

Chapter 4

Electric Potential and Potential Energy

First Edition

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Physics 2

Physics for Technical and Engineering Students -
Volume II

Electricity, Magnetism, Waves, Optics, Relativity,
Quantum Physics, and Nuclear Physics

1 September 2026

Contents

Preface	iii
Conventions and notation	iv
4 Electric Potential and Potential Energy	1
4.1 Motivation and roadmap	1
4.2 From work to electric potential energy	2
4.2.1 What is held fixed when a charge moves?	2
4.2.2 Electrical work and external work	3
4.2.3 Why an energy method is useful	3
4.3 Electric potential difference and voltage	4
4.3.1 A line integral is a sum of projected displacements	5
4.3.2 Potential difference is an endpoint quantity	5
4.3.3 Integrating a nonuniform field with piecewise regions	6
4.4 Reference potential and sign conventions	7
4.4.1 Changing a reference and grounding an object are different operations	8
4.4.2 When infinity cannot be the zero	8
4.5 Equipotential surfaces and field geometry	9
4.5.1 Reading the spacing of contours	10
4.5.2 Where the contour picture needs qualification	10
4.6 Potential of a point charge and scalar superposition	11
4.6.1 A geometry check before adding charges	12
4.6.2 Locating zero potential without locating zero field	13
4.7 Electric dipoles and far-field potential	13
4.7.1 Exact potential and the origin of the far-field approximation	14
4.7.2 A dipole in an external potential	15
4.8 Potential of continuous charge distributions	15
4.8.1 Source coordinates and observation coordinates	16
4.8.2 Ring geometry, symmetry and limits	17
4.8.3 A finite rod seen from its perpendicular bisector	17
4.8.4 A disk assembled from rings	18
4.9 Recovering the electric field from potential	19
4.9.1 Deriving the gradient relation one direction at a time	20
4.9.2 Reading a graph without differentiating a formula	20
4.9.3 Stationary potential and mechanical stability	21
4.10 Potential energy of a system of point charges	21
4.10.1 Deriving the pair sum by assembling the system	22
4.10.2 Interpreting the three-charge example	22
4.10.3 Moving one charge in an existing assembly	23
4.11 Energy conservation with electric potential	23
4.11.1 Accessible regions and turning points	24
4.11.2 Speed, trajectory and collision losses	25
4.11.3 An energy graph separates turning points from equilibria	25
4.11.4 Closest approach to a repulsive source	26
4.11.5 Escape from an attractive source	27
4.11.6 When both particles recoil	27

4.12	Conductors at constant potential	27
4.12.1	A spherical conductor: zero field does not mean zero potential	28
4.12.2	Local normal field and surface charge	29
4.12.3	Cavities, shielding and disconnected conductors	30
4.12.4	A bridge to capacitance	30
4.12.5	Potential in a cavity containing a charge	30
4.13	Laboratory and computational connection	31
4.13.1	What a voltmeter compares	31
4.13.2	Conductive paper as an analogue of a boundary-value problem	31
4.13.3	Estimating a field component and its uncertainty	32
4.13.4	Fitting a local potential instead of differencing one pair	33
4.13.5	A computational map with an analytic benchmark	33
4.14	Common mistakes and correction strategies	34
4.15	Main-chapter summary and formula sheet	34
4.16	Practice problems	35
4.17	Selected detailed solutions	38
4.18	Advanced Topics for Technical and Engineering Students	42
4.19	Poisson's equation and electrostatic boundary data	42
4.19.1	From flux to local curvature	42
4.19.2	Which boundary conditions are specified?	42
4.19.3	Uniqueness explains why boundary methods work	43
4.19.4	Bias voltage is not a complete field specification	43
4.20	Saddles, stability and the limits of static trapping	44
4.20.1	Two like charges provide a concrete saddle	44
4.20.2	Why a charge-free harmonic potential has no strict interior minimum	44
4.21	Multipoles and distant-source approximations	45
4.21.1	How a source reveals its structure with distance	45
4.21.2	The origin matters unless the net charge vanishes	46
4.22	A grounded plane and the method of images	46
4.22.1	Constructing the potential without guessing surface charge	46
4.22.2	Induced surface charge and attractive force	47
4.22.3	Why the interaction energy is not the real-image pair energy	47
4.23	Energy stored in the electrostatic field	48
4.23.1	From assembly work to a spatial energy density	48
4.23.2	Positive field energy and negative pair energy	48
4.24	Numerical potentials: approximation, solution and validation	49
4.24.1	A softened Coulomb potential changes the physical source	49
4.24.2	Finite differences turn Laplace's equation into neighbour averages	50
4.24.3	An analytic boundary problem for testing a solver	50
4.24.4	A reproducible calculation	50
4.24.5	Turning a numerical map into a defensible model	51
4.25	How electrode detail fades away from a boundary	52
4.25.1	A sinusoidal boundary produces an exponential spatial decay	52
4.25.2	Superposition connects simple modes to structured electrodes	53
4.26	When the electrostatic description must be extended	53
4.27	Advanced problems and solution checks	53

Preface

This chapter introduces electric potential and electric potential energy as part of the second volume of *Physics for Technical and Engineering Students*. It develops the scalar, energy-based language of electrostatics needed for capacitance, current, circuits, electronic instrumentation, detector physics, electromagnetic induction, and later quantum physics.

The preceding electrostatics chapters described charge, force, electric field, flux, and Gauss's law. Those ideas remain essential, but they are not always the most efficient way to solve problems. A scalar potential can encode electrostatic field information with less vector algebra, provided the electric field is conservative. The central task of this chapter is therefore to translate electrostatics into the language of work, energy, and voltage.

The word voltage is used constantly in laboratories and engineering. Students meet it on a multimeter, an oscilloscope, a battery label, a microcontroller input, a detector bias supply, a power supply, and a data-acquisition board. Nevertheless, voltage is often misunderstood. Voltage does not flow. Voltage is not a substance. Voltage is a potential difference: an energy difference per unit charge.

The Guideline

Read every voltage statement as a comparison between two points. Ask which point is final, which point is initial, which reference has been chosen, and what sign the moving charge has. This discipline prevents most sign mistakes in electric-potential problems.

Pedagogical boundary

This chapter treats electrostatic potential. In electrostatics the line integral of $\mathbf{E} \cdot d\mathbf{l}$ is path independent and a scalar potential can be defined. Time-dependent magnetic fields produce induced electric fields that require a more general treatment later in the course. Capacitance and circuit voltages are also postponed until after the meaning of potential difference is secure.

Conventions and notation

Vectors are written in bold notation. Cartesian unit vectors are $\hat{i}$, $\hat{j}$, and $\hat{k}$. Electric potential is denoted by V , electric potential energy by U , electric field by $\mathbf{E}$, and electric force by $\mathbf{F}$. The charge being moved is usually q , while source charges are often Q or q_i .

The potential difference from point a to point b is

$$\Delta V = V_b - V_a.$$

The potential-energy change of a particle of charge q is

$$\Delta U = U_b - U_a = q(V_b - V_a) = q\Delta V.$$

In electrostatics,

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\mathbf{l}.$$

Symbol	Meaning	SI unit
V	electric potential	$\text{V} = \text{J C}^{-1}$
ΔV	potential difference	V
U	electric potential energy	J
q, Q	electric charge	C
$\mathbf{E}$	electric field	$\text{N C}^{-1} = \text{V m}^{-1}$
$\mathbf{F}$	electric force	N
$d\mathbf{l}$	path displacement element	m
k_e	Coulomb constant $1/(4\pi\epsilon_0)$	$\text{N m}^2 \text{C}^{-2}$
λ	linear charge density	C m^{-1}
σ	surface charge density	C m^{-2}
ρ	volume charge density	C m^{-3}
$\mathbf{p}$	electric dipole moment	C m

Note

The symbol V denotes electric potential. When a volume element is needed, this chapter uses $d\tau$ instead of dV to avoid ambiguity.

Note

The units V/m and N/C are equivalent electric-field units. The unit V alone is a potential unit, not a field unit.

Chapter 4

Electric Potential and Potential Energy

4.1 Motivation and roadmap

Electric field is a vector language: it tells us the force per unit positive test charge. Electric potential is a scalar energy language: it tells us the potential energy per unit charge. Both descriptions are necessary. The field is local and directional. The potential is often easier to measure, easier to plot, and easier to use in energy conservation.

Figure 4.1 connects the chapter's main ideas. The most important practical word in this chapter is voltage. A voltmeter does not measure a mysterious electrical fluid. It compares two terminals and reports the electric potential difference between them. A supply labelled 5 V maintains a potential difference of about 5 J/C. A particle accelerator voltage sets an energy scale. A detector bias voltage shapes a potential landscape that guides charges to electrodes. The same concept appears across scales.

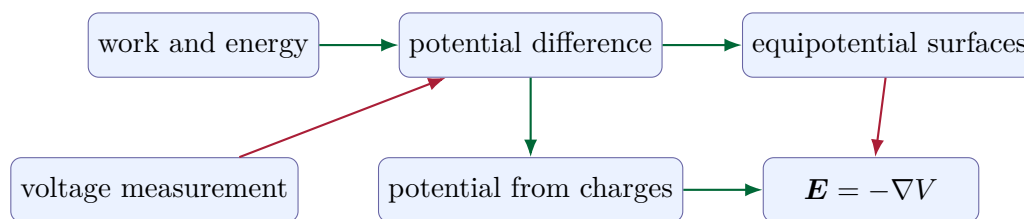


Figure 4.1: Roadmap for Chapter 4. The chapter begins with work and energy, defines potential difference, studies equipotentials, computes potentials from charge distributions, and recovers the field from the scalar potential.

Learning goals

After studying this chapter, the student should be able to:

- define electric potential energy and electric potential difference;
- explain voltage as energy per unit charge rather than as a flowing substance;
- calculate ΔV from $\mathbf{E}$ using a line integral;
- use $\Delta U = q\Delta V$ with the algebraic sign of q ;
- identify and interpret equipotential surfaces;
- compute potentials due to point charges and simple continuous charge distributions;
- recover electric field components from $\mathbf{E} = -\nabla V$;
- calculate the potential energy of a system of point charges;
- explain why a conductor in electrostatic equilibrium is an equipotential;
- connect potential methods to laboratory mapping, sensors, electrodes, and computational visualization.

Key message

Potential is not a replacement for the electric field. It is a complementary scalar description that is especially powerful for energy, voltage, conductors, and boundary-value problems.

4.2 From work to electric potential energy

The electrostatic force on a particle of charge q in an electric field is

$$\mathbf{F} = q\mathbf{E}. \quad (4.1)$$

During a small displacement $d\mathbf{l}$, the electric force does work

$$dW_{\text{elec}} = \mathbf{F} \cdot d\mathbf{l} = q\mathbf{E} \cdot d\mathbf{l}. \quad (4.2)$$

For an electrostatic field, the work done between two points is path independent. Therefore the electric force is conservative and we may introduce electric potential energy U by

$$\Delta U = U_b - U_a = -W_{\text{elec}}(a \rightarrow b). \quad (4.3)$$

Substituting the work integral gives

$$U_b - U_a = -q \int_a^b \mathbf{E} \cdot d\mathbf{l}. \quad (4.4)$$

Definition 4.1: Electric potential energy

Electric potential energy is the energy associated with the configuration of charges in an electrostatic system. For a charge q moving from a to b in an electrostatic field, $\Delta U = -W_{\text{elec}}$.

Caution

Potential energy belongs to an interacting system. The phrase “the charge has potential energy” is a useful shorthand, but the precise statement is that the charge and the source charges form a configuration with electric potential energy.

Guided checks**Work and energy checks**

Check. If the electric force does positive work, what happens to electric potential energy?

Answer. It decreases by the same amount.

Check. Does a negative charge necessarily move toward lower electric potential?

Answer. No. Its electric force points toward decreasing potential energy, hence toward increasing potential. Its velocity also depends on its initial conditions.

Check. Why is electrostatics easier than a general time-dependent electric field?

Answer. Because the electrostatic force is conservative and a scalar potential can be defined.

4.2.1 What is held fixed when a charge moves?

The expression $\mathbf{F} = q\mathbf{E}$ separates a source from a probe. In this chapter the source charges are fixed, or the electrodes maintain a specified static field, and the moving charge is small enough not to change that field appreciably. The potential used in $U = qV$ is the *external* potential, excluding

the charge's own field. A point charge's self-potential is singular and must never be included in its own force or interaction energy.

If the source charges also move, energy conservation still applies, but their kinetic energies and all changing pair separations must be included. If a metal electrode redistributes its charge in response to the moving particle, the induced charges are not a fixed source. A simple substitution into qV can then miscount the work. The method of images in the enrichment section gives an explicit example of that distinction.

A fixed support can exert a force without doing work: the point of application on an immobile source does not move. This is why a moving test charge can exchange kinetic and interaction potential energy while the supported source remains at rest. The approximation fails if source motion, collisions, radiation, or time dependence carries appreciable energy away.

4.2.2 Electrical work and external work

Imagine moving a positive charge slowly toward another positive charge. The electric force points away from the source, whereas the imposed displacement points toward it. Consequently the electric work is negative, and the interaction potential energy increases. The external agent must supply positive work to keep the speed nearly constant. Reversing the displacement reverses both works.

The work–energy theorem makes the bookkeeping explicit:

$$\Delta K = W_{\text{elec}} + W_{\text{other}}, \quad \Delta(K + U) = W_{\text{other}}. \quad (4.5)$$

Here W_{other} is the sum of work by forces not already represented by U . Gravity can either be included through gravitational potential energy or treated as work, but not both. For quasistatic transport with no change in kinetic energy and no other dissipative process, $W_{\text{app}} = \Delta U$. The word “slow” is a practical way to make ΔK negligible; the exact condition is equality of initial and final kinetic energies.

Example 4.1: A charge moved against the electric force

A fixed field does -0.72 mJ of work while a charge is carried from a to b . Its kinetic energy rises by 0.08 mJ . Determine the potential-energy change and the external work.

