

Chapter 22

Nuclear Structure and Radioactivity

First Edition

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Preface

This chapter introduces the atomic nucleus as a quantum system with characteristic femtometre dimensions, MeV energies and probabilistic decay processes. Coulomb repulsion, mass–energy conversion, conservation laws and quantum tunnelling meet here in a setting relevant to medicine, materials analysis, archaeology, radiation instrumentation and energy technology.

The core develops scattering evidence, nuclide notation, nuclear size and density, mass defect, binding and qualitative stability. It then derives the exponential decay law and treats alpha, beta, electron-capture and gamma processes with consistent mass conventions. Activity, measured counts and dose remain distinct. Worked examples and practice problems support independent study as well as tutorial teaching.

The enrichment section extends the same ideas through mass models, form factors, barrier penetration, decay chains and quantitative counting. Detector geometry, calibration, resolution and likelihood inference connect nuclear processes to observations. A reproducible synthetic half-life experiment makes the statistical ideas concrete. The material is independently written for technical, engineering, computer-science and applied-physics students; supplied reference books guide coverage and pedagogical comparison without reproduction of their pages or artwork.

The Guideline

For every nuclear calculation, begin by separating three levels of description: *what nucleus is present*, *what process is occurring*, and *what quantity is being measured*. Write the parent and daughter nuclides explicitly, check conservation of charge and nucleon number, and only then insert masses, half-lives or detector data. This habit prevents most sign, unit and isotope-identification errors.

Pedagogical boundary

This chapter develops nuclear structure and radioactivity. It introduces the binding-energy curve and reaction Q values because they are needed for decay energetics, but a systematic treatment of fission, fusion, reactors and stellar energy production belongs to the following chapter on nuclear energy. Elementary-particle classification and the Standard Model are likewise postponed to the later particle-physics chapter.

Conventions and notation

The neutron number $N = A - Z$ and the time-dependent radioactive population $N(t)$ use the same traditional letter in different contexts. The argument t or a population subscript distinguishes the latter; neither is an activity.

A nuclide is written

$${}^A_ZX, \quad (0.1)$$

where X is the chemical symbol, Z is the proton number (atomic number), A is the integer mass number, and

$$N = A - Z \quad (0.2)$$

is the neutron number. The symbol A is reserved for mass number. To avoid collision with this notation, radioactive activity is denoted by $\mathcal{A}$ rather than by A .

Nuclear lengths are usually expressed in femtometres,

$$1 \text{ fm} = 10^{-15} \text{ m}, \quad (0.3)$$

and nuclear energies are commonly expressed in electronvolts or mega-electronvolts,

$$1 \text{ eV} = 1.602176634 \times 10^{-19} \text{ J}, \quad 1 \text{ MeV} = 10^6 \text{ eV}. \quad (0.4)$$

The atomic mass unit is denoted by u and satisfies, to the precision needed in this course,

$$1 u c^2 \simeq 931.5 \text{ MeV}. \quad (0.5)$$

For mass–energy calculations we retain c explicitly rather than adopting natural units. This makes dimensions transparent and avoids confusion between a mass quoted in u , an energy quoted in MeV and a rest-energy quoted in MeV.

The decay constant is λ with SI unit s^{-1} . The mean lifetime is $\tau = 1/\lambda$, and the half-life is $T_{1/2} = \ln 2/\lambda$. If $N(t)$ denotes the expected number of undecayed nuclei in a macroscopic ensemble, then the activity is

$$\mathcal{A}(t) = -\frac{dN}{dt} = \lambda N(t), \quad (0.6)$$

measured in becquerels,

$$1 \text{ Bq} = 1 \text{ s}^{-1}. \quad (0.7)$$

A becquerel measures nuclear transformations per second, not detector counts per second and not biological dose.

Symbol	Meaning	Typical unit
A	mass number, total number of nucleons	dimensionless
Z	proton number or atomic number	dimensionless
$N = A - Z$	neutron number	dimensionless
R	characteristic nuclear radius	fm
R_0	empirical radius parameter, approximately 1.2 fm	fm
m_p, m_n, m_e	proton, neutron and electron masses	kg or u
$M(A, Z)$	mass of a specified nucleus or neutral atom, as stated	kg or u
B	nuclear binding energy	J or MeV

B/A	binding energy per nucleon	MeV
Q	energy release of a decay or reaction	J or MeV
λ	decay constant	s^{-1}
τ	mean lifetime, $1/\lambda$	s
$T_{1/2}$	half-life	s or other time unit
$\mathcal{A}$	activity, λN	Bq
D	absorbed dose	$\text{Gy} = \text{J kg}^{-1}$
H	equivalent dose	Sv

Caution

Mass tables may list either *nuclear masses* or *neutral-atomic masses*. The binding-energy formula must match the convention. With nuclear masses, use proton and neutron masses. With neutral-atomic masses, the convenient combination uses the mass of the neutral hydrogen atom rather than the bare proton mass. Electron masses do not simply disappear; they cancel only when the bookkeeping is done consistently.

Note

Nuclear physics often uses energy units to quote masses, for example $m_p c^2 \approx 938 \text{ MeV}$. In this chapter we write mc^2 when we mean a rest energy and m when we mean a mass. This small notational discipline prevents dimensional mistakes.

Chapter 22

Nuclear Structure and Radioactivity

22.1 Motivation and roadmap

Atomic physics established that electrons occupy quantum states around a positively charged centre, but the atomic energy scale alone does not explain where almost all atomic mass resides or why nuclei with different neutron-to-proton ratios have very different stabilities. Nuclear physics begins by changing scale. A typical atom has a radius of order 10^{-10} m, whereas a medium-mass nucleus has a radius of only a few 10^{-15} m. The nucleus is therefore roughly five orders of magnitude smaller in linear size, yet it contains more than 99.9% of the atomic mass. Matter is mostly empty space on the atomic scale, but the compact nuclear region is extraordinarily dense.

The energy scale changes just as dramatically. Atomic transitions are commonly measured in electronvolts. Nuclear separation energies, decay energies and excitation energies are commonly measured in mega-electronvolts. The factor of a million between eV and MeV is not an arbitrary unit choice; it reflects the much stronger interactions and much smaller distances characteristic of the nucleus.

This chapter follows the conceptual path shown in Fig. 22.1. Scattering experiments motivate the compact-nucleus picture. Nuclide notation and the empirical size law provide the geometry. Mass defect and binding energy provide the energy language. Stability then leads naturally to radioactivity, and radioactivity leads to the statistical decay law, radiation detection, dosimetry and dating.

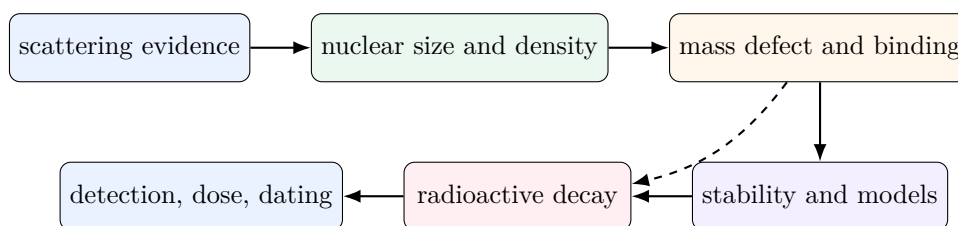


Figure 22.1: Conceptual roadmap for Chapter 22. Structural evidence, characteristic scales and binding-energy ideas prepare the statistical and experimental treatment of radioactivity.

Learning goals

After studying the main chapter, the student should be able to:

- explain how scattering evidence supports a compact nuclear model of the atom;
- use nuclide notation ${}^A_ZX$ and distinguish isotopes, isotones and isobars;
- use the empirical law $R = R_0A^{1/3}$ and explain why it implies nearly constant nuclear density;
- calculate mass defect, binding energy and binding energy per nucleon using a consistent mass convention;
- interpret the broad binding-energy-per-nucleon curve without confusing binding energy with

energy “stored inside” a nucleus;

- describe qualitatively the short-range nuclear interaction and the purpose of liquid-drop and shell models;
- derive and use $N(t) = N_0 e^{-\lambda t}$, $\mathcal{A} = \lambda N$, $T_{1/2} = \ln 2 / \lambda$ and $\tau = 1 / \lambda$;
- identify daughter nuclei in alpha, beta-minus, beta-plus, electron-capture and gamma processes;
- calculate simple decay Q values and explain how conservation laws constrain the final state;
- distinguish activity, detector count rate, absorbed dose and equivalent dose;
- analyze simple radiometric-dating and detector-counting data;
- connect nuclear physics to medicine, sensors, materials analysis and radiation instrumentation.

Key message

Three ideas organize the chapter. First, nuclear matter occupies a tiny volume and has an approximately constant density across a wide range of nuclei. Second, nuclear binding is measured by a mass difference through $E = mc^2$. Third, the time of decay of one unstable nucleus is unpredictable, while the statistics of a large ensemble are described with remarkable accuracy by an exponential law.

Guided checks

Before continuing, answer qualitatively.

1. If nuclear dimensions are about 10^5 times smaller than atomic dimensions, by what factor is the corresponding volume scale smaller?
2. Why can an energy difference of only 0.001 u c^2 already be large on an atomic scale?
3. Does the statement “a sample has a half-life of one hour” imply that every nucleus survives exactly one hour?

Answers: the volume factor is of order 10^{15} ; $0.001 \text{ u c}^2 \approx 0.93 \text{ MeV}$; and half-life is an ensemble property, not a deterministic clock for one nucleus.

22.2 From the atom to the nucleus: scattering evidence

At the beginning of the twentieth century, a successful atomic model had to explain two facts that pull in opposite directions. Atoms are electrically neutral, so positive and negative charge must balance. At the same time, electrons are very light compared with atoms, so the positive part must carry almost all the mass. The decisive geometrical question was therefore whether the positive charge and mass were spread throughout the atomic volume or concentrated in a much smaller region.

The Geiger–Marsden scattering experiments used energetic alpha particles incident on thin metal foils. Alpha particles are positively charged and much more massive than electrons. If the atom’s positive charge were smoothly distributed through the full atomic volume, an alpha particle would normally experience many weak deflections and very large scattering angles would be exceedingly difficult to obtain. The observation that a small fraction of projectiles underwent large-angle deflections indicated that they had encountered a very intense electric field localized in a very small region. Rutherford’s interpretation was a compact, positively charged nucleus surrounded by electrons.

Figure 22.2 shows the essential geometry. Most projectiles pass far enough from a nucleus that their

directions change only slightly. A projectile with a small impact parameter can experience a strong Coulomb repulsion and turn through a large angle. A nearly head-on projectile is decelerated as its kinetic energy is converted into electric potential energy and reaches a minimum distance before reversing direction.

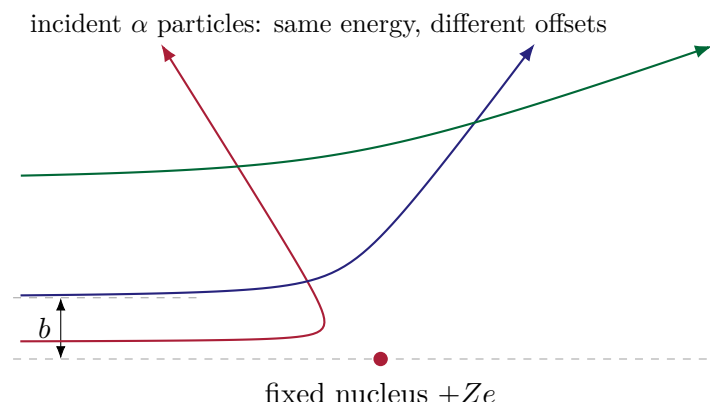


Figure 22.2: Repulsive Coulomb trajectories for three impact parameters. The force bends each path away from the positive nucleus; a small offset can produce backscattering. Paths are calculated from the repulsive inverse-square force in dimensionless coordinates. The nucleus symbol is enlarged and no nuclear-surface contact is implied.

For a head-on encounter, a simple classical estimate gives the distance of closest approach. Let an alpha particle of charge $+2e$ and initial kinetic energy K_0 approach a stationary nucleus of charge $+Ze$. Ignoring nuclear recoil and any short-range nuclear force until the turning point, conservation of mechanical energy gives

$$K_0 = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{r_{\min}}, \quad (22.1)$$

so

$$r_{\min} = \frac{1}{4\pi\epsilon_0} \frac{2Ze^2}{K_0}. \quad (22.2)$$

This estimate is useful because it translates a measured projectile energy into a length scale that the particle probes. If the projectile turns around at a distance much larger than the nuclear radius, the interaction is almost purely Coulombic. If the trajectory reaches the nuclear surface, short-range nuclear effects become important and the simple Coulomb model ceases to be sufficient.

Example 22.1: Closest approach in a head-on alpha scattering event

An alpha particle with kinetic energy 5.0 MeV approaches a gold nucleus ($Z = 79$) head-on. Estimate the classical distance of closest approach, treating the gold nucleus as fixed.

Strategy. Use Eq. (22.2). A convenient nuclear-physics form of the Coulomb constant is

$$\frac{e^2}{4\pi\epsilon_0} \simeq 1.44 \text{ MeV fm}.$$

Therefore

$$\begin{aligned} r_{\min} &= \frac{2Z(1.44 \text{ MeV fm})}{5.0 \text{ MeV}} \\ &= \frac{2(79)(1.44)}{5.0} \text{ fm} \simeq 45.5 \text{ fm}. \end{aligned}$$

A gold nucleus has a radius of only about 7 fm, so the turning point is well outside the nuclear surface. Coulomb repulsion is therefore the dominant interaction in this idealized head-on estimate.

Reasonableness check. Increasing the alpha-particle kinetic energy would reduce $r_{\min}$, exactly as expected for a more energetic probe.

Caution

The Rutherford experiment did not show that nuclei are classical hard spheres. It established that positive charge and most atomic mass are concentrated in a region much smaller than the atom. Modern nuclei are quantum many-body systems with diffuse surfaces and internal structure. The hard-sphere picture is only a geometrical first approximation.

Guided checks

1. Why cannot collisions with very light atomic electrons readily reverse the motion of a massive alpha projectile? How does this differ from using an electron beam to probe nuclear charge?
2. In Eq. (22.2), what happens to $r_{\min}$ if the target atomic number doubles at fixed projectile energy?
3. Why does observing a few large-angle events carry more structural information than observing many small-angle events?

22.3 Nuclides, isotopes and nuclear bookkeeping

The nucleus is composed of protons and neutrons, collectively called *nucleons*. The proton carries charge $+e$ and the neutron has zero net electric charge. A nuclide is specified by its proton number Z and neutron number N , or equivalently by Z and mass number $A = Z + N$. The notation

$${}^A_ZX \quad (22.3)$$

encodes this information compactly. Because the chemical element is determined by Z , the subscript is sometimes omitted when the chemical symbol is unambiguous, for example ${}^{14}\text{C}$ instead of ${}^{14}_6\text{C}$.

Nuclei with the same Z but different N are *isotopes*. They belong to the same chemical element but have different nuclear masses and often different stability. Nuclei with the same neutron number N are *isotones*. Nuclei with the same mass number A are *isobars*. These categories are useful because different nuclear processes preserve or change A , Z and N in characteristic ways.

A simple bookkeeping table makes the pattern visible:

Nuclide	Z	N	A	Classification example	Comment
${}^{12}_6\text{C}$	6	6	12	isotope of carbon	stable
${}^{13}_6\text{C}$	6	7	13	isotope of carbon	stable
${}^{14}_6\text{C}$	6	8	14	isotope of carbon	radioactive
${}^{14}_7\text{N}$	7	7	14	isobar of ${}^{14}\text{C}$	stable

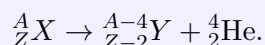
The distinction between atomic mass and nuclear mass is crucial. A neutral atom contains Z electrons in addition to its nucleus. If M_{atom} is the neutral-atomic mass and M_{nuc} is the nuclear mass, then schematically

$$M_{\text{atom}}c^2 = M_{\text{nuc}}c^2 + Zm_e c^2 - E_{\text{el,bind}}, \quad (22.4)$$

where $E_{\text{el,bind}}$ is the total electronic binding energy. On nuclear MeV scales, electronic binding energies are usually small, but they are not conceptually zero. Mass-table formulas are arranged so that electron masses cancel cleanly to the needed precision.

Method

When identifying a daughter nucleus, do not guess from the element name. Write a symbolic conservation line. For example, in alpha decay,



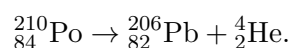
Only after the new Z is known should you use the periodic table to identify Y .

Example 22.2: Reading and transforming nuclide notation

Consider ${}_{84}^{210}\text{Po}$.

1. The proton number is $Z = 84$.
2. The mass number is $A = 210$.
3. The neutron number is $N = A - Z = 126$.
4. If it emits an alpha particle, the daughter has $A' = 206$ and $Z' = 82$, which is lead: ${}_{82}^{206}\text{Pb}$.

Thus



The mass numbers balance, $210 = 206 + 4$, and the proton numbers balance, $84 = 82 + 2$.

Guided checks

1. Two nuclei have the same A but different Z . Are they isotopes or isobars?
2. What happens to N in a beta-minus decay in which Z increases by one while A stays fixed?
3. Why can two isotopes have almost identical chemistry but very different radioactive behavior?

22.4 Nuclear size, the $A^{1/3}$ law and nuclear density

Scattering measurements show that a useful empirical scale for many nuclei is

$$R = R_0 A^{1/3}, \quad R_0 \approx 1.2 \text{ fm}. \quad (22.5)$$

The coefficient R_0 is not a universal exact constant. Real nuclear charge and matter distributions have diffuse edges, deformation can occur, and fitted values depend somewhat on the observable and model. Nevertheless, Eq. (22.5) captures one of the most important systematics of nuclear structure.

The exponent $1/3$ has immediate physical meaning. If the nuclear volume is approximated by that of a sphere,

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi R_0^3 A. \quad (22.6)$$

The volume is approximately proportional to the number of nucleons. Because the nuclear mass is also approximately proportional to A , the average density is nearly independent of A :

$$\rho_{\text{nuc}} \approx \frac{Am_N}{(4/3)\pi R_0^3 A} = \frac{3m_N}{4\pi R_0^3}, \quad (22.7)$$

where m_N is a representative nucleon mass. The cancellation of A is the key result.

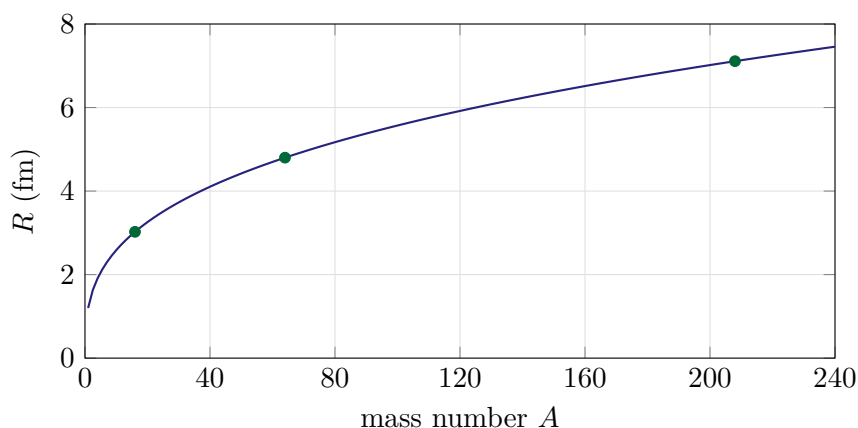


Figure 22.3: Calibrated prediction of the empirical size model $R = (1.2 \text{ fm})A^{1/3}$. Markers at $A = 16, 64, 208$ are model evaluations, not measured charge radii. The flattening curve expresses volume growth proportional to A .

Example 22.3: Radius and average density of a medium-mass nucleus

Estimate the radius of a nucleus with $A = 64$ and the order of magnitude of nuclear mass density.

Since $64^{1/3} = 4$,

$$R = (1.2 \text{ fm})(4) = 4.8 \text{ fm} = 4.8 \times 10^{-15} \text{ m}.$$

Using Eq. (22.7) with $m_N \approx 1.67 \times 10^{-27} \text{ kg}$,

$$\begin{aligned} \rho_{\text{nuc}} &\approx \frac{3(1.67 \times 10^{-27})}{4\pi(1.2 \times 10^{-15})^3} \\ &\approx 2.3 \times 10^{17} \text{ kg m}^{-3}. \end{aligned}$$

The result is not a statement that every point inside every nucleus has exactly this density. It is an average scale derived from the empirical radius law.

Reasonableness check. The factor A cancels, so repeating the estimate for a much heavier nucleus gives a similar density, as expected from the $A^{1/3}$ law.

Key message

The empirical relation $R \propto A^{1/3}$ says more than “heavy nuclei are larger.” It says nuclear volume grows approximately one nucleon-volume at a time. This saturation behavior is a clue that the effective nucleon–nucleon interaction is short-ranged rather than an unscreened long-range attraction acting equally between every pair of nucleons.

Caution

Do not compare nuclear density with ordinary material density by imagining that a macroscopic object can simply be compressed to nuclear density without new physics. Ordinary matter is supported mainly by electromagnetic and quantum effects at atomic scales; bulk nuclear matter is a qualitatively different state of matter.

Guided checks

1. If A increases by a factor of 8, by what factor does R increase?
2. If A increases by a factor of 27, by what factor does the nuclear volume increase?
3. Which step in the density derivation would fail if the radius scaled as $A^{1/2}$ instead?

22.5 The nuclear interaction: why nuclei can exist

A nucleus containing more than one proton faces an immediate electrostatic problem: every proton repels every other proton. Gravity is negligible on the femtometre scale. A stable or metastable nucleus therefore requires an attractive interaction strong enough, over nuclear distances, to overcome Coulomb repulsion. The effective interaction between nucleons is usually called the *nuclear force* or nucleon–nucleon interaction.

At the fundamental level, protons and neutrons are composite hadrons made of quarks and gluons, and their interactions arise from quantum chromodynamics (QCD). At the scale of introductory nuclear structure, however, it is useful to speak of a residual strong interaction between nucleons, much as neutral atoms can interact through residual electromagnetic forces even though each atom is overall neutral. This analogy is only qualitative, but it helps separate the fundamental QCD description from the effective nuclear-force description used in low-energy nuclear models.

Several broad properties are important:

- the attraction is strong on distances of order a few femtometres and falls rapidly at larger separation;
- the interaction is approximately charge independent after electromagnetic effects are separated, so proton–proton, neutron–neutron and proton–neutron strong interactions are closely related;
- it depends on spin and on the quantum state of the nucleon pair, so it is not a simple central force of one universal sign;
- at very short distances the effective interaction becomes strongly repulsive, preventing unlimited collapse in simple pictures;
- the short range produces saturation: a nucleon interacts most strongly with nearby nucleons rather than equally with all nucleons in a large nucleus.

No single classical potential reproduces the full nuclear interaction. The purpose of the preceding list is therefore not to define a force law to insert into $F = ma$, but to provide physical intuition for nuclear size, binding and saturation.

