{\pi w_0^2}{\lambda}, \quad (20.142)$$

where  $w_0$  is the waist radius and  $z_R$  is the Rayleigh range, as shown in Fig. 20.14. Far from the waist,

$$\theta \simeq \frac{\lambda}{\pi w_0} \quad (20.143)$$

for the ideal half-angle divergence. Real beams are often characterized by a beam-quality factor  $M^2 \geq 1$ ,

$$\theta \simeq M^2 \frac{\lambda}{\pi w_0}. \quad (20.144)$$

The product of spot size and angular spread is therefore constrained by diffraction. Focusing a beam to a smaller waist necessarily increases its divergence. For a real beam,  $M^2$  is rigorously defined using second-moment widths. When a Gaussian-envelope approximation is appropriate, use  $w(z)^2 \simeq w_0^2 + (\theta z)^2$  with the measured  $M^2$ ; a non-Gaussian profile need not have a unique  $1/e^2$  radius.

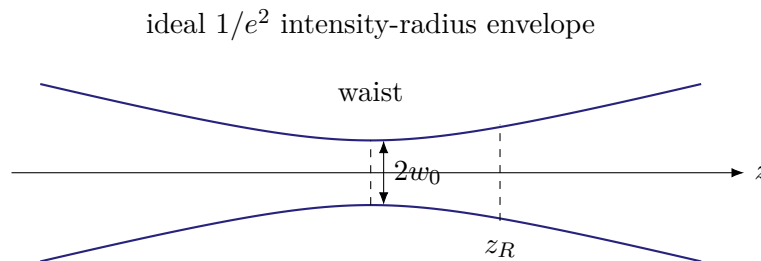


**Figure 20.14:** Schematic Gaussian-beam envelope. The Rayleigh range marks the distance at which the beam radius has grown to  $\sqrt{2} w_0$ .

**Example 20.37: Choosing a beam waist**

A  $\lambda = 1064$  nm laser should have an ideal far-field half-angle divergence no larger than 0.50 mrad. For an  $M^2 = 1$  beam,

$$w_0 \geq \frac{\lambda}{\pi \theta} = \frac{1.064 \times 10^{-6}}{\pi(5.0 \times 10^{-4})} \simeq 6.8 \times 10^{-4} \text{ m}.$$

Thus the waist radius should be at least about 0.68 mm. If the measured beam has  $M^2 = 1.5$ , the required radius grows to about 1.0 mm for the same divergence target.

### 20.24.16 Transverse resonator modes and two-mirror cavity stability

Longitudinal modes describe the standing-wave condition along the optical axis. Real resonators also support transverse field structures. In a paraxial two-mirror resonator, these can be represented by Hermite–Gaussian or Laguerre–Gaussian mode families. The fundamental mode has a Gaussian transverse profile, while higher-order modes contain nodes and broader spatial structure.

For a cavity of length  $L$  with mirror radii of curvature  $\mathcal{R}_1$  and  $\mathcal{R}_2$ , positive for the concave mirrors considered here, define the geometry parameters

$$g_{c1} = 1 - \frac{L}{\mathcal{R}_1}, \quad g_{c2} = 1 - \frac{L}{\mathcal{R}_2}. \quad (20.145)$$

An ideal paraxial two-mirror resonator is geometrically stable when

$$0 < g_{c1}g_{c2} < 1, \quad (20.146)$$

as a sufficient criterion away from degenerate geometries. Equality cases must be examined separately using the round-trip ray matrix; they are not all physically equivalent. The criterion means that a small transverse displacement or angle does not grow without bound after repeated round trips.

Several familiar geometries fit inside this classification. A symmetric confocal cavity has  $\mathcal{R}_1 = \mathcal{R}_2 = L$ , giving  $g_{c1} = g_{c2} = 0$  at a degenerate boundary of this strict criterion. Exact symmetric confocal geometry supports finite Gaussian modes, but small changes of the two mirror curvatures can move it into either a stable or unstable region. A symmetric cavity with  $\mathcal{R}_1 = \mathcal{R}_2 = 2L$ , for example, has  $g_{c1} = g_{c2} = 1/2$  and lies comfortably in the stable region.

For the positive- $g_{c1}, g_{c2}$  branch of a stable empty two-mirror resonator, the transverse-mode frequencies can be written in the paraxial approximation as

$$\nu_{qmn} = \Delta\nu_{\text{FSR}} \left[ q + \frac{m+n+1}{\pi} \cos^{-1}(\sqrt{g_{c1}g_{c2}}) \right], \quad (20.147)$$

where  $q$  is the longitudinal index and  $m, n$  are nonnegative transverse indices. The second term is a Gouy-phase contribution. For the negative- $g_{c1}, g_{c2}$  branch, a different Gouy-phase branch is required; the positive-square-root expression is not asserted for that branch. It explains why transverse modes are generally not degenerate with the fundamental mode even when they share the same approximate longitudinal wavelength.

#### Caution

Cavity-stability formulas assume the paraxial approximation, ideal mirror curvature and a simple two-mirror geometry. Thermal lensing, astigmatism, apertures and intracavity optical elements modify the effective resonator. For precision design, use an ABCD-matrix model or full optical simulation appropriate to the device.

#### Example 20.38: Checking a symmetric cavity

A cavity has length  $L = 0.30$  m and two identical concave mirrors with  $\mathcal{R} = 0.50$  m. Then

$$g_{c1} = g_{c2} = 1 - \frac{0.30}{0.50} = 0.40, \quad g_{c1}g_{c2} = 0.16.$$

Because  $0 < 0.16 < 1$ , the ideal cavity is stable. The transverse Gouy factor is

$$\frac{1}{\pi} \cos^{-1}(\sqrt{0.16}) = \frac{1}{\pi} \cos^{-1}(0.40) \simeq 0.369.$$

Thus increasing  $m + n$  by one shifts a transverse mode by about 0.369 of one FSR relative

to the corresponding longitudinal index. A scanning cavity can therefore show additional resonances if higher transverse modes are excited by imperfect mode matching.

### Key message

A laser resonator selects both frequency and spatial structure. Longitudinal mode indices describe phase accumulation along the axis; transverse mode indices describe diffraction and focusing across the beam. Good laser engineering requires control of both.

#### 20.24.17 Spot size, irradiance and detector linearity

Optical power alone does not determine how strongly a detector or material is illuminated. Irradiance depends on area. For a Gaussian beam with total power  $P$  and  $1/e^2$  radius  $w$ , the transverse intensity distribution is

$$I(r) = I_0 e^{-2r^2/w^2}, \quad I_0 = \frac{2P}{\pi w^2}. \quad (20.148)$$

The peak irradiance is twice the power divided by the circle area  $\pi w^2$  because the Gaussian is concentrated near the axis.

#### Example 20.39: Same power, very different peak irradiance

A  $P = 10$  mW beam first has  $w = 1.0$  mm. Its on-axis irradiance is

$$I_0 = \frac{2(0.010)}{\pi(1.0 \times 10^{-3})^2} \simeq 6.37 \times 10^3 \text{ W/m}^2.$$

If a lens focuses the same power to  $w = 20$   $\mu\text{m}$ ,

$$I_0 \simeq \frac{0.020}{\pi(20 \times 10^{-6})^2} \simeq 1.59 \times 10^7 \text{ W/m}^2.$$

Reducing beam radius by a factor of 50 increases peak irradiance by a factor of  $50^2 = 2500$ .

This scaling explains why focusing matters in laser processing, microscopy and detector protection. It also explains why safety cannot be inferred from power alone: wavelength, exposure time, beam diameter, divergence, pulse structure and access geometry all matter. In a laboratory, optical attenuation should be chosen using characterized components intended for the wavelength and power regime. Improvised absorbers can heat, scatter or fail.

**Detector linearity.** If a photodetector saturates, the recorded signal no longer scales with incident power. Spectroscopic line ratios, pulse shapes and linewidths can then be distorted. A good measurement therefore checks linearity by repeating the measurement at several attenuation levels. If fitted spectral parameters change with attenuation, the detector or acquisition chain may be nonlinear.

#### 20.24.18 Gain cross section, inversion density and threshold

A microscopic way to parameterize laser gain introduces an effective stimulated-emission cross section  $\sigma(\nu)$ . For an isolated transition with equally occupied magnetic sublevels and matching absorption/emission line shapes, with upper and lower population densities  $n_2$  and  $n_1$ , an idealized small-signal gain coefficient can be written

$$g(\nu) = \sigma(\nu)\Delta n, \quad \Delta n = n_2 - \frac{g_2}{g_1}n_1, \quad (20.149)$$

where the degeneracy factor ensures that zero net gain corresponds to equal population per substate. More detailed models distinguish absorption and emission cross sections, overlap factors, polarization and line shapes.

Combining Eq. (20.149) with the cavity threshold  $g_{\text{th}}$  gives the threshold inversion density

$$\Delta n_{\text{th}} = \frac{g_{\text{th}}}{\sigma}. \quad (20.150)$$

This relation is important in engineering because it separates two design questions: how difficult it is to create an inversion in the material, and how much inversion the optical cavity requires.