Strategy. Keep the two energy statements separate: conservative work determines ΔU , while total work determines ΔK .

$$\Delta U = -W_{\text{elec}} = +0.72 \text{ mJ}, \quad (4.6)$$

$$W_{\text{app}} = \Delta K - W_{\text{elec}} = 0.08 \text{ mJ} + 0.72 \text{ mJ} = 0.80 \text{ mJ}. \quad (4.7)$$

Of the supplied energy, 0.72 mJ increases the interaction energy and 0.08 mJ increases the particle's kinetic energy. The external work is not equal to ΔU in this transport because the speed changes. All terms have units of energy; no voltage is required until the charge is specified.

4.2.3 Why an energy method is useful

Newton's second law gives acceleration at every point and can determine a trajectory as a function of time. An energy equation usually gives less information, but often answers the requested question with much less calculation. If the final speed is needed, the endpoints and the change in potential may suffice even when the force changes continuously along the path. Energy also determines which positions are inaccessible because reaching them would require negative kinetic energy.

The analogy with gravitational potential is useful up to a point. Near Earth's surface, $U_g = mgy$ and g is approximately uniform. An electric uniform field gives a potential linear in the coordinate

along the field. However, mass in the elementary gravitational problem is positive, while electric charge can have either sign. A single electrical landscape V therefore corresponds to opposite potential-energy landscapes for positive and negative test charges.

4.3 Electric potential difference and voltage

Electric potential difference is defined as electric potential-energy difference per unit charge:

$$V_b - V_a = \frac{U_b - U_a}{q}. \quad (4.8)$$

Equivalently,

$$\Delta U = q\Delta V. \quad (4.9)$$

The SI unit is the volt,

$$1 \text{ V} = 1 \text{ J/C}. \quad (4.10)$$

Thus a potential difference of 12 V means 12 J of energy per coulomb of charge.

Combining the definition of V with the work integral gives the central electrostatic relation

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\mathbf{l}. \quad (4.11)$$

The minus sign is not optional. It encodes the fact that the electric field points in the direction of decreasing potential.

Definition 4.2: Electric potential and voltage

Electric potential is electric potential energy per unit charge after a reference has been chosen. Voltage is a potential difference between two points. In electrostatic problems, voltage is calculated by the negative line integral of the electric field.

Caution

Voltage does not flow. Charge flows. Current flows. Energy can be transferred. Voltage is a difference in energy per unit charge between two points.

Example 4.2: Potential difference in a uniform electric field

Two parallel plates are separated by $d = 4.0 \text{ mm}$. Between them the electric field is approximately uniform, $\mathbf{E} = 2.5 \times 10^4 \hat{\mathbf{i}} \text{ V/m}$. Point a is at the left plate and point b is at the right plate. Find $V_b - V_a$.

The line element along the straight path is $d\mathbf{l} = dx \hat{\mathbf{i}}$. Therefore

$$V_b - V_a = - \int_0^d E dx = -Ed. \quad (4.12)$$

Numerically,

$$V_b - V_a = -(2.5 \times 10^4 \text{ V/m})(4.0 \times 10^{-3} \text{ m}) = -100 \text{ V}. \quad (4.13)$$

The potential decreases in the direction of the electric field. A positive charge released from rest would accelerate toward the right plate; an electron would accelerate toward the left plate.

Example 4.3: Energy of an electron accelerated through a voltage

An electron moves from point a to point b , where $V_b - V_a = +250 \text{ V}$. Find ΔU and the kinetic-energy change if no non-electrostatic work is done.

For an electron, $q = -e = -1.602 \times 10^{-19} \text{ C}$. Hence

$$\Delta U = q\Delta V = (-1.602 \times 10^{-19} \text{ C})(250 \text{ V}) = -4.01 \times 10^{-17} \text{ J}. \quad (4.14)$$

Energy conservation gives $\Delta K = -\Delta U = +4.01 \times 10^{-17} \text{ J} = 250 \text{ eV}$. The electron gains kinetic energy while moving to a higher electric potential because its charge is negative.

4.3.1 A line integral is a sum of projected displacements

A line integral is needed when either the field or the path changes direction. Break the path into short displacements $\Delta \mathbf{l}_j$. Over a sufficiently short segment the field is nearly constant, and its contribution to voltage is

$$\Delta V_j \simeq -\mathbf{E}_j \cdot \Delta \mathbf{l}_j = -E_j \Delta l_j \cos \theta_j. \quad (4.15)$$

Add the contributions and make the segments progressively shorter. This is precisely the limiting operation denoted by the integral. The component of displacement perpendicular to the field does not contribute. The path length alone is therefore insufficient: $-E \times \text{path length}$ is valid only for a uniform field and motion directly along it.

For a spatially uniform field, its components can be taken outside the integral:

$$\Delta V = -\mathbf{E} \cdot (\mathbf{r}_b - \mathbf{r}_a) = -E_x \Delta x - E_y \Delta y - E_z \Delta z. \quad (4.16)$$

The result depends on net displacement, not distance travelled. A long curved path and the straight path between the same endpoints give the same electrostatic voltage. Their transit times and the external forces needed to follow them may differ.

Example 4.4: Two paths through the same uniform field

Let $\mathbf{E} = (300\hat{i} - 200\hat{j}) \text{ V/m}$, and let $a = (0, 0)$ and $b = (0.040, 0.030) \text{ m}$. Compare a straight path with a path that first travels horizontally and then vertically.

Straight path. Parametrize the path by $\mathbf{r}(t) = t\mathbf{r}_b$, $0 \leq t \leq 1$. Then

$$\Delta V = - \int_0^1 [300(0.040) - 200(0.030)] \text{ V } dt = -6.0 \text{ V}. \quad (4.17)$$

Broken path. The horizontal segment contributes -12 V . The vertical segment contributes $+6.0 \text{ V}$, because the field's y component is negative. Their sum is again -6.0 V .

For a $+2.0 \text{ nC}$ test charge, $\Delta U = -12 \text{ nJ}$ and $W_{\text{elec}} = +12 \text{ nJ}$. For the same negative charge magnitude, both signs reverse. The units check is $(\text{V/m})\text{m} = \text{V}$ before multiplying by charge. The equality of the two path integrals tests the field description, not the sign of the test charge.

4.3.2 Potential difference is an endpoint quantity

If two paths from a to b gave different electrostatic works, a charge could travel along one and return along the other, gaining net kinetic energy on every circuit while all source charges remained unchanged. The electrostatic interaction does not permit that. Reversing one path closes the loop, giving

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0. \quad (4.18)$$

Conversely, if this equality holds for every closed path in the region, the integral from a reference point to any other point is independent of the chosen path. It then defines a single-valued potential. These are two ways of expressing the same conservative structure.

The definition $\Delta V = \Delta U/q$ assumes a nonzero probe charge in a fixed external field. The resulting voltage is independent of the probe's charge and mass. Doubling q doubles the electric work and ΔU , leaving their ratio to q unchanged. For an uncharged probe, one determines V from the sources or field instead of dividing by zero.

4.3.3 Integrating a nonuniform field with piecewise regions

The uniform-field result is only a special case of the line integral. If the field varies along a line, the voltage is the negative signed area under the $E_x(x)$ curve. Regions with opposite signs can cancel in the integral, even though the local field is nowhere small except near the reversal. This geometric reading is often more useful than looking for a memorized potential formula.

When a source has an interior and an exterior region, first identify where the expression for E_x changes. Integrate each interval using the field valid there, carry the boundary value into the next interval, and enforce continuity of potential for ordinary finite charge densities and surface charge sheets. A separate arbitrary constant in each region is not an additional freedom once continuity and the reference are imposed.

Consider an infinite insulating slab occupying $-a < x < a$, with uniform positive volume charge density ρ . From planar symmetry and Gauss's law, its field is

$$E_x(x) = \begin{cases} -\rho a/\epsilon_0, & x < -a, \\ \rho x/\epsilon_0, & |x| \leq a, \\ +\rho a/\epsilon_0, & x > a. \end{cases} \quad (4.19)$$

The field is outward on both sides. Choose $V(0) = 0$, because this infinite planar model has no finite infinity reference. For $0 \leq x \leq a$, direct integration gives $V(x) = -\rho x^2/(2\epsilon_0)$. At the face $V(a) = -\rho a^2/(2\epsilon_0)$. For $x > a$, integrate from that face:

$$V(x) = V(a) - \frac{\rho a}{\epsilon_0}(x - a) = -\frac{\rho a}{\epsilon_0} \left(x - \frac{a}{2} \right). \quad (4.20)$$

Reflection symmetry makes the potential even, so the complete result is

$$V(x) = \begin{cases} -\rho x^2/(2\epsilon_0), & |x| \leq a, \\ -(\rho a/\epsilon_0)(|x| - a/2), & |x| \geq a. \end{cases} \quad (4.21)$$

Both the potential and its first derivative match at the faces. The second derivative changes because the volume density changes. The central potential is a maximum under the chosen reference, while a positive test charge released just off centre accelerates outward. The field at the centre is zero by symmetry; zero field at one point does not make the entire slab an equipotential.

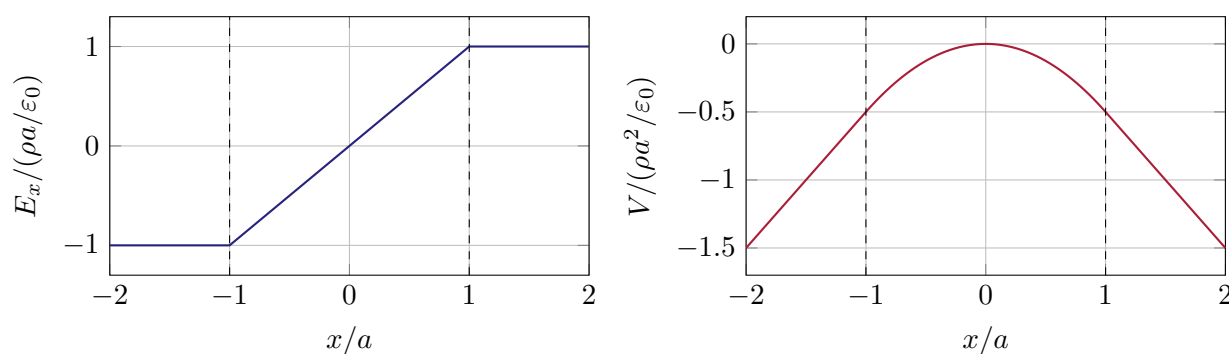


Figure 4.2: Calculated field and potential for an infinite uniformly charged slab, with $V(0) = 0$. Dashed lines locate the slab faces. The potential's slope is the negative field; a constant exterior field produces a linear exterior potential.

Figure 4.2 connects three pieces of physics: symmetry determines the field's sign, Gauss's law determines its spatial dependence, and a signed integral determines the voltage. At a finite charge-density discontinuity the field can be continuous. A separate ideal surface charge would instead add a finite jump in its normal component.

Example 4.5: Voltage across the interior and exterior of a slab

Take the ideal slab with $a = 5.0 \text{ mm}$ and $\rho = 2.0 \times 10^{-7} \text{ C/m}^3$. Find $V(2a) - V(0)$ and the electrical work on $+3.0 \text{ nC}$ moved from the centre to $x = 2a$.

Strategy. Integrate the linear interior field to a , then the constant exterior field to $2a$. The exterior field scale is $E_s = \rho a / \epsilon_0 = 113 \text{ V/m}$. Therefore

$$\Delta V = -\frac{1}{2}E_s a - E_s a = -\frac{3}{2}E_s a = -0.847 \text{ V}, \quad (4.22)$$

so $W_{\text{elec}} = -q\Delta V = +2.54 \text{ nJ}$. The triangular area under the interior field accounts for one third of this voltage drop; the exterior rectangular area accounts for two thirds. Using the exterior field throughout would predict -1.13 V , overestimating the magnitude by one third. A field-value estimate without the correct spatial profile can therefore miss the work even when the endpoint separation is known.

4.4 Reference potential and sign conventions

A potential value is meaningful only after a reference has been chosen. For isolated localized charge distributions, the usual convention is $V(\infty) = 0$. In circuits, one node is often labelled ground and assigned $V = 0$. In numerical electrostatics, boundary conditions choose the reference.

Adding a constant to all potentials does not change the electric field:

$$V'(\mathbf{r}) = V(\mathbf{r}) + C \implies -\nabla V' = -\nabla V. \quad (4.23)$$

Therefore forces, accelerations, and energy differences are unchanged. A potential value must be reported with its reference; potential differences are invariant under this common shift. Here C is an arbitrary constant potential, not a capacitance.

Method

Reliable sign procedure for voltage problems.

1. Write $\Delta V = V_f - V_i$ before substituting numbers.

2. Insert the algebraic charge q , including its sign.
3. Use $\Delta U = q\Delta V$.
4. Use $\Delta K = -\Delta U$ only when the electric force is the only force doing work.
5. Check the result with a physical sentence, not only a number.

Key message

The electric field tells where a positive test charge is pushed. The potential tells the potential energy per unit positive charge. A negative charge reverses the relation between V and its own U .

Guided checks

Reference checks

Check. Can all potentials be shifted by $+50\text{ V}$ without changing the field?

Answer. Yes, if the same constant is added everywhere.

Check. Why is $V(\infty) = 0$ convenient for a point charge?

Answer. Because the point-charge potential approaches zero as distance becomes very large.

Check. Why is $V(\infty) = 0$ not natural for an ideal infinite uniform field?

Answer. Because the potential changes without bound over an infinite distance.

4.4.1 Changing a reference and grounding an object are different operations

Changing the reference is a relabelling. If $V_A = 12\text{ V}$ and $V_B = 5\text{ V}$ in one convention, subtracting 5 V everywhere gives $V'_A = 7\text{ V}$ and $V'_B = 0$. No charge needs to move. The difference $V_A - V_B = 7\text{ V}$ is unchanged.