The competition between short-range attraction, proton–proton Coulomb repulsion, quantum shell structure and neutron–proton balance determines nuclear stability. Light stable nuclei tend to have comparable neutron and proton numbers. For heavier stable nuclei, additional neutrons are favored because they contribute attractive nuclear interactions without adding Coulomb repulsion. This trend will reappear when we discuss the chart of nuclides and beta decay.

Pedagogical boundary

The phrase “strong force” can refer to different levels of description. QCD is the fundamental theory of quarks and gluons. The force binding protons and neutrons into nuclei is an effective residual strong interaction. This chapter uses the nucleon-level description; quark and gluon dynamics belong to the later particle-physics course.

Guided checks

1. Why does Coulomb repulsion become increasingly important as Z grows?
2. How does a short-range attractive interaction help explain nearly constant nuclear density?
3. Why can adding neutrons help bind a heavy nucleus without directly increasing its Coulomb energy?

22.6 Mass defect and nuclear binding energy

The most useful quantitative measure of nuclear binding is obtained from mass. Consider a nucleus containing Z protons and $N = A - Z$ neutrons. If its mass were exactly the sum of the masses of the free nucleons, no energy would be required to separate it into those nucleons. Experimentally, a bound nucleus has a smaller mass than that sum. The difference is the *mass defect*.

Using nuclear masses,

$$\Delta m = Zm_p + Nm_n - M_{\text{nuc}}(A, Z), \quad (22.8)$$

and the binding energy is

$$B = \Delta m c^2. \quad (22.9)$$

This is the energy that must be supplied, in the idealized separation process, to produce free nucleons at rest infinitely far apart. Equivalently, it is the energy released when those free nucleons form the bound nucleus while conserving energy and momentum through emitted radiation or other products.

When neutral-atomic masses are used, it is convenient to use the mass m_{H} of a neutral hydrogen atom:

$$B \simeq [Zm_{\text{H}} + Nm_n - M_{\text{atom}}(A, Z)] c^2. \quad (22.10)$$

The electronic binding-energy correction is much smaller than typical nuclear binding energies and is usually negligible at the precision of introductory examples. Equation (22.10) works because the Z electron masses in Z neutral hydrogen atoms cancel the Z electron masses in the neutral target atom. More explicitly, let $E_{\text{el}}(A, Z) > 0$ be the total electronic binding energy of the neutral atom and $E_{\text{H}} > 0$ that of hydrogen. Substitution of the atomic-to-nuclear mass relation gives

$$B = [Zm_{\text{H}} + Nm_n - M_{\text{atom}}] c^2 - E_{\text{el}}(A, Z) + ZE_{\text{H}}. \quad (22.11)$$

The nuclear energy is obtained by subtracting the excess electronic binding included in the atomic-mass expression. This correction is usually negligible for a rounded total binding energy, but can matter when precision or a very small threshold is at issue.

Caution

Binding energy is *not* an energy stored inside the nucleus waiting to leak out. A bound system has lower total energy than its separated constituents. The positive number B is the energy required to dismantle the bound system into free nucleons, or the magnitude of the energy released when the bound state is formed from those constituents.

Example 22.4: Binding energy and binding energy per nucleon

Use the following approximate atomic masses to calculate the binding energy of ${}^4_2\text{He}$:

$$m_{\text{H}} = 1.007825 \, u, \quad m_n = 1.008665 \, u, \quad M({}^4\text{He}) = 4.002603 \, u.$$

For ${}^4\text{He}$, $Z = 2$ and $N = 2$. Using atomic masses,

$$\begin{aligned}\Delta m &= 2m_{\text{H}} + 2m_n - M({}^4\text{He}) \\ &= 2(1.007825) + 2(1.008665) - 4.002603 \\ &= 0.030377 \text{ u}.\end{aligned}$$

Therefore

$$\begin{aligned}B &= (0.030377 \text{ u})(931.5 \text{ MeV/u}) \\ &\approx 28.3 \text{ MeV}.\end{aligned}$$

The binding energy per nucleon is

$$\frac{B}{A} \approx \frac{28.3}{4} \text{ MeV} \approx 7.07 \text{ MeV}.$$

Interpretation. The neutral helium atom is lighter than two neutral hydrogen atoms plus two free neutrons by about $0.0304u$. The corresponding energy difference is tens of MeV, illustrating why small relative mass differences are physically important in nuclear systems.

Method

For a binding-energy calculation from a mass table:

1. determine whether the table gives nuclear masses or neutral-atomic masses;
2. compute $N = A - Z$;
3. use either Eq. (22.8) or Eq. (22.10), not a mixture;
4. convert u to MeV using $1u c^2 \simeq 931.5 \text{ MeV}$;
5. divide by A only if the problem asks for binding energy per nucleon;
6. check that the result is positive for a bound stable nucleus and that B/A is of order several MeV for ordinary nuclei.

Guided checks

1. A bound nucleus has less mass than its free constituent nucleons. Does this violate mass conservation?
2. Why is B/A often more useful than B when comparing nuclei of very different sizes?
3. In an atomic-mass calculation, why is m_{H} more convenient than m_p ?

22.7 Binding energy per nucleon and the landscape of stability

Total binding energy generally grows as more nucleons are added, so comparing B alone would make heavy nuclei appear automatically “more bound.” A more informative measure of average binding is the binding energy per nucleon B/A . Its broad dependence on A is one of the central organizing plots in nuclear physics.

For very light nuclei, B/A rises rapidly as nucleons are assembled into increasingly favorable bound configurations. Across medium-mass nuclei it reaches values near 8–9 MeV per nucleon, with the broad maximum in the iron–nickel region around $A \sim 60$. For still heavier nuclei, B/A decreases slowly because long-range Coulomb repulsion between protons becomes increasingly costly while the attractive nuclear interaction remains short-ranged and saturating.

Figure 22.4 distinguishes a calculated smooth model from mass-based light-nucleus markers. It should not be interpreted as a smooth exact function for every nuclide. Real nuclear masses show pairing effects, shell effects and local structure. The broad shape nevertheless explains why energy can be released both by combining very light nuclei and by splitting very heavy nuclei into medium-mass fragments. Those energy-production processes are developed systematically in the next chapter.

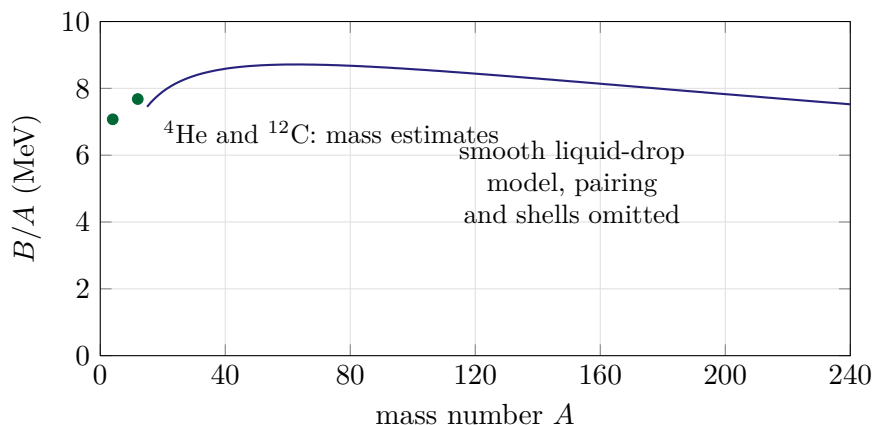


Figure 22.4: A quantitative illustration of average binding. The smooth line uses the four-term liquid-drop model discussed in the enrichment section, with coefficients (15.7, 17.8, 0.71, 23.6) MeV and continuous $Z = A/[2 + (0.71/(2 \times 23.6))A^{2/3}]$. It is not a measured mass curve. The helium and carbon markers follow the atomic-mass examples; local shell and pairing effects are absent from the line.

It is tempting to call the nucleus with the largest B/A “the most stable nucleus,” but the word *stable* has several meanings. A nuclide can be stable against one type of separation yet unstable to beta decay. Kinematic stability requires that no energetically allowed decay channel with suitable conservation laws is available. Dynamical stability additionally depends on the transition probability: a process may be energetically allowed but strongly suppressed, leading to an extremely long lifetime. Thus mass energetics and decay rates answer different questions.

The neutron-to-proton ratio is another major stability coordinate. For light nuclei, stable nuclides lie close to $N \approx Z$. As Z increases, stable nuclei require $N > Z$. Nuclei with too many neutrons for their proton number often move toward stability by beta-minus decay, converting a neutron into a proton. Proton-rich nuclei may undergo beta-plus decay or electron capture, converting a proton into a neutron. Very heavy nuclei can also become unstable to alpha emission or fission.

Key message

The binding-energy curve is an energy landscape, not a complete decay chart. It tells us about average binding and available energy, but a decay rate also depends on quantum transition probabilities, angular momentum, barrier penetration and the detailed initial and final nuclear states.

Example 22.5: Energy-scale comparison from a mass difference

Two nuclear configurations differ in mass by $0.0040 u$. What rest-energy difference does this represent?

$$\Delta E = (0.0040 u)c^2 = (0.0040)(931.5 \text{ MeV}) \approx 3.73 \text{ MeV}.$$

An apparently tiny mass difference at the level of four thousandths of an atomic mass unit therefore corresponds to several MeV. This is comparable with typical nuclear transition and decay energies and vastly larger than typical visible-photon energies of a few eV.

Guided checks

1. Why does B/A rather than B reveal a broad maximum near medium mass?
2. If the nuclear attraction were long-ranged and every nucleon attracted every other nucleon equally, would saturation and nearly constant density be natural expectations?
3. Why is an energetically allowed decay not necessarily fast?

22.8 Qualitative nuclear models: liquid drop and shell structure

A nucleus contains many interacting quantum particles. There is no single picture that makes every property intuitive. Nuclear physics therefore uses complementary models, each designed to organize a different class of observations. Two of the most influential are the liquid-drop model and the shell model.

The *liquid-drop model* emphasizes collective and average properties. The roughly constant nuclear density resembles the nearly constant density of a liquid drop. Short-range attraction gives a volume-like binding contribution; nucleons at the surface have fewer neighbors and therefore contribute a surface correction; proton–proton repulsion gives a Coulomb correction; neutron–proton imbalance costs energy; and pairing produces an additional even–odd effect. In the advanced section we turn this qualitative picture into the semi-empirical mass formula.

The *shell model* emphasizes single-particle quantum states in an average nuclear potential. Just as atomic electrons fill quantum shells, nucleons occupy quantized nuclear states. Particularly stable configurations occur when proton or neutron shells are filled. The resulting “magic numbers”

$$2, 8, 20, 28, 50, 82, 126 \quad (22.12)$$

appear repeatedly in nuclear systematics. Here 126 is the established neutron closure; the shorter sequence through 82 includes the familiar proton closures. The analogy with atomic shells is useful, but one must not push it too far: nuclear nucleons interact strongly with one another, proton and neutron states differ, and nuclear spin–orbit coupling is essential to the observed shell sequence.

The two models are not mutually exclusive. A heavy nucleus can exhibit collective vibrations and rotations reminiscent of a deformable drop while also displaying shell effects associated with quantized nucleon motion. Model choice is guided by the observable of interest.

Note

A physical model is not “true” or “false” in the same sense as a conservation law. It has a domain of usefulness. The liquid-drop model captures smooth trends in binding and collective deformation; the shell model captures quantized level structure, magic numbers and many spin/parity patterns. Good nuclear physics often comes from knowing which model should be trusted for which question.

Guided checks

1. Which feature of nuclei motivates the liquid-drop analogy most directly?
2. Why can a shell correction create local deviations from the smooth binding-energy curve?
3. What observation would tell you that a purely smooth liquid-drop description is incomplete?

22.9 Radioactivity as a stochastic physical process

Some nuclei are unstable even when left completely undisturbed. They transform spontaneously into other nuclear states while emitting particles or electromagnetic radiation. This phenomenon is

radioactivity. The word “spontaneous” does not mean uncaused in the sense of violating physics. It means that no external trigger such as heating, illumination or collision is required for the decay of an isolated unstable nucleus.

The essential conceptual point is probabilistic. For a prepared unstable nucleus, quantum theory does not normally predict a deterministic decay time. Instead, a decay constant λ specifies the probability per unit time for decay, provided the state and environment remain unchanged. If dt is short enough that $\lambda dt \ll 1$, then

$$P(\text{decay during } dt) \simeq \lambda dt. \quad (22.13)$$

The probability is approximately proportional to the duration of the short observation interval and, for a memoryless decay, does not depend on how long the nucleus has already survived.

This last statement often feels counterintuitive. A nucleus that has survived ten half-lives is not “overdue” in the way a mechanical component with wear and fatigue might be. Conditional on still being present in the same unstable state, it has the same decay probability per unit time as before. The exponential decay law is the macroscopic expression of this memoryless behavior. It is an excellent working approximation for an unchanged unstable state over ordinary observation times, rather than a theorem for all quantum times. Electron-capture rates can depend on electron availability; changing the ionization state can change that channel. The fixed-constant model assumes the relevant environment is unchanged.

Let $N(t)$ be the expected number of undecayed nuclei in a large ensemble. During a short time dt , the expected number of decays is $\lambda N dt$, so

$$dN = -\lambda N dt. \quad (22.14)$$

Dividing by dt gives the differential equation

$$\frac{dN}{dt} = -\lambda N. \quad (22.15)$$

Separating variables,

$$\frac{dN}{N} = -\lambda dt, \quad (22.16)$$

and integrating from N_0 at $t = 0$ to $N(t)$ at time t gives

$$\ln\left(\frac{N(t)}{N_0}\right) = -\lambda t. \quad (22.17)$$

Hence

$$\boxed{N(t) = N_0 e^{-\lambda t}}. \quad (22.18)$$

The activity is the expected decay rate,

$$\mathcal{A}(t) = -\frac{dN}{dt} = \lambda N(t) = \mathcal{A}_0 e^{-\lambda t}, \quad (22.19)$$

where $\mathcal{A}_0 = \lambda N_0$. Activity therefore obeys the same exponential time dependence as the number of radioactive nuclei.

Key message

Radioactive decay is random at the level of a single nucleus but highly predictable statistically for a large ensemble. The exponential curve is an expectation value. A finite sample exhibits counting fluctuations around that expectation.

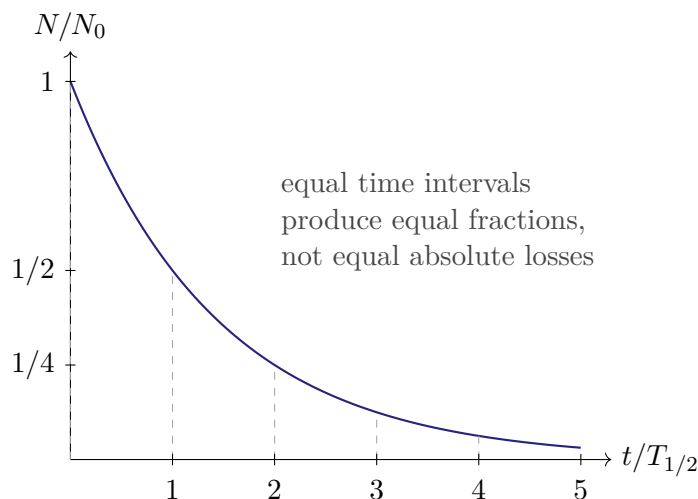


Figure 22.5: Exponential radioactive decay plotted in units of the half-life. Each additional half-life multiplies the expected surviving population by $1/2$.

Caution

The activity $\mathcal{A}$ is not generally equal to the count rate displayed by a detector. A detector records only some emitted quanta, may have less than 100% efficiency, may subtend only part of the solid angle, and also records background. Detector dead time can further reduce the observed rate.

22.10 Half-life, mean lifetime and logarithmic forms

The *half-life* $T_{1/2}$ is defined as the time for the expected number of undecayed nuclei to decrease by a factor of two:

$$N(T_{1/2}) = \frac{N_0}{2}. \quad (22.20)$$

Substituting Eq. (22.18),

$$\frac{1}{2} = e^{-\lambda T_{1/2}}, \quad (22.21)$$

so

$$T_{1/2} = \frac{\ln 2}{\lambda}. \quad (22.22)$$

The *mean lifetime* is

$$\tau = \frac{1}{\lambda}. \quad (22.23)$$

Therefore

$$T_{1/2} = \tau \ln 2 \approx 0.693\tau, \quad \tau \approx 1.443T_{1/2}. \quad (22.24)$$

The mean lifetime is longer than the half-life because the exponential distribution has a long tail of nuclei that survive for several half-lives.

The decay law can also be written directly in terms of half-life:

$$N(t) = N_0 2^{-t/T_{1/2}}, \quad \mathcal{A}(t) = \mathcal{A}_0 2^{-t/T_{1/2}}. \quad (22.25)$$

This form is especially efficient when the elapsed time is an integer or simple multiple of $T_{1/2}$.

Taking a logarithm linearizes the exponential law:

$$\ln[N(t)/N_0] = -\lambda t, \quad (22.26)$$

so a plot of $\ln N$ versus t is a straight line with slope $-\lambda$ for an ideal single-component decay. In laboratory work, this linearization is useful for visualization but is not automatically the statistically optimal fitting method, especially when counts are small and uncertainties are Poissonian. The advanced section discusses this point in more detail.

Example 22.6: Activity after several half-lives

A sealed calibration source has initial activity $\mathcal{A}_0 = 12.0 \text{ kBq}$ and half-life $T_{1/2} = 6.0 \text{ h}$. What activity remains after 18.0 h ?

The elapsed time is three half-lives:

$$\frac{t}{T_{1/2}} = \frac{18.0}{6.0} = 3.$$

Therefore

$$\mathcal{A} = \mathcal{A}_0 \left(\frac{1}{2}\right)^3 = 12.0 \text{ kBq} \times \frac{1}{8} = 1.50 \text{ kBq}.$$

Reasonableness check. The activity must be smaller than the initial value, and three successive halvings $12 \rightarrow 6 \rightarrow 3 \rightarrow 1.5 \text{ kBq}$ confirm the result without exponentials.

Example 22.7: Finding a decay constant from a half-life

A radionuclide has $T_{1/2} = 2.50 \text{ d}$. Calculate its decay constant in s^{-1} and its mean lifetime in days.

Convert the half-life to seconds:

$$T_{1/2} = 2.50(86400) \text{ s} = 2.16 \times 10^5 \text{ s}.$$

Then

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.6931}{2.16 \times 10^5} \text{ s}^{-1} \approx 3.21 \times 10^{-6} \text{ s}^{-1}.$$

The mean lifetime is

$$\tau = \frac{T_{1/2}}{\ln 2} \approx \frac{2.50}{0.6931} \text{ d} \approx 3.61 \text{ d}.$$

Method

Choose the most efficient decay formula for the data given.

- If the time is expressed in half-lives, use $2^{-t/T_{1/2}}$.
- If λ is given, use $e^{-\lambda t}$.
- If two measurements of the same ideal sample are given, divide the equations so that the unknown initial amount cancels.
- When solving for time, take a logarithm only after forming a dimensionless ratio such as N/N_0 or $\mathcal{A}/\mathcal{A}_0$.

Guided checks

1. After two half-lives, what fraction has decayed and what fraction remains?
2. Can a half-life be longer than the mean lifetime for an exponential decay?
3. What would curvature in a plot of $\ln \mathcal{A}$ versus t suggest about the assumptions of a single

ideal decay component?

22.11 Activity, number of nuclei and specific activity

Because $\mathcal{A} = \lambda N$, the same measured activity can correspond to very different numbers of radioactive nuclei depending on the half-life. A short-lived nuclide has a large λ , so relatively few nuclei can produce a large activity. A long-lived nuclide may require an enormous number of nuclei to produce the same activity.

If a pure sample of a radionuclide has mass m_s , molar mass M_m and contains $n = m_s/M_m$ moles, then the approximate number of radioactive nuclei is

$$N = nN_A = \frac{m_s}{M_m} N_A, \quad (22.27)$$

where N_A is Avogadro's constant. The activity is therefore

$$\mathcal{A} = \lambda \frac{m_s}{M_m} N_A. \quad (22.28)$$

The activity per unit mass is called the *specific activity*. For a pure radionuclide,

$$a_s \equiv \frac{\mathcal{A}}{m_s} = \frac{\lambda N_A}{M_m}. \quad (22.29)$$

This relation shows why short-lived radionuclides tend to have high specific activity.

Example 22.8: Number of radioactive nuclei from a measured activity

A sample has activity 2.0 kBq and half-life 8.0 h. Estimate the number of radioactive nuclei present.

First,

$$\lambda = \frac{\ln 2}{8.0 \times 3600 \text{ s}} \approx 2.41 \times 10^{-5} \text{ s}^{-1}.$$

Using $N = \mathcal{A}/\lambda$,

$$N = \frac{2.0 \times 10^3 \text{ s}^{-1}}{2.41 \times 10^{-5} \text{ s}^{-1}} \approx 8.3 \times 10^7.$$

The number is large on a counting scale but microscopically tiny compared with Avogadro's number.

Caution

A high specific activity does not by itself determine biological hazard. Hazard depends on radiation type and energy, route of exposure, chemical behavior, spatial distribution, biological retention and shielding, among other factors. Activity is only one part of a dosimetric assessment.

22.12 Decay modes and conservation-law bookkeeping

The most common introductory decay modes are alpha, beta and gamma decay. Their phenomenology differs because the emitted radiation and the underlying interactions differ. The bookkeeping rules, however, are unified by conservation laws.

At the level used in this chapter, check the following quantities for every nuclear process:

- electric charge;
- total energy, including rest energies and kinetic energies;
- linear momentum;
- angular momentum;
- nucleon number A in ordinary low-energy nuclear transformations;
- lepton number when beta processes are written with electrons, positrons and neutrinos.

More advanced particle physics refines some of these statements and explains which conservation laws are fundamental or approximate. For nuclear-decay bookkeeping, the list above is sufficient.

Figure 22.6 summarizes the corresponding changes on a chart of neutron and proton numbers.

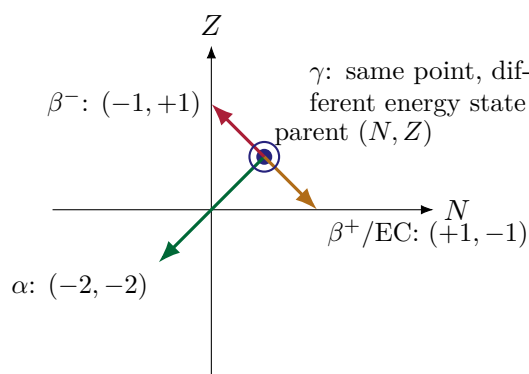


Figure 22.6: Local displacement on a nuclide chart with neutron number horizontal and proton number vertical. Labels give $(\Delta N, \Delta Z)$. Beta transitions move along fixed $A = N + Z$; alpha decay reduces A by four. Gamma emission changes only excitation, so it remains at the same chart point. Absolute axis origins are schematic.

The energy release of a decay is measured by its Q value. For a parent at rest decaying into specified products,

$$Q = (M_{\text{initial}} - M_{\text{final}}) c^2, \quad (22.30)$$

provided all masses are defined consistently. If $Q > 0$, the decay is energetically allowed in the sense that rest-energy decrease can appear as kinetic energy and, where relevant, photon energy. If $Q < 0$, an isolated parent cannot undergo that channel spontaneously.

