

Chapter 3

Electric Flux and Gauss's Law

First Edition

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Preface

This chapter develops electric flux and Gauss's law for first-cycle technical and engineering students. It builds on Chapter 2's electric field, Coulomb's law, and superposition. The new idea is an oriented surface integral: its signed total over a closed surface is fixed by the algebraic enclosed charge. Symmetry then turns this integral law into an efficient field-calculation method for spheres, cylinders, and planes.

The core route develops the geometry of flux before applying it to charge distributions, conductors, cavities, and engineering measurements. Worked examples show not only which expression to use, but why it applies, how direction enters, and what limiting cases reveal. A runnable numerical experiment tests the same law on a cube where the field is not uniform.

The clearly marked advanced enrichment section is optional for the first-cycle route. It introduces divergence, solid angle, nonuniform sources, finite-source error estimates, and boundary-value reasoning at advanced undergraduate or early MSc level. Its brief potential preview supports a shielding proof; the systematic study of potential remains in Chapter 4. Capacitance and dielectric polarization belong to Chapter 5, and current-carrying conductors to Chapter 6.

The Guideline

Begin with the physical source, its symmetry, and the chosen normal. A convenient mathematical surface cannot create a symmetry that the sources and boundary conditions do not possess. When the shortcut fails, the flux law remains exact.

Pedagogical boundary

Field calculations use vacuum, or air approximated as vacuum, unless explicitly stated otherwise. In matter, Gauss's law for $\mathbf{E}$ still uses ε_0 and total charge, including polarization charge. The separate formulation using $\mathbf{D}$ and free charge is deferred to Chapter 5. No general replacement of ε_0 by a material permittivity is implied here.

Conventions and notation

Vectors are written in bold, for example $\mathbf{E}$ for electric field and $\mathbf{A}$ for an area vector. Unit vectors are written as $\hat{\mathbf{r}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{i}}, \hat{\mathbf{j}},$ and $\hat{\mathbf{k}}$. The scalar $E = |\mathbf{E}|$ denotes a nonnegative magnitude. The component $E_r = \mathbf{E} \cdot \hat{\mathbf{r}}$ is signed: it is negative for an inward radial field. Similarly, $E_\perp$ is signed along a stated normal. The symbol Φ_E denotes electric flux. The subscript E is often included to distinguish electric flux from magnetic flux introduced later.

For a small surface patch, the vector area element is

$$d\mathbf{A} = \hat{\mathbf{n}} dA,$$

where $\hat{\mathbf{n}}$ is a unit normal to the patch and dA is the scalar area of the patch. For a closed surface the normal convention is outward. This convention fixes the sign of the flux: outward flux is positive and inward flux is negative.

The electric flux through an open surface S is

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A},$$

where the choice of normal orientation must be specified. The net electric flux through a closed surface S is

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A},$$

where the circle on the integral sign reminds us that the surface is closed.

Symbol	Meaning	SI unit or dimension
$\mathbf{E}$	electric field	N C^{-1} or V m^{-1}
Φ_E	electric flux	$\text{N m}^2 \text{C}^{-1}$
$d\mathbf{A}$	vector area element $\hat{\mathbf{n}}dA$	m^2
$\hat{\mathbf{n}}$	unit normal to a surface patch	dimensionless
S	surface of integration	—
$\oint_S$	integral over a closed surface	—
q_{enc}	algebraic charge enclosed by a closed surface	C
Q	total charge of a distribution or conductor	C
ρ	volume charge density	C m^{-3}
σ	surface charge density	C m^{-2}
λ	linear charge density	C m^{-1}
ϵ_0	vacuum permittivity	$\text{C}^2 \text{N}^{-1} \text{m}^{-2}$
R	radius of a sphere, cylinder, or conductor	m
r	distance from a symmetry axis or centre	m
A	scalar area	m^2

Note

The word “flux” in this chapter does not mean that charges cross the surface. Electric flux measures the signed amount of electric field piercing a mathematical or physical surface. A Gaussian surface is usually an imaginary surface chosen for calculation.

Chapter 3

Electric Flux and Gauss's Law

3.1 Motivation and roadmap

In the electric-field chapter, the field of an extended charge distribution was obtained by summing many small contributions. For a line of charge, a disk of charge, or a ring of charge, this means splitting the source into elements dq , calculating $d\mathbf{E}$ from each element, resolving components, and integrating. This is an important method because it is general. However, it is not always the most illuminating method.

Many physical systems possess symmetry. A uniformly charged spherical shell looks the same after rotation about its centre. An ideal infinitely long charged cylinder looks the same after translation along its axis and rotation about that axis. An ideal infinite charged sheet looks the same after translations parallel to the sheet and rotations in the plane. In such cases, the field direction and the possible coordinate dependence are strongly constrained before any calculation begins. Gauss's law turns these symmetry constraints into a powerful field-calculation method.

The conceptual roadmap is shown in Fig. 3.1. The chapter begins with area vectors and flux, then introduces closed surfaces and the sign convention for net flux, then states Gauss's law and interprets its content. After that we use the law for spherical, cylindrical, and planar systems, and finally apply it to conductors and shielding.

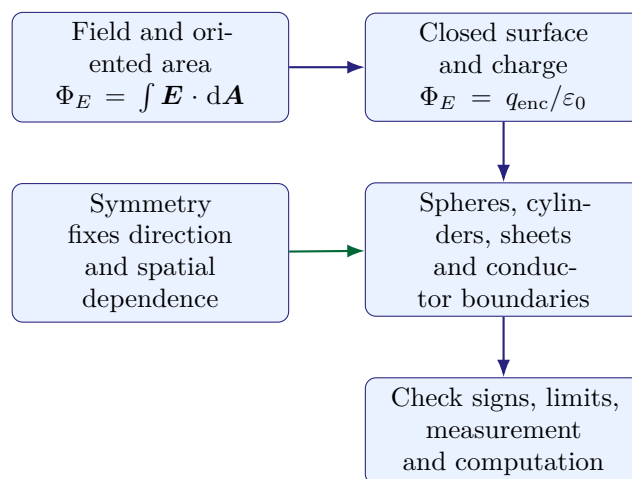


Figure 3.1: The chapter's reasoning sequence. The flux law supplies one constraint; symmetry or conductor equilibrium supplies the information needed to turn it into a field calculation.

Learning goals

After studying this chapter, the student should be able to:

- define the vector area element $d\mathbf{A}$ and explain its orientation for open and closed surfaces;
- compute electric flux through flat and curved surfaces in simple electric fields;

- distinguish electric flux from electric-field magnitude and from charge flow;
- state Gauss's law in integral form and interpret q_{enc} as algebraic enclosed charge;
- explain why external charges contribute zero net flux through a closed surface while still contributing to the local field;
- use symmetry to choose spherical, cylindrical, and planar Gaussian surfaces;
- derive electric fields for a charged spherical shell, a uniformly charged solid sphere, a long charged line or cylinder, and an infinite sheet;
- state the electrostatic equilibrium conditions for conductors;
- use a Gaussian pillbox to derive the field just outside a conducting surface;
- analyse common conceptual traps such as “zero flux means zero field”;
- connect Gauss's law to shielding, Faraday cages, coaxial geometries, and numerical flux estimation.

Key message

Gauss's law is an exact electric-field law, but it is not always a shortcut for calculating the field. It becomes a shortcut only when symmetry allows the flux integral to be evaluated without already knowing the detailed field everywhere on the surface.

3.2 From field vectors to area vectors

Flux begins with a simple but powerful idea: a surface patch can be represented by a vector. The magnitude of the vector is the area of the patch, and the direction of the vector is perpendicular to the patch. For a flat surface of area A , we write

$$\mathbf{A} = A\hat{\mathbf{n}}, \quad (3.1)$$

where $\hat{\mathbf{n}}$ is a chosen unit normal. For a curved surface, no single normal describes the entire surface, so we divide the surface into small patches and write

$$d\mathbf{A} = \hat{\mathbf{n}} dA. \quad (3.2)$$

The direction $\hat{\mathbf{n}}$ changes from patch to patch on a curved surface.

Figure 3.2 shows the geometry. The electric field component perpendicular to the surface contributes to flux. The component tangent to the surface does not pierce the surface and therefore gives zero contribution to the dot product.

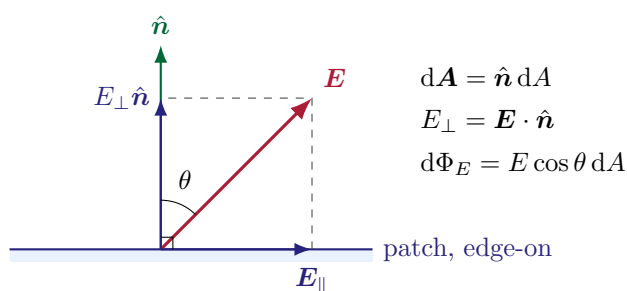


Figure 3.2: A common-tail decomposition of the field into components normal and tangent to a patch. The angle is measured from the normal, not from the surface. The area vector and the field component have different units and are not added to one another.

Definition 3.1: Vector area element

For a small surface patch of scalar area dA , the vector area element is

$$d\mathbf{A} = \hat{\mathbf{n}} dA, \quad (3.3)$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the patch. On a closed surface, $\hat{\mathbf{n}}$ is by convention the outward normal.

Caution

A surface has two possible normal directions. For an open surface, the problem must specify or imply the chosen normal. For a closed surface, the outward normal is the standard convention. A sign error in flux problems is often a normal-direction error, not an algebra error.

Guided checks

1. A flat square lies in the xy plane. What are the two possible unit normals? They are $+\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$.
2. For the top face of a closed cube centred at the origin, which normal is outward? $+\hat{\mathbf{z}}$.
3. For the bottom face of the same cube, which normal is outward? $-\hat{\mathbf{z}}$.
4. If $\mathbf{E}$ is tangent to a patch, what is $\mathbf{E} \cdot d\mathbf{A}$? It is zero.

3.3 Electric flux through an open surface

For a uniform electric field and a flat surface, the flux is the dot product

$$\Phi_E = \mathbf{E} \cdot \mathbf{A} = EA \cos \theta, \quad (3.4)$$

where θ is the angle between the electric field and the chosen area vector. The sign of the flux depends on whether the field points more along the chosen normal or opposite to it. If $\theta < 90^\circ$, the flux is positive; if $\theta > 90^\circ$, the flux is negative; if $\theta = 90^\circ$, the field skims the surface and the flux is zero.

For a nonuniform field or a curved surface, the surface must be divided into patches:

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A}. \quad (3.5)$$

This is a surface integral. At this stage the essential point is not to become lost in formalism. Equation (3.5) simply says: add the normal component of the electric field over the surface.

Definition 3.2: Electric flux through a surface

The electric flux through an oriented surface S is

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{A}. \quad (3.6)$$

Flux has SI unit $\text{N m}^2/\text{C}$. It is positive when the field has a net component along the chosen normal and negative when it has a net component opposite to the chosen normal.

Example 3.1: Flux through a tilted rectangular sensor plate

A rectangular plate of area $A = 0.120 \text{ m}^2$ is placed in a uniform electric field of magnitude $E = 850 \text{ N/C}$. The chosen area vector makes an angle 35° with the electric field. Find the electric flux through the plate.

Solution. Because the field is uniform and the plate is flat,

$$\Phi_E = EA \cos \theta. \quad (3.7)$$

Therefore

$$\Phi_E = (850 \text{ N/C})(0.120 \text{ m}^2) \cos 35^\circ = 83.6 \text{ N m}^2/\text{C}. \quad (3.8)$$

The result is positive because the angle between $\mathbf{E}$ and the chosen area vector is less than 90° . If the opposite normal had been chosen, the same physical plate would have flux $-83.6 \text{ N m}^2/\text{C}$.

Example 3.2: Flux through a square in component notation

A square of side 0.50 m lies in the plane $x = 2.0 \text{ m}$, with its area vector chosen as $+\hat{x}A$. The electric field in the region is

$$\mathbf{E} = (40x \hat{x} + 15 \hat{y}) \text{ N/C}, \quad (3.9)$$

where x is measured in metres. Find the flux through the square.

Solution. On the square $x = 2.0 \text{ m}$, so the normal component is

$$E_x = 40(2.0) = 80 \text{ N/C}. \quad (3.10)$$

The y component is tangent to the square and gives no flux. The area is

$$A = (0.50 \text{ m})^2 = 0.250 \text{ m}^2. \quad (3.11)$$

Thus

$$\Phi_E = E_x A = (80 \text{ N/C})(0.250 \text{ m}^2) = 20.0 \text{ N m}^2/\text{C}. \quad (3.12)$$

The calculation illustrates why flux is not the same as field magnitude: only the component normal to the chosen surface contributes.

Method

For flux through an open flat surface in a uniform field:

1. choose or identify the normal direction;
2. compute the scalar area A ;
3. find the component of $\mathbf{E}$ along the chosen normal;
4. multiply: $\Phi_E = E_\perp A$;
5. attach the sign from the normal convention.

3.3.1 What the surface integral actually adds

The sum defining flux contains three ingredients: a field component, a patch area, and an orientation. Divide a smooth surface into sufficiently small patches, evaluate the field at a representative point on each patch, and form

$$\Phi_E \simeq \sum_i (\mathbf{E}_i \cdot \hat{\mathbf{n}}_i) \Delta A_i. \quad (3.13)$$

The integral is the limit of this sum as the patch sizes tend to zero. A curved surface does not have one normal shared by every patch. A nonuniform field does not have one value shared by every patch. Either variation can prevent the shortcut $EA \cos \theta$.

For a uniform field, $A \cos \theta$ is the *signed projected area* of a flat patch onto a plane perpendicular to the field. A patch turned edge-on has zero projected area. Rotating it past 90° changes the sign of the oriented projection. This is a geometrical sign, not a statement that the field magnitude is negative.

The dimensions give a useful first check:

$$[\Phi_E] = [E][A] = \text{N m}^2/\text{C}, \quad [q/\epsilon_0] = \text{N m}^2/\text{C}. \quad (3.14)$$

Neither expression has the units of charge per time. The analogy with fluid flowing through a surface helps explain the dot product, but an electrostatic field can have flux without any material crossing the surface.

Example 3.3: Integrating a field that varies across the surface

A mathematical rectangular surface lies at $z = 0$, with $0 \leq x \leq a$ and $0 \leq y \leq b$. Its normal is $+\hat{z}$. In its neighbourhood the field is

$$\mathbf{E} = \alpha z \hat{x} + (E_0 + \alpha x) \hat{z}, \quad E_0 = 200 \text{ N/C}, \quad \alpha = 600 \text{ N}/(\text{C m}). \quad (3.15)$$

Find the flux for $a = 0.30 \text{ m}$ and $b = 0.20 \text{ m}$.

Strategy. The normal component varies with x , so integrate it across the rectangle. The x component is tangent and contributes nothing.

$$\Phi_E = \int_0^b \int_0^a (E_0 + \alpha x) dx dy \quad (3.16)$$

$$= b \left(E_0 a + \frac{\alpha a^2}{2} \right) = 17.4 \text{ N m}^2/\text{C}. \quad (3.17)$$

Check and interpretation. The normal field ranges from 200 to 380 N/C; its area average is 290 N/C. Multiplying by $ab = 0.060 \text{ m}^2$ reproduces the integral. Evaluating the field at the midpoint works here because the variation is linear; it is not an exact rule for arbitrary fields. The term $\alpha z \hat{x}$ makes this model consistent with an electrostatic potential in the surrounding region, although it does not affect the flux at $z = 0$.

3.4 Closed surfaces and net flux

A closed surface divides space into an inside and an outside. A sphere, a cube, a cylinder with end caps, and an irregular balloon are all closed surfaces. A disk, a single square, and the curved side of a cylinder without caps are open surfaces.

For a closed surface, the net electric flux is

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A}, \quad (3.18)$$

with $d\mathbf{A}$ outward everywhere. Figure 3.3 shows radial examples of the sign convention. Outward field contribution is positive; inward field contribution is negative. A field line picture can be useful, provided it is not taken too literally: field lines leaving a closed surface contribute positive flux; field lines entering contribute negative flux; lines that enter and leave contribute zero net flux overall.

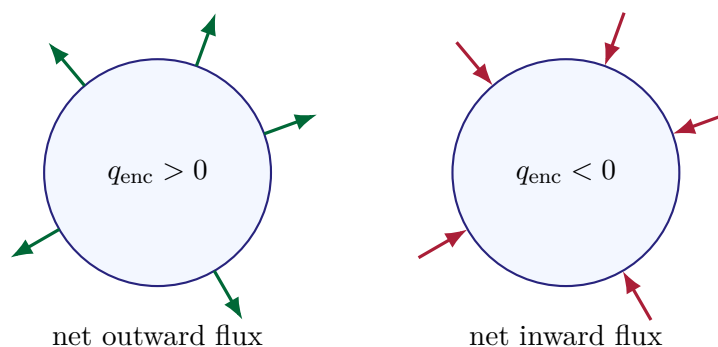


Figure 3.3: Radial examples illustrating the sign convention for closed-surface flux. The normal is outward; general fields may have both entering and leaving contributions. A positive enclosed charge produces positive net flux; a negative enclosed charge produces negative net flux.

Example 3.4: Net flux through a cube in a simple nonuniform field

A cube of side $L = 0.40$ m occupies $0 \leq x \leq L$, $0 \leq y \leq L$, $0 \leq z \leq L$. The electric field is

$$\mathbf{E} = ax \hat{\mathbf{x}}, \quad a = 300 \text{ N/(C m)}. \quad (3.19)$$

Find the net flux through the cube.

Solution. The field is perpendicular only to the faces normal to the x axis. On the left face, $x = 0$, the field is zero, so $\Phi_{\text{left}} = 0$. On the right face, $x = L$, the outward normal is $+\hat{\mathbf{x}}$ and $E_x = aL$. The area is L^2 , so

$$\Phi_{\text{right}} = aL L^2 = aL^3. \quad (3.20)$$

The other four faces have normals perpendicular to $\hat{\mathbf{x}}$, so $\mathbf{E} \cdot d\mathbf{A} = 0$ there. Hence

$$\Phi_E = aL^3 = (300)(0.40)^3 = 19.2 \text{ N m}^2/\text{C}. \quad (3.21)$$

The positive result means that the field has more outward flux through the right face than inward flux through the left face. In a later mathematical language, this corresponds to a positive divergence in the volume.

Caution

Zero net flux does not imply zero electric field. A uniform electric field through a closed box enters one side and leaves the opposite side, giving zero net flux even though the field is nonzero everywhere inside the box.

Guided checks

1. A uniform electric field points in the $+x$ direction through a closed cube. Is the net flux positive, negative, or zero? Zero.
2. Is the flux through the right face of the cube zero? No. It is positive if the right face normal is $+\hat{\mathbf{x}}$.
3. Is the flux through the left face zero? No. It is negative.
4. What cancels in the total? The positive outward flux through one face cancels the negative inward flux through the opposite face.