Grounding, by contrast, makes an electrical connection to a large conducting reservoir or a specified reference electrode. Charge can flow through that connection until the connected conductor reaches the reservoir's potential. In a model with ground fixed at zero, this imposes the physical boundary condition $V = 0$ on the conductor. The redistribution can change the field everywhere nearby. A grounded conductor need not be neutral; its induced charge is determined by the rest of the configuration.

Different isolated conductors need not have the same potential. Joining them by a conducting wire allows charge to redistribute until the connected assembly reaches a common potential. Assigning both conductors the label zero before making that connection would erase a physically meaningful voltage difference. One arbitrary additive constant can be chosen for the whole connected field description, not independently for every point.

4.4.2 When infinity cannot be the zero

For localized finite charge distributions, E decreases rapidly enough that the work integral from infinity converges. An ideal infinite line of charge provides a useful counterexample. Recalling its radial field from Gauss's law,

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r}, \quad V(r) - V(r_0) = -\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r}{r_0}\right). \quad (4.24)$$

The logarithm has a dimensionless argument. The voltage between two finite radii is finite, but the potential does not approach a finite value as $r \rightarrow \infty$. One must choose a finite reference radius. The same issue occurs for a uniform field extended throughout space: its potential grows linearly in some directions.

This is a limitation of the infinite-source idealization, not a failure of voltage. A real cable has a return conductor and finite boundaries; these supply physical voltage references and alter the far field. Useful local approximations should not be extrapolated to infinity without checking their convergence.

Guided checks

Check. A calculation sets $V = 0$ on an isolated conductor. Does this prove that it is grounded?

Answer. No. It may only specify the reference. Grounding adds a connection that allows charge exchange and imposes a potential relative to the environment.

4.5 Equipotential surfaces and field geometry

An equipotential surface is a surface on which the electric potential has one constant value. If a charge moves between two points on the same equipotential surface, then $\Delta V = 0$ and $\Delta U = 0$. The electric field does no net work for displacement along the surface.

The uniform-field geometry is shown in Figure 4.3. At regular points, equipotential surfaces are perpendicular to electrostatic field lines. If the electric field had a tangential component along an equipotential, then a small displacement along the surface would give $\mathbf{E} \cdot d\mathbf{l} \neq 0$, and Eq. (4.11) would imply a nonzero potential difference along a constant-potential surface. That is impossible.

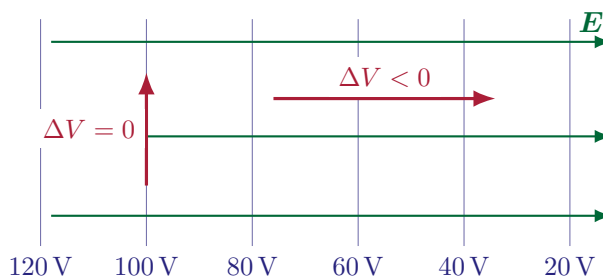


Figure 4.3: Equal voltage increments give equally spaced equipotentials in a uniform field. Red arrows indicate possible displacements, not trajectories. Only the component of displacement parallel to the field contributes to the voltage change.

Caution

An equipotential surface is not a field-free surface. It is a surface along which the field has no tangential component. The field may be large and perpendicular to the surface.

Guided checks

Equipotential checks

Check. Can $\mathbf{E}$ be nonzero at points on an equipotential?

Answer. Yes. It can be perpendicular to the surface.

Check. For the same potential increment, are neighbouring equipotentials closer in strong or weak fields?

Answer. In strong fields, because the potential changes more rapidly with distance.

Check. Can a conductor surface in electrostatic equilibrium be an equipotential?

Answer. Yes. Otherwise charges would move along the surface.

4.5.1 Reading the spacing of contours

Equipotential maps resemble topographic maps, but their labels are voltages rather than heights. Let Δn be the small distance measured *normal* to neighbouring contours. Then the local field magnitude is approximately

$$E \simeq \frac{|\Delta V|}{\Delta n}. \quad (4.25)$$

The approximation becomes better when the field changes little across the spacing. Distances measured obliquely are larger than the normal separation and underestimate the magnitude if inserted into this expression. Spacing can be compared directly only when the voltage increments are the same: a tightly packed set of 1 V contours does not automatically imply a stronger field than widely spaced 100 V contours.

A voltage measured between two distant points yields an integral constraint. It does not determine the field at each intermediate point. For example, many nonuniform functions $E_x(x)$ have the same integral over a gap. Reconstructing a local field requires a spatially resolved map, a symmetry argument, or a solved boundary-value problem.

Example 4.6: Field and work from an equipotential map

Near a point P , locally parallel contours have values 18.0, 15.0 and 12.0 V, with consecutive normal separations of 6.0 mm. The unit normal $\hat{n}$ points from the 18.0 V contour toward the 12.0 V contour. Estimate the field and the electrical work on a charge $q = -3.0$ nC moving between these outer contours.

Model. The contours are sufficiently close for a locally uniform field approximation. Over either interval,

$$\mathbf{E} \simeq \frac{3.0 \text{ V}}{6.0 \times 10^{-3} \text{ m}} \hat{n} = 500 \hat{n} \text{ V/m}. \quad (4.26)$$

For the full transfer, $\Delta V = -6.0$ V, so

$$\Delta U = (-3.0 \text{ nC})(-6.0 \text{ V}) = +18 \text{ nJ}, \quad W_{\text{elec}} = -18 \text{ nJ}. \quad (4.27)$$

The local field estimate depends on the contour spacing; the work depends only on the two endpoint voltages. The negative charge is displaced against its electric force, so negative electrical work is physically consistent.

4.5.2 Where the contour picture needs qualification

Distinct equipotential values cannot intersect, because the potential at a point has a single value. However, different branches of the *same* level set can meet at a stationary point. For example, $V = A(x^2 - y^2)$ has the zero-potential lines $y = x$ and $y = -x$, which meet at the origin where $\mathbf{E} = 0$. At that point the level set is not a smooth surface with a unique normal. The statement that field and equipotential are perpendicular applies at regular points where $\nabla V \neq 0$.

Following an equipotential path gives zero electrical work even if that path is curved. Such motion usually requires a constraint or a suitably chosen velocity; a freely released charge is not generally confined to an equipotential. Likewise, a field line specifies the local force direction on a positive charge, not its velocity. With an initial transverse velocity, a particle can cross many field lines.

4.6 Potential of a point charge and scalar superposition

Figure 4.4 shows spherical equipotentials. For a fixed source charge q in vacuum, the radial field is $E_r = k_e q / r^2$. Along a radial path from a finite reference radius r_b to r ,

$$V(r) - V(r_b) = -k_e q \int_{r_b}^r \frac{dr'}{r'^2} = k_e q \left(\frac{1}{r} - \frac{1}{r_b} \right). \quad (4.28)$$

The integration variable r' is a distance, and the sign of the source charge remains in q . Taking $r_b \rightarrow \infty$ makes the second term vanish. For a point charge q , with the reference $V(\infty) = 0$, the electric potential at distance r is

$$V(r) = k_e \frac{q}{r}, \quad k_e = \frac{1}{4\pi\epsilon_0}. \quad (4.29)$$

With this particular reference, the sign of V is the sign of q . Potential is a scalar, so several point-charge contributions add algebraically:

$$V(\mathbf{r}) = k_e \sum_i \frac{q_i}{r_i}, \quad (4.30)$$

where r_i is the distance from source charge q_i to the field point.

Method

Point-charge potential method.

1. Mark the field point.
2. Compute the distance from the field point to every source charge.
3. Add $k_e q_i / r_i$ algebraically.
4. Do not draw direction arrows for scalar potential terms.
5. Use vector methods or gradients only when the field is requested.

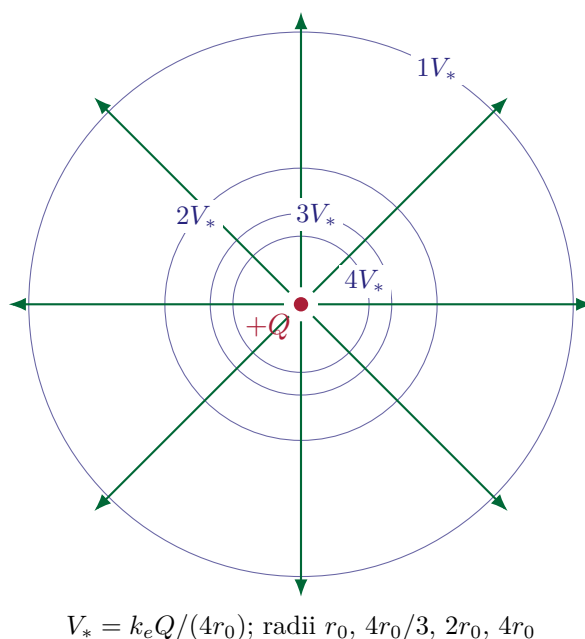


Figure 4.4: A plane through a positive point charge intersects its equipotential spheres in circles. The radii are calculated from $V = k_e Q / r$: equal potential increments crowd closer to the charge. Green arrows show radial field direction, not field magnitude.

Example 4.7: Potential at the centre of a square of charges

Four charges lie at the corners of a square of side $a = 0.20$ m: $+2.0$ nC, -2.0 nC, $+3.0$ nC, and $+1.0$ nC. Find the potential at the centre.

All corners are at distance $r = a/\sqrt{2} \simeq 0.1414$ m from the centre. Since potential is scalar,

$$V = \frac{k_e}{r}(q_1 + q_2 + q_3 + q_4). \quad (4.31)$$

The algebraic sum is 4.0 nC, so

$$V = \frac{(8.99 \times 10^9)(4.0 \times 10^{-9})}{0.1414} \text{ V} \simeq 2.54 \times 10^2 \text{ V}. \quad (4.32)$$

The electric field at the same point would require vector addition and would not be obtained by simply summing signed charge values.

Example 4.8: Zero potential does not mean zero electric field

Charges $+Q$ and $-Q$ are fixed at $x = -a$ and $x = +a$. At the origin,

$$V(0) = k_e \frac{Q}{a} + k_e \frac{-Q}{a} = 0. \quad (4.33)$$

However, the field due to $+Q$ points in the $+x$ direction at the origin, and the field due to $-Q$ also points in the $+x$ direction. Thus

$$\mathbf{E}(0) = 2 \frac{k_e Q}{a^2} \hat{\mathbf{i}}. \quad (4.34)$$

The potential value is zero, but the slope of the potential is not zero. Electric field is determined by spatial change, not by the value of the potential itself.

4.6.1 A geometry check before adding charges

The distance in a potential contribution is the positive separation between the source and the observation point. The sign comes from the charge, not from the coordinate difference. For source q_i at $\mathbf{r}_i$,

$$V_i(\mathbf{r}) = \frac{k_e q_i}{|\mathbf{r} - \mathbf{r}_i|}. \quad (4.35)$$

Writing $r_i = x - x_i$ without an absolute value on a one-dimensional axis can produce a spurious change of sign when the observation point passes to the other side of a charge. At the source itself the ideal point-charge formula is undefined.

For the square example, Figure 4.5 makes the common distance explicit. The potential at the centre does not depend on which corner receives which charge, because all distances are equal. The field can change when the charges are permuted. Thus a configuration that is simple for potential may still require careful vector reasoning for force.

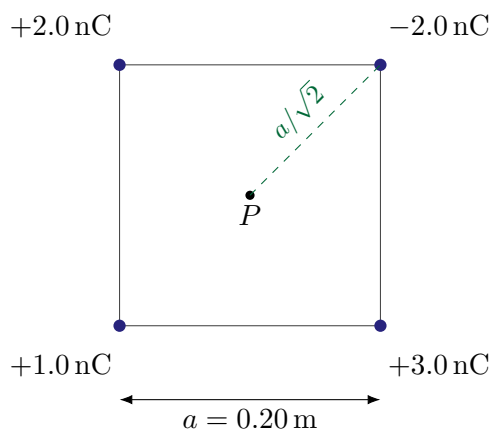


Figure 4.5: Geometry of Example 4.7. Each corner lies the same distance from P . Source signs enter an algebraic sum for V ; electric fields must also include their directions.

4.6.2 Locating zero potential without locating zero field

The condition $V = 0$ is one scalar equation. The condition $\mathbf{E} = 0$ imposes the vanishing of all field components. There is no reason for these conditions to identify the same points. Moreover, a reference change moves a zero-potential contour while leaving every zero-field point unchanged.

Example 4.9: Zero-potential points for unequal opposite charges

Place $+Q$ at $x = 0$ and $-2Q$ at $x = d$, with $Q > 0$ and $d > 0$. Find all finite points on the x axis where $V = 0$, using zero potential at infinity.

Strategy. Use positive distances and divide the axis at the source positions. The potential equation is

$$\frac{1}{|x|} - \frac{2}{|x - d|} = 0, \quad |x - d| = 2|x|. \quad (4.36)$$

For $x < 0$, $d - x = -2x$, giving $x = -d$. For $0 < x < d$, $d - x = 2x$, giving $x = d/3$. For $x > d$, the equation becomes $x - d = 2x$, which has no solution in that region. The two points are therefore $-d$ and $d/3$.

At $x = d/3$, the two field contributions both point toward $+x$, so the field certainly does not vanish. At $x = -d$, their magnitudes are $k_e Q/d^2$ to the left and $k_e Q/(2d^2)$ to the right; the net field is leftward. Neither zero-potential point is a force-free point. In three dimensions the equation $|\mathbf{r} - d\hat{\mathbf{i}}| = 2|\mathbf{r}|$ describes a sphere; the two axial solutions are only its intersections with the axis.

The potential falls as $1/r$, while the field magnitude falls as $1/r^2$. These different powers are required by differentiation. Far away from a compact set of charges with nonzero total charge, the leading potential is $k_e Q_{\text{total}}/r$. If the total charge is zero, this leading contribution cancels, but the potential need not vanish at finite distance. The dipole is the simplest example.