Energy conservation alone does not determine whether a decay is rapid. A transition can be energetically open yet have a very long lifetime because of quantum tunnelling, weak-interaction strength, angular-momentum barriers or selection rules.

22.13 Alpha decay

An alpha particle is a ${}^4_2\text{He}$ nucleus containing two protons and two neutrons. In alpha decay,



The parent loses four units of mass number and two units of proton number. Alpha decay is common among sufficiently heavy nuclei because emitting a tightly bound helium nucleus can reduce Coulomb energy while moving the daughter toward a more favorable binding configuration.

Using neutral-atomic masses, the electron bookkeeping is especially simple because the daughter atom has $Z - 2$ electrons and the neutral helium atom has two. Therefore

$$Q_\alpha = [M_{\text{atom}}(X) - M_{\text{atom}}(Y) - M_{\text{atom}}({}^4\text{He})] c^2. \quad (22.32)$$

For a two-body decay of a parent initially at rest, momentum conservation requires the daughter nucleus and alpha particle to recoil in opposite directions with equal momentum magnitude. The lighter alpha particle usually receives most of the kinetic energy because for nonrelativistic recoil $K = p^2/(2m)$.

Classically, alpha decay presents a paradox. The alpha particle is bound by the short-range nuclear attraction, while outside the nucleus it faces a positive Coulomb barrier. The measured alpha-particle energy can be lower than the barrier maximum, so a classical particle could not escape. Quantum tunnelling resolves the paradox. The advanced section develops the barrier-penetration picture and explains qualitatively why alpha-decay half-lives are extraordinarily sensitive to alpha-particle energy.

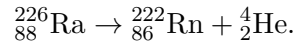
Example 22.9: Identifying an alpha-decay daughter and energy partition

Suppose $^{226}_{88}\text{Ra}$ undergoes alpha decay.

The daughter has

$$A' = 226 - 4 = 222, \quad Z' = 88 - 2 = 86.$$

Element $Z = 86$ is radon, so



If the total kinetic-energy release were, for illustration, $Q = 4.9 \text{ MeV}$ and both products are nonrelativistic, the common momentum p gives

$$\frac{K_\alpha}{K_D} = \frac{M_D}{M_\alpha} \approx \frac{222}{4} = 55.5.$$

Thus the alpha particle carries approximately

$$K_\alpha = Q \frac{M_D}{M_D + M_\alpha} \approx 4.9 \text{ MeV} \frac{222}{226} \approx 4.81 \text{ MeV},$$

while the daughter recoil receives only about 0.09 MeV. The numerical Q value here is used only to illustrate two-body energy partition; the isotope identification follows purely from conservation of A and Z .

Key message

Alpha decay changes the identity of the element because Z decreases by two. Its quantum-mechanical signature is barrier tunnelling: the emitted alpha particle can escape even when its energy is below the classical Coulomb-barrier height.

22.14 Beta decay, positron emission and electron capture

Beta processes alter the neutron-to-proton balance of a nucleus without changing the mass number A . They are governed by the weak interaction. The three basic modes are beta-minus decay, beta-plus decay and electron capture.

In beta-minus decay, a neutron in the nuclear environment transforms according to

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad (22.33)$$

so the nuclear transformation is

$$^A_Z X \rightarrow ^A_{Z+1} Y + e^- + \bar{\nu}_e. \quad (22.34)$$

The mass number remains fixed and Z increases by one. The emitted electron is created in the weak transition; it is not a pre-existing electron trapped inside the nucleus.

In beta-plus decay,

$$p \rightarrow n + e^+ + \nu_e \quad (22.35)$$

within a suitable nucleus, giving

$${}_Z^AX \rightarrow {}_{Z-1}^AY + e^+ + \nu_e. \quad (22.36)$$

The mass number remains fixed and Z decreases by one. A free proton does not spontaneously undergo this process because the final rest mass exceeds the proton mass, but nuclear binding-energy differences can make beta-plus decay energetically possible inside some nuclei.

Electron capture competes with beta-plus decay in many proton-rich nuclei:

$$p + e^- \rightarrow n + \nu_e, \quad (22.37)$$

so

$${}_Z^AX + e^- \rightarrow {}_{Z-1}^AY + \nu_e. \quad (22.38)$$

The captured electron is usually from an inner atomic shell, after which the atomic electron cloud rearranges and can emit characteristic x rays or Auger electrons.

The neutrino is essential. Early beta-decay measurements showed that emitted electron energies form a continuous spectrum rather than a single fixed energy expected for a simple two-body decay. The missing energy and momentum are carried by the neutrino or antineutrino. The beta particle and neutrino share the available decay energy, together with the small daughter recoil.

Using neutral-atomic masses, the commonly used introductory Q -value formulas are

$$Q_{\beta^-} = [M_{\text{atom}}(X) - M_{\text{atom}}(Y)] c^2, \quad (22.39)$$

$$Q_{\beta^+} = [M_{\text{atom}}(X) - M_{\text{atom}}(Y) - 2m_e] c^2, \quad (22.40)$$

$$Q_{\text{EC}} \simeq [M_{\text{atom}}(X) - M_{\text{atom}}(Y)] c^2, \quad (22.41)$$

where atomic electron binding-energy corrections are omitted at the introductory precision. The nuclide equations above use nuclear charges, whereas the tabulated M_{atom} values refer to neutral atoms. Electron rest-mass cancellation is a bookkeeping conversion, not a claim that the immediate daughter is neutral. Atomic ionization, excitation and subsequent relaxation require separate accounting at high precision. In electron capture, the neutrino energy for a particular shell and daughter level is approximately $Q_{\text{EC}} - E_x - E_{\text{hole}} - K_Y$; a positive total mass difference does not guarantee every level or capture shell is accessible.

The $2m_e$ term in beta-plus decay is a common source of mistakes: one electron mass accounts for the emitted positron, and the second arises from the difference between the numbers of electrons included in the neutral-atomic masses.

Example 22.10: Daughter nuclei in beta processes

Determine the daughter nuclide for each transformation.

(a) **Beta-minus decay of ${}^{14}_6\text{C}$.** Here A stays 14 and Z increases to 7:

$${}^{14}_6\text{C} \rightarrow {}^{14}_7\text{N} + e^- + \bar{\nu}_e.$$

(b) **Beta-plus decay of ${}^{22}_{11}\text{Na}$.** Here A stays 22 and Z decreases to 10:

$${}^{22}_{11}\text{Na} \rightarrow {}^{22}_{10}\text{Ne} + e^+ + \nu_e.$$

(c) **Electron capture by ${}^7_4\text{Be}$.** Again A stays 7 and Z decreases to 3:

$${}^7_4\text{Be} + e^- \rightarrow {}^7_3\text{Li} + \nu_e.$$

These transformations illustrate that beta processes change the proton–neutron balance rather than removing nucleons from the nucleus.

Caution

Do not omit the neutrino from beta decay. Without it, the observed continuous beta spectrum and event-by-event conservation of energy, momentum and angular momentum cannot be described correctly.

Guided checks

1. In beta-minus decay, which quantity stays fixed: A , Z or N ?
2. Why does beta-plus decay require more available mass energy than electron capture when atomic masses are used?
3. Why is the beta-particle energy spectrum continuous while a simple two-body alpha decay produces nearly discrete alpha energies for a fixed transition?

22.15 Gamma decay and nuclear excited states

A nucleus can be produced in an excited quantum state after a preceding alpha or beta decay, a nuclear reaction or electromagnetic excitation. It can then move to a lower-energy state by emitting a gamma-ray photon:



The asterisk denotes an excited nuclear state. Neither A nor Z changes. The emitted photon energy is approximately the nuclear level spacing, reduced slightly because the recoiling nucleus must carry momentum.

For a transition energy ΔE and a heavy daughter nucleus of mass M , momentum conservation gives photon momentum $p_\gamma = E_\gamma/c$ and recoil energy approximately

$$K_{\text{recoil}} \approx \frac{p_\gamma^2}{2M} = \frac{E_\gamma^2}{2Mc^2}. \quad (22.43)$$

Thus

$$\Delta E = E_\gamma + K_{\text{recoil}}. \quad (22.44)$$

The recoil correction is usually small because Mc^2 is many GeV for a medium or heavy nucleus, while E_γ is often keV to MeV.

Figure 22.7 illustrates direct and cascade de-excitation paths. The total excitation-energy decrease is the same for either route, although a cascade emits more than one photon per parent de-excitation.

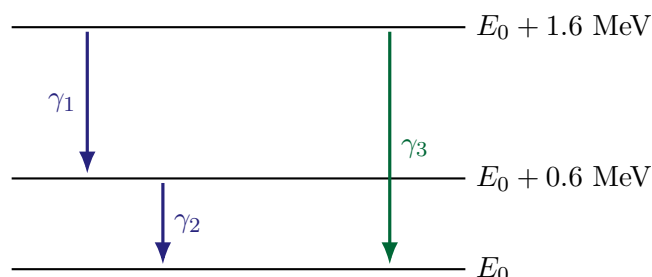


Figure 22.7: Illustrative excitation levels of one nuclide. A cascade releases approximately 1.0 and 0.6 MeV photons; a direct transition releases approximately 1.6 MeV. Recoil is neglected in the indicated differences, and no isotope-specific level scheme or branching fractions are asserted. A and Z remain unchanged for every arrow.

Some excited nuclear states have unusually long lifetimes because direct electromagnetic transitions are hindered by angular-momentum or parity constraints. Such states are called *metastable* or

isomeric states and are often denoted by an “m,” as in ^{99m}Tc . Nuclear isomers are not different chemical isotopes; they have the same A and Z but different internal nuclear excitation states.

Gamma emission is not the only way an excited nucleus can lose energy. In *internal conversion*, the nucleus transfers its excitation energy to an atomic electron, which is ejected. This mechanism is useful to know because it reminds us that “gamma decay” is shorthand for nuclear electromagnetic de-excitation and does not guarantee that every de-excitation emits a gamma photon.

Example 22.11: Gamma energy and recoil scale

A nucleus of mass approximately $100u$ emits a 1.00 MeV gamma ray. Estimate the nuclear recoil energy.

The nuclear rest energy is approximately

$$Mc^2 \approx 100(931.5\text{ MeV}) = 9.315 \times 10^4\text{ MeV}.$$

Then

$$K_{\text{recoil}} \approx \frac{E_\gamma^2}{2Mc^2} = \frac{(1.00\text{ MeV})^2}{2(9.315 \times 10^4\text{ MeV})} \approx 5.4 \times 10^{-6}\text{ MeV} \approx 5.4\text{ eV}.$$

The recoil energy is tiny compared with 1 MeV , but it is not exactly zero. Precision gamma spectroscopy must account for recoil and other line-shift effects.

Guided checks

1. Does gamma emission change the chemical element?
2. Why must an emitting nucleus recoil even if the recoil energy is very small?
3. How can two nuclear states have the same A and Z but different energies?

22.16 Decay chains, branching and daughter activity

Many radioactive nuclei do not decay directly to a stable daughter. Instead, the daughter is itself radioactive, producing a *decay chain*. A simple chain has the form

$$X_1 \xrightarrow{\lambda_1} X_2 \xrightarrow{\lambda_2} X_3 \rightarrow \dots \quad (22.45)$$

For the parent,

$$\frac{dN_1}{dt} = -\lambda_1 N_1. \quad (22.46)$$

For the daughter,

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2. \quad (22.47)$$

The first term produces daughter nuclei and the second removes them. Unlike an isolated single nuclide, the daughter population need not decrease monotonically; it can initially grow and later decline.

Some nuclei have more than one possible decay channel. If channel i has partial decay constant λ_i , then the total decay constant is

$$\lambda = \sum_i \lambda_i. \quad (22.48)$$

The *branching fraction* for channel i is

$$b_i = \frac{\lambda_i}{\lambda}, \quad \sum_i b_i = 1. \quad (22.49)$$

For a macroscopic sample, approximately a fraction b_i of all decays follow that branch, subject to counting fluctuations and detection efficiencies.

Note

A decay chain is a system of coupled differential equations. This provides a direct link between nuclear physics and mathematical methods used in chemical kinetics, compartment models, epidemiology, signal processing and dynamical systems. The physics differs, but the equation structure is shared.

Guided checks

1. Can the number of daughter nuclei increase even while every individual daughter nucleus is unstable?
2. If two decay branches have equal partial decay constants, what are their branching fractions?
3. Why is the total half-life determined by the sum of partial decay constants rather than the sum of partial half-lives?

22.17 How nuclear radiation interacts with matter

Alpha particles, beta particles and gamma rays have different charges, masses and interaction mechanisms, so their penetration and energy-deposition patterns differ strongly. The familiar statement that alpha radiation is “least penetrating” and gamma radiation “most penetrating” is useful only as a first orientation. Real attenuation depends on particle energy, material composition, thickness and geometry.

An alpha particle has charge $+2e$ and a mass near $4u$. It interacts strongly with atomic electrons and loses energy rapidly through ionization and excitation. In ordinary matter its range is comparatively short. Consequently alpha radiation is usually easy to stop externally, but an alpha-emitting radionuclide can be important if taken into the body because the energy can be deposited over a short microscopic path.

Beta particles are electrons or positrons. They are much lighter and undergo repeated scattering by atomic electrons and nuclei. Their paths can be tortuous, and their range is longer than that of typical alpha particles of comparable nuclear-decay energies. When energetic electrons are decelerated in the electric fields of nuclei, they can emit bremsstrahlung x rays. For this reason, shielding design for beta radiation is not captured by the simplistic instruction “use the densest material available.”

Gamma rays are photons. They carry no electric charge and interact probabilistically through processes such as photoelectric absorption, Compton scattering and, at sufficiently high energy, pair production. Gamma intensity therefore tends to decrease approximately exponentially in a uniform absorber:

$$I(x) = I_0 e^{-\mu x}, \quad (22.50)$$

where μ is the linear attenuation coefficient for the specified photon energy and material. The half-value layer is

$$x_{1/2} = \frac{\ln 2}{\mu}. \quad (22.51)$$

This equation describes attenuation of a narrow beam under specified conditions. Scattered radiation, buildup, broad-beam geometry and secondary radiation require more detailed treatment.

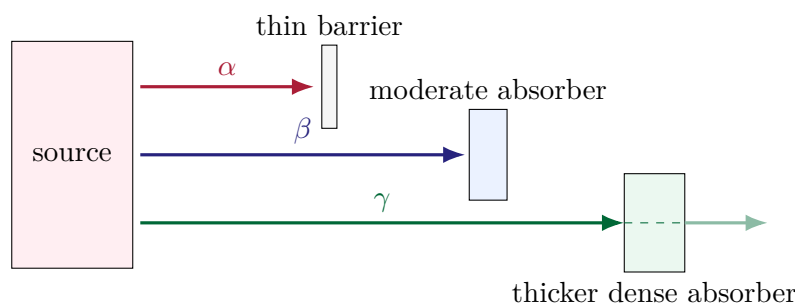


Figure 22.8: Qualitative comparison of penetration. The sketch is deliberately schematic: actual attenuation depends on energy and material, and beta shielding can generate bremsstrahlung.

Caution

Radiation shielding is an engineering calculation, not a memorized material list. The safe choice depends on radiation type and energy, source strength, geometry, occupancy, secondary radiation and regulatory limits. Classroom comparisons such as “paper stops alpha” are qualitative illustrations, not complete safety procedures.

22.18 Radiation detection: from ionization to electronic signals

Radiation is detected indirectly through the energy it deposits in matter. A detector converts microscopic interactions into a measurable electrical or optical signal. The detector type determines what information is available: some devices mainly count events, while others measure energy, timing, position or combinations of these quantities.

A gas-filled detector uses ionization in a gas. In a Geiger–Müller tube, a sufficiently energetic ionizing event initiates a gas avalanche that produces a large standardized pulse. This makes the device robust for counting but poor for measuring the original particle energy because pulse height carries little energy information.

A scintillation detector converts deposited energy into visible or ultraviolet photons. A photomultiplier tube, silicon photomultiplier or photodiode converts the light pulse into an electrical signal. Pulse height can be approximately related to deposited energy after calibration.

A semiconductor detector creates electron–hole pairs when radiation deposits energy in a depleted region. Because the energy required to produce one charge pair is small, semiconductor detectors can provide excellent energy resolution. Germanium detectors are widely used for precision gamma spectroscopy, while silicon detectors are common in charged-particle and x-ray measurements.

A detector measurement is characterized by more than its sensing material. Relevant concepts include intrinsic efficiency, geometric acceptance, energy resolution, timing resolution, dead time, threshold, background and calibration. If a source emits at rate $\mathcal{A}$ and the detector has overall efficiency ϵ for the radiation and geometry of interest, then a simplified expected signal count rate is

$$R_s \approx \epsilon b \mathcal{A}, \quad (22.52)$$

where b is the relevant emission or branching probability. A background rate R_b adds to the measured rate,

$$R_{\text{meas}} = R_s + R_b, \quad (22.53)$$

provided dead-time losses are negligible.

Example 22.12: Efficiency and background correction

A source has activity 8.0 kBq and emits the gamma line of interest in 60% of decays. The total photopeak efficiency for the measurement geometry is 2.5%. The background in the same photopeak window is 3.0 counts/s. Estimate the measured count rate. The expected source photopeak rate is

$$R_s = (0.025)(0.60)(8.0 \times 10^3 \text{ s}^{-1}) = 120 \text{ s}^{-1}.$$

Therefore

$$R_{\text{meas}} \approx 120 + 3 = 123 \text{ s}^{-1}.$$

This example illustrates why activity and count rate are different physical quantities.

Guided checks

1. Why can a Geiger counter count radiation well while providing little spectroscopic information?
2. If the solid angle or source–detector distance changes, should the intrinsic detector efficiency necessarily change?
3. Why is a background measurement needed even when no radioactive source has been intentionally placed near the detector?

22.19 Activity, absorbed dose and equivalent dose are different quantities

Three quantities that students often mix are source activity, absorbed dose and equivalent dose. They answer different questions.

The *activity* $\mathcal{A}$ counts nuclear transformations per unit time and is measured in becquerels. It is a property of the radioactive source population.

The *absorbed dose* D measures energy deposited per unit mass:

$$D = \frac{E_{\text{dep}}}{m}, \quad 1 \text{ Gy} = 1 \text{ J kg}^{-1}. \quad (22.54)$$

Dose therefore depends on how much of the emitted radiation reaches and deposits energy in the material of interest.

The *equivalent dose* to an organ or tissue T uses the absorbed dose $D_{T,R}$ averaged over that tissue from radiation component R :

$$H_T = \sum_R w_R D_{T,R}, \quad H = w_R D \quad \text{for one component.} \quad (22.55)$$

The effective dose further combines tissue contributions,

$$E_{\text{eff}} = \sum_T w_T H_T, \quad (22.56)$$

using tissue weighting factors w_T . Both quantities are measured in sieverts, with the same SI dimensions as the gray. Their weighting conventions support radiological protection; effective dose is not a personalized prediction of biological injury. Absorbed dose remains the underlying physical energy-per-mass quantity [5].

A large activity does not imply a large dose unless the radiation reaches the target and deposits energy there. Conversely, a small amount of a radionuclide can be important if its radiation energy, biological localization and residence time lead to concentrated energy deposition in sensitive tissue. This distinction is central to medical imaging, therapy and radiation protection.

Example 22.13: A simple absorbed-dose estimate

Suppose a small material sample of mass 20 g absorbs 3.0 mJ of ionizing-radiation energy during an exposure. Estimate the absorbed dose.

Convert units:

$$m = 20 \text{ g} = 2.0 \times 10^{-2} \text{ kg}, \quad E_{\text{dep}} = 3.0 \times 10^{-3} \text{ J}.$$

Then

$$D = \frac{3.0 \times 10^{-3}}{2.0 \times 10^{-2}} \text{ J kg}^{-1} = 0.15 \text{ Gy}.$$

This calculation determines absorbed dose only. It does not by itself determine equivalent dose, biological outcome or medical risk.

Caution

Do not use dose units as interchangeable labels. The gray is energy per mass. The sievert is a radiation-protection quantity that includes weighting factors. The becquerel is decays per second. The becquerel has dimension inverse time. Gray and sievert both have SI dimensions J kg^{-1} , but represent different quantities; their dimensional equality does not make them interchangeable.

22.20 Radiometric dating and radioactive clocks

Radioactive decay can be used as a clock when an initial condition or reference ratio can be established. The basic mathematics follows directly from the decay law, but the reliability of a real dating method depends on geochemical or archaeological assumptions in addition to nuclear physics.

Radiocarbon dating is the standard introductory example. Living organisms exchange carbon with their environment, including a small fraction of radioactive ^{14}C . After death, exchange ceases and the existing ^{14}C inventory decreases by beta-minus decay. The half-life is approximately 5730 y. If a sample's present ^{14}C activity per unit carbon is a fraction f of the appropriate modern reference, the idealized age is

$$t = -\frac{1}{\lambda} \ln f = -\frac{T_{1/2}}{\ln 2} \ln f. \quad (22.57)$$

The real method requires calibration for historical variations in atmospheric ^{14}C , contamination, reservoir effects and laboratory backgrounds. Equation (22.57) is therefore the nuclear-physics core, not the entire dating methodology.

Longer-lived radionuclide systems are used for geological timescales. Depending on the method, one measures parent abundance, daughter abundance or isotope ratios and uses mineralogical information to infer the initial composition and whether the system has remained closed.

Example 22.14: Idealized radiocarbon age

An archaeological sample has corrected ^{14}C activity equal to 35.0% of the relevant living-reference value. Using $T_{1/2} = 5730 \text{ y}$, estimate the idealized radiocarbon age.

Here $f = 0.350$, so

$$\begin{aligned} t &= -\frac{5730 \text{ y}}{\ln 2} \ln(0.350) \\ &\approx (8267 \text{ y})(1.0498) \\ &\approx 8.68 \times 10^3 \text{ y.} \end{aligned}$$

The calculation gives about 8.7 kyr. A professional archaeological age would require calibration and uncertainty analysis rather than direct use of the idealized exponential alone.

Method

A radiometric-dating problem has two layers. First solve the nuclear decay mathematics. Then ask what physical or geochemical assumption makes the initial reference known. If the second layer is missing, the exponential equation alone does not determine an age.

Guided checks

1. Why does a dating method need information about an initial abundance or ratio?
2. Why does contamination by modern carbon tend to make an old sample appear younger in a simple radiocarbon analysis?
3. What mathematical feature limits the usefulness of a radionuclide after many half-lives have elapsed?

22.21 Engineering, medical and scientific applications

Nuclear physics is unusually broad in its technical reach because radiation can carry information out of otherwise inaccessible systems. The same underlying processes that define radioactivity are used as probes, tracers, clocks and energy-deposition tools.

22.21.1 Nuclear medicine and tracer principles

A radioactive tracer can be attached to or incorporated into a biologically active compound. Its distribution can then be inferred from emitted radiation. The ideal tracer activity is large enough for statistically useful detection but small enough to limit unnecessary dose and avoid perturbing the biological process. This is an optimization problem involving half-life, photon or particle energy, chemistry, detector sensitivity and timing.