3.5 Statement and interpretation of Gauss's law

Gauss's law states that the net electric flux through any closed surface is equal to the algebraic charge enclosed by that surface divided by the vacuum permittivity:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0}. \quad (3.22)$$

The surface S is often called a Gaussian surface. It may be a physical surface, such as a conductor boundary, but in most calculations it is an imaginary surface chosen for mathematical convenience.

Several features of Eq. (3.22) deserve careful attention.

1. The integral is over a closed surface.
2. The field $\mathbf{E}$ in the integral is the total electric field at the surface, produced by all charges, both inside and outside.
3. The charge q_{enc} is only the algebraic sum of charges inside the surface.
4. A positive q_{enc} corresponds to positive net outward flux; a negative q_{enc} corresponds to negative net outward flux (a net inward field contribution).
5. The law is exact, with total enclosed charge, but it is easy to use for finding $\mathbf{E}$ only when symmetry is strong enough.

Theoretical point 3.1: Gauss's law as a source law

Gauss's law says that electric charge is the source of electric flux. It does not say that electric field exists only where charge is present. A point in empty space can have a nonzero electric field due to charges elsewhere. The law constrains the net flux through closed surfaces, not the local value of $\mathbf{E}$ at a point by itself.

Example 3.5: Using Gauss's law to infer enclosed charge

A closed surface has measured net electric flux

$$\Phi_E = 2.40 \times 10^5 \text{ N m}^2/\text{C}. \quad (3.23)$$

Find the algebraic charge enclosed by the surface.

Solution. Gauss's law gives

$$q_{\text{enc}} = \epsilon_0 \Phi_E. \quad (3.24)$$

Using $\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)$,

$$q_{\text{enc}} = (8.854 \times 10^{-12})(2.40 \times 10^5) = 2.13 \times 10^{-6} \text{ C}. \quad (3.25)$$

Thus the enclosed charge is $+2.13 \mu\text{C}$. The sign is positive because the measured net flux is outward.

Caution

Do not replace q_{enc} by “the nearest charge” or by “the total charge producing the field.” Gauss's law contains only charges inside the chosen closed surface on the right-hand side. Charges outside the surface affect the local field $\mathbf{E}$ on the left-hand side but contribute zero to the net closed-surface flux.

3.5.1 The connection with Coulomb's law

The inverse-square dependence from Chapter 2 explains why a point charge has a radius-independent flux through concentric spheres. For an isolated point charge q , write the field as a signed radial component,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}. \quad (3.26)$$

The outward normal on a concentric sphere is $\hat{\mathbf{r}}$, so

$$\Phi_E = \frac{q}{4\pi\epsilon_0 r^2} \int_S dA = \frac{q}{4\pi\epsilon_0 r^2} (4\pi r^2) = \frac{q}{\epsilon_0}. \quad (3.27)$$

The r^{-2} field and the r^2 area cancel. This argument proves the result for a concentric sphere. Extending it to an arbitrary closed surface requires tracking the projected areas seen from the charge, as developed using solid angle in Sec. 3.19.4. Superposition then adds the signed contributions from many charges. A charge outside the surface contributes zero to the integral; one inside contributes q/ϵ_0 .

Conversely, Gauss's law together with spherical symmetry gives Coulomb's field. The symmetry is an essential input. One integral constraint cannot by itself determine an arbitrary vector field.

Note

Gauss's law for $\mathbf{E}$ retains the form $\oint \mathbf{E} \cdot d\mathbf{A} = q_{\text{enc}}/\epsilon_0$ in matter when q_{enc} includes *all* charge, including polarization charge. It also remains a Maxwell equation for time-dependent fields. The present chapter restricts the field calculations to electrostatics and mostly to vacuum, with air approximated as vacuum. Chapter 5 introduces polarization and the distinct free-charge formulation using $\mathbf{D}$; one must not simply replace ϵ_0 by a material permittivity in the general total-charge law.

Example 3.6: Two point charges and three Gaussian surfaces

Two point charges $q_1 = +6.0 \text{ nC}$ and $q_2 = -2.0 \text{ nC}$ are fixed in space. Find the net flux through three closed surfaces: S_A encloses only q_1 , S_B encloses only q_2 , and S_C encloses both charges.

Solution. Gauss's law depends only on algebraic enclosed charge. Therefore

$$\Phi_A = \frac{6.0 \times 10^{-9}}{\epsilon_0} = 6.78 \times 10^2 \text{ N m}^2/\text{C}, \quad (3.28)$$

$$\Phi_B = \frac{-2.0 \times 10^{-9}}{\epsilon_0} = -2.26 \times 10^2 \text{ N m}^2/\text{C}, \quad (3.29)$$

$$\Phi_C = \frac{4.0 \times 10^{-9}}{\epsilon_0} = 4.52 \times 10^2 \text{ N m}^2/\text{C}. \quad (3.30)$$

The surfaces can have arbitrary shapes. The local field pattern may be complicated, but the total flux is fixed by the enclosed charge.

Figure 3.4 illustrates how to close an open surface mathematically.

Example 3.7: Open hemisphere in a uniform electric field

A hemisphere of radius R is placed in a uniform electric field $\mathbf{E} = E_0 \hat{\mathbf{z}}$. The hemisphere is the curved upper half of a sphere, open at the equatorial plane. Its area vector is chosen outward from the missing full sphere. Find the flux through the curved hemispherical surface.

Solution. Direct integration over the curved surface is possible, but there is a simpler idea.

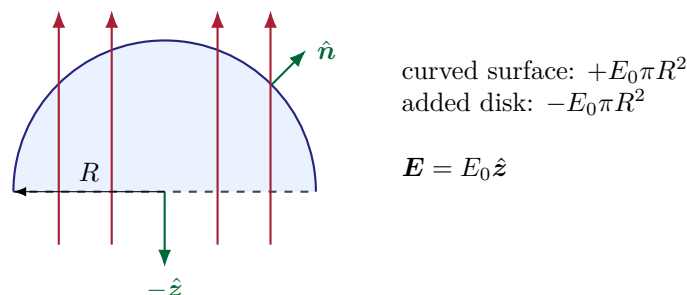


Figure 3.4: An open upper hemisphere closed mathematically by a disk. The diagram is a meridional section. The disk's outward normal is downward, making its flux negative. The curved area is $2\pi R^2$, but its projected area is only πR^2 .

Close the hemisphere by adding the flat circular disk at the equator. The closed half-sphere encloses no charge in a uniform external field, so the total flux through the curved surface plus the disk is zero. The outward normal of the flat disk points downward, $-\hat{z}$, so

$$\Phi_{\text{disk}} = \mathbf{E} \cdot (-\hat{z}\pi R^2) = -E_0\pi R^2. \quad (3.31)$$

Therefore

$$\Phi_{\text{curved}} = +E_0\pi R^2. \quad (3.32)$$

The result equals the flux through the projected area, not the larger curved area $2\pi R^2$.

3.5.2 A curved surface: the normal changes from point to point

The hemisphere construction can also be checked by direct integration. This is worth doing once: it shows why a curved area cannot generally be multiplied by one field magnitude. Parameterize the upper hemisphere by polar angle $0 \leq \theta \leq \pi/2$ measured from $+z$ and azimuth $0 \leq \phi < 2\pi$. A narrow band has meridional width $R d\theta$ and circumferential length $2\pi R \sin \theta$. Its area is therefore

$$dA_{\text{band}} = 2\pi R^2 \sin \theta d\theta. \quad (3.33)$$

The normal is radial. Although the uniform field has the same magnitude everywhere, its normal component is $E_0 \cos \theta$: it is largest at the top and vanishes at the equator. Consequently,

$$\Phi_{\text{curved}} = 2\pi E_0 R^2 \int_0^{\pi/2} \cos \theta \sin \theta d\theta \quad (3.34)$$

$$= 2\pi E_0 R^2 \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = E_0 \pi R^2. \quad (3.35)$$

This reproduces the closure argument. The area-weighted average normal component over the hemisphere is $\Phi/(2\pi R^2) = E_0/2$. The field itself is not weaker at the equator; only its projection on the local normal becomes zero.

For any surface in a uniform field, one can write

$$\Phi_E = \mathbf{E}_0 \cdot \mathbf{A}_S, \quad \mathbf{A}_S = \int_S \hat{n} dA. \quad (3.36)$$

The vector area $\mathbf{A}_S$ differs from the scalar area $\int_S dA$. Contributions from differently oriented patches can cancel in the vector sum. For a closed surface, $\mathbf{A}_S = \mathbf{0}$: each projection onto a coordinate plane counts an outward-facing contribution and an oppositely oriented contribution. A uniform field therefore has zero net flux through a closed surface of any shape, not just a cube.

This also explains a useful open-surface result. If two oriented surfaces have the same boundary, and one surface together with the reversed other surface forms a closed boundary, they have the same flux in a uniform field. A bowed membrane and a flat disk spanning the same rim thus have equal oriented flux even though their scalar areas differ. For nonuniform fields, equality still holds when the intervening closed volume contains zero net charge, but it need not hold when charge lies between the surfaces. The field and the enclosed region must be specified; the rim alone is not enough in general.

Guided checks

If the hemisphere's radius is doubled in the same uniform field, its flux increases by four. If the field is reversed, the flux changes sign. If both field and chosen normal are reversed, the flux is unchanged. These three checks test scale, orientation, and the dot product independently.

3.6 Why outside charges give zero net flux

One of the most important conceptual points in Gauss's law is also one of the easiest to misread. A charge outside a closed surface can create a strong electric field at the surface. It may even dominate the local field. Nevertheless, it contributes zero net flux through the closed surface. Field lines from an external positive charge that enter the surface must also leave it, because no field line begins or ends inside the surface unless charge is enclosed.

This is not merely a field-line story. See Fig. 3.5. It is a statement about the surface integral. For a point charge outside a closed surface, the solid angle subtended by entering and leaving parts of the surface cancels algebraically. In advanced language, the divergence of the point-charge field is zero everywhere except at the position of the charge. If the singular point is outside the volume, the net flux through the boundary is zero.

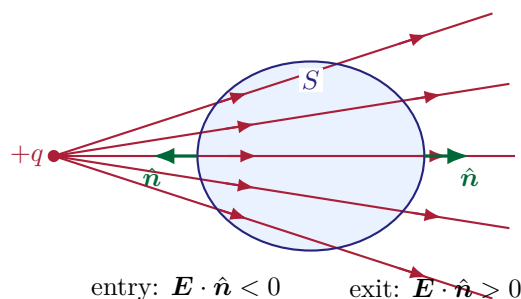


Figure 3.5: Straight radial field lines of an external positive point charge. The closed Gaussian surface does not bend the field. Each narrow bundle entering it also leaves it, so the two signed fluxes cancel even though the field strength changes along the bundle.

Guided checks

1. Can $\mathbf{E}$ be nonzero at every point of a closed surface while the net flux is zero? Yes.
2. Can $q_{\text{enc}} = 0$ while $\mathbf{E}$ on the surface is nonzero? Yes, for example in a uniform external field.
3. Can a charge outside a Gaussian surface affect the integrand $\mathbf{E} \cdot d\mathbf{A}$? Yes.
4. Can that outside charge change the total closed-surface flux? No, not by itself.

3.7 Gauss's law as a calculation method

Gauss's law becomes useful for field calculation when the surface integral can be simplified. The ideal situation is one in which symmetry tells us that:

1. the field is perpendicular to some parts of the Gaussian surface and tangent to others;
2. the field magnitude is constant over the parts where the field is perpendicular;
3. the enclosed charge is easy to express in terms of the Gaussian-surface size.

Then the integral becomes something like

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = EA, \quad (3.37)$$

where A is the relevant area. This is not magic. It is symmetry doing the work.

Method

Gauss-law strategy.

1. Identify the symmetry of the charge distribution: spherical, cylindrical, planar, or none.
2. Choose a closed Gaussian surface adapted to that symmetry.
3. Determine the direction of $\mathbf{E}$ on the surface from symmetry.
4. Decide where $\mathbf{E} \cdot d\mathbf{A}$ is zero and where E is constant.
5. Evaluate the flux integral.
6. Compute the algebraic enclosed charge q_{enc} .
7. Apply $\oint \mathbf{E} \cdot d\mathbf{A} = q_{\text{enc}}/\epsilon_0$.
8. Solve for the field magnitude and attach the direction.

Caution

A Gaussian surface is not usually a physical shell, membrane, or boundary. It is a mathematical surface chosen by us. The charge distribution does not have to lie on the Gaussian surface, and the Gaussian surface does not block or guide field lines.

3.7.1 Symmetry belongs to the physical system

A spherical surface that we draw does not make the field spherically symmetric. The symmetry must belong to the sources *and the boundary conditions*. A ball whose upper and lower halves carry different charge densities is geometrically spherical but does not have spherical charge symmetry. Likewise, a spherical conductor next to a charged object generally acquires a nonuniform surface charge.

For the field of a prescribed charge distribution alone, rotations and reflections of the source give the corresponding field symmetries. When other fields or conductors are present, test the symmetry of the complete configuration. It may be useful to compute the field of one symmetric contribution and then add the remaining contributions by superposition, but do not apply the symmetric shortcut directly to a total field that lacks that symmetry.

For an infinitely long uniform line, translation along the axis prevents any dependence on axial position. Rotation prevents azimuthal dependence. Reflection in a plane containing the axis excludes an azimuthal field, and reflection in a plane perpendicular to the line excludes an axial field. These statements leave a radial field depending only on distance from the axis. This explicit argument explains why “long and straight” is a physical approximation, not a licence to ignore the ends of any finite object.

Example 3.8: One charge at the centre of a cube

A point charge q is fixed at the centre of a mathematical cube. Find the flux through one face without integrating the nonuniform field.

Strategy. Gauss's law gives the total flux; the rotational symmetry of the charge-cube arrangement makes the six face integrals equal. Therefore

$$\Phi_{\text{face}} = \frac{1}{6} \Phi_{\text{cube}} = \frac{q}{6\epsilon_0}. \quad (3.38)$$

The field is *not* constant on a face: points near its corners are farther from the charge than its centre, and the angle with the face normal changes. Equality of the six integrated fluxes is enough; replacing each integral by one local value times the area would not be justified. Moving the charge off centre preserves the total flux but destroys equal sharing among the faces.

3.7.2 Choosing a surface without assuming the answer

A useful Gaussian surface turns a vector integral into a known geometric area times an unknown field component. The decision is therefore made before solving for the field. Ask which transformations leave the physical source unchanged, which components they forbid, and which coordinates they eliminate. Then choose a surface whose normal has a simple relation to the surviving component.

Table 3.1: What symmetry contributes to the Gaussian-surface method.

Physical source	Information supplied by symmetry	Useful closed surface and flux
Spherically symmetric charge	Radial field; the same signed E_r at all points with the same r .	Concentric sphere; $4\pi r^2 E_r$.
Infinite uniform line or radial cylinder	Radial field; no axial or azimuthal dependence.	Coaxial cylinder; $2\pi r L E_r$ on its side and zero on its caps.
Isolated infinite uniform sheet	Normal field with equal magnitudes and opposite directions on the two sides.	Symmetric pillbox; two nonzero cap terms, $2EA$ for positive sheet charge.
Locally smooth conductor surface	Equilibrium makes the interior field zero and removes the exterior tangential field.	Infinitesimal pillbox; one nonzero cap, $E_{\perp} A$. Global source symmetry is unnecessary for this local relation.
Finite rod, ring, or disk	Some directions may cancel, but the field is not constant on a suitable whole closed boundary.	Gauss's law gives a constraint; direct integration or a boundary-value solution is generally needed to find the local field.

For example, a charged ring has axial symmetry, but a sphere centred on the ring does not have a constant radial field over its surface. The field at its poles and near its equator has different geometry. A cylinder also fails to produce a simple total flux: both its caps and side can contribute unknown, spatially varying components. Knowing the ring's total charge determines the flux of either enclosing surface, but it does not determine the field at a chosen point on the axis.

An analogous failure occurs for a finite rod. Away from its perpendicular bisector, the field can have an axial component; even on a coaxial cylindrical side its magnitude varies with axial position. The caps then cannot simply be discarded. The infinite-line answer can be a good approximation in a restricted central region, but that is a claim about the finite model's error, not an exact consequence

of the chosen cylindrical surface. The finite-source calculations in the enrichment section quantify one such approximation.

3.8 Spherical symmetry

Unless stated otherwise, the symmetric examples below describe the field of the specified source alone, with no external applied field or nearby conductor breaking the symmetry.

Spherical symmetry is the cleanest setting for Gauss's law. If a charge distribution is centred at a point and looks the same after any rotation about that point, the electric field must be radial and its signed radial component can depend only on the distance r from the centre:

$$\mathbf{E}(\mathbf{r}) = E_r(r)\hat{\mathbf{r}}. \quad (3.39)$$

A sphere of radius r centred on the distribution is the natural Gaussian surface. See Fig. 3.6. On that surface, $d\mathbf{A} = \hat{\mathbf{r}}dA$, so

$$\mathbf{E} \cdot d\mathbf{A} = E_r(r)dA. \quad (3.40)$$

Because $E_r(r)$ is constant over the spherical surface,

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = E_r(r) \oint_S dA = E_r(r)4\pi r^2. \quad (3.41)$$

Gauss's law then gives

$$E_r(r)4\pi r^2 = \frac{q_{\text{enc}}(r)}{\epsilon_0}. \quad (3.42)$$

Thus

$$E_r(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{enc}}(r)}{r^2}. \quad (3.43)$$

This compact equation is valid for spherically symmetric charge distributions, with $q_{\text{enc}}(r)$ the charge inside radius r .

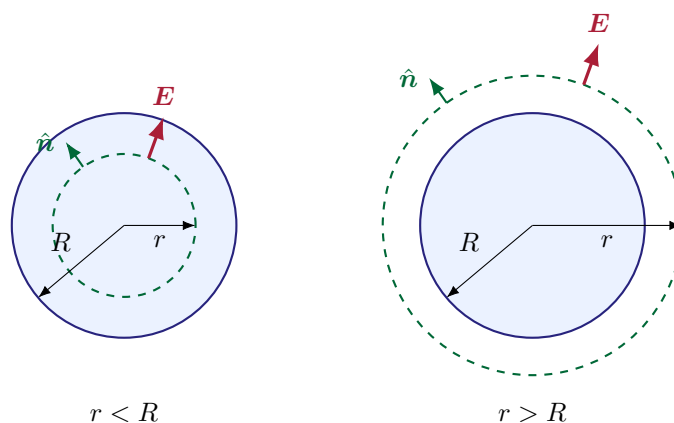


Figure 3.6: Cross-sections of concentric Gaussian spheres (dashed) inside and outside a uniformly charged solid sphere (blue), shown for $\rho > 0$. The source radius R is fixed; the Gaussian radius r selects the field point. Only the charge within the dashed surface enters $q_{\text{enc}}(r)$.

