

Chapter 23

Nuclear Energy: Fission, Fusion, and Reactors

First Edition

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Physics 2

Physics for Technical and Engineering Students -
Volume II

Electricity, Magnetism, Waves, Optics, Relativity,
Quantum Physics, and Nuclear Physics

1 September 2026

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Preface

Nuclear-energy physics is where several ideas developed earlier in this volume meet in a particularly direct way. Mass–energy equivalence, nuclear binding, reaction Q -values, radioactive decay, thermal energy, power, particle collisions and quantum tunnelling all become parts of one connected story. The central question is simple to state: under what circumstances can a rearrangement of nuclei release energy, and how can that microscopic energy release be converted into a controlled macroscopic power flow?

This chapter develops the physics of nuclear fission, neutron chain reactions, fission reactors, thermonuclear fusion and stellar energy production. The emphasis is deliberately sober and quantitative. A reactor is treated first as a physical system governed by neutron balance and heat transfer, not as a political symbol. Fusion is treated first as a nuclear-reaction and plasma-confinement problem, not as a promise about future technology. Safety, fuel-cycle and societal questions are discussed only after the underlying physics is separated clearly from engineering implementation and policy choices.

The chapter is written for Technical Physics, engineering, computer-science, space-science and applied-science students. For that audience, microscopic quantities such as MeV per reaction are repeatedly translated into reaction rates, fuel-consumption rates, thermal power, electric power and measurable engineering quantities. Original TikZ diagrams are used for the binding-energy landscape, a fission chain, a generic thermal reactor, fusion barriers, stellar fusion and confinement concepts. A computational section develops a simple branching-process model for a neutron chain and shows where that model stops being realistic.

The treatment assumes that students have already met nuclear masses, binding energy, radioactive decay and elementary quantum tunnelling. These ideas are briefly recalled where needed, but the purpose here is application and synthesis rather than repetition. The advanced enrichment section goes further into neutron economy, moderation, cross sections, point kinetics, decay heat, the Lawson criterion, Gamow-peak reasoning, confinement physics, diagnostics and simple computational modelling.

All examples, figures, exercises and computational tasks in this chapter are original teaching material prepared for this lecture-note series. The supplied textbooks were compared for scientific coverage, derivations, visual explanations and problem-solving approaches; the exposition and diagrams here are independently developed.

The Guideline

For every nuclear-energy calculation, keep four levels separate: *nuclear reaction*, *reaction rate*, *thermal power* and *useful output power*. A Q -value is an energy per reaction. Multiplying by reactions per second gives power. Converting thermal power into electricity introduces engineering efficiency and heat rejection, which are separate from nuclear binding-energy physics.

Conventions and notation

Nuclei are written as ${}^A_ZX$, where Z is the proton number, A the nucleon number and $N = A - Z$ the neutron number. A neutron is ${}_0^1n$, a proton may be written ${}_1^1p$, and the deuteron and triton are denoted d and t when no ambiguity arises. Atomic masses may be used in reaction Q -value calculations when the electron bookkeeping is consistent on both sides; otherwise nuclear masses must be used consistently.

The reaction energy is

$$Q = (m_{\text{initial}} - m_{\text{final}}) c^2. \quad (0.1)$$

A positive Q means that the total rest mass decreases and the corresponding mass-energy appears as kinetic energy, electromagnetic radiation, neutrino energy and/or later decay energy, depending on the reaction. It is more precise to say that rest mass is converted into other forms of energy than to say that “mass is destroyed.”

Nuclear energies are commonly expressed in electronvolts. We use

$$1 \text{ eV} = 1.602\,176\,634 \times 10^{-19} \text{ J}, \quad 1 \text{ MeV} = 1.602\,176\,634 \times 10^{-13} \text{ J}. \quad (0.2)$$

The atomic mass unit satisfies, to the precision needed here,

$$1 \text{ u } c^2 \approx 931.5 \text{ MeV}. \quad (0.3)$$

For fission chains, k or k_{eff} denotes the effective multiplication factor: the ratio of neutron population in one generation to that in the preceding generation in a simplified generation picture. The states $k_{\text{eff}} < 1$, $= 1$ and > 1 are called subcritical, critical and supercritical. In real reactor kinetics, delayed neutrons and time-dependent feedback are essential; therefore the generation model is introduced as a conceptual model, not as a complete reactor-dynamics equation.

For fusion plasmas, n denotes a particle number density, T a temperature, τ_E an energy-confinement time, $\sigma(E)$ a reaction cross section and $\langle \sigma v \rangle$ the Maxwellian-averaged reactivity. Magnetic field is written $\mathbf{B}$. We retain SI units for macroscopic quantities while using eV or keV for particle energies and temperatures when this is standard and pedagogically clearer. When $k_B T$ is quoted in keV, the factor k_B is already included in the energy-like temperature scale.

Symbols are defined as they are introduced; the detailed notation dictionary is collected in Appendix A.

Caution

The symbol σ is used in nuclear and particle physics for a reaction cross section, whereas earlier electrostatics chapters used σ for surface charge density. Context makes the meaning clear, but the units are completely different: a nuclear cross section has units of area.

Note

The barn is a non-SI unit used for nuclear cross sections:

$$1 \text{ barn} = 10^{-28} \text{ m}^2. \quad (0.4)$$

It is convenient because nuclear interaction cross sections are often of this order or larger, but all dimensional checks in this chapter can be performed in SI units.

Chapter 23

Nuclear Energy: Fission, Fusion, and Reactors

23.1 Motivation and roadmap

The binding-energy curve is one of the most informative graphs in nuclear physics. Light nuclei can move toward greater binding per nucleon by combining; very heavy nuclei can move toward greater binding per nucleon by splitting into medium-mass fragments. These two directions correspond, respectively, to fusion and fission. In both cases the final products can have a lower total rest mass than the initial system, so energy is released.

The important engineering question is not merely whether a reaction has $Q > 0$. A useful energy system must also achieve an adequate reaction rate, transfer the released energy to a controllable heat or particle stream, sustain or replenish its fuel, remove waste heat, protect materials and maintain stable operation. Fission and fusion therefore connect microscopic nuclear physics to macroscopic transport, control and thermodynamics.

Figure 23.1 distinguishes the steps needed to turn nuclear binding into usable power.

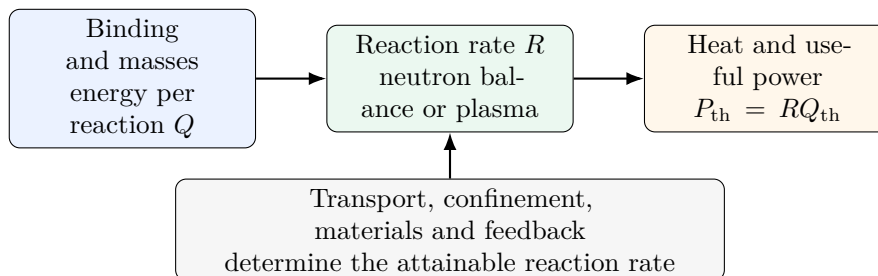


Figure 23.1: The progression from an energy per event to usable power. Feedback changes the reacting system and its rate; it does not alter the nuclear binding-energy landscape.

Learning goals

After studying this chapter, the student should be able to:

- explain fission and fusion using the binding-energy-per-nucleon curve;
- compute reaction Q -values with consistent mass conventions;
- convert MeV per reaction into a reaction rate and macroscopic thermal power;
- describe why neutrons are effective projectiles for fission and why moderation can matter;
- explain a fission chain reaction and interpret subcritical, critical and supercritical states;
- distinguish reactor criticality from an explosion and explain the role of delayed neutrons qualitatively;
- identify the functions of fuel, moderator, coolant, control absorbers, heat exchangers and containment in a generic fission power system;

- explain the physical origin of decay heat and why shutdown does not make reactor heat production vanish instantaneously;
- explain how natural conditions once supported self-sustained fission in the Oklo deposits;
- calculate representative fusion Q -values and estimate a Coulomb-barrier scale;
- explain why thermonuclear fusion relies on high temperature and quantum tunnelling;
- describe the proton–proton chain at a physically appropriate level and connect stellar fusion to the binding-energy curve;
- distinguish magnetic and inertial confinement and state the physical meaning of a Lawson-type confinement requirement;
- construct and interpret a simple branching-process simulation of neutron multiplication;
- discuss fission, fusion, safety and fuel-cycle questions with clear separation between physical facts, engineering choices and policy judgments.

Key message

A positive nuclear Q -value is necessary for an energy-producing reaction, but it is not sufficient for a practical power source. A power system additionally requires an adequate and controllable reaction rate, energy extraction, materials that survive the environment, and a stable way to maintain the reacting state.

Guided checks

1. Why can both splitting a very heavy nucleus and combining light nuclei release energy without contradiction?
2. What extra quantity must be known, in addition to energy released per reaction, to determine power?
3. Why is “fusion has a positive Q -value” not an argument that fusion must be technologically easy?

23.2 From nuclear binding to usable energy

The nucleus is a quantum many-body system held together primarily by the strong nuclear interaction, with the electrostatic repulsion of protons acting in the opposite direction. The mass of a bound nucleus is smaller than the sum of the rest masses of its separated nucleons. The corresponding mass difference is the binding energy,

$$B = [Zm_p + (A - Z)m_n - m_{\text{nucleus}}] c^2, \quad (23.1)$$

when nuclear masses are used. With atomic masses, electron masses must be handled consistently. The binding energy per nucleon, B/A , is often more useful than B itself because it compares nuclei of different size.

For light nuclei, adding nucleons generally increases B/A strongly. For medium-mass nuclei near the iron–nickel region, B/A is close to its maximum. For very heavy nuclei, electrostatic repulsion between many protons lowers the binding per nucleon. The qualitative consequence is the central energy argument of this chapter:

- light nuclei can release energy by *fusion*, moving upward on the B/A curve toward more tightly bound nuclei;
- very heavy nuclei can release energy by *fission*, splitting into fragments with larger B/A .

The curve should not be interpreted as saying that every conceivable fusion or fission reaction happens spontaneously. It only indicates an energetic direction. Reaction barriers, conservation laws, quantum selection rules, available states and interaction cross sections determine whether a reaction proceeds at an appreciable rate.

Figure 23.2 displays the broad energetic trend.

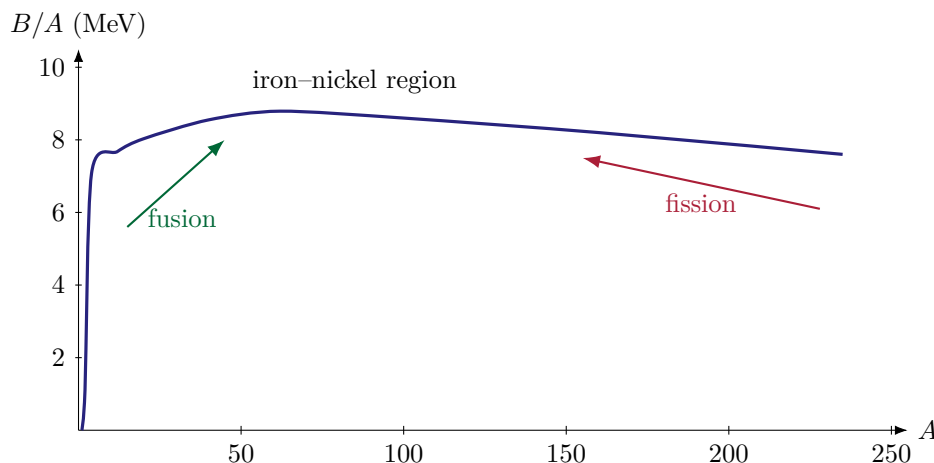


Figure 23.2: Schematic broad trend of binding energy per nucleon. The smooth line is an orientation diagram, not an interpolation of evaluated masses; individual light nuclei show strong structure. A particular reaction releases energy only when its total final binding exceeds its total initial binding.

23.2.1 Energy accounting and the reaction Q -value

For a reaction

$$a + b \longrightarrow c + d + \cdots, \quad (23.2)$$

the rest-energy balance is summarized by

$$Q = (m_a + m_b - m_c - m_d - \cdots) c^2. \quad (23.3)$$

If $Q > 0$, the reaction is exoergic: the final total rest mass is smaller. If the initial kinetic energy is small and no excited products remain, most of Q appears as final kinetic energy. In realistic nuclear systems some energy can appear in gamma rays, neutrinos, excitation of daughter nuclei or later radioactive decays.

A useful conservation statement is

$$E_{\text{rest},i} + K_i = E_{\text{rest},f} + K_f + E_\gamma + E_\nu + \cdots, \quad (23.4)$$

where each term is included only when relevant. A Q -value is therefore not a mysterious extra energy source; it is the change in rest-energy bookkeeping permitted by the reaction.

Caution

When atomic masses are used, do not add or remove electron masses casually. Atomic-mass tables refer to neutral atoms. For a reaction in which the total number of atomic electrons can be paired consistently on both sides, atomic masses can often be used directly. Otherwise write the electron bookkeeping explicitly or use nuclear masses throughout.

Example 23.1: A fission estimate that counts emitted neutrons

A heavy nucleus with $A = 235$ and average binding energy 7.60 MeV per nucleon captures a slow neutron. Suppose the final fragments contain 233 bound nucleons in total with average binding energy 8.45 MeV per nucleon; three neutrons emerge unbound. Estimate the immediate reaction energy.

Strategy and assumptions. Check $235 + 1 = 233 + 3$. For a reaction conserving proton and neutron numbers, the free-nucleon rest-mass terms cancel, so $Q = B_{\text{final}} - B_{\text{initial}}$. Free neutrons have no nuclear binding energy; the incident neutron has negligible kinetic energy.

$$Q \approx 233(8.45 \text{ MeV}) - 235(7.60 \text{ MeV}) \quad (23.5)$$

$$= 182.85 \text{ MeV} \approx 1.8 \times 10^2 \text{ MeV}, \quad (23.6)$$

$$Q \approx 2.9 \times 10^{-11} \text{ J}. \quad (23.7)$$

Interpretation. This rough immediate release is consistent with the scale of fission energy. Later fragment decays and neutron captures contribute additional energy to an operating reactor. Their contributions, and energy carried away by antineutrinos, must be included when specifying recoverable heat.

Check. Multiplying all 236 nucleons by a fragment binding energy would incorrectly treat the three free neutrons as bound. A smooth binding-energy estimate is useful for scale, but a particular channel requires actual masses.

23.2.2 From microscopic energy to macroscopic power

If reactions occur at a rate R reactions per second and each reaction releases an average usable thermal energy Q_{th} , the thermal power is

$$P_{\text{th}} = RQ_{\text{th}}. \quad (23.8)$$

If Q is quoted in MeV, convert it to joules before using watts. Conversely,

$$R = \frac{P_{\text{th}}}{Q_{\text{th}}}. \quad (23.9)$$

This equation is one of the most important bridges in applied nuclear physics.

Method

For any “power from nuclear reactions” problem:

1. identify whether the stated power is thermal or electric;
2. identify the average energy deposited as heat per reaction;
3. convert MeV to joules;
4. use $R = P/Q$;
5. if a fuel-consumption rate is requested, multiply the reaction rate by the mass per fuel nucleus;
6. state clearly which idealizations are being made, for example that every consumed fissile nucleus fissions and that a specified Q is fully thermalized.

Example 23.2: Fission rate for a 1.0 GW thermal reactor

Assume each fission deposits an average of 200 MeV as recoverable heat. Estimate the fission rate needed to produce 1.0 GW of thermal power.

Solution. Convert the energy per fission:

$$Q = (200 \text{ MeV})(1.602 \times 10^{-13} \text{ J/MeV}) = 3.204 \times 10^{-11} \text{ J}. \quad (23.10)$$

Then

$$R = \frac{1.0 \times 10^9 \text{ J/s}}{3.204 \times 10^{-11} \text{ J}} \approx 3.12 \times 10^{19} \text{ s}^{-1}. \quad (23.11)$$

Thus roughly 3×10^{19} fissions per second are required.

If each event consumes one ^{235}U nucleus, the corresponding idealized mass-consumption rate is

$$\dot{m} = R(235u) \approx (3.12 \times 10^{19})(235)(1.6605 \times 10^{-27}) \text{ kg/s} \approx 1.22 \times 10^{-5} \text{ kg/s}. \quad (23.12)$$

This is about

$$\dot{m} \approx 1.05 \text{ kg/day}. \quad (23.13)$$

The actual fuel-management problem is more complicated because fuel contains several isotopes, not every neutron absorption causes fission, fissile isotopes are bred and depleted, and the reactor does not consist of pure ^{235}U .

Guided checks

1. A reaction releases twice as much energy per event at the same reaction rate. What happens to power?
2. A plant has 3 GW thermal output but 1 GW electric output. Which number belongs in $P = RQ$ when Q is heat deposited per fission?
3. Why can “one fission releases only 10^{-11} J” and “a reactor produces gigawatts” both be true?

23.3 Neutrons as nuclear projectiles

The neutron has no electric charge. That simple fact has major consequences for fission physics. A proton or alpha particle approaching a heavy positive nucleus encounters strong Coulomb repulsion. A neutron does not. It can therefore approach the nuclear surface without first overcoming an electrostatic barrier. This makes neutron-induced reactions especially important in heavy nuclei.

A neutron moving through matter can scatter elastically from nuclei, scatter inelastically, be captured, or induce fission when the target and neutron energy permit it. The probabilities are described by energy-dependent cross sections. Cross section is not literally the geometric size of the nucleus; it is an effective area describing interaction probability for a specified process.

23.3.1 Fast and thermal neutrons

Fission neutrons are typically born with kinetic energies in the MeV range. In many thermal-reactor concepts, these neutrons are slowed through collisions with light nuclei. After sufficient scattering, their energy distribution approaches thermal equilibrium with the moderator. The characteristic kinetic-energy scale is then of order $k_B T$, which at room temperature is a few hundredths of an electronvolt.

The reason light nuclei are effective moderators can be understood from collision kinematics. A neutron can transfer a large fraction of its energy in an elastic collision when the target mass

is comparable with the neutron mass. Hydrogen is therefore very effective per collision. Other engineering considerations, including neutron absorption, temperature, chemistry and structural requirements, determine which moderator is useful in a particular reactor design.

Example 23.3: Thermal-neutron energy scale

Estimate the order of magnitude of the mean translational kinetic energy of a neutron in thermal equilibrium at $T = 300$ K using the classical expression $\langle K \rangle = \frac{3}{2}k_B T$.

Solution.

$$\langle K \rangle = \frac{3}{2}(1.381 \times 10^{-23} \text{ J/K})(300 \text{ K}) = 6.21 \times 10^{-21} \text{ J.} \quad (23.14)$$

Converting to electronvolts,

$$\langle K \rangle \approx \frac{6.21 \times 10^{-21}}{1.602 \times 10^{-19}} \text{ eV} \approx 3.9 \times 10^{-2} \text{ eV.} \quad (23.15)$$

This is many orders of magnitude below the MeV energy with which fission neutrons are commonly born. Moderation is therefore a large energy-degradation process, not a small correction.

Key message

Moderation changes neutron energy, not neutron charge or identity. Its purpose in a thermal reactor is to reshape the neutron-energy spectrum so that the balance of fission, capture and leakage supports the desired chain reaction.

Caution

“Slow neutron” does not mean mechanically slow on a human scale. A thermal neutron at room temperature moves at kilometres per second. The adjective “thermal” refers to its energy distribution being comparable with thermal energies of the surrounding material.

23.3.2 Cross section, number density and mean free path: a preview

At a specified neutron energy, if target nuclei have number density n_t and channel r has microscopic cross section σ_r , its macroscopic cross section is

$$\Sigma_r = n_t \sigma_r, \quad (23.16)$$

with unit m^{-1} . For a monoenergetic narrow beam in a uniform material, removal from the *uncollided* beam depends on the total interaction cross section $\Sigma_t = \Sigma_s + \Sigma_a$. It gives

$$I_{\text{unc}}(x) = I_0 e^{-\Sigma_t x}, \quad (23.17)$$

and the corresponding mean free path is

$$\lambda_{\text{mfp}} = \frac{1}{\Sigma_t}. \quad (23.18)$$

This language is developed more carefully in the enrichment section. It already shows why reactor physics depends on both nuclear probabilities and material composition.

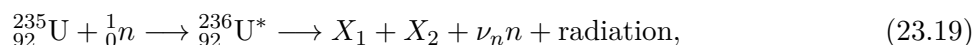
Guided checks

1. Why can a neutron enter a heavy nucleus more easily than a proton of the same low kinetic energy?
2. Why is hydrogen efficient at removing neutron energy in elastic collisions?
3. If a material has a large scattering cross section but also a very large absorption cross section, why might it be a poor moderator for some reactor purposes?

23.4 Nuclear fission: the basic process

When a fissile heavy nucleus absorbs a neutron, it can form an excited compound nucleus. The excitation can deform the nucleus. If the deformation develops beyond the restoring influence of the nuclear surface energy, the system can separate into two principal fragments, accompanied by several neutrons and gamma radiation. The exact fragment masses vary from event to event; fission is not one unique two-body reaction.

A schematic channel may be written as



where X_1 and X_2 represent neutron-rich fission fragments and ν_n denotes the number of prompt neutrons in that particular event. Charge and nucleon number are conserved in every allowed channel. The symbol Q denotes an energy balance, not another emitted particle; the integer ν_n here is an event multiplicity, whereas the mean yield $\bar{\nu}$ is generally noninteger.

The fragments are typically born with large kinetic energy because, immediately after scission, they are positively charged and repel one another strongly. As they slow in the fuel and surrounding material, their kinetic energy becomes heat. Prompt neutrons and gamma rays also transfer energy to matter. The neutron-rich fragments subsequently undergo beta decays, producing additional energy over longer timescales and giving rise to decay heat after shutdown.

The prompt and delayed energy channels are separated in Fig. 23.3.

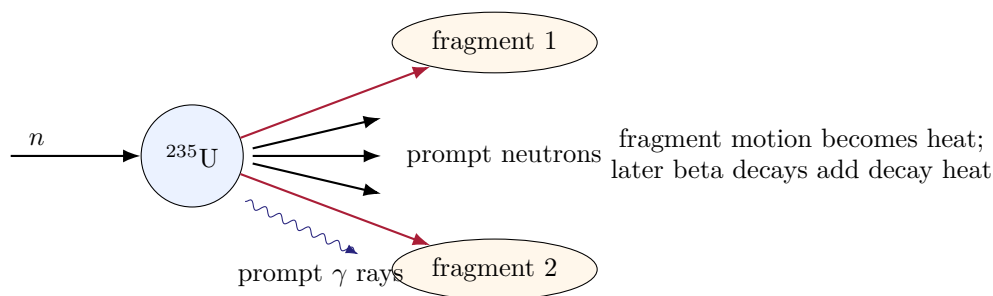


Figure 23.3: Neutron-induced fission and its energy channels. Fragment identities and neutron multiplicity vary. The diagram distinguishes prompt energy transfer from later radioactive heating.