#### Example 20.40: Threshold inversion density

Suppose a laser transition has effective stimulated-emission cross section  $\sigma = 3.0 \times 10^{-20} \text{ m}^2$  and the resonator requires  $g_{\text{th}} = 0.12 \text{ m}^{-1}$ . Then

$$\Delta n_{\text{th}} = \frac{0.12}{3.0 \times 10^{-20}} = 4.0 \times 10^{18} \text{ m}^{-3}.$$

If the active-ion density is  $2.0 \times 10^{24} \text{ m}^{-3}$ , the required net inversion is only about  $2 \times 10^{-6}$  of the total active-ion density in this deliberately simplified example. Real thresholds also depend on mode overlap, internal losses and spectroscopic details.

#### 20.24.19 Two-variable laser rate equations and relaxation dynamics

A laser is a coupled dynamical system: photons deplete the upper-state population, while that population controls gain. Assume an effective four-level system with a rapidly emptied lower laser state. Let  $D$  be the number of atoms in the upper state (approximately the inversion in this limit),  $S$  the photon number in one modeled cavity mode, and  $R_p$  the net pump transfer in atoms per second. A consistent reduced model is

$$\frac{dD}{dt} = R_p - \frac{D}{\tau} - GDS, \quad (20.151)$$

$$\frac{dS}{dt} = GDS - \frac{S}{\tau_p} + \beta \frac{D}{\tau}. \quad (20.152)$$

Here  $G$  is a coupling rate per atom per photon,  $\tau$  the upper-state lifetime,  $\tau_p$  the photon-number decay time, and  $\beta$  the fraction of spontaneous decay feeding the modeled mode. Stimulated emission removes one excited atom and adds one photon, explaining the opposite signs of  $GDS$ . If  $D$  instead meant  $N_2 - N_1$  in a closed two-level system, a stimulated event would change it by two; the same equations could not be used unchanged.

For a passive uniform cavity with round-trip time  $t_{\text{rt}}$  and surviving round-trip intensity fraction  $\mathcal{R} < 1$ , photon decay obeys  $e^{-t_{\text{rt}}/\tau_p} = \mathcal{R}$ , so  $\tau_p = -t_{\text{rt}}/\ln \mathcal{R}$ . This connects the rate model to the earlier mirror-loss balance. With  $\beta = 0$ , the below-threshold state has  $S = 0$ ,  $D = R_p\tau$ . Its photon perturbation grows only when  $GD > 1/\tau_p$ . Thus

$$D_{\text{th}} = \frac{1}{G\tau_p}, \quad R_{p,\text{th}} = \frac{1}{G\tau_p\tau}. \quad (20.153)$$

Above threshold,  $D = D_{\text{th}}$  and  $S = \tau_p(R_p - R_{p,\text{th}})$ . Inversion is clamped while output grows with pump. A small  $\beta > 0$  provides a spontaneous seed and smooths the sharp threshold of the ideal  $\beta = 0$  model.

### 20.24.20 A numerical experiment: integrating laser rate equations

Normalize by

$$\theta = t/\tau, \quad n = G\tau_p D, \quad s = G\tau S, \quad p = G\tau_p \tau R_p, \quad \epsilon = \tau/\tau_p.$$

Then

$$\frac{dn}{d\theta} = p - n - ns, \quad (20.154)$$

$$\frac{ds}{d\theta} = \epsilon[(n-1)s + \beta n]. \quad (20.155)$$

The  $n, s$  here are normalized population and photon variables, distinct from atomic quantum numbers and electron spin. For  $\beta = 0$ , the steady state is  $(n, s) = (p, 0)$  below threshold and  $(1, p-1)$  above threshold. Linearization about the above-threshold state gives

$$J = \begin{pmatrix} -p & -1 \\ \epsilon(p-1) & 0 \end{pmatrix}, \quad \lambda_{\pm} = \frac{-p \pm \sqrt{p^2 - 4\epsilon(p-1)}}{2}.$$

Complex eigenvalues predict damped relaxation oscillations. For  $p = 2$  and  $\epsilon = 30$ ,  $\lambda_{\pm} = -1 \pm i\sqrt{29}$  in units of  $1/\tau$ . The envelope damps over  $\tau$  while the small-perturbation period is  $2\pi\tau/\sqrt{29} \simeq 1.17\tau$ .

Integrate the model for  $p = 0.5, 1, 2$ ,  $\epsilon = 30$  and  $\beta = 10^{-5}$  from  $(n, s) = (0, 10^{-6})$ . Use an adaptive solver or fourth-order Runge–Kutta with a step small relative to  $1/\epsilon$ . Plot populations and photons separately; their scales need not match. Compare the computed steady state with the positive solution of

$$(1 - \beta)n^2 - (p + 1)n + p = 0, \quad s = p/n - 1.$$

At zero pump the steady state is  $(n, s) = (0, 0)$ ; the expression  $s = p/n - 1$  is used for  $p > 0$ . Check positivity and repeat with tighter tolerance or half the step. A plot of a few oscillations is not enough to verify the steady state or convergence.

Figure 20.15 illustrates turn-on dynamics, while Fig. 20.16 shows the resulting pump dependence.

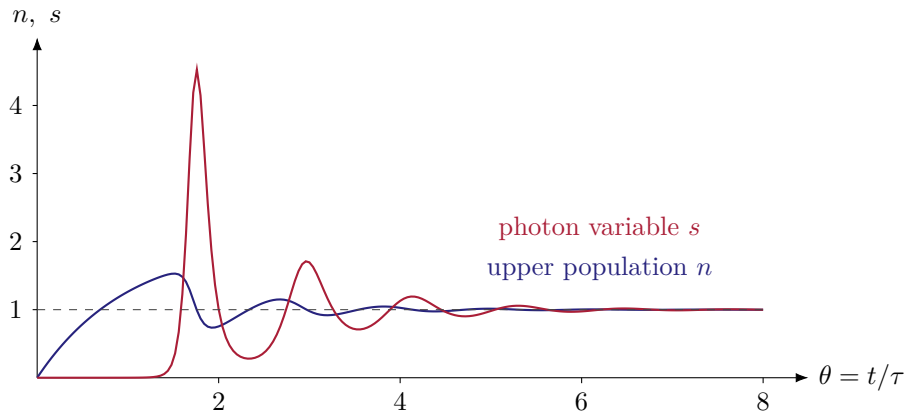


**Figure 20.15:** Computed laser turn-on for  $p = 2$ ,  $\epsilon = 30$ ,  $\beta = 10^{-5}$  and initial state  $(n, s) = (0, 10^{-6})$ . Population builds before photons grow; stimulated depletion then produces damped relaxation oscillations. Time and populations are dimensionless model quantities. The curves were integrated with a converged adaptive solver.

#### Note

Population and photon-number equations discard optical phase. They describe intensity dynamics but cannot reproduce coherent Rabi oscillations, mode locking, interference, or laser frequency noise. The mathematical model must match the measured quantity.

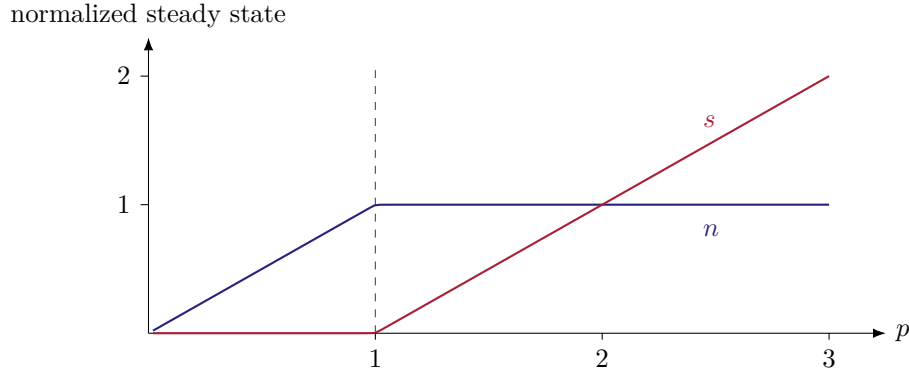


**Figure 20.16:** Steady states of the normalized four-level rate model, with  $\beta = 10^{-5}$ . Below  $p \simeq 1$ , the upper population grows with pump and photons remain small. Above that scale,  $n$  is nearly clamped at one while  $s$  grows. Curves are calculated from the physical nonnegative root of the quadratic stated in the text; they are not measured laser data.