Single-photon imaging detects gamma rays emitted directly by radionuclides. Positron-emission tomography (PET) instead uses beta-plus emitters. After slowing in matter, the emitted positron annihilates with an electron, producing two photons with energies close to 511 keV each. Momentum conservation makes them nearly back-to-back in the centre-of-momentum frame. Coincident detection of the two photons constrains the annihilation to a line of response, and many such lines are combined computationally to reconstruct an image. The advanced section develops this geometrical idea further.

22.21.2 Industrial gauges and materials analysis

Radiation transmission can measure thickness, density or composition without direct contact. A source and detector placed on opposite sides of a sheet or pipe provide a signal that depends on attenuation. X-ray fluorescence, gamma spectroscopy and neutron activation methods can identify

elements through characteristic radiation signatures. Such techniques require calibration and careful control of geometry, backgrounds and detector response.

22.21.3 Smoke detection and ionization

Some smoke detectors use a small sealed alpha source to ionize air in a chamber. The resulting ion current changes when smoke particles alter the transport of ions. The device illustrates a useful engineering principle: radiation need not be measured directly as a spectrum; it can create an intermediate physical effect that a simple electrical circuit senses.

22.21.4 Space and high-radiation environments

Spacecraft electronics experience ionizing radiation from trapped particles, solar energetic particles and cosmic rays. Radiation can cause single-event upsets, displacement damage and long-term total ionizing dose. Nuclear and particle detectors are therefore relevant not only to physics experiments but also to spacecraft reliability, dosimetry and fault-tolerant computer architecture.

22.21.5 Environmental monitoring and safeguards

Gamma spectroscopy can identify radionuclides by characteristic line energies, while count-rate trends can indicate changes in source strength or geometry. Environmental measurements must distinguish natural background, detector response and statistical fluctuations from genuine source changes. Nuclear safeguards similarly rely on quantitative signatures rather than visual inspection alone.

Key message

In applications, the central problem is rarely “which decay equation should I use?” The real task is to connect source physics, transport through matter, detector response, statistical inference and the engineering question being asked.

22.22 Laboratory and computational connection: stochastic decay and half-life fitting

A computer simulation is an excellent way to reconcile two statements that initially seem contradictory: individual decay times are random, but the ensemble follows a smooth exponential. Consider N_0 independent unstable nuclei. During a short time step Δt , each surviving nucleus decays with probability

$$p = 1 - e^{-\lambda\Delta t} \quad (22.58)$$

rather than the approximate $p \approx \lambda\Delta t$ if the step is not infinitesimal.

A simple Monte Carlo algorithm is:

1. initialize $N = N_0$ and $t = 0$;
2. for each surviving nucleus, generate a uniform random number $u \in [0, 1)$;
3. mark the nucleus as decayed if $u < p$;
4. update N and $t \rightarrow t + \Delta t$;
5. repeat until the desired duration has elapsed;
6. repeat the entire simulation many times and compare the ensemble mean with $N_0 e^{-\lambda t}$.

For large N_0 , a more efficient simulation can draw the number of decays in each step from a binomial distribution with parameters N and p rather than testing every nucleus separately.

An alternative event-driven method samples each lifetime directly. If u is uniformly distributed on $(0, 1)$, then an exponential lifetime can be generated from

$$t_d = -\frac{1}{\lambda} \ln u. \quad (22.59)$$

This follows by inverting the cumulative distribution. The method is computationally efficient and illustrates a general Monte Carlo technique used throughout scientific computing.

For a laboratory half-life measurement, suppose counts n_i are recorded in equal-duration intervals around times t_i , together with a background measurement. A simple model is

$$\mu_i = \int_{t_i - \Delta t/2}^{t_i + \Delta t/2} (R_0 e^{-\lambda t} + R_b) dt, \quad (22.60)$$

where t_i is the bin centre and μ_i is the expected number of counts in bin i . The midpoint approximation $\mu_i \approx \Delta t [R_0 e^{-\lambda t_i} + R_b]$ requires suitably small $\lambda \Delta t$. At high counts, weighted least squares can be adequate. At low counts, a Poisson likelihood is more appropriate. The advanced section develops both approaches and discusses confidence intervals.

Example 22.15: Designing a decay simulation time step

A nuclide has half-life $T_{1/2} = 10$ s. A student wants a discrete Monte Carlo step with decay probability below 1% per step. What upper bound on Δt satisfies this requirement?

The decay constant is

$$\lambda = \frac{\ln 2}{10 \text{ s}} = 0.06931 \text{ s}^{-1}.$$

Require

$$1 - e^{-\lambda \Delta t} < 0.01.$$

At the limiting value,

$$e^{-\lambda \Delta t} = 0.99, \quad \Delta t = -\frac{\ln 0.99}{\lambda} \approx \frac{0.01005}{0.06931} \text{ s} \approx 0.145 \text{ s}.$$

The strict inequality requires $\Delta t < 0.145$ s; the limiting value is a supremum rather than an included maximum. A step of 0.1 s satisfies the requirement comfortably.

Note

A computational model should be checked against an analytic limit whenever possible. For the decay simulation, verify that the mean surviving population follows $N_0 e^{-\lambda t}$, that the mean lifetime approaches $1/\lambda$, and that the simulated half-life approaches $\ln 2/\lambda$ as the number of trials grows.

22.23 Common mistakes and how to avoid them

1. **Treating half-life as the lifetime of every nucleus.** Half-life describes an ensemble survival probability. Individual decay times fluctuate randomly.
2. **Confusing activity with number of nuclei.** Activity is λN , so the same N gives different activity for different half-lives.

3. **Confusing activity with detector count rate.** Efficiency, branching, geometry, background and dead time intervene between source and measured counts.
4. **Confusing activity with dose.** Becquerel, gray and sievert answer different physical questions.
5. **Mixing nuclear and atomic masses.** Decide which convention the mass table uses before calculating binding energy or Q value.
6. **Forgetting the neutrino in beta decay.** This breaks conservation-law bookkeeping and contradicts the continuous beta spectrum.
7. **Using the wrong beta-plus Q formula with atomic masses.** The $2m_e$ threshold term is required.
8. **Assuming gamma emission changes A or Z .** Gamma decay changes the nuclear energy state, not the nuclide identity.
9. **Calling binding energy stored energy.** A bound system has lower total energy than its separated constituents.
10. **Interpreting $R = R_0 A^{1/3}$ as an exact sharp radius.** It is an empirical scale; real nuclear distributions have diffuse surfaces and possible deformation.
11. **Assuming energetically allowed means fast.** Decay probabilities depend on quantum matrix elements, tunnelling and selection rules.
12. **Using an exponential attenuation law without geometry assumptions.** Scattered radiation and buildup can invalidate a simple narrow-beam model.
13. **Fitting logged counts without thinking about statistics.** Taking logarithms changes error distributions and fails for zero counts; Poisson methods are often more appropriate at low counts.

Caution

A useful self-check is to ask whether your answer respects all three layers: *identity* (correct daughter), *energy* (correct mass convention and Q sign), and *statistics* (correct interpretation of half-life or count data). Many apparently algebraic mistakes are actually failures to separate these layers.

22.24 Guided checks for the main chapter

Guided checks

1. Why did large-angle alpha scattering imply a compact concentration of positive charge?
2. Show verbally why $R \propto A^{1/3}$ leads to approximately constant average nuclear density.
3. What does a positive binding energy mean physically?
4. Why can B/A decrease for very heavy nuclei even though the total B continues to increase?
5. What is the difference between $T_{1/2}$ and τ ?
6. A sample survives for three half-lives. What fraction remains, and what fraction has decayed?
7. Which decay mode changes A by four? Which modes leave A unchanged?
8. Why is a neutrino required in beta decay?

9. Why does a gamma-emitting nucleus recoil?
10. What additional information is needed to convert an activity into a detector count rate?
11. What additional information is needed to convert source activity into absorbed dose?
12. Why is radiometric dating partly a geochemical problem rather than only a nuclear-decay calculation?

22.25 Main chapter summary

Summary

- Rutherford scattering established that positive charge and nearly all atomic mass are concentrated in a compact nucleus.
- A nuclide ${}^A_ZX$ has Z protons, $N = A - Z$ neutrons and mass number A .
- Nuclear radii follow the empirical scale $R \approx R_0 A^{1/3}$ with $R_0 \sim 1.2 \text{ fm}$, implying nearly constant average nuclear density.
- Nuclear binding arises from a short-range residual strong interaction competing with Coulomb repulsion and quantum-structure effects.
- Binding energy is the mass defect multiplied by c^2 . Atomic and nuclear mass conventions must not be mixed.
- The broad maximum of B/A near the iron–nickel region organizes nuclear energetics but does not by itself determine decay rates.
- Radioactive decay is stochastic for individual nuclei and exponential for an ensemble: $N = N_0 e^{-\lambda t}$.
- Activity is $\mathcal{A} = \lambda N$, half-life is $T_{1/2} = \ln 2 / \lambda$, and mean lifetime is $\tau = 1 / \lambda$.
- Alpha decay changes $(A, Z) \rightarrow (A - 4, Z - 2)$; beta processes keep A fixed and change Z by one; gamma decay leaves both A and Z unchanged.
- Beta decay requires neutrinos for conservation of energy, momentum and angular momentum and produces a continuous beta spectrum.
- Activity, detector count rate, absorbed dose and equivalent dose are distinct quantities connected only through additional physical information.
- Radioactivity underlies dating, nuclear medicine, radiation detection, materials analysis and many engineering measurements.

Formula summary

$$N = A - Z,$$

$$R = R_0 A^{1/3},$$

$$B = [Zm_p + Nm_n - M_{\text{nuc}}] c^2,$$

$$B \simeq [Zm_{\text{H}} + Nm_n - M_{\text{atom}}] c^2 \quad (\text{atomic masses}),$$

$$\frac{dN}{dt} = -\lambda N,$$

$$N(t) = N_0 e^{-\lambda t} = N_0 2^{-t/T_{1/2}},$$

$$\mathcal{A}(t) = \lambda N(t) = \mathcal{A}_0 e^{-\lambda t},$$

$$T_{1/2} = \frac{\ln 2}{\lambda}, \quad \tau = \frac{1}{\lambda},$$

$$Q = (M_{\text{initial}} - M_{\text{final}}) c^2,$$

$$I(x) = I_0 e^{-\mu x}, \quad x_{1/2} = \frac{\ln 2}{\mu},$$

$$D = \frac{E_{\text{dep}}}{m}, \quad H = w_R D.$$

22.26 Advanced Topics for Technical and Engineering Students

The core results are sufficient for introductory calculations. The following modules extend them in three directions: nuclear structure and quantum decay, coupled populations, and quantitative measurements. Each module states a model and its limits. Students may study the structure modules after binding energy and the measurement modules after the decay law; matrix methods and likelihood inference are optional extensions rather than prerequisites for the core chapter.

22.26.1 The semi-empirical mass formula and liquid-drop energetics

The liquid-drop picture can be made quantitative through the semi-empirical mass formula (SEMF), also called the Weizsäcker formula. In one common convention, the nuclear binding energy is approximated by

$$B(A, Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A, Z). \quad (22.61)$$

The coefficients a_v, a_s, a_c, a_a and the pairing scale entering δ are fitted empirically. Their precise fitted values depend on the selected data set and convention, so we focus on structure rather than memorizing numerical coefficients.

The *volume term* $a_v A$ expresses saturation. Each interior nucleon has a roughly fixed number of strongly interacting neighbors, so the total attractive binding grows approximately in proportion to the number of nucleons.

The *surface term* $-a_s A^{2/3}$ corrects for nucleons near the boundary. The surface area scales as $R^2 \propto A^{2/3}$, and surface nucleons have fewer nearby partners than interior nucleons.

The *Coulomb term* is negative because electrostatic repulsion reduces binding. A uniformly charged sphere has electrostatic self-energy scaling approximately as $Z^2/R \propto Z^2/A^{1/3}$. Writing $Z(Z-1)$ rather than Z^2 removes the unphysical self-interaction of each proton with itself.

The *asymmetry term*

$$-a_a \frac{(A-2Z)^2}{A} \quad (22.62)$$

penalizes large neutron–proton imbalance. Its origin is quantum mechanical: protons and neutrons fill fermionic states separately, so forcing many more particles of one species into the nucleus requires occupation of higher momentum states.

Finally, the *pairing term* models the empirical tendency for even numbers of like nucleons to be more strongly bound:

$$\delta(A, Z) \sim \begin{cases} +a_p A^{-1/2}, & Z, N \text{ both even,} \\ 0, & A \text{ odd,} \\ -a_p A^{-1/2}, & Z, N \text{ both odd,} \end{cases} \quad (22.63)$$

where the power of A used in phenomenological fits can vary by convention.

The SEMF is not a microscopic derivation from QCD. It is a successful phenomenological organization of smooth mass trends. Shell closures, deformation and detailed pairing correlations produce deviations that are themselves physically informative.

Theoretical point 22.1: Why the Coulomb term grows faster than the volume attraction

For nuclei near a fixed proton fraction $Z \propto A$, the volume term grows as A , while the Coulomb

term scales roughly as

$$\frac{Z^2}{A^{1/3}} \propto A^{5/3}.$$

The proton fraction does not actually remain fixed for stable heavy nuclei, but this scaling shows why Coulomb repulsion becomes increasingly important and why very large nuclei cannot be stabilized simply by adding nucleons in the same proportion.

Example 22.16: Maximum binding and minimum atomic mass are different

At fixed A , ignore pairing, electronic binding and the difference between $Z(Z-1)$ and Z^2 . The smooth binding terms are

$$B_Z = -a_c Z^2 A^{-1/3} - a_a (A - 2Z)^2 / A.$$

Setting $\partial B_Z / \partial Z = 0$ gives

$$Z_B = \frac{A}{2 + (a_c/2a_a)A^{2/3}}. \quad (22.64)$$

This maximizes binding, but beta stability concerns the *atomic mass*

$$M_{\text{atom}} c^2 = A m_n c^2 + Z(m_H - m_n) c^2 - B(A, Z).$$

With $\Delta = (m_n - m_H) c^2 > 0$, differentiation gives

$$0 = -\Delta + 2a_c Z A^{-1/3} - 4a_a (A - 2Z) / A.$$

Therefore the continuous atomic-mass minimum is

$$Z_M = \frac{A(4a_a + \Delta)}{8a_a + 2a_c A^{2/3}}. \quad (22.65)$$

Example and check. With $A = 100$, illustrative $a_c = 0.71$ MeV, $a_a = 23.6$ MeV and $\Delta \simeq 0.782$ MeV, $Z_B \simeq 43.03$ while $Z_M \simeq 43.38$. Both favour neutron excess. The offset is small for a broad trend but can matter when testing neighbouring integer isobars. Pairing, shell effects and the positron threshold must still be considered before assigning an actual stable nuclide.

Caution

Do not use the SEMF to predict high-precision nuclear masses or detailed decay schemes. It captures smooth trends. Near closed shells or strongly deformed nuclei, local quantum effects can shift masses by amounts that are decisive for whether a decay channel is open.

22.26.2 Separation energies: the cost of removing one nucleon

The total binding energy gives a global measure of nuclear cohesion, but many processes are governed by a more local question: how much energy is required to remove one particular nucleon? This motivates the neutron and proton separation energies. In terms of nuclear masses,

$$S_n(A, Z) = [M(A-1, Z) + m_n - M(A, Z)] c^2, \quad (22.66)$$

$$S_p(A, Z) = [M(A-1, Z-1) + m_p - M(A, Z)] c^2. \quad (22.67)$$

Equivalently, because $B(A, Z)$ is the energy required to separate all nucleons,

$$S_n(A, Z) = B(A, Z) - B(A - 1, Z), \quad (22.68)$$

$$S_p(A, Z) = B(A, Z) - B(A - 1, Z - 1). \quad (22.69)$$

A positive separation energy means that energy must be supplied to remove the nucleon. A vanishing or negative value signals that the corresponding nucleon is not bound against spontaneous emission in the simple ground-state energy accounting.

Separation energies are especially useful because shell closures and pairing effects that look subtle on a smooth B/A curve become much more visible in differences of neighbouring masses. In modern nuclear physics, systematic plots of S_n and S_p help delineate the neutron and proton drip lines, beyond which adding another neutron or proton does not produce a bound ground state.

Example 22.17: A separation-energy estimate from binding energies

Suppose two neighbouring nuclei have measured binding energies

$$B(A, Z) = 492.0 \text{ MeV}, \quad B(A - 1, Z) = 484.6 \text{ MeV}.$$

The neutron separation energy is

$$S_n = 492.0 - 484.6 = 7.4 \text{ MeV}.$$

Thus at least 7.4 MeV must be supplied in the ground-state energy balance to separate one neutron, before considering kinetic-energy distributions or excitation of the residual nucleus.

Guided checks

1. Why can two nuclei have similar values of B/A but noticeably different neutron separation energies?
2. If $S_n < 0$, what does that imply about the proposed nucleus as a bound ground state?
3. Why are separation energies naturally sensitive to shell closures?

22.26.3 The beta-stability valley from a mass-parabola viewpoint

For a fixed mass number A , the semi-empirical mass formula makes the nuclear mass approximately a quadratic function of Z . This produces a family of “mass parabolas.” Beta decay changes Z by one unit while leaving A unchanged, so it moves a nucleus horizontally along one of these parabolas.

Ignoring pairing for the moment, the minimum of the neutral-atomic mass parabola identifies a smooth candidate for beta stability at that A , using Eq. (22.65). Nuclei on the neutron-rich side can lower their mass through beta-minus decay, which converts a neutron into a proton. Nuclei on the proton-rich side can move toward lower mass by beta-plus decay or electron capture, provided the relevant Q value is positive.

Pairing modifies this picture. For even A , even–even and odd–odd nuclei can lie on different parabolas because of the pairing term. An even–even isobar can be stable against single beta decay to a higher-mass odd–odd neighbour while a lower-mass even–even isobar exists two charge units away. A second-order double-beta transition can then be energetically possible. This is different from two energetically allowed successive single-beta decays.

Key message

Beta stability is an energy statement at fixed A . The neutron-to-proton ratio is not governed by a single universal number: the competition between Coulomb energy and neutron–proton asymmetry shifts the stable ratio toward neutron-rich compositions as Z increases.

22.26.4 Shell structure, magic numbers and spin–orbit coupling

The shell model treats each nucleon as moving approximately independently in an average potential generated by all other nucleons. The independent-particle picture is an approximation, but it reveals why certain proton and neutron numbers produce enhanced stability and characteristic spins.

A naive three-dimensional harmonic-oscillator potential produces shell closures, but not the observed nuclear magic sequence. A crucial ingredient is strong spin–orbit coupling. For a nucleon with orbital angular momentum ℓ and intrinsic spin s , the total angular momentum is

$$\mathbf{j} = \ell + \mathbf{s}, \quad (22.70)$$

so for spin-1/2 nucleons the allowed values are

$$j = \ell \pm \frac{1}{2}. \quad (22.71)$$

A spin–orbit term proportional to $\ell \cdot s$ splits these states. The ordering and strength of this splitting reorganize the level sequence and generate the observed closures at

$$2, 8, 20, 28, 50, 82, 126. \quad (22.72)$$

For even–even nuclei, nucleons tend to pair with opposite angular momenta, and the ground state often has total spin $J = 0$. In an odd- A nucleus, the ground-state spin is frequently dominated by the unpaired nucleon. This simple rule is not universal, because residual interactions and deformation can mix configurations, but it provides a useful first interpretation of nuclear spins.

The shell picture also clarifies why separation energies and decay energies can show abrupt changes near magic numbers. Adding a nucleon just beyond a closed shell may require occupation of a substantially higher single-particle level.

Note

Atomic and nuclear shell models share quantum language but differ physically. Atomic shells arise mainly from electrons in the Coulomb field of the nucleus plus electron–electron corrections. Nuclear shells arise from strongly interacting nucleons in a self-consistent average potential with a prominent spin–orbit interaction.

22.26.5 Isospin as a symmetry language for protons and neutrons

Because the strong interaction between two nucleons is approximately insensitive to whether a nucleon is a proton or neutron, nuclear physics often introduces an abstract two-state quantum number called *isospin*. The proton and neutron are treated as two components of a nucleon doublet, analogous mathematically to two spin states. We choose the nuclear convention

$$t = \frac{1}{2}, \quad t_z = +\frac{1}{2} \text{ for the neutron}, \quad t_z = -\frac{1}{2} \text{ for the proton}, \quad (22.73)$$

using the nuclear convention $T_z = (N - Z)/2$. Reversing every third-component sign is an alternative convention and does not change the physics.

For a nucleus, the third component of total isospin is related to the neutron–proton imbalance. Isospin organizes mirror nuclei and selection rules and makes approximate proton–neutron symmetry visible. Electromagnetism and the up–down quark mass difference break the symmetry, so isospin is not exact.

Key message

Isospin is not a new spatial direction and not literal mechanical spin. It is an internal symmetry label that packages the approximate similarity of proton and neutron strong interactions into the language of angular-momentum algebra.

22.26.6 Nuclear charge distributions and elastic form factors

The radius law treats a nucleus as though it had a sharp boundary, but a real quantum nucleus has a spatially distributed charge density $\rho_{\text{ch}}(\mathbf{r})$. Elastic scattering can probe that distribution. In the simplest Born-approximation picture, the scattering amplitude contains the Fourier transform of the charge density. A normalized charge form factor can be written schematically as

$$F(\mathbf{q}) = \frac{1}{Ze} \int \rho_{\text{ch}}(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}/\hbar} d^3r, \quad F(0) = 1, \quad (22.74)$$

where $\mathbf{q}$ is the momentum transfer. The normalization $F(0) = 1$ states that a probe with wavelength much larger than the nucleus sees only the total charge.

As $|\mathbf{q}|$ increases, the probe resolves finer spatial structure and $F(q)$ departs from unity. For a spherically symmetric density,

$$F(q) = \frac{4\pi}{Ze} \int_0^\infty \rho_{\text{ch}}(r) \frac{\sin(qr/\hbar)}{qr/\hbar} r^2 dr. \quad (22.75)$$

Diffraction-like minima in elastic electron-scattering data therefore carry information about nuclear size and surface diffuseness. This is a beautiful connection between wave optics and nuclear structure: spatial information is encoded in a Fourier pattern in momentum space.

For small momentum transfer, expanding the exponential gives

$$F(q) \approx 1 - \frac{q^2 \langle r^2 \rangle}{6\hbar^2} + \dots, \quad (22.76)$$

so the slope of the form factor near $q = 0$ determines the mean-square charge radius $\langle r^2 \rangle$.

Pedagogical boundary

Equation (22.74) is introduced as a structural idea, not as a full scattering theory. A precision treatment requires relativistic electron wave functions, Coulomb distortions, magnetic contributions and careful experimental normalization. The pedagogical message is the Fourier connection between spatial density and momentum-transfer dependence.

Guided checks

1. Why does a very long-wavelength probe fail to resolve the internal charge distribution?
2. What qualitative change in a scattering pattern would you expect when the probe wavelength becomes comparable with the nuclear radius?
3. Which earlier Physics 2 topic makes diffraction minima a familiar idea?