23.4.1 Fissile, fissionable and fertile nuclei

A *fissionable* nucleus can undergo fission under suitable conditions. A *fissile* nuclide can support fission induced by low-energy neutrons; ${}^{235}\text{U}$ and ${}^{239}\text{Pu}$ are important examples. A *fertile* nuclide can be converted by neutron capture and subsequent decays into a fissile nuclide, as ${}^{238}\text{U}$ can become ${}^{239}\text{Pu}$. These terms describe different properties. In particular, ${}^{238}\text{U}$ is fertile and can fission with sufficiently energetic neutrons, but it is not fissile with thermal neutrons.

The incident neutron need not supply the fission-barrier energy as kinetic energy. When it is

captured, the compound nucleus acquires excitation

$$E^* \simeq S_n + E_n^{\text{cm}}, \quad S_n = [m(X) + m_n - m(X + n)]c^2, \quad (23.20)$$

where S_n is the neutron separation energy of the compound nucleus and E_n^{cm} is the initial relative kinetic energy. Even a thermal neutron can therefore produce several MeV of nuclear excitation. Whether fission competes effectively with gamma emission depends on this excitation, the barrier and nuclear structure.

Example 23.4: Why a slow neutron can excite a nucleus strongly

Use the rounded atomic masses $m(^{235}\text{U}) = 235.043922 \text{ u}$ and $m(^{236}\text{U}) = 236.045562 \text{ u}$, together with $m_n = 1.008665 \text{ u}$. Estimate the excitation following capture of a neutron with negligible kinetic energy.

Strategy. Both uranium atoms have 92 electrons, so electron masses cancel. Compare the unbound initial system with the compound-nucleus ground state:

$$S_n = [235.043922 + 1.008665 - 236.045562](931.5) \text{ MeV} \quad (23.21)$$

$$\approx 6.54 \text{ MeV}. \quad (23.22)$$

The energy is initially stored as compound-nucleus excitation, rather than necessarily being emitted as one photon. Fission and radiative capture are competing decay paths. The calculation explains the relevant energy scale; it does not determine their branching probabilities.

23.4.2 Why the fragments are usually neutron rich

Stable medium-mass nuclei have a smaller neutron-to-proton ratio than very heavy stable nuclei. When a heavy nucleus splits quickly, the fragments inherit roughly the parent neutron richness and therefore tend to lie on the neutron-rich side of the valley of stability. They commonly beta-minus decay toward more stable isotopes. This provides a direct link from fission physics to the radioactive-decay chapter.

The delayed beta and gamma energy is operationally important. Even when the fission chain reaction is shut down, radioactive fission products continue to decay and produce heat. Thus “reactor shutdown” means rapid suppression of the chain reaction, not instantaneous disappearance of all heat sources.

Caution

Do not describe reactor shutdown as “turning off nuclear reactions completely.” The fission chain can be terminated quickly, but radioactive decays in the accumulated fission products continue. Removing this decay heat is a central thermal-engineering requirement after shutdown.

23.4.3 A liquid-drop picture of fission

A useful qualitative model treats the nucleus as a charged liquid drop. The short-range nuclear attraction favours a compact surface, whereas Coulomb repulsion between protons favours separation of charge. Excitation following neutron capture can deform the nucleus. Small deformations may oscillate back toward a compact shape; sufficiently large deformations can cross a fission barrier and proceed to scission.

The model is not a complete microscopic theory. Shell effects and quantum structure matter strongly for actual fission probabilities and fragment distributions. Nevertheless, the competition between surface energy and Coulomb energy gives valuable intuition for why very heavy nuclei are susceptible to fission.

Guided checks

1. Why are fission fragments commonly beta-minus radioactive?
2. Why does the energy of fragment motion eventually appear mainly as heat in a reactor?
3. What does the liquid-drop picture explain well, and what important nuclear-structure physics does it omit?

23.5 Chain reactions and neutron multiplication

A fission event can release more than one neutron. Some of these neutrons may cause further fissions. This creates the possibility of a branching chain. The simplest generation model assumes that every generation is characterized by an effective multiplication factor k :

$$N_{g+1} = kN_g, \quad (23.23)$$

so that

$$N_g = N_0 k^g. \quad (23.24)$$

The model leads immediately to three regimes:

$$k < 1 : \quad \text{subcritical: the chain decreases,} \quad (23.25)$$

$$k = 1 : \quad \text{critical: each generation replaces itself on average,} \quad (23.26)$$

$$k > 1 : \quad \text{supercritical: the chain grows.} \quad (23.27)$$

The adjective *critical* often causes misunderstanding in everyday language because it sounds like “dangerous” or “near failure.” In reactor physics it has a precise technical meaning: the neutron population is steady on average under the stated conditions. A power reactor operating steadily is designed to be critical in this sense.

Figure 23.4 gives an explicit critical balance in a teaching model.

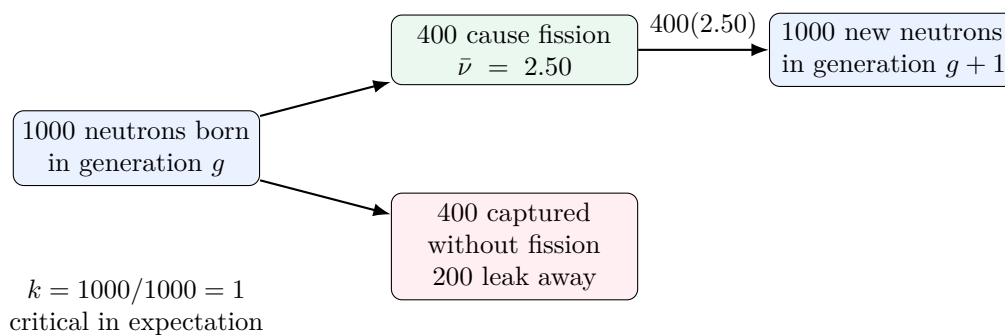


Figure 23.4: Illustrative critical neutron bookkeeping. Scattering can occur before an eventual fission, capture or leakage and is not a mutually exclusive final fate. The counts are assumed for teaching and are not data for a real reactor.

23.5.1 What determines the effective multiplication factor?

The value of k_{eff} is not a property of the fissile isotope alone. It depends on the complete system: fuel composition, geometry, neutron spectrum, moderator, absorbers, structural materials, temperature and neutron leakage. In a simplified balance, neutrons born from fission can have several fates:

- cause another fission;
- be captured without causing fission;

- scatter and change energy;
- leak out of the active region.

A reactor designer seeks a state in which these competing processes give $k_{\text{eff}} = 1$ at the desired operating power, together with stable feedback and adequate thermal margins.

Example 23.5: Generation growth in a toy chain model

A simplified branching model has $N_0 = 100$ neutrons in generation zero. Compare the expected population after 20 generations for $k = 0.95$, 1.00 and 1.05.

Solution. Use $N_g = N_0 k^g$. For $k = 0.95$,

$$N_{20} = 100(0.95)^{20} \approx 35.8. \quad (23.28)$$

For $k = 1$,

$$N_{20} = 100. \quad (23.29)$$

For $k = 1.05$,

$$N_{20} = 100(1.05)^{20} \approx 265. \quad (23.30)$$

A five-percent change in multiplication per generation has a large cumulative effect after many generations. This illustrates why reactivity control is important. It does *not* predict the time evolution of a real power reactor, because real kinetics depend strongly on delayed neutrons, feedback and generation times.

23.5.2 Prompt and delayed neutrons

Most fission neutrons are emitted promptly, essentially at the time of fission. A small fraction are emitted later by radioactive precursors among the fission products. These delayed neutrons are crucial to reactor controllability because they introduce much longer timescales into the neutron population dynamics.

A steady reactor can therefore be adjusted using mechanical and thermal control systems on humanly manageable timescales. The enrichment section introduces a minimal point-kinetics model that shows this mathematically. The main conceptual point is enough here: treating all neutrons as prompt gives a misleading picture of reactor control.

Key message

The chain reaction in a power reactor is not maintained by “using all the neutrons.” Most neutrons follow other paths. Criticality requires only that, on average, exactly enough neutrons from one generation produce fissions that replace that generation.

Caution

Criticality and explosion are not synonyms. A reactor at steady power is critical by design. The word describes neutron balance, not mechanical destruction. Real accident analysis requires kinetics, feedback, coolant behaviour, materials and engineered safety systems; the scalar condition $k_{\text{eff}} > 1$ by itself is not an accident model.

Guided checks

1. If every fission emits several neutrons, why is a critical reactor not required to have k equal to that emitted-neutron number?
2. Why does changing reactor geometry change k_{eff} even if fuel composition is unchanged?
3. Why do delayed neutrons matter disproportionately even though they are only a small fraction of all fission neutrons?

23.6 A controlled fission reactor: neutron balance and power conversion

A nuclear power reactor is an engineered system that maintains a self-sustained fission chain reaction while transferring the released nuclear energy to a working fluid and, ultimately, to useful electrical or process power. The word *reactor* refers to the nuclear system in which the chain reaction occurs; a complete power station also contains heat exchangers, turbines or other conversion machinery, generators, condensers, pumps, instrumentation, shielding and containment structures.

The functions required in a generic thermal-neutron reactor are easier to understand if they are separated physically:

- the **fuel** contains fissile nuclides and is the source of fission energy;
- the **moderator**, when used, reduces neutron kinetic energies through scattering so that thermal-neutron fission becomes more probable in fuels such as ^{235}U ;
- **neutron absorbers** provide a means of changing the neutron balance and therefore the power level;
- the **coolant** removes thermal energy from the core;
- **structural materials** maintain geometry while tolerating temperature, stress and irradiation;
- **shielding and containment** reduce radiation outside the intended region and provide physical barriers to radioactive material.

These functions need not be performed by six different materials. In a light-water reactor, for example, water can serve both as moderator and coolant. Other reactor concepts make different choices. The physics discussed here is therefore intentionally functional rather than tied to one commercial design.

Figure 23.5 shows how separate fluid loops transport heat in one reactor concept.

23.6.1 Moderator and coolant are different concepts

A common source of confusion is the statement that “water slows neutrons.” In reactor physics this is a statement about the *moderating* role of collisions. The same water may also carry energy away from the core, but neutron moderation and heat transport are distinct physical processes. Heavy water, graphite and other materials can be used as moderators in appropriate reactor concepts; gases, liquid metals, molten salts and water can serve as coolants in different designs. The chapter does not attempt a taxonomy of reactor engineering, but students should always ask which physical function a material is performing.

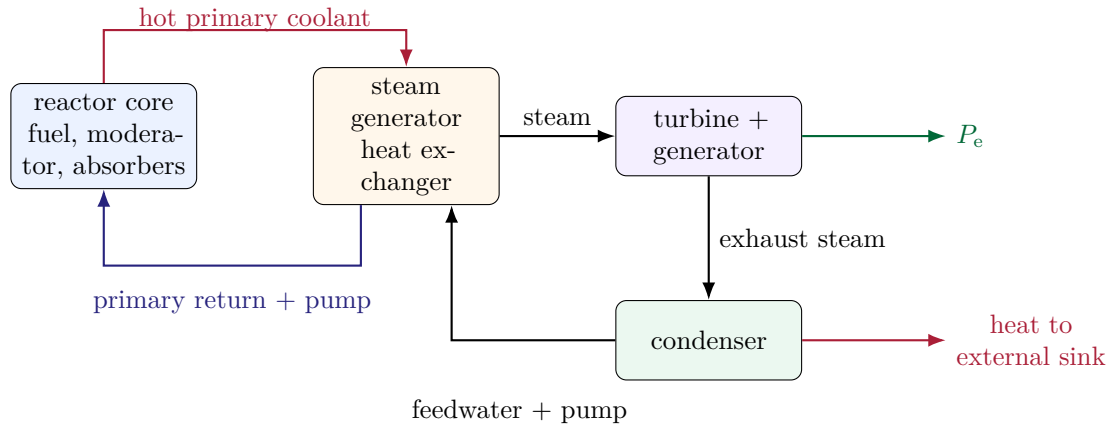


Figure 23.5: Simplified pressurized-water arrangement showing two separate fluid loops and an external heat sink. Energy crosses the steam-generator boundary; primary and secondary water do not mix. Other reactor concepts have different layouts.

Guided checks

1. Could a material be an excellent coolant but a poor moderator? Explain what properties the two functions depend on.
2. Why is a moderator most useful for a reactor concept whose fission probability is enhanced at low neutron energy?
3. If more neutrons leak out of a core without causing fission, what happens to k_{eff} ?

23.6.2 From thermal power to electrical power

The fission rate determines the thermal power deposited in the core. The power-conversion system then operates as a heat engine. Define an overall thermal-to-electric efficiency

$$\eta_e = \frac{P_e}{P_{\text{th}}}, \quad 0 < \eta_e < 1, \quad (23.31)$$

where P_{th} is the reactor thermal power and P_e is the delivered electrical power. The rejected thermal power in a steady energy balance is approximately

$$P_{\text{reject}} = P_{\text{th}} - P_e = P_{\text{th}}(1 - \eta_e), \quad (23.32)$$

neglecting smaller auxiliary terms when making a first estimate. Equation (23.31) is a thermodynamic relation, not a nuclear one. The second law of thermodynamics prevents all reactor heat from being converted into work.

Example 23.6: Thermal and electrical power in a generating station

A power station produces 3.0 GW of thermal power in its reactor and produces 1.0 GW of gross electrical power. Neglect auxiliary loads and losses outside the thermal cycle. Estimate its gross thermal-to-electric efficiency and the rate at which energy must be rejected as heat.

Solution. From Eq. (23.31),

$$\eta_e = \frac{1.0}{3.0} = 0.333 \approx 33\%. \quad (23.33)$$

The remaining power is

$$P_{\text{reject}} = 3.0 \text{ GW} - 1.0 \text{ GW} = 2.0 \text{ GW}. \quad (23.34)$$

This large rejected-heat rate is not evidence that the nuclear reaction is “inefficient.” It is a consequence of converting heat to work. Fossil-fuel and solar-thermal stations face the same thermodynamic restriction.

Key message

The reactor core is a heat source. Fission determines the microscopic energy release and neutron population; the turbine-generator or other conversion system determines how a fraction of the resulting heat becomes useful power.

23.6.3 Control absorbers and neutron economy

Materials with large neutron-absorption probabilities can reduce the fraction of neutrons available to cause additional fissions. Moving absorber elements or changing other control variables therefore changes k_{eff} . A power reactor is designed so that routine operation remains close to $k_{\text{eff}} = 1$, while shutdown systems can introduce sufficient negative reactivity to stop the self-sustaining chain reaction.

The word *stop* needs care. Terminating the chain reaction removes the dominant fission heat source, but radioactive fission products continue to decay and release *decay heat*. Heat removal therefore remains an engineering requirement after shutdown. This distinction is central to reactor-safety physics.

Caution

A control rod does not “cool the reactor” directly. Its primary nuclear role is to absorb neutrons and change the chain-reaction rate. Temperature changes later because the fission power changes and because the thermal system transports stored and decay heat.

23.7 Criticality, reactivity, feedback and shutdown

The multiplication factor k_{eff} is the most direct first-level descriptor of neutron balance, but engineers often use the dimensionless *reactivity*

$$\rho = \frac{k_{\text{eff}} - 1}{k_{\text{eff}}}. \quad (23.35)$$

Thus $\rho = 0$ at criticality, $\rho < 0$ when the system is subcritical and $\rho > 0$ when it is supercritical. For k_{eff} very close to unity, $\rho \approx k_{\text{eff}} - 1$, but Eq. (23.35) is the conventional exact definition within this simplified treatment.

23.7.1 Why feedback matters

The multiplication factor is not generally a fixed constant independent of reactor state. It can depend on fuel temperature, moderator density, coolant void fraction, isotopic composition and geometry. Reactor designers therefore analyse *reactivity coefficients*, which describe how reactivity changes when a state variable changes. Many power-reactor systems are intentionally designed so that important operating feedbacks are negative over their normal operating range: an increase in temperature tends to reduce reactivity and oppose the original power increase. The sign and magnitude, however, are design-dependent and must never be generalized blindly from one reactor type to another.

A simple linearized local relation can be written schematically as

$$\Delta\rho \approx \alpha_T \Delta T, \quad (23.36)$$

where α_T is a temperature coefficient of reactivity. Equation (23.36) is a local engineering model, not a universal law. If $\alpha_T < 0$, a positive temperature change produces negative reactivity.

Example 23.7: Interpreting a reactivity coefficient

A simplified reactor model near one operating point has a temperature coefficient $\alpha_T = -4.0 \times 10^{-5} \text{ K}^{-1}$. If the relevant effective temperature increases by 50 K, estimate the reactivity change within the linear model.

Solution.

$$\Delta\rho = \alpha_T \Delta T = (-4.0 \times 10^{-5} \text{ K}^{-1})(50 \text{ K}) = -2.0 \times 10^{-3}. \quad (23.37)$$

The negative sign means that this particular assumed feedback acts against the temperature increase. The numerical coefficient was specified as part of the model; it should not be transferred to an unspecified real reactor.

23.7.2 Shutdown does not mean zero thermal power

Immediately after fission power is terminated, unstable fission products and activation products continue to decay. Their radiation deposits energy in fuel and structures. A schematic model for a selected component of decay heat might be written

$$P_d(t) = P_{d0} e^{-t/\tau_d}, \quad (23.38)$$

where τ_d is an effective time constant. Real decay heat is a superposition of many decay chains and is not described by one exponential over all times. Equation (23.38) is therefore useful only as a toy model for exercises or data fitting.

Example 23.8: A toy decay-heat estimate

Suppose a simplified single-exponential model has $P_{d0} = 80 \text{ MW}$ and $\tau_d = 3.0 \text{ h}$. What power does the model predict after 6.0 h?

Solution.

$$P_d(6 \text{ h}) = (80 \text{ MW})e^{-6/3} = 80e^{-2} \text{ MW} \approx 10.8 \text{ MW}. \quad (23.39)$$

The calculation illustrates a timescale, not an empirical decay-heat correlation. In engineering work, validated decay-heat data or standards are required.

Method

When analysing a reactor transient conceptually, separate four questions:

1. What changes the neutron population and therefore the fission rate?
2. What feedbacks change reactivity as temperatures or densities change?
3. What thermal energy is already stored in fuel, coolant and structures?
4. What non-fission heat sources, especially radioactive decay, remain after shutdown?

Conflating these questions leads to incorrect statements such as “the reactor is shut down, therefore no cooling is needed.”

Guided checks

1. Evaluate the sign of ρ for $k_{\text{eff}} = 0.995, 1.000$ and 1.005 .
2. Why can a negative temperature coefficient contribute to passive stabilization?
3. Why is one exponential generally inadequate to describe total decay heat for long time intervals?

23.8 Energy flow, fuel use and system-level accounting

Nuclear energy problems span an unusually wide range of scales. A single fission event releases energy of order 10^{-11} J, while a power station may operate at 10^9 J/s. Connecting those scales is an excellent exercise in dimensional reasoning.

If each useful fission releases energy Q_f and the fission rate is R_f , then

$$P_{\text{th}} = R_f Q_f. \quad (23.40)$$

The idealized mass-consumption rate of a pure fissile nuclide of molar mass M is

$$\dot{m} = R_f \frac{M}{N_A}, \quad (23.41)$$

provided every counted nucleus is assumed to fission. A real fuel assembly contains fertile material, structural materials and fission products; fuel management is therefore more complicated than Eq. (23.41).

23.8.1 Annual energy, fissioned mass and mass equivalent

Suppose a reactor operates at $P_{\text{th}} = 2.5$ GW for one calendar year at a capacity factor of 90%. For a transparent estimate take $Q_f = 200$ MeV per fission.

The actual operating time represented by the annual average is

$$t = (0.90)(365.25 \times 86400 \text{ s}) \approx 2.84 \times 10^7 \text{ s}. \quad (23.42)$$

The thermal energy is therefore

$$E = P_{\text{th}} t \approx (2.5 \times 10^9)(2.84 \times 10^7) \approx 7.1 \times 10^{16} \text{ J}. \quad (23.43)$$

One fission releases

$$Q_f \approx 3.20 \times 10^{-11} \text{ J}, \quad (23.44)$$

so the total fission count is

$$N_f = \frac{E}{Q_f} \approx 2.2 \times 10^{27}. \quad (23.45)$$

If every fission were of ^{235}U , the corresponding fissioned mass would be

$$m_f = \frac{N_f}{N_A} (0.235 \text{ kg/mol}) \approx 8.6 \times 10^2 \text{ kg}. \quad (23.46)$$

This final number is not the annual fresh-fuel loading. Real fuel contains mostly fertile material, some fissile material is created during irradiation, not all heavy nuclei fission, and discharge criteria depend on burnup and materials limits. The tutorial therefore distinguishes a clean nuclear-physics quantity—fissioned heavy-metal mass—from a fuel-management quantity.

If the plant's net electrical efficiency were 33%, the net annual electricity corresponding to the same operating period would be approximately

$$E_e \approx 0.33E \approx 2.3 \times 10^{16} \text{ J} \approx 6.5 \text{ TWh}. \quad (23.47)$$

The conversion uses $1 \text{ TWh} = 3.6 \times 10^{15} \text{ J}$.

Guided checks

Why does the capacity factor multiply the calendar time rather than the fission energy per event? Which result in this tutorial would change if thermal efficiency improved while reactor thermal power remained fixed?

The mass equivalent of the emitted energy is a third quantity:

$$\Delta m_{\text{eq}} = \frac{E}{c^2} \approx 0.79 \text{ kg}. \quad (23.48)$$

Thus about 860 kg of nuclei fission in this model, but only about 0.79 kg of rest mass is converted into recoverable thermal energy. Almost all the original nuclear rest mass remains in fragments and emitted neutrons. If a specified fraction $c_\gamma = R_{\text{capture}}/R_f$ of additional fissile nuclei is lost by non-fission capture, fissile depletion is $(1 + c_\gamma)R_f$, not R_f alone. Fresh-fuel loading depends on composition and achievable burnup as well.

23.8.2 Nuclear energy density versus chemical energy density

Chemical reactions rearrange electrons and typically involve energies of order electronvolts per molecule. Nuclear reactions change nuclear binding and typically involve megaelectronvolts per nucleus. That difference of roughly six orders of magnitude per microscopic event explains why nuclear fuels can have extremely high energy density.

A fair comparison must state what is being compared. “One kilogram of uranium” can mean natural uranium, enriched fuel, or pure ^{235}U ; not all nuclei in a fuel assembly fission; and a power plant does not convert all released heat into electricity. Therefore broad energy-density statements should be framed as order-of-magnitude comparisons rather than universal fuel-cycle numbers.

Example 23.9: Coal equivalent of a kilogram of fissile fuel

Assume complete fission of 1.00 kg of ^{235}U , with 200 MeV released per fission. Compare with a representative coal heating value $q_c = 24 \text{ MJ/kg}$. This is an illustrative energy-density comparison, not a life-cycle analysis.