### 20.24.21 Magnetic resonance beyond the two-level sketch: Bloch dynamics

The simplest magnetic-resonance discussion treats a spin flip as absorption of a photon with  $hf = \Delta E$ . A complementary macroscopic description tracks the sample magnetization  $\mathbf{M}$ . Neglecting relaxation for a moment, a magnetic moment in a magnetic field precesses according to

$$\frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B}, \quad (20.156)$$

where  $\gamma$  is the gyromagnetic ratio. The precession angular frequency is the Larmor frequency

$$\omega_0 = |\gamma|B_0. \quad (20.157)$$

Real samples relax toward thermal equilibrium. A phenomenological Bloch model for a static field  $B_0\hat{\mathbf{z}}$  adds two timescales:

$$\frac{dM_x}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_x - \frac{M_x}{T_2}, \quad (20.158)$$

$$\frac{dM_y}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_y - \frac{M_y}{T_2}, \quad (20.159)$$

$$\frac{dM_z}{dt} = \gamma(\mathbf{M} \times \mathbf{B})_z - \frac{M_z - M_0}{T_1}. \quad (20.160)$$

The longitudinal time  $T_1$  characterizes recovery of  $M_z$  toward equilibrium, while  $T_2$  characterizes loss of transverse phase coherence. In the simple Markovian spin-1/2 relaxation model,  $1/T_2 = 1/(2T_1) + 1/T_\phi$ , so  $T_2 \leq 2T_1$ ; additional dephasing often makes  $T_2$  shorter than  $T_1$ , but observed signals may decay faster still because of field inhomogeneity; that effective time is often denoted  $T_2^*$ .

#### Caution

$T_1$ ,  $T_2$  and  $T_2^*$  are not interchangeable.  $T_2$  describes intrinsic transverse dephasing processes in the phenomenological Bloch picture, while  $T_2^*$  also includes reversible-looking dephasing caused by static field inhomogeneity and ensemble spread.

### 20.24.22 How magnetic-field gradients encode position in MRI

Magnetic resonance becomes an imaging technique when controlled field gradients make the resonance frequency position-dependent. Suppose

$$B(z) = B_0 + G_z z. \quad (20.161)$$

For protons,

$$f(z) = \frac{\gamma_p}{2\pi} (B_0 + G_z z), \quad (20.162)$$

so the frequency shift relative to the center is

$$\Delta f(z) = \frac{\gamma_p}{2\pi} G_z z. \quad (20.163)$$

Frequency therefore carries spatial information. Real MRI combines slice selection, frequency encoding, phase encoding and Fourier reconstruction, but Eq. (20.163) captures the essential physical bridge from spin resonance to imaging.

#### Example 20.41: Frequency encoding with a field gradient

For protons,  $\gamma_p/(2\pi) \simeq 42.58$  MHz/T. A gradient  $G_z = 20$  mT/m is applied. Two regions separated by  $\Delta z = 5.0$  cm have resonance-frequency separation

$$\Delta f = \left( 42.58 \times 10^6 \frac{\text{Hz}}{\text{T}} \right) (20 \times 10^{-3} \frac{\text{T}}{\text{m}}) (0.050 \text{ m}) \simeq 4.26 \times 10^4 \text{ Hz}.$$

Thus a 5 cm spatial separation maps to about 42.6 kHz under this gradient.

### 20.24.23 Atomic clocks and resonance as a frequency standard

Atomic clocks reverse the usual spectroscopy question. Instead of asking “what is the unknown transition frequency?”, they use a well-defined atomic transition to stabilize an oscillator. An interrogation field probes atoms near a reference frequency. The measured transition probability generates an error signal, and a feedback loop steers a microwave or optical oscillator toward the atomic resonance.

The SI second is defined by fixing the unperturbed ground-state hyperfine transition frequency of cesium-133 to exactly [4]

$$9\,192\,631\,770 \text{ Hz}. \quad (20.164)$$

This is a metrological definition, not a claim that every practical cesium clock realizes the frequency without correction. Real clocks must account for magnetic fields, blackbody shifts, collisions, motion, microwave or optical field imperfections and many other systematic effects.

For any clock transition,

$$f_0 = \frac{\Delta E}{h}. \quad (20.165)$$

Precision is often expressed as a fractional frequency uncertainty or instability,

$$\frac{\delta f}{f_0}. \quad (20.166)$$

A 1 Hz error is enormous for some optical-clock applications but tiny for an audio oscillator; the fractional measure makes the comparison meaningful.

**Ramsey’s separated-field idea.** Rather than interrogating atoms continuously with one long interaction, two coherent interactions separated by a free-evolution time can create an interference pattern in transition probability versus frequency. Increasing the separation narrows the central feature, much as increasing path difference improves phase sensitivity in an interferometer. This is one of the deep connections between atomic spectroscopy and wave interference.

**Key message**

An atomic clock is not a device that “counts atomic orbits.” It is a feedback-controlled frequency standard based on a reproducible quantum transition and a carefully characterized interrogation protocol.

**Example 20.42: Atomic clock fractional error**

Suppose an oscillator intended to realize the cesium hyperfine frequency is high by 0.10 Hz. The fractional offset is

$$\frac{\delta f}{f} = \frac{0.10}{9.192631770 \times 10^9} \simeq 1.09 \times 10^{-11}.$$

If such a fractional rate error remained constant, a simple accumulated-time estimate over one day would be

$$(86400 \text{ s})(1.09 \times 10^{-11}) \simeq 9.4 \times 10^{-7} \text{ s},$$

about one microsecond per day. Precision timing translates tiny fractional frequency errors into measurable long-term timing offsets.

**20.24.24 Optical pumping and state preparation**

Light can do more than excite atoms randomly. If transition frequencies and polarizations are chosen carefully, repeated absorption and spontaneous-emission cycles can redistribute population among magnetic or hyperfine substates. This process is called *optical pumping*. It is widely used to prepare atomic ensembles before spectroscopy, magnetometry and precision measurements.

A simple conceptual example considers two ground substates, only one of which is resonantly excited by a selected light polarization. Atoms in the addressed state absorb light and decay back into both ground substates. Population gradually accumulates in the ground state that does not interact with the chosen light. That state becomes “dark” to the pumping beam. The process is a practical demonstration that optical selection rules can be used as a control tool, not only as a classification scheme.

**Note**

The phrase “dark state” has several precise meanings in modern atomic physics. In this introductory context it simply denotes a state that is not coupled by the chosen optical excitation. Coherent dark-state phenomena such as electromagnetically induced transparency require a more advanced treatment.

**20.24.25 Laser cooling: using photon momentum to control atoms**

A photon carries momentum

$$p_\gamma = \frac{h}{\lambda} = \hbar k. \quad (20.167)$$

When an atom absorbs a photon from a laser beam, its momentum changes by approximately  $+\hbar\mathbf{k}$  in the beam direction. Spontaneous emission also gives recoil, but if many emissions occur in random directions their average momentum contribution can approach zero while the absorption kicks remain directional.

Velocity selectivity arises from the Doppler effect. Suppose a laser frequency is tuned slightly below an atomic resonance. An atom moving toward the laser sees the light Doppler shifted upward, closer

to resonance, and absorbs more strongly. With counterpropagating beams, atoms moving either way can experience a force opposing their motion. This is the basis of Doppler laser cooling.

For a simple two-level model, the characteristic Doppler temperature scale is

$$T_D = \frac{\hbar\Gamma}{2k_B}, \quad (20.168)$$

where  $\Gamma = 1/\tau$  is the excited-state population decay rate in  $\text{s}^{-1}$ , equal numerically to the natural FWHM in angular frequency for a stable lower state. The frequency FWHM is  $\Gamma/(2\pi)$ . This Doppler limit assumes a closed two-level transition, low saturation and optimal red detuning; multilevel sub-Doppler cooling can reach lower temperatures.

Laser cooling is a beautiful synthesis of topics from this volume: photon momentum, Doppler shifts, resonance, spontaneous emission and statistical motion. It also prepares the conceptual ground for modern atomic clocks and quantum technologies.

#### Example 20.43: Photon recoil scale

An atom of mass  $87u$  absorbs a  $780 \text{ nm}$  photon. The photon momentum is

$$p_\gamma = \frac{h}{\lambda} \simeq 8.50 \times 10^{-28} \text{ kg m/s.}$$

The atomic velocity change from one absorption is

$$\Delta v = \frac{p_\gamma}{M} = \frac{8.50 \times 10^{-28}}{87(1.6605 \times 10^{-27})} \simeq 5.9 \times 10^{-3} \text{ m/s.}$$

One photon changes the velocity by only millimetres per second, but millions of repeated scattering events can strongly alter an atomic beam. This is the scale behind radiation-pressure cooling and trapping.

#### 20.24.26 Fourier limit for pulses: short time implies broad spectrum

A continuous monochromatic wave extends indefinitely in time. A pulse exists only over a finite time interval and therefore requires a range of frequencies. This is a Fourier property rather than an atomic peculiarity. For a transform-limited pulse, duration  $\Delta t$  and spectral width  $\Delta\nu$  obey a relation of the form

$$\Delta\nu \Delta t \geq C, \quad (20.169)$$

where the constant  $C$  depends on the definition of width and pulse shape. For a Gaussian intensity pulse using FWHM widths,

$$\Delta\nu \Delta t \simeq 0.44 \quad (20.170)$$

for a transform-limited pulse.