4.7 Electric dipoles and far-field potential

The opposite-charge geometry in Figure 4.6 introduces the simplest neutral source. A dipole consists of charges $+q$ and $-q$ separated by a displacement vector $\mathbf{d}$ from the negative charge to the positive charge. Its dipole moment is

$$\mathbf{p} = q\mathbf{d}. \quad (4.37)$$

Far from the dipole, where $r \gg d$ and θ is measured from $\mathbf{p}$, the potential is approximately

$$V(r, \theta) = k_e \frac{p \cos \theta}{r^2}. \quad (4.38)$$

The $1/r$ monopole contribution cancels because the total charge is zero. The dipole potential falls as $1/r^2$, and the dipole field falls as $1/r^3$.

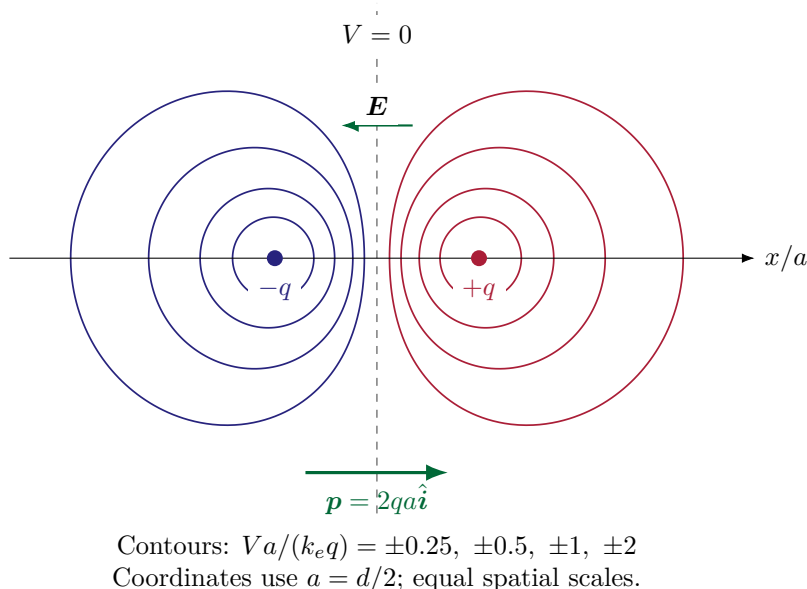


Figure 4.6: Calculated equipotential contours of the exact two-charge dipole. Positive contours surround $+q$ and negative contours surround $-q$; the perpendicular bisector has zero potential. At its regular points the field crosses from the positive side toward the negative side. The green field arrow shows direction only.

Note

The dipole formula is a far-field approximation. Near the charges, use the exact sum of the two point-charge potentials.

4.7.1 Exact potential and the origin of the far-field approximation

For the dipole in Figure 4.6, place $-q$ at $x = -d/2$ and $+q$ at $x = d/2$. Let r be the distance from the midpoint to the observation point and θ its angle from the positive x axis, which is the direction of $\mathbf{p}$. The distances to the positive and negative sources are

$$r_{\pm} = \sqrt{r^2 + \frac{d^2}{4} \mp r d \cos \theta}, \quad V = k_e q \left(\frac{1}{r_+} - \frac{1}{r_-} \right). \quad (4.39)$$

This expression is exact for two ideal point charges, away from their positions. It distinguishes the observation radius r from the two source-to-point distances $r_{\pm}$.

For $d/r \ll 1$, factor r out of each square root and use $(1 + u)^{-1/2} = 1 - u/2 + \dots$. To first order in d/r ,

$$\frac{1}{r_{\pm}} \simeq \frac{1}{r} \left(1 \pm \frac{d \cos \theta}{2r} \right), \quad V \simeq \frac{k_e q d \cos \theta}{r^2}. \quad (4.40)$$

The two $1/r$ terms cancel. This derivation explains both the extra inverse power of distance and the angular factor. The approximation must not be used close to either source. On the perpendicular bisector $\cos \theta = 0$, the exact potential is zero by symmetry, not merely small in the far field.

The exact contours separate into positive loops around the positive charge, negative loops around the negative charge, and the zero-potential perpendicular bisector. A curve enclosing both charges

cannot simply be assigned one nonzero dipole potential without checking the defining equation. The plotted contours in Figure 4.6 are generated from Eq. (4.39), rather than guessed from the shape of field lines.

Example 4.10: How far away is far enough for a dipole?

For $q = 2.0 \text{ nC}$, $d = 1.0 \text{ cm}$, evaluate the exact and dipole potentials at $x = r = 5.0 \text{ cm}$ on the positive axis. Estimate the relative approximation error.

Strategy. Along this axis the distances are $r - d/2$ and $r + d/2$. Combining the fractions gives

$$V_{\text{exact}} = \frac{k_e q d}{r^2 - d^2/4}, \quad V_{\text{dip}} = \frac{k_e q d}{r^2}. \quad (4.41)$$

With $k_e = 8.99 \times 10^9 \text{ N m}^2/\text{C}^2$, the values are $V_{\text{exact}} = 72.6 \text{ V}$ and $V_{\text{dip}} = 71.9 \text{ V}$ to three displayed figures. The approximate value is smaller. Its fractional error relative to the exact value is

$$\frac{V_{\text{exact}} - V_{\text{dip}}}{V_{\text{exact}}} = \frac{d^2}{4r^2} = 0.010. \quad (4.42)$$

The error is 1.0% in this geometry. This is a model-error estimate, not an experimental uncertainty. It also shows why the expansion parameter is a ratio of lengths. Relative errors are unhelpful near a zero of potential, where absolute error or field error is the more useful measure.

4.7.2 A dipole in an external potential

The potential *created by* a dipole should be distinguished from its energy *in an external field*. For a small fixed dipole with midpoint $\mathbf{r}_0$, the latter is

$$U_{\text{ext}} = qV_{\text{ext}}(\mathbf{r}_0 + \mathbf{d}/2) - qV_{\text{ext}}(\mathbf{r}_0 - \mathbf{d}/2) \quad (4.43)$$

$$\simeq q\mathbf{d} \cdot \nabla V_{\text{ext}}(\mathbf{r}_0) = -\mathbf{p} \cdot \mathbf{E}_{\text{ext}}(\mathbf{r}_0). \quad (4.44)$$

The dipole's internal interaction energy is constant if its separation is fixed and has been omitted from this orientation-dependent energy. In a uniform external field, $U_{\text{ext}} = -pE \cos \alpha$, where α is the angle between $\mathbf{p}$ and that field. Alignment minimizes the energy. Differentiating with respect to angle gives the restoring torque component $\tau_\alpha = -pE \sin \alpha$. This recovers the torque result from the electric-field chapter by an energy method. A uniform field exerts no net force on the ideal dipole even though it can exert torque.

4.8 Potential of continuous charge distributions

For a continuous charge distribution, the point-charge sum becomes an integral:

$$V(\mathbf{r}) = k_e \int \frac{dq}{R}, \quad (4.45)$$

where R is the distance from the charge element dq to the field point. The charge element is written as $dq = \lambda dl$ for a line distribution, $dq = \sigma dA$ for a surface distribution, and $dq = \rho d\tau$ for a volume distribution.

Potential integrals are frequently easier than field integrals because V is scalar. There are no vector components to cancel. The modelling work remains: choose coordinates, express R , write dq , set limits, and check limiting cases.

Method**Continuous-potential calculation.**

1. Draw the distribution and the field point.
2. Choose a charge element dq .
3. Write the distance R from the element to the field point.
4. Integrate $k_e dq/R$ over the full distribution.
5. Check units and limiting cases.

Example 4.11: Potential on the axis of a uniformly charged ring

A ring of radius R carries total charge Q uniformly. At a point on the axis a distance z from the centre, every charge element is the same distance

$$R_z = \sqrt{R^2 + z^2} \quad (4.46)$$

from the field point. Therefore

$$V(z) = k_e \int \frac{dq}{R_z} = \frac{k_e Q}{\sqrt{R^2 + z^2}}. \quad (4.47)$$

The axial electric field follows from $E_z = -dV/dz$:

$$E_z = \frac{k_e Q z}{(R^2 + z^2)^{3/2}}. \quad (4.48)$$

The potential method obtains the result with a scalar integral; the vector field calculation requires explicit cancellation of transverse components.

Example 4.12: Potential of a finite uniformly charged rod

A rod extends from $x = 0$ to $x = L$ and has linear charge density λ . Find the potential at $x = a > L$ on the same axis. A charge element is $dq = \lambda dx$, and its distance from the field point is $a - x$. Thus

$$V(a) = k_e \int_0^L \frac{\lambda dx}{a - x} = k_e \lambda \ln \left(\frac{a}{a - L} \right). \quad (4.49)$$

The result diverges as $a \rightarrow L^+$ because the ideal line-charge model places charge arbitrarily close to the field point. A real rod has finite radius and cannot be treated as a mathematical line at such distances.

4.8.1 Source coordinates and observation coordinates

The integration variable labels the source, not the observation point. A notation that makes this distinction explicit is

$$V(\mathbf{r}) = k_e \int \frac{\rho(\mathbf{r}') d\tau'}{|\mathbf{r} - \mathbf{r}'|}. \quad (4.50)$$

During integration $\mathbf{r}$ is held fixed while $\mathbf{r}'$ ranges over the charged object. For a thin wire replace $\rho d\tau'$ with $\lambda dl'$; for a thin sheet use $\sigma dA'$. These models specify the charge density. A conducting wire or plate held at fixed potential will generally have a nonuniform charge density and requires a different calculation.

The expressions here assume vacuum, or air approximated as vacuum, with no additional polarizable

bodies. Dielectric interfaces require the material treatment of Chapter 5. They cannot generally be included by inserting one arbitrary dielectric constant into a vacuum result.

4.8.2 Ring geometry, symmetry and limits

Figure 4.7 shows why the ring integral is simple: all elements of one ring are equally far from the axial observation point. A uniform ring has rotational symmetry, which also makes its transverse field components vanish on the axis. Even a nonuniformly charged ring has the same *axial potential* $k_e Q / \sqrt{R^2 + z^2}$, because the distance remains constant. Its transverse field components need not vanish. Knowledge of V only on a line determines the derivative along that line, not all derivatives off it.

For the uniform ring result, the limits are instructive:

$$V(0) = \frac{k_e Q}{R}, \quad E_z(0) = 0, \quad (4.51)$$

$$V(z) \simeq \frac{k_e Q}{|z|} \quad (|z| \gg R), \quad E_z(z) \simeq \frac{k_e Q z}{|z|^3} \quad (|z| \gg R). \quad (4.52)$$

At the centre the potential is nonzero but its axial slope vanishes. Far away, the ring looks like a point charge. Near the centre, a binomial expansion gives

$$V(z) = \frac{k_e Q}{R} \left[1 - \frac{z^2}{2R^2} + \mathcal{O}\left(\frac{z^4}{R^4}\right) \right]. \quad (4.53)$$

For $Q > 0$ this is a maximum along the axis. A negative test charge constrained to that axis therefore has a local *minimum* of $U = qV$. This is only a constrained stability statement; the enrichment examines transverse displacements.

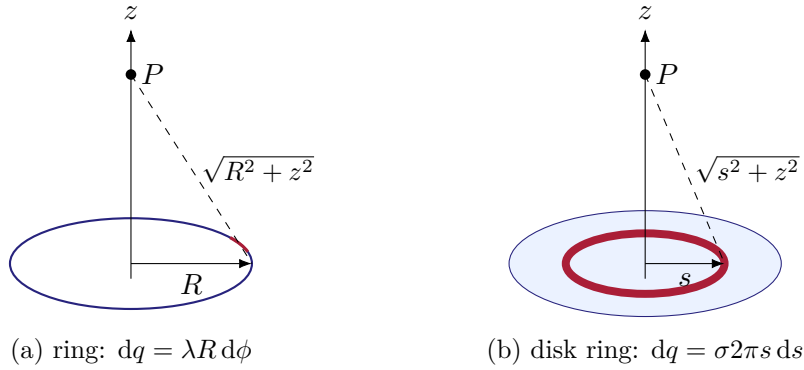


Figure 4.7: Perspective schematics for axial potential integrals. The disk is decomposed into annuli of radius s and radial width ds . The axial coordinate z remains fixed while s is integrated.

4.8.3 A finite rod seen from its perpendicular bisector

The axial rod in Example 4.12 has distances linear in the source coordinate. A perpendicular observation introduces a square root and illustrates a second common integration geometry. Let a uniformly charged rod extend from $x' = -L/2$ to $x' = L/2$, and observe it at $(0, b)$, $b > 0$. Then

$$V(b) = k_e \lambda \int_{-L/2}^{L/2} \frac{dx'}{\sqrt{x'^2 + b^2}} \quad (4.54)$$

$$= 2k_e \lambda \ln \left(\frac{L/2 + \sqrt{(L/2)^2 + b^2}}{b} \right). \quad (4.55)$$

The factor two follows from an even integrand. One antiderivative is $\ln[x' + \sqrt{x'^2 + b^2}]$; its logarithm of a dimensional quantity is only an intermediate shorthand. The definite integral becomes a logarithm of a dimensionless ratio, as required.

For $b \gg L$, the logarithm gives $V \simeq k_e \lambda L / b = k_e Q / b$. As $L \rightarrow \infty$ at fixed λ , the potential itself diverges. Differences between two finite b values remain finite and approach the infinite-line formula in Eq. (4.24). In the ideal line model, approaching the charged line also creates a logarithmic divergence. A finite wire radius limits that model near the material.

Example 4.13: A rod measurement with a point-charge estimate

A thin insulating rod of length 0.40 m carries a uniformly distributed charge 8.0 nC. Calculate the potential 0.15 m from its centre on the perpendicular bisector. Is a point-charge model adequate there?