Solution. The number of uranium nuclei is

$$N = \frac{1.00}{0.235} N_A \approx 2.56 \times 10^{24}. \quad (23.49)$$

The ideal nuclear energy is

$$E_f = NQ_f \approx (2.56 \times 10^{24})(3.204 \times 10^{-11}) \approx 8.2 \times 10^{13} \text{ J.} \quad (23.50)$$

The coal mass giving the same heat would be

$$m_c = \frac{8.2 \times 10^{13}}{24 \times 10^6} \approx 3.4 \times 10^6 \text{ kg.} \quad (23.51)$$

Thus the microscopic binding-energy scale creates a million-fold-scale difference in ideal fuel energy density. Real systems require additional corrections for fuel composition, burnup, conversion efficiency and the entire fuel cycle.

Caution

High energy density does not by itself decide whether an energy technology is economically, environmentally or socially preferable. Those questions require system boundaries, resource assumptions, construction and decommissioning, waste handling, reliability, regulation and many other non-nuclear-physics inputs.

23.9 A natural fission reactor: the Oklo case study

In the early 1970s, isotopic anomalies in uranium ore from Oklo in present-day Gabon were recognized as evidence that self-sustaining natural fission reactions had occurred roughly two billion years ago [4]. The case is physically instructive because it combines isotope evolution, neutron moderation, geometry and feedback without any engineered reactor.

The key historical condition was the larger fraction of ^{235}U in natural uranium at that time. Because ^{235}U has a shorter half-life than ^{238}U , its abundance decreases faster. Looking backward in time therefore increases the $^{235}\text{U}/^{238}\text{U}$ ratio.

For two isotopes with decay constants λ_{235} and λ_{238} , the ratio evolves backward by

$$\left(\frac{N_{235}}{N_{238}}\right)_{\text{past}} = \left(\frac{N_{235}}{N_{238}}\right)_{\text{now}} \exp[(\lambda_{235} - \lambda_{238})\Delta t]. \quad (23.52)$$

This calculation isolates isotope decay only. Geological transport and ore formation require separate evidence.

Example 23.10: How much larger was the ^{235}U fraction two billion years ago?

Use approximate half-lives $T_{1/2,235} = 7.04 \times 10^8 \text{ y}$ and $T_{1/2,238} = 4.47 \times 10^9 \text{ y}$. Take the present atom fraction of ^{235}U as $x_{235} \approx 0.0072$. Estimate the fraction $2.0 \times 10^9 \text{ y}$ ago, neglecting minor isotopes.

Solution. The present ratio is

$$r_0 = \frac{x_{235}}{1 - x_{235}} \approx \frac{0.0072}{0.9928} = 7.25 \times 10^{-3}. \quad (23.53)$$

The decay constants are

$$\lambda_{235} = \frac{\ln 2}{7.04 \times 10^8 \text{ y}} \approx 9.85 \times 10^{-10} \text{ y}^{-1}, \quad (23.54)$$

$$\lambda_{238} = \frac{\ln 2}{4.47 \times 10^9 \text{ y}} \approx 1.55 \times 10^{-10} \text{ y}^{-1}. \quad (23.55)$$

Therefore

$$r_{\text{past}} = r_0 \exp[(8.30 \times 10^{-10})(2.0 \times 10^9)] \approx 0.038. \quad (23.56)$$

The corresponding fraction is

$$x_{235,\text{past}} = \frac{r_{\text{past}}}{1 + r_{\text{past}}} \approx 0.0367 \approx 3.7\%. \quad (23.57)$$

This is comparable to the few-percent fissile enrichment used in many thermal-reactor fuels, although a natural geological system is not equivalent to an engineered reactor.

Groundwater can act as a neutron moderator in suitable ore geometry. One qualitative model of Oklo operation therefore contains a natural negative feedback: heating reduces or redistributes water, moderation weakens, and the chain reaction becomes less effective; after cooling and renewed water penetration, conditions may again become favourable. The detailed geological history is richer than this simple picture, but the example shows why criticality is a system property involving composition, geometry and material state.

Key message

Oklo is valuable pedagogically because it demonstrates that a chain reaction does not require human technology. Criticality follows from nuclear composition, neutron balance, geometry and moderation.

Guided checks

1. Why does the ^{235}U fraction increase when we extrapolate backward in time?
2. What role could groundwater play in a natural uranium deposit?
3. Why is a sufficiently high fissile fraction necessary but not sufficient for a self-sustaining chain reaction?

23.10 Fusion: favourable binding energy, difficult reaction conditions

For light nuclei, moving toward larger binding energy per nucleon generally releases energy. Fusion therefore looks attractive from the binding-energy curve. The challenge is that two positively charged nuclei repel each other electrically before the short-range strong interaction can bind them.

23.10.1 Representative fusion reactions

Important terrestrial candidates include



The deuterium–tritium reaction has a comparatively large reaction probability at temperatures reachable in laboratory plasmas and is therefore central to much controlled-fusion research.

Example 23.11: Q-value of deuterium–tritium fusion

Use atomic masses

$$\begin{aligned} m(^2\text{H}) &= 2.01410178 \text{ u}, & m(^3\text{H}) &= 3.01604928 \text{ u}, \\ m(^4\text{He}) &= 4.00260325 \text{ u}, & m_n &= 1.00866492 \text{ u}. \end{aligned}$$

Calculate the Q-value of $^2\text{H} + ^3\text{H} \rightarrow ^4\text{He} + n$.

Solution. Atomic masses may be used consistently because there are two electrons on the initial side and two in a neutral helium atom on the final side. The mass difference is

$$\Delta m = (2.01410178 + 3.01604928) \quad (23.61)$$

$$- (4.00260325 + 1.00866492) \text{ u} \quad (23.62)$$

$$= 0.01888289 \text{ u}. \quad (23.63)$$

Using $1 \text{ u } c^2 \approx 931.5 \text{ MeV}$,

$$Q = \Delta m c^2 \approx 17.59 \text{ MeV}. \quad (23.64)$$

The positive sign means the final particles carry approximately 17.6 MeV of kinetic energy in the centre-of-mass frame, distributed according to momentum conservation.

Where the D–T energy goes

For a collision with small initial centre-of-mass kinetic energy, the neutron and alpha emerge with equal and opposite momenta. Figure 23.6 separates this momentum statement from the unequal energy sharing. In a nonrelativistic estimate,

$$\mathbf{p}_n = -\mathbf{p}_\alpha, \quad K_n = \frac{p^2}{2m_n}, \quad K_\alpha = \frac{p^2}{2m_\alpha}, \quad \frac{K_n}{K_\alpha} = \frac{m_\alpha}{m_n}. \quad (23.65)$$

Using $m_\alpha \simeq 4m_n$ and $K_n + K_\alpha \simeq Q$ gives

$$K_n \simeq \frac{4}{5}Q \simeq 14.1 \text{ MeV}, \quad K_\alpha \simeq \frac{1}{5}Q \simeq 3.5 \text{ MeV}. \quad (23.66)$$

The lighter particle takes most of the kinetic energy even though neither has more momentum. Actual masses and relativistic two-body kinematics refine these approximate values. A significant incident energy adds to the available centre-of-mass energy; transformation to the laboratory frame can also make the measured product energy depend on emission direction.

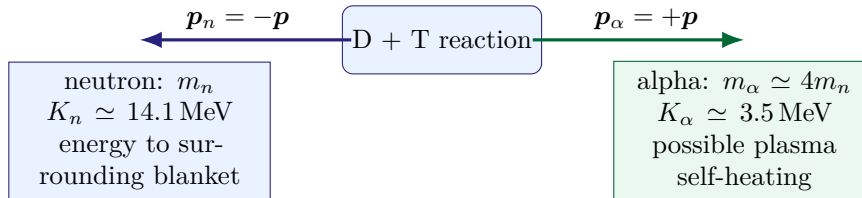


Figure 23.6: D–T energy sharing in the centre-of-mass frame. Equal arrow lengths indicate equal momentum magnitudes, not equal energies. Neutrons are not confined by the magnetic field; useful alpha heating requires retention and slowing down in the plasma.

23.10.2 The classical Coulomb barrier

Outside the range of the strong interaction, the electrostatic potential energy of two nuclei with charges Z_1e and Z_2e is approximately

$$U_C(r) = \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{r} \approx (1.44 \text{ MeV fm}) \frac{Z_1 Z_2}{r [\text{fm}]} \quad (23.67)$$

If one naively requires the kinetic energy to exceed this potential at a few femtometres, the required energy is hundreds of kiloelectronvolts for hydrogen isotopes. Typical thermal energies in fusion plasmas are much smaller. Fusion nevertheless occurs because quantum mechanics permits tunnelling through the barrier and because a thermal plasma contains a distribution of particle energies.

The turning points and the forbidden interval are marked in Fig. 23.7.

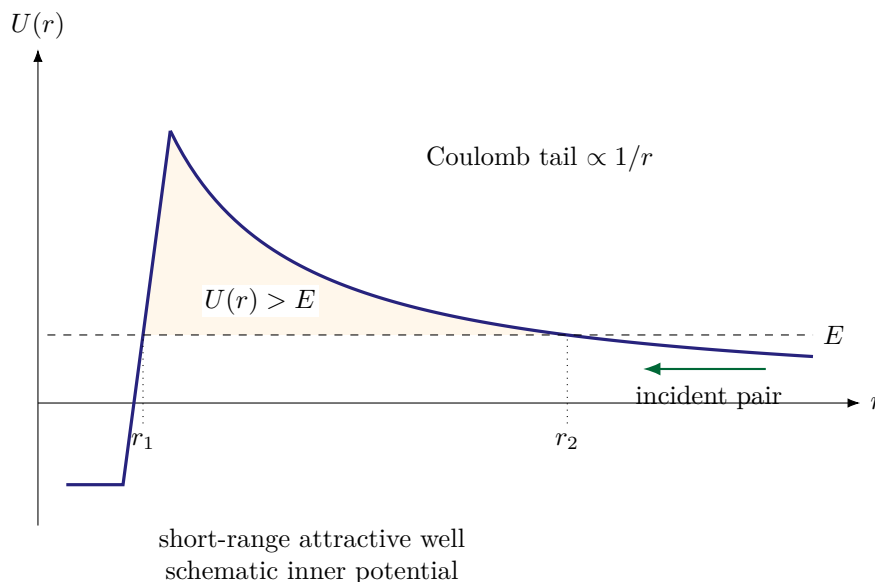


Figure 23.7: One schematic total potential, with the forbidden interval bounded by its intersections with the collision energy. The Coulomb tail is plotted as $5.6/r$ in arbitrary consistent plot units; the short-range segment is qualitative. Sub-barrier fusion requires tunnelling across $r_1 < r < r_2$, not classical motion through that interval.

Example 23.12: Barrier scale for two hydrogen isotopes

Estimate the Coulomb potential energy for two singly charged nuclei at $r = 3.5$ fm. Compare it with $k_B T$ at $T = 1.5 \times 10^8$ K.

Solution. With $Z_1 = Z_2 = 1$,

$$U_C \approx \frac{1.44 \text{ MeV fm}}{3.5 \text{ fm}} \approx 0.41 \text{ MeV} = 410 \text{ keV}. \quad (23.68)$$

The thermal energy scale is

$$k_B T = (8.617 \times 10^{-5} \text{ eV/K})(1.5 \times 10^8 \text{ K}) \quad (23.69)$$

$$\approx 1.29 \times 10^4 \text{ eV} = 12.9 \text{ keV}. \quad (23.70)$$

Thus $k_B T$ is far below the simple barrier estimate. Fusion at such temperatures relies on the high-energy tail of the velocity distribution and, crucially, quantum tunnelling.

Caution

The phrase “fusion requires a temperature high enough to overcome the Coulomb barrier” is an oversimplification. In laboratory and stellar fusion, most reacting nuclei do not classically climb over the full barrier. Tunnelling is essential.

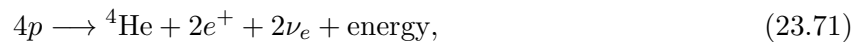
Guided checks

1. Why is a positive Q -value not sufficient to make a fusion reaction proceed rapidly?
2. How does increasing $Z_1 Z_2$ affect the Coulomb barrier?
3. Why can a plasma with $k_B T \ll U_C$ still undergo fusion?

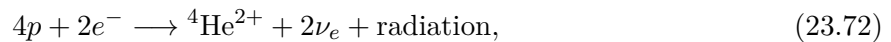
23.11 Fusion in the Sun and in stars

The Sun is a gravitationally confined thermonuclear plasma. Its central temperature is of order 10^7 K, and its enormous size and density provide enough interacting particles for very slow weak-interaction steps to sustain the observed luminosity over billions of years.

For the Sun, the dominant net conversion is

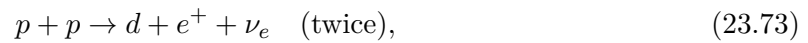


The bare-nuclear reaction has $Q \approx 24.7$ MeV. Each positron subsequently annihilates with an ambient electron. Including those two electrons in the initial system gives the net balance



with $Q \approx 26.7$ MeV. Equivalently, to nuclear accuracy, $Q = [4m({}^1\text{H}) - m({}^4\text{He})]c^2$ using neutral atomic masses. The two extra electron masses included this way supply the annihilation bookkeeping; the small electronic binding energies are negligible on the MeV scale. A small fraction is carried away by neutrinos; most ultimately thermalizes and is transported outward before being radiated from the solar surface.

One important branch of the proton–proton chain can be summarized by



The first reaction is slow because it involves the weak interaction. That slowness is essential to the long lifetime of main-sequence stars like the Sun.

The required multiplicities are shown explicitly in Fig. 23.8.

23.11.1 Solar luminosity and mass-equivalent rate

Einstein's relation $E = mc^2$ lets us express any energy flow as a mass-equivalent flow. If the solar luminosity is $L_\odot$, the mass-equivalent rate associated with the emitted electromagnetic energy is

$$\dot{m}_{\text{eq}} = \frac{L_\odot}{c^2}. \quad (23.76)$$

This does not mean that the Sun converts all reacting mass to radiation. Only a small fraction of the initial nuclear rest energy appears as released energy.

Example 23.13: Mass-equivalent rate of solar radiation

Take $L_\odot = 3.83 \times 10^{26}$ W. Find the equivalent mass loss per second associated with this luminosity.

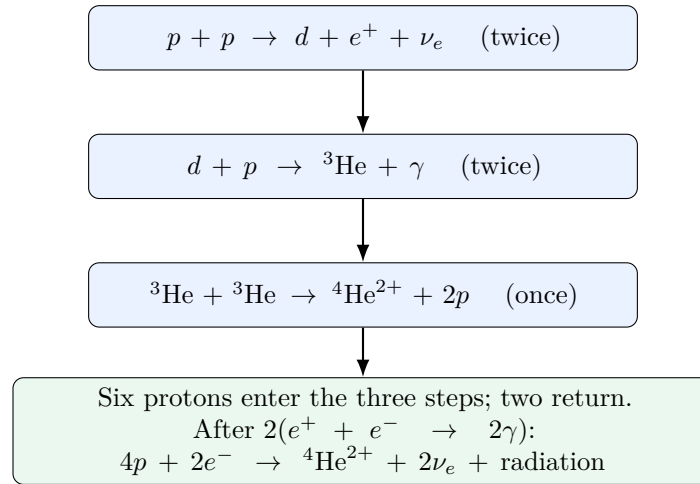


Figure 23.8: The pp-I branch with reaction multiplicities and electron accounting. The first two reactions occur twice for each final helium nucleus. Positron annihilation raises the complete energy release to approximately 26.7 MeV; escaping neutrinos reduce heat deposition.

Solution.

$$\dot{m}_{\text{eq}} = \frac{3.83 \times 10^{26}}{(2.998 \times 10^8)^2} \approx 4.26 \times 10^9 \text{ kg/s.} \quad (23.77)$$

The number looks enormous on human scales, but it is tiny compared with the solar mass $M_{\odot} \approx 1.99 \times 10^{30} \text{ kg}$. It represents the mass equivalent of electromagnetic radiation, not total hydrogen consumption. Neutrino emission and the solar wind are additional contributions to the Sun's total mass-energy loss.

23.11.2 Fuel consumption and stellar lifetime

The Sun radiates enormous power, yet it has remained stable for billions of years. This is possible because the energy reservoir is enormous and the microscopic proton–proton reaction is extraordinarily slow at solar conditions.

A useful global timescale estimate is

$$\tau \sim \frac{\epsilon f M_{\odot} c^2}{L_{\odot}}, \quad (23.78)$$

where f is the fraction of solar mass that effectively participates in core hydrogen burning over the main-sequence lifetime and $\epsilon \approx 0.007$ is the fractional rest-mass release when hydrogen becomes helium. Taking $f \sim 0.1$ gives

$$\tau \sim \frac{(0.007)(0.1)(1.99 \times 10^{30})(2.998 \times 10^8)^2}{3.83 \times 10^{26}} \quad (23.79)$$

$$\sim 3.3 \times 10^{17} \text{ s} \sim 10^{10} \text{ y.} \quad (23.80)$$

This order-of-magnitude estimate explains the main-sequence lifetime scale without solving a stellar-structure model.

Why is the first proton–proton step slow? Two protons must tunnel close enough to interact, and one proton must change into a neutron through the weak interaction to form deuterium. The weak interaction makes this event far rarer than an ordinary strong nuclear reaction once the particles are close. If proton fusion proceeded with a typical strong-interaction rate under solar conditions, the Sun's lifetime would be vastly shorter and its structure would be different.

Guided checks

In Eq. (23.78), what happens to the estimated lifetime if luminosity doubles while all other quantities are held fixed? Why is f less than one for a main-sequence star like the Sun?

23.11.3 Why stellar fusion does not release energy indefinitely

The binding-energy curve explains why fusion can power stars only while reactions move nuclei toward more tightly bound configurations. Fusion of light nuclei up to the iron-group region can release energy in appropriate reaction networks. Beyond the maximum in binding energy per nucleon, simple fusion of progressively heavier nuclei is no longer an energy source. Many nuclei heavier than iron are formed instead through neutron-capture processes in evolved stars and violent astrophysical events, followed by beta decays.

Key message

Stars do not shine merely because they are hot. Their luminosity reflects a coupled system: gravity establishes high density and temperature, quantum tunnelling permits charged-particle reactions, nuclear binding provides energy, and energy transport carries that energy to the surface.

23.12 Controlled thermonuclear fusion on Earth

A terrestrial fusion-energy system must create a plasma in which fusion reactions occur frequently enough that the released fusion energy can compensate plasma losses and, eventually, power the whole facility. Three quantities therefore appear repeatedly: particle density n , temperature T , and an energy-confinement time τ_E . Increasing any one can help, but the optimum combination depends on the confinement concept and fuel.

23.12.1 Reaction rate and reactivity

For two distinct reacting ion species with number densities n_1 and n_2 , the fusion reaction rate per unit volume is

$$\mathcal{R} = n_1 n_2 \langle \sigma v \rangle, \quad (23.81)$$

where $\langle \sigma v \rangle$ is the velocity-averaged product of cross section and relative speed. For an equal deuterium–tritium mixture with total ion density $n = n_D + n_T$, one has $n_D = n_T = n/2$ and therefore

$$\mathcal{R}_{DT} = \frac{n^2}{4} \langle \sigma v \rangle_{DT}. \quad (23.82)$$

The fusion power density is then

$$p_f = \mathcal{R}_{DT} Q_{DT}. \quad (23.83)$$

The result can also be derived by counting pairs: one deuteron reacts at rate $n_T \langle \sigma v \rangle$, and there are $V n_D$ deuterons. With deuterium fraction $x = n_D/n$, the rate density is

$$\mathcal{R} = x(1-x)n^2 \langle \sigma v \rangle. \quad (23.84)$$

At fixed total ion density and temperature, $x(1-x)$ is maximized at $x = 1/2$. For identical reactants such as D–D, each pair must instead be counted once:

$$\mathcal{R}_{ii} = \frac{1}{2} n_i^2 \langle \sigma v \rangle. \quad (23.85)$$

These are dilute-plasma binary-collision models. Temperature enters through reactivity; spatially varying plasmas require $R = \int n_D(\mathbf{r}) n_T(\mathbf{r}) \langle \sigma v \rangle [T(\mathbf{r})] d^3r$, so multiplying separately averaged quantities can bias the inferred power.

Example 23.14: Fusion power density from a specified reactivity

Consider an equal D–T plasma with total ion density $n = 1.0 \times 10^{20} \text{ m}^{-3}$. At a chosen temperature, suppose $\langle \sigma v \rangle = 1.0 \times 10^{-22} \text{ m}^3/\text{s}$. Estimate the fusion reaction-rate density and total fusion power density. Use $Q_{DT} = 17.6 \text{ MeV}$.

Solution.

$$\mathcal{R}_{DT} = \frac{(10^{20})^2}{4}(10^{-22}) = 2.5 \times 10^{17} \text{ m}^{-3}\text{s}^{-1}. \quad (23.86)$$

The event energy is

$$Q_{DT} = 17.6 \times 10^6 (1.602 \times 10^{-19}) \approx 2.82 \times 10^{-12} \text{ J}. \quad (23.87)$$

Hence

$$p_f = (2.5 \times 10^{17})(2.82 \times 10^{-12}) \approx 7.1 \times 10^5 \text{ W/m}^3. \quad (23.88)$$

The assumed reactivity is part of the problem statement. In a real analysis it must be obtained from measured or evaluated cross-section data and the plasma velocity distribution.

23.12.2 Lawson-type confinement requirement

A useful first measure of fusion performance combines density and confinement time. For a specified fuel and operating temperature, one can derive a threshold condition of the schematic form

$$n\tau_E \gtrsim \text{required value at the chosen } T, \quad (23.89)$$

or use the triple product $nT\tau_E$. The numerical threshold is not universal: it depends on the fuel, temperature, assumptions about energy deposition, radiation losses and the definition of the gain condition. The enrichment section derives a simplified D–T form.

Caution

Do not treat “the Lawson criterion” as one temperature-independent magic number. Textbook estimates are useful for scale, but modern fusion performance is usually discussed with a temperature-dependent criterion or the triple product $nT\tau_E$.

23.12.3 Magnetic confinement

Because plasma particles are charged, magnetic fields can strongly modify their trajectories. A particle with perpendicular speed $v_\perp$ in a locally uniform magnetic field has a Larmor radius

$$r_L = \frac{mv_\perp}{|q|B}. \quad (23.90)$$

Strong magnetic fields can therefore help keep hot particles away from material walls. In a tokamak, a toroidal magnetic configuration combines externally produced and plasma-current-related fields to create nested magnetic surfaces. Confinement is not perfect: collisions, turbulence, instabilities and edge processes cause transport and energy loss.