#### Example 20.44: Bandwidth of a femtosecond pulse

An ideal Gaussian pulse has intensity FWHM  $\Delta t = 100 \text{ fs}$ . The minimum frequency FWHM is approximately

$$\Delta\nu \simeq \frac{0.44}{100 \times 10^{-15}} = 4.4 \times 10^{12} \text{ Hz.}$$

At central wavelength  $800 \text{ nm}$ , a small-bandwidth conversion gives

$$\Delta\lambda \simeq \frac{\lambda^2}{c} \Delta\nu \simeq 9.4 \text{ nm.}$$

Thus an ultrashort pulse can be highly coherent in a deterministic phase sense while being far from monochromatic. Narrow spectrum and short duration are competing requirements.

If different frequency components acquire different phases during propagation, the pulse becomes *chirped* and can lengthen beyond the transform limit. Dispersion compensation in ultrafast laser systems is therefore a phase-engineering problem.

### 20.24.27 Optical frequency combs: linking optical and microwave scales

A mode-locked laser can produce a periodic train of ultrashort pulses. In the frequency domain, that periodicity corresponds to a set of narrow spectral lines—an optical frequency comb. The frequencies can be written

$$\nu_n = n f_{\text{rep}} + f_{\text{CEO}}, \quad (20.171)$$

where  $f_{\text{rep}}$  is the pulse repetition frequency and  $f_{\text{CEO}}$  is the carrier-envelope offset frequency. The large integer  $n$  places the comb tooth in the optical-frequency range.

The power of a frequency comb is that optical frequencies of hundreds of terahertz become connected to radio or microwave frequencies that electronics can count directly. If  $f_{\text{rep}}$  and  $f_{\text{CEO}}$  are stabilized to references, every comb tooth is known. Conversely, an unknown optical transition can be measured by beating it against the nearest comb tooth and measuring the radiofrequency difference.

This technology unifies several ideas developed in the chapter:

- cavity-like mode spacing appears as  $f_{\text{rep}}$ ;
- phase coherence across many modes allows short pulses;
- atomic transitions provide optical references;
- feedback stabilizes repetition and offset frequencies;
- electronic frequency counting turns optical spectroscopy into precision metrology.

#### Key message

A frequency comb is not simply a broadband laser spectrum. Its defining feature is a phase-coherent, evenly structured set of optical frequencies whose positions can be related to countable radiofrequency references.

### 20.24.28 Advanced case study: choosing a linewidth model from physical scales

Suppose a gas-cell transition near 600 nm has radiative lifetime 30 ns, atoms of mass  $40u$ , temperature 400 K, and an instrument resolution of 1.0 GHz. Before fitting any data, estimate the scales.

The natural width is roughly

$$\Delta\nu_{\text{nat}} \sim \frac{1}{2\pi(30 \times 10^{-9})} \simeq 5.3 \text{ MHz}.$$

The optical frequency is  $\nu_0 \simeq 5.0 \times 10^{14}$  Hz. The Doppler FWHM is

$$\Delta\nu_D = \nu_0 \sqrt{\frac{8k_B T \ln 2}{mc^2}} \simeq 1.13 \text{ GHz}.$$

The Doppler width is about 210 times the natural width and is comparable to the instrumental resolution. A first model should include a dominant Gaussian contribution plus instrument response. A pure Lorentzian would be poorly motivated unless pressure broadening were independently known to be large.

As a mathematical comparison, reduce the temperature parameter from 400 K to 4 K while holding all other model inputs fixed. A real gas may condense or cease to have the assumed equilibrium vapor density; this is not a proposed laboratory cooling procedure. Doppler width falls by  $\sqrt{4/400} = 0.1$ , to about 113 MHz in this hypothetical scaling exercise. The 1 GHz instrument would then become the dominant limitation. Improving atomic conditions without improving the spectrometer would no longer improve the measured linewidth. This is a classic example of a measurement becoming instrument-limited.

### Method

Estimate all relevant physical widths *before* choosing a fitting model. Comparing scales can tell you whether natural, Doppler, pressure, laser or instrumental broadening is likely to dominate and therefore which parameters can realistically be extracted from the data.

## 20.24.29 Advanced case study: from photon shot noise to signal-to-noise scaling

Photodetection is quantized because light arrives in discrete energy quanta. For ideal counting of a coherent beam with independent detection events and mean detected count  $N$  during an interval, Poisson statistics give

$$\sigma_N = \sqrt{N}. \quad (20.172)$$

The relative counting uncertainty is therefore

$$\frac{\sigma_N}{N} = \frac{1}{\sqrt{N}}. \quad (20.173)$$

If signal is proportional to photon number and shot noise dominates, the signal-to-noise ratio scales as  $\sqrt{N}$  rather than  $N$ .

### Example 20.45: Why four times the photons give twice the shot-noise SNR

If a measurement detects  $N = 10^6$  photons, the ideal Poisson standard deviation is  $10^3$ , so the counting SNR is  $10^3$ . To double that SNR to  $2 \times 10^3$ , one needs

$$\sqrt{N'} = 2\sqrt{N} \quad \Rightarrow \quad N' = 4N = 4 \times 10^6.$$

Thus improving a shot-noise-limited measurement by a factor of two in SNR requires four times as many detected photons, obtainable through more power, longer integration, better collection or higher detector efficiency—subject to saturation and systematic limits.

Real detectors also have dark current, read noise, gain noise and technical intensity noise. At high photon flux, systematic drift may dominate long before the Poisson limit is reached. Nevertheless, Eq. (20.173) provides a fundamental reference scale for spectroscopy and optical sensing.

## 20.24.30 Atomic magnetometry: converting resonance frequency into field

A magnetic resonance can be used not only to identify a transition but also to measure an unknown field. In the simplest linear regime,

$$f(B) = f_0 + \alpha B, \quad (20.174)$$

where the coefficient  $\alpha$  is determined by the relevant magnetic moments and quantum numbers. A measured frequency shift then gives

$$\hat{B} = \frac{f - f_0}{\alpha}. \quad (20.175)$$

If the line center can be determined with standard uncertainty  $\sigma_f$ , and the calibration coefficient is treated as known, the corresponding field uncertainty is

$$\sigma_B \simeq \frac{\sigma_f}{|\alpha|}. \quad (20.176)$$

This simple formula reveals why narrow spectral features are valuable for sensing: they can permit a more precise estimate of line center. But linewidth alone does not set sensitivity. Signal amplitude, slope, detection noise, averaging time and technical drift also matter.

Suppose a resonance signal  $S(f)$  is operated on a steep dispersive slope. A small field change produces

$$\delta S = \frac{dS}{df} \frac{df}{dB} \delta B. \quad (20.177)$$

If the measurement noise over the relevant bandwidth is  $\delta S_{\text{noise}}$ , the smallest resolvable field scale is approximately

$$\delta B_{\min} \sim \frac{\delta S_{\text{noise}}}{|dS/df| |df/dB|}. \quad (20.178)$$

Equation (20.178) is deliberately generic. It emphasizes that sensitivity improves with a steeper resonance, stronger magnetic response and lower detection noise.

**Bandwidth–sensitivity trade-off.** A long interrogation time can narrow a resonance and improve frequency precision, but it also slows the sensor response. A device intended to measure rapidly changing magnetic fields cannot average indefinitely. Atomic magnetometer design therefore balances sensitivity, dynamic range and bandwidth just as classical sensors do.

**Environmental systematics.** Temperature changes vapor density and collision rates. Laser frequency and intensity can shift or distort the atomic resonance. Magnetic gradients make different atoms sample different fields. Nearby ferromagnetic objects and current-carrying wires can create offsets. Thus a quantum magnetometer requires careful shielding, calibration and control of classical environmental variables.

#### Example 20.46: Frequency resolution translated into field resolution

A resonance has magnetic coefficient  $\alpha = 7.0 \text{ Hz/nT}$ . If repeated fits determine its center with standard uncertainty  $\sigma_f = 0.35 \text{ Hz}$ , then

$$\sigma_B \simeq \frac{0.35}{7.0} \text{ nT} = 0.050 \text{ nT} = 50 \text{ pT}.$$

This is the uncertainty implied by the line-center fit alone. A complete sensor specification must additionally include field calibration, systematic offsets, bandwidth and long-term drift.

### 20.24.31 Uncertainty propagation for a Zeeman measurement

Suppose a Zeeman splitting is measured as a frequency difference

$$\Delta f = \frac{gJ\mu_B}{h} B. \quad (20.179)$$

Solving for the Landé factor gives

$$gJ = \frac{h\Delta f}{\mu_B B}. \quad (20.180)$$

If  $\Delta f$  and  $B$  have independent standard uncertainties  $\sigma_f$  and  $\sigma_B$ , the first-order relative uncertainty is

$$\left(\frac{\sigma_g}{g_J}\right)^2 \simeq \left(\frac{\sigma_f}{\Delta f}\right)^2 + \left(\frac{\sigma_B}{B}\right)^2. \quad (20.181)$$

Here uncertainties in  $h$  and  $\mu_B$  are negligible for a typical teaching laboratory compared with instrumental uncertainties.