22.26.7 Root-mean-square radius versus a sharp-sphere radius

Experimental nuclear radii are often quoted as root-mean-square charge radii,

$$r_{\text{rms}} = \sqrt{\langle r^2 \rangle}, \quad \langle r^2 \rangle = \frac{1}{Ze} \int r^2 \rho_{\text{ch}}(\mathbf{r}) d^3r. \quad (22.77)$$

This is not identical to the geometric radius R of an ideal hard sphere. For a uniform sphere of radius R ,

$$\langle r^2 \rangle = \frac{3}{5} R^2, \quad r_{\text{rms}} = \sqrt{\frac{3}{5}} R. \quad (22.78)$$

Thus a statement such as “the nuclear radius is 5 fm” is incomplete unless the radius convention is clear. For a uniform density $\rho_{\text{ch}} = 3Ze/(4\pi R^3)$ inside R , direct integration gives $\langle r^2 \rangle = 3R^2/5$. Inserting the same density into Eq. (22.75), set $x = qR/\hbar$ and integrate $r \sin(qr/\hbar)$ by parts to obtain

$$F(x) = 3 \frac{\sin x - x \cos x}{x^3}, \quad F(0) = 1. \quad (22.79)$$

Expansion yields $F = 1 - x^2/10 + \dots$, matching $1 - q^2 \langle r^2 \rangle / (6\hbar^2)$. Figure 22.9 shows the resulting diffraction structure. For a specified spin-independent charge contribution in the plane-wave Born approximation, $d\sigma/d\Omega$ is a point-charge cross section multiplied by $|F(q)|^2$; magnetic and distortion effects require a fuller treatment.

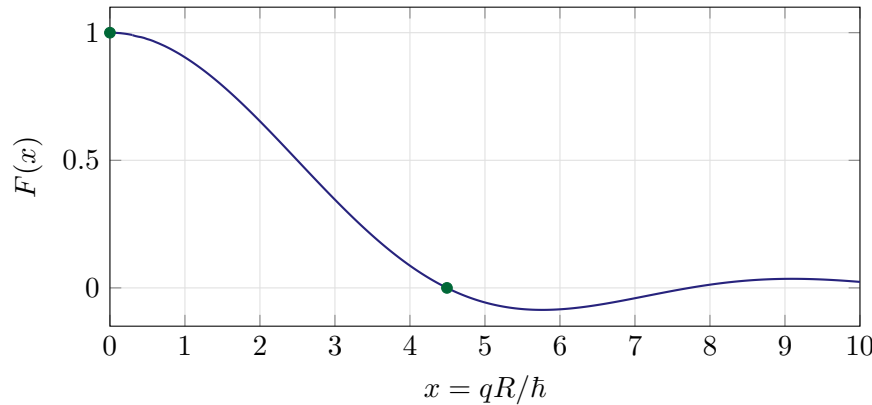


Figure 22.9: Normalized charge form factor calculated for a uniform sphere. The first zero occurs at $qR/\hbar \simeq 4.493$. The sign change is an amplitude effect; an unpolarized charge contribution to a Born cross section contains $|F|^2$, which is nonnegative.

Example 22.18: Converting an rms radius to a uniform-sphere radius

A nucleus has measured charge rms radius $r_{\text{rms}} = 4.00$ fm. If it were approximated as a uniform sharp sphere, what sphere radius would reproduce the same $\langle r^2 \rangle$?

$$R = \sqrt{\frac{5}{3}} r_{\text{rms}} = \sqrt{\frac{5}{3}} (4.00 \text{ fm}) \approx 5.16 \text{ fm}.$$

The two radius numbers describe the same idealized distribution using different conventions.

22.26.8 Rutherford scattering beyond closest approach

The head-on closest-approach calculation uses only energy conservation. A fuller scattering analysis relates the measured angular distribution to the Coulomb potential. For a projectile of charge ze and kinetic energy E scattering from a heavy nucleus of charge Ze , the classical Rutherford differential cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2}{16\pi\epsilon_0 E} \right)^2 \frac{1}{\sin^4(\theta/2)}. \quad (22.80)$$

The differential cross section has dimensions of area per unit solid angle and describes how strongly events are distributed in scattering angle θ .

The striking $\sin^{-4}(\theta/2)$ dependence predicts many more small-angle than large-angle events. It also predicts that the cross section scales as Z^2 and decreases as E^{-2} . Deviations from this law at sufficiently high projectile energy or small distance can reveal finite nuclear size, screening by atomic electrons, nuclear interactions or quantum effects.

To connect the cross section to a detector, suppose a thin target contains areal number density n_t nuclei per unit area and the incident fluence is Φ particles per unit area. A small detector subtending solid angle $\Delta\Omega$ near angle θ records an expected number

$$N_{\text{det}} \approx \Phi A_{\text{beam}} n_t \left. \frac{d\sigma}{d\Omega} \right|_{\theta} \Delta\Omega \epsilon, \quad (22.81)$$

where A_{beam} is the illuminated area and ϵ is detection efficiency. This equation is schematic; real experiments also include finite target thickness, energy loss, multiple scattering and detector acceptance.

Example 22.19: Angular dependence of Rutherford scattering

For the same beam, target and detector acceptance, compare the Rutherford count rate at 30° with that at 60° .

Because all other factors cancel,

$$\frac{R(30^\circ)}{R(60^\circ)} = \left[\frac{\sin(60^\circ/2)}{\sin(30^\circ/2)} \right]^4 = \left[\frac{\sin 30^\circ}{\sin 15^\circ} \right]^4.$$

Using $\sin 30^\circ = 0.5$ and $\sin 15^\circ \approx 0.2588$,

$$\frac{R(30^\circ)}{R(60^\circ)} \approx (1.932)^4 \approx 13.9.$$

The dramatic increase at smaller angle is a characteristic feature of Coulomb scattering.

22.26.9 Quantum tunnelling and the alpha-decay lifetime

The qualitative alpha-decay picture can be sharpened by modelling the alpha particle as preformed within a nuclear potential well. Outside the nuclear surface, the dominant potential is approximately Coulombic,

$$V(r) \approx \frac{1}{4\pi\epsilon_0} \frac{2(Z-2)e^2}{r}, \quad r \gtrsim R, \quad (22.82)$$

where Z is the parent proton number. For a typical alpha decay, the alpha-daughter relative energy $E = Q$ for a ground-state channel is below the Coulomb barrier near the nuclear surface. Classically, the particle would remain trapped. Quantum mechanically, the wave function penetrates the forbidden region and has a nonzero amplitude beyond the outer turning point.

In the WKB approximation, the barrier transmission probability has the form

$$P \sim \exp \left[-2 \int_{r_1}^{r_2} \kappa(r) dr \right], \quad \kappa(r) = \frac{\sqrt{2\mu[V(r) - E]}}{\hbar}, \quad (22.83)$$

where μ is the alpha-daughter reduced mass and r_1, r_2 are classical turning points. The decay constant can be estimated schematically as

$$\lambda \sim f P P_\alpha, \quad (22.84)$$

where f is an “assault frequency” describing how often the alpha cluster encounters the barrier and P_α is a preformation factor representing the probability that an alpha-like cluster configuration exists inside the parent.

Figure 22.10 defines the relative-coordinate geometry. Because μ is the reduced mass, the energy in the radial equation is the *total* relative kinetic energy $E = K_\alpha + K_D$, not K_α alone. For a daughter excitation E_x , replace E by $Q - E_x$. A nonzero relative orbital angular momentum adds $\hbar^2\ell(\ell+1)/(2\mu r^2)$ to the barrier.

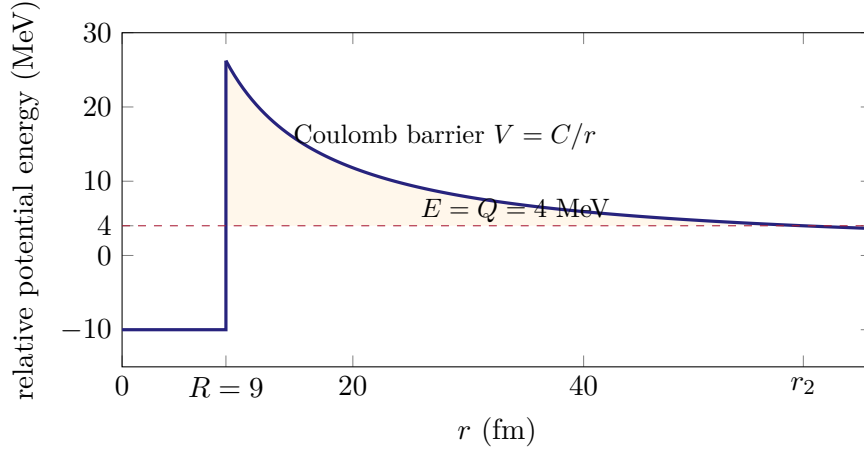


Figure 22.10: Illustrative alpha–daughter relative-coordinate potential with daughter charge 82, $C = 2(82)(1.44) = 236.16$ MeV fm and contact radius $R = 9$ fm. The interior square well is schematic; the exterior Coulomb curve is calculated. Tunnelling occurs where $R < r < r_2 = C/E = 59.04$ fm and $V > E$; only the portion above the dashed energy line is forbidden.

In the simplified $\ell = 0$ exterior model, take $r_1 = R$, $r_2 = C/E$ and $0 < R/r_2 < 1$. Substituting $r = r_2 \sin^2 \theta$ into the WKB integral gives

$$G = \frac{2C\sqrt{2\mu c^2}}{\hbar c\sqrt{E}} \left[\arccos \sqrt{R/r_2} - \sqrt{(R/r_2)(1 - R/r_2)} \right], \quad P \sim e^{-G}. \quad (22.85)$$

Use C in MeV fm, μc^2 and E in MeV, and $\hbar c \simeq 197.327$ MeV fm; G is dimensionless. The contact radius is a separation of two extended clusters and need not equal the parent sharp-sphere radius.

Example 22.20: Energy sensitivity of a Coulomb barrier

For the model in Fig. 22.10, use $\mu c^2 = (4 \times 206/210)(931.5) = 3655$ MeV. Equation (22.85) gives approximately $G = 82.9$ at $E = 4$ MeV and $G = 66.5$ at $E = 5$ MeV. Holding preformation and assault frequency fixed,

$$\frac{\lambda(5)}{\lambda(4)} \sim e^{82.9-66.5} \sim 1.3 \times 10^7.$$

This predicts a dramatic rate change within a model, not the measured half-life of a particular isotope. Nuclear structure, the interior potential and angular momentum can change the prefactor or the barrier itself.

The exponential dependence on the integral is the key physics. A modest change in E changes the width and height of the forbidden region and can alter the half-life by many orders of magnitude. This explains the empirical Geiger–Nuttall correlation between alpha energy and half-life.

Theoretical point 22.2: Why a small energy change can produce a huge lifetime change

If the decay constant contains a factor $\lambda \propto e^{-G(E)}$, then

$$\ln T_{1/2} = \text{constant} + G(E).$$

A moderate change in G becomes a multiplicative exponential change in $T_{1/2}$. Barrier tunnelling therefore naturally generates lifetimes ranging from microseconds to geological times even when alpha energies lie within a comparatively narrow MeV range.

Caution

The tunnelling model explains the dominant energy dependence but does not imply that an alpha particle is a rigid classical object bouncing inside every nucleus. Cluster preformation, nuclear structure and angular momentum can all modify the decay probability.

22.26.10 Beta-decay kinematics and endpoint energy

Beta decay is a three-body final-state problem when the daughter recoil is included. For beta-minus decay,

$$X \rightarrow Y + e^- + \bar{\nu}_e, \quad (22.86)$$

energy conservation for a parent initially at rest can be written

$$Q = K_e + K_\nu + K_Y, \quad (22.87)$$

ignoring the tiny neutrino rest mass at introductory precision. Because the electron and neutrino can share the available energy continuously while satisfying momentum conservation, the electron kinetic energy ranges from near zero up to an endpoint close to Q .

The endpoint is particularly important experimentally. Its value constrains the decay energy and, in precision beta spectroscopy, the detailed shape near the endpoint contains information about neutrino mass. This is an example of how a simple conservation-law observation becomes a high-precision modern experiment.

For beta-plus decay using atomic masses, the threshold $Q_{\beta+} > 0$ requires

$$M_{\text{atom}}(X) - M_{\text{atom}}(Y) > 2m_e. \quad (22.88)$$

In energy units the rest-energy threshold is $2m_e c^2 \approx 1.022 \text{ MeV}$. Electron capture does not require creating a positron and therefore can remain allowed when beta-plus emission is forbidden.

Example 22.21: Beta-plus threshold from an atomic-mass difference

Suppose the parent–daughter neutral-atomic mass difference is $0.00150 u$. Is beta-plus emission energetically allowed?

The available atomic-mass difference corresponds to

$$\Delta M c^2 = (0.00150)(931.5 \text{ MeV}) \approx 1.397 \text{ MeV}.$$

Subtract the positron-emission atomic-mass threshold:

$$Q_{\beta+} \approx 1.397 \text{ MeV} - 1.022 \text{ MeV} = 0.375 \text{ MeV} > 0.$$

Thus beta-plus emission is energetically allowed, although the actual decay rate still depends on nuclear matrix elements and available phase space.

22.26.11 Beta spectra: phase space, endpoint and spectral shape

The continuous beta spectrum follows from three-body kinematics, but its detailed shape also contains weak-interaction physics. Figure 22.11 makes the phase-space effect explicit. In a simplified allowed beta decay, the electron spectrum can be written schematically as

$$\frac{dN}{dE_e} \propto F(Z, E_e) p_e E_e (Q - T_e)^2, \quad (22.89)$$

where T_e is the electron kinetic energy, $E_e = T_e + m_e c^2$, p_e is its momentum and $F(Z, E_e)$ accounts approximately for Coulomb attraction or repulsion between the emitted beta particle and daughter nucleus. Nuclear matrix elements and forbiddenness can modify the shape. Here $p_e c = \sqrt{T_e(T_e + 2m_e c^2)}$, so the displayed factors are functions of the kinetic energy even though E_e in Eq. (22.89) is the total electron energy. With $F = 1$, the phase-space density vanishes at zero kinetic energy and at the endpoint; the physical Coulomb factor can change the low-energy behaviour.

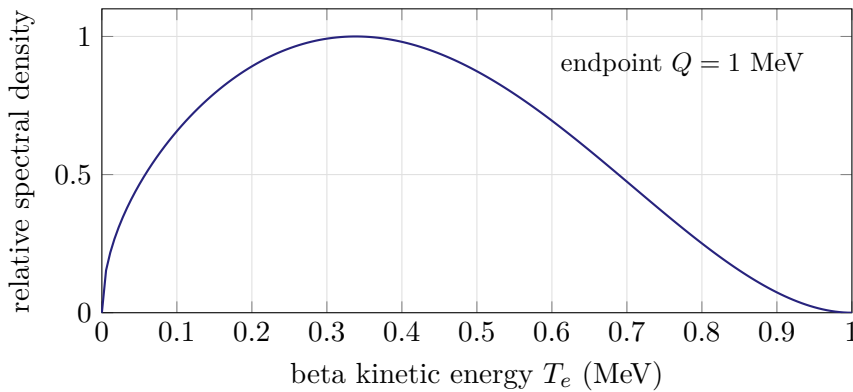


Figure 22.11: Calculated phase-space illustration for an allowed beta spectrum with $Q = 1$ MeV, $m_e c^2 = 0.511$ MeV, negligible recoil and neutrino mass, constant nuclear factor and $F = 1$. Density is normalized to its maximum. This is a teaching model rather than an isotope-specific spectrum; Coulomb and forbidden-transition corrections can alter its shape.

Near the endpoint, the neutrino carries very little energy. Precision measurements of the spectral endpoint can therefore probe the neutrino mass, although the required treatment goes far beyond Eq. (22.89). Historically, the continuous beta spectrum was decisive evidence that an unseen neutral particle must participate if energy and momentum conservation are to remain valid event by event.

Pedagogical boundary

The expression above is schematic and should not be used for precision beta-spectrum calculations. Its role is to display the phase-space factors and explain why the spectrum is continuous. Modern analyses include nuclear matrix elements, radiative corrections, molecular final states where relevant and detector-response effects.

22.26.12 Gamma transitions, multipolarity and selection-rule intuition

A nuclear gamma transition must carry away not only energy but also angular momentum and parity information. Electromagnetic radiation can be classified into electric and magnetic multipoles, denoted $E1$, $M1$, $E2$, $M2$, and so on. The multipole order L indicates how much angular momentum the photon field can carry in the transition.

At an introductory qualitative level, lower multipoles are generally less hindered than higher multipoles because the electromagnetic field can couple more strongly to simpler changes in the nuclear charge and current distribution. Large changes in nuclear angular momentum may force a

high multipolarity and therefore a longer lifetime. This is one reason nuclear isomeric states can persist much longer than a naive “electromagnetic processes are fast” argument would suggest.

For initial and final spins J_i, J_f , a single real photon has $L \geq 1$ and must satisfy $|J_i - J_f| \leq L \leq J_i + J_f$. In particular, a $0 \rightarrow 0$ transition cannot emit one gamma photon. The parity relation is $\pi_i \pi_f = (-1)^L$ for electric EL and $(-1)^{L+1}$ for magnetic ML radiation. Thus the allowed multipoles depend on both spin and parity. Internal conversion or other channels can compete, so a measured level lifetime need not equal a partial gamma lifetime. A full treatment requires quantum angular-momentum algebra. The practical lesson here is that gamma lifetime is not determined by transition energy alone. Nuclear spin and parity can suppress otherwise energetically allowed electromagnetic transitions.

22.26.13 Lifetime and spectral width: an energy–time connection

An excited state with finite mean lifetime τ is not infinitely sharp in energy. A decaying amplitude of the form

$$\psi(t) \propto e^{-iE_0 t/\hbar} e^{-t/(2\tau)} \quad (t \geq 0) \quad (22.90)$$

Fourier transforms into a resonance line with characteristic width

$$\Gamma \sim \frac{\hbar}{\tau}. \quad (22.91)$$

For the ideal exponential amplitude, the intensity line shape is Lorentzian and the full width at half maximum is $\Gamma = \hbar/\tau$ under the standard convention.

This relation is sometimes loosely called an “energy–time uncertainty relation.” It should not be interpreted exactly like the position–momentum uncertainty relation because time is not an operator in elementary nonrelativistic quantum mechanics in the same way position is. Here the connection follows concretely from the Fourier transform of a finite-lived state.

Example 22.22: Natural width of a short-lived level

Estimate the natural energy width of a level with mean lifetime $\tau = 1.0 \text{ ps}$. Using $\hbar \approx 6.58 \times 10^{-16} \text{ eV s}$,

$$\Gamma \approx \frac{6.58 \times 10^{-16}}{1.0 \times 10^{-12}} \text{ eV} = 6.58 \times 10^{-4} \text{ eV}.$$

This is much narrower than typical keV-scale detector resolution. The experimentally observed peak width would therefore usually be dominated by the instrument, not by the natural linewidth.

22.26.14 The Mössbauer effect: recoil-free gamma resonance

When an isolated nucleus emits a gamma photon, momentum conservation gives the nucleus a recoil energy approximately

$$E_R \approx \frac{E_\gamma^2}{2Mc^2}. \quad (22.92)$$

Although E_R may be tiny in absolute terms, it can be enormous compared with the natural linewidth of a long-lived nuclear transition. This recoil would normally prevent resonant absorption by an identical free nucleus because emission and absorption energies no longer match exactly.

In a solid, however, the entire lattice can sometimes take up the recoil momentum without creating a phonon. The effective recoil energy of the macroscopic lattice is then negligible. Recoil-free gamma emission and absorption form the basis of the Mössbauer effect, one of the most sensitive resonance techniques in physics. Tiny energy shifts can probe chemical environments, magnetic hyperfine fields and gravitational redshift.

The Mössbauer effect is a striking synthesis of nuclear structure and solid-state physics: a nuclear transition becomes experimentally sharper because the nucleus is embedded in a many-atom crystal.

Example 22.23: Free-nucleus recoil scale

Estimate the recoil energy for a 14.4 keV gamma ray emitted by a nucleus of mass number $A \approx 57$. Taking $Mc^2 \approx 57(931.5 \text{ MeV})$,

$$E_R \approx \frac{(14.4 \times 10^3 \text{ eV})^2}{2(57)(931.5 \times 10^6 \text{ eV})} \approx 1.95 \times 10^{-3} \text{ eV}.$$

A millielectronvolt recoil is tiny on ordinary nuclear-energy scales, yet it can be huge compared with the natural width of a very narrow gamma transition.

22.26.15 Coupled decay chains and the Bateman solution

Consider a two-member chain

$$1 \xrightarrow{\lambda_1} 2 \xrightarrow{\lambda_2} \text{stable}.$$

The parent obeys

$$N_1(t) = N_{10}e^{-\lambda_1 t}. \quad (22.93)$$

If initially $N_2(0) = 0$, the daughter equation is

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_{10}e^{-\lambda_1 t}. \quad (22.94)$$

Multiplying by the integrating factor $e^{\lambda_2 t}$,

$$\frac{d}{dt} (N_2 e^{\lambda_2 t}) = \lambda_1 N_{10} e^{(\lambda_2 - \lambda_1)t}. \quad (22.95)$$

For $\lambda_1 \neq \lambda_2$, integration gives

$$N_2(t) = \frac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}). \quad (22.96)$$

This is the simplest Bateman solution. For initial daughter inventory N_{20} and feeding branch b , add $N_{20}e^{-\lambda_2 t}$ and multiply the production part by b . The denominator in Eq. (22.96) is not a physical singularity. Taking the limit $\lambda_2 \rightarrow \lambda_1 = \lambda$ gives

$$N_2(t) = b\lambda N_{10}te^{-\lambda t} + N_{20}e^{-\lambda t}. \quad (22.97)$$

For $b = 1$, $N_{20} = 0$, the maximum occurs at $t = 1/\lambda$ and $N_2 = N_{10}/e$. This limit is an important independent check for numerical implementations.

The daughter activity is

$$\mathcal{A}_2(t) = \lambda_2 N_2(t). \quad (22.98)$$

It begins at zero, rises as the parent feeds the daughter, reaches a maximum, and then eventually declines. The time of maximum daughter population follows from $dN_2/dt = 0$, equivalent to production balancing removal:

$$\lambda_1 N_1 = \lambda_2 N_2. \quad (22.99)$$

Using Eq. (22.96), one finds

$$t_{\max} = \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}. \quad (22.100)$$

The expression remains positive for either ordering of λ_1 and λ_2 because numerator and denominator change sign together.

Example 22.24: Growth and decay of a daughter radionuclide

A parent has half-life 10 h and its radioactive daughter has half-life 2.0 h. Initially no daughter is present. Find the time when the daughter population is maximal.

The decay constants are

$$\lambda_1 = \frac{\ln 2}{10 \text{ h}} = 0.06931 \text{ h}^{-1}, \quad \lambda_2 = \frac{\ln 2}{2.0 \text{ h}} = 0.3466 \text{ h}^{-1}.$$

Then

$$\begin{aligned} t_{\max} &= \frac{\ln(0.3466/0.06931)}{0.3466 - 0.06931} \text{ h} \\ &= \frac{\ln 5}{0.2773} \text{ h} \approx 5.80 \text{ h}. \end{aligned}$$

At this time the daughter is being produced and removed at equal rates, as shown in Fig. 22.12.