23.12.4 Inertial confinement

In inertial confinement, a small fuel capsule is compressed and heated rapidly. Fusion must occur before the compressed fuel disassembles. The characteristic confinement time is therefore extremely short, compensated by very high transient density. Laser-driven systems use intense radiation to implode a capsule; other driver concepts are possible. The key contrast with magnetic confinement is not “strong versus weak” confinement but *low-density/longer-time* versus *high-density/short-time* strategies.

Figure 23.9 compares the two density–time strategies.

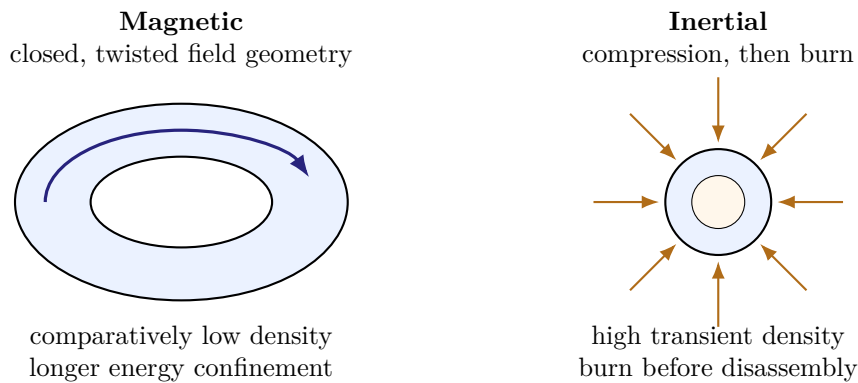


Figure 23.9: Confinement strategies at the level of physical mechanisms. The magnetic panel shows toroidal orientation, not a complete field-line map; an actual configuration also needs field-line twist. The inertial panel denotes inward compression rather than literal magnetic forces.

23.12.5 Why net fusion power is difficult

A successful fusion power plant must solve more than the plasma-physics problem. It must also handle heat exhaust, energetic neutron loads for D–T fuel, plasma-facing components, tritium inventory and breeding, magnet or driver efficiency, remote maintenance, materials activation, power conversion and plant availability. Moreover, different gain factors answer different questions. A plasma gain $Q_{\text{plasma}} = P_{\text{fusion}}/P_{\text{external heating}}$ is not the same as net electrical gain for an entire plant.

Key message

A positive nuclear Q -value says that a reaction releases energy once it occurs. A useful fusion power plant additionally requires a sufficiently high reaction rate, adequate confinement, manageable losses, durable materials and a favourable whole-plant energy balance.

Guided checks

1. In Eq. (23.82), why does doubling total density at fixed temperature increase the reaction-rate density by a factor of four?
2. Why can inertial confinement tolerate a much shorter confinement time than magnetic confinement?
3. Why is $Q_{\text{plasma}} > 1$ not automatically equivalent to net electrical power production?

23.13 Engineering, measurement and applied-science connections

Nuclear-energy systems are unusually interdisciplinary. Nuclear physics determines reaction rates and radiation fields, but reliable operation depends on thermal engineering, materials science, control and instrumentation, fluid mechanics, electrical engineering, computational modelling and statistical measurement.

23.13.1 Neutron and radiation instrumentation

Neutrons carry no electric charge, so they are not detected directly by ordinary ionization from their passage in the same way as charged particles. Detectors instead use neutron-induced reactions that produce charged particles, scintillation light, capture gamma rays or activation products. Reactor

instrumentation may use different detector technologies for different flux ranges. The measured count rate must be related to physical neutron flux through efficiency, geometry and energy response. For a detector with efficiency ϵ exposed to a particle rate R , an idealized count rate is

$$C = \epsilon R. \quad (23.91)$$

If counts are independent, the counting uncertainty is approximately Poissonian: for N counts, $\sigma_N \approx \sqrt{N}$, so the relative statistical uncertainty is

$$\frac{\sigma_N}{N} \approx \frac{1}{\sqrt{N}}. \quad (23.92)$$

This simple relation links nuclear instrumentation directly to statistical data analysis.

Example 23.15: Counting statistics for a neutron monitor

A detector records $N = 25,000$ counts during a fixed measurement interval. Estimate the one-standard-deviation counting uncertainty and the relative statistical uncertainty.

Solution.

$$\sigma_N \approx \sqrt{25,000} \approx 158, \quad \frac{\sigma_N}{N} \approx 6.3 \times 10^{-3} = 0.63\%. \quad (23.93)$$

Systematic uncertainties in efficiency, background and calibration may be larger and are not included in this statistical estimate.

23.13.2 Calorimetry and heat balance

Thermal power can be inferred from measured mass-flow rate $\dot{m}$, specific heat c_p and temperature rise ΔT in a coolant stream:

$$P_{\text{th}} \approx \dot{m} c_p \Delta T, \quad (23.94)$$

when phase changes and other energy terms are negligible. More complete plant measurements include pressure, enthalpy, pump work and phase-change effects. Equation (23.94) is nevertheless a useful bridge between nuclear reaction rates and ordinary thermal instrumentation.

Example 23.16: Coolant-flow power estimate

Water flows at $\dot{m} = 1.5 \times 10^4$ kg/s through a heat-transfer system and its average temperature rises by 35 K. Using $c_p = 4.2$ kJ/(kg K), estimate the transported thermal power.

Solution.

$$P \approx (1.5 \times 10^4)(4.2 \times 10^3)(35) \approx 2.2 \times 10^9 \text{ W} = 2.2 \text{ GW}. \quad (23.95)$$

The estimate assumes single-phase flow and a representative constant c_p . Engineering thermo-hydraulics normally uses enthalpy data rather than this simplified form.

23.13.3 Materials under irradiation

Energetic neutrons and charged particles can displace atoms from lattice sites, create gas by transmutation, change microstructure and alter mechanical properties. Reactor and fusion materials must therefore be evaluated not only at high temperature but also under irradiation. “Radiation resistance” is not one scalar property; swelling, embrittlement, creep, corrosion and activation can evolve differently with dose, temperature and material composition.

23.13.4 Computer-science and control connections

Modern nuclear systems use layered instrumentation and control, but the underlying computational ideas are familiar: filtering noisy sensor data, estimating state variables, detecting anomalies, solving

differential equations, propagating uncertainties and validating simulation against measurements. Safety-related control software adds demanding verification and redundancy requirements. For students of computer science, the important lesson is that numerical output is only as trustworthy as the physical model, input data, validation and uncertainty treatment behind it.

23.14 Computational laboratory: branching, power and uncertainty

This section develops a small computational project that can be implemented in Python, Julia, MATLAB, C++ or a spreadsheet. It is intentionally a pedagogical model rather than reactor-design software.

23.14.1 Part A: stochastic neutron branching

Let each neutron in one generation produce a random number of successor neutrons that cause fission in the next generation. A particularly simple model uses a Poisson distribution with mean k . If X_i is the offspring number from neutron i , then

$$N_{g+1} = \sum_{i=1}^{N_g} X_i, \quad X_i \sim \text{Poisson}(k). \quad (23.96)$$

The expected population obeys $\mathbb{E}[N_{g+1}|N_g] = kN_g$, reproducing the deterministic relation on average, but individual histories fluctuate and may go extinct even when the mean initially grows.

Note

Minimal algorithm.

1. Choose N_0 , k , the number of generations and a random seed.
2. For each generation, draw the total successor count as a Poisson random number with mean kN_g . This is equivalent to summing independent Poisson offspring variables.
3. Stop if $N_g = 0$, otherwise continue.
4. Repeat the whole history hundreds or thousands of times.
5. Plot individual histories, the ensemble mean and the extinction fraction.

The model contains no neutron-energy dependence, geometry, delayed neutrons or thermal feedback. Its purpose is to teach stochastic branching, not reactor kinetics.

For $k < 1$, eventual extinction has probability 1. For $k = 1$, the mean population stays constant but fluctuations grow, and eventual extinction still has probability 1 in this nondegenerate model. For $k > 1$, there is a nonzero probability of sustained growth. This difference between the mean trajectory and individual stochastic histories is a useful general lesson in probabilistic modelling.

An exact extinction check

The state $N_g = 0$ is absorbing: without an external source an extinct chain cannot restart. Let q_g be the probability that a chain starting from one ancestor has reached zero by generation g . Since a Poisson variable has probability-generating function $\mathbb{E}[z^X] = \exp[k(z - 1)]$, independence of the descendant chains gives

$$q_0 = 0, \quad q_{g+1} = \exp[k(q_g - 1)], \quad \Pr(N_g = 0 | N_0) = q_g^{N_0}. \quad (23.97)$$

This recurrence requires no random draws and provides a stringent check on a Monte Carlo program. The eventual extinction probability is the smallest root of $q = \exp[k(q - 1)]$ in $[0, 1]$. For $k \leq 1$ that

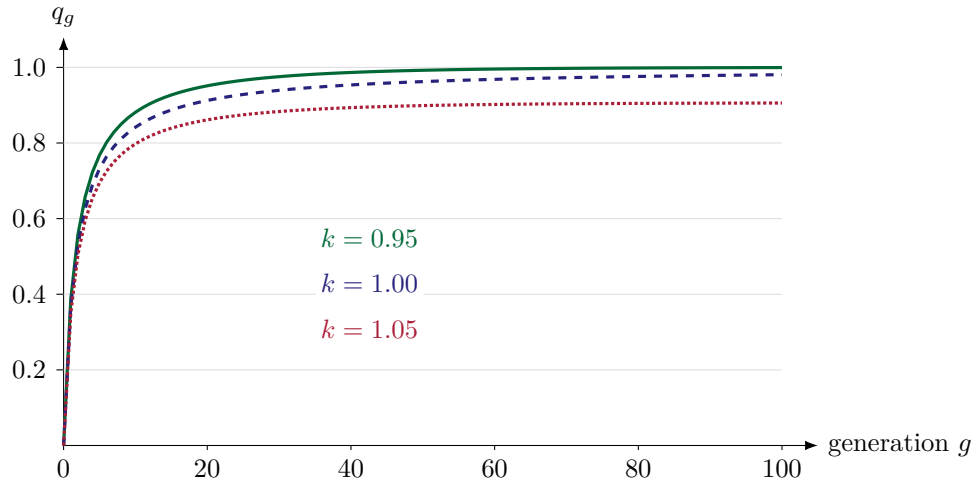


Figure 23.10: Probability of extinction by generation g for one initial ancestor, evaluated from Eq. (23.97). Curves connect values at every integer generation. The critical process has constant mean population but eventual extinction probability 1; the supercritical process has a positive probability of indefinite survival in this ideal model.

root is 1; for $k > 1$ it is smaller than 1. These statements assume independent offspring, unlimited growth and no immigration. They do not describe the shutdown probability of a controlled reactor.

Example 23.17: A constant mean does not imply a surviving chain

For $k = 1$ and $N_0 = 1$, evaluate Eq. (23.97) for 100 generations and compare extinction with the ensemble mean.

Strategy. Iterate the deterministic recurrence and separately use $\mathbb{E}[N_g] = N_0 k^g$. The recurrence gives $q_{100} = 0.9806471$, so about 98.1% of these histories have died out by generation 100, although $\mathbb{E}[N_{100}] = 1$. Since extinct histories contribute zero, the conditional mean among survivors is

$$\mathbb{E}[N_{100} \mid N_{100} > 0] = \frac{\mathbb{E}[N_{100}]}{1 - q_{100}} \simeq 51.7. \quad (23.98)$$

A few large surviving histories compensate for the many extinct ones. Starting instead with 100 independent ancestors gives extinction probability $q_{100}^{100} \simeq 0.1417$. The ancestor count matters as much as the multiplication factor when interpreting a finite simulation.

The following reproducible NumPy example checks the critical case. Zero populations remain zero without a special stopping rule.

```
import numpy as np
rng = np.random.default_rng(2301)
k, N0, G, M = 1.0, 100, 100, 10000
N = np.full(M, N0, dtype=np.int64)
q = 0.0
for g in range(G):
    N = rng.poisson(k * N)
    q = np.exp(k * (q - 1.0))
p_exact = q ** N0
p_MC = np.mean(N == 0)
se_MC = np.sqrt(p_exact * (1 - p_exact) / M)
print("extinction: MC, exact, standard error",
      p_MC, p_exact, se_MC)
print("population mean: MC, exact",
      N.mean(), N0 * k ** G)
```

Compare discrepancies with statistical uncertainty rather than requiring exact agreement with a random sample. Repeat with larger M or independent seeds: an uncertainty proportional to $M^{-1/2}$ should emerge. Figure 23.10 is calculated from the recurrence, so it is not subject to Monte Carlo noise.

23.14.2 Part B: convert fission counts to thermal power

If a simulation represents a physical time interval Δt and counts N_f fissions, estimate

$$P = \frac{N_f Q_f}{\Delta t}. \quad (23.99)$$

The branching variable counts effective chain initiators, not an independently measured fission tally. Before identifying N_f with any simulated count, specify that each initiator represents one fission and assign a generation duration. This imposed Δt does not follow from k and cannot reproduce real prompt or delayed neutron kinetics. Students should explicitly convert Q_f from MeV to joules. A dimensional check should be printed by the program or written next to the spreadsheet formula.

23.14.3 Part C: propagate a simple measurement uncertainty

Suppose thermal power is inferred from Eq. (23.94), with independent uncertainties in $\dot{m}$, c_p and ΔT . For small relative uncertainties,

$$\left(\frac{\sigma_P}{P}\right)^2 \approx \left(\frac{\sigma_{\dot{m}}}{\dot{m}}\right)^2 + \left(\frac{\sigma_{c_p}}{c_p}\right)^2 + \left(\frac{\sigma_{\Delta T}}{\Delta T}\right)^2. \quad (23.100)$$

A Monte Carlo uncertainty propagation can then be compared with the linearized expression.

Method

A good computational report should contain more than code. State the model assumptions, units, random seed or reproducibility method, convergence test, at least one analytic cross-check, and a short discussion of what physics is missing.

Guided checks

1. Why can a critical branching process have many realizations that die out even though its mean population is constant?
2. What unit conversion error would occur if 200 MeV were inserted directly into Eq. (23.99) as though it were joules?
3. Which uncertainty dominates Eq. (23.100) if the temperature-rise measurement is much less precise than the others?

23.15 Safety physics, barriers and defence in depth

Safety in nuclear engineering is not one device or one equation. It is a systems problem built around controlling reactivity, removing heat, confining radioactive material and maintaining these functions under disturbances. A useful physics-level organization is to separate the *source term*, the *barriers*, and the *heat-removal paths*.

23.15.1 Source term and radioactive inventory

During operation, fission creates a large inventory of unstable nuclei. Some decay on short timescales and dominate early decay heat; others persist much longer. The activity of nuclide i is

$$A_i = \lambda_i N_i, \quad (23.101)$$

so a large number of nuclei does not automatically imply large activity: the decay constant matters. Conversely, a small inventory of a short-lived isotope can have substantial activity.

The total radioactive inventory is not the same as the material that could reach the environment in a hypothetical event. Release depends on chemical form, temperature, transport paths and containment. This distinction between *inventory* and *release* is essential in safety analysis.

23.15.2 Multiple physical barriers

Commercial reactor concepts use multiple barriers between radioactive material and the environment. At a schematic level these include the fuel matrix, fuel cladding, the pressure boundary or primary-system boundary where applicable, and a containment structure. The exact barrier arrangement differs across reactor types.

The physical reasoning behind multiple barriers is probabilistic and defensive. No engineered component is assumed to be perfectly infallible. Independent or partially independent barriers reduce the probability that one failure directly produces a release. This is the essence of *defence in depth*: prevention, control, mitigation and emergency preparedness are layered rather than assigned to a single mechanism.

The layered-barrier idea is illustrated schematically in Fig. 23.11.

23.15.3 Reactivity control and heat removal are different safety functions

Making a reactor subcritical suppresses the self-sustaining fission chain, but does not instantaneously remove stored thermal energy or radioactive decay heat. Conversely, excellent cooling does not guarantee subcriticality. The two functions must therefore be analysed separately.

A lumped thermal balance after shutdown can be written

$$C_T \frac{dT}{dt} = P_d(t) - H[T(t) - T_s], \quad (23.102)$$

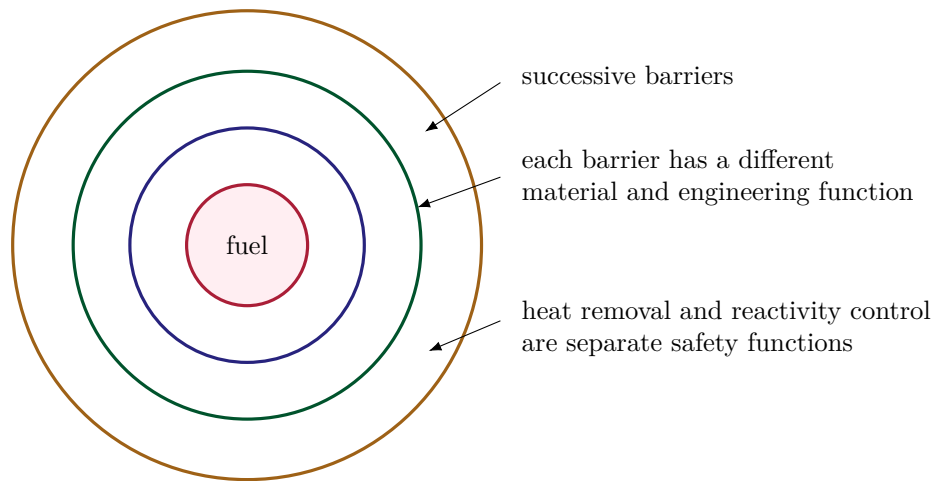


Figure 23.11: Conceptual defence-in-depth picture. The circles are not a literal reactor geometry; they emphasize that safety relies on multiple functions and barriers rather than one perfect component.

where $P_d(t)$ is decay heat, C_T an effective heat capacity, H a heat-removal coefficient and T_s a sink temperature. Even this simple ODE shows two routes to temperature rise: a large heat source or inadequate heat removal.

Method

When reading an accident description, avoid the vague phrase “the reactor failed.” Ask which physical function was lost or degraded: neutron control, heat removal, coolant inventory, electrical supply, pressure boundary, containment, instrumentation, or some combination.

23.15.4 Passive and active features

An *active* safety function requires powered or actuated equipment such as pumps, valves or control drives. A *passive* function relies primarily on physical effects such as gravity, natural circulation, stored pressure, thermal expansion or negative feedback. Real systems combine both. Passive does not mean “cannot fail,” and active does not mean “unsafe”; they are descriptions of how a function is accomplished.

Natural circulation provides a useful mechanics/thermodynamics connection. Heating reduces fluid density, buoyancy drives upward flow, and cooler denser fluid returns downward. The available driving pressure scale is approximately

$$\Delta P_{\text{buoy}} \sim gH \Delta \rho, \quad (23.103)$$

where H is a vertical height scale and $\Delta \rho$ a density difference. Flow resistance must be lower than the available buoyancy head for sustained circulation.

Guided checks

1. Why does reactor shutdown address reactivity but not automatically solve the heat-removal problem?
2. Why is activity in becquerels not equivalent to absorbed dose in grays or biological effect in sieverts?
3. What is the engineering logic behind several independent barriers rather than one extremely strong barrier?
4. Give one example each of an active and a passive physical process that could contribute to heat removal.

23.16 Fuel cycles, radioactive waste and decommissioning: a physics view

A reactor does not consume an abstract substance called “nuclear fuel.” It uses specific isotopic mixtures embedded in engineered materials. The *fuel cycle* describes the sequence from resource extraction and fuel preparation through irradiation, storage, possible recycling and final waste management. This section stays at the physics level and does not advocate one fuel-cycle policy.

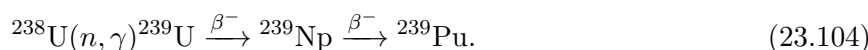
23.16.1 Fresh fuel, irradiated fuel and transmutation

Fresh uranium fuel contains mostly ^{238}U with a smaller fissile fraction. During irradiation, several classes of change occur simultaneously:

- fissile nuclides disappear by fission and non-fission capture;
- ^{238}U can capture a neutron and, after beta decays, produce plutonium isotopes;
- hundreds of fission-product nuclides accumulate and decay;
- structural and cladding materials can become activated by neutron reactions.

Thus spent fuel is physically and isotopically different from fresh fuel even if its macroscopic shape is unchanged.

A simplified fertile-to-fissile chain is



The detailed isotope balance depends on neutron spectrum, flux and irradiation history.

23.16.2 Why waste changes with time

For a single radionuclide,

$$N(t) = N_0 e^{-\lambda t}, \quad A(t) = A_0 e^{-\lambda t}. \quad (23.105)$$

A real waste inventory contains many nuclides. Short-lived fission products can dominate early activity and heat generation, whereas longer-lived fission products and actinides become relatively more important later. There is therefore no single “half-life of nuclear waste.”

23.16.3 Heat generation and storage times

Immediately after discharge, spent fuel produces substantial decay heat. Cooling requirements decrease with time as short-lived nuclides decay. This is why time itself is an engineering resource in radioactive-material management: interim storage allows heat generation and activity from many short-lived species to decrease before later handling stages.

The relationship between half-life and specific activity is worth emphasizing. For a pure radionuclide of molar mass M , the activity per unit mass is

$$\frac{A}{m} = \frac{\ln 2}{T_{1/2}} \frac{N_A}{M}. \quad (23.106)$$

Short half-life generally means high specific activity, while long half-life generally means lower activity per kilogram but longer persistence.

23.16.4 Waste volume, mass, activity and hazard are different quantities

Technical discussions can become confused when one scalar quantity is used as a proxy for all others. Waste *volume* matters for storage and repository engineering; *mass* matters for material handling; *activity* measures decays per second; *decay heat* determines thermal load; and radiological *hazard* additionally depends on radiation type, energy, mobility and exposure pathway. These measures can evolve differently with time.

Caution

Statements such as “this waste remains dangerous for a million years” or “the waste disappears after ten half-lives” are too vague for technical work. Specify the nuclides, quantity of interest, exposure scenario and threshold being discussed.

23.16.5 Decommissioning as an applied-physics problem

At the end of facility life, activation products and surface contamination must be characterized. Radiation mapping, spectroscopy, remote handling, shielding calculations and materials classification are all direct applications of the physics developed in this volume. Decommissioning therefore illustrates that nuclear engineering spans the full lifecycle of materials, not only reactor operation.

23.17 Reactor concepts and neutron spectra: what changes and what stays universal

The same conservation laws and neutron-balance ideas apply across many reactor concepts, but the implementation can differ greatly. A useful classification begins with neutron spectrum.

23.17.1 Thermal-spectrum systems

Thermal reactors slow most fission neutrons before they induce new fissions. This makes the large low-energy fission cross section of nuclides such as ^{235}U especially useful. Thermal systems therefore require a moderator with favourable scattering-to-absorption behaviour. Light-water, heavy-water and graphite-moderated concepts are examples of different material choices.