**Example 20.47: Which instrument limits a  $g$ -factor measurement?**

A splitting is measured as  $\Delta f = (8.20 \pm 0.02)$  GHz at a field  $B = (0.440 \pm 0.005)$  T. The relative frequency uncertainty is

$$\frac{0.02}{8.20} \simeq 2.4 \times 10^{-3},$$

while the relative field uncertainty is

$$\frac{0.005}{0.440} \simeq 1.14 \times 10^{-2}.$$

The combined relative uncertainty is therefore about

$$\sqrt{(2.4 \times 10^{-3})^2 + (1.14 \times 10^{-2})^2} \simeq 1.17 \times 10^{-2}.$$

Improving the frequency counter would barely help; magnetic-field calibration dominates. This is an engineering conclusion, not just an error-bar calculation.

### 20.24.32 Advanced summary

#### Summary

The advanced description organizes orbital and spin angular momentum into total  $J$ , introduces Landé factors and weak-field Zeeman splitting, and clarifies how selection rules follow from angular-momentum and parity structure. Einstein coefficients place absorption, spontaneous emission and stimulated emission in one rate framework and show why stimulated emission is required by thermal equilibrium. Finite lifetimes, thermal motion and collisions give spectral linewidths; resonators add mode spacing, finesse and quality factor; Gaussian-beam diffraction limits directionality and focusing.

Laser rate equations show the feedback between inversion and photon number, while gain cross sections connect microscopic transition strength to macroscopic threshold. Bloch dynamics and field gradients extend magnetic resonance toward NMR and MRI. Atomic clocks demonstrate how a quantum transition becomes a feedback-controlled frequency reference. Optical pumping and laser cooling show that light can prepare and control atomic states. Finally, spectroscopy and atomic sensing illustrate how quantum physics must be combined with calibration, electronics, numerical fitting and uncertainty analysis in real engineering systems.

## 20.25 Original Practice Problems

The problems in this section are original and are arranged by purpose rather than by a single increasing difficulty scale. Unless stated otherwise, use  $h = 6.62607015 \times 10^{-34}$  J s,  $c = 2.99792458 \times 10^8$  m/s,  $e = 1.602176634 \times 10^{-19}$  C,  $\mu_B = 9.2740 \times 10^{-24}$  J/T,  $k_B = 1.380649 \times 10^{-23}$  J/K and  $1 \text{ eV} = 1.602176634 \times 10^{-19}$  J when needed. Numerical answers should normally be reported with physically meaningful significant figures rather than with every digit shown by a calculator.

### Conceptual questions

- C1.** Explain why a line spectrum is strong evidence for discrete atomic energies. What additional assumptions connect a measured wavelength to an energy difference?
- C2.** An electron has a well-defined value of  $L^2$  and  $L_z$ . Explain why it is misleading to imagine that the full vector  $\mathbf{L}$  has a fixed classical direction in space.
- C3.** A student says, “spin-up means the electron is physically spinning counterclockwise when seen from above.” Identify two problems with this statement.
- C4.** Why does a Stern–Gerlach magnet need a field gradient rather than merely a strong uniform magnetic field to split a neutral atomic beam spatially?
- C5.** Two electrons occupy the same spatial orbital. What must be different about their quantum states, and why?
- C6.** Explain why the Pauli exclusion principle is not a consequence of the electrostatic repulsion between electrons.
- C7.** The Aufbau filling mnemonic is useful but not exact. Give a physical reason why the energy ordering of subshells can change across atoms and ions.
- C8.** Distinguish an x-ray tube’s continuous bremsstrahlung spectrum from its characteristic line spectrum in terms of the microscopic processes that produce them.
- C9.** Compare absorption, spontaneous emission and stimulated emission in terms of the initial atomic state, need for an incident photon and properties of the emitted light.
- C10.** Explain why ordinary positive-temperature thermal equilibrium cannot produce population inversion for two nondegenerate levels with equal statistical weights.
- C11.** A laser cavity has many mathematically allowed longitudinal modes. Why does the laser not necessarily oscillate on all of them?
- C12.** Explain why a highly precise atomic transition does not automatically produce a highly accurate sensor or clock.

### Analytical and numerical problems

- A1.** A spectral line has wavelength 486.1 nm. Calculate the corresponding photon energy in joules and electronvolts and the optical frequency.
- A2.** An atom emits a photon of wavelength 620 nm in a transition to a level at 1.20 eV above the ground state. Find the energy of the initial level relative to the ground state.
- A3.** For  $\ell = 3$ , list all allowed  $m_\ell$  values, calculate  $L/\hbar$ , and determine the largest possible magnitude of  $L_z$ .

- A4.** An electron is in a state with  $\ell = 2$  and  $m_\ell = -1$ . Calculate  $\mu_{L,z}$  in units of  $\mu_B$  and in SI units. State the direction of the magnetic moment component relative to  $L_z$ .
- A5.** For an electron spin in a magnetic field  $B = 1.20$  T, estimate the spin Zeeman splitting using  $g_s = 2.0023$ . Express the answer in eV and GHz.
- A6.** A neutral atom has magnetic-moment projection  $\mu_z = \mu_B$  and passes through a region with  $\partial B_z / \partial z = 600$  T/m. Calculate the magnetic force. Compare it with the weight of a silver atom of mass  $108u$ .
- A7.** A silver atomic beam travels at  $450$  m/s through a Stern–Gerlach magnet of length  $4.0$  cm. Take  $|\mu_z| = \mu_B$ , field gradient  $800$  T/m and atomic mass  $108u$ . Neglect the drift region after the magnet. Calculate the vertical separation between the two spin components when they leave the magnet.
- A8.** Use  $\gamma_p / (2\pi) = 42.58$  MHz/T to calculate a proton's resonance frequency in a  $2.00$  T field. What frequency change results from a  $1.0$  mT change in field?
- A9.** How many one-electron states are available in the  $n = 5$  shell? How many are in the  $5f$  subshell? Explain both counts.
- A10.** Write the ground-state configurations, using noble-gas core notation where appropriate, for Al ( $Z = 13$ ), K ( $Z = 19$ ), Sc ( $Z = 21$ ) and Zn ( $Z = 30$ ) using the standard introductory filling order. Identify the outermost occupied subshell in each.
- A11.** An x-ray tube operates at  $V = 50.0$  kV. Calculate the ideal minimum wavelength of the continuous spectrum.
- A12.** Use the screened-hydrogen estimate  $E_{K_\alpha} \simeq 10.2$  eV  $(Z - 1)^2$  to estimate the  $K_\alpha$  energy and wavelength for  $Z = 26$ . Compare the energy with the maximum photon energy of a  $20$  kV x-ray tube and estimate the  $K$ -vacancy threshold using  $13.6$  eV  $(Z - 1)^2$ . Is the voltage sufficient? Explain why the characteristic photon energy alone cannot establish an excitation threshold or efficiency.
- A13.** Two levels separated by  $1.50$  eV have equal degeneracy. Calculate  $N_2/N_1$  at  $T = 300$  K and at  $T = 3000$  K. Explain why neither result is a population inversion.
- A14.** A cavity of optical length  $n_r L = 0.450$  m operates near  $632.8$  nm. Calculate its free spectral range and estimate the number of longitudinal resonances contained inside a  $1.5$  GHz-wide gain interval.
- A15.** A uniformly filled linear laser cavity has  $L = 0.25$  m, mirror reflectivities  $R_1 = 0.998$  and  $R_2 = 0.970$ , and distributed internal intensity loss  $\alpha_i = 0.015$  m $^{-1}$ . Estimate the threshold gain coefficient
- $$g_{\text{th}} = \alpha_i + \frac{1}{2L} \ln \left( \frac{1}{R_1 R_2} \right).$$
- A16.** A  $5.0$  mW laser emits at  $532$  nm. Calculate the photon emission rate corresponding to the output power. If the detector records  $8.0 \times 10^{14}$  photons per second, estimate the overall photon-detection efficiency.
- A17.** A transition lifetime is  $18$  ns. Estimate the natural linewidth using  $\Delta\nu \sim (2\pi\tau)^{-1}$ . Compare it with a Doppler width of  $1.1$  GHz.
- A18.** A laser has central wavelength  $780$  nm and linewidth  $500$  kHz. Estimate its coherence time using  $\tau_c \sim 1/\Delta\nu$  and its coherence length in air using  $L_c \sim c\tau_c$ . State why numerical factors of order unity depend on linewidth convention.

- A19.** A resonator at  $\nu_0 = 3.8 \times 10^{14}$  Hz has FSR 750 MHz and resonance FWHM 5.0 MHz. Find the finesse and  $Q$ .
- A20.** An ideal Gaussian beam at 1064 nm has waist radius 0.40 mm. Calculate the Rayleigh range and far-field half-angle divergence. Repeat the divergence for  $M^2 = 1.8$ .