Example 3.9: Uniformly charged spherical shell

A thin spherical shell of radius R carries total charge Q uniformly over its surface. Find the electric field for $r > R$ and for $r < R$.

Solution. Use a concentric spherical Gaussian surface of radius r .

For $r > R$, the Gaussian surface encloses the full charge Q . Spherical symmetry gives

$$E_r(4\pi r^2) = \frac{Q}{\varepsilon_0}, \quad (3.44)$$

so

$$\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}, \quad r > R. \quad (3.45)$$

The shell acts externally like a point charge at its centre.

For $r < R$, the Gaussian surface encloses no charge, so

$$E_r(4\pi r^2) = 0. \quad (3.46)$$

Therefore

$$\mathbf{E}(r) = \mathbf{0}, \quad r < R. \quad (3.47)$$

The vanishing field inside the uniformly charged shell follows from symmetry and Gauss's law, not from the absence of charges on the shell.

Example 3.10: Uniformly charged nonconducting sphere

A nonconducting sphere of radius R has uniform volume charge density ρ and total charge $Q = \rho(4\pi R^3/3)$. Find $\mathbf{E}(r)$ inside and outside the sphere.

Solution. Again use a concentric spherical Gaussian surface.

For $r > R$, the full charge is enclosed:

$$E_r(4\pi r^2) = \frac{Q}{\varepsilon_0}, \quad \mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}. \quad (3.48)$$

For $r < R$, only the charge inside radius r is enclosed:

$$q_{\text{enc}}(r) = \rho \frac{4\pi r^3}{3} = Q \frac{r^3}{R^3}. \quad (3.49)$$

Then

$$E_r(4\pi r^2) = \frac{Qr^3}{\varepsilon_0 R^3}, \quad (3.50)$$

so

$$\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Qr}{R^3} \hat{\mathbf{r}} = \frac{\rho r}{3\varepsilon_0} \hat{\mathbf{r}}, \quad r < R. \quad (3.51)$$

The field grows linearly from zero at the centre to the surface value $Q/(4\pi\varepsilon_0 R^2)$ at $r = R$.

Guided checks

1. Why is a concentric spherical Gaussian surface appropriate for a uniformly charged sphere? Because the charge distribution is rotationally invariant.
2. Does Gauss's law say the field inside every charged shell is zero? No. The conclusion requires a uniformly charged spherical shell.
3. For a uniformly charged solid sphere, does q_{enc} scale as r , r^2 , or r^3 inside? It scales as r^3 .
4. Why does E inside the solid sphere scale as r ? Because $E_r 4\pi r^2 = q_{\text{enc}}/\varepsilon_0 \propto r^3$.

3.8.1 Reading a field profile

For positive charge, Fig. 3.7 compares a uniform volume distribution with a thin surface shell of equal total charge and radius. The exterior fields coincide because both systems are spherically symmetric and enclose the same Q . Their interior fields differ because $q_{\text{enc}}(r)$ differs. An exterior measurement alone therefore cannot distinguish these two radial charge distributions.

For the uniform sphere, the linear rise near the centre has a simple interpretation. The enclosed charge grows as r^3 , while the flux-carrying area grows as r^2 . Their ratio grows as r . At $r = 0$ the field is zero by symmetry; the radial unit vector is undefined at that single point, so the vector result there is specified by its zero limit.

A finite volume density changes abruptly to zero at the surface, but there is no infinitely thin layer carrying a finite charge per area. The field is therefore continuous at R , though its derivative need not be. For the thin shell, all charge sits in an ideal surface layer. The radial component jumps from zero to $Q/(4\pi\epsilon_0 R^2)$. The notation R^- and R^+ means approaching from inside and outside; it avoids assigning an ambiguous field directly on an ideal sheet.

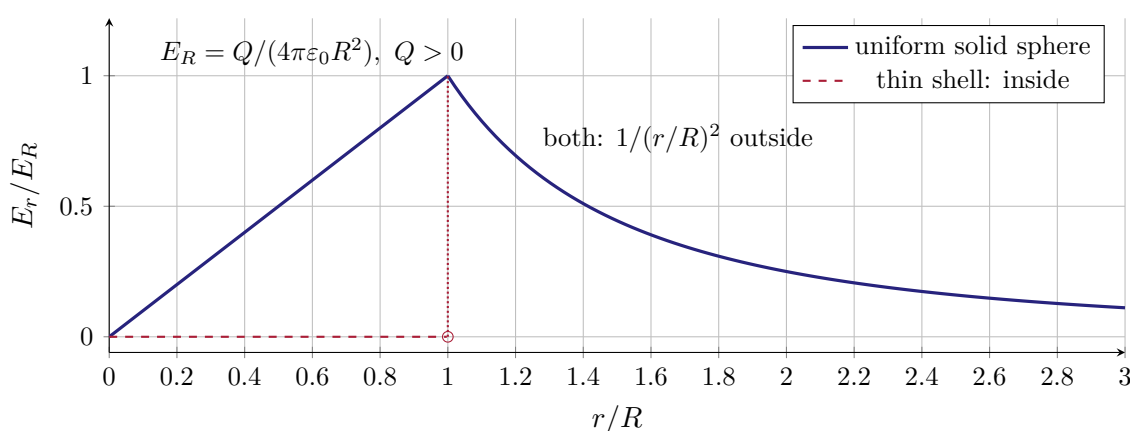


Figure 3.7: Calculated field profiles for equal positive Q and radius R . A uniform volume charge gives a continuous field at R ; an ideal surface shell gives a jump. The dotted vertical segment indicates a discontinuity, not field values assigned at the singular sheet.

Example 3.11: Designing a spherical field model from measurements

An isolated spherically symmetric source lies within $R = 0.10$ m. A radial field of 900 N/C is measured at $r = 0.20$ m, directed outward. Determine the total charge. Would this measurement establish that the source is a conducting sphere?

Solution. Spherical symmetry makes the field uniform in radial component on a sphere through the measurement point:

$$Q = 4\pi\epsilon_0 r^2 E_r = 4.01 \times 10^{-9} \text{ C} \simeq 4.0 \text{ nC.} \quad (3.52)$$

The positive sign follows from the outward field. The dimensions of $\epsilon_0 r^2 E_r$ are charge. A conductor, a thin uniformly charged insulating shell, and many nonuniform spherical volume distributions can all give this exterior field. To distinguish them, one needs further physical information or interior measurements. If the symmetry were unknown, a single local field measurement would not determine Q .

3.9 Cylindrical symmetry

For a very long charged line or cylinder, the idealized symmetry is cylindrical. The field points radially away from the axis for positive charge and radially inward for negative charge. Its magnitude depends only on the perpendicular distance r from the axis.

The natural Gaussian surface is a cylinder of radius r and length L , coaxial with the charge distribution. See Fig. 3.8. The closed surface consists of the curved side and two end caps. On the curved side, $\mathbf{E}$ is parallel or antiparallel to $d\mathbf{A}$ and has constant signed radial component. On the end caps, $\mathbf{E}$ is tangent to the caps, so $\mathbf{E} \cdot d\mathbf{A} = 0$. Therefore

$$\oint \mathbf{E} \cdot d\mathbf{A} = E_r(2\pi rL). \quad (3.53)$$

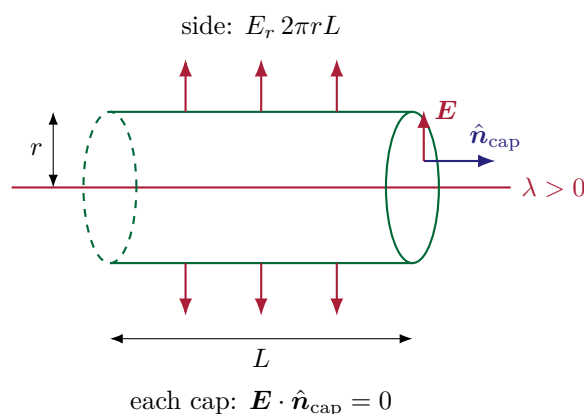


Figure 3.8: A closed coaxial Gaussian cylinder. The charged line continues beyond both caps. Radial fields are tangent to the caps; the cap normal shown is axial. The nonzero flux comes from the curved side.

Example 3.12: Electric field of a long line of charge

An ideal infinitely long line carries uniform linear charge density λ . Find the electric field at distance r from the line.

Solution. Choose a coaxial cylindrical Gaussian surface of radius r and length L . The enclosed charge is

$$q_{\text{enc}} = \lambda L. \quad (3.54)$$

The flux is only through the curved side:

$$\Phi_E = E_r(2\pi rL). \quad (3.55)$$

Gauss's law gives

$$E_r(2\pi rL) = \frac{\lambda L}{\epsilon_0}, \quad (3.56)$$

so

$$\mathbf{E}(r) = \frac{\lambda}{2\pi\epsilon_0 r} \hat{\mathbf{r}}. \quad (3.57)$$

For $\lambda > 0$ the direction is radially outward; for $\lambda < 0$ it is radially inward.

Example 3.13: Uniformly charged nonconducting cylinder

A very long nonconducting cylinder of radius R has uniform volume charge density ρ . Find the electric field inside and outside the cylinder.

Solution. For a coaxial Gaussian cylinder of radius r and length L , the flux is $E_r(2\pi rL)$. For $r > R$, the enclosed charge is

$$q_{\text{enc}} = \rho\pi R^2 L. \quad (3.58)$$

Thus

$$E_r(2\pi rL) = \frac{\rho\pi R^2 L}{\varepsilon_0}, \quad \mathbf{E}(r) = \frac{\rho R^2}{2\varepsilon_0 r} \hat{\mathbf{r}}. \quad (3.59)$$

Equivalently, with $\lambda = \rho\pi R^2$,

$$\mathbf{E}(r) = \frac{\lambda}{2\pi\varepsilon_0 r} \hat{\mathbf{r}}. \quad (3.60)$$

For $r < R$, the enclosed charge is

$$q_{\text{enc}} = \rho\pi r^2 L. \quad (3.61)$$

Therefore

$$E_r(2\pi rL) = \frac{\rho\pi r^2 L}{\varepsilon_0}, \quad \mathbf{E}(r) = \frac{\rho r}{2\varepsilon_0} \hat{\mathbf{r}}. \quad (3.62)$$

The field grows linearly inside and falls as $1/r$ outside.

3.9.1 Interior versus exterior scaling

The cylindrical result can be written as $E_r(r) = \lambda_{\text{enc}}(r)/(2\pi\varepsilon_0 r)$, where λ_{enc} is the enclosed charge per unit length. Inside a uniform cylinder it grows as r^2 , so the field grows linearly. Outside, it is fixed, so the field falls as $1/r$. The arbitrary Gaussian length L cancels because both enclosed charge and side area are proportional to L .

This cancellation does not assert that the actual object has length L . In the infinite model, the physical cylinder extends beyond both end caps. For a finite cylinder, the approximation is best away from its ends and at observation distances small compared with the distance to either end. The quantitative rod test in Sec. 3.19.13 shows how to turn this condition into an error estimate.

At a sharp boundary between two finite volume densities, the radial field remains continuous unless an additional surface charge is present. For the uniform cylinder, both expressions at $r = R$ give $\rho R/(2\varepsilon_0)$. This is a useful check on the factor of two: the corresponding uniform sphere gives $\rho R/(3\varepsilon_0)$ because its enclosed volume and Gaussian area have different geometry.

3.10 Planar symmetry

An ideal infinite sheet with uniform surface charge density σ has planar symmetry. The field must be perpendicular to the sheet, and by symmetry it must have the same magnitude at equal distances on the two sides. A Gaussian pillbox straddling the sheet is the natural choice. See Fig. 3.9. The flux through the side wall of the pillbox is zero because the field is tangent to that wall. For $\sigma > 0$, the flux through the two flat caps is $EA + EA = 2EA$; for negative charge both cap contributions reverse sign.

Example 3.14: Electric field of an infinite charged sheet

An ideal infinite sheet carries uniform surface charge density σ . Find the electric field on either side.

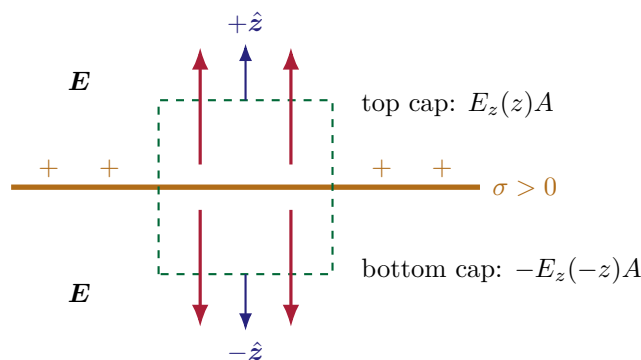


Figure 3.9: Side view of a pillbox straddling an isolated uniform sheet. The outward normals on the two caps point in opposite directions. For positive charge, both cap fluxes are positive. The dashed boundary is a mathematical surface, not an enclosure made of metal.

Solution. First take $\sigma > 0$, so E is the outward field magnitude on each side. Use a pillbox of cap area A crossing the sheet. The enclosed charge is

$$q_{\text{enc}} = \sigma A. \quad (3.63)$$

The field is perpendicular to the sheet and has the same magnitude on both sides. Thus the net flux is

$$\Phi_E = EA + EA = 2EA. \quad (3.64)$$

Gauss's law gives

$$2EA = \frac{\sigma A}{\epsilon_0}, \quad (3.65)$$

so

$$E = \frac{\sigma}{2\epsilon_0}. \quad (3.66)$$

For either sign, the magnitude is $|\sigma|/(2\epsilon_0)$. For $\sigma > 0$, the field points away from the sheet on both sides; for $\sigma < 0$, it points toward the sheet on both sides. At $z \neq 0$ the signed component is $E_z = \sigma \operatorname{sgn}(z)/(2\epsilon_0)$ for a sheet at $z = 0$.

Example 3.15: Field between two oppositely charged large plates

Two large parallel sheets carry surface charge densities $+\sigma$ and $-\sigma$, with $\sigma > 0$. Neglect edge effects. Find the field between the plates and outside them.

Solution. A single infinite positive sheet produces field magnitude $\sigma/(2\epsilon_0)$ away from itself. A single infinite negative sheet produces field magnitude $\sigma/(2\epsilon_0)$ toward itself.

Between the sheets, the two fields point in the same direction, from the positive sheet toward the negative sheet. Therefore

$$E_{\text{between}} = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}. \quad (3.67)$$

Outside the pair, the two fields are equal in magnitude and opposite in direction, so

$$E_{\text{outside}} = 0 \quad (3.68)$$

in the ideal infinite-sheet approximation. This result is the starting point for the parallel-plate capacitor model developed later.

Guided checks

1. Why does an infinite sheet have field perpendicular to the sheet? Any parallel component would choose a preferred direction in the plane, violating symmetry.
2. Why do both caps of the pillbox contribute positive flux for a positive sheet? The outward normal points upward on the top cap and downward on the bottom cap, matching the field directions.
3. Why is the field of one sheet independent of distance in the ideal model? The pillbox result contains no distance variable; this is a consequence of infinite planar symmetry.
4. Does a real finite sheet have exactly distance-independent field everywhere? No. The approximation works near the central region when the distance from edges is large compared with the plate separation or observation distance.

3.10.1 The factor of two and the meaning of surface density

For a uniform insulating sheet in vacuum, σ is the charge per area of that one mathematical sheet. Equal flux emerges on its two sides. For one face of a conductor, σ is the local charge per area *on that face*, and the field on the metal side is zero. These are different boundary conditions, not competing versions of Gauss's law.

For example, an isolated broad conducting plate carrying total excess charge Q and face area A has, in the symmetric infinite-plate approximation, $Q/2$ on each face. Each face then has density $Q/(2A)$ and exterior field magnitude $|Q|/(2\epsilon_0 A)$. If two such plates carry $+Q$ and $-Q$ and are brought close together, charge redistributes onto the facing surfaces in the ideal parallel-plate model. Their facing densities become $\pm Q/A$, and the gap field has magnitude $|Q|/(\epsilon_0 A)$. Always identify which physical face a quoted σ describes.

Example 3.16: Unequal charged sheets: a sign table prevents mistakes

Two prescribed, infinite insulating sheets lie at $z = 0$ and $z = d$. Their fixed surface charge densities are $\sigma_1 = +6.0 \text{ nC/m}^2$ and $\sigma_2 = -2.0 \text{ nC/m}^2$. Determine the field in all three regions, with no externally applied field.

Strategy. Each sheet produces $\sigma_i/(2\epsilon_0)$ in the direction of increasing z above it and the negative of that signed value below it. Sum components, not magnitudes:

$$E_z = \begin{cases} -(\sigma_1 + \sigma_2)/(2\epsilon_0), & z < 0, \\ (\sigma_1 - \sigma_2)/(2\epsilon_0), & 0 < z < d, \\ (\sigma_1 + \sigma_2)/(2\epsilon_0), & z > d. \end{cases} \quad (3.69)$$

Numerically the three values are -226 , $+452$, and $+226 \text{ N/C}$, respectively. The outside field does not vanish because the two sheets have nonzero total charge per area. Across the first sheet the component increases by $678 \text{ N/C} = \sigma_1/\epsilon_0$; across the second it decreases by $226 \text{ N/C} = |\sigma_2|/\epsilon_0$. These jumps provide an independent check. The fields belong to fixed insulating sheets; conducting plates would generally redistribute their charges.

Example 3.17: Surface discontinuity across an ideal charged sheet

An ideal surface charge density σ lies in the plane $z = 0$. Use a pillbox to find the discontinuity in the normal electric field across the sheet.

Solution. Let the pillbox have cap area A and negligible thickness. The side flux vanishes in

the thin limit. If E_z^+ is the field just above and E_z^- just below, the outward flux is

$$\Phi_E = E_z^+ A - E_z^- A. \quad (3.70)$$

The enclosed charge is σA . Gauss's law gives

$$E_z^+ - E_z^- = \frac{\sigma}{\varepsilon_0}. \quad (3.71)$$

This boundary condition is more general than the isolated infinite-sheet result.

3.10.2 From a sheet to a volume: a uniform slab

A sheet idealizes charge with negligible thickness. A slab instead spreads charge through a finite thickness, while retaining infinite extent in the other two directions. Let uniform density $\rho > 0$ occupy $-a < x < a$, with no additional surface sheets or applied field. Translation within the slab planes eliminates dependence on y and z . Reflection through the midplane makes $E_x(-x) = -E_x(x)$ and therefore $E_x(0) = 0$.