23.17.2 Fast-spectrum systems

Fast reactors avoid strong moderation and sustain fission primarily with higher-energy neutrons. The neutron cross sections, leakage, breeding possibilities, coolant requirements and materials environment therefore differ from thermal reactors. The physical point is not that one spectrum is universally superior; it is that changing the neutron-energy distribution changes every reaction probability in the neutron economy.

Define a spectrum-weighted one-group cross section by

$$\bar{\sigma} = \frac{\int_0^\infty \sigma(E)\phi(E) dE}{\int_0^\infty \phi(E) dE}. \quad (23.107)$$

Two reactor systems made from the same nuclide can therefore have very different effective reaction rates if their neutron spectra differ.

23.17.3 Coolant choice as coupled physics

Coolant properties affect more than heat transfer. They can influence neutron moderation or absorption, chemical compatibility, corrosion, operating pressure, pumping power and accident response. For example, water has strong thermal and moderating roles; liquid metals can transfer

heat with little neutron moderation; gases can operate at high temperature but have low density and therefore require different pumping arrangements. These are coupled design trade-offs rather than isolated material properties.

For single-phase forced convection, a rough pumping power scale is

$$P_{\text{pump}} \sim \Delta P \dot{V}, \quad (23.108)$$

where ΔP is pressure drop and $\dot{V}$ volumetric flow. Higher flow can improve heat transfer but increases pumping demand. This is one reason whole-plant efficiency depends on auxiliary systems.

23.17.4 Generation labels and technology maturity

Reactor technologies are sometimes grouped into “generations.” These labels summarize broad historical and design families but are not fundamental physics categories. A scientifically useful discussion should instead identify concrete features: neutron spectrum, fuel form, coolant, moderator, power-conversion temperature, passive or active safety functions, fuel-cycle strategy and materials limits.

Key message

A reactor type is a coupled choice of neutron spectrum, fuel, moderator, coolant, geometry, materials and safety architecture. Changing one component often changes several physical balances at once.

23.18 Fusion engineering interfaces beyond the plasma core

A fusion plasma is only the central energy-producing region of a future fusion plant. Surrounding systems must capture heat, protect magnets and structures, maintain fuel inventory, remove impurities and convert thermal energy into electricity.

23.18.1 The first wall and divertor

In magnetic-confinement devices, particles and heat leaving the confined plasma are directed toward plasma-facing components. The divertor is designed to handle a large fraction of exhaust power and remove helium “ash” and impurities. Heat flux rather than total power is often the limiting variable:

$$q'' = \frac{P}{A}. \quad (23.109)$$

A modest area receiving a large power can experience extreme MW/m²-scale heat flux. Materials must simultaneously withstand thermal gradients, sputtering, erosion and neutron damage.

23.18.2 Blanket functions

A D–T fusion blanket can perform several functions at once: absorb neutron energy, breed tritium from lithium, shield superconducting magnets, and carry heat to a power-conversion system. These functions can compete. Increasing neutron absorption may improve shielding but alter tritium breeding; changing materials for thermal performance changes activation and nuclear response.

23.18.3 Magnets and cryogenic power

High-field superconducting magnets reduce electrical resistive losses in the windings but require cryogenic refrigeration. Refrigeration at low temperature has a significant electrical cost because

removing one joule of heat at cryogenic temperature requires more than one joule of work at room temperature. An ideal Carnot refrigerator has coefficient of performance

$$\text{COP}_{\text{Carnot}} = \frac{T_c}{T_h - T_c}, \quad (23.110)$$

where T_c is the cold temperature and T_h the hot reservoir temperature. Real refrigeration is less efficient. Thus magnet technology enters whole-plant power balance through both electromagnetic and thermodynamic constraints.

23.18.4 Vacuum and neutral-particle exhaust

Hot plasma cannot be allowed to exchange freely with ordinary gas. Vacuum systems reduce neutral density, while pumps and cryopumps remove injected fuel, helium ash and impurities. The plasma is therefore an open system with continuous particle and energy throughput, even when its macroscopic state appears steady.

23.18.5 Remote maintenance and activation

Energetic fusion neutrons activate structural materials and can make direct human access impractical during maintenance periods. Remote robotics, modular components and maintainability are therefore not secondary conveniences: they can become fundamental availability constraints. This creates a strong link between nuclear physics, mechanical engineering, robotics and computer vision.

Guided checks

1. Why can heat flux be a more severe engineering constraint than total heat power?
2. Name three distinct functions of a D–T blanket.
3. Why can superconducting magnets still impose a substantial electrical auxiliary load?
4. Why is remote maintenance relevant even if the fusion fuel products are mainly helium and neutrons?

23.19 Main chapter summary

Summary

- Nuclear energy release is governed by differences in total rest energy, conveniently expressed through the reaction Q-value $Q = (m_i - m_f)c^2$.
- The binding-energy-per-nucleon curve explains the broad trend: heavy nuclei can release energy by fission, whereas light nuclei can release energy by fusion.
- A typical ^{235}U fission releases energy of order 200 MeV and several neutrons, enabling a chain reaction.
- The effective multiplication factor k_{eff} classifies neutron balance: subcritical for $k_{\text{eff}} < 1$, critical for $k_{\text{eff}} = 1$, and supercritical for $k_{\text{eff}} > 1$.
- Reactor criticality is a steady neutron-balance condition, not a synonym for explosion. Delayed neutrons and feedback make controlled power operation possible.
- A reactor power system separates nuclear heat generation from heat removal and thermodynamic conversion. Shutdown ends the self-sustaining chain reaction but not radioactive decay heat.
- The Oklo natural reactors demonstrate that criticality can arise from suitable fissile abundance, moderation and geometry in nature.
- Fusion of light nuclei is energetically favourable but suppressed by Coulomb repulsion. Quantum tunnelling and the thermal velocity distribution are essential.
- The Sun is powered mainly by proton–proton fusion, whose weak-interaction bottleneck supports a long stellar lifetime.
- Controlled fusion requires sufficiently favourable density, temperature and energy confinement. Magnetic and inertial confinement pursue different combinations of these quantities.
- Fusion power density depends on $n_1 n_2 \langle \sigma v \rangle Q$, while useful plant power additionally depends on heating, losses, materials, fuel cycle and conversion efficiency.
- Nuclear-energy analysis should separate physical facts from engineering design choices and from broader economic or policy judgments.

23.20 Advanced Topics for Technical and Engineering Students

This section develops a deeper quantitative framework for strong students. It is not required for a first pass through the chapter. The emphasis remains on physical modelling: every advanced equation is accompanied by assumptions, scale estimates and links to measurable quantities.

Pedagogical boundary

The enrichment section introduces standard reactor-physics and fusion-plasma ideas at an educational level. It is not a reactor-design manual, operational procedure or safety analysis. Numerical design work requires evaluated nuclear data, validated multiphysics software, material-property databases and regulatory methods well beyond this course.

23.20.1 The semi-empirical mass formula and the energetic tendency toward fission

The smooth trends of nuclear binding can be described approximately by the semi-empirical mass formula (SEMF), also called the liquid-drop formula. For a nucleus with mass number A and proton number Z , a common form is

$$B(A, Z) = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A, Z). \quad (23.111)$$

The coefficients are fitted to nuclear masses; different parameter sets give slightly different numbers. The terms have transparent physical interpretations:

- $a_v A$: a volume term reflecting the short-range saturation of the nuclear force;
- $-a_s A^{2/3}$: a surface penalty because nucleons at the surface have fewer neighbours;
- $-a_c Z(Z-1)A^{-1/3}$: Coulomb repulsion among protons;
- $-a_a (A-2Z)^2/A$: an asymmetry penalty related to the quantum-state filling of neutrons and protons;
- δ : a pairing correction favouring even–even nuclei and disfavoring odd–odd nuclei.

The SEMF is not a detailed model of fission dynamics, shell effects or barrier penetration, but it explains a key trend. For very heavy nuclei the Coulomb term grows rapidly because many protons repel one another. Splitting a large nucleus reduces the electrostatic self-energy enough that the total binding can increase, even after paying additional surface-energy cost for two fragments.

Theoretical point 23.1: Scaling argument for fission tendency

Imagine, only for scaling, a nucleus split into two equal fragments with the same proton fraction. The original Coulomb term scales roughly as $Z^2/A^{1/3}$. Each fragment has $Z/2$ and $A/2$, so the total Coulomb contribution scales as

$$2 \frac{(Z/2)^2}{(A/2)^{1/3}} = 2^{-2/3} \frac{Z^2}{A^{1/3}} \approx 0.63 \frac{Z^2}{A^{1/3}}. \quad (23.112)$$

Thus splitting can substantially reduce Coulomb repulsion. The surface term moves in the opposite direction: the two fragments have more total surface area. Fission energetics emerge from this competition, modified importantly by shell structure.

A useful dimensionless indicator in liquid-drop discussions is proportional to Z^2/A : nuclei with large Z^2/A have stronger Coulomb stress relative to surface binding. Students should not mistake

this smooth criterion for a prediction of spontaneous-fission half-life, which can vary dramatically because quantum barriers and shell effects matter.

Guided checks

Why does the surface term oppose fission while the Coulomb term tends to favour it for very heavy nuclei? Why can two nuclei with similar macroscopic SEMF energies nevertheless have very different fission lifetimes?

23.20.2 Microscopic cross sections, macroscopic cross sections and mean free path

A microscopic cross section $\sigma(E)$ characterizes the probability of a specified interaction between one incident particle and one target nucleus. Its SI unit is square metre, but nuclear data are commonly quoted in barns,

$$1 \text{ barn} = 10^{-28} \text{ m}^2. \quad (23.113)$$

The cross section is generally a strong function of neutron energy and reaction channel. A material containing n_t target nuclei per unit volume has a macroscopic cross section

$$\Sigma = n_t \sigma, \quad [\Sigma] = \text{m}^{-1}. \quad (23.114)$$

For a uniform material, take $\Sigma = \Sigma_t$ at the specified energy. The probability of travelling a distance x without any interaction is

$$P_{\text{surv}}(x) = e^{-\Sigma x}. \quad (23.115)$$

The mean free path is

$$\ell = \frac{1}{\Sigma}. \quad (23.116)$$

This exponential law resembles radioactive decay mathematically, but the independent variable is distance rather than time. Both arise from a constant hazard probability per infinitesimal interval.

Example 23.18: Mean free path from an assumed cross section

A material has number density $n_t = 5.0 \times 10^{28} \text{ m}^{-3}$. At a specified neutron energy, suppose the total microscopic interaction cross section is 2.0 barn. Find Σ and the mean free path.

Solution.

$$\Sigma = (5.0 \times 10^{28})(2.0 \times 10^{-28}) = 10 \text{ m}^{-1}, \quad (23.117)$$

so

$$\ell = \frac{1}{10} = 0.10 \text{ m}. \quad (23.118)$$

Here the given cross section is taken to be the total interaction cross section at the stated energy. Real neutron transport includes multiple channels and energy changes; absorption alone does not determine the uncollided-beam attenuation if scattering is present.

The scalar neutron flux is $\phi = n_n v$ in a one-speed model, or the angular integral of the angular flux more generally. It sums particle track length per volume per time; it is not the net current through a plane, which can vanish in an isotropic field. For a specified reaction channel, the reaction-rate density for a one-speed model is

$$\mathcal{R} = \Sigma \phi = n_t \sigma \phi. \quad (23.119)$$

This is the reactor-physics analogue of the fusion relation $n_1 n_2 \langle \sigma v \rangle$: both convert interaction probabilities into event rates.

23.20.3 Elastic-scattering moderation: why light nuclei slow neutrons efficiently

Consider an elastic head-on collision of a neutron of mass m with a stationary moderator nucleus of mass $M = Am$. In one dimension, conservation of momentum and kinetic energy gives a final neutron speed

$$v' = \frac{1 - A}{1 + A}v. \quad (23.120)$$

The ratio of final to initial neutron kinetic energy for a head-on collision is therefore

$$\frac{K'}{K} = \left(\frac{A - 1}{A + 1} \right)^2. \quad (23.121)$$

For hydrogen, $A = 1$, so a head-on elastic collision can transfer essentially all the neutron kinetic energy to the proton. For a very heavy nucleus, $A \gg 1$, the fraction remains close to unity and little energy is lost in one collision.

Example 23.19: Head-on energy loss on carbon and hydrogen

Compare the minimum neutron energy fraction after a head-on collision with hydrogen ($A = 1$) and carbon ($A = 12$).

Solution. For hydrogen,

$$(K'/K)_H = 0. \quad (23.122)$$

For carbon,

$$(K'/K)_C = \left(\frac{11}{13} \right)^2 \approx 0.716. \quad (23.123)$$

Thus even the most effective head-on carbon collision leaves about 72% of the neutron energy, whereas hydrogen can remove nearly all of it in one ideal head-on collision. Real collisions have a range of scattering angles, so moderation theory uses angular averages.

A commonly used moderation measure is the average logarithmic energy decrement ξ , which describes the mean reduction in $\ln E$ per collision. For isotropic elastic scattering in the centre-of-mass frame,

$$\xi = 1 + \frac{\alpha \ln \alpha}{1 - \alpha}, \quad \alpha = \left(\frac{A - 1}{A + 1} \right)^2. \quad (23.124)$$

The result follows by averaging $\ln(E/E')$ over a uniform distribution of $u = E'/E$ on $[\alpha, 1]$:

$$\xi = \frac{1}{1 - \alpha} \int_{\alpha}^1 -\ln u \, du. \quad (23.125)$$

Here isotropy in the centre-of-mass frame and a stationary target are assumed. The hydrogen limit $\alpha \rightarrow 0$ gives $\xi = 1$ because $\alpha \ln \alpha \rightarrow 0$. The heavy-target limit gives $\xi \simeq 2/A$. Near thermal equilibrium, target motion can also upscatter neutrons, so the one-way slowing-down approximation eventually fails. The approximate number of collisions needed to reduce energy from E_0 to E is

$$N_{\text{coll}} \approx \frac{\ln(E_0/E)}{\xi}. \quad (23.126)$$

This logarithmic viewpoint is useful because neutron energies span many orders of magnitude.

23.20.4 The four-factor formula: a structured neutron-accounting model

For an ideal infinite thermal reactor with no neutron leakage, a classic decomposition of the multiplication factor is

$$k_{\infty} = \eta \epsilon p f. \quad (23.127)$$

The symbols represent physically distinct stages of the neutron life cycle:

- η : reproduction factor, the number of fast neutrons produced per thermal neutron absorbed in fuel;
- ϵ : fast-fission factor, accounting for additional fissions before complete moderation;
- p : resonance-escape probability, the fraction of neutrons that slow to thermal energies without being lost to resonance absorption;
- f : thermal-utilization factor, the fraction of thermal-neutron absorptions that occur in fuel rather than other materials.

For a finite system, leakage reduces multiplication. One writes schematically

$$k_{\text{eff}} = k_{\infty} P_f P_t, \quad (23.128)$$

where P_f and P_t are fast and thermal non-leakage probabilities. The resulting product is often called the six-factor formula.

This decomposition is useful pedagogically because it turns a single number k_{eff} into a diagnostic chain. Increasing moderation, for example, may improve one factor while worsening another through absorption or geometry. Reactor design is therefore an optimization problem with coupled effects rather than a search for “maximum moderation.”

Example 23.20: Sensitivity of multiplication to one factor

Suppose an illustrative model has $\eta = 1.85$, $\epsilon = 1.03$, $p = 0.88$, $f = 0.70$, $P_f = 0.97$ and $P_t = 0.96$. Estimate k_{eff} . Then increase p by 1% relative while keeping all other factors fixed.

Solution.

$$k_{\text{eff}} = (1.85)(1.03)(0.88)(0.70)(0.97)(0.96) \quad (23.129)$$

$$\approx 1.09. \quad (23.130)$$

A 1% relative increase in p produces a 1% relative increase in the product, so k_{eff} becomes approximately 1.10. Real factors are not independent, so this one-at-a-time sensitivity is only a local mathematical exercise.

23.20.5 Diffusion intuition and critical size

Neutron transport is fundamentally an integro-differential problem in position, direction, energy and time. Under suitable conditions, repeated scattering makes the angular distribution nearly isotropic and a diffusion approximation becomes useful. A one-group steady diffusion equation can be written schematically as

$$-D\nabla^2\phi + \Sigma_a\phi = \frac{\nu\Sigma_f}{k_{\text{eff}}}\phi, \quad (23.131)$$

where D is a diffusion coefficient, Σ_a is the macroscopic absorption cross section, $\nu\Sigma_f\phi$ represents neutron production by fission, and ϕ is the scalar neutron flux.

The Laplacian term describes leakage driven by spatial gradients. This immediately explains why geometry matters: a small system has a high surface-to-volume ratio and therefore large leakage relative to production. Increasing size can reduce the fractional leakage and raise k_{eff} .

For a bare homogeneous system, separation of variables leads to a geometric buckling B_g^2 determined by shape and dimensions. Criticality is reached when the material's ability to support neutron multiplication matches the leakage associated with that geometry. The exact formulas depend on boundary conditions and extrapolation lengths; the important course-level lesson is the scaling:

$$\text{leakage strength} \sim \frac{D}{L^2}, \quad (23.132)$$

where L is a characteristic size. Larger systems leak a smaller fraction of their neutrons.

Guided checks

Why does the leakage term contain spatial derivatives rather than simply being a constant loss? What qualitative change occurs if every linear dimension of a similar reactor core is doubled while material properties remain fixed?

Diffusion length and a critical-size scaling

Consider the one-group diffusion equation in a homogeneous multiplying slab,

$$-D \frac{d^2 \phi}{dx^2} + \Sigma_a \phi = \frac{\nu \Sigma_f}{k} \phi, \quad (23.133)$$

with idealized zero-flux boundaries at $x = \pm a$. The even fundamental mode is

$$\phi(x) = \phi_0 \cos\left(\frac{\pi x}{2a}\right). \quad (23.134)$$

Since

$$-\frac{1}{\phi} \frac{d^2 \phi}{dx^2} = \left(\frac{\pi}{2a}\right)^2, \quad (23.135)$$

substitution yields

$$k(a) = \frac{\nu \Sigma_f}{\Sigma_a + D(\pi/2a)^2}. \quad (23.136)$$

The denominator contains absorption plus an effective leakage term. Criticality $k = 1$ occurs when

$$D \left(\frac{\pi}{2a_c}\right)^2 = \nu \Sigma_f - \Sigma_a, \quad (23.137)$$

so

$$a_c = \frac{\pi}{2} \sqrt{\frac{D}{\nu \Sigma_f - \Sigma_a}}. \quad (23.138)$$

This formula is not a design equation for a real reactor: the boundary condition is idealized, energy dependence is collapsed into one group, and geometry is one-dimensional. It nevertheless makes the competing scales explicit.

Define a diffusion length

$$L = \sqrt{\frac{D}{\Sigma_a}}. \quad (23.139)$$

It measures how far a neutron population spreads by diffusion before absorption in a nonmultiplying medium. The critical-size result shows the same square-root structure: distance scales emerge from a transport coefficient divided by a loss or production-rate coefficient.

Example 23.21: Critical half-width in a toy one-group slab

Take $D = 1.2 \text{ cm}$, $\nu \Sigma_f = 0.080 \text{ cm}^{-1}$, and $\Sigma_a = 0.060 \text{ cm}^{-1}$. Estimate the critical half-width in the idealized slab model.

Solution.

$$a_c = \frac{\pi}{2} \sqrt{\frac{1.2}{0.020}} \text{ cm} = \frac{\pi}{2} \sqrt{60} \text{ cm} \approx 12.2 \text{ cm}. \quad (23.140)$$

A real critical dimension would differ because of extrapolated boundaries, energy groups, heterogeneity, anisotropic scattering and three-dimensional geometry.

23.20.6 Point kinetics and the special role of delayed neutrons

The deterministic generation model $N_{g+1} = kN_g$ ignores time. A minimal time-dependent reactor model treats the neutron population as spatially averaged and introduces delayed-neutron precursor populations. With one effective precursor group,

$$\frac{dn}{dt} = \frac{\rho - \beta}{\Lambda} n + \lambda C + S, \quad (23.141)$$

$$\frac{dC}{dt} = \frac{\beta}{\Lambda} n - \lambda C. \quad (23.142)$$

Here n is proportional to neutron population or power (with a consistent normalization), ρ is reactivity, β is the effective delayed-neutron fraction, Λ is a prompt-neutron generation time, C is the effective precursor population in the same normalization, λ is its decay constant and S represents an external source.

The striking feature is the combination $\rho - \beta$. When $0 < \rho < \beta$, the system is supercritical overall but still depends on delayed neutrons for its growth timescale. This is the regime in which mechanical control systems can act. At $\rho = \beta$ it is *prompt critical*; for $\rho > \beta$ it is *prompt supercritical* in this point model and the characteristic timescale would become dramatically shorter.

Caution

The prompt-critical concept is introduced here to clarify reactor kinetics, not to provide an operational threshold for a real facility. Values of β , Λ and feedback coefficients depend on fuel, spectrum and reactor state and must come from validated analysis.

At exact critical steady state with $S = 0$ and constant n , Eqs. (23.141)–(23.142) give

$$C_{\text{eq}} = \frac{\beta}{\lambda \Lambda} n. \quad (23.143)$$

This precursor inventory is why the delayed-neutron contribution does not disappear immediately after a control change.

Theoretical point 23.2: Deriving the one-group reactor period

For constant coefficients and $S = 0$, assume $n, C \propto e^{\omega t}$. The precursor equation gives $C/n = \beta/[\Lambda(\omega + \lambda)]$. Substituting into the neutron equation yields

$$\rho = \Lambda\omega + \frac{\beta\omega}{\omega + \lambda}, \quad (23.144)$$

or

$$\omega^2 + \left[\lambda + \frac{\beta - \rho}{\Lambda} \right] \omega - \frac{\lambda\rho}{\Lambda} = 0. \quad (23.145)$$

At $\rho = 0$ one mode is steady and the other decays. For small positive reactivity with $\omega \ll \lambda$, $\omega \simeq \rho/(\Lambda + \beta/\lambda)$. The delayed contribution β/λ can greatly exceed Λ , explaining the long growth timescale. This approximation fails as growth becomes fast. A growing mode has period $T_{\text{per}} = 1/\omega$; it is an e-folding time, not a periodic oscillation.

23.20.7 Temperature feedback and coupled neutron–thermal dynamics

A simple multiphysics model couples neutron power P to a lumped temperature T . One may write, schematically,

$$\rho(T) = \rho_0 + \alpha_T(T - T_0), \quad (23.146)$$

$$C_T \frac{dT}{dt} = P - H(T - T_c), \quad (23.147)$$

where C_T is an effective heat capacity, H a heat-transfer coefficient and T_c a coolant or sink temperature. If $\alpha_T < 0$, increasing temperature inserts negative reactivity, creating a stabilizing feedback loop.