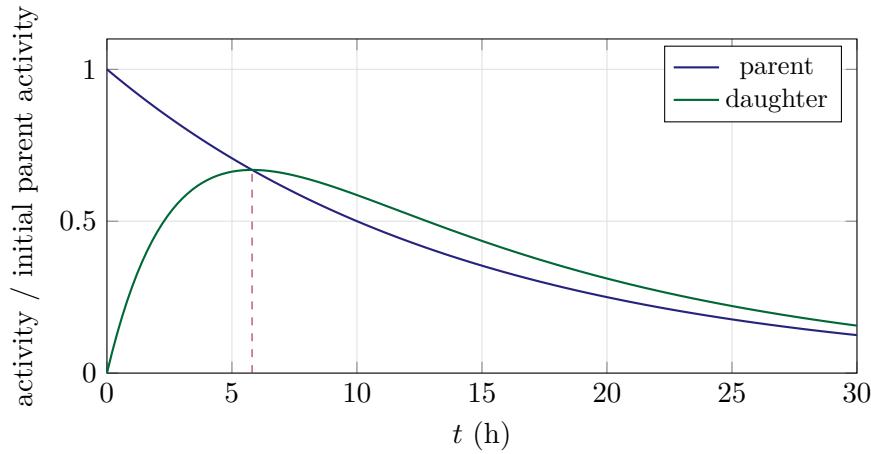


Figure 22.12: Calculated activities for the example with parent half-life 10 h, daughter half-life 2 h, unit feeding and no initial daughter. At $t_{\max} = 5.80 \text{ h}$, daughter production equals removal and the activities cross. At later times their ratio approaches $\lambda_2/(\lambda_2 - \lambda_1) = 1.25$, the transient-equilibrium ratio.

22.26.16 Secular and transient radioactive equilibrium

When a long-lived parent feeds a shorter-lived daughter, the daughter can approach an approximate equilibrium with the parent. Suppose $\lambda_1 \ll \lambda_2$. After several daughter half-lives, the term $e^{-\lambda_2 t}$ in Eq. (22.96) becomes small while the parent has hardly changed. Then

$$N_2(t) \approx \frac{\lambda_1}{\lambda_2} N_1(t), \quad (22.101)$$

and therefore

$$\mathcal{A}_2 = \lambda_2 N_2 \approx \lambda_1 N_1 = \mathcal{A}_1. \quad (22.102)$$

This is *secular equilibrium*: the daughter activity approximately tracks the parent activity.

If the parent is only moderately longer lived than the daughter, a *transient equilibrium* can occur in which the ratio of activities approaches a constant greater than one while both decay with approximately the parent's effective time dependence. For $\lambda_2 > \lambda_1$ after the short transient,

$$\frac{\mathcal{A}_2}{\mathcal{A}_1} \longrightarrow b \frac{\lambda_2}{\lambda_2 - \lambda_1}. \quad (22.103)$$

The secular value is b when $\lambda_1/\lambda_2 \rightarrow 0$. If the daughter is longer lived than the parent, this equilibrium interpretation does not apply. These regimes are important in generator systems used in nuclear medicine and in environmental decay chains.

Guided checks

1. In secular equilibrium, why can parent and daughter activities be nearly equal even though the daughter contains far fewer nuclei?
2. What hierarchy of half-lives is needed for secular equilibrium?
3. If a daughter is chemically separated from its parent, what happens immediately to its production term in Eq. (22.47)?

22.26.17 Activation and saturation under a constant production rate

Radioactive nuclei need not be present initially; they can be produced continuously by a reaction or by irradiation. If radioactive nuclei are created at a constant rate P and decay with constant λ , then

$$\frac{dN}{dt} = P - \lambda N. \quad (22.104)$$

For $N(0) = 0$ the solution is

$$N(t) = \frac{P}{\lambda} (1 - e^{-\lambda t}), \quad \mathcal{A}(t) = P (1 - e^{-\lambda t}). \quad (22.105)$$

At short times the inventory grows approximately linearly, $N \approx Pt$. At long times production and decay balance and the activity approaches the production rate,

$$\mathcal{A}_{\text{sat}} = P. \quad (22.106)$$

This is called saturation.

If production stops at time t_i , the subsequent activity simply decays:

$$\mathcal{A}(t) = \mathcal{A}(t_i) e^{-\lambda(t-t_i)}, \quad t > t_i. \quad (22.107)$$

Example 22.25: How quickly is saturation approached?

What fraction of the saturation activity is reached after three half-lives of constant production?

$$\frac{\mathcal{A}}{\mathcal{A}_{\text{sat}}} = 1 - e^{-\lambda t} = 1 - 2^{-t/T_{1/2}} = 1 - 2^{-3} = \frac{7}{8} = 0.875.$$

After three half-lives the activity has reached 87.5% of saturation. After ten half-lives it exceeds 99.9%.

Caution

Saturation does not mean that individual nuclei stop decaying. At saturation, creation and removal continue dynamically at equal average rates. The radioactive inventory becomes statistically stationary only in its mean.

22.26.18 A matrix formulation for general decay networks

For several coupled radionuclides, the Bateman equations can be written compactly as a linear system

$$\frac{d\mathbf{N}}{dt} = \mathbf{M}\mathbf{N} + \mathbf{P}(t), \quad (22.108)$$

where $\mathbf{N}$ contains the nuclide populations, $\mathbf{P}$ contains external production terms and the matrix $\mathbf{M}$ contains negative total decay constants on its diagonal and positive feeding terms off the diagonal.

For time-independent $\mathbf{M}$ and no external production,

$$\mathbf{N}(t) = e^{\mathbf{M}t} \mathbf{N}(0). \quad (22.109)$$

The matrix exponential is therefore the natural generalization of the simple scalar exponential $e^{-\lambda t}$. This formulation is valuable computationally because complicated decay trees can be assembled algorithmically from a database of half-lives and branching fractions.

For numerical work, however, the matrix exponential is not always the only or best method. Stiff ODE solvers can be preferable when half-lives span many orders of magnitude or when production rates vary with time. The correct numerical choice depends on scale separation, required precision and the sparsity of the network.

22.26.19 Poisson statistics of radioactive counting

Radioactive counting experiments are intrinsically statistical. Even if the activity and detector response remain perfectly constant, repeated measurements over equal time intervals do not return exactly the same number of counts. For a Poisson-process approximation to independent registered events, the number of counts n in a fixed interval is described by the Poisson distribution

$$P(n | \mu) = \frac{\mu^n e^{-\mu}}{n!}, \quad n = 0, 1, 2, \dots, \quad (22.110)$$

where $\mu = \int R(t)dt$ is the expected number of counts in the interval. A constant rate is not required for an inhomogeneous Poisson model. However, counts from a fixed finite number of decaying nuclei are exactly binomial for one interval; the Poisson approximation requires rare registration per nucleus and sufficiently many nuclei. Dead time and correlated cascades can invalidate independent Poisson counting.

For a Poisson distribution,

$$\langle n \rangle = \mu, \quad \text{Var}(n) = \mu, \quad \sigma_n = \sqrt{\mu}. \quad (22.111)$$

In practice, when n is not too small, the measured count itself is often used to estimate the standard uncertainty, $\sigma_n \approx \sqrt{n}$. The *relative* statistical uncertainty therefore scales as

$$\frac{\sigma_n}{n} \sim \frac{1}{\sqrt{n}}. \quad (22.112)$$

Collecting four times as many counts improves the purely statistical relative uncertainty only by a factor of two.

For a measured rate $R = n/\Delta t$ over a fixed counting interval Δt , the corresponding approximate statistical uncertainty is

$$\sigma_R \approx \frac{\sqrt{n}}{\Delta t} = \sqrt{\frac{R}{\Delta t}}. \quad (22.113)$$

Longer counting reduces statistical noise but can be problematic if the source is decaying appreciably during the interval or if environmental conditions drift.

Example 22.26: Counting time needed for a target statistical precision

A detector records a net signal rate of approximately 25 s^{-1} . Ignoring background uncertainty, how long should one count to make the Poisson relative uncertainty about 2%?

Require

$$\frac{1}{\sqrt{n}} \approx 0.02,$$

so

$$n \approx \frac{1}{0.02^2} = 2500.$$

At 25 s^{-1} ,

$$\Delta t = \frac{2500}{25 \text{ s}^{-1}} = 100 \text{ s}.$$

The result illustrates the square-root law: achieving a factor-two improvement in precision requires four times as many counts.

Caution

The rule $\sigma \approx \sqrt{n}$ describes Poisson counting fluctuations, not every source of experimental uncertainty. Calibration, background subtraction, dead time, geometry, detector efficiency and source positioning can dominate once counting statistics become precise.

22.26.20 Finite populations, observed counts and bin integration

For N_0 independent nuclei present at time zero, let $S = e^{-\lambda t}$. The actual surviving number $n(t)$ is binomial:

$$P[n(t) = k] = \binom{N_0}{k} S^k (1 - S)^{N_0 - k}, \quad \langle n \rangle = N_0 S, \quad \text{Var}(n) = N_0 S(1 - S). \quad (22.114)$$

After one half-life, the expectation is $N_0/2$, but the standard deviation is $\sqrt{N_0}/2$. For only 100 nuclei, a realization need not leave exactly 50. At different times survivors are correlated: the late population is a subset of the early population. Drawing independent Poisson survivor populations at each time would violate that nesting.

For a counting interval $[a, b]$, a given parent has probability $p = e^{-\lambda a} - e^{-\lambda b}$ to decay within that interval. If at most one selected count can be registered per decay with probability ϵb_γ , the registration probability is $p_d = \epsilon b_\gamma p$. The count is binomial with mean $N_0 p_d$ and variance $N_0 p_d(1 - p_d)$. A Poisson model is accurate when $p_d \ll 1$; its mean still uses the integrated time dependence.

For a signal rate $R_0 e^{-\lambda t}$ in a bin of width h centred at t_c ,

$$\mu_s = h R_0 e^{-\lambda t_c} \frac{\sinh(\lambda h/2)}{\lambda h/2}. \quad (22.115)$$

The ratio to the midpoint approximation is $1 + (\lambda h)^2/24 + \dots$. For a start-time approximation the correction appears already at first order in λh . Equal-width bins are special: the midpoint factor is constant across bins, so a free amplitude can absorb it for a single exponential. It need not bias the fitted decay constant. Unequal bin widths, fixed amplitude, or a more complicated signal remove that cancellation.

Example 22.27: A wide bin changes amplitude before it changes slope

Take $T_{1/2} = 100 \text{ s}$, $h = 50 \text{ s}$ and $R_0 = 120 \text{ s}^{-1}$, for a bin starting at zero. Then $\lambda h = 0.34657$

and

$$\mu_s = \frac{120}{0.006931}(1 - 2^{-1/2}) \simeq 5071.$$

Using the start rate gives 6000, an overestimate of about 18.3%. The midpoint estimate is $50(120)2^{-1/4} \simeq 5045$, low by about 0.50%. Integrating removes both approximations. If every bin has this same width and the amplitude is fitted freely, the midpoint factor rescales the amplitude while leaving the exact exponential slope unchanged.

22.26.21 Background subtraction and uncertainty propagation

Suppose a source-plus-background measurement gives n_{sb} counts in time t_{sb} , while a separate background measurement gives n_b counts in time t_b . The estimated net rate is

$$R_s = \frac{n_{sb}}{t_{sb}} - \frac{n_b}{t_b}. \quad (22.116)$$

If the two counts are independent and Poisson distributed, approximate variance propagation gives

$$\sigma_{R_s}^2 \approx \frac{n_{sb}}{t_{sb}^2} + \frac{n_b}{t_b^2}. \quad (22.117)$$

A background measurement therefore contributes its own statistical uncertainty. Subtracting a number does not subtract its uncertainty.

When the signal is weak compared with background, increasing only the source measurement time may be inefficient if the background itself is poorly known. Experimental design should allocate enough time to both measurements. In more advanced analyses, background may be fitted simultaneously with source parameters rather than subtracted beforehand.

Example 22.28: Net rate with background uncertainty

A source-plus-background run records 1250 counts in 100 s. A background run records 320 counts in 200 s. Estimate the net rate and its counting-statistical standard uncertainty.

The rates are

$$R_{sb} = 12.50 \text{ s}^{-1}, \quad R_b = 1.60 \text{ s}^{-1},$$

so

$$R_s = 10.90 \text{ s}^{-1}.$$

The uncertainty is

$$\begin{aligned} \sigma_{R_s} &\approx \sqrt{\frac{1250}{100^2} + \frac{320}{200^2}} \text{ s}^{-1} \\ &= \sqrt{0.125 + 0.0080} \text{ s}^{-1} \\ &\approx 0.365 \text{ s}^{-1}. \end{aligned}$$

Thus one may report the purely statistical result as approximately

$$R_s = (10.90 \pm 0.37) \text{ s}^{-1}.$$

Systematic uncertainties in efficiency or geometry would be additional.

22.26.22 Detector dead time and rate distortion

After recording an event, many detectors and electronic systems require a short recovery time during which another event may not be recorded correctly. This interval is called the *dead time*. At low

rates the effect is negligible, but at high rates it can produce significant count losses and distort spectra.

In a simple nonparalyzable model with dead time τ_d , an event arriving during the dead period is lost but does not extend that period. If the true rate is R and the measured rate is r , then

$$r = \frac{R}{1 + R\tau_d}. \quad (22.118)$$

Solving for the true rate,

$$R = \frac{r}{1 - r\tau_d}, \quad (22.119)$$

which is meaningful only when $r\tau_d < 1$.

A paralyzable model behaves differently: an event arriving during dead time extends the unavailable period, leading to

$$r = Re^{-R\tau_d}. \quad (22.120)$$

Neither output is generally a Poisson process: electronics introduces correlations between registered arrivals. Correcting a mean rate does not restore the original Poisson variance. The two models coincide approximately when $R\tau_d \ll 1$ but differ strongly at high rate. Real instruments may require more detailed models.

Example 22.29: Dead-time correction

A detector following the nonparalyzable model has dead time $\tau_d = 20 \mu\text{s}$ and reports $r = 1.00 \times 10^4 \text{ s}^{-1}$. Estimate the true rate.

First,

$$r\tau_d = (1.00 \times 10^4)(20 \times 10^{-6}) = 0.20.$$

Then

$$R = \frac{1.00 \times 10^4}{1 - 0.20} \text{ s}^{-1} = 1.25 \times 10^4 \text{ s}^{-1}.$$

The measured system loses about 20% of the events relative to this corrected estimate.

Note

Dead time is a good example of why a detector is part of the physics model. The source may decay perfectly exponentially while the *measured* rate deviates from an exponential if the detector response is nonlinear at high count rate.

22.26.23 Fitting a half-life from data

Suppose measured counts n_i are obtained in bins with start and end times a_i, b_i ; bin widths may differ. A physically motivated model is

$$\mu_i(\theta) = R_0 \frac{e^{-\lambda a_i} - e^{-\lambda b_i}}{\lambda} + R_b(b_i - a_i), \quad \theta = (R_0, \lambda, R_b). \quad (22.121)$$

Here R_0 is the initial detected signal rate, and R_b is a constant detected background rate. At large counts, one may minimize a weighted chi-squared,

$$\chi^2(\theta) = \sum_i \frac{[n_i - \mu_i(\theta)]^2}{\sigma_i^2}, \quad (22.122)$$

with suitable variances σ_i^2 . For genuinely Poisson counts, a more natural likelihood is

$$\mathcal{L}(\theta) = \prod_i \frac{\mu_i(\theta)^{n_i} e^{-\mu_i(\theta)}}{n_i!}. \quad (22.123)$$

It is numerically convenient to minimize the negative log-likelihood, omitting terms independent of the parameters:

$$-\ln \mathcal{L}(\theta) = \sum_i [\mu_i(\theta) - n_i \ln \mu_i(\theta)] + \text{constant}. \quad (22.124)$$

The fitted half-life is then

$$T_{1/2} = \frac{\ln 2}{\lambda}. \quad (22.125)$$

If the fitted uncertainty on λ is σ_λ and linear error propagation is valid,

$$\sigma_{T_{1/2}} \approx \frac{\ln 2}{\lambda^2} \sigma_\lambda. \quad (22.126)$$

For $n_i = 0$, its contribution is simply μ_i ; implement $n_i \ln \mu_i = 0$ by continuity rather than evaluating $0 \ln 0$ numerically. Require $R_0 > 0$, $\lambda > 0$, $R_b \geq 0$ and a positive expectation for any bin with events. An independent background run contributes its own Poisson likelihood with mean $R_b t_b$; it should not be treated as an exact subtraction.

Parameter correlations matter: uncertainty in background can correlate strongly with λ when the observation window is short or the late-time data are sparse.

Method

For decay-data analysis, plot the raw counts and the fitted model on the original scale before inspecting a logarithmic plot. A log plot can make exponential behavior visually obvious, but it can hide zero-count bins, distort symmetric error bars and overemphasize low-count fluctuations.

22.26.24 Attenuation, half-value layer and shielding design variables

For a narrow monoenergetic photon beam passing through a homogeneous absorber, the probability of surviving an infinitesimal thickness dx without interaction leads to

$$\frac{dI}{dx} = -\mu I, \quad (22.127)$$

with solution $I = I_0 e^{-\mu x}$. The mathematics is identical to radioactive decay because both are memoryless survival processes with a constant probability per unit independent variable: time for decay and path length for attenuation.

The mass attenuation coefficient

$$\frac{\mu}{\rho} \quad (22.128)$$

has units of area per mass and is useful for comparing materials independent of density. In a mixture or compound, approximate mass-weighted combination rules can be used over suitable energy ranges, though accurate shielding work relies on evaluated databases and radiation-transport calculations.

For n half-value layers,

$$\frac{I}{I_0} = 2^{-n}. \quad (22.129)$$

Ten half-value layers reduce a narrow-beam uncollided intensity by approximately a factor of $2^{10} \approx 10^3$.

Shielding design must also consider geometry. A point-like isotropic source produces fluence scaling approximately as $1/r^2$ in free space. Combining geometric dilution with exponential attenuation gives a simplified form

$$I(r, x) \propto \frac{e^{-\mu x}}{r^2}. \quad (22.130)$$

This expression is useful for scaling arguments but omits scattering, source size and buildup.

Example 22.30: Combining distance and attenuation

A simplified photon field is measured at distance $r_1 = 1.0\text{ m}$ with no absorber. The detector is moved to $r_2 = 2.0\text{ m}$ and two half-value layers of shielding are inserted. Ignoring scattering and source size, what fraction of the original intensity remains?

Geometric dilution gives

$$\left(\frac{r_1}{r_2}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Two half-value layers give another factor

$$2^{-2} = \frac{1}{4}.$$

Thus

$$\frac{I_2}{I_1} \approx \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.$$

The example illustrates multiplicative mitigation mechanisms but should not be used as a complete shielding design model.

22.26.25 Gamma-ray spectroscopy: calibration before interpretation

A pulse-height spectrum is not automatically an energy spectrum. The detector and electronics first map deposited energy to a channel number or digital pulse amplitude. An energy calibration is therefore required. Over a restricted range, a common model is

$$E = a + bC, \quad (22.131)$$

where C is channel number and a, b are obtained from known reference peaks. If nonlinearity is significant, a higher-order calibration may be needed, but additional parameters should be justified by data rather than added automatically.

Peak analysis then separates several ingredients: a smooth background or continuum, the peak shape, finite resolution and possible overlapping lines. A simple isolated full-energy peak is often approximated by a Gaussian,

$$g(E) = A_p \exp \left[-\frac{(E - E_0)^2}{2\sigma_E^2} \right] + B(E), \quad (22.132)$$

with $\text{FWHM} = 2\sqrt{2\ln 2}\sigma_E$. Real detector response can contain asymmetric tails, escape peaks, Compton structures and pile-up; the Gaussian-plus-background model is therefore a starting point, not a universal law.

Method

For spectrum analysis, separate three questions: *where* is the peak (energy calibration), *how broad* is it (resolution), and *how many events* does it contain (efficiency and activity information). A single fitted curve may estimate all three, but the physical meanings are different.

Example 22.31: Two-point energy calibration

Two reference gamma lines of 356.0 keV and 661.7 keV appear at channels 420 and 782. Assuming a linear calibration, find a and b .

The slope is

$$b = \frac{661.7 - 356.0}{782 - 420} \approx 0.8445 \text{ keV/channel.}$$

Then

$$a = 356.0 - b(420) \approx 1.3 \text{ keV.}$$

A peak at channel 600 would therefore correspond to approximately

$$E \approx 1.3 + 0.8445(600) \approx 508 \text{ keV.}$$

The result should not be extrapolated far beyond the calibrated range without checking detector linearity.

22.26.26 Energy resolution and spectral peak width

A monoenergetic gamma ray does not appear as an infinitely narrow line in a real detector. Statistical fluctuations in charge or light production, electronic noise, incomplete charge collection and calibration effects broaden the response. A common measure of energy resolution is

$$\mathcal{R} = \frac{\text{FWHM}}{E_0}, \quad (22.133)$$

where E_0 is the peak energy and FWHM is the full width at half maximum.

For an approximately Gaussian peak with standard deviation σ_E ,

$$\text{FWHM} = 2\sqrt{2\ln 2}\sigma_E \approx 2.355\sigma_E. \quad (22.134)$$

Better resolution means a smaller value of $\mathcal{R}$. Semiconductor detectors often achieve substantially better gamma-energy resolution than scintillators, while scintillators may offer higher efficiency, faster timing or lower cost. Detector choice is therefore an engineering tradeoff rather than a single ranking.

Example 22.32: Converting FWHM to Gaussian sigma

A 662 keV gamma peak has FWHM 2.0 keV. The relative resolution is

$$\mathcal{R} = \frac{2.0}{662} \approx 3.0 \times 10^{-3} = 0.30\%.$$

If the peak is Gaussian,

$$\sigma_E = \frac{2.0 \text{ keV}}{2.355} \approx 0.85 \text{ keV.}$$

These two numbers describe the same width in different conventions.

22.26.27 Detector efficiency, solid angle and inverse-square intuition

For an isotropic point source, the fraction of emitted radiation that reaches a detector of area A_d at distance r and approximately normal incidence is, when the detector dimensions are small compared with r ,

$$f_\Omega \approx \frac{A_d}{4\pi r^2}. \quad (22.135)$$

This is the geometric efficiency. The intrinsic efficiency ϵ_{int} is the probability that a quantum entering the active detector produces the required signal. The full efficiency is approximately

$$\epsilon \approx f_\Omega \epsilon_{\text{int}} \quad (22.136)$$

for the simple geometry.

For a circular detector of radius a whose centre is on the source axis at perpendicular distance r , the rim subtends half-angle θ_0 with $\cos \theta_0 = r/\sqrt{r^2 + a^2}$. Integrating $d\Omega = 2\pi \sin \theta d\theta$ gives

$$f_\Omega = \frac{1 - \cos \theta_0}{2} = \frac{1}{2} \left(1 - \frac{r}{\sqrt{r^2 + a^2}} \right). \quad (22.137)$$

Expansion for $a/r \ll 1$ recovers $a^2/(4r^2) = A_d/(4\pi r^2)$. For $a = r$, the exact fraction is 0.146, while the small-detector approximation gives 0.25 and is unsuitable.