For $0 < x < a$, choose a closed pillbox with caps at $-x$ and $+x$ and cap area A . The field on each cap is outward, with magnitude $E_x(x)$. The pillbox encloses volume $2xA$, so

$$2AE_x(x) = \frac{\rho(2xA)}{\varepsilon_0}, \quad E_x(x) = \frac{\rho x}{\varepsilon_0}. \quad (3.72)$$

For $x > a$, increasing the pillbox thickness encloses no additional charge: the charged volume inside it is always $2aA$. Its cap area is also unchanged. Hence the exterior field is constant, $E_x = \rho a/\varepsilon_0$ on the right and the negative of that value on the left. Combining the regions gives

$$E_x(x) = \begin{cases} -\rho a/\varepsilon_0, & x < -a, \\ \rho x/\varepsilon_0, & -a \leq x \leq a, \\ +\rho a/\varepsilon_0, & x > a. \end{cases} \quad (3.73)$$

Figure 3.10 emphasizes the difference between a finite density discontinuity and an ideal surface charge. The density drops abruptly at $\pm a$, so the slope of the field changes abruptly. The field itself remains continuous. Only a finite charge per area concentrated in a vanishing thickness creates a jump in the normal field.

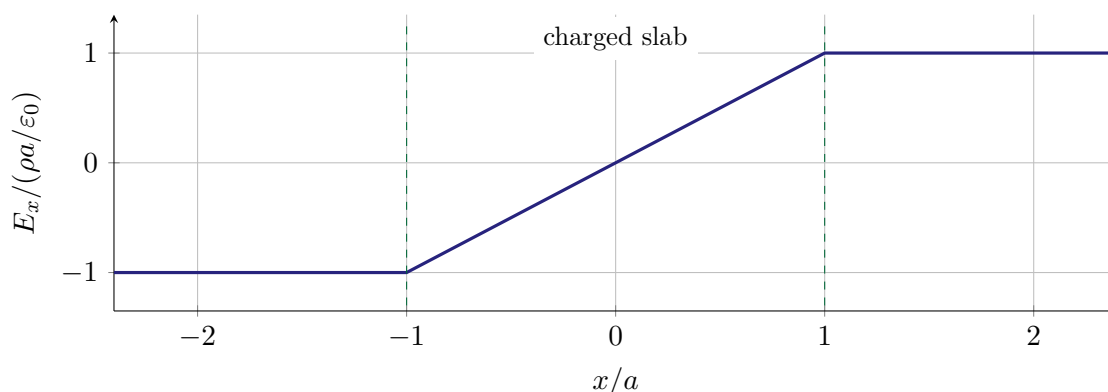


Figure 3.10: Signed field of a uniformly charged infinite slab, with $\rho > 0$ and no applied background field. The dashed lines mark the material boundaries. Negative E_x on the left means the field points toward decreasing x , away from the positive slab.

In the thin-slab limit, keep the charge per area $\sigma = 2a\rho$ fixed while reducing a . The exterior values remain $\pm\sigma/(2\varepsilon_0)$, whereas the linear transition is compressed into an ever smaller interval. This

recovers the sheet result and its field jump. By contrast, reducing a while keeping ρ fixed sends the charge per area and the exterior field to zero. A limiting argument must say which physical quantity is held fixed.

The slab also clarifies why dimensional reasoning helps but is not sufficient. The product $\rho x/\epsilon_0$ has field units, yet dimensional analysis cannot supply the boundary condition $E_x(0) = 0$ or distinguish a possible added uniform field. Those come from the source symmetry and the absence of an applied background. An asymmetric density profile requires a different argument, developed in Sec. 3.19.10.

3.11 Conductors in electrostatic equilibrium

A conductor in electrostatic equilibrium has no sustained motion of free charge. This requires the electric field inside the conducting material to be zero. If a nonzero electric field existed inside, free charges would accelerate and rearrange until the field was cancelled. Several important consequences follow.

1. The electric field inside conducting material is zero in electrostatic equilibrium.
2. Any excess charge on an isolated conductor resides on its surface.
3. The electric field just outside a conducting surface is perpendicular to the surface.
4. The normal field just outside a conducting surface is related to the local surface charge density by

$$E_{\perp} = \frac{\sigma}{\epsilon_0}. \quad (3.74)$$

The last relation follows from a small Gaussian pillbox crossing the conductor surface. See Fig. 3.11. Inside the conductor, $E = 0$. If the pillbox is thin, the side-wall flux is negligible. The normal is directed out of the metal, and $E_{\perp}$ is signed along it. The outside cap contributes $E_{\perp}A$, and the enclosed charge is σA , so

$$E_{\perp}A = \frac{\sigma A}{\epsilon_0}, \quad (3.75)$$

which gives Eq. (3.74).

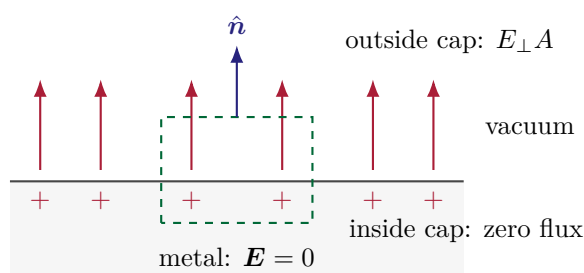


Figure 3.11: Local conductor boundary, shown for $\sigma > 0$. One cap of the thin Gaussian pillbox lies in the metal, where the macroscopic field vanishes. The other lies in vacuum. In general $E_{\perp} = \sigma/\epsilon_0$ is a signed component; the field magnitude is $|\sigma|/\epsilon_0$.

Example 3.18: Surface field near a charged conductor

At a point on the surface of a conductor in electrostatic equilibrium, the surface charge density is $\sigma = 3.0 \text{ nC/m}^2$. Find the electric field just outside the conductor.

Solution. The field just outside is perpendicular to the surface and has magnitude

$$E = \frac{\sigma}{\epsilon_0} = \frac{3.0 \times 10^{-9} \text{ C/m}^2}{8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)} = 3.39 \times 10^2 \text{ N/C.} \quad (3.76)$$

If σ is positive, the field points outward from the conductor. If σ is negative, it points inward toward the conductor.

Caution

The condition $\mathbf{E} = 0$ inside a conductor applies to the conducting material in electrostatic equilibrium. It does not imply a zero field in a cavity containing charge. A completely enclosed charge-free cavity has zero field for the stronger reason discussed next.

The conductor statements refer to the macroscopic field, averaged over atomic distances, in the elementary ideal-conductor model. They do not claim that microscopic fields at atomic positions vanish. Nor do they apply to a resistive wire carrying a steady current: that wire is in a different physical state, studied in Chapter 6.

To see why no excess bulk charge can remain, choose a small closed surface wholly in the metal. Its flux is zero because the field is zero. Its enclosed net charge must therefore vanish. Since the small volume can be selected anywhere in the homogeneous conducting bulk, excess charge must reside at interfaces. The equilibrium condition supplies $\mathbf{E} = 0$ first; a zero charge integral by itself would not prove a zero field.

For a curved surface, choose a pillbox whose lateral size is small compared with the local curvature radius and the scale over which σ changes. Take its height to zero while keeping the cap area fixed and small. Bounded side fields then give side flux proportional to the vanishing height. The cap field becomes the local normal component. At a sharp ideal corner the field may be singular, so this smooth-surface argument is understood as a limit from nearby smooth points.

3.12 Cavities, shielding, and Faraday cages

Conductors can rearrange charge so that the electric field inside their material vanishes. This is the basis of electrostatic shielding. A closed conducting shell can protect its interior from external static electric fields, provided the shell remains conducting and the situation is electrostatic. Charges induced on the outer surface arrange themselves to cancel the field within the conducting material and, in an empty cavity, to keep the field zero.

If a charge is placed inside a cavity in a conductor, the situation changes. A Gaussian surface drawn within the conducting material around the cavity has zero flux because $\mathbf{E} = 0$ in the conductor. Therefore the net charge on the inner cavity wall must be the negative of the charge inside the cavity. If the conductor as a whole has net charge, the remaining charge resides on the outer surface.

Example 3.19: Charge inside a conducting cavity

An isolated, initially neutral conducting shell contains a cavity. A point charge $+q$ is placed inside the cavity without touching the conductor. What is the net charge induced on the inner cavity surface and on the outer conductor surface?

Solution. Draw a Gaussian surface entirely within the conducting material, enclosing the

cavity but not crossing the outer surface. Since the field in the conducting material is zero,

$$\oint \mathbf{E} \cdot d\mathbf{A} = 0. \quad (3.77)$$

Gauss's law gives $q_{\text{enc}} = 0$. The enclosed charge consists of the point charge $+q$ plus the induced charge q_{inner} on the cavity wall:

$$q + q_{\text{inner}} = 0. \quad (3.78)$$

Thus

$$q_{\text{inner}} = -q. \quad (3.79)$$

The conductor was initially neutral, so the total conductor charge remains zero. Therefore the outer surface must carry

$$q_{\text{outer}} = +q. \quad (3.80)$$

The distribution on the inner surface need not be uniform unless the charge is symmetrically placed, but the net induced inner charge is fixed by Gauss's law.

Key message

Electrostatic shielding is not the absence of electric field everywhere. It is the rearrangement of conductor charges so that the field inside the conducting material, and inside a fully enclosed charge-free cavity, satisfies the electrostatic equilibrium conditions.

Figure 3.12 separates three regions that must not be confused: the cavity, the conducting material, and the exterior. A cavity containing charge normally has a nonzero field. The metal surrounding it still has zero field. The external field depends on the outer boundary conditions and the net charge exposed on the outer surface.

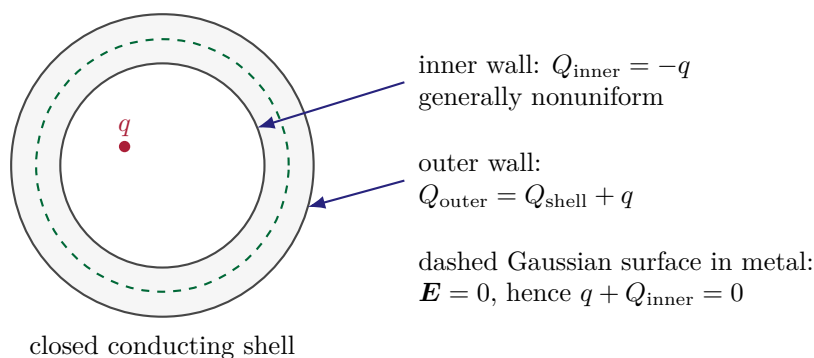


Figure 3.12: Charge accounting for an isolated shell, with no contact between the internal charge and the metal. The Gaussian surface lies entirely in the conducting material. Its zero flux determines the total charge on the inner wall, even for an off-centre charge. No guessed field lines are needed.

3.12.1 Empty cavities and grounding

For a completely enclosed, charge-free cavity bounded by one connected conductor in electrostatic equilibrium, the field is zero. Gauss's law alone gives only zero total charge on the cavity wall; it cannot exclude positive and negative patches individually. The stronger result uses the conductor's constant potential and the uniqueness of an electrostatic solution in a charge-free cavity. Sec. 3.19.12 gives a short proof after introducing the required potential notation as enrichment.

Grounding is a separate condition. An isolated conductor has a specified conserved net charge; a

grounded conductor can exchange charge with a reservoir to maintain a specified potential. For a cavity charge q , the inner-wall total remains $-q$ in either case. For an isolated conductor of net charge Q_{shell} , the outer-wall total is $Q_{\text{shell}} + q$. For a grounded enclosure with no external sources and zero potential at infinity, the exterior field vanishes and the outer-wall charge is zero. The reservoir supplies the required net conductor charge $-q$.

An off-centre charge can produce a nonuniform inner-wall distribution while the exterior of an otherwise isolated spherical shell remains radial. This is a shielding and boundary-uniqueness result: one should not claim that the entire charge configuration is rotationally symmetric. If external charges are added, the outer-wall distribution generally becomes nonuniform while an empty, fully enclosed cavity remains shielded.

Caution

A cage with holes, an open pail, and a completely closed conducting shell are not the same boundary-value problem. Openings permit field penetration, and the accuracy of shielding depends on their geometry. The exact statements here concern closed ideal conductors and static equilibrium. Shielding of time-varying electromagnetic fields requires the induction and wave physics of later chapters.

Example 3.20: Spherical conductor with a central cavity charge

A conducting spherical shell has inner radius a , outer radius b , and net charge Q_{shell} . A point charge q is placed at the centre of the spherical cavity. Assume no external sources and an isolated conductor. Determine the net charges on the inner and outer conducting surfaces and find the field in each radial region.

Solution. In the conducting material $a < r < b$, the electric field is zero. A Gaussian sphere in the metal, with $a < r < b$, has zero flux. It encloses the central charge q and the charge Q_{inner} on the inner surface, so

$$q + Q_{\text{inner}} = 0, \quad Q_{\text{inner}} = -q. \quad (3.81)$$

The conductor has total charge Q_{shell} , so

$$Q_{\text{outer}} = Q_{\text{shell}} - Q_{\text{inner}} = Q_{\text{shell}} + q. \quad (3.82)$$

Because the charge is central, spherical symmetry applies. Thus

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}, \quad 0 < r < a. \quad (3.83)$$

Inside the conductor,

$$\mathbf{E} = \mathbf{0}, \quad a < r < b. \quad (3.84)$$

Outside,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q_{\text{shell}} + q}{r^2} \hat{\mathbf{r}}, \quad r > b. \quad (3.85)$$

The outside world sees the total charge of the whole system.

3.13 Engineering models and charge measurement

3.13.1 Coaxial conductors: locating charge on each surface

Coaxial geometry is useful because it separates a controlled field region from conducting boundaries. Consider a solid inner conductor of radius a , and an outer conducting shell with inner radius b and outer radius c . The system is treated as infinitely long, in vacuum, in electrostatic equilibrium, and without external sources. Let the inner conductor carry line charge λ , and let the *net* line charge of the outer conductor be λ_{out} .

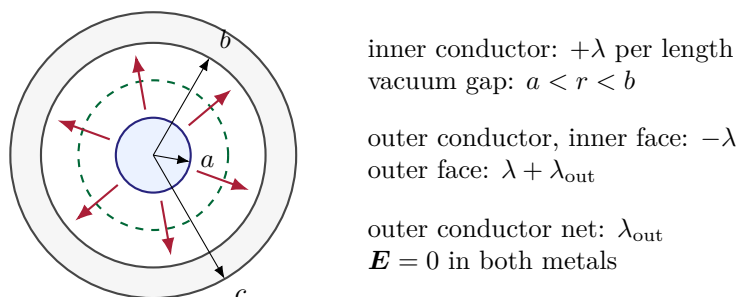


Figure 3.13: Cross-section of an ideal infinitely long coaxial system. Arrows show the gap field for $\lambda > 0$. The exterior field vanishes when $\lambda_{\text{out}} = -\lambda$ and no external field is applied. The dashed circle represents a coaxial cylindrical Gaussian surface.

The reasoning in Fig. 3.13 proceeds from the metal outward. A Gaussian cylinder lying in the outer metal has zero flux. Thus the charge per length on the inner face of the outer shell is $-\lambda$. The remaining charge per length on its outer face is $\lambda_{\text{out}} + \lambda$. The complete signed field is

$$\mathbf{E}(r) = \begin{cases} \mathbf{0}, & r < a, \\ \frac{\lambda}{2\pi\epsilon_0 r} \hat{\mathbf{r}}, & a < r < b, \\ \mathbf{0}, & b < r < c, \\ \frac{\lambda + \lambda_{\text{out}}}{2\pi\epsilon_0 r} \hat{\mathbf{r}}, & r > c. \end{cases} \quad (3.86)$$

Consequently, equal and opposite conductor charges confine the electrostatic field to the gap. Zero exterior flux gives zero exterior field here only because cylindrical symmetry is also established. A real cable's ends, apertures, dielectric filling, and time-dependent signals require additional modelling.

Example 3.21: Maximum gap field in a vacuum coaxial geometry

Take $a = 1.0$ mm, $b = 5.0$ mm, and $\lambda = 2.0$ nC/m, with a compensating outer-conductor charge. Locate the largest gap field and calculate its magnitude.

Solution. The positive radial component falls as $1/r$, so its largest limiting value in the gap occurs at the inner conductor:

$$E(a^+) = \frac{2.0 \times 10^{-9}}{2\pi(8.854 \times 10^{-12})(1.0 \times 10^{-3})} = 3.60 \times 10^4 \text{ N/C}, \quad (3.87)$$

$$E(b^-) = 7.19 \times 10^3 \text{ N/C}. \quad (3.88)$$

The ratio is $E(a^+)/E(b^-) = b/a = 5$, a dimensionless check independent of λ . The inner-conductor surface density is $\sigma_a = \lambda/(2\pi a)$, and σ_a/ϵ_0 gives the same inner field. On the inner face of the outer conductor, the normal from metal into the gap is $-\hat{\mathbf{r}}$. Its negative surface density therefore produces a field in $+\hat{\mathbf{r}}$, consistent with the gap solution. No material breakdown threshold is assumed by this electrostatic calculation.

3.13.2 Rounded terminals: what a simple model predicts

For an isolated conducting sphere, $E(R^+) = Q/(4\pi\epsilon_0 R^2)$. At fixed Q , doubling R reduces the surface field by a factor of four. This gives a useful geometric design trend, but it is not a general local curvature law for arbitrary conductors. The surface charge on an irregular object depends on the complete shape and nearby boundaries. A protruding tip often enhances the field, yet Gauss's law alone cannot calculate that enhancement.

For a terminal with $R = 0.30\text{ m}$ and $Q = 1.0\text{ }\mu\text{C}$, the spherical approximation gives $E(R^+) \simeq 1.0 \times 10^5\text{ N/C}$. Whether discharge occurs also depends on the surrounding medium, pressure, gap geometry, and surface condition. The calculation predicts a field for a stated idealization; it does not supply a universal operating limit. Potential-controlled terminals require another comparison, developed in Chapter 4.

3.13.3 Faraday pail: induction versus charge transfer

Lower a small charged conducting object into a deep metal pail without touching the wall. In the closed-enclosure idealization, a cavity charge $+q$ induces $-q$ on the inside and $+q$ on the outside of an initially neutral, isolated enclosure. Removing the object without contact restores the original uncharged state. Thus an electrical indication during insertion need not imply that charge has been transferred to the pail.

If the object touches the inside and forms one connected conductor with the enclosure, the combined conductor carries net charge q . With no remaining separately charged object in the cavity, the excess charge resides on the exterior. Disconnecting and removing the small object after it has equilibrated within the ideal enclosed cavity leaves the pail charged. The distinction between these two procedures is a direct experimental test of induction and charge conservation.

A real pail is open. The induced-charge relation approaches the closed-shell result when the object is sufficiently far below the opening; measuring the reading versus insertion depth tests that approximation. Other useful controls are zeroing the electrometer, checking leakage versus time, avoiding accidental contact, and repeating the insertion with both charge signs. An electrometer may sense an induced electrode charge or a calibrated voltage; it does not directly count drawn field lines.