Linear stability analysis around an operating point is a standard engineering tool. If precursor dynamics have been consistently eliminated in a reduced model, perturbations δn and δT produce a coupled system

$$\frac{d}{dt} \begin{pmatrix} \delta n \\ \delta T \end{pmatrix} = \mathbf{A} \begin{pmatrix} \delta n \\ \delta T \end{pmatrix}. \quad (23.148)$$

With the one-group point-kinetics model retained, δC must be included and the matrix is at least 3×3 . The real parts of its eigenvalues determine growth or decay; imaginary parts describe oscillation. A negative temperature coefficient alone is not proof of stability when transport delays and other feedbacks are present. This connects reactor physics directly to control theory and dynamical-systems analysis.

23.20.8 Decay heat as a superposition of many radioactive chains

The main chapter used one exponential only as a toy model. A more realistic mathematical structure is

$$P_d(t) = \sum_i E_i \lambda_i N_i(t), \quad (23.149)$$

where $N_i(t)$ are populations of many radioactive fission products and daughters and E_i is the mean *deposited* energy per decay, excluding escaping neutrino energy. Coupled decay chains obey Bateman equations. For a simple parent–daughter chain,

$$\frac{dN_1}{dt} = -\lambda_1 N_1, \quad (23.150)$$

$$\frac{dN_2}{dt} = b\lambda_1 N_1 - \lambda_2 N_2, \quad (23.151)$$

with branching fraction b . For $N_2(0) = 0$ and $\lambda_1 \neq \lambda_2$,

$$N_2(t) = \frac{b\lambda_1 N_1(0)}{\lambda_2 - \lambda_1} (e^{-\lambda_1 t} - e^{-\lambda_2 t}). \quad (23.152)$$

For equal decay constants the continuous limit is $N_2(t) = b\lambda_1 N_1(0)t e^{-\lambda_1 t}$. A daughter inventory can initially rise even though its parent decreases. The daughter solution contains two exponentials. Large fission-product inventories contain many such contributions, explaining why total decay heat approximately follows a slowly changing power-law-like envelope over broad times rather than one fixed half-life.

23.20.9 Neutron activation and cooldown curves

Suppose a stable target nuclide is activated by neutron capture at production rate $R_p = \phi\sigma N_T$, producing radioactive nuclei N^* with decay constant λ . Assume negligible depletion of target N_T , negligible capture of the activated product, constant flux and one radioactive product. During constant irradiation,

$$\frac{dN^*}{dt} = R_p - \lambda N^*. \quad (23.153)$$

For $N^*(0) = 0$,

$$N^*(t) = \frac{R_p}{\lambda} (1 - e^{-\lambda t}), \quad (23.154)$$

and the activity is

$$A(t) = R_p (1 - e^{-\lambda t}). \quad (23.155)$$

After a long irradiation, activity approaches the production rate: this is *saturation activity*. After irradiation stops at $t = t_i$, activity decays as

$$A(t) = A(t_i)e^{-\lambda(t-t_i)}. \quad (23.156)$$

Activation, irradiation and cooldown scheduling

For constant production rate R_p , activation during irradiation for time t_i gives end-of-irradiation activity

$$A_{\text{EOI}} = R_p(1 - e^{-\lambda t_i}). \quad (23.157)$$

After a cooldown t_c ,

$$A_{\text{cool}} = A_{\text{EOI}}e^{-\lambda t_c} = R_p(1 - e^{-\lambda t_i})e^{-\lambda t_c}. \quad (23.158)$$

This expression reveals two saturation effects. Irradiating much longer than several half-lives barely increases activity because decay balances production. Cooling for several half-lives strongly reduces activity for that nuclide.

Example 23.22: Irradiation followed by cooldown

A component produces a radionuclide at $R_p = 5.0 \times 10^7 \text{ s}^{-1}$ during a 30-day irradiation. The nuclide half-life is 10 days. Find the activity at shutdown and after a further 20-day cooldown.

Solution. Thirty days is three half-lives, so

$$A_{\text{EOI}} = R_p(1 - 2^{-3}) = 0.875R_p = 4.38 \times 10^7 \text{ Bq}. \quad (23.159)$$

Twenty days is two more half-lives, giving

$$A_{\text{cool}} = A_{\text{EOI}}2^{-2} \approx 1.09 \times 10^7 \text{ Bq}. \quad (23.160)$$

A real component contains many activation products with different half-lives, so the total cooldown curve is a sum of such terms.

23.20.10 Fuel evolution, burnup and neutron capture

Fuel composition changes continuously under irradiation. Fissile nuclei disappear by fission and capture; fertile nuclei can capture neutrons and later transform into fissile species; fission products accumulate; some fission products strongly absorb neutrons. A minimal nuclide-depletion equation has the form

$$\frac{dN_i}{dt} = - \left(\lambda_i + \phi \sum_r \sigma_{i,r} \right) N_i + \sum_j (b_{j \rightarrow i} \lambda_j + \phi \sigma_{j \rightarrow i}) N_j. \quad (23.161)$$

Here N_i must all denote populations in a fixed volume, or all denote number densities; the production coefficients include decay branches and reaction yields, including fission yields where needed. This is a large linear ODE system when flux and cross sections are fixed, and a nonlinear multiphysics problem when they depend on the evolving reactor state.

The engineering quantity *burnup* measures energy extracted per unit initial fuel mass, commonly in MW d/kg or GW d/t. It should not be confused with the fraction of atoms that have literally burned. It is an energy-per-initial-mass bookkeeping variable.

Example 23.23: Burnup unit conversion

A fuel sample reaches 45 GW d/t. Convert this to joules per kilogram.

Solution.

$$45 \text{ GW d/t} = 45 \times 10^9 \text{ W} \times 86400 \text{ s} / (1000 \text{ kg}) \quad (23.162)$$

$$\approx 3.89 \times 10^{12} \text{ J/kg}. \quad (23.163)$$

This number is the thermal energy attributed per kilogram of initial heavy-metal fuel, not an assertion that every nucleus in that kilogram underwent fission.

23.20.11 Xenon poisoning as an example of isotope–kinetics feedback

A classic reactor-kinetics example is ^{135}Xe , which has an unusually large thermal-neutron absorption cross section. It is produced both directly in fission and through decay of precursors such as ^{135}I . A schematic pair of equations is

$$\frac{dI}{dt} = Y_I \Sigma_f \phi - \lambda_I I, \quad (23.164)$$

$$\frac{dX}{dt} = Y_X \Sigma_f \phi + \lambda_I I - (\lambda_X + \sigma_{a,X} \phi) X. \quad (23.165)$$

Here I and X are number densities, Y_I is the yield feeding iodine (including its rapid precursors), and Y_X is the direct xenon yield. When power changes, xenon concentration does not respond instantaneously because the iodine precursor inventory persists. The result can be a delayed reactivity transient.

This is a useful systems lesson: a reactor contains state variables with very different time constants. Some respond on neutron-generation times, others on thermal times, and others on radioactive-decay times. Control strategies must account for this hierarchy.

23.20.12 Radiation shielding: attenuation, buildup and secondary radiation

For a narrow monoenergetic photon beam in a uniform material, the uncollided intensity follows

$$I(x) = I_0 e^{-\mu x}, \quad (23.166)$$

where μ is the linear attenuation coefficient. The half-value layer is

$$x_{1/2} = \frac{\ln 2}{\mu}. \quad (23.167)$$

Real shielding calculations are more complex because scattered photons contribute a buildup component, neutrons change energy by scattering and can generate capture gamma rays, and material activation may create delayed sources.

Example 23.24: Half-value-layer estimate

For a specified photon energy, assume $\mu = 0.50 \text{ cm}^{-1}$. Find the half-value layer and the uncollided fraction after 10 cm.

Solution.

$$x_{1/2} = \frac{0.693}{0.50} \approx 1.39 \text{ cm}. \quad (23.168)$$

Also,

$$I/I_0 = e^{-5} \approx 6.7 \times 10^{-3}. \quad (23.169)$$

This is a narrow-beam attenuation calculation; it is not a complete shielding design because scattered radiation and secondary sources are omitted.

23.20.13 Radiation measurements: count rates, dead time and uncertainty

At low rates, Poisson statistics often dominate counting uncertainty. At high rates, detector and electronics dead time can bias the measured count rate. For a simple nonparalyzable model with dead time τ_d , true rate R and observed rate r ,

$$r = \frac{R}{1 + R\tau_d}, \quad R = \frac{r}{1 - r\tau_d}. \quad (23.170)$$

The correction becomes unstable as $r\tau_d \rightarrow 1$, showing why a detector should not be operated arbitrarily close to saturation.

Example 23.25: Dead-time correction

A counter with $\tau_d = 10 \mu\text{s}$ records $r = 2.0 \times 10^4 \text{ s}^{-1}$. Estimate the true rate in the nonparalyzable model.

Solution.

$$r\tau_d = (2.0 \times 10^4)(10^{-5}) = 0.20, \quad (23.171)$$

so

$$R = \frac{2.0 \times 10^4}{0.80} = 2.5 \times 10^4 \text{ s}^{-1}. \quad (23.172)$$

Different detector systems can follow different dead-time models, so the correction model itself must be validated experimentally.

23.20.14 Thermal hydraulics: from nuclear heat source to temperature fields

The local volumetric heat-generation rate in fuel may be denoted q''' in W/m^3 . Heat conduction in a stationary solid with thermal conductivity k_t obeys

$$\rho_m c_p \frac{\partial T}{\partial t} = \nabla \cdot (k_t \nabla T) + q'''. \quad (23.173)$$

In steady one-dimensional planar geometry with constant k_t ,

$$k_t \frac{d^2 T}{dx^2} + q''' = 0. \quad (23.174)$$

For a slab of half-thickness L with symmetry at $x = 0$ and surface temperature T_s , the solution is

$$T(x) = T_s + \frac{q'''}{2k_t}(L^2 - x^2). \quad (23.175)$$

The centreline temperature rise is therefore

$$T(0) - T_s = \frac{q''' L^2}{2k_t}. \quad (23.176)$$

This simple result shows why power density, thermal conductivity and length scale matter strongly to fuel temperature.

Example 23.26: Temperature rise across a heat-generating slab

A teaching model has $q''' = 2.0 \times 10^8 \text{ W/m}^3$, $L = 4.0 \text{ mm}$, and $k_t = 3.0 \text{ W/(m K)}$. Estimate the centre-to-surface temperature difference.

Solution.

$$\Delta T = \frac{(2.0 \times 10^8)(4.0 \times 10^{-3})^2}{2(3.0)} \approx 5.3 \times 10^2 \text{ K.} \quad (23.177)$$

The large rise emphasizes that thermal conductivity and geometry can limit allowable volumetric power. Real cylindrical fuel has temperature-dependent properties, gap conduction and convective boundary conditions.

Convective heat transfer to coolant is often represented by

$$q'' = h(T_w - T_b), \quad (23.178)$$

where h is a heat-transfer coefficient, T_w a wall temperature and T_b a bulk-fluid temperature. Determining h is a fluid-mechanics problem involving Reynolds, Prandtl and Nusselt numbers.

23.20.15 Fusion cross sections, the astrophysical S-factor and the Gamow picture

At nonrelativistic energies, a charged-particle fusion cross section is often written in the form

$$\sigma(E) = \frac{S(E)}{E} \exp[-2\pi\eta(E)], \quad (23.179)$$

where $S(E)$ is the astrophysical S-factor and

$$\eta(E) = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 \hbar v} = Z_1 Z_2 \alpha \frac{c}{v} \quad (23.180)$$

for relative speed v . The exponential factor represents Coulomb tunnelling in a simple semiclassical treatment. The S-factor removes much of the rapidly varying penetration factor so that the remaining nuclear-structure dependence may vary more slowly with energy, except near resonances.

A thermal plasma supplies a Maxwellian distribution that decreases roughly as $e^{-E/k_B T}$, while tunnelling becomes easier as energy increases. Their product produces a finite energy window—the *Gamow window*—in which most fusion reactions occur.

For a nonresonant reaction, the relevant exponential is approximately

$$\exp \left[-\frac{E}{k_B T} - \sqrt{\frac{E_G}{E}} \right], \quad (23.181)$$

where

$$E_G = 2\mu c^2 (\pi\alpha Z_1 Z_2)^2. \quad (23.182)$$

Here α is the fine-structure constant and $E = \mu v^2/2$ is centre-of-mass relative energy. The nonresonant, slowly varying $S(E)$ approximation is useful for stellar reasoning; it is not a quantitative fit for the resonant D–T reaction. The maximum occurs at

$$E_0 = \left(\frac{E_G (k_B T)^2}{4} \right)^{1/3}. \quad (23.183)$$

Thus reacting particles typically come from an energy range above the simple mean thermal energy but still far below the classical Coulomb-barrier height.

Figure 23.12 plots a defined toy example whose peak can be checked analytically.

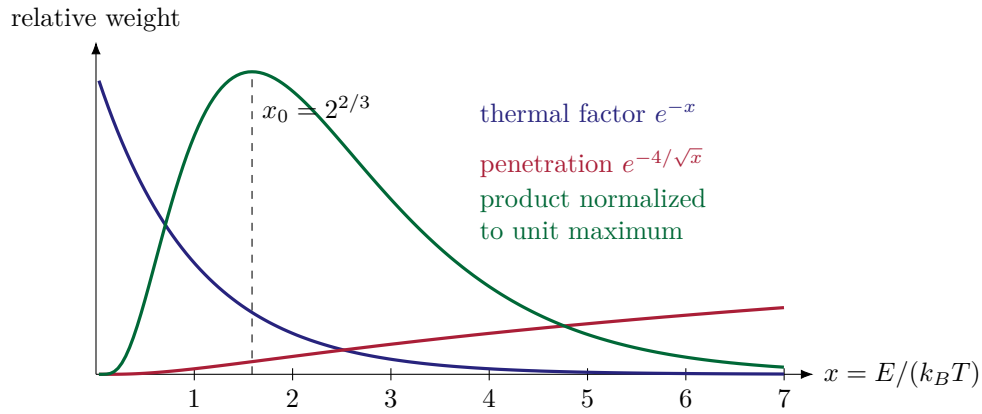


Figure 23.12: Analytical illustration with $E_G/(k_B T) = 16$ and constant $S(E)$. The maximum of $\exp[-x - 4/\sqrt{x}]$ occurs at $x_0 = 2^{2/3} \simeq 1.5874$. The green product alone is rescaled to unit peak; this is a specified toy model, not measured D–T reactivity.

Key message

The Gamow picture resolves an apparent paradox: stellar temperatures are too low for most nuclei to classically overcome the Coulomb barrier, yet the combination of a thermal high-energy tail and quantum tunnelling gives a finite reaction rate.

Locating the gamow peak

For a nonresonant charged-particle reaction, focus on the dominant exponential factor

$$F(E) = \exp[-f(E)], \quad f(E) = \frac{E}{k_B T} + \sqrt{\frac{E_G}{E}}. \quad (23.184)$$

The peak occurs where $f(E)$ is minimized. Differentiate:

$$\frac{df}{dE} = \frac{1}{k_B T} - \frac{1}{2} \sqrt{E_G} E^{-3/2}. \quad (23.185)$$

Setting $df/dE = 0$ gives

$$E_0^{3/2} = \frac{1}{2} \sqrt{E_G} k_B T, \quad (23.186)$$

therefore

$$E_0 = \left(\frac{E_G (k_B T)^2}{4} \right)^{1/3}, \quad (23.187)$$

which reproduces Eq. (23.183).

The width of the effective energy window can be estimated by expanding $f(E)$ to second order around E_0 . Writing

$$f(E) \approx f(E_0) + \frac{1}{2} f''(E_0) (E - E_0)^2, \quad (23.188)$$

turns the integrand locally into a Gaussian. The resulting characteristic width scales approximately as

$$\Delta E_{1/e} = 4 \sqrt{E_0 k_B T / 3}. \quad (23.189)$$

This is the full width between the points where the local Gaussian falls to $1/e$ of its maximum, since $f''(E_0) = 3/(2E_0 k_B T)$. It is an asymptotic estimate, reliable when the window is narrow enough for the quadratic expansion.

This saddle-point method is widely useful in physics: identify a rapidly varying exponential, locate the dominant point and expand around it. Similar reasoning appears in statistical mechanics, optics and quantum field theory.

23.20.16 Maxwellian-averaged fusion reactivity

For two Maxwellian species at the same temperature, the velocity-averaged reactivity is

$$\langle \sigma v \rangle = \left(\frac{8}{\pi \mu} \right)^{1/2} \frac{1}{(k_B T)^{3/2}} \int_0^\infty \sigma(E) E e^{-E/k_B T} dE, \quad (23.190)$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. Equation (23.190) is a bridge between microscopic nuclear measurements and macroscopic plasma power.

Several important modelling lessons follow:

1. The reactivity is not simply $\sigma(k_B T) \times$ a thermal speed. It is an integral over the distribution.
2. Resonances in $\sigma(E)$ can strongly alter the temperature dependence.
3. Non-Maxwellian plasma distributions require a more general velocity-space average.
4. Tabulated reactivities are normally used in engineering calculations rather than recomputing Eq. (23.190) from raw cross-section data each time.

Method

When using a tabulated $\langle \sigma v \rangle$, always record the fuel reaction, temperature unit, source of the fit or table, and whether the distribution is Maxwellian. A numerical value without those qualifiers is not reproducible physics.

23.20.17 A simplified Lawson criterion from power balance

Consider an equal D–T plasma with total ion density n , ion and electron temperatures comparable, and a thermal energy density approximated as

$$w \approx 3nk_B T. \quad (23.191)$$

For a fully ionized D–T mixture, quasineutrality gives $n_e = n_D + n_T = n$. With $T_i = T_e = T$, ions contribute $3nk_B T/2$ and electrons contribute the same amount. The thermal energy excludes magnetic-field energy and the energy of nonthermal fast particles. For a uniform plasma, define the loss power density by

$$p_{\text{loss}} = \frac{w}{\tau_E}. \quad (23.192)$$

For D–T fusion, the reaction-rate density is Eq. (23.82). The alpha particle carries $E_\alpha \approx 3.5$ MeV, while the neutron carries about 14.1 MeV. In a magnetically confined plasma, the charged alpha can deposit its energy directly in the plasma, whereas the neutron largely escapes the magnetic confinement region and deposits energy in surrounding structures.

At idealized self-heating or ignition, alpha heating balances losses:

$$\frac{n^2}{4} \langle \sigma v \rangle E_\alpha \gtrsim \frac{3nk_B T}{\tau_E}. \quad (23.193)$$

Rearranging gives

$$n\tau_E \gtrsim \frac{12k_B T}{\langle \sigma v \rangle E_\alpha}. \quad (23.194)$$

The right-hand side depends on temperature through both T and $\langle \sigma v \rangle$. This is why there is an optimum temperature range and why one fixed $n\tau_E$ number is incomplete.

Multiplying by T motivates the triple product

$$nT\tau_E, \quad (23.195)$$

which is a common experimental performance metric. Different communities may quote temperature in kelvin or energy units such as keV, so units must always accompany the number.

Guided checks

If $\langle\sigma v\rangle$ were independent of temperature, how would the simplified Lawson requirement in Eq. (23.194) scale with T ? Why is that hypothetical scaling not the real D–T behaviour?

Example 23.27: An explicit D–T self-heating estimate

For equal D and T densities, take $k_B T = 10$ keV, $\langle\sigma v\rangle = 1.0 \times 10^{-22} \text{ m}^3/\text{s}$ and $E_\alpha = 3.5$ MeV. Estimate the ideal ignition requirement and the corresponding τ_E at $n = 10^{20} \text{ m}^{-3}$.

Strategy. Use Eq. (23.194); the energy ratio can be evaluated in the same units:

$$n\tau_E \gtrsim \frac{12(0.010)}{(10^{-22})(3.5)} = 3.43 \times 10^{20} \text{ m}^{-3}\text{s}. \quad (23.196)$$

Thus $\tau_E \gtrsim 3.4$ s under this model. The right-hand side has units $(\text{m}^3/\text{s})^{-1}$. It assumes complete alpha-energy deposition, a uniform Maxwellian plasma with equal ion/electron temperatures, and no separately omitted loss. It is not a specification for a particular experiment.

If radiation is treated separately from transport loss, write

$$p_\alpha + p_{\text{ext}} = w/\tau_{\text{tr}} + p_{\text{rad}}; \quad (23.197)$$

do not then include the same radiation again in a confinement time based on total loss. With total losses included in τ_E , complete alpha deposition and finite plasma gain, the threshold becomes

$$n\tau_E \gtrsim \frac{12k_B T}{\langle\sigma v\rangle(E_\alpha + Q_{DT}/Q_{\text{plasma}})}. \quad (23.198)$$

Ignition is the ideal limit $p_{\text{ext}} \rightarrow 0$ or $Q_{\text{plasma}} \rightarrow \infty$; plasma breakeven $Q_{\text{plasma}} = 1$ is a different condition. Nonuniform profiles, helium ash, impurities and incomplete alpha deposition change the balance.

23.20.18 Charged-particle motion in a fusion plasma

The Lorentz force

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (23.199)$$

produces helical motion around magnetic-field lines in a uniform $\mathbf{B}$. The cyclotron angular frequency is

$$\Omega_c = \frac{|q|B}{m}, \quad (23.200)$$

and the Larmor radius is Eq. (23.90). For thermal particles, a characteristic perpendicular speed is of order

$$v_\perp \sim \sqrt{\frac{2k_B T}{m}}, \quad (23.201)$$

so

$$r_L \sim \frac{\sqrt{2mk_B T}}{|q|B}. \quad (23.202)$$

Thus stronger magnetic field decreases orbit size, while higher temperature increases it.

Example 23.28: Thermal deuteron Larmor radius

Estimate the Larmor radius of a deuteron with perpendicular kinetic energy 10 keV in a 5.0 T magnetic field. Take $m_D = 3.34 \times 10^{-27}$ kg and $|q| = e$.

Solution. From $K_{\perp} = \frac{1}{2}mv_{\perp}^2$,

$$v_{\perp} = \sqrt{\frac{2(10^4 \text{ eV})(1.602 \times 10^{-19} \text{ J/eV})}{3.34 \times 10^{-27}}} \approx 9.8 \times 10^5 \text{ m/s.} \quad (23.203)$$

Then

$$r_L = \frac{(3.34 \times 10^{-27})(9.8 \times 10^5)}{(1.602 \times 10^{-19})(5.0)} \approx 4.1 \times 10^{-3} \text{ m.} \quad (23.204)$$

A millimetre-scale gyroradius is much smaller than metre-scale device dimensions, but it does not guarantee perfect confinement because drifts, collisions and turbulence also transport particles.

Magnetic drifts and why a torus is not automatically confining

In a nonuniform or curved magnetic field, gyro-orbits develop guiding-centre drifts. For example, a perpendicular gradient in field strength produces a grad- B drift with schematic magnitude

$$v_{\nabla B} \sim \frac{mv_{\perp}^2}{2|q|B^3} |\mathbf{B} \times \nabla B|. \quad (23.205)$$

Curvature of magnetic field lines produces an additional drift. Ions and electrons drift in opposite directions because their charges have opposite signs. In a simple toroidal field these drifts would lead to charge separation and loss.