### Computational and data-analysis problems

- N1.** Write a program that accepts  $n$  and enumerates every allowed  $(\ell, m_\ell, m_s)$  state. Verify numerically for  $n = 1, \dots, 8$  that the total number is  $2n^2$ . Present one table and one plot.
- N2.** Simulate a Stern–Gerlach beam with a Gaussian distribution of longitudinal speeds. Assume two magnetic projections  $\pm\mu_B$ , a fixed field gradient, finite initial beam width and a specified drift distance. Use a positive-speed distribution (or a truncated Gaussian with a small relative width). Vary mean speed and fractional speed spread separately and compare peak separation with width; a Gaussian extending to zero speed is unsuitable because deflection scales as  $v^{-2}$ .
- N3.** Generate synthetic Gaussian spectral data centered near 589.0 nm, add random measurement noise, fit the line center and width, and repeat the experiment many times. Compare the spread of fitted centers with the uncertainty reported by a single least-squares fit.
- N4.** Convolve a Gaussian Doppler profile with a Lorentzian natural profile numerically. Plot the resulting Voigt-like shape for three ratios of Gaussian to Lorentzian width. Explain which part of the curve is most sensitive to Lorentzian wings.
- N5.** Numerically integrate the dimensionless laser rate equations in Eqs. (20.154)–(20.155) for at least three pump values. Demonstrate convergence with respect to solver tolerances and identify any relaxation oscillations.
- N6.** Create a Monte Carlo uncertainty propagation for  $g_J = h\Delta f/(\mu_B B)$  using normally distributed measurement uncertainties in  $\Delta f$  and  $B$ . Compare the simulated standard deviation with the first-order propagation formula in Eq. (20.181).

### Engineering and applied-science problems

- E1.** A compact spectrometer must resolve two lines separated by 0.20 nm near 600 nm. What minimum resolving power  $R = \lambda/\Delta\lambda$  is required? Discuss at least two reasons why the instrument should be specified with margin beyond this ideal value.
- E2.** A cavity must have FSR at least 2.0 GHz in air. What is the maximum mirror separation for a simple linear cavity? If mechanical design requires  $L = 12$  cm, does the specification hold?
- E3.** A 633 nm alignment laser has measured  $M^2 = 1.4$  and waist radius 0.75 mm. Estimate the far-field divergence and spot-radius increase after 20 m using a Gaussian-beam approximation. Discuss why atmospheric turbulence may invalidate the simple result outdoors.
- E4.** An MRI system uses a gradient of 25 mT/m over a 0.40 m field of view. Estimate the proton frequency span due to that gradient. What sampling bandwidth must the receiver exceed in an idealized one-dimensional frequency-encoding picture?
- E5.** A laboratory wants to infer gas temperature from Doppler linewidth. List the instrument properties and systematic effects that must be characterized before interpreting the fitted width thermometrically.

- E6.** A narrow-line laser is frequency-locked to an atomic transition. Sketch a feedback loop containing reference, detector, error signal, controller and actuator. Explain what can limit accuracy even if the loop suppresses short-term frequency noise strongly.

### Challenge problems

- H1.** Starting from  $\mathbf{J} = \mathbf{L} + \mathbf{S}$ , derive Eq. (20.110). Use it to compare the sign of  $\mathbf{L} \cdot \mathbf{S}$  for  $j = \ell + 1/2$  and  $j = \ell - 1/2$ .
- H2.** Use the Landé formula to derive  $g_J$  for  $^2D_{3/2}$  and  $^2D_{5/2}$ . At  $B = 0.30$  T, calculate the spacing of adjacent  $m_j$  Zeeman components in frequency for each multiplet.
- H3.** Starting from the Einstein rate balance in thermal equilibrium and the Boltzmann population ratio, derive the form of Planck's spectral energy density up to the two Einstein-coefficient relations. State clearly the convention for  $\rho(\nu)$ .
- H4.** A gas transition at 500 nm belongs to atoms of mass  $20u$  at 400 K and has lifetime 10 ns. Estimate Doppler and natural FWHM scales. Which dominates? At what approximate temperature would the Doppler width equal the natural width, using the same formulas?
- H5.** A linear cavity of length  $L$  has mirror intensity reflectivities  $R_1$  and  $R_2$  and a distributed internal intensity-loss coefficient  $\alpha_i$ . Derive the round-trip threshold condition and the expression for  $g_{\text{th}}$ . Identify where the factor  $2L$  enters.
- H6.** Linearize a simple two-variable laser rate model about a steady state and explain qualitatively how complex eigenvalues correspond to damped relaxation oscillations. A full closed-form expression is not required, but the Jacobian and physical interpretation are.

## 20.26 Selected Solutions to Representative Problems

The solutions below are intentionally more explicit than an answer key. They demonstrate reasoning, dimensional checks and interpretation.

### Solution to A1: photon energy of a 486.1 nm line

Use

$$f = \frac{c}{\lambda}, \quad E = hf = \frac{hc}{\lambda}.$$

With  $\lambda = 486.1 \times 10^{-9}$  m,

$$f = \frac{2.9979 \times 10^8}{486.1 \times 10^{-9}} \simeq 6.17 \times 10^{14} \text{ Hz}.$$

Then

$$E = (6.6261 \times 10^{-34})(6.17 \times 10^{14}) \simeq 4.09 \times 10^{-19} \text{ J}.$$

In electronvolts,

$$E \simeq \frac{4.09 \times 10^{-19}}{1.6022 \times 10^{-19}} \simeq 2.55 \text{ eV}.$$

A visible blue-green photon having energy of a few eV is physically reasonable.

### Solution to A5: spin splitting at 1.20 T

For electron spin,

$$\Delta E \simeq g_s \mu_B B.$$

Therefore

$$\Delta E = (2.0023)(9.2740 \times 10^{-24})(1.20) \simeq 2.23 \times 10^{-23} \text{ J}.$$

In electronvolts,

$$\Delta E \simeq 1.39 \times 10^{-4} \text{ eV}.$$

The resonance frequency is

$$f = \frac{\Delta E}{h} \simeq 3.36 \times 10^{10} \text{ Hz} = 33.6 \text{ GHz}.$$

This lies in the microwave region, consistent with electron-spin resonance scales in tesla-level fields.

### Solution to A7: Stern–Gerlach separation at the magnet exit

The force magnitude on each component is

$$F = \mu_B \frac{dB}{dz} = (9.274 \times 10^{-24})(800) = 7.42 \times 10^{-21} \text{ N}.$$

The atomic mass is

$$M = 108u \simeq 108(1.6605 \times 10^{-27}) = 1.79 \times 10^{-25} \text{ kg}.$$

Thus

$$a_z = \frac{F}{M} \simeq 4.14 \times 10^4 \text{ m/s}^2.$$

The transit time through a 0.040 m magnet is

$$t = \frac{L}{v} = \frac{0.040}{450} = 8.89 \times 10^{-5} \text{ s}.$$

The vertical displacement of one component is

$$y = \frac{1}{2}a_z t^2 \simeq 1.64 \times 10^{-4} \text{ m}.$$

The two components move symmetrically, so their separation is

$$\Delta y = 2y \simeq 3.27 \times 10^{-4} \text{ m} = 0.327 \text{ mm}.$$

A downstream drift region would increase the separation further because the atoms leave the magnet with opposite vertical velocities.

### Solution to A9: shell and subshell capacities

For a shell with principal quantum number  $n$ , the total number of spin-orbitals is

$$N_n = 2n^2.$$

For  $n = 5$ ,

$$N_5 = 2(25) = 50.$$

For a  $5f$  subshell,  $\ell = 3$ . There are  $2\ell + 1 = 7$  possible  $m_\ell$  values, each with two spin projections, so

$$N_{5f} = 2(2\ell + 1) = 2(7) = 14.$$

The shell count includes the  $5s$ ,  $5p$ ,  $5d$ ,  $5f$  and  $5g$  subshells allowed by  $\ell = 0, 1, 2, 3, 4$ .

### Solution to A11: x-ray cutoff at 50.0 kV

The maximum photon energy is the electron kinetic energy  $eV$ :

$$E_{\max} = 50.0 \text{ keV}.$$

Using  $hc \simeq 1240 \text{ eV nm}$ ,

$$\lambda_{\min} = \frac{1240 \text{ eV nm}}{50.0 \times 10^3 \text{ eV}} = 2.48 \times 10^{-2} \text{ nm} = 24.8 \text{ pm}.$$

The target material can strongly alter the characteristic peaks and overall efficiency, but the ideal cutoff follows the accelerating voltage.

### Solution to A13: thermal population ratios

For equal degeneracies,

$$\frac{N_2}{N_1} = e^{-\Delta E/(k_B T)}.$$

It is convenient to use  $k_B = 8.6173 \times 10^{-5} \text{ eV/K}$ . At 300 K,

$$k_B T \simeq 0.02585 \text{ eV}, \quad \frac{N_2}{N_1} = e^{-1.50/0.02585} \simeq 6.3 \times 10^{-26}.$$

At 3000 K,

$$k_B T \simeq 0.2585 \text{ eV}, \quad \frac{N_2}{N_1} = e^{-1.50/0.2585} \simeq 3.0 \times 10^{-3}.$$

Heating increases the excited fraction dramatically, but the ratio remains below one for any positive temperature in this two-level equal-degeneracy equilibrium model. Population inversion requires nonequilibrium pumping.