Example 3.22: A calibrated charge measurement with uncertainty

A charge-measuring amplifier has calibration $q_{\text{ind}} = C_f V_{\text{out}}$ after the sign convention has been established experimentally. Its calibration constant is $C_f = (100.0 \pm 1.0)\text{ pF}$ and its baseline-subtracted reading is $V_{\text{out}} = (0.250 \pm 0.003)\text{ V}$. Treat the two quoted uncertainties as independent standard uncertainties. Find q_{ind} and the ideal flux associated with that enclosed charge.

Solution. Since $Q = CV$,

$$q_{\text{ind}} = 25.0\text{ pC}, \quad \frac{u_q}{|q_{\text{ind}}|} = \sqrt{\left(\frac{1.0}{100.0}\right)^2 + \left(\frac{0.003}{0.250}\right)^2} = 0.0156. \quad (3.89)$$

Thus $q_{\text{ind}} = (25.0 \pm 0.4)\text{ pC}$. If the calibrated indication equals the enclosed charge, Gauss's law predicts

$$\Phi_E = (2.82 \pm 0.04)\text{ N m}^2/\text{C}. \quad (3.90)$$

The uncertainty in ϵ_0 is negligible at this precision. Aperture leakage, charge loss, and incomplete collection are systematic effects not included in these two calibration uncertainties. The example converts a calibrated electrical measurement into a flux prediction; it does not claim that the apparatus measured a surface integral directly.

3.14 Computing flux through a cube

The cube experiment tests a precise physical claim: moving a point charge inside a closed surface changes the local field but not the integrated flux. It also tests whether an implementation uses the outward normal correctly. Unlike a centred sphere, a cube offers no constant-field shortcut over an individual face.

Let the cube occupy $-a \leq x, y, z \leq a$, and place a point charge $q \neq 0$ at $\mathbf{r}_q$, away from its boundary. At each observation point use the Coulomb field

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_q}{|\mathbf{r} - \mathbf{r}_q|^3}. \quad (3.91)$$

Subdivide each face into $N \times N$ equal squares. With midpoint locations $\mathbf{r}_{ij}$ and area $\Delta A = (2a/N)^2$, calculate

$$\Phi_N = \sum_{\text{six faces}} \sum_{i,j} \mathbf{E}(\mathbf{r}_{ij}) \cdot \hat{\mathbf{n}}_{\text{face}} \Delta A. \quad (3.92)$$

The normals are $-\hat{\mathbf{x}}$ on $x = -a$, $+\hat{\mathbf{x}}$ on $x = +a$, and similarly for y and z . Do not take absolute values of the dot products. An external charge requires cancellation of substantial positive and negative face contributions.

3.14.1 A dimensionless implementation

Use coordinates measured in units of a and report

$$f_N = \frac{\epsilon_0 \Phi_N}{q}. \quad (3.93)$$

The exact value is one for a charge inside and zero for a charge outside. Rescaling all lengths by the same factor leaves f_N unchanged because $E \propto a^{-2}$ and $\Delta A \propto a^2$. This normalization separates geometry and numerical error from the numerical values of q and ϵ_0 .

The following Python code uses only the standard library. The argument $\mathbf{r}_q$ is the dimensionless charge position. The order of the face loop ensures that both signs of each coordinate normal are included.

```
from math import fsum, pi

def cube_flux(n, rq):
    h = 2.0/n
    u = [-1.0+(i+0.5)*h for i in range(n)]
    face_flux = []
    for axis in range(3):
        other = [j for j in range(3) if j != axis]
        for sign in (-1, 1):
            terms = []
            for a in u:
                for b in u:
                    r = [0.0, 0.0, 0.0]
                    r[axis] = float(sign)
                    r[other[0]], r[other[1]] = a, b
                    d = [r[j]-rq[j] for j in range(3)]
                    d2 = fsum(x*x for x in d)
                    terms.append(sign*d[axis]*h*h /
                                (4*pi*d2**1.5))
            face_flux.append(fsum(terms))
    return fsum(face_flux)

for n in (8, 16, 32, 64, 128):
    print(n, cube_flux(n, (0.6, 0.2, -0.3)))
```

3.14.2 Convergence and a near-boundary test

Table 3.2 gives values calculated with this midpoint method. They are synthetic numerical results, not measurements. The displaced interior charge tests broken symmetry; the near-face charge tests resolution; the exterior charge tests signed cancellation.

N	centre (0, 0, 0)	off centre (0.6, 0.2, -0.3)	near face (0.95, 0, 0)	outside (1.5, 0, 0)
8	1.003829	1.003797	0.743690	-0.001234
16	1.000957	1.000956	0.895752	-0.000310
32	1.000239	1.000239	0.988639	-0.000077
64	1.000060	1.000060	0.999938	-0.000019
128	1.000015	1.000015	1.000006	-0.000005
exact	1	1	1	0

Table 3.2: Computed normalized flux f_N through the cube $[-1, 1]^3$. Each face has N^2 sample points. Values are rounded to six decimal places.

For the centred and ordinary off-centre cases, doubling N reduces the error by approximately a factor of four. That is the expected second-order behaviour of midpoint quadrature for a smooth integrand. The near-face case behaves differently on coarse grids. The closest distance from the charge to the surface is $d = 0.05a$, while the patch width for $N = 8$ is $0.25a$. A sharply peaked normal field cannot be resolved by patches wider than its characteristic scale. Refinement first resolves the peak; only then should one expect the usual asymptotic convergence.

The small negative result for the exterior charge is not negative enclosed charge. It is the quadrature error left after cancellation. A relative error formed by dividing by the exact exterior flux is undefined because that flux is zero. Use the dimensionless absolute residual $|f_N|$ instead. For an interior charge, the natural normalized error is $|f_N - 1|$.

Example 3.23: Interpreting a numerical discrepancy

For $q = 5.00 \text{ nC}$ inside the cube, the exact flux is

$$\Phi_E = \frac{5.00 \times 10^{-9}}{8.854 \times 10^{-12}} = 5.65 \times 10^2 \text{ N m}^2/\text{C}. \quad (3.94)$$

At the near-face position with $N = 8$, the normalized result is 0.743690. The estimate is about $420 \text{ N m}^2/\text{C}$, or 25.6% below the exact value. Increasing to $N = 128$ reduces the normalized discrepancy to approximately 5.7×10^{-6} .

The charge and the physical law did not change. Only the numerical resolution changed. Before interpreting a field-map residual as missing charge, check mesh convergence, normals, units, and the distance of singular sources from the sampled boundary. A point charge exactly on the boundary is outside the assumptions of this elementary test.

For several charges, sum their field contributions before forming each face flux, or use the linearity of the flux integral to sum separate charge results. A useful further test adds an external charge while holding all interior charges fixed. The total exact flux must remain unchanged. Passing that test verifies a global constraint; it does not prove that every local field component is accurate.

3.15 Common misconceptions and diagnostic tests

Gauss's law is compact, and compact equations can hide conceptual traps. The most common difficulties are listed in Table 3.3. Students should use these as diagnostic tests while solving problems.

Table 3.3: Common misconceptions about flux and Gauss's law.

Misconception	Why it is wrong	Repair strategy
Flux is the same as field magnitude.	Flux includes area, orientation, and sign.	Always write $\mathbf{E} \cdot d\mathbf{A}$.
Zero net flux means zero field.	Positive and negative flux can cancel.	Test with a uniform field through a closed cube.
All charges producing $\mathbf{E}$ go into q_{enc} .	Only enclosed charges appear on the right side.	Draw the Gaussian surface before writing q_{enc} .
Any closed surface makes Gauss's law easy.	Poor surfaces may leave an impossible integral.	Choose surfaces adapted to symmetry.
A Gaussian surface is a physical boundary.	It is usually imaginary.	Do not assign material properties to it unless it is a real conductor surface.
The field is always constant on a Gaussian surface.	This is true only for selected symmetric surfaces.	Justify constancy from symmetry before factoring E out of the integral.
A conductor always has no field everywhere inside its outline.	A cavity containing charge is not part of the conducting material.	Distinguish conducting material from empty cavity.

Guided checks

1. If a Gaussian sphere encloses $+3\mu\text{C}$ and $-3\mu\text{C}$, what is the net flux? Zero.
2. Does zero net flux prove that no charges are inside? No. It proves only that the algebraic enclosed charge is zero.
3. If E is larger on one side of a closed surface than another because of external charges, can Gauss's law still hold? Yes.
4. In a conductor, where does excess charge reside in electrostatic equilibrium? On surfaces.

3.16 Chapter summary and formula guide

Electric flux is an oriented surface integral. For an open surface its sign depends on the chosen normal; for a closed surface the normal is outward. Gauss's law constrains the net closed-surface flux using the algebraic enclosed charge. External charges contribute to the local field, although their net flux contribution is zero. A field calculation needs additional information from symmetry, superposition, or conductor equilibrium.

Table 3.4: Core results and their conditions. Radial components and surface densities are signed.

System or quantity	Result	Required conditions
Oriented flux	$\Phi_E = \int_S \mathbf{E} \cdot \hat{\mathbf{n}} dA$	Normal specified; closed surfaces use outward normals.
Uniform field, flat patch	$\Phi_E = EA \cos \theta$	θ is measured from the normal.
Gauss's law	$\oint_S \mathbf{E} \cdot d\mathbf{A} = q_{\text{enc}}/\epsilon_0$	Closed surface; total field and total enclosed charge.
Spherical source	$E_r = q_{\text{enc}}(r)/(4\pi\epsilon_0 r^2)$	Complete spherical symmetry; concentric Gaussian sphere.
Uniform insulating ball	$E_r = \rho r/(3\epsilon_0)$ inside; kQ/r^2 outside	Radius R , $Q = 4\pi\rho R^3/3$; no additional sources.
Infinite cylindrical source	$E_r = \lambda_{\text{enc}}(r)/(2\pi\epsilon_0 r)$	Axial translation, rotation, and reflection symmetries.
Uniform insulating cylinder	$E_r = \rho r/(2\epsilon_0)$ inside; $\rho R^2/(2\epsilon_0 r)$ outside	Infinite cylinder of radius R ; no applied field.
Isolated infinite sheet	$ \mathbf{E} = \sigma /(2\epsilon_0)$	Equal magnitudes on both sides; direction set by charge sign.
Ideal surface charge	$E_n^+ - E_n^- = \sigma/\epsilon_0$	One fixed normal from the minus side to the plus side.
Conductor boundary	$\mathbf{E}_{\text{out}} = (\sigma/\epsilon_0)\hat{\mathbf{n}}$	Electrostatic equilibrium; smooth vacuum interface; normal out of metal.
Charge in closed cavity	$Q_{\text{inner}} = -q$	Gaussian surface entirely in metal; q is cavity charge.
Isolated conductor	$Q_{\text{outer}} = Q_{\text{conductor}} + q$	Conserved conductor charge, one charged cavity.

Method

A final check before accepting a field calculation. Identify the physical source and the mathematical surface separately. State why the normal component is constant where it is factored out. Compute the enclosed charge for the actual radial region. Check signs, units, boundary jumps or continuity, and limiting behaviour. If those tests fail, revisit the physical assumptions before changing the algebra.

3.17 Practice problems

Problems C1–C10 test interpretation; P1–P18 develop calculation; E1–E8 connect the model to engineering and computation; H1–H6 integrate ideas. Selected solutions follow. Unless stated otherwise, sources are in vacuum, conductors are in electrostatic equilibrium, and ideal infinite sources have no applied background field. Use $\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)$. A requested field includes direction and all stated regions.

Conceptual questions

- C1.** A uniform electric field threads a closed cube. Explain why the flux is zero even though the field is nonzero on every face. Which faces need not individually have zero flux?
- C2.** A closed surface encloses a dipole with charges $+q$ and $-q$. What does its flux tell you about the enclosed charges? What does it fail to tell you about the field?

- C3.** A positive charge approaches a closed surface from outside without crossing it. Discuss changes in the local field and in the total flux. How would the answer change if the surface were an open disk?
- C4.** A flat sensor plate has zero flux. Give two physically different explanations. If its normal is reversed, what happens to its flux and to the physical electric field?
- C5.** A point charge lies off-centre inside a Gaussian sphere. Is the field radial relative to the sphere centre? Is its magnitude constant there? Can the total flux nevertheless be found exactly?
- C6.** In the infinite-line construction, why do the end caps have zero flux? Explain why that argument generally fails near the end of a finite charged rod.
- C7.** Reconcile $|\sigma|/(2\epsilon_0)$ for an isolated sheet with $|\sigma|/\epsilon_0$ just outside a conductor. What physical surface does each σ describe?
- C8.** A Gaussian surface within the metal around a cavity has zero flux. Why does this determine the net cavity-wall charge but not its point-by-point distribution?
- C9.** Compare an isolated neutral conductor containing charge q in a cavity with a grounded one. Which induced charge is fixed by Gauss's law, and which charge needs a boundary condition or charge-conservation argument?
- C10.** A computer gives the exact total flux to six digits. Does that prove its electric field is accurate everywhere? Propose an additional test.

Analytical and numerical problems

- P1.** A circular disk of radius 0.20 m has its chosen normal at 60° to a uniform field of magnitude 600 N/C. Find its flux and the flux for the reversed normal.
- P2.** A rectangle in the yz plane at $x = 0.40$ m spans $0 < y < 0.30$ m and $0 < z < 0.20$ m. Its normal is $+\hat{x}$. For $\mathbf{E} = (a + by)\hat{x} + bx\hat{y}$, where $a = 100$ N/C and $b = 200$ N/(C m), integrate its flux. Why does its tangential field not contribute?
- P3.** A cube occupies $0 < x, y, z < L$. In it, $\mathbf{E} = ax\hat{x} + by\hat{y} + E_0\hat{z}$. Compute each face flux and infer its enclosed charge. Evaluate for $L = 0.20$ m, $a = 150$ N/(C m), $b = -50$ N/(C m), and $E_0 = 80$ N/C.
- P4.** A closed surface has net outward flux -1.80×10^4 N m²/C. Find its algebraic enclosed charge. Can an external positive charge change that answer without charge crossing the surface?
- P5.** A charge +12.0 nC is at the centre of a cube of side 0.40 m. Find the flux through one face. Explain why dividing the face flux by its area gives an average normal component, rather than the field at every point.
- P6.** A uniformly charged thin spherical shell has radius 0.15 m and charge -8.0 nC. Find the fields at $r = 0.10$ m and 0.30 m. Give the one-sided values at the shell.
- P7.** A uniformly charged insulating sphere has $R = 0.10$ m and $Q = 12.0$ nC. Find its density and fields at $r = 0.05$ m, 0.10 m, and 0.20 m. Sketch the signed radial field.
- P8.** An infinite line has $\lambda = -3.0$ nC/m. Find the field at $r = 0.040$ m and the flux through a coaxial closed cylinder of length 0.25 m. Explain why the flux is independent of cylinder radius.
- P9.** An infinite insulating cylinder has radius $R = 0.020$ m and uniform density $\rho = 4.0$ $\mu\text{C}/\text{m}^3$. Find the fields at $r = R/2$ and $r = 2R$. Check continuity at R and explain the different interior

and exterior scaling.

- P10.** Fixed insulating sheets at $x = 0$ and $x = d$ carry $+4.0$ and -1.0 nC/m². Find E_x in all three regions. Check each signed jump. Would equal and opposite charge densities make the exterior field vanish?
- P11.** Immediately below an ideal sheet at $z = 0$, $E_z^- = -120$ N/C. The sheet carries $\sigma = 2.00$ nC/m². Find E_z^+ . Explain why the isolated-sheet formula is insufficient here.
- P12.** A conductor has local surface density -4.50 nC/m². Find the signed outward normal component and the magnitude of the field just outside. What is the tangential component?
- P13.** An isolated conductor has net charge -3.0 nC and a closed cavity containing a point charge $+5.0$ nC. Find the inner and outer surface charges. Which conclusions survive displacement of the point charge away from the cavity centre?
- P14.** An isolated spherical conductor of radius 0.080 m carries $+6.0$ nC with no external sources. Find its surface density, the field just outside, and the field at $r = 0.16$ m. State what fails if a nearby charged object is introduced.
- P15.** A sphere of radius R has $\rho(r) = \rho_0 r/R$ inside and no charge outside. Derive its total charge and field in both regions. Check the centre and the boundary. A thin spherical shell is *not* present at R .
- P16.** A thick insulating shell occupies $a < r < b$ with uniform density ρ . Derive the field in all regions, and show explicitly that it is continuous at a and b . Recover the uniformly charged solid sphere as $a \rightarrow 0$.
- P17.** An infinite thin cylindrical shell of radius R carries uniform surface density σ . Find its interior and exterior fields and the radial jump at R . Distinguish σ from its charge per unit length.
- P18.** An infinite insulating slab occupies $-a < x < a$ and has uniform density $\rho > 0$. Derive the signed field $E_x(x)$ in all regions. Sketch it, check continuity, and take the thin-slab limit holding $2a\rho$ fixed.

Engineering and computational problems

- E1.** A long coaxial assembly has inner conductor radius $a = 1.0$ mm and outer conductor inner/outer radii $b = 4.0$ mm, $c = 5.0$ mm. The inner conductor carries $+2.0$ nC/m and the outer conductor has net -1.0 nC/m. Find the charge per length on each outer-conductor face, all radial field regions, and the field at $r = 2.0$ mm and 10 mm. The gap is vacuum.
- E2.** Two isolated spherical terminals have radii 0.10 m and 0.30 m, each carrying 1.0 μ C. Compare their surface fields. Explain why this calculation does not establish a universal law linking the local curvature and field of an arbitrary electrode.
- E3.** A cube of side $L = 0.100$ m has measured outward face-averaged normal fields

$$(40, 20, -10, 5, -15, 0) \text{ N/C}.$$

Estimate its enclosed charge. Each face average has independent standard uncertainty 2.0 N/C; neglect uncertainty in L . Estimate charge uncertainty. What changes if the same additive calibration error affects all six readings?

- E4.** A flux program assigns $+\hat{x}$ to both faces $x = +1$ and $x = -1$. Predict its result for a uniform field $E_0\hat{x}$ and explain how this test isolates the error. Propose one additional test using an enclosed point charge.

- E5.** A charged ball is lowered deeply into an initially neutral isolated conducting pail: (i) without contact, (ii) then removed without contact, and (iii) in a separate trial allowed to touch and transfer all its charge before removal. Track the pail's net charge and describe the ideal induced charge pattern at each stage. Identify a limitation of the open-pail model.
- E6.** At the perpendicular bisector of a uniformly charged rod of length 2ℓ , the exact finite/infinite field ratio is $1/\sqrt{1 + (r/\ell)^2}$. For $\ell = 0.50$ m and $r = 0.10$ m, find the percentage error of the infinite-line approximation relative to the exact field. Find the largest r for error below 2%.
- E7.** Run the cube-flux code for a charge at $(0.95, 0, 0)$ with $N = 8, 16, 32, 64, 128$. Explain why doubling N does not initially produce a uniform factor-four reduction in error. Repeat at $(1.05, 0, 0)$ and judge convergence using absolute normalized error.
- E8.** A spherical field map gives $E_r = Ar$ for $r < R$ and $E_r = AR^3/r^2$ for $r > R$, where $A > 0$ is constant. Infer $q_{\text{enc}}(r)$ and the volume density without assuming a material model. Is there a surface charge at R ? What extra physical information would distinguish an insulator from other possible charge-support mechanisms?