Tokamaks and stellarators mitigate this problem by using twisted field-line geometry so that particles sample regions above and below the midplane and the secular drifts are greatly reduced. This is a conceptual reason why magnetic-confinement devices require more than a doughnut-shaped solenoid.

Note

The guiding-centre formulas assume that field variation over one gyro-orbit is small. This requires a separation of scales $r_L \ll L_B$, where L_B is the characteristic magnetic-field variation length.

23.20.19 Plasma beta and pressure balance

Magnetic confinement must balance plasma pressure. A useful dimensionless parameter is plasma beta,

$$\beta_{\text{pl}} = \frac{p}{B^2/(2\mu_0)}, \quad (23.206)$$

where p is plasma pressure and $B^2/(2\mu_0)$ is magnetic energy density or magnetic pressure. A higher beta means more plasma pressure for a given magnetic field, which can be favourable for compact power density, but stability limits constrain how large beta can be in a particular configuration.

For a fully ionized hydrogenic plasma with comparable ion and electron temperatures,

$$p \approx (n_i + n_e)k_B T \approx 2n_i k_B T. \quad (23.207)$$

Thus high density and high temperature both increase the pressure the magnetic field must confine.

Example 23.29: Magnetic pressure scale

Calculate $B^2/(2\mu_0)$ for $B = 5.0$ T.

Solution.

$$\frac{B^2}{2\mu_0} = \frac{25}{2(4\pi \times 10^{-7})} \approx 9.95 \times 10^6 \text{ Pa} \approx 10 \text{ MPa.} \quad (23.208)$$

This is an energy-density/pressure scale associated with the field. The mechanical stresses in actual magnet structures require detailed electromagnetic and structural analysis.

Energy confinement, particle confinement and transport

Three timescales should be distinguished:

- **particle confinement time:** how long particles remain before being lost or replaced;
- **energy confinement time τ_E :** stored thermal energy divided by power loss;
- **current-relaxation or resistive times:** timescales associated with magnetic-field diffusion and plasma current evolution.

They are not generally equal.

Classical collisional transport alone often predicts better confinement than experiments observe. Turbulence driven by gradients can dominate radial heat and particle transport. Magnetic-confinement studies therefore use diagnostics and large-scale numerical simulations to connect microscopic fluctuations with macroscopic confinement.

A generic diffusion model for temperature is

$$\frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \chi \frac{\partial T}{\partial r} \right) + S_T, \quad (23.209)$$

where χ is an effective thermal diffusivity and S_T represents heating and losses in a simplified cylindrical geometry. In real plasmas, χ is not a universal material constant; it can depend on profiles and turbulence.

23.20.20 Bremsstrahlung losses and why fuel charge matters

Electrons accelerated in Coulomb fields radiate bremsstrahlung. In an optically thin fully ionized plasma, a rough nonrelativistic scaling is

$$p_{\text{br}} \propto n_e \sum_j n_j Z_j^2 \sqrt{T_e} = Z_{\text{eff}}^2 n_e^2 \sqrt{T_e}, \quad (23.210)$$

where $Z_{\text{eff}} = \sum_j n_j Z_j^2 / n_e$ and quasineutrality gives $n_e = \sum_j Z_j n_j$. The omitted coefficient contains physical constants and a Gaunt factor. For one fully ionized ion species the equivalent form is $p_{\text{br}} \propto Z^2 n_e n_i \sqrt{T_e}$; one must not substitute Z_{eff}^2 into the mixture formula. Fusion power scales with reactant densities and $\langle \sigma v \rangle$. High- Z impurities can therefore be particularly damaging: they increase radiative losses strongly while diluting fusion fuel.

This is one reason plasma-facing material erosion matters to core performance. If high- Z wall atoms enter the hot plasma, even a small concentration can significantly increase radiation losses. Divertor and edge-plasma physics therefore couple directly to the central fusion power balance.

For advanced fuels such as proton–boron reactions, the larger nuclear charge makes the Coulomb barrier higher and bremsstrahlung losses more severe. A high Q -value alone does not determine fuel attractiveness.

23.20.21 Fusion gain: several definitions answer different questions

The word *gain* is potentially ambiguous. Define a plasma physics gain

$$Q_{\text{plasma}} = \frac{P_{\text{fusion}}}{P_{\text{ext,plasma}}}, \quad (23.211)$$

where $P_{\text{ext,plasma}}$ is externally supplied heating delivered to the plasma. A plant-level engineering gain might instead compare electrical output with total electrical input to magnets, cryogenics, lasers, pumps, heating systems and other auxiliaries.

If a driver has wall-plug efficiency η_d , then providing plasma heating P_h requires at least

$$P_{\text{driver,e}} = \frac{P_h}{\eta_d}. \quad (23.212)$$

Similarly, fusion heat converted back to electricity has efficiency $\eta_e < 1$. Even a large Q_{plasma} can therefore coexist with poor whole-plant energy balance if driver or conversion efficiencies are low.

A whole-plant fusion recirculating-power model

Let a fusion plant produce fusion power P_f . A fraction f_r is recoverable as heat and converted to gross electricity with efficiency η_e :

$$P_g = \eta_e f_r P_f. \quad (23.213)$$

Suppose the plasma requires external heating

$$P_h = \frac{P_f}{Q_{\text{plasma}}}, \quad (23.214)$$

provided by a system with wall-plug efficiency η_h . Its electrical demand is

$$P_{h,e} = \frac{P_f}{Q_{\text{plasma}} \eta_h}. \quad (23.215)$$

Let all other auxiliary loads be P_0 . Then a simple net-power model is

$$P_{\text{net}} = \eta_e f_r P_f - \frac{P_f}{Q_{\text{plasma}} \eta_h} - P_0. \quad (23.216)$$

The condition for positive net output is

$$Q_{\text{plasma}} > \frac{1}{\eta_h (\eta_e f_r - P_0/P_f)}, \quad (23.217)$$

provided the denominator is positive.

Example 23.30: Minimum plasma gain in a hypothetical plant model

Take $\eta_e = 0.40$, $f_r = 0.90$, $\eta_h = 0.50$, and $P_0/P_f = 0.10$. Find the minimum Q_{plasma} for positive net output in this simplified model.

Solution.

$$\eta_e f_r - P_0/P_f = (0.40)(0.90) - 0.10 = 0.26. \quad (23.218)$$

Hence

$$Q_{\text{plasma}} > \frac{1}{(0.50)(0.26)} \approx 7.69. \quad (23.219)$$

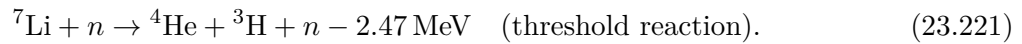
Thus $Q_{\text{plasma}} > 1$ is far from sufficient in this assumed whole-plant model. Different auxiliary loads and efficiencies change the threshold substantially.

Caution

Equation (23.216) counts recovered fusion heat only. It omits recovery of deposited external heating and additional nuclear heat from blanket reactions. If those are included, replace $f_r P_f$ in the gross-output term by the specified total recoverable thermal power, while retaining the electrical heating demand. The threshold is therefore specific to the declared energy boundary and is not a forecast for a particular fusion concept.

23.20.22 Tritium breeding as a fuel-cycle constraint

Tritium is radioactive and scarce in nature, so a D–T power plant would need to produce tritium from lithium-containing blanket material. Representative reactions include



A key systems parameter is the tritium breeding ratio (TBR), conceptually

$$\text{TBR} = \frac{\text{tritium atoms produced}}{\text{tritium atoms consumed}}. \quad (23.222)$$

A self-sufficient fuel cycle requires a practical margin above unity to cover radioactive decay, processing losses, inventory growth and downtime. The required design margin is an engineering question, not fixed by Eq. (23.222) alone.

This topic illustrates a recurring systems principle: fuel availability is not determined only by terrestrial abundance of deuterium. A D–T plant also needs a closed tritium cycle integrated with neutron transport, materials and safety systems.

23.20.23 Inertial confinement: compression, areal density and disassembly time

For an inertially confined capsule of characteristic radius R and mass density ρ_m , a useful quantity is areal density

$$\rho_m R, \quad (23.223)$$

which measures how much material a charged fusion product encounters while escaping. Large areal density helps alpha particles deposit their energy and heat the fuel.

The confinement time is linked to the hydrodynamic disassembly time,

$$\tau_{\text{hyd}} \sim \frac{R}{c_s}, \quad (23.224)$$

where c_s is a characteristic sound speed in the hot compressed fuel. Increasing density reduces the required confinement time in a Lawson-type product; this is the core physical logic of inertial confinement.

Ignition requires the central hot region to generate enough alpha heating that burn propagates into surrounding compressed fuel. The actual target physics involves radiation hydrodynamics, instabilities, opacity, equation of state and laser–plasma coupling, far beyond the simple scalings here.

Hydrodynamic instabilities in compressed targets

When a dense fluid is accelerated by a lighter fluid or effective pressure, the interface can be Rayleigh–Taylor unstable. In an ideal incompressible model, a perturbation with wavenumber k grows approximately as

$$\gamma_{RT} = \sqrt{A_t k g}, \quad (23.225)$$

where $A_t = (\rho_2 - \rho_1)/(\rho_2 + \rho_1)$ is the Atwood number and g is the effective acceleration. Real inertial-confinement implosions include ablative stabilization, compressibility, spherical convergence and other effects, so Eq. (23.225) is only a conceptual baseline.

The lesson is broader than fusion: small surface imperfections can grow under acceleration and spoil symmetry. Precision target fabrication and pulse shaping are therefore as important as peak driver energy.

23.20.24 Numerical integration of a simple reaction network

Consider a toy sequence



The abundances satisfy

$$\dot{N}_A = -\lambda_1 N_A, \quad (23.227)$$

$$\dot{N}_B = \lambda_1 N_A - \lambda_2 N_B, \quad (23.228)$$

$$\dot{N}_C = \lambda_2 N_B. \quad (23.229)$$

The sum $N_A + N_B + N_C$ is conserved. This conservation law is an excellent numerical diagnostic.

An explicit Euler scheme uses

$$\mathbf{N}_{j+1} = \mathbf{N}_j + \Delta t \mathbf{f}(\mathbf{N}_j). \quad (23.230)$$

If Δt is too large compared with $1/\lambda_i$, the method can become inaccurate or even produce negative populations. Stiff reaction networks require more stable integration methods. This provides a natural connection between nuclear physics and numerical analysis.

Method

For any reaction-network code, check at least three things: conservation laws, positivity of populations, and convergence under timestep reduction. A plot that looks smooth is not by itself evidence of numerical correctness.

23.20.25 Sensitivity, uncertainty and parameter correlations

Suppose an output $y = f(x_1, \dots, x_n)$ depends on measured or model parameters. For small uncertainties, the covariance form of linear uncertainty propagation is

$$\sigma_y^2 \approx \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \text{Cov}(x_i, x_j). \quad (23.231)$$

The off-diagonal covariance terms matter when inputs are correlated. Nuclear-data evaluations provide covariance information precisely because cross sections, fission yields and decay data are not known independently and exactly.

A normalized sensitivity coefficient

$$S_i = \frac{x_i}{y} \frac{\partial y}{\partial x_i} \quad (23.232)$$

answers the local question: what fractional change in y follows from a fractional change in x_i ? Such coefficients help prioritize measurements and model improvements.

23.20.26 Advanced synthesis example: from microscopic reaction to plant power

Example 23.31: A multi-scale D–T power estimate

A hypothetical uniform D–T plasma occupies $V = 1000 \text{ m}^3$, has total ion density $n = 8.0 \times 10^{19} \text{ m}^{-3}$, and at its operating temperature has $\langle \sigma v \rangle = 8.0 \times 10^{-23} \text{ m}^3/\text{s}$. Assume equal D and T densities. Estimate (a) the reaction rate, (b) total fusion power, (c) neutron power, and (d) gross electrical power if 90% of neutron plus alpha energy is recovered as heat and the thermal-to-electric efficiency is 40%.

Solution. The rate density is

$$\mathcal{R} = \frac{n^2}{4} \langle \sigma v \rangle \quad (23.233)$$

$$= \frac{(8.0 \times 10^{19})^2}{4} (8.0 \times 10^{-23}) \quad (23.234)$$

$$= 1.28 \times 10^{17} \text{ m}^{-3} \text{ s}^{-1}. \quad (23.235)$$

Hence the total reaction rate is

$$R = V\mathcal{R} = 1.28 \times 10^{20} \text{ s}^{-1}. \quad (23.236)$$

Using $Q = 17.6 \text{ MeV} = 2.82 \times 10^{-12} \text{ J}$,

$$P_f = RQ \approx 3.61 \times 10^8 \text{ W} = 361 \text{ MW}. \quad (23.237)$$

The neutron carries fraction $14.1/17.6$, so

$$P_n \approx 289 \text{ MW}. \quad (23.238)$$

If 90% of the total fusion energy becomes recoverable plant heat,

$$P_h = 0.90(361) \approx 325 \text{ MW}. \quad (23.239)$$

The gross electric output is then

$$P_e = 0.40P_h \approx 130 \text{ MW}. \quad (23.240)$$

This result does *not* establish net power. External plasma heating, magnets, cryogenics, pumps and other auxiliary loads have not been subtracted.

23.21 Original practice problems

The problems below are original exercises for this lecture-note series. Numerical constants are supplied when needed so that the focus remains on physical reasoning rather than data lookup. Unless stated otherwise, use

$$c = 2.998 \times 10^8 \text{ m/s}, \quad 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}, \quad 1 \text{ u } c^2 = 931.5 \text{ MeV}, \quad N_A = 6.022 \times 10^{23} \text{ mol}^{-1}.$$

Conceptual questions

1. **Binding-energy direction.** Draw a qualitative binding-energy-per-nucleon curve. Mark a light-nucleus region, an iron-group maximum and a very-heavy-nucleus region. Use arrows to show why fusion and fission can each release energy.

2. **Q-value language.** A nuclear reaction has $Q = +5 \text{ MeV}$. State precisely what this tells you and list two things it does *not* tell you about the reaction rate.
3. **Atomic versus nuclear masses.** Give a procedure for deciding whether tabulated atomic masses can be used directly in a Q-value calculation.
4. **Why neutrons?** Compare a slow neutron and a slow proton approaching a heavy nucleus. Why can the neutron reach the nuclear surface more readily?
5. **Criticality.** Explain the physical meaning of $k_{\text{eff}} = 0.98$, 1.00 and 1.02 . Why does $k_{\text{eff}} = 1$ not mean one neutron is emitted per fission?
6. **Moderator versus absorber.** Explain how a material can be effective at slowing neutrons but undesirable if it also absorbs too many of them.
7. **Moderator versus coolant.** Give one reason the two functions may be performed by the same material and one reason they should still be kept conceptually separate.
8. **Delayed neutrons.** Why can a small delayed-neutron fraction have a large influence on reactor controllability?
9. **Shutdown heat.** A reactor is made deeply subcritical. Explain why cooling may still be required even though the self-sustaining chain reaction has ended.
10. **Oklo.** State three physical conditions that were important for the natural reactor phenomenon at Oklo.
11. **Fusion barrier.** Why is the statement “the plasma must be heated until nuclei can classically climb over the Coulomb barrier” inaccurate?
12. **Stellar fusion.** Why is the proton–proton reaction network suitable for the Sun but unattractive as the primary reaction for a terrestrial power reactor?
13. **Confinement strategies.** Compare magnetic and inertial confinement using the three variables density, temperature and confinement time.
14. **Gain definitions.** Explain why $Q_{\text{plasma}} = 10$ is not enough information to conclude that a fusion plant exports net electricity.

Analytical and quantitative problems

15. **Fission event energy.** Convert 200 MeV into joules. How many such fissions are required to release 1.00 kWh of thermal energy?
16. **Fission rate at specified thermal power.** A reactor produces 1.50 GW thermal, with 202 MeV deposited per fission. Find the fission rate.
17. **Ideal fissile consumption.** Using the result of Problem 16 and molar mass 235 g/mol , estimate the ideal mass of ^{235}U fissioned in 30 days.
18. **Generational multiplication.** Start with $N_0 = 500$ effective neutrons. Find N_{30} for $k = 0.97$, 1.00 and 1.03 . Comment on sensitivity to small changes in k .
19. **Reactivity.** Calculate $\rho = (k - 1)/k$ for $k = 0.995$, 1.000 , 1.005 . Compare ρ with the approximation $k - 1$.
20. **Linear feedback.** A hypothetical model has $\alpha_T = -3.0 \times 10^{-5} \text{ K}^{-1}$. Find the reactivity change for a 70 K temperature increase. State why the result should be interpreted locally.

21. **Mean free path.** A material contains $6.0 \times 10^{28} \text{ m}^{-3}$ nuclei and has a 1.5-barn total interaction cross section at a specified energy. Find Σ_t and the interaction mean free path.
22. **Attenuation.** For the material in Problem 21, what fraction of an uncollided monoenergetic neutron beam survives 20 cm?
23. **Moderation kinematics.** Use Eq. (23.121) to find the minimum energy fraction after a head-on neutron collision with deuterium ($A = 2$), carbon ($A = 12$) and lead ($A = 207$).
24. **Q-value of D–D fusion.** Using atomic masses $m_D = 2.01410178 \text{ u}$, $m_{^3\text{He}} = 3.01602932 \text{ u}$ and $m_n = 1.00866492 \text{ u}$, calculate Q for $D + D \rightarrow ^3\text{He} + n$.
25. **D–T energy partition.** Treat the final neutron and alpha nonrelativistically and show from momentum conservation that their kinetic energies are inversely proportional to their masses. Use this to estimate the neutron and alpha energies from a 17.59 MeV release.
26. **Coulomb barrier scaling.** Estimate U_C at $r = 4.0 \text{ fm}$ for D–T ($Z_1 Z_2 = 1$) and for $^3\text{He} + ^3\text{He}$ ($Z_1 Z_2 = 4$).
27. **Thermal scale.** Find the temperature corresponding to $k_B T = 10 \text{ keV}$. Use $k_B = 8.617 \times 10^{-5} \text{ eV/K}$.
28. **Fusion reaction-rate density.** An equal D–T plasma has total ion density $n = 1.5 \times 10^{20} \text{ m}^{-3}$ and $\langle \sigma v \rangle = 8.0 \times 10^{-23} \text{ m}^3/\text{s}$. Find the reaction-rate density.
29. **Fusion power density.** Use Problem 28 and $Q = 17.6 \text{ MeV}$ to find the fusion power density.
30. **Larmor radius.** Find the Larmor radius of a 15 keV perpendicular-energy triton in a 6 T field. Use $m_T = 5.01 \times 10^{-27} \text{ kg}$.

Computational and data-analysis problems

31. **Branching Monte Carlo.** Simulate a Poisson branching process with $N_0 = 100$ for 100 generations. Compare $k = 0.98, 1.00, 1.02$. For each case estimate the extinction fraction over 10^4 histories and compare the ensemble mean with $N_0 k^g$.
32. **Random free paths.** Generate 10^5 free paths from $x = -\ln U/\Sigma$ with $\Sigma = 4 \text{ m}^{-1}$. Verify numerically that the sample mean approaches $1/\Sigma$ and that the survival fraction beyond x follows $e^{-\Sigma x}$.
33. **Oklo time evolution.** Plot the inferred ^{235}U fraction from today to 4 Gyr ago using the half-lives in the worked example. At what look-back time does the model predict a 3% fraction?
34. **Decay-heat components.** Model $P(t) = \sum_{i=1}^4 A_i e^{-t/\tau_i}$ with $A_i = (40, 20, 5, 1) \text{ MW}$ and $\tau_i = (20 \text{ s}, 20 \text{ min}, 2 \text{ h}, 2 \text{ d})$. These are teaching parameters, not an evaluated decay-heat curve. Plot each component and the total on linear and logarithmic time axes. Identify which component dominates in each interval.
35. **Calorimetric uncertainty.** Let $\dot{m} = 1.20 \times 10^4 \pm 1.0\% \text{ kg/s}$, $c_p = 4200 \pm 0.5\% \text{ J/(kg K)}$, and $\Delta T = 30.0 \pm 0.4 \text{ K}$. Compute P and its linearized uncertainty, then verify with a Monte Carlo propagation.
36. **Toy Maxwellian reactivity.** Use $E_G = 16E_*$, constant $S_0 > 0$, $x = E/E_*$ and $\theta = k_B T/E_*$. After removing a positive temperature-independent prefactor from Eq. (23.190), the reactivity is $F(\theta) = \theta^{-3/2} \int_0^\infty \exp[-x/\theta - 4/\sqrt{x}] dx$. Evaluate F for $0.1 \leq \theta \leq 100$ and locate its maximum. Vary the upper integration limit and numerical tolerances to demonstrate convergence. This synthetic nonresonant model is not D–T nuclear data.

37. **Two-step decay chain.** Integrate Eq. (23.151) with $b = 1$, $\lambda_1 = 0.10 \text{ s}^{-1}$, $\lambda_2 = 0.020 \text{ s}^{-1}$, $N_1(0) = 10^6$ and $N_2(0) = 0$ for $0 \leq t \leq 300 \text{ s}$. Compare with the analytic solution and demonstrate convergence as the timestep is reduced. For a forward-Euler method, also check that the chosen timestep keeps the populations nonnegative.
38. **Detector counting statistics.** Generate 1000 Poisson measurements for mean counts $\mu = 10, 100, 1000$ and 10000 . Verify that the standard deviation approaches $\sqrt{\mu}$ and that the relative spread follows the expected inverse-square-root scaling.

Engineering and applied problems

39. **Heat exchanger estimate.** A primary coolant transfers 2.5 GW while changing temperature by 28 K . Taking $c_p = 4.5 \text{ kJ}/(\text{kg K})$, estimate the required mass-flow rate in the single-phase approximation.
40. **Instrument resolution.** A neutron monitor must measure a count rate with less than 0.5% statistical uncertainty. Approximately how many counts are required? How long is the measurement if the rate is 5000 s^{-1} ?
41. **Thermal-to-electric chain.** A plant has $P_f = 2800 \text{ MW}$, heat-transfer efficiency 0.985 , thermal-cycle efficiency 0.36 and auxiliary load 80 MW . Find gross and net electrical output and net efficiency relative to P_f .
42. **Shield thickness.** A simplified narrow-beam gamma calculation has $\mu = 0.12 \text{ cm}^{-1}$. What thickness gives attenuation by factors $10, 100$ and 1000 ? Explain why these are not complete shielding-design answers.
43. **Fusion blanket loading.** A hypothetical D–T device produces 1.5 GW of fusion power. Estimate neutron power using the $14.1/17.6$ energy fraction. If an effective wall area is 800 m^2 , find average neutron power loading in MW/m^2 .
44. **Tritium inventory decay.** Tritium has a half-life of 12.3 y . What fraction of a stored inventory decays in one year? In five years? Explain why radioactive decay matters to fuel-cycle accounting even without plasma operation.
45. **Heat storage after shutdown.** A component of mass $2.0 \times 10^5 \text{ kg}$ with average heat capacity $500 \text{ J}/(\text{kg K})$ receives 20 MW net heat for 60 s before heat removal catches up. Estimate its average temperature rise in a lumped model.
46. **Sensor calibration.** A detector efficiency is 0.180 ± 0.006 , and it records 9000 counts in 60 s with negligible background. Estimate the incident particle rate and combine counting and efficiency uncertainties approximately.