**Solution to A15: cavity threshold gain**

Use

$$g_{\text{th}} = \alpha_i + \frac{1}{2L} \ln \left( \frac{1}{R_1 R_2} \right).$$

The mirror product is

$$R_1 R_2 = (0.998)(0.970) = 0.96806.$$

Thus

$$\frac{1}{2L} \ln \left( \frac{1}{R_1 R_2} \right) = \frac{1}{0.50} \ln(1.0330) \simeq 0.0649 \text{ m}^{-1}.$$

Adding internal loss,

$$g_{\text{th}} \simeq 0.015 + 0.0649 = 0.0799 \text{ m}^{-1}.$$

The result is small because the cavity mirrors are highly reflective and the assumed internal loss is modest.

**Solution to A17: natural linewidth**

With  $\tau = 18 \text{ ns}$ ,

$$\Delta\nu_{\text{nat}} \sim \frac{1}{2\pi\tau} = \frac{1}{2\pi(18 \times 10^{-9})} \simeq 8.8 \text{ MHz}.$$

The stated Doppler width is 1.1 GHz, about 125 times larger. Under these conditions the observed line width would be dominated by Doppler broadening unless a Doppler-free method or very low temperature were used.

**Solution to A20: Gaussian beam parameters**

With  $w_0 = 0.40 \text{ mm} = 4.0 \times 10^{-4} \text{ m}$  and  $\lambda = 1.064 \times 10^{-6} \text{ m}$ ,

$$z_R = \frac{\pi w_0^2}{\lambda} = \frac{\pi(4.0 \times 10^{-4})^2}{1.064 \times 10^{-6}} \simeq 0.472 \text{ m}.$$

The ideal divergence is

$$\theta = \frac{\lambda}{\pi w_0} \simeq 8.47 \times 10^{-4} \text{ rad} = 0.847 \text{ mrad}.$$

For  $M^2 = 1.8$ ,

$$\theta \simeq 1.52 \text{ mrad}.$$

The larger  $M^2$  means the real beam spreads faster than a diffraction-limited Gaussian with the same waist.

**Solution to E2: cavity length from FSR**

For a linear air cavity,

$$\Delta\nu_{\text{FSR}} = \frac{c}{2L}.$$

Requiring at least 2.0 GHz gives

$$L \leq \frac{c}{2\Delta\nu_{\text{FSR}}} = \frac{2.998 \times 10^8}{4.0 \times 10^9} \simeq 7.49 \times 10^{-2} \text{ m} = 7.49 \text{ cm}.$$

A 12 cm cavity has

$$\Delta\nu_{\text{FSR}} \simeq 1.25 \text{ GHz},$$

so it does not meet the 2.0 GHz requirement.

**Solution to E4: MRI gradient bandwidth**

The total field variation across a 0.40 m field of view is

$$\Delta B = G \Delta z = (25 \times 10^{-3})(0.40) = 0.010 \text{ T}.$$

The proton frequency span is

$$\Delta f = \frac{\gamma_p}{2\pi} \Delta B = (42.58 \times 10^6)(0.010) \simeq 4.26 \times 10^5 \text{ Hz} = 426 \text{ kHz}.$$

In an idealized direct frequency-encoding picture, the receiver must support a bandwidth greater than this span, with additional engineering margin for filters and sampling.

**Solution to H1: the  $L \cdot S$  identity**

Square Eq. (20.101):

$$J^2 = (\mathbf{L} + \mathbf{S}) \cdot (\mathbf{L} + \mathbf{S}) = L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S}.$$

Rearranging gives

$$\mathbf{L} \cdot \mathbf{S} = \frac{1}{2}(J^2 - L^2 - S^2),$$

which is Eq. (20.110). In an eigenstate of  $J^2, L^2, S^2$ ,

$$\langle \mathbf{L} \cdot \mathbf{S} \rangle = \frac{\hbar^2}{2} [j(j+1) - \ell(\ell+1) - s(s+1)].$$

For  $s = 1/2$  and  $j = \ell + 1/2$ , this quantity is positive; for  $j = \ell - 1/2$ , it is negative. The semiclassical language “more parallel” and “more antiparallel” is useful intuition, but the eigenvalue relation is the precise statement.

**Solution to H5: deriving the cavity threshold**

Let the intensity at some point be  $I_0$ . During one pass through active length  $L$ , gain multiplies intensity by  $e^{gL}$  and distributed loss multiplies it by  $e^{-\alpha_i L}$ . A round trip contains two passes, giving

$$e^{2(g-\alpha_i)L}.$$

The two mirror reflections multiply intensity by  $R_1 R_2$ . Therefore

$$I_{\text{after}} = I_0 R_1 R_2 e^{2(g-\alpha_i)L}.$$

At threshold the field reproduces itself in intensity after each round trip:

$$R_1 R_2 e^{2(g_{\text{th}}-\alpha_i)L} = 1.$$

Taking the natural logarithm,

$$\ln(R_1 R_2) + 2(g_{\text{th}} - \alpha_i)L = 0,$$

so

$$g_{\text{th}} = \alpha_i + \frac{1}{2L} \ln \left( \frac{1}{R_1 R_2} \right).$$

The factor  $2L$  appears because a complete round trip traverses the active length twice.

### Solution to N4: constructing a Voigt profile numerically

Use detuning  $x = \nu - \nu_0$ . Define zero-centered normalized profiles

$$G(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}, \quad L(x) = \frac{1}{\pi} \frac{\gamma_L}{x^2 + \gamma_L^2}.$$

Here the Gaussian FWHM is  $2\sqrt{2\ln 2}\sigma$  and the Lorentzian FWHM is  $2\gamma_L$ . Their convolution is

$$V(x) = \int_{-\infty}^{\infty} G(u)L(x-u)du.$$

Plot the observed spectrum as  $V(\nu - \nu_0)$ . Centering both input functions on  $\nu_0$  and convolving them would incorrectly place the peak at  $2\nu_0$ . Numerically, choose a frequency interval many widths larger than both profiles, sample it finely, and use either direct numerical convolution or a fast-Fourier-transform method. Normalize the final area to unity and verify that increasing the grid size does not change the central width or wings appreciably.

When  $\sigma \gg \gamma_L$ , the central region looks nearly Gaussian, but the far wings can still retain Lorentzian influence. When  $\gamma_L \gg \sigma$ , the line is predominantly Lorentzian. When the widths are comparable, neither elementary profile describes the full curve adequately. Plotting the same profile on both linear and logarithmic vertical scales is instructive: a linear plot emphasizes the core, while a logarithmic plot reveals the slowly decaying Lorentzian wings much more clearly.

### Solution to H3: Einstein coefficients and Planck consistency

At thermal equilibrium, detailed balance requires

$$N_1 B_{12} \rho(\nu) = N_2 B_{21} \rho(\nu) + N_2 A_{21}.$$

Solve for the spectral energy density:

$$\rho(\nu) = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}}.$$

Using

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu/(k_B T)},$$

we obtain

$$\rho(\nu) = \frac{(g_2/g_1) e^{-h\nu/k_B T} A_{21}}{B_{12} - (g_2/g_1) e^{-h\nu/k_B T} B_{21}}.$$

For this to have the Planck denominator  $e^{h\nu/k_B T} - 1$  for all temperatures, the stimulated coefficients must satisfy

$$g_1 B_{12} = g_2 B_{21}.$$

After using that relation and multiplying numerator and denominator by  $e^{h\nu/k_B T}$ ,

$$\rho(\nu) = \frac{A_{21}/B_{21}}{e^{h\nu/k_B T} - 1}.$$

Comparison with the Planck spectral energy density per unit ordinary frequency,

$$\rho(\nu) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/k_B T} - 1},$$

gives

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3}.$$

The derivation demonstrates two points at once: absorption and stimulated emission have related strengths after degeneracy is accounted for, and spontaneous emission must scale relative to stimulated emission in a frequency-dependent way that reproduces blackbody radiation. The result is tied to the stated convention for  $\rho(\nu)$ ; changing from frequency to angular-frequency density changes the numerical coefficient.

## 20.27 Bridge to Chapter 21: Electrical Conduction in Solids

The next conceptual step is to place many atoms close enough that their electronic states cannot be treated as independent copies of isolated-atom orbitals. Atomic energy levels broaden and reorganize into bands. Pauli exclusion remains essential because it determines how electrons occupy the available states. The distinction between filled and partially filled states becomes a distinction between insulating, semiconducting and conducting behavior.

Several ideas from this chapter therefore carry forward directly:

- **Quantum states and occupancy:** one-electron states remain the basic bookkeeping language.
- **Pauli exclusion:** state filling determines which electronic responses are possible.
- **Energy differences:** optical absorption and emission in solids are still governed by energy conservation, now between bands and defect states as well as discrete levels.
- **Magnetic moments and spin:** electron spin remains central to magnetic materials and spin-dependent transport.
- **Lasers:** semiconductor lasers reinterpret population inversion and stimulated emission using band populations rather than isolated atomic levels.