Synthesis and challenge problems

- H1.** A sphere has $\rho(r) = \rho_0(r/R)^n$ for $0 < r < R$. Determine the ranges of n for finite total charge, bounded density, and bounded field at the centre. Explain why these are different conditions.
- H2.** A spherical cavity of radius a is removed from an otherwise uniformly charged insulating sphere of density ρ . The displacement from the large sphere's centre to the cavity centre is $\mathbf{b}$, with $a + |\mathbf{b}| < R$. Use superposition to find the field throughout the cavity. Why is the conducting-cavity argument inapplicable?
- H3.** An isolated conductor of net charge Q has two disjoint closed cavities containing charges q_1 and q_2 . Determine the net charge on each cavity wall and on the external surface. What information is still needed to calculate each local surface density?
- H4.** A uniform insulating sphere of radius R and charge $Q > 0$ is concentric with a thin insulating shell of radius $2R$ and charge $-Q$. Find and sketch $E_r(r)$ in all regions, including its jump at $2R$. Identify a region with zero field even though the entire system contains nonzero charges.
- H5.** A point charge is at the centre of a cube. One face is divided into four equal squares. Find the flux through one square by symmetry. Why is the same quarter-face statement invalid for an off-centre charge?
- H6.** A proposed static field map has $\mathbf{E} = \kappa(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}})$. Show that every closed box has zero net flux. Evaluate its circulation around a circle and explain why passing the Gauss-law test does not establish electrostatic consistency.

3.18 Selected solutions

The solutions below expose the choice of surface, sign convention, and checks. A numerical answer without those choices is incomplete.

Example 3.24: P1: an oriented disk

The disk is flat and the field uniform, so $\Phi_E = EA \cos \theta$, with $A = \pi R^2$. Hence

$$\Phi_E = 600\pi(0.20)^2 \cos 60^\circ = 37.7 \text{ N m}^2/\text{C}.$$

The reversed normal makes angle 120° and gives $-37.7 \text{ N m}^2/\text{C}$. The area is unchanged. As a check, the largest possible magnitude is $EA = 75.4 \text{ N m}^2/\text{C}$, so the answer is half that maximum.

Example 3.25: P3: flux face by face

The normals on opposite faces have opposite signs. At $x = 0$ the x component vanishes; at $x = L$ it is aL . The same reasoning applies to the y faces. The constant z component gives equal and opposite fluxes:

face	$x = 0$	$x = L$	$y = 0$	$y = L$	$z = 0$	$z = L$
Φ_E	0	aL^3	0	bL^3	$-E_0L^2$	E_0L^2

Thus $\Phi_E = (a + b)L^3 = 0.800 \text{ N m}^2/\text{C}$ and $q_{\text{enc}} = \epsilon_0 \Phi_E = 7.08 \text{ pC}$. Individually the z fluxes are $\mp 3.20 \text{ N m}^2/\text{C}$: discarding them is justified only after cancellation. The implied uniform density is $\epsilon_0(a + b)$; multiplying it by L^3 independently checks the charge.

Example 3.26: P4: negative net flux

Use the outward-normal convention without taking an absolute value:

$$q_{\text{enc}} = \epsilon_0 \Phi_E = (8.854 \times 10^{-12})(-1.80 \times 10^4) = -1.59 \times 10^{-7} \text{ C}.$$

The net enclosed charge is $-0.159 \mu\text{C}$. An external charge changes the local integrand but contributes zero to its closed-surface integral. No statement about the individual enclosed charges follows from their algebraic sum alone.

Example 3.27: P7: interior and exterior of a charged ball

The uniform density is $\rho = 3Q/(4\pi R^3) = 2.86 \times 10^{-6} \text{ C/m}^3$. A concentric Gaussian sphere encloses $Q(r/R)^3$ inside and Q outside. Therefore

$$E_r(r) = \begin{cases} kQr/R^3, & 0 \leq r \leq R, \\ kQ/r^2, & r \geq R, \end{cases} \quad k = \frac{1}{4\pi\epsilon_0}.$$

The requested values are 5.39×10^3 , 1.08×10^4 , and $2.70 \times 10^3 \text{ N/C}$, all outward. The graph rises linearly to $r = R$ and then decreases as r^{-2} . Both branches give kQ/R^2 at R , because no ideal surface charge is present there. The field at $2R$ is one quarter of the surface field, not one half.

Example 3.28: P12 and P13: local field versus net induced charge

For P12, use the normal directed from metal into vacuum:

$$E_\perp = \frac{-4.50 \times 10^{-9}}{8.854 \times 10^{-12}} = -508 \text{ N/C}.$$

The magnitude is 508 N/C , the direction is inward toward the metal, and the tangential component is zero.

For P13, a Gaussian surface entirely in the metal has zero flux. It encloses the cavity charge and the inner wall, so $Q_{\text{inner}} = -5.0 \text{ nC}$. Conservation of the isolated conductor's charge gives

$$Q_{\text{outer}} = Q_{\text{conductor}} - Q_{\text{inner}} = -3.0 - (-5.0) = +2.0 \text{ nC}.$$

Both totals remain true when the charge moves off-centre. The local inner-wall density then generally changes and need not be uniform. Its value at an individual point cannot be inferred from the net charge.

Example 3.29: P15: integrate charge before applying Gauss's law

A thin shell of radius r' and thickness dr' has volume $4\pi r'^2 dr'$. Thus

$$q_{\text{enc}}(r) = 4\pi \int_0^r \rho_0 \frac{r'}{R} r'^2 dr' = \frac{\pi \rho_0 r^4}{R}, \quad Q = \pi \rho_0 R^3.$$

Since $4\pi r^2 E_r = q_{\text{enc}}/\epsilon_0$,

$$E_r(r) = \begin{cases} \rho_0 r^2 / (4\epsilon_0 R), & r < R, \\ \rho_0 R^3 / (4\epsilon_0 r^2), & r > R. \end{cases}$$

The field vanishes quadratically at the centre and has the same one-sided value $\rho_0 R / (4\epsilon_0)$ at R . Replacing $\rho(r')$ by its value at the Gaussian boundary would overcount the interior charge; it would give a factor 4/3 error.

Example 3.30: P17: an ideal cylindrical surface charge

For a coaxial cylinder of length L , the cap flux is zero and the side flux is $2\pi r L E_r$. There is no enclosed charge for $r < R$, whereas for $r > R$ the enclosed charge is $\sigma(2\pi R L)$. Consequently

$$E_r(r) = \begin{cases} 0, & r < R, \\ \sigma R / (\epsilon_0 r), & r > R. \end{cases}$$

The charge per unit length is $\lambda = 2\pi R \sigma$, with units C/m, not C/m². The one-sided radial jump is $E_r(R^+) - E_r(R^-) = \sigma/\epsilon_0$, as required by the pillbox condition. A field value exactly on an ideal zero-thickness charged surface is not assigned by either limiting formula.

Example 3.31: E1: an unbalanced coaxial assembly

Inside the outer metal, $b < r < c$, a Gaussian cylinder has zero flux. Therefore the inner face of that metal must carry -2.0 nC/m . Its net charge is -1.0 nC/m , leaving $+1.0 \text{ nC/m}$ on its outer face. The fields are

$$E_r(r) = \begin{cases} 0, & r < a, \\ (2.0 \times 10^{-9}) / (2\pi\epsilon_0 r), & a < r < b, \\ 0, & b < r < c, \\ (1.0 \times 10^{-9}) / (2\pi\epsilon_0 r), & r > c. \end{cases}$$

At 2.0 mm, $E_r = 1.80 \times 10^4 \text{ N/C}$; at 10 mm, $E_r = 1.80 \times 10^3 \text{ N/C}$. Both are outward. A zero field outside a coaxial cable requires zero total charge per length, which this assembly does not have. Fields in the two metal regions still vanish.

Example 3.32: E3: charge and measurement uncertainty

The readings already use each face's outward normal; do not reverse any signs again. Their sum is 40 N/C, so

$$\Phi_E = L^2 \sum_i \bar{E}_{n,i} = 0.400 \text{ N m}^2/\text{C}, \quad q = 3.54 \text{ pC}.$$

Independent standard uncertainties add in quadrature:

$$u(q) = \varepsilon_0 L^2 \sqrt{6} (2.0 \text{ N/C}) = 0.434 \text{ pC}.$$

Thus $q = (3.54 \pm 0.43) \text{ pC}$, where the uncertainty is one standard uncertainty, not a guaranteed bound. If the same additive error of standard uncertainty 2.0 N/C affects all six signed readings, it adds coherently: $u(q) = \varepsilon_0 L^2 (6)(2.0) = 1.06 \text{ pC}$. Face averaging must itself resolve variation across each face; a single central sample is not automatically a face average.

Example 3.33: E6: a quantified long-line approximation

Let $x = r/\ell$. Since $E_{\text{finite}} = E_{\infty}/\sqrt{1+x^2}$, the relative overestimate with the exact field in the denominator is

$$\delta = \frac{E_{\infty} - E_{\text{finite}}}{E_{\text{finite}}} = \sqrt{1+x^2} - 1.$$

At $x = 0.20$, $\delta = 0.01980$, or 1.98%. Requiring $\delta < 0.02$ gives

$$r < \ell \sqrt{1.02^2 - 1} = 0.1005 \text{ m}.$$

This bound applies at the perpendicular bisector of a prescribed uniformly charged rod. A conducting rod generally has a nonuniform equilibrium charge density, so its error cannot be estimated by inserting its average density into this model.

Example 3.34: H4: neutral total charge with a nonzero internal field

Spherical symmetry allows the radial field to be found from the enclosed charge in each region:

$$E_r(r) = \begin{cases} kQr/R^3, & 0 \leq r < R, \\ kQ/r^2, & R < r < 2R, \\ 0, & r > 2R. \end{cases}$$

The negative shell contributes zero field everywhere inside itself. The field is continuous at R but drops from $kQ/(4R^2)$ to zero at $2R$. Its jump is $-Q/(16\pi\varepsilon_0 R^2)$, equal to $\sigma_{\text{shell}}/\varepsilon_0$ because $\sigma_{\text{shell}} = -Q/[4\pi(2R)^2]$. The exterior field is zero by *spherical symmetry plus* zero enclosed total charge; zero charge alone would not establish it.

3.19 Advanced Topics for Technical and Engineering Students

This optional section is intended for advanced undergraduate and early MSc study. The core chapter requires elementary vector algebra, integration, and symmetry; the enrichment adds partial derivatives, divergence, and a brief potential-based proof. Each extension addresses a limitation encountered earlier: nonuniform sources, nonuniform surface fields, incomplete boundary information, or finite experimental geometry.

3.19.1 From integral Gauss law to local divergence

Before using the divergence theorem, it is helpful to derive its local content. Consider a small rectangular box of volume $\Delta x \Delta y \Delta z$ centred at (x, y, z) . The two faces normal to x contribute

$$\Delta \Phi_x = \left[E_x \left(x + \frac{\Delta x}{2}, y, z \right) - E_x \left(x - \frac{\Delta x}{2}, y, z \right) \right] \Delta y \Delta z \quad (3.95)$$

$$\simeq \frac{\partial E_x}{\partial x} \Delta x \Delta y \Delta z. \quad (3.96)$$

The minus sign on the second face comes from its outward normal. Adding the other two pairs gives

$$\frac{\Delta \Phi_E}{\Delta V} \rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \equiv \nabla \cdot \mathbf{E}. \quad (3.97)$$

Divergence is thus outgoing flux per volume in a shrinking-volume limit. The derivatives compare the same field component on opposite faces; they do not add the sizes of all field components. For a differentiable field, this local relation becomes the integral divergence theorem when many boxes are combined: internal face contributions cancel in pairs, leaving the outer boundary.

The field need only be smooth where the ordinary derivative is used. At a point charge the Coulomb field is singular. It has zero ordinary divergence everywhere except the charge, but a surrounding sphere still has finite flux. The point-source contribution is represented by a delta distribution,

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^3} \right) = 4\pi \delta^{(3)}(\mathbf{r}), \quad (3.98)$$

understood through integration over a volume. Omitting that singular contribution would incorrectly predict zero flux around a point charge. An ideal sheet similarly requires a jump condition rather than an ordinary derivative exactly on the sheet.

The integral form of Gauss's law is

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\varepsilon_0} \int_V \rho dV, \quad (3.99)$$

where ∂V is the closed boundary surface of a volume V . The divergence theorem states that for a sufficiently smooth vector field,

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \int_V (\nabla \cdot \mathbf{E}) dV. \quad (3.100)$$

Combining this with Eq. (3.99) gives

$$\int_V \left(\nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} \right) dV = 0 \quad (3.101)$$

for arbitrary volumes. Therefore the local form is

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}} \quad (3.102)$$

This equation says that charge density is the local source of electric-field divergence.

Pedagogical boundary

Equation (3.102) is introduced here as enrichment. A full vector-calculus treatment of divergence, curl, boundary conditions, and Maxwell's equations belongs to a later course. In this chapter, the differential form is used only to clarify the local meaning of flux and to support computational thinking.

Theoretical point 3.2: Local source statement

If $\rho = 0$ in a region, then $\nabla \cdot \mathbf{E} = 0$ there. This does not imply $\mathbf{E} = 0$. It means that, locally, field lines do not begin or end in that charge-free region. They may pass through it because their sources lie elsewhere.

3.19.2 Divergence in Cartesian examples

For a vector field

$$\mathbf{E} = E_x(x, y, z)\hat{\mathbf{x}} + E_y(x, y, z)\hat{\mathbf{y}} + E_z(x, y, z)\hat{\mathbf{z}}, \quad (3.103)$$

its divergence is

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}. \quad (3.104)$$

Thus the field $\mathbf{E} = ax\hat{\mathbf{x}}$ has divergence a . If this field existed in electrostatics throughout a region, the implied charge density would be

$$\rho = \varepsilon_0 a. \quad (3.105)$$

This is exactly what the cube-flux example suggested: a net outward flux through every small box corresponds to positive charge density in the volume.

Example 3.35: Charge density implied by a model field

Suppose in a region the electric field is modelled by

$$\mathbf{E} = (\alpha x\hat{\mathbf{x}} + \beta y\hat{\mathbf{y}} - 2\alpha z\hat{\mathbf{z}}), \quad (3.106)$$

where $\alpha = 100 \text{ N}/(\text{C m})$ and $\beta = 40 \text{ N}/(\text{C m})$. Find the charge density required by Gauss's law.

Solution. The divergence is

$$\nabla \cdot \mathbf{E} = \alpha + \beta - 2\alpha = \beta - \alpha = -60 \text{ N}/(\text{C m}). \quad (3.107)$$

Therefore

$$\rho = \varepsilon_0(\nabla \cdot \mathbf{E}) = (8.854 \times 10^{-12})(-60) = -5.31 \times 10^{-10} \text{ C/m}^3. \quad (3.108)$$

The negative sign means that the field has local net convergence rather than local net spreading.

3.19.3 Why divergence is not a complete field specification

Gauss's law fixes a divergence, not all three components of a field. For example, adding any uniform field $\mathbf{E}_b$ leaves $\nabla \cdot \mathbf{E}$ unchanged. It changes the local field and the flux through individual open surfaces, while contributing zero to every closed-surface flux. Boundary conditions are needed to decide whether such a background field is present.

Electrostatics adds another independent condition: the electric field is conservative. Its work per unit test charge around a closed loop is zero,

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0, \quad (3.109)$$

or locally $\nabla \times \mathbf{E} = \mathbf{0}$ in a smooth, simply connected charge-free region. The potential formulation of this statement is developed in Chapter 4. It is mentioned here to prevent a common overinterpretation of the divergence equation.

Example 3.36: A field that passes Gauss's law but is not electrostatic

Consider $\mathbf{E} = \kappa(-y\hat{x} + x\hat{y})$, where κ has units N/(C m). Its divergence is zero because $\partial E_x/\partial x = \partial E_y/\partial y = 0$. Hence its flux through every smooth closed surface is zero.

On a circle of radius R in the xy plane, however, the field is tangent to the circle with component κR in the positive azimuthal direction. Therefore

$$\oint_C \mathbf{E} \cdot d\mathbf{\ell} = (\kappa R)(2\pi R) = 2\pi\kappa R^2 \neq 0 \quad (\kappa \neq 0). \quad (3.110)$$

This field satisfies the charge-free Gauss constraint but cannot be produced by static charges alone. A time-dependent magnetic field can produce circulating electric fields, as discussed in the induction chapter. Checking flux conservation in a simulation is essential, but additional equations and boundary conditions must also be satisfied.

3.19.4 Solid angle and flux through an open disk

Solid angle is the three-dimensional counterpart of an ordinary angle. An area dA at distance s from a source projects to $\cos\theta dA$ perpendicular to the source-to-patch direction. Dividing that projected area by s^2 gives the oriented solid-angle element

$$d\Omega = \frac{\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}}{s^2} dA. \quad (3.111)$$

It is dimensionless, conventionally measured in steradians. Its sign follows the chosen surface orientation. For a point charge q at the source,

$$d\Phi_E = \frac{q}{4\pi\epsilon_0} d\Omega, \quad \Phi_E = \frac{q\Omega}{4\pi\epsilon_0}. \quad (3.112)$$

Figure 3.14 shows the cone geometry for a disk.

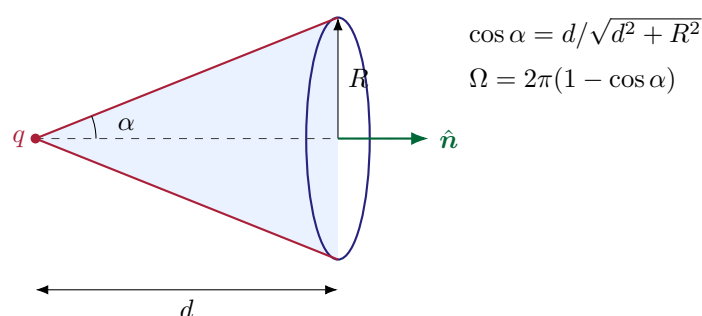


Figure 3.14: A point charge views an axial disk through a cone of half-angle α . The chosen disk normal points away from the charge, giving positive oriented solid angle. The shaded meridional section is a geometric construction, not a conducting object.

This gives a geometric explanation for the cancellation of flux from an external charge. A narrow cone of rays cuts an entry patch and an exit patch. Although the patch areas and distances differ, their signed projected areas divided by s^2 have equal magnitudes and opposite signs. More complicated nonconvex surfaces can produce several crossings; the signed contributions still cancel in pairs when the source is outside.