Here $\lambda = 20 \text{ nC/m}$, and the endpoint distance is $\sqrt{0.20^2 + 0.15^2} = 0.25 \text{ m}$. Thus

$$V = 2(8.99 \times 10^9)(20 \times 10^{-9}) \ln(3) \text{ V} = 395 \text{ V}. \quad (4.56)$$

Replacing the rod by a point charge at its centre gives $k_e Q / b = 479 \text{ V}$, about 21% larger. Most elements of the rod are farther away than the centre, so concentrating them all at the centre overestimates their potential. The observation distance is not large compared with the rod length, and the discrepancy is physically expected. The two models converge as the observation point is moved outward.

4.8.4 A disk assembled from rings

Consider a thin *insulating* disk of radius R with prescribed uniform surface charge density σ . On its symmetry axis, a ring of radius s and width ds contributes

$$dV = k_e \frac{2\pi\sigma s ds}{\sqrt{s^2 + z^2}}. \quad (4.57)$$

Integrating from the centre to the rim gives

$$V(z) = 2\pi k_e \sigma \int_0^R \frac{s ds}{\sqrt{s^2 + z^2}} \quad (4.58)$$

$$= \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2 + z^2} - |z| \right). \quad (4.59)$$

The absolute value is essential: $\sqrt{z^2} = |z|$, so the potential is even in z . Choosing zero at infinity fixes the integration constant. For stable numerical evaluation at large $|z|$, rationalize the difference:

$$V(z) = \frac{\sigma R^2}{2\epsilon_0 [\sqrt{R^2 + z^2} + |z|]}. \quad (4.60)$$

This form avoids subtracting nearly equal large numbers.

For $z > 0$, differentiation yields

$$E_z = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{R^2 + z^2}} \right). \quad (4.61)$$

For $z < 0$ the field reverses by symmetry. At the ideal sheet the two one-sided limits differ by σ/ϵ_0 . The potential is continuous but has a cusp, so a unique classical derivative at the sheet is not defined. This is a boundary idealization, not an infinite jump in potential.

Far away, $\sqrt{R^2 + z^2} - |z| \simeq R^2/(2|z|)$, giving $V \simeq k_e Q/|z|$ with $Q = \pi R^2 \sigma$. Close to the central surface, $0 < z \ll R$, the field approaches $\sigma/(2\epsilon_0)$, the infinite-sheet value. Taking $R \rightarrow \infty$ at fixed

σ makes V diverge even though E_z remains finite. This again explains why infinite sheets require a finite voltage reference.

Example 4.14: How close to a disk is the infinite-sheet field accurate?

A uniformly charged disk has $R = 0.10$ m and $\sigma = 4.0$ nC/m². Find V and E_z at $z = 0.020$ m and compare the field with the infinite-sheet estimate.

Using $\epsilon_0 \simeq 8.854 \times 10^{-12}$ F/m gives $\sigma/(2\epsilon_0) = 226$ V/m. Equations (4.59) and (4.61) give

$$V = (226 \text{ V/m})(0.10198 - 0.02000) \text{ m} = 18.5 \text{ V}, \quad (4.62)$$

$$E_z = (226 \text{ V/m}) \left(1 - \frac{0.02000}{0.10198} \right) = 182 \text{ V/m}. \quad (4.63)$$

The exact field is about 19.6% below the infinite-sheet value. More generally this deficit, expressed as a fraction of the infinite-sheet value, is $z/\sqrt{R^2 + z^2}$. Requiring it to be below 5% gives $z/R < 0.0501$. The phrase “near the disk” therefore needs a quantitative tolerance. This calculation concerns a fixed uniform surface charge; a conducting disk held at one voltage does not generally meet that assumption.

4.9 Recovering the electric field from potential

Figure 4.8 relates the potential slope to field direction. Electric potential is not merely an energy bookkeeping device. It contains the electric field information. The field is the negative gradient of the potential:

$$\mathbf{E} = -\nabla V. \quad (4.64)$$

In Cartesian coordinates,

$$\mathbf{E} = -\left(\frac{\partial V}{\partial x} \hat{\mathbf{i}} + \frac{\partial V}{\partial y} \hat{\mathbf{j}} + \frac{\partial V}{\partial z} \hat{\mathbf{k}} \right). \quad (4.65)$$

In one dimension,

$$E_x = -\frac{dV}{dx}. \quad (4.66)$$

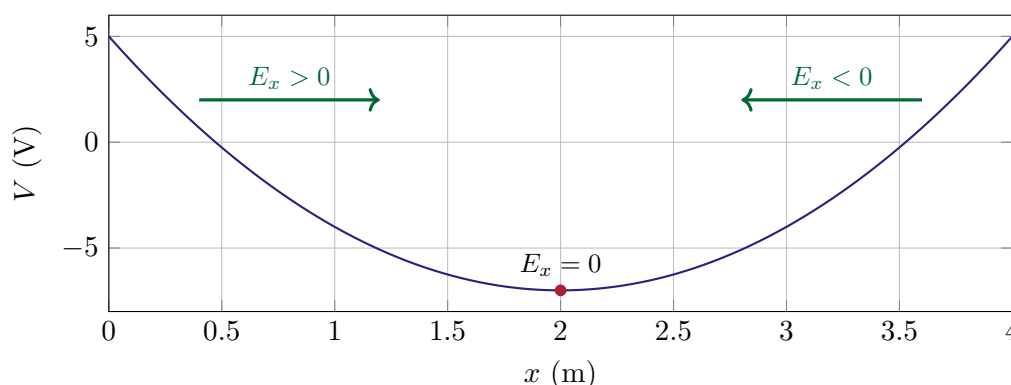


Figure 4.8: The potential in Example 4.15. The field component is the negative slope, and its direction is along the spatial x axis. A field vector is not a tangent vector drawn in the (x, V) plotting plane.

Example 4.15: Field from a polynomial potential

Let

$$V(x) = ax^2 + bx + c, \quad (4.67)$$

Worked example (continued)

with $a = 3.0 \text{ V/m}^2$, $b = -12 \text{ V/m}$, and $c = 5.0 \text{ V}$. Then

$$E_x = -\frac{dV}{dx} = -(2ax + b) = (12 \text{ V/m}) - (6.0 \text{ V/m}^2)x \quad (4.68)$$

with x a length. The field is zero at $x = 2.0 \text{ m}$. This is a stationary point of potential. Stability still depends on the sign of the charge and on the curvature of V .

4.9.1 Deriving the gradient relation one direction at a time

A partial derivative is an ordinary rate of change with the other coordinates held fixed. Move a small distance dx in the x direction while holding y and z fixed. Then $dV = -E_x dx$, so $E_x = -\partial V/\partial x$. Repeating in the other two directions gives the three Cartesian components. More generally,

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = \nabla V \cdot d\mathbf{r}. \quad (4.69)$$

Comparison with $dV = -\mathbf{E} \cdot d\mathbf{r}$ for every displacement establishes $\mathbf{E} = -\nabla V$. Thus the gradient is not a new physical law introduced by notation. It collects three local slope measurements into one vector.

For a displacement along a unit vector $\hat{\mathbf{s}}$, the slope is $dV/ds = \nabla V \cdot \hat{\mathbf{s}}$. The largest decrease per unit distance occurs for $\hat{\mathbf{s}}$ opposite to ∇V , namely along $\mathbf{E}$. Along an equipotential tangent this directional derivative vanishes. These observations connect the algebraic and geometric descriptions.

If a potential is known only as $V(x, 0, 0)$, differentiating gives only E_x on that line. It does not prove that E_y and E_z are zero. A symmetry argument or the full spatial dependence is required. For the uniform ring, rotational and reflection symmetry supply that information; for an arbitrarily charged ring they do not.

Example 4.16: A two-dimensional potential and a directional voltage change

In a small region, let

$$V(x, y) = A(x^2 - y^2) + Bx + V_0, \quad (4.70)$$

where $A = 20 \text{ V/m}^2$, $B = -6.0 \text{ V/m}$ and $V_0 = 4.0 \text{ V}$. Find the field at $(0.10, 0.05) \text{ m}$ and estimate the potential change for displacement $\Delta\mathbf{r} = (1.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}) \text{ mm}$.

Differentiating while holding the other coordinate fixed gives

$$E_x = -2Ax - B, \quad E_y = +2Ay. \quad (4.71)$$

At the specified point, $\mathbf{E} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}) \text{ V/m}$, with magnitude 2.83 V/m and direction 45° above the x axis. The local estimate is

$$\Delta V \simeq -\mathbf{E} \cdot \Delta\mathbf{r} = -0.0060 \text{ V}. \quad (4.72)$$

The exact polynomial difference adds $A[(\Delta x)^2 - (\Delta y)^2] = -0.000060 \text{ V}$, so the exact change is -6.06 mV . The small correction quantifies the error from treating the field as constant over the displacement. Changing V_0 affects neither the field nor this voltage difference.

4.9.2 Reading a graph without differentiating a formula

A plot of V against one coordinate gives the corresponding field component through its slope. Where the graph rises toward increasing x , $E_x < 0$; where it falls, $E_x > 0$. A horizontal tangent implies

$E_x = 0$, but establishes a zero *total* field only when other components vanish. The vertical location of the curve has no effect on the field.

A finite voltage drop over a segment gives the average field component,

$$\overline{E}_x = -\frac{V(x_2) - V(x_1)}{x_2 - x_1}, \quad (4.73)$$

when the path holds y, z fixed. This average need not equal the local field at either endpoint. For a linear potential the slope is constant, making the average exact everywhere. A curved potential requires smaller segments or an analytic derivative to resolve local variations.

4.9.3 Stationary potential and mechanical stability

An equilibrium position for a freely moving test charge requires $\mathbf{F} = -\nabla U = 0$. To decide whether it is stable, consider nearby displacements. In one constrained coordinate, expand about x_0 :

$$U(x) = U(x_0) + \frac{1}{2}qV''(x_0)(x - x_0)^2 + \cdots, \quad (4.74)$$

where $V'(x_0) = 0$. If $qV''(x_0) > 0$, the force is restoring for sufficiently small displacements; if it is negative, the force pushes farther away. If the second derivative vanishes, higher orders must be examined.

For the polynomial in Example 4.15, $V'' = 6.0 \text{ V/m}^2$. A positive charge constrained to move along x is stable at $x = 2.0 \text{ m}$; a negative charge is unstable there. A three-dimensional charge-free region cannot have this purely one-dimensional quadratic potential throughout it: its nonzero second derivative implies charge density through Poisson's equation. A claim of unconstrained electrostatic trapping must therefore examine all directions and the sources, as developed in the enrichment.

4.10 Potential energy of a system of point charges

The electrostatic potential energy of point charges assembled from infinity is

$$U = k_e \sum_{i < j} \frac{q_i q_j}{r_{ij}}. \quad (4.75)$$

The sum is over unordered pairs. The notation $i < j$ avoids double counting. Alternatively, if V_i is the potential at charge q_i due to all other charges, then

$$U = \frac{1}{2} \sum_i q_i V_i. \quad (4.76)$$

The factor $1/2$ is necessary because each pair appears twice in $\sum_i q_i V_i$.

Caution

The sign of U matters. Like-charge pairs contribute positive potential energy. Opposite-charge pairs contribute negative potential energy. A negative interaction potential energy means the configuration is below the separated-charge reference. It does not, by itself, prove a stable equilibrium or a bound many-particle trajectory; kinetic energy and possible escape channels also matter.

Example 4.17: Work required to assemble three charges

Let $q_1 = +2.0 \mu\text{C}$ at the origin, $q_2 = -3.0 \mu\text{C}$ at $x = 0.40 \text{ m}$, and $q_3 = +1.0 \mu\text{C}$ at $y = 0.30 \text{ m}$. The separations are $r_{12} = 0.40 \text{ m}$, $r_{13} = 0.30 \text{ m}$, and $r_{23} = 0.50 \text{ m}$. Therefore

$$U = k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \quad (4.77)$$

$$\simeq -0.129 \text{ J}. \quad (4.78)$$

The negative sign means that, relative to infinitely separated charges, the final configuration has lower electrostatic potential energy.

4.10.1 Deriving the pair sum by assembling the system

Bring the charges from infinite separation to their final positions one at a time, keeping the already placed charges fixed. Bringing in the first charge changes no interaction energy. Bringing in the second requires quasistatic external work $k_e q_1 q_2 / r_{12}$. The third charge sees the potential of both earlier charges, so its contribution is

$$W_3 = q_3 \left(\frac{k_e q_1}{r_{13}} + \frac{k_e q_2}{r_{23}} \right). \quad (4.79)$$

Summing the assembly steps includes each pair once. The order of assembly changes the intermediate work contributions but not their total, since every final pair separation is the same. For N charges there are $N(N-1)/2$ distinct pairs. This count is a useful check on a handwritten sum or a computer loop.

The alternative $U = \frac{1}{2} \sum_i q_i V_i$ uses V_i due to all *other* charges. It is not the sum of the work needed to bring every charge independently into an already completed configuration. That imagined procedure includes every interaction twice. Nor may a self-potential be inserted into V_i : the point-charge interaction model omits self-energies.

4.10.2 Interpreting the three-charge example

For Example 4.17, the pair terms, rounded only for display, are

$$U_{12} = -0.13485 \text{ J}, \quad U_{13} = +0.05993 \text{ J}, \quad U_{23} = -0.05394 \text{ J}. \quad (4.80)$$

Their sum is -0.12886 J , consistent with the quoted -0.129 J . During quasistatic assembly an external agent extracts net energy. If the assembly is performed by allowing free motion instead, some of that decrease becomes kinetic energy, and the charges will not automatically stop at the specified positions.

Figure 4.9 records the geometry. The 0.50 m diagonal is required in the interaction between q_2 and q_3 ; using one of the coordinate lengths for that separation is a common mistake.