The conceptual move is therefore not from “quantum atoms” back to classical matter. It is from a few quantized states in isolated systems to an enormous number of quantized states in a collective solid.

# Appendix A

## Notation Dictionary

| Symbol                               | Meaning                                                              | Typical SI unit                                                             |
|--------------------------------------|----------------------------------------------------------------------|-----------------------------------------------------------------------------|
| $n$                                  | principal quantum number                                             | dimensionless                                                               |
| $\ell$                               | orbital angular-momentum quantum number                              | dimensionless                                                               |
| $m_\ell$                             | orbital magnetic quantum number                                      | dimensionless                                                               |
| $s$                                  | spin quantum number; $s = 1/2$ for an electron                       | dimensionless                                                               |
| $m_s$                                | spin magnetic quantum number                                         | dimensionless                                                               |
| $j, m_j$                             | total one-electron angular-momentum quantum number and projection    | dimensionless                                                               |
| $\mathbf{L}, \mathbf{S}, \mathbf{J}$ | orbital, spin and total angular momentum                             | J s                                                                         |
| $\hbar$                              | reduced Planck constant $h/(2\pi)$                                   | J s                                                                         |
| $\mu_L$                              | orbital magnetic dipole moment                                       | J/T                                                                         |
| $\mu_S$                              | spin magnetic dipole moment                                          | J/T                                                                         |
| $\mu_B$                              | Bohr magneton $e\hbar/(2m_e)$                                        | J/T                                                                         |
| $g_s$                                | electron spin $g$ factor                                             | dimensionless                                                               |
| $g_J$                                | Landé factor for a $J$ level                                         | dimensionless                                                               |
| $B$                                  | magnetic-field magnitude                                             | T                                                                           |
| $\gamma$                             | gyromagnetic ratio                                                   | rad s <sup>-1</sup> T <sup>-1</sup> or,<br>when divided by<br>$2\pi$ , Hz/T |
| $E_i, E_f$                           | initial and final atomic energies                                    | J or eV                                                                     |
| $\nu, f$                             | ordinary frequency                                                   | Hz                                                                          |
| $\omega$                             | angular frequency                                                    | rad/s                                                                       |
| $\lambda$                            | wavelength                                                           | m                                                                           |
| $A_{21}$                             | spontaneous-emission Einstein coefficient                            | s <sup>-1</sup>                                                             |
| $B_{12}, B_{21}$                     | Einstein absorption and stimulated-emission coefficients             | convention-<br>dependent                                                    |
| $\rho(\nu)$                          | spectral electromagnetic energy density per unit frequency           | J m <sup>-3</sup> Hz <sup>-1</sup>                                          |
| $N_i$                                | number of atoms in level $i$                                         | dimensionless<br>count                                                      |
| $n_i$                                | number density in level $i$                                          | m <sup>-3</sup>                                                             |
| $g_i$                                | degeneracy of level $i$                                              | dimensionless                                                               |
| $g(\nu)$                             | optical gain coefficient                                             | m <sup>-1</sup>                                                             |
| $\alpha_i$                           | distributed internal optical loss coefficient                        | m <sup>-1</sup>                                                             |
| $R_1, R_2$                           | mirror intensity reflectivities                                      | dimensionless                                                               |
| $L, \ell_g$                          | cavity length and active-medium length                               | m                                                                           |
| $n_r, n_g$                           | phase refractive index and group index                               | dimensionless                                                               |
| $\mathcal{R}_1, \mathcal{R}_2$       | mirror curvature radii (distinct from reflectivities)                | m                                                                           |
| $g_{c1}, g_{c2}$                     | cavity geometry parameters                                           | dimensionless                                                               |
| $D, S$                               | upper-state atom number and cavity photon number in four-level model | counts                                                                      |

| Symbol                   | Meaning                                                                   | Typical SI unit |
|--------------------------|---------------------------------------------------------------------------|-----------------|
| $\tau, \tau_p$           | upper-state population lifetime and photon-number lifetime                | s               |
| $n, s$ (rate model)      | normalized population and photon variables; distinct from quantum numbers | dimensionless   |
| $p, \epsilon, \beta$     | normalized pump, lifetime ratio, spontaneous-mode fraction                | dimensionless   |
| $\Omega, \Delta$         | angular Rabi frequency and angular-frequency detuning                     | rad/s           |
| $\Gamma$                 | excited-state population decay rate in cooling model                      | $\text{s}^{-1}$ |
| $\Delta\nu_{\text{FSR}}$ | free spectral range                                                       | Hz              |
| $Q$                      | resonator quality factor                                                  | dimensionless   |
| $\mathcal{F}$            | cavity finesse                                                            | dimensionless   |
| $w_0$                    | Gaussian-beam waist radius                                                | m               |
| $z_R$                    | Rayleigh range                                                            | m               |
| $M^2$                    | beam-quality factor                                                       | dimensionless   |
| $T_1$                    | longitudinal magnetic-resonance relaxation time                           | s               |
| $T_2$                    | transverse magnetic-resonance relaxation time                             | s               |
| $G_z$                    | magnetic-field gradient along $z$                                         | T/m             |

# Appendix B

## Compact Formula Sheet

### Formula summary

#### Photon and transition relations

$$E_\gamma = h\nu = \frac{hc}{\lambda}, \quad \Delta E = |E_i - E_f|.$$

#### Orbital and spin angular momentum

$$\begin{aligned} L &= \sqrt{\ell(\ell+1)}\hbar, & L_z &= m_\ell\hbar, \\ S &= \sqrt{s(s+1)}\hbar, & s &= \frac{1}{2}, & S_z &= m_s\hbar, \quad m_s = \pm\frac{1}{2}. \end{aligned}$$

#### Magnetic moments

$$\begin{aligned} \mu_B &= \frac{e\hbar}{2m_e}, & \mu_L &= -\mu_B \frac{\mathbf{L}}{\hbar}, & \mu_{L,z} &= -m_\ell\mu_B, \\ \mu_S &= -g_s\mu_B \frac{\mathbf{S}}{\hbar}, & U &= -\boldsymbol{\mu} \cdot \mathbf{B}. \end{aligned}$$

#### Stern–Gerlach and magnetic resonance

$$F_z = \mu_z \frac{\partial B_z}{\partial z}, \quad \omega_0 = |\gamma|B.$$

#### Pauli capacities

$$N_{n\ell} = 2(2\ell+1), \quad N_n = 2n^2.$$

#### X rays

$$\lambda_{\min} = \frac{hc}{eV}, \quad E_{K_\alpha} \simeq 10.2 \text{ eV}(Z-1)^2 \quad (\text{simple screened model}).$$

#### Thermal populations and laser gain

$$\begin{aligned} \frac{N_2}{N_1} &= \frac{g_2}{g_1} e^{-\Delta E/(k_B T)}, & \text{inversion: } \frac{N_2}{g_2} &> \frac{N_1}{g_1}, \\ I(z) &= I_0 e^{gz}, & g_{\text{th}} &= \alpha_i + \frac{1}{2L} \ln\left(\frac{1}{R_1 R_2}\right). \end{aligned}$$

#### Cavity and beam formulas

$$\begin{aligned} \Delta\nu_{\text{FSR}} &= \frac{c}{2n_r L}, & Q &= \frac{\nu_0}{\delta\nu}, & \mathcal{F} &= \frac{\Delta\nu_{\text{FSR}}}{\delta\nu}, \\ w(z) &= w_0 \sqrt{1 + (z/z_R)^2}, & z_R &= \frac{\pi w_0^2}{\lambda}, & \theta &\simeq M^2 \frac{\lambda}{\pi w_0}. \end{aligned}$$

**Advanced angular momentum**

$$\begin{aligned} \mathbf{J} &= \mathbf{L} + \mathbf{S}, & j &= |\ell - s|, \dots, \ell + s, \\ g_J &= 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}, & \Delta E_Z &= \mu_B g_J m_j B. \end{aligned}$$

**Linewidth and MRI estimates**

$$\begin{aligned} \Delta\nu_{\text{nat}} &\sim \frac{1}{2\pi\tau}, & \Delta\nu_D &= \nu_0 \sqrt{\frac{8k_B T \ln 2}{mc^2}}, \\ \Delta f(z) &= \frac{\gamma_p}{2\pi} G_z z. \end{aligned}$$

**Caution**

Formula sheets are memory aids, not replacements for model selection. The threshold expression above assumes a uniformly filled cavity and intensity coefficients; for a shorter gain medium use Eq. (20.83). The Gaussian envelope and  $z_R$  formulas assume  $M^2 = 1$  in free space. The natural-width estimate is the frequency FWHM for exponential decay to a stable lower level. Weak-field Zeeman formulas fail outside their coupling regime, and the screened  $K_\alpha$  formula is approximate.

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