Challenge problems

47. **SEMF split estimate.** Neglect asymmetry and pairing changes and compare the surface and Coulomb terms for a nucleus (A, Z) splitting into two equal fragments. Derive the factors by which each term changes and discuss when Coulomb-energy reduction can outweigh surface-energy increase.
48. **One-group criticality.** For the one-dimensional slab equation $-D d^2\phi/dx^2 + \Sigma_a\phi = (\nu\Sigma_f/k)\phi$, assume $\phi(\pm a) = 0$ and fundamental shape $\phi \propto \cos(\pi x/2a)$. Derive $k(a)$ and show how increasing a reduces leakage.
49. **Point-kinetics eigenvalues.** Set $S = 0$ in Eqs. (23.141)–(23.142) and assume $n, C \propto e^{\omega t}$. Derive the quadratic equation for ω . Examine qualitatively how the roots change as ρ passes

through zero and approaches β .

50. **Simplified Lawson optimum.** Suppose a toy reactivity is $\langle\sigma v\rangle = AT^2e^{-T/T_s}$ over a limited range, with $A > 0$ carrying the required units and $T_s > 0$. Insert it into Eq. (23.194) and derive the temperature condition that minimizes required $n\tau_E$. State why the result is specific to the toy fit.
51. **Coupled uncertainty.** Fusion power is modelled as $P = Cn^2g(T)$. Let n and T have covariance $\text{Cov}(n, T) \neq 0$. Derive the linearized fractional variance of P including the covariance term. State the signs of the sensitivities required for a negative correlation to reduce output uncertainty.

23.22 Problem-solving workshop

The workshop problems combine several ideas and are intended for auditorium classes or guided self-study.

Workshop A: from fission energy to grid electricity

A hypothetical station operates at 2.8 GW thermal. Assume 201 MeV deposited per fission, a gross thermal-cycle efficiency of 35%, and an auxiliary electrical load of 90 MW.

1. Find the fission rate.
2. Find the idealized ^{235}U mass fissioned per day.
3. Find gross and net electrical output.
4. Find the net efficiency relative to reactor thermal power.
5. Draw an energy-flow diagram and identify where the second law of thermodynamics enters.

Workshop B: D–T plasma power balance

An equal D–T plasma has $n = 1.2 \times 10^{20} \text{ m}^{-3}$, $V = 700 \text{ m}^3$, and $\langle\sigma v\rangle = 1.1 \times 10^{-22} \text{ m}^3/\text{s}$.

1. Find the reaction-rate density and total reaction rate.
2. Find total fusion power and split it into neutron and alpha power.
3. If 70% of alpha power returns to the thermal plasma while all neutron power leaves the plasma, estimate direct alpha self-heating.
4. State what additional quantities are needed to decide whether the plasma is self-sustaining.

Workshop C: radiation measurement

A detector records sample and background counts in equal 100-s intervals: $N_s = 25,600$ and $N_b = 3,600$.

1. Find the background-subtracted count rate.
2. Assuming independent Poisson statistics, find the uncertainty of $N_s - N_b$.
3. Find the relative statistical uncertainty of the net rate.
4. If detector efficiency is $0.25 \pm 2\%$ relative, combine the statistical and efficiency uncertainties approximately.

Workshop D: thermal feedback as a dynamical system

Consider a toy model

$$\begin{aligned}\dot{P} &= a[\rho_0 + \alpha(T - T_0)]P, \\ C_T \dot{T} &= P - H(T - T_c).\end{aligned}$$

1. Find a steady state with $P > 0$.
2. Linearize about the steady state.
3. Discuss qualitatively how the sign of α affects stability.
4. Explain why this model omits delayed neutrons and therefore should not be interpreted as a realistic transient model.

Workshop E: whole-plant fusion gain

A hypothetical fusion system has $P_{\text{fusion}} = 2.0$ GW, external plasma heating $P_h = 150$ MW, heater efficiency 45%, thermal conversion efficiency 40%, fixed auxiliary load 250 MW, and 90% recoverable fusion heat.

1. Calculate Q_{plasma} .
2. Calculate electrical power required by the heating system.
3. Calculate gross electrical output.
4. Calculate net electrical output after the stated auxiliaries.
5. Repeat for heater efficiencies 30% and 60%; discuss sensitivity.

23.23 Selected solutions to representative problems

Solution to Problem 15: fission event energy

$$Q = 200 \times 10^6 (1.602 \times 10^{-19}) = 3.204 \times 10^{-11} \text{ J}, \quad (23.241)$$

$$1 \text{ kWh} = 3.6 \times 10^6 \text{ J}, \quad (23.242)$$

$$N = \frac{3.6 \times 10^6}{3.204 \times 10^{-11}} \approx 1.12 \times 10^{17} \text{ fissions}. \quad (23.243)$$

A unit check gives joules divided by joules per fission, hence number of fissions.

Solution to Problem 16: fission rate at 1.50 GW

The event energy is

$$Q = (202 \times 10^6)(1.602 \times 10^{-19}) = 3.236 \times 10^{-11} \text{ J}. \quad (23.244)$$

Therefore

$$R = \frac{1.50 \times 10^9}{3.236 \times 10^{-11}} \approx 4.64 \times 10^{19} \text{ s}^{-1}. \quad (23.245)$$

The order of magnitude 10^{19} – 10^{20} s^{-1} is characteristic of gigawatt-scale fission power because each event supplies only $\sim 10^{-11} \text{ J}$.

Solution to Problem 17: ideal fissile consumption over 30 days

From Problem 16, the number of fissions in $t = 30 \times 86400$ s is

$$N = Rt \approx (4.64 \times 10^{19})(2.592 \times 10^6) \approx 1.20 \times 10^{26}. \quad (23.246)$$

The corresponding ideal ^{235}U mass is

$$m = \frac{N}{N_A}(0.235 \text{ kg/mol}) \approx 46.9 \text{ kg}. \quad (23.247)$$

Again, this is the mass actually fissioned in the idealized calculation, not total fresh-fuel loading.

Solution to Problem 19: reactivity

Using $\rho = (k - 1)/k$,

$$k = 0.995 : \quad \rho = -5.03 \times 10^{-3}, \quad (23.248)$$

$$k = 1.000 : \quad \rho = 0, \quad (23.249)$$

$$k = 1.005 : \quad \rho = +4.98 \times 10^{-3}. \quad (23.250)$$

Near unity, $k - 1$ differs only at second order in the small departure from criticality.

Solution to Problem 21: mean free path

$$\Sigma = (6.0 \times 10^{28})(1.5 \times 10^{-28}) = 9.0 \text{ m}^{-1}, \quad \ell = 0.111 \text{ m}. \quad (23.251)$$

Solution to Problem 22: uncollided attenuation

From Problem 21, $\Sigma = 9.0 \text{ m}^{-1}$. For $x = 0.20 \text{ m}$,

$$P_{\text{surv}} = e^{-\Sigma x} = e^{-1.8} \approx 0.165. \quad (23.252)$$

Thus about 16.5% remain uncollided. This is not the same as saying 83.5% of all neutrons disappear; scattered neutrons may remain in the system.

Solution to Problem 23: moderation kinematics

$$A = 2 : \quad \alpha = (1/3)^2 = 0.111, \quad (23.253)$$

$$A = 12 : \quad \alpha = (11/13)^2 = 0.716, \quad (23.254)$$

$$A = 207 : \quad \alpha = (206/208)^2 \approx 0.981. \quad (23.255)$$

The trend makes the physical advantage of light moderator nuclei clear.

Solution to Problem 24: D-D Q-value

The initial atomic mass is $2m_D = 4.02820356 \text{ u}$. The final mass is

$$m_f = 3.01602932 + 1.00866492 = 4.02469424 \text{ u}. \quad (23.256)$$

Thus

$$\Delta m = 0.00350932 \text{ u}, \quad Q \approx 0.00350932(931.5) = 3.27 \text{ MeV}. \quad (23.257)$$

Electron numbers cancel: two atomic electrons are present initially and two in the neutral ^3He atom finally.

Solution to Problem 25: D–T two-body energy sharing

In the centre-of-mass frame with negligible initial kinetic energy,

$$\mathbf{p}_n = -\mathbf{p}_\alpha, \quad |p_n| = |p_\alpha| = p. \quad (23.258)$$

Then

$$K_n = \frac{p^2}{2m_n}, \quad K_\alpha = \frac{p^2}{2m_\alpha}, \quad (23.259)$$

so

$$\frac{K_n}{K_\alpha} = \frac{m_\alpha}{m_n} \approx 4. \quad (23.260)$$

With $K_n + K_\alpha = 17.59 \text{ MeV}$,

$$K_n \approx \frac{4}{5}(17.59) = 14.1 \text{ MeV}, \quad K_\alpha \approx 3.52 \text{ MeV}. \quad (23.261)$$

Actual masses and a relativistic two-body treatment refine the estimate; the familiar values remain near 14.1 and 3.5 MeV.

Solution to Problem 28: reaction-rate density

For an equal D–T mixture,

$$\mathcal{R} = \frac{n^2}{4} \langle \sigma v \rangle \quad (23.262)$$

$$= \frac{(1.5 \times 10^{20})^2}{4} (8.0 \times 10^{-23}) \quad (23.263)$$

$$\approx 4.5 \times 10^{17} \text{ m}^{-3} \text{ s}^{-1}. \quad (23.264)$$

The corresponding power density in Problem 29 is

$$p_f = \mathcal{R}(17.6 \times 10^6)(1.602 \times 10^{-19}) \approx 1.27 \times 10^6 \text{ W/m}^3. \quad (23.265)$$

Solution to Problem 30: triton Larmor radius

A 15 keV perpendicular kinetic energy corresponds to

$$v_\perp = \sqrt{\frac{2K}{m_T}} = \sqrt{\frac{2(15 \times 10^3)(1.602 \times 10^{-19})}{5.01 \times 10^{-27}}} \approx 9.79 \times 10^5 \text{ m/s}. \quad (23.266)$$

Then

$$r_L = \frac{m_T v_\perp}{eB} \approx \frac{(5.01 \times 10^{-27})(9.79 \times 10^5)}{(1.602 \times 10^{-19})(6)} \approx 5.1 \times 10^{-3} \text{ m}. \quad (23.267)$$

Solution to Problem 40: instrument resolution

For Poisson counting,

$$\frac{1}{\sqrt{N}} \leq 0.005 \quad \Rightarrow \quad N \geq 4.0 \times 10^4. \quad (23.268)$$

At 5000 s^{-1} , the minimum measurement time is approximately

$$t = \frac{4.0 \times 10^4}{5000} = 8.0 \text{ s}. \quad (23.269)$$

This excludes background and systematic uncertainties.

Solution to Problem 41: thermal-to-electric chain

Heat delivered to the cycle is

$$P_h = (0.985)(2800) = 2758 \text{ MW}. \quad (23.270)$$

Gross electrical power is

$$P_g = (0.36)(2758) = 993 \text{ MW}. \quad (23.271)$$

Net power after 80 MW auxiliaries is

$$P_{\text{net}} = 913 \text{ MW}. \quad (23.272)$$

Relative to the original 2800 MW thermal source,

$$\eta_{\text{net}} = 913/2800 \approx 32.6\%. \quad (23.273)$$

Solution to Problem 43: fusion blanket loading

The neutron fraction is $14.1/17.6 \approx 0.801$, so

$$P_n \approx 0.801(1.5 \text{ GW}) = 1.20 \text{ GW}. \quad (23.274)$$

Then

$$q_n = \frac{1.20 \times 10^9}{800} \approx 1.50 \times 10^6 \text{ W/m}^2 = 1.50 \text{ MW/m}^2. \quad (23.275)$$

The real spatial loading would not be uniform.

Solution to Problem 44: tritium inventory decay

The surviving fraction after time t is

$$f = 2^{-t/12.3 \text{ y}}. \quad (23.276)$$

After one year,

$$1 - f \approx 1 - 2^{-1/12.3} \approx 5.5\%. \quad (23.277)$$

After five years,

$$1 - f \approx 1 - 2^{-5/12.3} \approx 24.5\%. \quad (23.278)$$

Fuel-cycle inventories therefore change measurably even without consumption in fusion reactions.

Solution to Problem 45: stored heat

The deposited energy in 60 s is

$$E = (20 \times 10^6)(60) = 1.2 \times 10^9 \text{ J}. \quad (23.279)$$

The lumped heat capacity is

$$mc = (2.0 \times 10^5)(500) = 1.0 \times 10^8 \text{ J/K}. \quad (23.280)$$

Hence

$$\Delta T = \frac{E}{mc} = 12 \text{ K}. \quad (23.281)$$

This ignores spatial gradients, phase changes and temperature-dependent material properties.

Solution to Problem 46: detector calibration uncertainty

The measured count rate is

$$r = 9000/60 = 150 \text{ s}^{-1}. \quad (23.282)$$

The inferred incident rate is

$$R = \frac{r}{\epsilon} = \frac{150}{0.180} \approx 833 \text{ s}^{-1}. \quad (23.283)$$

Counting relative uncertainty is $1/\sqrt{9000} = 1.05\%$. Efficiency relative uncertainty is

$$\frac{0.006}{0.180} = 3.33\%. \quad (23.284)$$

Assuming independence,

$$\frac{\sigma_R}{R} \approx \sqrt{0.0105^2 + 0.0333^2} \approx 0.0349 = 3.49\%. \quad (23.285)$$

Thus $R \approx 833 \pm 29 \text{ s}^{-1}$ statistically plus calibration uncertainty as combined here. The efficiency dominates the error budget.

Solution outline to Problem 48: one-group criticality

For $\phi \propto \cos(\pi x/2a)$,

$$-\frac{d^2\phi}{dx^2} = \left(\frac{\pi}{2a}\right)^2 \phi. \quad (23.286)$$

Substitution gives

$$D \left(\frac{\pi}{2a}\right)^2 + \Sigma_a = \frac{\nu\Sigma_f}{k}, \quad (23.287)$$

so

$$k(a) = \frac{\nu\Sigma_f}{\Sigma_a + D(\pi/2a)^2}. \quad (23.288)$$

As a increases, the leakage term proportional to a^{-2} decreases and k rises toward the infinite-medium limit $\nu\Sigma_f/\Sigma_a$.

Solution to Problem 50: a defined Lawson optimum

At fixed fuel and the other assumptions of Eq. (23.194),

$$n\tau_E \propto \frac{T}{AT^2 e^{-T/T_s}} = \frac{e^{T/T_s}}{AT}. \quad (23.289)$$

The logarithmic derivative is $1/T_s - 1/T$, which vanishes at $T = T_s$. The function decreases below T_s and increases above it, so the stationary point is a minimum if it lies within the fit domain. This toy optimum results from competing factors and is not a prediction for measured D–T reactivity.

23.24 Bridge to the next modern-physics topics

Nuclear energy closes a conceptual loop that began with mass–energy equivalence, quantum tunnelling and nuclear binding. It also opens the door to particle and astroparticle physics. Beta decay already requires neutrinos and the weak interaction; nuclear reactions depend on quantum scattering amplitudes and conservation laws; stellar nucleosynthesis connects microscopic particles with cosmic evolution. The next stage of modern physics asks what the constituents of nucleons are, how the fundamental interactions are organized, and how high-energy experiments reveal them.

Chapter A

Notation dictionary for Chapter 23

Symbol	Meaning in this chapter	Typical unit
A	mass number; also area when stated explicitly	1 or m ²
Z	proton number	dimensionless
N	neutron number or event/count population, by context	dimensionless
B	nuclear binding energy; magnetic-field magnitude when used in plasma sections	J, MeV; T
B/A	binding energy per nucleon	MeV/nucleon
Q	reaction Q-value	J, eV, MeV
R	reaction or event rate	s ⁻¹
P	power	W
P_{th}	thermal power	W
P_e	electrical power	W
P_f	fission or fusion power, as defined locally	W
k_{eff}	effective neutron multiplication factor	dimensionless
ρ	reactivity, $(k_{\text{eff}} - 1)/k_{\text{eff}}$	dimensionless
α_T	temperature coefficient of reactivity in a local model	K ⁻¹
β	effective delayed-neutron fraction; plasma beta written β_{pl}	dimensionless
Λ	prompt-neutron generation time in point kinetics	s
C	delayed-neutron precursor concentration or population	model dependent
λ	radioactive-decay constant	s ⁻¹
σ	microscopic nuclear cross section	m ² , barn
Σ	macroscopic cross section, $n_t\sigma$	m ⁻¹
ϕ	scalar neutron flux	m ⁻² s ⁻¹
n	number density	m ⁻³
$\mathcal{R}$	reaction-rate density	m ⁻³ s ⁻¹
$\langle\sigma v\rangle$	fusion reactivity	m ³ /s
T	thermodynamic/plasma temperature	K or energy unit with $k_B T$ explicit
τ_E	plasma energy-confinement time	s
r_L	Larmor radius	m
Ω_c	cyclotron angular frequency	rad/s
μ	reduced mass in fusion; attenuation coefficient where explicitly stated	kg; m ⁻¹
η	efficiency; Sommerfeld parameter in specified context	dimensionless
η_e	thermal-to-electric efficiency	dimensionless
N_A	Avogadro constant	mol ⁻¹
ϵ_0	vacuum permittivity	F/m
μ_0	vacuum permeability	N/A ²
e	elementary charge magnitude	C
c	speed of light in vacuum	m/s
k_B	Boltzmann constant	J/K or eV/K
$\hbar$	reduced Planck constant	J s

Chapter B

Compact formula sheet

Mass–energy and nuclear reactions

$$Q = (m_i - m_f)c^2, \quad (\text{B.1})$$

$$B = (Zm_p + Nm_n - m_{\text{nucleus}})c^2, \quad (\text{B.2})$$

$$1 \text{ u } c^2 \approx 931.5 \text{ MeV}, \quad (\text{B.3})$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}. \quad (\text{B.4})$$

Rates and power

$$P = RQ, \quad (\text{B.5})$$

$$\eta_e = P_e/P_{\text{th}}, \quad (\text{B.6})$$

$$\dot{m} = R_f M/N_A, \quad (\text{B.7})$$

$$P_{\text{th}} \approx \dot{m} c_p \Delta T. \quad (\text{B.8})$$

Fission and reactor quantities

$$N_g = N_0 k^g, \quad (\text{B.9})$$

$$\rho = (k_{\text{eff}} - 1)/k_{\text{eff}}, \quad (\text{B.10})$$

$$\Sigma = n_t \sigma, \quad (\text{B.11})$$

$$P_{\text{unc}}(x) = e^{-\Sigma_t x}, \quad (\text{B.12})$$

$$\ell = 1/\Sigma_t, \quad (\text{B.13})$$

$$\mathcal{R} = \Sigma \phi, \quad (\text{B.14})$$

$$k_{\infty} = \eta \epsilon p f, \quad (\text{B.15})$$

$$k_{\text{eff}} = k_{\infty} P_f P_t. \quad (\text{B.16})$$

Moderation

$$\alpha = \left(\frac{A-1}{A+1} \right)^2, \quad (\text{B.17})$$

$$\xi = 1 + \frac{\alpha \ln \alpha}{1 - \alpha}, \quad (\text{B.18})$$

$$N_{\text{coll}} \approx \frac{\ln(E_0/E)}{\xi}. \quad (\text{B.19})$$

Fusion and plasma quantities

$$U_C(r) \approx \frac{1}{4\pi\epsilon_0} \frac{Z_1 Z_2 e^2}{r}, \quad (\text{B.20})$$

$$\mathcal{R} = n_1 n_2 \langle \sigma v \rangle, \quad (\text{B.21})$$

$$\mathcal{R}_{DT} = \frac{n^2}{4} \langle \sigma v \rangle_{DT} \quad (\text{equal D-T mixture}), \quad (\text{B.22})$$

$$p_f = \mathcal{R}Q, \quad (\text{B.23})$$

$$r_L = \frac{mv_\perp}{|q|B}, \quad (\text{B.24})$$

$$\Omega_c = \frac{|q|B}{m}, \quad (\text{B.25})$$

$$\beta_{pl} = \frac{p}{B^2/(2\mu_0)}, \quad (\text{B.26})$$

$$n\tau_E \gtrsim \frac{12k_B T}{\langle \sigma v \rangle E_\alpha} \quad (\text{simplified D-T ignition model}). \quad (\text{B.27})$$

Counting and attenuation

$$\sigma_N \approx \sqrt{N}, \quad (\text{B.28})$$

$$\frac{\sigma_N}{N} \approx N^{-1/2}, \quad (\text{B.29})$$

$$I(x) = I_0 e^{-\mu x}, \quad (\text{B.30})$$

$$x_{1/2} = \frac{\ln 2}{\mu}, \quad (\text{B.31})$$

$$r = \frac{R}{1 + R\tau_d} \quad (\text{nonparalyzable dead-time model}). \quad (\text{B.32})$$

Caution

A formula sheet is not a substitute for model selection. Before using any equation, identify its assumptions: one-speed or energy-dependent neutrons, homogeneous or heterogeneous material, Maxwellian or non-Maxwellian plasma, narrow-beam or buildup radiation field, steady state or transient dynamics.

Chapter C

Useful physical constants and unit reminders

Quantity	Value used for course estimates
Speed of light	$c = 2.99792458 \times 10^8 \text{ m/s}$
Elementary charge	$e = 1.602176634 \times 10^{-19} \text{ C}$
Electronvolt	$1 \text{ eV} = 1.602176634 \times 10^{-19} \text{ J}$
Atomic mass unit energy equivalent	$1 \text{ u } c^2 \approx 931.494 \text{ MeV}$
Boltzmann constant	$k_B = 1.380649 \times 10^{-23} \text{ J/K} = 8.61733 \times 10^{-5} \text{ eV/K}$
Avogadro constant	$N_A = 6.02214076 \times 10^{23} \text{ mol}^{-1}$
Coulomb energy constant	$e^2/(4\pi\epsilon_0) \approx 1.44 \text{ MeV fm}$
Barn	$1 \text{ b} = 10^{-28} \text{ m}^2$
Day	86400 s
Kilowatt-hour	$1 \text{ kWh} = 3.6 \times 10^6 \text{ J}$

The number of significant digits retained in a worked problem should be consistent with the input data. Constants above are listed more accurately than most pedagogical estimates require.

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