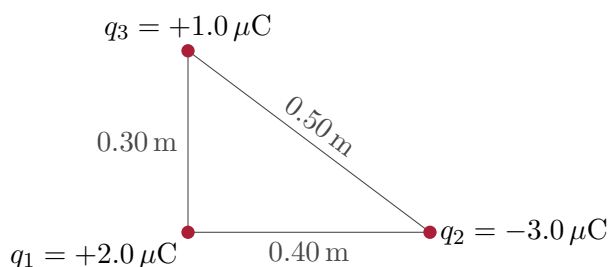


Figure 4.9: Three pair separations in the assembly calculation. The source positions are fixed while the interaction energy is evaluated.

4.10.3 Moving one charge in an existing assembly

Suppose q_1 and q_2 remain fixed while q_3 moves. The pair energy U_{12} does not change and cancels when final and initial energies are subtracted. The remaining change is

$$\Delta U = q_3 [V_{12}(\mathbf{r}_f) - V_{12}(\mathbf{r}_i)], \quad (4.81)$$

where V_{12} is the potential of the two fixed sources. This connects the full pair sum with the familiar test-charge relation. The latter is efficient precisely because the unchanged source–source energy can be ignored.

If two particles of comparable masses are released, both generally acquire kinetic energy. Setting all the released interaction energy equal to the kinetic energy of just one particle is then incorrect. Total momentum, as well as energy, can be needed. The fixed-source limit is justified when a support holds the source or when its recoil energy is negligible.

4.11 Energy conservation with electric potential

When only the electrostatic force does work,

$$K_i + U_i = K_f + U_f. \quad (4.82)$$

For a charge q moving through ΔV ,

$$\Delta K = -q\Delta V. \quad (4.83)$$

If an applied force does work W_{app} , then

$$\Delta K = -q\Delta V + W_{\text{app}}. \quad (4.84)$$

For slow motion with no change in kinetic energy,

$$W_{\text{app}} = q\Delta V. \quad (4.85)$$

Note

The electronvolt is an energy unit:

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}. \quad (4.86)$$

It is the magnitude of the energy change of a charge of magnitude e moving through 1 V.

Example 4.18: A sign-sensitive energy problem

A particle has charge $q = -4.0 \text{ nC}$ and mass $m = 2.0 \text{ mg}$. It moves from a to b , where $V_b - V_a = -600 \text{ V}$. Its initial speed is 0.80 m/s . No non-electrostatic work is done.

The potential-energy change is

$$\Delta U = q\Delta V = (-4.0 \times 10^{-9})(-600) \text{ J} = +2.4 \times 10^{-6} \text{ J}. \quad (4.87)$$

The initial kinetic energy is

$$K_i = \frac{1}{2}(2.0 \times 10^{-6})(0.80)^2 \text{ J} = 6.4 \times 10^{-7} \text{ J}. \quad (4.88)$$

The required increase in potential energy is larger than the initial kinetic energy. The particle cannot reach point b under the electric force alone. If it is constrained to a one-dimensional path with a monotonically rising potential energy, it reaches a turning point before b . In more general geometry the energy test excludes b but does not specify the trajectory.

4.11.1 Accessible regions and turning points

In a time-independent external potential, write the constant mechanical energy as $\mathcal{E} = K + qV$. Then

$$K(\mathbf{r}) = \mathcal{E} - qV(\mathbf{r}) \geq 0. \quad (4.89)$$

This inequality rules out positions whose potential energy exceeds the available total energy. It does not specify a unique route through the allowed region. In one-dimensional motion, a boundary satisfying $\mathcal{E} = qV$ is a turning point if the force reverses the velocity there. If a particle starts exactly at an unstable equilibrium with zero speed, the ideal mathematical solution can remain there; a perturbation changes that situation.

The example with the negative bead has $K_i = 0.64 \mu\text{J}$. At a turning point the required potential change is

$$\Delta V_{\text{turn}} = \frac{K_i}{q} = -160 \text{ V}. \quad (4.90)$$

The proposed drop of 600 V is therefore inaccessible. If the potential falls uniformly along the chosen one-dimensional path, the turning point lies $160/600$ of the way to the proposed endpoint. Without the spatial potential function, only the turning *voltage*, not its position, is known.

Example 4.19: Retarding an electron beam

Electrons enter a one-dimensional retarding region with kinetic energy 180 eV . The entry electrode is at $+40 \text{ V}$. What electrode potential is sufficient to stop electrons with this energy, neglecting collisions and time variation?

Strategy. Set $K_f = 0$ in $\Delta K = -q\Delta V$. For an electron, $q = -e$, hence

$$-180 \text{ eV} = e(V_f - 40 \text{ V}), \quad V_f = -140 \text{ V}. \quad (4.91)$$

The electron moves toward a lower potential, increasing its potential energy. A rise of 180 V would accelerate it instead. The stopping *difference* is -180 V ; the final electrode value depends on the chosen reference.

This result is an energy threshold. At exact threshold an electron may approach a smooth barrier top asymptotically, so it need not reach the electrode in finite time. A distribution of entry energies is progressively rejected as the retarding voltage is increased. If transverse motion is present and the potential varies only along the beam direction, the threshold concerns the initial longitudinal kinetic energy, because the transverse kinetic energy remains unchanged.

4.11.2 Speed, trajectory and collision losses

For a nonrelativistic particle whose kinetic energy is entirely translational,

$$v_f^2 = v_i^2 - \frac{2q}{m}(V_f - V_i). \quad (4.92)$$

A negative right-hand side indicates an inaccessible endpoint, not an imaginary physical speed. The kinetic-energy relation remains the correct starting point at high energy, but the expression $K = mv^2/2$ must then be replaced by its relativistic form. For electrons, a useful check is $K \ll m_e c^2 \simeq 511 \text{ keV}$. The 250 eV example lies comfortably in that regime, giving $v \simeq 9.38 \times 10^6 \text{ m/s}$, about 0.031 times the speed of light.

A voltage alone does not determine a travel time. Two gaps with the same voltage but different lengths give the same collisionless energy gain and different accelerations. Nor does $\Delta K = -q\Delta V$ describe the final kinetic energy of a carrier crossing a material while colliding repeatedly. The electric field still does work $-q\Delta V$, but part of that work is transferred to the material. Detector drift and collisionless acceleration must be distinguished.

Example 4.20: A charged bead and gravity

A bead with $q = +5.0 \text{ nC}$ and $m = 2.0 \text{ mg}$ is released from rest and constrained to move vertically upward through $h = 3.0 \text{ cm}$. The upper endpoint is 120 V below the lower endpoint. Neglect friction. Can it reach the top?

Model. Include gravitational energy as well as electric potential energy. The electrical change is $q\Delta V = -0.600 \mu\text{J}$, whereas the gravitational increase is

$$mgh = (2.0 \times 10^{-6})(9.81)(0.030) \text{ J} = 0.589 \mu\text{J}. \quad (4.93)$$

Therefore $K_f = 0.0114 \mu\text{J} > 0$ and $v_f \simeq 0.107 \text{ m/s}$. In the uniform-field model, $E = 4000 \text{ V/m}$ upward and the net acceleration is only $qE/m - g = 0.19 \text{ m/s}^2$. This agrees with $v_f^2 = 2ah$. Ignoring gravity would predict 0.775 m/s and would miss the near cancellation. The result is sensitive to small changes in the assumed charge and mass, so the quoted values are an idealized example rather than a precision prediction for a real bead. The threshold is $q|\Delta V| = mgh$.

4.11.3 An energy graph separates turning points from equilibria

For constrained one-dimensional motion, draw $U(x) = qV(x)$ and a horizontal line at the conserved mechanical energy $\mathcal{E}$. The vertical gap between the two is $K(x)$. An intersection with U identifies zero kinetic energy; a horizontal tangent to U identifies zero force. These are different conditions. A particle can turn where the force is nonzero, and it can pass through a force-free point with substantial speed.

Figure 4.10 uses a local quadratic potential around a minimum. It is a model of a constrained motion, not an assertion that a charge-free three-dimensional field has a stable minimum. The additive energy reference is arbitrary, but the gap $\mathcal{E} - U$ is not.

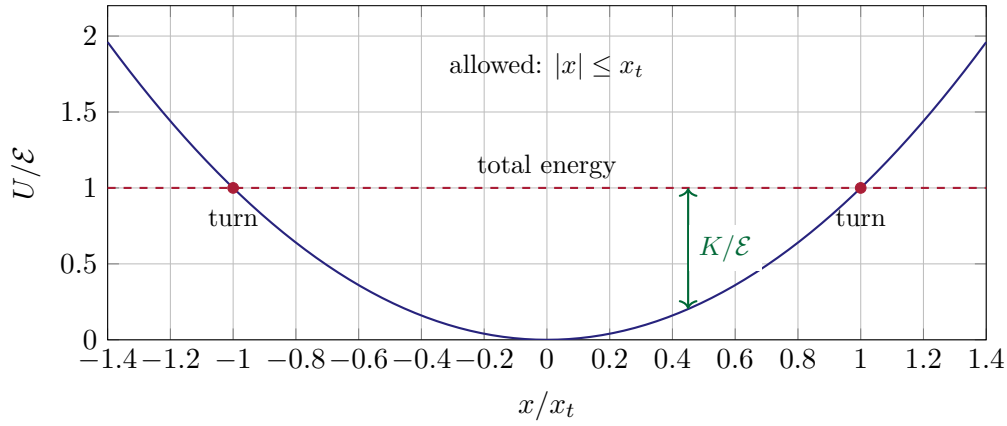


Figure 4.10: A constrained quadratic potential-energy landscape $U = \mathcal{E}(x/x_t)^2$. Turning points satisfy $K = 0$; equilibrium is at $x = 0$, where the force vanishes. The energy gap gives speed but not the sign of velocity.

For the same energy and position, a particle can move left or right. The energy equation determines v^2 , while the initial velocity selects the direction until a turning point is reached. To obtain a time of travel in a known direction one may integrate

$$t_f - t_i = \int_{x_i}^{x_f} \frac{dx}{v(x)}, \quad |v(x)| = \sqrt{\frac{2}{m}[\mathcal{E} - qV(x)]}. \quad (4.94)$$

The appropriate branch and intervals must be chosen. At a smooth barrier maximum, the threshold-energy transit time can diverge; finite kinetic energy at the starting point does not guarantee crossing every accessible-looking stationary boundary in finite time.

4.11.4 Closest approach to a repulsive source

A particle with positive charge q approaching a fixed positive source Q slows because $U = k_e Qq/r$ increases as r decreases. For head-on motion from a large distance, with initial kinetic energy K_∞ , the radial speed vanishes when

$$K_\infty = \frac{k_e Qq}{r_{\min}}, \quad r_{\min} = \frac{k_e Qq}{K_\infty}. \quad (4.95)$$

This calculation uses the nonrelativistic Coulomb model and neglects source recoil. With nonzero angular momentum, radial motion can turn while tangential kinetic energy remains; the simple head-on expression no longer gives the same closest approach. A finite-size source also requires the exterior point-charge formula to remain valid at the predicted radius.

Example 4.21: A collisionless ion turned back by a charged sphere

A small ion of charge $+e$ approaches head-on from far away with kinetic energy 120 eV. The source is an isolated conducting sphere of radius $R = 2.0$ cm and charge $Q = 1.0$ nC. Treat the ion as a nonperturbing test charge and determine whether it reaches the surface.

Strategy. Compare the ion's available energy with the surface barrier before predicting a closest distance. The surface potential is $V_s = k_e Q/R = 450$ V, corresponding to a barrier $eV_s = 450$ eV. The ion has too little energy to reach the surface. Its turning potential is $K_\infty/e = 120$ V, so

$$r_{\min} = \frac{k_e Q}{120 \text{ V}} = 0.0749 \text{ m}. \quad (4.96)$$

This exceeds R , validating use of the exterior spherical potential. The direction reverses at a nonzero electric field. If the ion energy were above 450 eV, the exterior model would

Worked example (continued)

predict arrival at the surface with remaining kinetic energy, but would not describe collisions or absorption inside the material. Near a conductor, a larger test charge can induce significant polarization; that source-response effect is excluded here.

4.11.5 Escape from an attractive source

For a particle of negative charge $-q$ in the field of a fixed positive source Q , the interaction energy is $U = -k_e Qq/r$. In radial motion, the threshold to escape from radius r_0 to infinity with zero final speed is

$$K_{\text{esc}} = \frac{k_e Qq}{r_0}, \quad v_{\text{esc}} = \sqrt{\frac{2k_e Qq}{mr_0}}. \quad (4.97)$$

The speed expression assumes nonrelativistic motion. At threshold, the particle's kinetic energy falls toward zero as it recedes; it reaches arbitrarily large radii only over arbitrarily long times. Below threshold, radial outward motion reaches a finite turning radius. Above threshold, the asymptotic kinetic energy is the positive excess $K_i - k_e Qq/r_0$.

A negative *potential* is not by itself evidence of a bound particle. The conserved total energy and available motions determine escape. Even a negative interaction energy does not guarantee binding when initial kinetic energy is large. This distinction resolves a common misuse of the sign of U in charge-assembly problems.

4.11.6 When both particles recoil

If two isolated particles repel and both are free, the released interaction energy is shared. Let two charges initially be at rest at separation r_0 , with masses m_1, m_2 and positive product $q_1 q_2$. Neglect radiation and all other interactions. At very large separation,

$$K_1 + K_2 = \frac{k_e q_1 q_2}{r_0}. \quad (4.98)$$

Conservation of momentum gives equal and opposite momenta, so $m_1 v_1 = m_2 v_2$ in the nonrelativistic regime. Hence

$$\frac{K_1}{K_2} = \frac{m_2}{m_1}, \quad K_1 = \frac{m_2}{m_1 + m_2} \frac{k_e q_1 q_2}{r_0}, \quad K_2 = \frac{m_1}{m_1 + m_2} \frac{k_e q_1 q_2}{r_0}. \quad (4.99)$$

The lighter particle receives the larger kinetic energy even though their momentum magnitudes are equal. For equal masses, each receives half. As $m_2/m_1 \rightarrow \infty$, K_1 approaches the full released energy and source recoil becomes negligible, recovering the fixed-source approximation. This limit states precisely when treating the source as immobile is reasonable without a mechanical support.