The inverse-square law is therefore a statement about solid angle, not a property of radioactive decay itself. A large nearby detector, an extended source or a collimated beam can violate the point-source approximation even though no new physics is involved.

Guided checks

1. If detector distance doubles while the point-source approximation remains valid, what happens to geometric efficiency?
2. Why can intrinsic efficiency depend strongly on photon energy even when detector geometry is unchanged?
3. Why should one not interpret a change in count rate caused by moving the detector as a change in source activity?

22.26.28 Coincidence counting and accidental coincidences

Coincidence measurements identify events observed in two detectors within a time window. They are powerful because a physical process can create genuinely correlated signals, whereas many background events are independent.

If two independent detectors have singles rates r_1 and r_2 , the accidental coincidence rate is approximately

$$r_{\text{acc}} \approx 2\tau_c r_1 r_2, \quad (22.138)$$

for a symmetric coincidence window of total width $2\tau_c$, provided the rates and window are small enough that multiple events are rare. Exact conventions differ between electronics systems, so the factor multiplying the time window must always be matched to the implementation.

The equation illustrates an engineering trade-off. A wider coincidence window captures more true pairs when timing resolution is imperfect, but it also increases accidental coincidences. Improved timing allows a narrower window and therefore improves background rejection.

22.26.29 PET coincidence geometry and image reconstruction intuition

In positron-emission tomography, a positron emitted in beta-plus decay loses kinetic energy and eventually annihilates with an electron. In the centre-of-momentum frame, two-photon annihilation produces photons of energy

$$E_\gamma = m_e c^2 \approx 511 \text{ keV} \quad (22.139)$$

traveling in opposite directions. A ring of detectors records pairs of photons within a coincidence time window. A valid coincidence identifies a *line of response* joining the two detector elements.

Figure 22.13 shows why one coincidence does not identify a unique point along that line. Image reconstruction combines many lines of response from many annihilation events. In simple filtered back-projection, each event contributes information along a line and mathematical filtering corrects the blurring that naive back-projection would produce. Modern PET systems often use iterative statistical reconstruction methods that incorporate detector geometry, attenuation, scatter and timing information.

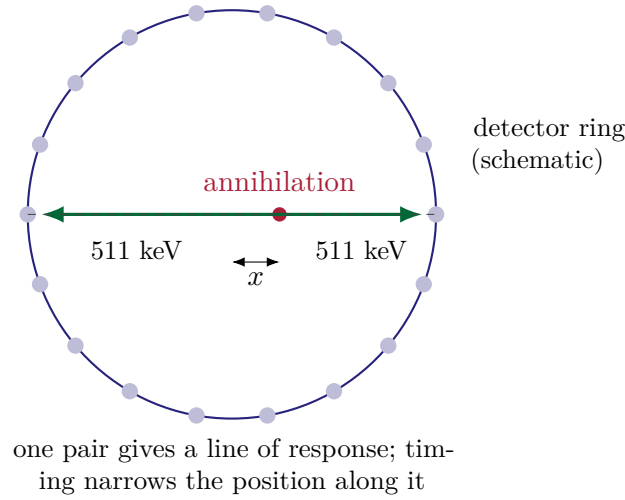


Figure 22.13: PET line-of-response geometry for approximately two-photon annihilation. If the event is displaced by x to the right of the midpoint, the left arrival time minus the right arrival time is $2x/c$. The detector ring is schematic; positron range, photon non-collinearity and scattering are omitted.

If the system measures the difference in arrival times Δt accurately, time-of-flight PET can localize the annihilation along the line of response with spatial resolution in the same width convention approximately

$$\Delta x \sim \frac{c \Delta t}{2}. \quad (22.140)$$

The factor $1/2$ appears because moving the source by Δx shortens one photon path and lengthens the other by the same amount, changing the path-length difference by $2\Delta x$.

Example 22.33: Time-of-flight localization scale

A PET detector system has coincidence time-difference FWHM $\Delta t = 400$ ps. Estimate the corresponding line-of-response localization scale.

$$\Delta x \sim \frac{(3.00 \times 10^8 \text{ m/s})(400 \times 10^{-12} \text{ s})}{2} \approx 6.0 \times 10^{-2} \text{ m} = 6.0 \text{ cm}.$$

The result is a 6 cm FWHM along this single line of response. It does not mean the final image resolution is 6 cm; reconstruction combines many events and is also limited by detector size, positron range, non-collinearity and other effects. The estimate isolates only the timing contribution.

22.26.30 Uncertainty propagation in a binding-energy calculation

Precision mass measurements translate directly into precision nuclear energies. Suppose a binding energy computed from independent measured masses has the form

$$B = c^2 (Zm_H + Nm_n - M). \quad (22.141)$$

If the standard mass uncertainties are σ_H , σ_n and σ_M and correlations can be neglected, then

$$\sigma_B = c^2 \sqrt{Z^2 \sigma_H^2 + N^2 \sigma_n^2 + \sigma_M^2}. \quad (22.142)$$

When masses are quoted in atomic mass units, the factor c^2 is conveniently represented by $931.5 \text{ MeV}/u$.

Correlations matter in high-precision mass evaluations because several reported masses can share common reference measurements. Treating correlated inputs as independent can overestimate or

underestimate the final uncertainty. This is a general lesson extending well beyond nuclear physics: uncertainty propagation depends on the covariance structure, not only on individual error bars.

22.26.31 A generic parent–daughter dating model

Radiometric dating need not rely only on the fraction of a parent isotope remaining. Some methods infer time from the accumulation of a daughter product. For a closed system with one radioactive parent P decaying to stable daughter D ,

$$N_P(t) = N_{P0}e^{-\lambda t}, \quad (22.143)$$

$$N_D(t) = N_{D0} + N_{P0}(1 - e^{-\lambda t}). \quad (22.144)$$

Eliminating N_{P0} using $N_{P0} = N_P e^{\lambda t}$ gives

$$N_D(t) = N_{D0} + N_P(t)(e^{\lambda t} - 1). \quad (22.145)$$

The challenge is that the initial daughter abundance N_{D0} is generally unknown. Real geochronology overcomes this in different ways, for example by using mineral systems, reference isotopes or isochron methods. The physics lesson is that a decay clock is only as reliable as the assumption that the relevant material has remained a closed system and that initial conditions are controlled.

Caution

A numerical age from an exponential formula is not automatically a reliable geological age. Contamination, gain or loss of parent or daughter material, reservoir effects and calibration curves can dominate the uncertainty. The physical half-life is only one part of the inference problem.

22.26.32 A measurement-design calculation: dividing sample and background time

Let s and b be constant expected net signal and background rates. For a fixed total time $T = t_s + t_b$ divided between independent sample-plus-background and background-only runs, the variance of the estimated net rate is

$$V(t_s) = \frac{s+b}{t_s} + \frac{b}{T-t_s}. \quad (22.146)$$

Differentiating and setting the derivative to zero gives

$$\frac{t_s}{t_b} = \sqrt{\frac{s+b}{b}}, \quad t_s = T \frac{\sqrt{s+b}}{\sqrt{s+b} + \sqrt{b}}. \quad (22.147)$$

The second derivative is positive for positive rates and $0 < t_s < T$, so this is a minimum. With $s = 10 \text{ s}^{-1}$, $b = 2 \text{ s}^{-1}$ and $T = 300 \text{ s}$, choose about $t_s = 213 \text{ s}$ and $t_b = 87 \text{ s}$; the predicted standard uncertainty is 0.282 s^{-1} . An equal split gives 0.306 s^{-1} . This design result assumes stationary rates and only Poisson statistics. A decaying source, changing geometry or dominant calibration error requires a different optimization.

22.27 Problem-solving workshop

These longer cases connect several steps rather than repeat isolated substitutions. They complement the core examples and the practice problems.

Example 22.34: identifying a decay chain

A nucleus ${}^A_ZX$ undergoes one alpha decay, followed by two beta-minus decays, and then a gamma transition. Determine the final mass number and atomic number in terms of A and Z .

Solution. Alpha decay changes $(A, Z) \rightarrow (A - 4, Z - 2)$. Each beta-minus decay converts a neutron to a proton, leaving A unchanged and increasing Z by one. Two beta-minus decays therefore change $(A - 4, Z - 2) \rightarrow (A - 4, Z)$. Gamma decay changes neither A nor Z .

The final nuclide is therefore

$${}^{A-4}_ZY,$$

where Y is the element with the same atomic number as the original parent. The chemical element can return to the original Z even though the isotope has changed because the mass number is four units smaller.

Example 22.35: half-life from two activity measurements

A corrected activity is 9.60 kBq at $t = 0$ and 2.40 kBq at $t = 14.0$ h. Find the decay constant, half-life and mean lifetime.

Solution. The ratio is $2.40/9.60 = 1/4 = 2^{-2}$. Thus two half-lives have elapsed in 14.0 h, so

$$T_{1/2} = 7.00 \text{ h}.$$

Then

$$\lambda = \frac{\ln 2}{7.00 \text{ h}} = 0.0990 \text{ h}^{-1} = 2.75 \times 10^{-5} \text{ s}^{-1},$$

and

$$\tau = \frac{1}{\lambda} = 10.1 \text{ h}.$$

A useful check is $\tau = T_{1/2}/\ln 2 \approx 1.443T_{1/2}$, which agrees.

Example 22.36: Q value with atomic masses

A hypothetical alpha decay has neutral-atom masses

$$M_p = 226.02540 \text{ u}, \quad M_d = 222.01760 \text{ u}, \quad M_\alpha = 4.00260 \text{ u}.$$

Find the Q value.

Solution. Atomic electron numbers balance in alpha decay when the helium mass is taken as the neutral helium-atom mass, so

$$Q = (M_p - M_d - M_\alpha)c^2.$$

The mass difference is

$$\Delta m = 226.02540 - 222.01760 - 4.00260 = 0.00520 \text{ u}.$$

Therefore

$$Q = (0.00520)(931.5) \text{ MeV} \approx 4.84 \text{ MeV}.$$

The positive value means the decay is energetically allowed. It does not by itself determine the half-life; barrier tunnelling controls the decay probability.

Example 22.37: activation followed by cooling

A radionuclide is produced at constant rate $P = 2.0 \times 10^4 \text{ s}^{-1}$ for four half-lives and then production stops. What is the activity immediately at shutdown, and what fraction of P remains after two additional half-lives?

Solution. At shutdown,

$$\mathcal{A}_i = P(1 - 2^{-4}) = \frac{15}{16}P = 1.875 \times 10^4 \text{ Bq}.$$

Two half-lives later the activity has fallen by a factor four:

$$\mathcal{A} = \frac{1}{4}\mathcal{A}_i = \frac{15}{64}P.$$

Numerically,

$$\mathcal{A} \approx 4.69 \times 10^3 \text{ Bq}.$$

The build-up and decay stages are governed by the same decay constant but different differential equations because the production term is present only during activation.

Example 22.38: diagnosing a poor half-life fit

A student fits $\ln n_i$ versus time with an unweighted straight line. The late-time bins contain only a few counts and several bins contain zero. The fitted half-life changes substantially when the last two bins are removed. Diagnose the problem and propose a better analysis.

Solution. Taking logarithms makes zero-count bins undefined and produces strongly nonuniform uncertainties because a fluctuation of one count is far more important when n is small. An unweighted straight-line fit therefore gives inappropriate influence to different bins. If background is present, the logarithm of signal plus background is not linear either. A better approach is to fit the original integer counts with expected values

$$\mu_i = \int_{t_i}^{t_i+\Delta t} (R_0 e^{-\lambda t} + R_b) dt$$

using a Poisson likelihood. The fit should include the background rate if it is not independently known, and stability should be checked with residuals, toy Monte Carlo studies and reasonable variations of the fit interval. Sensitivity to arbitrary bin removal is a warning sign that the assumed model or uncertainty treatment is inadequate.

Example 22.39: When more counts stop helping

A measurement has 1% statistical uncertainty, 4% efficiency uncertainty and 1.5% branching-fraction uncertainty. The combined relative uncertainty is

$$\sqrt{0.01^2 + 0.04^2 + 0.015^2} \approx 4.4\%.$$

Collecting four times as many counts would reduce the statistical term to 0.5%, giving

$$\sqrt{0.005^2 + 0.04^2 + 0.015^2} \approx 4.3\%.$$

The total barely improves because the efficiency calibration dominates. The most effective next step is therefore better calibration, not simply longer counting.

22.28 Practice problems

The 48 independently written problems progress from interpretation to calculation, data analysis and advanced derivation. Unless stated otherwise, use $R_0 = 1.2$ fm, $uc^2 = 931.5$ MeV, $m_e c^2 = 0.511$ MeV and $N_A = 6.022 \times 10^{23}$ mol⁻¹. All mass data needed for the numerical problems are supplied. Detector models refer to the specified geometry and selected count channel.

Conceptual reasoning

- C1.** Rutherford scattering is sometimes summarized by saying that alpha particles “hit the nucleus.” Why is that wording misleading for most scattering events?
- C2.** Distinguish isotope, isotone and isobar. Give a symbolic example of each relationship.
- C3.** Why does the empirical radius law $R = R_0 A^{1/3}$ imply approximately constant nuclear density?
- C4.** A nucleus has a positive binding energy. Does this mean its nucleons possess extra rest mass compared with free nucleons? Explain.
- C5.** What is meant by the statement that radioactive decay is random for an individual nucleus but predictable for a macroscopic ensemble?
- C6.** Does a nucleus become “more likely” to decay because it has already survived for many half-lives? Explain for ideal exponential decay.
- C7.** Explain why beta-plus decay has an atomic-mass threshold involving $2m_e c^2$.
- C8.** Activity and absorbed dose both concern radiation. Why can neither be inferred from the other without additional information?
- C9.** Under what assumptions is a selected detector count rate smaller than the source activity? Explain how multiple photons per decay, daughter decays and background can change that comparison.
- C10.** Why does taking a logarithm of count data not automatically make an exponential-fit problem statistically simpler?

Analytical and numerical problems

- A1.** Estimate the radius of nuclei with $A = 16$, $A = 64$ and $A = 208$. Compare the ratios of the radii with the ratios of the mass numbers.
- A2.** For a hypothetical nuclide with $Z = 8$, $N = 8$, neutral-atom mass $15.9905 u$, and using $m_H = 1.007825 u$, $m_n = 1.008665 u$, calculate the binding energy and binding energy per nucleon.
- A3.** Two neighbouring nuclei have binding energies $B(A, Z) = 612.4$ MeV and $B(A - 1, Z) = 604.1$ MeV. Find the neutron separation energy.
- A4.** A radionuclide has half-life 4.0 h. What fraction remains after 1.0 h, 8.0 h and 20.0 h?
- A5.** A sample initially contains 2.0×10^{12} radioactive nuclei with $T_{1/2} = 6.0$ h. Find the initial activity and the activity after 18 h.
- A6.** A detector-corrected activity falls from 15.0 kBq to 3.75 kBq in 8.0 h. Find the half-life.
- A7.** A pure radioactive sample has mass $2.0 \mu\text{g}$, molar mass 60 g/mol and half-life 10 d. Estimate its initial activity.

- A8.** A source has activity 5.0 kBq. A radiation line occurs with branching fraction 0.80 and is detected with total efficiency 0.030. Estimate the signal count rate.
- A9.** A gamma beam has attenuation coefficient $\mu = 0.18 \text{ cm}^{-1}$. Find the transmitted fraction through 2.0 cm and the half-value layer.
- A10.** A 10 g sample absorbs 0.50 mJ of radiation energy. Find the absorbed dose.
- A11.** A two-body alpha decay releases $Q = 5.50 \text{ MeV}$. If the daughter mass number is 210, estimate how the kinetic energy is divided between the alpha particle and daughter by nonrelativistic momentum conservation.
- A12.** Neutral atomic masses in a beta-minus decay are $M_p = 14.0032 u$ and $M_d = 14.0031 u$. Estimate the Q value. Treat the supplied masses as rounded hypothetical inputs; do not identify a measured isotope decay from them.
- A13.** Neutral atomic masses in a beta-plus candidate are $M_p - M_d = 0.00140 u$. Determine whether positron emission is energetically allowed using $m_e c^2 = 0.511 \text{ MeV}$. Treat the supplied masses as rounded hypothetical inputs; do not identify a measured isotope decay from them.
- A14.** Complete the alpha decay ${}_{92}^{238}\text{U} \rightarrow ? + {}_2^4\text{He}$.

Data and computation

- N1.** Write pseudocode that simulates the decay of N_0 independent nuclei in time steps Δt using the exact finite-step probability $p = 1 - e^{-\lambda \Delta t}$.
- N2.** Generate, conceptually or in code, 10^4 exponential lifetimes using $t = -\ln u/\lambda$. Which two sample summaries would you compare with analytic expectations to validate the generator?
- N3.** A counting experiment records $n = 400$ events in 20 s. Assuming Poisson statistics and negligible background, estimate the count rate and its standard statistical uncertainty.
- N4.** A sample-plus-background measurement gives 960 counts in 60 s; a background measurement gives 180 counts in 90 s. Estimate the net signal rate and its standard uncertainty.
- N5.** A nonparalyzable detector has dead time $15 \mu\text{s}$ and records $2.0 \times 10^4 \text{ s}^{-1}$. Estimate the corrected true rate.
- N6.** Calibration lines at 122.1 keV and 661.7 keV appear at channels 150 and 820. Find a linear calibration $E = a + bC$ and predict the energy at channel 500.
- N7.** Five non-overlapping 10 s bins, starting at $t = 0$, contain raw counts 808, 574, 408, 289, 205. Background is negligible. Estimate the half-life from a diagnostic semilog slope, then formulate a Poisson likelihood with integrated bin means. Explain why equal bin widths change the fitted amplitude but not the ideal exponential slope.
- N8.** For a source with $T_{1/2} = 20 \text{ s}$, choose a simulation step such that the finite-step decay probability is at most 0.5%. Compute the largest acceptable Δt .
- N9.** Construct a 3×3 response matrix with 80% retention and 10% migration to each adjacent bin. At the outer boundaries, let the outward 10% leave the recorded range, rather than wrap to the opposite edge. Use columns for true bins, apply it to $(0, 1000, 0)^T$, and explain why edge-column sums are 0.9.
- N10.** Write a likelihood function for binned decay data with unknown initial rate, half-life and constant background. Identify the three fit parameters and the physical constraints they should satisfy.

Engineering and measurement

- E1.** A detector with area A_d has linear dimensions small compared with its distance r from an approximately isotropic point source. Derive the geometric efficiency $\epsilon_\Omega \approx A_d/(4\pi r^2)$ and discuss when the approximation fails.
- E2.** A PET system improves coincidence timing time-difference FWHM from 600 ps to 300 ps. Compare the simple time-of-flight localization scales $c\Delta t/2$.
- E3.** A gamma counting system has two singles rates $r_1 = 2.0 \times 10^4 \text{ s}^{-1}$ and $r_2 = 1.5 \times 10^4 \text{ s}^{-1}$. Using a symmetric half-window $\tau_c = 5 \text{ ns}$, estimate the accidental coincidence rate with Eq. (22.138).
- E4.** A production process creates a radionuclide at $P = 5.0 \times 10^5 \text{ s}^{-1}$ with half-life 2.0 h. Find the activity after 1, 2, 4 and 10 half-lives of continuous production as fractions of saturation. Assume no initial population of the produced radionuclide.
- E5.** A semiconductor detector resolves two nearby lines. Explain qualitatively how improving energy resolution can reduce systematic uncertainty in the fitted peak areas even when the total number of counts is unchanged.
- E6.** A counting system must reach a purely statistical relative uncertainty of 2% at a net rate of 4 s^{-1} . Estimate the required number of net counts and, neglecting background, the corresponding measurement time.

Advanced challenges

- H1.** Starting from the survival probability $S(t) = e^{-\lambda t}$, derive the exponential lifetime density $f(t)$ and verify that its mean is $1/\lambda$.
- H2.** Show directly that an exponential lifetime distribution is memoryless: $P(T > s + t \mid T > s) = P(T > t)$.
- H3.** Derive the time of maximum daughter activity for a chain $1 \rightarrow 2$ with $N_2(0) = 0$ and $\lambda_1 \neq \lambda_2$.
- H4.** For a parent much longer lived than its daughter, derive the secular-equilibrium approximation $\mathcal{A}_2 \approx \mathcal{A}_1$ after the initial transient.
- H5.** Using the semi-empirical mass formula without pairing, derive the approximate Z that maximizes binding for fixed A .
- H6.** A detector has nonparalyzable dead time. Starting from the idea that each recorded event makes the detector insensitive for time τ_d , derive $r = R/(1 + R\tau_d)$.
- H7.** For a decay fit with expected counts $\mu_i(\theta)$, derive the Poisson log-likelihood up to additive constants and explain why bins with zero observed counts remain usable.
- H8.** Derive the uncertainty transformation from λ to $T_{1/2} = \ln 2/\lambda$ for small uncertainty σ_λ .

22.29 Selected solutions to representative problems

The following solutions are intentionally selective. They provide enough feedback for self-study while leaving the majority of the exercise set for independent work.

Example 22.40: Selected solution for A1: nuclear radii

Using $R = 1.2A^{1/3}$ fm,

$$R(16) \approx 1.2(2.52) = 3.02 \text{ fm},$$

$$R(64) = 1.2(4) = 4.80 \text{ fm},$$

$$R(208) \approx 1.2(5.93) = 7.12 \text{ fm}.$$

The mass number changes by factors much larger than the radius because the radius scales only as the cube root. For example, increasing A from 16 to 128 would increase the radius only by a factor 2.

Example 22.41: Selected solution for A2: binding energy

Here $Z = N = 8$. With atomic masses,

$$\Delta m = 8m_{\text{H}} + 8m_n - M_{\text{atom}}.$$

Substitution gives

$$\Delta m = 8(1.007825) + 8(1.008665) - 15.9905 = 0.14142 \text{ u}.$$

Therefore

$$B \approx 0.14142(931.5) = 131.7 \text{ MeV}, \quad B/A \approx 8.23 \text{ MeV}.$$

Because the supplied mass is hypothetical, the purpose is the mass accounting rather than comparison with a real nuclide table.

Example 22.42: Selected solution for A5: activity from number of nuclei

The decay constant is

$$\lambda = \frac{\ln 2}{6.0 \times 3600 \text{ s}} = 3.21 \times 10^{-5} \text{ s}^{-1}.$$

Initially,

$$\mathcal{A}_0 = \lambda N_0 = (3.21 \times 10^{-5})(2.0 \times 10^{12}) \approx 6.42 \times 10^7 \text{ Bq}.$$

After $18 \text{ h} = 3T_{1/2}$, the activity is one eighth as large:

$$\mathcal{A} = 8.03 \times 10^6 \text{ Bq}.$$

Example 22.43: Selected solution for A7: specific sample activity

The number of nuclei is

$$N = \frac{m}{M} N_A = \frac{2.0 \times 10^{-6} \text{ g}}{60 \text{ g/mol}} (6.022 \times 10^{23}) \approx 2.01 \times 10^{16}.$$

For $T_{1/2} = 10 \text{ d} = 8.64 \times 10^5 \text{ s}$,

$$\lambda = 8.02 \times 10^{-7} \text{ s}^{-1}.$$

Thus

$$\mathcal{A}_0 = \lambda N \approx 1.61 \times 10^{10} \text{ Bq}.$$

The large activity despite the tiny mass illustrates how strongly specific activity depends on half-life and molar mass.