If a simple closed surface encloses the charge, the net oriented solid angle is 4π . If it does not, the net is zero. These statements, combined with superposition, connect Coulomb's law with Gauss's law for arbitrary closed surfaces. They do not assume that $\mathbf{E}$ is uniform on those surfaces.

Example 3.37: Flux from an axial point charge through a disk

A point charge q lies on the axis of a circular disk of radius R , a perpendicular distance $d > 0$ from its plane. Choose the disk normal to point away from the charge. Find the flux.

Geometry. At radial distance s on the disk, the distance from the charge is $\sqrt{d^2 + s^2}$, and the normal projection supplies a factor $d/\sqrt{d^2 + s^2}$. Integrate over annuli of area $2\pi s ds$:

$$\Phi_E = \frac{q}{4\pi\epsilon_0} \int_0^R \frac{d 2\pi s ds}{(d^2 + s^2)^{3/2}} \quad (3.113)$$

$$= \frac{q}{2\epsilon_0} \left(1 - \frac{d}{\sqrt{d^2 + R^2}} \right). \quad (3.114)$$

Thus $\Omega = 2\pi(1 - d/\sqrt{d^2 + R^2})$ for this orientation. For $d \gg R$, expand the square root to obtain $\Phi_E \simeq [q/(4\pi\epsilon_0 d^2)]\pi R^2$, the nearly uniform-field result. For $R \gg d$, the disk approaches a half-space view and the flux tends to $q/(2\epsilon_0)$. This is an open-surface result, not a claim that the disk encloses half the charge. Reversing the normal reverses the flux.

The limit $d \rightarrow 0^+$ is one-sided. At $d = 0$, the point source lies on the integration surface, and the elementary nonsingular derivation does not apply.

3.19.5 An off-centre charge: the local flux density

For a sphere of radius R centred at the origin and a charge at $d\hat{\mathbf{z}}$ with $0 \leq d < R$, a point on the sphere has position $R\hat{\mathbf{r}}$. The outward normal is $\hat{\mathbf{r}}$, so direct use of Coulomb's field gives

$$\mathbf{E} \cdot \hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0} \frac{R - d \cos \theta}{(R^2 + d^2 - 2Rd \cos \theta)^{3/2}}. \quad (3.115)$$

At the nearest and farthest poles this becomes $q/[4\pi\epsilon_0(R - d)^2]$ and $q/[4\pi\epsilon_0(R + d)^2]$. If $d = R/2$, their ratio is nine. The field is plainly not uniform, even though the total flux is still q/ϵ_0 .

One can verify the integral without assuming the answer. Put $u = \cos \theta$. With the azimuthal integration performed,

$$\Phi_E = \frac{qR^2}{2\epsilon_0} \int_{-1}^1 \frac{R - du}{(R^2 + d^2 - 2Rdu)^{3/2}} du. \quad (3.116)$$

The integrand has antiderivative $(Ru - d)/[R^2\sqrt{R^2 + d^2 - 2Rdu}]$. For $d < R$, its endpoint difference is $2/R^2$, giving $\Phi_E = q/\epsilon_0$. For $d > R$, the endpoint difference is zero. At $d = R$ the charge is on the sphere and the nonsingular calculation breaks down. This explicit example separates three claims: nonuniform local field, an exact integral constraint, and a singular boundary case.

3.19.6 Nonuniform spherical charge distributions

For any spherically symmetric charge density $\rho(r)$, the enclosed charge is

$$q_{\text{enc}}(r) = 4\pi \int_0^r \rho(r') r'^2 dr'. \quad (3.117)$$

Then

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{\text{enc}}(r)}{r^2} \hat{\mathbf{r}}. \quad (3.118)$$

This formula is often the simplest route for nonuniform spherical systems.

Example 3.38: Sphere with linearly decreasing charge density

A sphere of radius R has charge density

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right), \quad 0 \leq r \leq R. \quad (3.119)$$

Find the total charge and the electric field inside.

Solution. The total charge is

$$Q = 4\pi \int_0^R \rho_0 \left(1 - \frac{r}{R}\right) r^2 dr \quad (3.120)$$

$$= 4\pi \rho_0 \left(\frac{R^3}{3} - \frac{R^3}{4}\right) = \frac{\pi \rho_0 R^3}{3}. \quad (3.121)$$

For $r < R$,

$$q_{\text{enc}}(r) = 4\pi \rho_0 \int_0^r \left(1 - \frac{r'}{R}\right) r'^2 dr' \quad (3.122)$$

$$= 4\pi \rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right). \quad (3.123)$$

Thus

$$\mathbf{E}(r) = \frac{\rho_0}{\varepsilon_0} \left(\frac{r}{3} - \frac{r^2}{4R}\right) \hat{\mathbf{r}}, \quad r < R. \quad (3.124)$$

Outside, the field is that of total charge Q at the centre:

$$\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}, \quad r > R. \quad (3.125)$$

3.19.7 Nonuniform cylindrical charge distributions

For a cylindrically symmetric volume charge density $\rho(r)$ inside a long cylinder, a coaxial Gaussian cylinder gives

$$q_{\text{enc}}(r) = 2\pi L \int_0^r \rho(r') r' dr' \quad (3.126)$$

and

$$E_r(r) 2\pi r L = \frac{q_{\text{enc}}(r)}{\varepsilon_0}. \quad (3.127)$$

Hence

$$E_r(r) = \frac{1}{\varepsilon_0 r} \int_0^r \rho(r') r' dr'. \quad (3.128)$$

This expression is useful for beams, charged fibres, and idealized cylindrical detector geometries.

Example 3.39: Long cylinder with parabolic charge density

A very long cylinder of radius R has

$$\rho(r) = \rho_0 \left(1 - \frac{r^2}{R^2}\right), \quad r < R. \quad (3.129)$$

Find the signed radial field inside and outside.

Solution. The enclosed charge for a Gaussian cylinder of radius $r < R$ and length L is

$$q_{\text{enc}}(r) = 2\pi L \rho_0 \int_0^r \left(1 - \frac{r'^2}{R^2}\right) r' dr' \quad (3.130)$$

$$= 2\pi L \rho_0 \left(\frac{r^2}{2} - \frac{r^4}{4R^2} \right). \quad (3.131)$$

Gauss's law gives

$$E_r(2\pi r L) = \frac{2\pi L \rho_0}{\varepsilon_0} \left(\frac{r^2}{2} - \frac{r^4}{4R^2} \right), \quad (3.132)$$

so

$$\mathbf{E}(r) = \frac{\rho_0}{\varepsilon_0} \left(\frac{r}{2} - \frac{r^3}{4R^2} \right) \hat{\mathbf{r}}, \quad r < R. \quad (3.133)$$

The total line charge is $\lambda = \pi \rho_0 R^2/2$. Hence outside,

$$E_r(r) = \frac{\rho_0 R^2}{4\varepsilon_0 r}, \quad r > R.$$

Both branches give $\rho_0 R/(4\varepsilon_0)$ at R . Near the axis the leading term is $\rho_0 r/(2\varepsilon_0)$, matching a uniform cylinder with the central density.

3.19.8 A radial density is not necessarily a uniform density

When a density depends only on radius, the angular symmetry survives. The extra work lies in computing enclosed charge. A thin spherical layer of radius r' and thickness dr' has volume $4\pi r'^2 dr'$; a cylindrical layer of length L has volume $2\pi r' L dr'$. The prime distinguishes the integration coordinate from the radius r where the field is wanted.

For a source supported within radius R , set $r_m = \min(r, R)$. The signed radial components are

$$E_r^{\text{sphere}}(r) = \frac{1}{\varepsilon_0 r^2} \int_0^{r_m} \rho(r') r'^2 dr', \quad (3.134)$$

$$E_r^{\text{cylinder}}(r) = \frac{1}{\varepsilon_0 r} \int_0^{r_m} \rho(r') r' dr'. \quad (3.135)$$

These equations presume no extra central point charge, line charge, or surface sheet. Such contributions must be added explicitly to q_{enc} ; they are not hidden inside a regular volume-density integral.

For the linearly decreasing spherical density derived above, the interior field has a maximum before the surface when $\rho_0 > 0$:

$$\frac{dE_r}{dr} = \frac{\rho_0}{\varepsilon_0} \left(\frac{1}{3} - \frac{r}{2R} \right) = 0 \implies r_{\text{max}} = \frac{2R}{3}. \quad (3.136)$$

The maximum value is $\rho_0 R/(9\varepsilon_0)$, whereas the surface value is $\rho_0 R/(12\varepsilon_0)$. The field maximum need not coincide with the maximum density: enclosed charge and the growing Gaussian area compete. For the parabolic cylindrical profile, differentiating its field gives $r_{\text{max}} = R\sqrt{2/3}$, again inside the source.

Example 3.40: Power-law density and regularity at the centre

Let a sphere have $\rho(r) = \rho_0(r/R)^n$ for $0 < r < R$ and no separate point charge at the origin. Determine the field and identify when the model is regular.

Calculation. A finite integrated charge requires $n > -3$. Then

$$q_{\text{enc}}(r) = \frac{4\pi\rho_0}{(n+3)R^n} r^{n+3}, \quad Q = \frac{4\pi\rho_0 R^3}{n+3}, \quad (3.137)$$

$$E_r(r) = \frac{\rho_0 r^{n+1}}{\varepsilon_0(n+3)R^n} \quad (0 < r < R), \quad E_r(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \quad (r > R). \quad (3.138)$$

The interior expression equals $[Q/(4\pi\varepsilon_0 R^2)](r/R)^{n+1}$. It joins continuously to the exterior field at R .

Regularity. The condition $n > -3$ guarantees finite charge, not a finite density or a continuous field at the origin. For $n \geq 0$, the density is bounded and the field tends continuously to zero. For $-1 < n < 0$, the density diverges but the field still tends to zero. At $n = -1$, the radial field magnitude tends to a nonzero constant, so the vector has no direction-independent limit at the centre. For $-3 < n < -1$, the field magnitude diverges. These singular models may be useful mathematical limits, but they must not be described as smooth physical charge distributions.

Example 3.41: Thick nonconducting spherical shell with uniform volume charge

A nonconducting spherical shell occupies $a < r < b$ and has uniform volume charge density ρ . Find the electric field in the three regions $r < a$, $a < r < b$, and $r > b$.

Solution. Spherical symmetry gives $E_r 4\pi r^2 = q_{\text{enc}}/\varepsilon_0$. For $r < a$, no charge is enclosed, so $\mathbf{E} = \mathbf{0}$. For $a < r < b$,

$$q_{\text{enc}}(r) = \rho \frac{4\pi}{3} (r^3 - a^3), \quad (3.139)$$

so

$$\mathbf{E}(r) = \frac{\rho}{3\varepsilon_0} \left(r - \frac{a^3}{r^2} \right) \hat{\mathbf{r}}. \quad (3.140)$$

For $r > b$, the full charge is

$$Q = \rho \frac{4\pi}{3} (b^3 - a^3), \quad (3.141)$$

and

$$\mathbf{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}. \quad (3.142)$$

3.19.9 Piecewise volume charge and reversal of the radial field

A source can be radially symmetric even when its density changes sign. The field direction is then determined by the cumulative enclosed charge, not by the density at the field point. This distinction is useful in interpreting layered charge distributions and field maps.

Example 3.42: A charged core surrounded by oppositely charged material

An insulating sphere has fixed density $+\rho_0$ for $0 < r < a$ and $-\rho_0$ for $a < r < 2a$, with $\rho_0 > 0$. There are no ideal surface sheets. Find the field in every region and locate any radius where it reverses direction.

Charge accounting. For $r < a$, the enclosed charge is $4\pi\rho_0 r^3/3$. Within the surrounding layer, add the full positive core to the negative charge between a and r :

$$q_{\text{enc}}(r) = \frac{4\pi\rho_0}{3}a^3 - \frac{4\pi\rho_0}{3}(r^3 - a^3) = \frac{4\pi\rho_0}{3}(2a^3 - r^3). \quad (3.143)$$

At $r = 2a$, the total charge is $Q = -8\pi\rho_0 a^3$. Gauss's law gives

$$E_r(r) = \frac{\rho_0}{3\varepsilon_0} \begin{cases} r, & 0 \leq r < a, \\ 2a^3/r^2 - r, & a < r < 2a, \\ -6a^3/r^2, & r > 2a. \end{cases} \quad (3.144)$$

The nonzero-radius zero is $r_0 = 2^{1/3}a$, inside the negative layer. Between a and r_0 , the local density is negative but the field still points outward: the positive core outweighs the negative charge enclosed so far. Beyond r_0 , the enclosed negative charge dominates and the field points inward.

Boundary checks. Both branches at a give $\rho_0 a/(3\varepsilon_0)$. Both branches at $2a$ give $-\rho_0 a/(2\varepsilon_0)$. The field is continuous at both density steps because neither boundary carries an ideal surface charge. The exterior has the field of the negative total charge and approaches zero from below. These checks would expose either a missing core charge or an incorrect shell volume.

Figure 3.15 displays the cumulative effect. The steep decline in the layer is not described by the exterior point-charge formula with the full Q : the enclosed charge is still changing there. Using the total charge at an interior point would even predict the wrong direction over part of the layer.

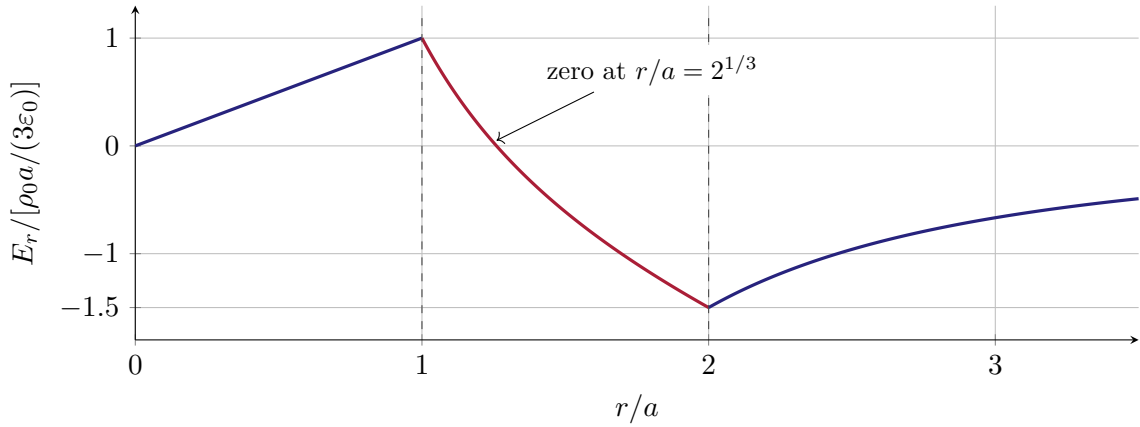


Figure 3.15: Signed radial field of the two-layer insulating sphere. The negative-density layer occupies $1 < r/a < 2$. Its field crosses zero before reaching the outer boundary, while remaining continuous at both density steps.

More generally, a radial field map determines $q_{\text{enc}}(r) = 4\pi\varepsilon_0 r^2 E_r(r)$. Wherever the density is a regular function, differentiating the enclosed charge gives

$$\rho(r) = \frac{\varepsilon_0}{r^2} \frac{d}{dr} [r^2 E_r(r)]. \quad (3.145)$$

At an extremum of E_r with $r > 0$, this becomes $\rho = 2\varepsilon_0 E_r/r$. Thus a field maximum need not coincide with a density maximum or with a material boundary. At a zero of the field, the cumulative enclosed charge is zero, but the local density can be nonzero, as in the layered example. Differentiation can amplify measurement noise, so the later field-reconstruction discussion distinguishes local density estimates from integrated charge estimates.

3.19.10 Planar volume charge and the missing boundary condition

For a planar distribution, write $\mathbf{E} = E_x(x)\hat{x}$. A thin box spanning x_1 to x_2 gives

$$E_x(x_2) - E_x(x_1) = \frac{1}{\varepsilon_0} \int_{x_1}^{x_2} \rho(x') dx'. \quad (3.146)$$

Gauss's law therefore determines field differences. An arbitrary uniform background field can be added without changing this equation. Reflection symmetry or a specified boundary condition is needed to select it.

For the uniform slab of Sec. 3.10.2, reflection symmetry selected $E_x(0) = 0$ and led to Eq. (3.73). The following profile has a different symmetry. The signed density is odd, so the source field is even, not odd. Treating every slab as if its two cap fluxes were equal and outward would miss that distinction.

Example 3.43: A neutral slab with nonzero internal field

Let $\rho(x) = \rho_0 x/a$ for $|x| < a$ and zero elsewhere, with $\rho_0 > 0$. Assume no externally applied uniform field. This odd density has zero total charge per area, but it separates positive charge on the right from negative charge on the left.

Boundary condition. Regard the source as a stack of infinite sheets. Outside the entire stack, each sheet contributes its signed density times $dx/(2\varepsilon_0)$. Since $\int_{-a}^a \rho(x)dx = 0$, the exterior field of this source alone is zero. Zero total charge by itself would leave open the possibility of an added uniform background field; our source-only condition excludes it.

Integrate Eq. (3.146) inward from a , where the field is continuous and $E_x(a) = 0$:

$$E_x(x) = -\frac{1}{\varepsilon_0} \int_x^a \frac{\rho_0 x'}{a} dx' = \frac{\rho_0}{2a\varepsilon_0} (x^2 - a^2), \quad |x| < a. \quad (3.147)$$

The component is negative throughout the interior, as expected for a field directed from the positive half toward the negative half. It is an *even* function of x , and $E_x(0) = -\rho_0 a/(2\varepsilon_0)$ is not zero. A symmetric Gaussian box encloses zero charge and has cancelling cap fluxes; it cannot determine the internal field by the shortcut $2EA$. This example shows exactly why one must establish the field symmetry before evaluating the flux.

3.19.11 An offset cavity in a charged insulator

Gauss's law can help solve a problem without matching the symmetry of the final system. The useful strategy is to decompose that system into symmetric source distributions and then superpose their fields. This is particularly effective for cavities in prescribed volume charge.

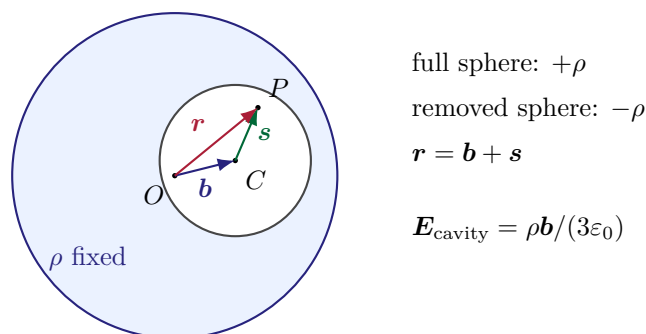


Figure 3.16: Cross-section of a cavity removed from a uniformly charged insulator. Vector addition relates positions measured from the two centres. The negative-density sphere is used only in the superposition calculation; the physical cavity contains no charge.