4.12 Conductors at constant potential

A conductor in electrostatic equilibrium is an equipotential object. Inside the conducting material, $\mathbf{E} = 0$. Therefore for any two points in the conductor,

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\mathbf{l} = 0. \quad (4.100)$$

Figure 4.11 illustrates the surface condition. The surface is also an equipotential. If a tangential electric field existed on the surface, mobile charge would move along the surface, contradicting electrostatic equilibrium.

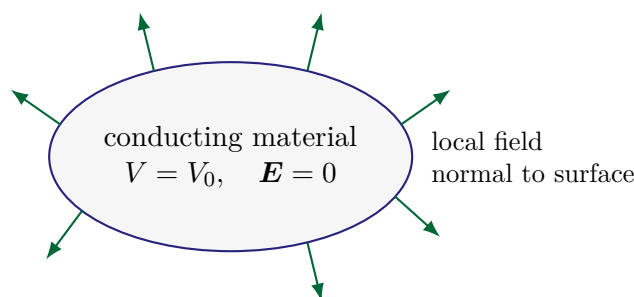


Figure 4.11: Surface field directions for a positively charged conductor in electrostatic equilibrium. The arrows use the normal to the illustrated ellipse; their lengths do not represent field magnitudes. No assumption of elliptical external equipotentials is made.

Key message

A conductor can have a complicated shape and still be at one potential in electrostatic equilibrium. The charge distribution adjusts until the tangential field vanishes.

4.12.1 A spherical conductor: zero field does not mean zero potential

For an isolated conducting sphere of radius R and total charge Q , with no nearby objects, spherical symmetry and Gauss's law give $E_r = k_e Q/r^2$ outside and zero inside the metal. Integrating inward from infinity gives

$$V(r) = \begin{cases} k_e Q/R, & 0 \leq r \leq R, \\ k_e Q/r, & r \geq R. \end{cases} \quad (4.101)$$

The interior value is fixed by continuity at the surface. It cannot be reset independently to zero without inserting an unphysical potential discontinuity. The field changes abruptly across the ideal surface charge layer, but the voltage across an infinitesimal thickness vanishes because the finite field is integrated over a vanishing distance.

A thin spherical shell with uniformly prescribed charge has the same potential function even if it is insulating: the result follows from its charge distribution and spherical symmetry. A solid insulating sphere with uniform volume charge has a different interior potential. In that case Gauss's law gives $E_r = k_e Q r/R^3$ for $r < R$. Starting from the known surface value,

$$V(r) - V(R) = - \int_R^r \frac{k_e Q r'}{R^3} dr' = \frac{k_e Q}{2R} \left(1 - \frac{r^2}{R^2} \right), \quad (4.102)$$

$$V(r) = \frac{k_e Q}{2R} \left(3 - \frac{r^2}{R^2} \right), \quad r \leq R. \quad (4.103)$$

Thus both fields vanish at the centre, but only the conductor has zero field throughout the interior. Figure 4.12 compares these models using the same $Q > 0$ and R .

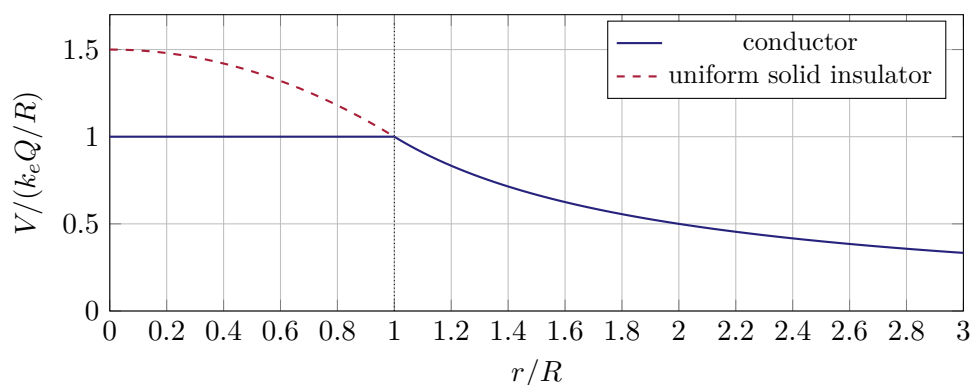


Figure 4.12: Calculated radial potentials for two spheres with the same radius and total charge. Both have $V(\infty) = 0$ and the same exterior potential. The conductor has constant interior potential; the uniform solid insulator has a quadratic interior potential.

Example 4.22: Potential and surface field of a conducting sphere

An isolated conducting sphere has $R = 5.0$ cm and $Q = 2.0$ nC. Find its potential, its surface field just outside, and the voltage change from the surface to $r = 20$ cm.

Spherical symmetry applies only because other objects are assumed remote. The results are

$$V_s = \frac{k_e Q}{R} = 360 \text{ V}, \quad (4.104)$$

$$E_s = \frac{k_e Q}{R^2} = 7.19 \times 10^3 \text{ V/m}, \quad (4.105)$$

$$\Delta V = k_e Q \left(\frac{1}{0.20 \text{ m}} - \frac{1}{0.050 \text{ m}} \right) = -270 \text{ V}. \quad (4.106)$$

A $+1.0$ pC probe moved outward has $\Delta U = -270$ pJ. Although the metal is at 360 V, its interior field is zero. The relation $E_s = V_s/R$ is a geometric result for this isolated sphere with this reference, not a universal relation between absolute potential and field.

4.12.2 Local normal field and surface charge

Immediately outside a conductor in vacuum, a thin Gaussian pillbox gives

$$\mathbf{E}_{\text{out}} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}, \quad -\frac{\partial V}{\partial n} \Big|_{\text{out}} = \frac{\sigma}{\epsilon_0}, \quad (4.107)$$

where $\hat{\mathbf{n}}$ points outward from the metal and σ is the signed surface charge density. The tangential field is zero. A negative σ gives an inward field. The derivative on the inside is zero, so the normal derivative can jump even though V is continuous.

Two points of one conductor can have the same potential and very different exterior field magnitudes. Their local surface charge densities then differ. A useful illustration is two widely separated conducting spheres joined by a thin wire. Neglecting their mutual potentials and the charge on the wire, equality of potentials requires $Q_1/R_1 = Q_2/R_2$. Their surface fields obey

$$\frac{E_1}{E_2} \simeq \frac{R_2}{R_1}. \quad (4.108)$$

The smaller sphere has the larger surface field. This supports intuition about field concentration at sharp features, but is not an exact local-curvature law for an arbitrary conductor. Nearby electrodes and the whole boundary geometry matter.

4.12.3 Cavities, shielding and disconnected conductors

An empty, fully enclosed cavity in a conductor at electrostatic equilibrium has constant potential and zero field if it contains no charge or internal electrode. A careful reason uses the uniqueness of the charge-free boundary-value problem: a constant potential satisfies the prescribed constant boundary value, and the solution is unique. The enrichment supplies the proof. Equal potentials at two endpoints alone would not prove that the field vanishes everywhere between them.

If a charge is placed inside the cavity, the metal is still an equipotential, but the cavity is no longer field free. Charges induced on the inner wall enforce the boundary condition. The electric field vanishes in the conducting material, not automatically in every empty region that lies geometrically inside the object. An aperture or a time-dependent field also changes the shielding problem and requires further analysis.

An isolated neutral conductor in an external field can have nonzero surface charge densities of both signs. Neutrality is a condition on the total charge, while equipotentiality is a condition on the potential throughout the metal. Neither should be substituted for the other.

4.12.4 A bridge to capacitance

For the isolated sphere above, $Q/V_s = 4\pi\epsilon_0 R$. Its value depends on geometry, not on the particular charge used in a linear electrostatic model. For a pair of conductors the useful comparison is between their potentials, and the next chapter defines capacitance from the charge associated with that voltage difference. The distinction between source charge, electrode voltage, local field and energy remains essential. Capacitance networks, dielectric polarization and capacitor charging are developed there rather than repeated here.

4.12.5 Potential in a cavity containing a charge

A spherically symmetric example makes the distinction between metal and cavity explicit. Place a point charge q at the centre of a neutral conducting spherical shell with inner radius a and outer radius b . Assume vacuum everywhere outside the metal, no other sources, and $V(\infty) = 0$. Gauss's law within the metal forces charge $-q$ on the inner surface. Neutrality then places $+q$ on the outer surface. Symmetry makes each surface charge distribution uniform.

The field is $k_e q/r^2$ in the cavity, zero for $a < r < b$, and $k_e q/r^2$ outside. Integrating from infinity gives the piecewise potential

$$V(r) = \begin{cases} k_e q \left(\frac{1}{r} - \frac{1}{a} + \frac{1}{b} \right), & 0 < r < a, \\ k_e q/b, & a \leq r \leq b, \\ k_e q/r, & r \geq b. \end{cases} \quad (4.109)$$

It is continuous at both surfaces. The cavity field is unchanged from that of the central point charge because a uniform spherical shell has zero interior field, but the shell adds a constant to the cavity potential. This constant matters when comparing a cavity point with infinity; it cancels between two cavity points.

Example 4.23: A neutral shell compared with a grounded shell

For the neutral shell just described, let $q = 1.0 \text{ nC}$, $a = 2.0 \text{ cm}$ and $b = 3.0 \text{ cm}$. Find the metal potential and the potential at $r = 1.0 \text{ cm}$. Then ground the shell while holding the central charge fixed.

Initially, the metal potential is $k_e q/b = 300 \text{ V}$. At the cavity point,

$$V = k_e q (100 - 50 + 33.33) \text{ m}^{-1} = 749 \text{ V}. \quad (4.110)$$

Worked example (continued)

After grounding, the inner surface still carries $-q$, as required by zero field in the metal. The outer surface now carries zero charge in this symmetric isolated environment. The potential becomes zero in the metal and outside, while the cavity potential is

$$V_{\text{grounded}}(r) = k_e q \left(\frac{1}{r} - \frac{1}{a} \right). \quad (4.111)$$

At $r = 1.0 \text{ cm}$ this is 450 V . Within the cavity the potential has shifted by -300 V while its field is unchanged. Outside, the field has changed from a Coulomb field to zero. Grounding has therefore made a physical change to the charge configuration; it is not a single constant relabelling of the potential everywhere in space.

This example also distinguishes specification by total charge from specification by voltage. For the neutral isolated shell, total shell charge is given and its common potential is determined. For the grounded shell, potential is given and its total charge becomes $-q$. Specifying both neutrality and zero potential would generally be inconsistent with the central charge and the infinity reference.

4.13 Laboratory and computational connection

4.13.1 What a voltmeter compares

An ideal voltmeter reports the potential at its positive input minus the potential at its reference input. If the red lead is connected at b and the reference lead at a , its reading is $V_b - V_a$. Reversing the leads reverses the sign. The instrument compares two points even when one is informally called “ground.” A reading at one terminal without specifying the other terminal is incomplete.

A real input draws a small current and has capacitance. Connecting it can change the charge distribution and the voltage being measured. In a resistive circuit a large input resistance reduces loading; on an isolated charged object input capacitance and charge redistribution also matter. The detailed circuit models belong to Chapters 6 and 7. Here the lesson is that a voltage map is measured under a stated configuration, rather than existing independently of all probes and connections.

4.13.2 Conductive paper as an analogue of a boundary-value problem

In an equipotential-mapping experiment, electrodes on a resistive sheet are connected to a low-voltage supply. A fixed reference lead stays at one electrode, and a movable probe locates points with the same reading. The mapped curves can then be used to estimate field directions and gradients. Figure 4.13 shows the measurement geometry.

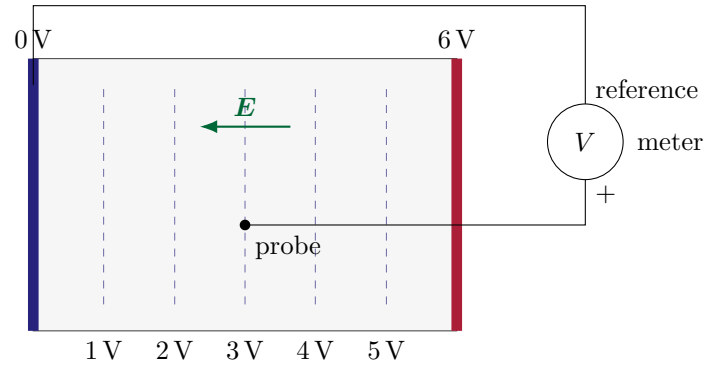


Figure 4.13: A low-voltage sheet-mapping arrangement. The central contours are drawn for an ideal uniform sheet with full-width parallel electrodes and insulating side edges. The meter reads 3 V at the shown probe position. The drawing specifies measurement connections, not a complete power-supply circuit.

The sheet is *not* a metal in electrostatic equilibrium: current flows through it. Its connection to vacuum electrostatics is mathematical. In a uniform isotropic sheet with steady current, $\mathbf{J} = \sigma_c \mathbf{E} = -\sigma_c \nabla V$ and charge conservation gives $\nabla \cdot \mathbf{J} = 0$. If the conductivity σ_c is constant, then

$$\nabla^2 V = 0. \quad (4.112)$$

This is the same equation obeyed by potential in a charge-free electrostatic region with matching boundary conditions. If conductivity or thickness varies, the sheet solves a different weighted equation. A planar sheet also models a two-dimensional field problem, not automatically the three-dimensional field of point electrodes in open space.

Practical analysis should record electrode positions, probe coordinates and readings, the reference connection, and the voltage increment between contours. Repeating readings near edges reveals where the parallel-plate approximation fails. Electrode contact resistance, sheet inhomogeneity, finite probe area and position uncertainty can all limit agreement with the ideal model. These effects should be separated from random reading scatter.