Example 22.44: Selected solution for A11: recoil in alpha decay

The alpha and daughter momenta have equal magnitudes p . Therefore

$$\frac{K_\alpha}{K_d} = \frac{M_d}{m_\alpha}.$$

With $M_d \approx 210u$ and $m_\alpha \approx 4u$,

$$K_\alpha = Q \frac{M_d}{M_d + m_\alpha} = 5.50 \frac{210}{214} \approx 5.40 \text{ MeV},$$

and

$$K_d = Q - K_\alpha \approx 0.10 \text{ MeV}.$$

The lighter particle receives most of the kinetic energy.

Example 22.45: Selected solution for N4: background subtraction

The sample-plus-background rate is $960/60 = 16.0 \text{ s}^{-1}$ and the background rate is $180/90 = 2.0 \text{ s}^{-1}$, giving

$$r_s = 14.0 \text{ s}^{-1}.$$

The variance is

$$\sigma_{r_s}^2 = \frac{960}{60^2} + \frac{180}{90^2} = 0.2667 + 0.0222 = 0.2889 \text{ s}^{-2},$$

so

$$\sigma_{r_s} = 0.537 \text{ s}^{-1}.$$

A concise statistical result is $r_s = (14.0 \pm 0.5) \text{ s}^{-1}$.

Example 22.46: Selected solution for N5: dead-time correction

For a nonparalyzable model,

$$R = \frac{r}{1 - r\tau_d}.$$

With $r = 2.0 \times 10^4 \text{ s}^{-1}$ and $\tau_d = 15 \mu\text{s}$,

$$r\tau_d = 0.300, \quad R = \frac{2.0 \times 10^4}{0.700} \approx 2.86 \times 10^4 \text{ s}^{-1}.$$

A 43% increase relative to the observed rate is already large enough that the validity of the simple model and the stability of the electronics should be checked carefully.

Example 22.47: Selected solution for E2: PET timing

Using $\Delta x \sim c\Delta t/2$,

$$\Delta x_{600} \approx \frac{(3.00 \times 10^8)(600 \times 10^{-12})}{2} = 9.0 \text{ cm},$$

while

$$\Delta x_{300} \approx 4.5 \text{ cm.}$$

Halving the timing uncertainty halves this particular localization scale. It does not imply a factor-two improvement in every component of reconstructed image resolution.

Example 22.48: Selected solution for E3: accidental coincidences

Using Eq. (22.138),

$$r_{\text{acc}} \approx 2(5 \times 10^{-9})(2.0 \times 10^4)(1.5 \times 10^4) = 3.0 \text{ s}^{-1}.$$

Narrowing the coincidence window would reduce accidental coincidences approximately linearly, provided true coincidences are not lost excessively.

Example 22.49: Selected solution for H1: mean lifetime

The lifetime density is

$$f(t) = -\frac{dS}{dt} = \lambda e^{-\lambda t}, \quad t \geq 0.$$

It is normalized because

$$\int_0^\infty \lambda e^{-\lambda t} dt = 1.$$

The mean is

$$\langle t \rangle = \int_0^\infty t \lambda e^{-\lambda t} dt.$$

Integrating by parts gives

$$\langle t \rangle = \frac{1}{\lambda} = \tau.$$

Therefore $T_{1/2} = \tau \ln 2 < \tau$.

Example 22.50: Selected solution for H2: memorylessness

For T exponentially distributed,

$$P(T > s + t \mid T > s) = \frac{P(T > s + t)}{P(T > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(T > t).$$

Survival for time s does not change the conditional survival law for the additional time t . This mathematical property is the precise meaning of “the nucleus does not age” within the ideal exponential-decay model.

Example 22.51: Selected solution for H8: half-life uncertainty

For

$$T = \frac{\ln 2}{\lambda},$$

first-order uncertainty propagation gives

$$\sigma_T \approx \left| \frac{dT}{d\lambda} \right| \sigma_\lambda = \frac{\ln 2}{\lambda^2} \sigma_\lambda.$$

Dividing by $T = \ln 2/\lambda$ gives the useful relation

$$\frac{\sigma_T}{T} \approx \frac{\sigma_\lambda}{\lambda}.$$

Thus the relative uncertainty is unchanged to first order by the reciprocal transformation.

22.30 Laboratory-style computational project: measuring a half-life from synthetic data

This project is designed for a computer laboratory or a two-hour guided tutorial followed by independent work. It requires no radioactive source. The data are generated numerically, but the analysis mirrors the logic of a real counting experiment closely enough to expose the main statistical and modelling issues.

22.30.1 Physical objective

Assume an unknown radionuclide produces a detectable line with time-dependent signal rate

$$R_s(t) = R_0 e^{-\lambda t}, \quad (22.148)$$

while the apparatus also records a constant background rate R_b . The task is to infer the half-life

$$T_{1/2} = \frac{\ln 2}{\lambda} \quad (22.149)$$

from binned integer counts and to demonstrate that the quoted uncertainty behaves sensibly under repeated simulated experiments.

The project deliberately separates four layers:

1. the physical decay model;
2. the detector/counting model;
3. the random data generator;
4. the inference procedure.

Keeping these layers separate is a central learning outcome.

22.30.2 Step 1: choose truth parameters

A convenient default configuration is

$$T_{1/2}^{\text{true}} = 40 \text{ s}, \quad R_0 = 120 \text{ s}^{-1}, \quad R_b = 5 \text{ s}^{-1}, \quad \Delta t = 5 \text{ s}. \quad (22.150)$$

Generate data for at least five half-lives. The exact expected count in bin i , beginning at t_i , is

$$\mu_i = R_0 \frac{e^{-\lambda t_i} - e^{-\lambda(t_i + \Delta t)}}{\lambda} + R_b \Delta t. \quad (22.151)$$

Do not replace this expression by $[R_0 e^{-\lambda t_i} + R_b] \Delta t$ until you have quantified the approximation.

22.30.3 Step 2: generate one synthetic experiment

For every bin draw

$$n_i \sim \text{Poisson}(\mu_i). \quad (22.152)$$

Store a table containing t_i , $t_i + \Delta t$, the bin centre, μ_i and the observed integer n_i . Plot the observed bin-average rate $n_i/\Delta t$ against time together with the bin-average expectation $\mu_i/\Delta t$ used by the generator, rather than an endpoint rate.

A good implementation should accept an explicit random-number seed. Repeating the program with the same seed should reproduce the same toy data exactly. Changing the seed should alter the statistical fluctuations but not the underlying expectation.

22.30.4 Step 3: perform two fits

Fit A: diagnostic semilog method. For bins with positive background-subtracted counts, form a semilog plot and obtain an approximate slope. Document which bins had to be omitted and why. Treat this as a visualization method, not as the final statistical result.

Fit B: Poisson likelihood. Fit the original counts with the model in Eq. (22.151). A suitable negative log-likelihood, omitting constants independent of the parameters, is

$$-\ln L(R_0, \lambda, R_b) = \sum_i [\mu_i - n_i \ln \mu_i]. \quad (22.153)$$

Constrain $R_0 > 0$, $\lambda > 0$ and $R_b \geq 0$. Convert the fitted λ to half-life only after the fit.

22.30.5 Step 4: inspect fit quality

A best-fit number is not enough. Produce at least two diagnostics:

1. a plot of observed counts with the fitted expectation;
2. a residual or pull-like plot showing where data deviate from the fit.

Comment on whether deviations look randomly distributed or display structure. If a late-time plateau is visible, explain how the fitted background term accounts for it.

22.30.6 Step 5: test statistical performance with repeated experiments

Repeat the full simulation and fit at least several hundred times. For each pseudo-experiment save the fitted half-life $\hat{T}_{1/2}$ and its estimated standard uncertainty σ_T . Then examine:

- the mean fitted half-life compared with $T_{1/2}^{\text{true}}$;
- the standard deviation of fitted half-lives;
- the mean reported uncertainty;
- the pull variable

$$p = \frac{\hat{T}_{1/2} - T_{1/2}^{\text{true}}}{\sigma_T}. \quad (22.154)$$

A well-behaved analysis should have a pull mean near zero and width near one, within the statistical precision of the test. Large deviations indicate bias, misestimated uncertainties or failed fits.

22.30.7 An executed reference experiment

To make the project reproducible, one simulated experiment uses NumPy's PCG64 generator with seed 220930 and the truth parameters above. The first eight observed bins contain 608, 561, 451, 480, 427, 390, 357 and 359 counts; the last four contain 47, 59, 40 and 39. All 40 observations are embedded in Fig. 22.14 and in the figure coordinates in the source.

An integrated Poisson fit, with all three parameters free and no truth information supplied to the optimizer, gives

$$\hat{T}_{1/2} = 40.25 \text{ s}, \quad \hat{R}_0 = 119.38 \text{ s}^{-1}, \quad \hat{R}_b = 4.73 \text{ s}^{-1}. \quad (22.155)$$

The local observed-likelihood curvature gives an approximate one-standard-deviation half-life uncertainty of 1.59 s. Thus this particular realization is compatible with the input 40 s. This agreement is one check, not a proof of unbiased inference. The Poisson deviance is approximately 48.0 for 40 bins and three fitted parameters; it is a diagnostic to interpret with the model and simulated repetitions, rather than a universal goodness-of-fit guarantee.

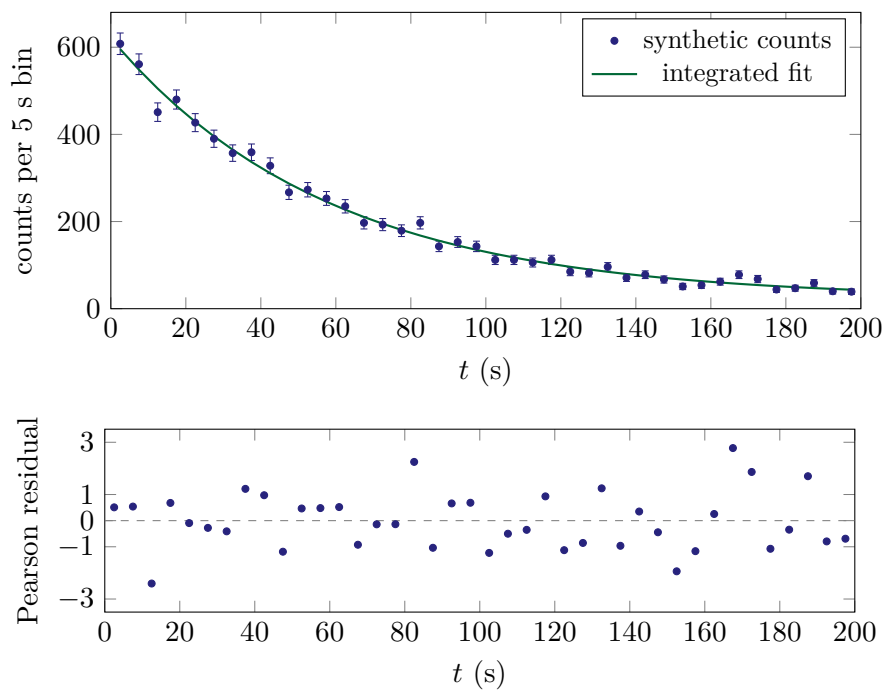


Figure 22.14: Executed synthetic experiment with input half-life 40 s, initial detected signal rate 120 s^{-1} and background 5 s^{-1} . Forty 5 s bins are generated with a PCG64 seed of 220930. Upper panel: integer counts and the fitted bin-integrated mean; error bars are the high-count $\sqrt{n}$ visualization. Lower panel: $(n_i - \mu_i)/\sqrt{\mu_i}$ using the fitted means. These are generated teaching data, not laboratory observations.

Repeating the experiment 300 times with independent PCG64 draws (seed 220931) gives a direct check of the pipeline. The ensemble results are reported below to the precision justified by 300 trials; they are not additional physical constants.

Diagnostic	Value in this simulation
Mean fitted half-life	40.16 s
Standard deviation of fitted half-lives	1.60 s
Mean curvature uncertainty	1.60 s
Pull mean and standard deviation	0.04, 0.99
Fraction within the quoted $\pm 1\sigma$ interval	67.7%

The nominal Gaussian one-standard-deviation coverage is about 68%; a 300-trial coverage fraction has Monte Carlo standard uncertainty of about 2.7 percentage points. Compare the observed

deviations with that finite-test precision before diagnosing the procedure. Curvature intervals can be inadequate near a boundary or with sparse counts; a profile-likelihood or calibrated interval is then preferable. The repeated experiment tests this stated configuration and does not validate every possible source or detector model.

22.30.8 Step 6: controlled model mismatch

Repeat the analysis after deliberately violating one assumption at a time. Useful tests include:

1. generate a slowly drifting background but fit a constant background;
2. add a second weak exponential component with a different half-life;
3. generate with finite dead time but fit a model that ignores it;
4. use unequal bin widths while fitting with a constant-rate approximation, then compare with equal-width bins and a free amplitude;
5. truncate the data after only one half-life.

For the equal-width single-exponential case, Eq. (22.115) predicts an amplitude shift without an ideal slope shift; do not claim a half-life bias solely from wide equal bins. Finite statistical information can still decrease through coarse binning. For dead-time studies, simulate arrival times and the electronics explicitly: independent Poisson draws from a corrected mean do not reproduce the registered-count distribution. For each test, identify whether the main effect is bias, larger uncertainty, poor goodness of fit, or a combination of these.

22.30.9 Step 7: engineering interpretation

Treat total acquisition time as a design variable. Compare the precision obtained when measuring for two, five and ten half-lives while keeping all other truth parameters fixed. Very late data contain little signal but remain useful for constraining background. There is therefore no universal statement that “longer is always proportionally better.” The information gained per unit time depends on both signal decay and background.

Likewise compare two bin widths. Narrow bins preserve time information but contain fewer counts per bin. Wide bins increase counts per bin but smear the time dependence. With the correct integrated Poisson likelihood, bin width can often be chosen for computational and visualization convenience within sensible limits; with approximate fitting methods it can matter much more.

22.30.10 Required project report

A concise report should contain:

1. the decay model and parameter definitions;
2. the exact expected-count formula;
3. one example synthetic data set;
4. the diagnostic semilog plot;
5. the Poisson-likelihood fit and fitted half-life;
6. uncertainty and residual diagnostics;
7. results of repeated pseudo-experiments;

8. one model-mismatch study;
9. a short discussion of limitations and possible improvements.

The report should make clear which quantities are truth inputs, which are observations and which are fitted estimates.

Method

A computational project is successful only when the analysis can fail visibly. Code that always returns the generated half-life is not evidence of correctness; it may simply be reusing truth information accidentally. Blind the fitter to the truth parameters, test it on many random samples, and inspect its residuals and pull distributions.

Guided checks

1. What does wide-bin integration change when all widths are equal and the signal amplitude is free? What changes when bin widths differ?
2. Why can zero-count bins be included naturally in a Poisson likelihood?
3. What would a pull distribution with mean $+0.8$ suggest?
4. What would a pull width of 1.5 suggest about the quoted uncertainties?
5. Why can late-time bins improve the background estimate even when the radioactive signal is small?

Take-home message

The project turns the decay law into a complete scientific workflow: formulate a physical model, generate stochastic data, estimate parameters, quantify uncertainty, diagnose model failure and test the analysis by simulation. This is the same logic used in much larger experiments, only with a model simple enough that every step can be understood from first principles.

22.31 Bridge to Chapter 23

This chapter has concentrated on nuclear structure and spontaneous radioactive change. The natural next question is what happens when nuclei are deliberately transformed in reactions and when binding-energy differences are exploited on a macroscopic scale. Chapter 23 will develop nuclear-reaction energetics more systematically and use the binding-energy curve to understand fission, fusion, chain reactions, reactor concepts and stellar energy production.

The conceptual bridge is the Q value. Radioactive decay already showed that a reduction in total rest mass appears as kinetic or electromagnetic energy. Nuclear-energy processes use exactly the same accounting, but reaction rates, neutron transport, criticality, plasma conditions and engineering constraints become central. Students should carry forward two habits from the present chapter: conserve the relevant quantum numbers before doing arithmetic, and distinguish an energetically possible process from a dynamically probable one.

Chapter A

Notation dictionary

Symbol	Meaning	Typical unit
A	mass number, number of nucleons	1
Z	atomic number, number of protons	1
$N = A - Z$	neutron number	1
${}^A_ZX$	nuclide notation	–
R	characteristic nuclear radius	fm
R_0	empirical radius constant, about 1.2	fm
m_p, m_n, m_e	proton, neutron and electron masses	kg or u
m_H	mass of neutral hydrogen atom	kg or u
$M(A, Z)$	specified nuclear or atomic mass	kg or u
B	nuclear binding energy	J or MeV
B/A	binding energy per nucleon	MeV/nucleon
S_n, S_p	neutron and proton separation energies	MeV
Q	reaction or decay energy release	J or MeV
λ	decay constant	s^{-1}
$T_{1/2}$	half-life	s
τ	mean lifetime, $1/\lambda$	s
$\mathcal{A}$	radioactive activity	Bq
b_i	branching fraction for channel i	1
ϵ	detection efficiency	1
R_s, R_b	signal and background count rates	s^{-1}
τ_d	detector dead time	s
μ	linear attenuation coefficient	m^{-1}
D	absorbed dose	Gy
H	equivalent dose	Sv
w_R	radiation weighting factor in equivalent dose	1
σ_E	Gaussian energy-resolution parameter	eV or keV
R	detector-response matrix	1
N_A	Avogadro constant	mol^{-1}

Chapter B

Compact formula sheet

Use these results with the assumptions stated in the relevant sections. The activation expressions assume constant production and $N(0) = 0$. The two-member chain expressions assume unit feeding, $N_2(0) = 0$ and unequal decay constants; the equal-constant limit is derived in the chapter. The shorthand $H = w_R D$ applies to one radiation type in a specified tissue. Beta-decay formulas use neutral atomic masses, with electronic-state and recoil qualifications as discussed in the text.

Formula summary

Nuclear geometry

$$R = R_0 A^{1/3}, \quad \rho_{\text{nuc}} \approx \frac{3m_N}{4\pi R_0^3}.$$

Binding energy

$$\begin{aligned} B &= [Zm_p + Nm_n - M_{\text{nuc}}]c^2, \\ B &\approx [Zm_{\text{H}} + Nm_n - M_{\text{atom}}]c^2, \\ S_n &= B(A, Z) - B(A-1, Z), \\ S_p &= B(A, Z) - B(A-1, Z-1). \end{aligned}$$

Radioactive decay

$$\begin{aligned} \frac{dN}{dt} &= -\lambda N, & N(t) &= N_0 e^{-\lambda t} = N_0 2^{-t/T_{1/2}}, \\ \mathcal{A} &= \lambda N, & T_{1/2} &= \frac{\ln 2}{\lambda}, & \tau &= \frac{1}{\lambda}. \end{aligned}$$

Decay and reaction energetics

$$\begin{aligned} Q &= (M_i - M_f)c^2, \\ Q_{\beta^-} &= [M_{\text{atom}}(p) - M_{\text{atom}}(d)]c^2, \\ Q_{\beta^+} &= [M_{\text{atom}}(p) - M_{\text{atom}}(d) - 2m_e]c^2. \end{aligned}$$

Attenuation and dose

$$\begin{aligned} I(x) &= I_0 e^{-\mu x}, & \text{HVL} &= \frac{\ln 2}{\mu}, \\ D &= \frac{E_{\text{dep}}}{m}, & H &= w_R D. \end{aligned}$$

Counting statistics and detector corrections

$$\begin{aligned}\sigma_n &= \sqrt{n} \quad (\text{Poisson estimate}), \\ r_s &= \frac{n_{sb}}{t_{sb}} - \frac{n_b}{t_b}, \\ \sigma_{r_s}^2 &\approx \frac{n_{sb}}{t_{sb}^2} + \frac{n_b}{t_b^2}, \\ r &= \frac{R}{1 + R\tau_d} \quad (\text{nonparalyzable}), \\ R &= \frac{r}{1 - r\tau_d}.\end{aligned}$$

Activation

$$\begin{aligned}\frac{dN}{dt} &= P - \lambda N, \\ N(t) &= \frac{P}{\lambda}(1 - e^{-\lambda t}), \\ \mathcal{A}(t) &= P(1 - e^{-\lambda t}).\end{aligned}$$

Two-member decay chain

$$\begin{aligned}N_1(t) &= N_{10}e^{-\lambda_1 t}, \\ N_2(t) &= \frac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}), \\ t_{\max} &= \frac{\ln(\lambda_2/\lambda_1)}{\lambda_2 - \lambda_1}.\end{aligned}$$

Semi-empirical mass formula, schematic form

$$B = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A, Z).$$

Scattering and finite-size structure

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_R &= \left(\frac{zZe^2}{16\pi\epsilon_0 E}\right)^2 \frac{1}{\sin^4(\theta/2)}, \\ F(\mathbf{q}) &= \frac{1}{Ze} \int \rho_{\text{ch}}(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}/\hbar} d^3r, \\ F(q) &\approx 1 - \frac{q^2 \langle r^2 \rangle}{6\hbar^2} + \dots.\end{aligned}$$

Spectroscopy and imaging scales

$$\begin{aligned}\Gamma &\approx \frac{\hbar}{\tau}, & E_R &\approx \frac{E_\gamma^2}{2Mc^2}, \\ \text{FWHM} &= 2\sqrt{2\ln 2} \sigma_E, & \Delta x &\sim \frac{c\Delta t}{2}.\end{aligned}$$

Chapter C

Reference data and useful constants

The rounded physical constants were checked against NIST CODATA values [4]. The following compact table is sufficient for the self-contained numerical examples in this chapter. More precise work should use an up-to-date evaluated mass table or constants database appropriate to the experiment.

Quantity	Value used in this chapter
speed of light c	$2.99792458 \times 10^8 \text{ m s}^{-1}$ (exact)
elementary charge e	$1.602176634 \times 10^{-19} \text{ C}$ (exact)
atomic mass unit u	$1.66054 \times 10^{-27} \text{ kg}$
uc^2	931.5 MeV
electron rest energy $m_e c^2$	0.511 MeV
hydrogen atomic mass m_{H}	approximately 1.007825 u
neutron mass m_n	approximately 1.008665 u
Avogadro constant N_A	$6.02214076 \times 10^{23} \text{ mol}^{-1}$ (exact)
representative nuclear radius constant R_0	approximately 1.2 fm
carbon-14 half-life used for illustrative dating	approximately 5730 y

Note

The constants above are rounded to a precision appropriate for introductory calculations. Binding energies inferred from modern precision mass data can require many more significant figures than shown here. Never combine a high-precision mass table with a low-precision conversion constant if the desired uncertainty is smaller than the rounding error.

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