Example 3.44: A uniform field inside an off-centre spherical cavity

A nonconducting sphere has fixed uniform volume density ρ . A spherical cavity of radius a is removed, entirely inside the original sphere. Let $\mathbf{b}$ point from the original centre O to the cavity centre C . Find the field at any point P inside the cavity. Assume the remaining charge does not rearrange.

Model. Replace the removed material mathematically by a smaller sphere of density $-\rho$ centred at C . Its superposition with the full original sphere has exactly the required charge distribution. At P , define $\mathbf{r} = \overrightarrow{OP}$ and $\mathbf{s} = \overrightarrow{CP}$, so $\mathbf{r} = \mathbf{b} + \mathbf{s}$.

The geometry is shown in Fig. 3.16.

Both fields are interior fields of uniform spheres. Using the Gauss-law result in vector form,

$$\mathbf{E}_{\text{full}} = \frac{\rho}{3\epsilon_0} \mathbf{r}, \quad \mathbf{E}_{\text{removed}} = -\frac{\rho}{3\epsilon_0} \mathbf{s}. \quad (3.148)$$

Their sum is

$$\mathbf{E}_{\text{cavity}} = \frac{\rho}{3\epsilon_0} \mathbf{b}. \quad (3.149)$$

It is uniform throughout the cavity. For positive ρ , it points from the original centre toward the cavity centre. With $\rho = 1.0 \mu\text{C}/\text{m}^3$ and $|\mathbf{b}| = 2.0 \text{ cm}$, its magnitude is $7.5 \times 10^2 \text{ N/C}$.

Checks. A concentric cavity has $\mathbf{b} = 0$ and therefore zero field. A nonzero offset gives nonzero field despite zero enclosed charge for any Gaussian surface wholly within the cavity. The units $\rho b/\epsilon_0$ are N/C. This is an insulating cavity, not a conducting enclosure: there are no mobile surface charges available to enforce shielding. The negative sphere is a calculation device, not an additional physical object.

3.19.12 Why an empty conducting cavity is field-free

This subsection introduces only the potential facts needed for a proof; Chapter 4 develops their meaning and applications. In electrostatics define $\mathbf{E} = -\nabla V$. Inside a charge-free cavity, Gauss's law gives $\nabla^2 V = 0$. A connected conductor has one potential V_0 , so the entire cavity boundary has $V = V_0$.

Put $u = V - V_0$, which vanishes on the cavity wall. The vector identity

$$\nabla \cdot (u \nabla u) = |\nabla u|^2 + u \nabla^2 u \quad (3.150)$$

and the divergence theorem give

$$\int_{\text{cavity}} |\nabla u|^2 d^3r = \oint_{\text{wall}} u \nabla u \cdot d\mathbf{A} - \int_{\text{cavity}} u \nabla^2 u d^3r = 0. \quad (3.151)$$

Both terms on the right vanish: $u = 0$ on the wall and $\nabla^2 u = 0$ in the cavity. The remaining integrand is nonnegative, so its integral can vanish only when $\nabla u = 0$ throughout the smooth cavity. Hence $\mathbf{E} = 0$.

The assumptions do real work. If charge is present in the cavity, the Laplace equation is replaced by Poisson's equation and the volume term need not vanish. If an aperture is present, the whole boundary is no longer held at one conductor potential. If two separate electrodes form part of the boundary at different potentials, u cannot vanish everywhere on it. None of these cases contradicts shielding for a closed charge-free cavity bounded by one connected conductor.

For several cavities in the same isolated conductor, each charged cavity independently induces the opposite net charge on its own wall. With charges q_1, q_2 in two cavities and conductor net charge Q , the wall totals are $-q_1, -q_2$, and the outer-surface total is $Q + q_1 + q_2$. An empty third cavity remains field-free. The metal separates the boundary-value problems even though its total charge accounting couples their surface totals.

3.19.13 Quantifying the infinite-source approximations

An infinite source is an ideal model for a finite system in a restricted observation region. A dimensionless error estimate is more informative than saying that a wire is “long” or a sheet is “large”. Here direct Coulomb integration supplies the finite-source answer; comparison with the Gauss-law result then identifies the usable range.

For a uniformly charged straight rod extending from $z = -\ell$ to $z = +\ell$, observe the field at distance r on its perpendicular bisector. The axial contributions cancel and the radial component is

$$E_{\text{rod}}(r) = \frac{\lambda}{4\pi\epsilon_0} \int_{-\ell}^{\ell} \frac{r \, dz}{(r^2 + z^2)^{3/2}} \quad (3.152)$$

$$= \frac{\lambda}{2\pi\epsilon_0 r} \frac{\ell}{\sqrt{\ell^2 + r^2}}. \quad (3.153)$$

For $\lambda > 0$, divide by the infinite-line magnitude to obtain

$$\frac{E_{\text{rod}}}{E_{\text{line}}} = \frac{1}{\sqrt{1 + (r/\ell)^2}} = 1 - \frac{1}{2} \left(\frac{r}{\ell} \right)^2 + \mathcal{O} \left(\frac{r^4}{\ell^4} \right). \quad (3.154)$$

At $r \gg \ell$, the finite rod instead behaves as a point charge $Q = 2\ell\lambda$: its field falls as $1/r^2$, not $1/r$. The infinite model must therefore fail in the far field.

For a disk of radius R with fixed uniform surface density $\sigma > 0$, observe on its axis at $z > 0$. Integrating concentric rings gives

$$E_{\text{disk}}(z) = \frac{1}{4\pi\epsilon_0} \int_0^R \frac{z \, \sigma \, 2\pi s \, ds}{(z^2 + s^2)^{3/2}} \quad (3.155)$$

$$= \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right). \quad (3.156)$$

The infinite-sheet value is $E_{\text{sheet}} = \sigma/(2\epsilon_0)$, so

$$\frac{E_{\text{disk}}}{E_{\text{sheet}}} = 1 - \frac{z/R}{\sqrt{1 + (z/R)^2}}. \quad (3.157)$$

For $z \ll R$, the leading fractional correction relative to the sheet value is z/R , rather than a squared distance ratio. For $z \gg R$, the finite-disk field tends to $Q/(4\pi\epsilon_0 z^2)$ with $Q = \pi R^2 \sigma$. Figure 3.17 compares the two calculated ratios.

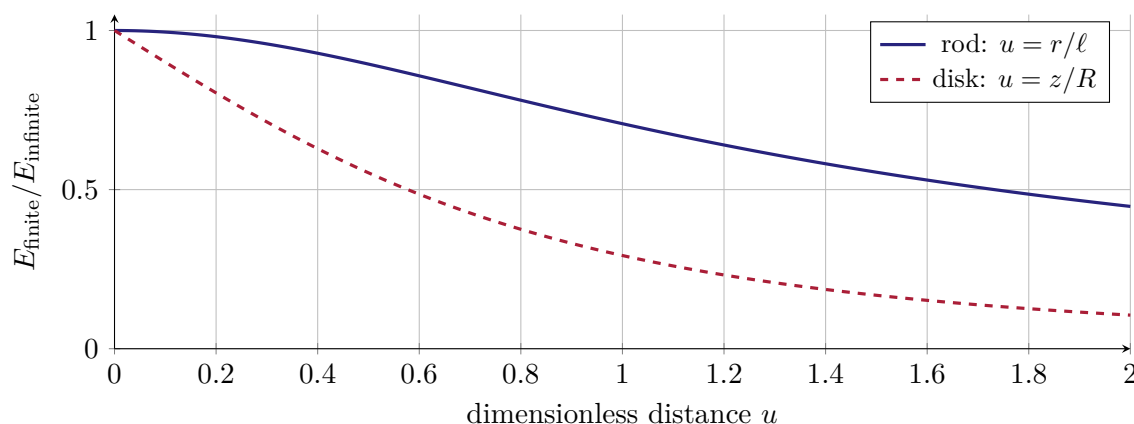


Figure 3.17: Calculated validity tests for two idealizations, with positive uniform source densities. A finite rod of length 2ℓ is observed on its perpendicular bisector; a disk of radius R is observed on its axis at $z > 0$. The two ratios approach one in different ways, so “near the source” requires a quantitative geometry check.

Example 3.45: A specified five-percent modelling tolerance

Require the relative overestimate of the infinite model, measured against the *finite-source field*, to be at most 5%:

$$\frac{E_{\text{infinite}} - E_{\text{finite}}}{E_{\text{finite}}} \leq 0.05. \quad (3.158)$$

For the rod, Eq. (3.154) gives

$$\sqrt{1 + (r/\ell)^2} - 1 \leq 0.05 \implies r/\ell \leq 0.320. \quad (3.159)$$

For the disk, write $u = z/R$. The inequality requires

$$1 - \frac{u}{\sqrt{1 + u^2}} \geq \frac{1}{1.05} \implies u \leq 0.0477. \quad (3.160)$$

A rod of total length 1.0 m meets this tolerance at its midpoint for $r \lesssim 0.16$ m. A disk of radius 0.20 m meets it on its axis for $z \lesssim 9.5$ mm. These limits apply to the specified geometries and fixed uniform densities. They do not quantify fringing for arbitrary conducting plates, whose surface charge redistributes.

3.19.14 Finite-volume balance and field-map diagnostics

A finite-volume discretization applies Gauss's law to each computational cell. For cell i with oriented faces f , define the integrated face flux F_{if} and require

$$\sum_{f \in \partial i} F_{if} = \frac{q_i}{\varepsilon_0}, \quad F_{if} \simeq (\mathbf{E}_f \cdot \hat{\mathbf{n}}_{if}) A_f. \quad (3.161)$$

For two adjacent cells, the shared face has the same physical field and opposite outward normals. A conservative implementation assigns equal and opposite shared fluxes. Summing over a group of cells then cancels internal fluxes exactly and leaves the external flux equal to the group's enclosed charge divided by ε_0 .

This cancellation is an algebraic property of the discretization, not an extra approximation to electrostatics. It explains why conservation should be checked both locally and globally. Two erroneous neighbouring cells can have residuals of opposite signs that cancel in a global test. Conversely, a locally conservative solution can still have an inaccurate field because of a coarse mesh or incorrect boundary data.

Example 3.46: Two cells with a common face

Two adjacent cubic cells have charges $q_1 = +3.0$ pC and $q_2 = -1.0$ pC. Let the flux through their common face be F outward from cell 1. Then it is $-F$ outward from cell 2. If Φ_{1e} and Φ_{2e} denote fluxes through their other faces,

$$\Phi_{1e} + F = \frac{q_1}{\varepsilon_0}, \quad \Phi_{2e} - F = \frac{q_2}{\varepsilon_0}. \quad (3.162)$$

Addition gives

$$\Phi_{\text{outer}} = \Phi_{1e} + \Phi_{2e} = \frac{2.0 \text{ pC}}{\varepsilon_0} = 0.226 \text{ N m}^2/\text{C}. \quad (3.163)$$

The value of F is not determined by these two balance equations alone. Nevertheless, it must cancel from the flux of the combined volume. Using the same outward normal for both cells would instead double-count it and violate conservation.

3.19.15 Inferring charge from sampled fields

If a complete boundary field is measured or computed, Gauss's law can infer charge:

$$\hat{q} = \varepsilon_0 \sum_i w_i E_{n,i}, \quad (3.164)$$

where w_i are patch-area quadrature weights and $E_{n,i}$ are signed outward-normal field samples. If these samples have covariance matrix C_{ij} , the propagated variance is

$$u_q^2 = \varepsilon_0^2 \sum_{i,j} w_i C_{ij} w_j. \quad (3.165)$$

For independent samples this reduces to $\varepsilon_0^2 \sum_i w_i^2 u_i^2$. Repeating more measurements can reduce independent random noise, but a shared calibration error produces off-diagonal covariance and does not average away in the same manner. Quadrature error and incomplete surface coverage must be assessed separately.

For a particularly simple example, assume each face of a cube of side $L = 0.10$ m has a measured *face-averaged* outward-normal component, each with independent standard uncertainty $u_E = 2.0$ N/C. Then

$$u_q = \varepsilon_0 L^2 \sqrt{6} u_E = 0.434 \text{ pC}. \quad (3.166)$$

One point sample per face can replace a face average only with a justified field model. For a strongly nonuniform field, that substitution can create a systematic error much larger than the instrument's random uncertainty.

Local reconstruction uses $\rho = \varepsilon_0 \nabla \cdot \mathbf{E}$. A centred derivative estimate, for example

$$\frac{\partial E_x}{\partial x} \simeq \frac{E_x(x+h) - E_x(x-h)}{2h}, \quad (3.167)$$

has random uncertainty $u_E/(\sqrt{2}h)$ when the two samples are independent and equally uncertain. Smaller h improves spatial resolution but amplifies noise. Integral reconstruction averages over a finite volume and often gives a more stable net-charge estimate, at the cost of losing information about the charge's detailed location. This is a physical resolution trade-off, not evidence that one form of Gauss's law is more exact than the other.

3.19.16 Advanced exercises

These eight problems belong to the enrichment route. They use the mathematics introduced in this section, including divergence, boundary data, and uncertainty propagation.

- A1.** A sphere has $\rho(r) = \rho_0(1 - r/R)$ for $0 < r < R$, with $\rho_0 > 0$. Starting from its field, locate its maximum and show that the maximum/surface ratio is $4/3$. Repeat the location calculation for the cylindrical density $\rho_0(1 - r^2/R^2)$.
- A2.** A disk of radius 0.20 m is normal to its axis. A charge $+3.0$ nC lies on that axis 0.10 m from the plane. For the normal directed away from the charge, calculate the flux and solid angle. Find the fractional error from assuming the field is uniform at its centre value.
- A3.** A point charge lies $d = 0.8R$ from the centre of a Gaussian sphere of radius R . Compare the normal fields at the nearest and farthest poles. Integrate the exact normal component to recover the total flux. Explain why a uniform angular grid becomes difficult as $d \rightarrow R^-$.
- A4.** A neutral infinite slab has $\rho(x) = \rho_0 x/a$ for $|x| < a$, and an applied uniform background field $E_b \hat{\mathbf{x}}$. Find the field everywhere. For $\rho_0 > 0$, classify when the interior field has two, one, or no zeros. Discuss the special boundary case $E_b = 0$ separately.

- A5.** Two adjacent cells have exact charges $+3.0$ and -1.0 pC. Computed residual charges, defined as $\epsilon_0 \sum_f F_{if} - q_i$, are $+0.20$ and -0.20 pC. What does a global flux test report? Propose a local residual measure that detects the errors, and explain why it is insufficient to determine the local field error.
- A6.** Six face-averaged outward fields on a cube have a common independent-noise standard uncertainty u_r and a shared additive calibration standard uncertainty u_c . Derive u_q in terms of L, u_r, u_c . Compare the effects of repeating each reading M times if the calibration error persists across repetitions.
- A7.** A point-charge field has ordinary divergence zero for $r > 0$. Apply the divergence theorem to a spherical shell $a < r < b$, keeping both boundary normals. Explain why shrinking a to zero does not allow the inner boundary contribution to be discarded.
- A8.** A smooth charged interface separates two vacuum regions. Use a curved, vanishing-height pillbox to derive the normal jump at one point. State which geometric length scales must remain large compared with the pillbox size. Explain why finite volume density contributes vanishing charge in the height limit but surface density does not.

3.19.17 Selected advanced solutions

Example 3.47: A2: an open disk subtends less than a hemisphere

The axial distance is half the disk radius. From Eq. (3.114),

$$\Phi_E = \frac{3.0 \times 10^{-9}}{2\epsilon_0} \left(1 - \frac{0.10}{\sqrt{0.10^2 + 0.20^2}} \right) = 93.7 \text{ N m}^2/\text{C}.$$

Its oriented solid angle is $\Omega = 2\pi(1 - 1/\sqrt{5}) = 3.47$ sr. The result is less than the half-space value $q/(2\epsilon_0) = 169 \text{ N m}^2/\text{C}$, as it must be for a finite disk viewed from a positive distance. The centre-field approximation gives $\Phi_{\text{centre}} = [q/(4\pi\epsilon_0 d^2)]\pi R^2 = 339 \text{ N m}^2/\text{C}$. Relative to the exact integral, its error is $(339 - 93.7)/93.7 \simeq 2.62$, or 262%. Here $R/d = 2$, so the field is far from uniform across the disk. Both its magnitude and its normal projection decrease away from the centre; a correct area alone cannot repair that approximation.

Example 3.48: A4: boundary data select the uniform part of a field

Superpose the background field on the source-only result:

$$E_x(x) = \begin{cases} E_b + \frac{\rho_0}{2a\epsilon_0}(x^2 - a^2), & |x| < a, \\ E_b, & |x| > a. \end{cases}$$

It is continuous at $\pm a$. Define $B = \rho_0 a/(2\epsilon_0) > 0$. The interior field is $E_b - B + B(x/a)^2$. If $0 < E_b < B$, it has two zeros,

$$x = \pm a \sqrt{1 - E_b/B}.$$

If $E_b = B$, the only zero is at $x = 0$. If $E_b < 0$ or $E_b > B$, there is no interior zero. For $E_b = 0$, the field is negative in the open interior, zero at both boundaries, and zero everywhere outside. At $E_b < 0$ the interior component is everywhere negative; at $E_b > B$ it is everywhere positive. All these fields have the same derivative $\rho(x)/\epsilon_0$ and therefore the same charge density. Gauss's law alone cannot distinguish their different applied boundary fields.

Example 3.49: A6: independent noise and correlated calibration

Write each reading as its true value plus an independent random error r_i and one shared calibration error c . The charge error is

$$\Delta q = \varepsilon_0 L^2 \left(\sum_{i=1}^6 r_i + 6c \right).$$

Independence of the random errors from each other and from c gives

$$u_q = \varepsilon_0 L^2 \sqrt{6u_r^2 + 36u_c^2}.$$

For averages of M independent repetitions, only the random contribution decreases:

$$u_q(M) = \varepsilon_0 L^2 \sqrt{\frac{6u_r^2}{M} + 36u_c^2}.$$

The limiting uncertainty is $6\varepsilon_0 L^2 u_c$. Reducing it requires improved calibration, not simply more repeated samples. This model assumes a common error in the six *signed outward readings*. A common error in one laboratory-coordinate field component would have different signs on opposite faces and requires a different covariance model.

3.20 Bridge to Chapter 4: electric potential

Gauss's law answers a source question: how does enclosed charge constrain the electric field? Symmetry made that constraint sufficient for several important geometries. More general configurations need more information. In electrostatics the field also has zero circulation, allowing it to be expressed using a scalar electric potential.

Chapter 4 develops potential difference, work, potential energy, and equipotential surfaces. The radial fields found here will become inputs to one-dimensional potential integrals. The conductor results will acquire a complementary interpretation: each connected conductor in equilibrium is an equipotential, and its surface field is normal to that equipotential. Chapter 5 then uses potential differences and separated charges to define capacitance and introduces dielectric polarization. These connections extend the present results; they do not change the need to distinguish total charge, enclosed charge, and boundary conditions.

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