4.13.3 Estimating a field component and its uncertainty

For a smooth potential along a line, a central estimate is

$$E_x(x) \simeq -\frac{V(x+h) - V(x-h)}{2h}. \quad (4.113)$$

The two sample positions are separated by $2h$, not h . Taylor expansion gives the more precise statement

$$-\frac{V(x+h) - V(x-h)}{2h} = -V'(x) - \frac{h^2}{6} V'''(x) + \mathcal{O}(h^4). \quad (4.114)$$

Thus the discretization error is second order in spacing if the function is sufficiently smooth. This argument does not apply across an ideal surface-charge discontinuity in the field.

If the two voltage readings have independent standard uncertainties u_V , and h is known accurately, error propagation gives

$$u(E_x) = \frac{\sqrt{u_V^2 + u_V^2}}{2h} = \frac{u_V}{\sqrt{2}h}. \quad (4.115)$$

Reducing h decreases truncation error but amplifies the effect of reading noise. A common voltage-offset error cancels in the difference; a calibration scale error does not. If the two readings are correlated, their covariance must be included instead of assuming independent errors.

Example 4.24: A measured field with a stated uncertainty

An illustrative pair of readings is $V(x + h) = 4.80 \text{ V}$ and $V(x - h) = 5.20 \text{ V}$ at $h = 2.0 \text{ mm}$. Each has independent standard uncertainty 0.02 V . Estimate the field component and its reading uncertainty.

Equation (4.113) gives

$$E_x = -\frac{4.80 - 5.20}{0.0040} \text{ V/m} = +100 \text{ V/m}, \quad u(E_x) = 7.1 \text{ V/m}. \quad (4.116)$$

The result may be reported as $E_x = (100 \pm 7) \text{ V/m}$, where the uncertainty is one standard uncertainty from the readings alone. Position error and finite-spacing bias have not been included. Potential decreases as x increases, agreeing with the positive field component. Halving h without improving the voltmeter uncertainty would double this noise contribution.

4.13.4 Fitting a local potential instead of differencing one pair

Several samples can make a field estimate more robust. For approximately linear data $V_j = \alpha + \beta x_j + \epsilon_j$ with equal independent reading uncertainties, a least-squares slope is

$$\hat{\beta} = \frac{\sum_j (x_j - \bar{x})(V_j - \bar{V})}{\sum_j (x_j - \bar{x})^2}, \quad \hat{E}_x = -\hat{\beta}. \quad (4.117)$$

The intercept absorbs the arbitrary voltage reference. If the standard reading uncertainty is known to be u_V , the slope uncertainty is $u_V / [\sum_j (x_j - \bar{x})^2]^{1/2}$. A wider fitting interval improves noise averaging but risks bias if the true potential is curved. Plotting the fit residuals is therefore part of the physical interpretation, not merely statistical decoration.

For illustration, the synthetic readings below are generated from a linear field with small assigned deviations. They are not reported measurements.

$x \text{ (mm)}$	-4	-2	0	2	4
$V \text{ (V)}$	5.41	5.19	5.01	4.80	4.59

Here $\bar{x} = 0$, $\bar{V} = 5.00 \text{ V}$ and the fitted slope is -101.5 V/m . With independent 0.02 V uncertainties, $u(\hat{E}_x) = 3.2 \text{ V/m}$. This improves the reading uncertainty over a single central pair, provided curvature is negligible across the 8 mm interval. No averaging procedure can remove a systematic error shared by every reading.

4.13.5 A computational map with an analytic benchmark

For charges lying in the xy plane, evaluate

$$V(x, y) = k_e \sum_n \frac{q_n}{\sqrt{(x - x_n)^2 + (y - y_n)^2}}, \quad (4.118)$$

excluding grid points close to a source. This is a planar slice through a three-dimensional Coulomb potential. It is not the Green function of a purely two-dimensional electrostatic problem, which has logarithmic distance dependence.

At each regular grid point compute the central-difference field, then compare it with the analytic components

$$(E_x, E_y) = k_e \sum_n \frac{q_n (x - x_n, y - y_n)}{[(x - x_n)^2 + (y - y_n)^2]^{3/2}}. \quad (4.119)$$

Use equal spatial scales on both axes so apparent angles remain meaningful. Test the midpoint of an opposite-charge pair, a point far from a nonneutral collection, and invariance after adding a

constant to all potentials. Refining the grid should reduce derivative error away from singularities. A mask must cover every neighbour used by a difference stencil; differentiating through masked or missing values is not a physical calculation.

Replacing r by $\sqrt{r^2 + a^2}$ makes an attractive smooth picture, but changes the source model. It does not simply repair a point-charge singularity while preserving the same physics. The enrichment derives the charge density implicit in that replacement and supplies a tested numerical relaxation example for electrode boundary conditions.

4.14 Common mistakes and correction strategies

1. **Saying voltage flows.** Replace it by: charge flows, current flows, energy is transferred, or a potential difference exists.
2. **Confusing V with U .** Remember $U = qV$ only after a charge is specified.
3. **Ignoring the sign of q .** The sign of the moving charge is essential in $\Delta U = q\Delta V$.
4. **Treating potential as a vector.** Potential contributions add algebraically.
5. **Forgetting the reference.** A potential value requires a zero; a potential difference does not.
6. **Thinking $V = 0$ means $\mathbf{E} = 0$.** The field depends on spatial derivatives of V .
7. **Double counting pair energies.** Use $\sum_{i < j}$ or include the factor $1/2$ in $\sum_i q_i V_i$.
8. **Using electrostatic potential in induction problems.** Time-varying magnetic fields require a later, more general treatment.

Take-home message

Potential is a scalar map of energy per unit charge. The electric field is the negative gradient of that map. The electric force points toward decreasing $U = qV$. A particle with initial kinetic energy can move temporarily toward increasing U ; an electric-field line is generally not its trajectory.

4.15 Main-chapter summary and formula sheet

Summary

- Electric potential energy changes according to $\Delta U = -W_{\text{elec}}$.
- Electric potential difference is $\Delta V = \Delta U/q$.
- In electrostatics, $V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{l}$.
- Equipotential surfaces are perpendicular to electric field lines.
- The point-charge potential in vacuum is $V = k_e q/r$ with $V(\infty) = 0$.
- Potentials superpose algebraically.
- The electric field is recovered from $\mathbf{E} = -\nabla V$.
- The energy of point charges is $U = k_e \sum_{i < j} q_i q_j / r_{ij}$.
- A conductor in electrostatic equilibrium is an equipotential.

Formula summary

$$\Delta U = q\Delta V, \quad (4.120)$$

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\mathbf{l}, \quad (4.121)$$

$$V_{\text{point}}(r) = k_e \frac{q}{r} \quad [V(\infty) = 0], \quad (4.122)$$

$$V(\mathbf{r}) = k_e \sum_i \frac{q_i}{r_i}, \quad (4.123)$$

$$V(\mathbf{r}) = k_e \int \frac{dq}{R}, \quad (4.124)$$

$$\mathbf{E} = -\nabla V, \quad (4.125)$$

$$E_x = -\frac{\partial V}{\partial x}, \quad (4.126)$$

$$U = k_e \sum_{i < j} \frac{q_i q_j}{r_{ij}}, \quad (4.127)$$

$$\Delta K = -q\Delta V \quad \text{if only electrostatic work occurs.} \quad (4.128)$$

4.16 Practice problems

Use electrostatic conditions and vacuum unless stated otherwise. In numerical work use $k_e = 8.99 \times 10^9 \text{ N m}^2/\text{C}^2$, $e = 1.602 \times 10^{-19} \text{ C}$, $m_e = 9.11 \times 10^{-31} \text{ kg}$ and $m_p = 1.673 \times 10^{-27} \text{ kg}$. Set $V(\infty) = 0$ for finite isolated charge distributions. State the sign of every requested voltage difference. Selected detailed solutions follow the set.

Concepts and interpretation

- P1.** Distinguish electric potential, potential energy, electric field and force by giving a definition and unit for each. Explain which can be specified before a test charge is selected. Replace “voltage flows” with two physically meaningful statements relevant to a sensor.
- P2.** A point is at -30 V relative to a chosen reference. Find $U = qV$ for $q = +2.0 \mu\text{C}$ and $q = -2.0 \mu\text{C}$. Then shift the reference by adding 50 V everywhere. Which numbers change, and which measurable quantities remain unchanged?
- P3.** A uniform field is $450\hat{\mathbf{j}} \text{ V/m}$. Find ΔV for displacements $+0.20\hat{\mathbf{j}} \text{ m}$, $-0.20\hat{\mathbf{j}} \text{ m}$ and $+0.20\hat{\mathbf{i}} \text{ m}$. Can a curved path between the same endpoints change the answers?
- P4.** Construct one configuration with $V = 0$ but $\mathbf{E} \neq 0$ and one with $\mathbf{E} = 0$ but $V \neq 0$. State the reference. Does shifting the reference change either zero-field condition?
- P5.** Diagnose the statement “equipotential lines can never cross.” Distinguish different potential values from branches of the same level set. Use $V = A(x^2 - y^2)$ with $A \neq 0$ as a test case.
- P6.** A negative charge has velocity initially directed toward lower electric potential. Must its potential energy decrease? Must its speed increase? Explain why the force direction alone does not determine the velocity direction.
- P7.** Two isolated conductors are assigned values 0 V and 80 V . A student says both can be set to zero because the reference is arbitrary. Explain the error. Compare this relabelling with joining the conductors by a wire.

- P8.** A hollow conductor is at constant potential. State separately what can be concluded about the field in the metal, in an empty fully enclosed cavity, and in a cavity containing a point charge.

Charges, geometry and potential integrals

- P9.** A $+6.0\text{ nC}$ source is at the origin. Find V at $r = 0.50\text{ m}$ and the work done by the field when a -1.0 nC test charge moves radially from 0.50 to 1.0 m .
- P10.** Place $+Q$ at $x = 0$ and $-3Q$ at $x = d$, with $Q, d > 0$. Find every finite zero-potential point on the axis. In three dimensions, determine the centre and radius of the zero-potential surface.
- P11.** A square of side a has $+q, +q, -q, -q$ at its corners. Compare two arrangements: (i) alternating signs around the perimeter; (ii) the two positive charges on the upper side and the two negative charges on the lower side. Find V and $\mathbf{E}$ at the centre in each case, taking $+y$ upward.
- P12.** A uniform ring has radius $R = 0.060\text{ m}$ and charge $Q = 3.0\text{ nC}$. Find V at $z = 0$ and $z = 0.080\text{ m}$, and E_z at the latter point. Check the source-to-point distance before differentiating.
- P13.** Find the axial position $z > 0$ where the positive field of a uniformly charged positive ring is largest. Obtain that maximum in terms of $k_e Q/R^2$. Explain why the field is zero at both the centre and infinity.
- P14.** A thin rod extends from $x' = 0$ to L , with prescribed nonuniform density $\lambda(x') = \lambda_0 x'/L$. Find its total charge and its potential at $x = a > L$. Check the far-field limit while keeping the total charge fixed.
- P15.** Derive the potential of a uniform rod on its perpendicular bisector, then differentiate to obtain the perpendicular field. Check both the distant point-charge limit and the long-rod limit at fixed density.
- P16.** An insulating annulus has inner radius R_1 , outer radius R_2 and uniform σ . Find its axial potential and field by subtracting two disk results. Is the axial field continuous at the centre of the hole? Explain.
- P17.** A disk of radius R has uniform positive surface density. At what positive ratio z/R does its axial field equal half the infinite-sheet field? Compare this with the condition for a 5% deficit used in the worked example.
- P18.** An infinite line has $\lambda = 5.0\text{ nC/m}$. Find $V(0.10\text{ m}) - V(0.020\text{ m})$. Explain why this difference is meaningful although $V(\infty) = 0$ is unavailable.
- P19.** A dipole has separation d . On its positive axis, find the minimum r/d for which the far-field potential has less than 2% relative error with respect to the exact potential. Would the same relative-error criterion be useful on the perpendicular bisector?

Fields, energy and engineering models

- P20.** Let $V(x) = (5.0\text{ V/m}^3)x^3 - (2.0\text{ V/m})x$. Find E_x , its zeros, and the stability of each equilibrium for positive and negative charges constrained to the x axis. State why the word “constrained” matters.
- P21.** In three dimensions $V = A(x^2 - y^2) + Cz$, with $A = 8.0\text{ V/m}^2$ and $C = 3.0\text{ V/m}$. Find $\mathbf{E}$ at $(0.20, 0.10, 0)\text{ m}$. Is the origin a zero-field point? Explain why a plot restricted to $z = 0$ might mislead.

- P22.** A piecewise-linear potential has values $V(0) = 8\text{ V}$, $V(2\text{ mm}) = 4\text{ V}$, $V(5\text{ mm}) = 4\text{ V}$ and $V(7\text{ mm}) = -2\text{ V}$. Find the field in each open interval and the work on $+2.0\text{ nC}$ across the entire path. What idealized physical feature is indicated by a jump in the field?
- P23.** Three charges $+q, +q, -q$ occupy an equilateral triangle of side a . Find the interaction energy and the external work required to remove the negative charge to infinity while the positive charges stay fixed. Explain why the two results have different signs.
- P24.** Four equal positive charges occupy the corners of a square of side a . Count the pairs and find the energy. Then find the energy change if one corner charge is removed to infinity, with the others fixed.
- P25.** An electron starts from rest at $V_i = -20\text{ V}$ and reaches $V_f = +60\text{ V}$. Find its final energy in electronvolts and joules and its nonrelativistic speed. Check whether that approximation is justified.
- P26.** A proton starts from rest and moves through $V_f - V_i = -500\text{ V}$. Find its final speed. Repeat the energy calculation, in electronvolts only, for an ion of charge $+3e$ across the same drop.
- P27.** A negative bead has $q = -4.0\text{ nC}$, $m = 2.0\text{ mg}$ and initial speed 0.80 m/s . It moves along a horizontal track where $V(x) = V_i - (2.0 \times 10^3\text{ V/m})x$. Find its turning position. Assume the track is frictionless and exerts no tangential force.
- P28.** A $+5.0\text{ nC}$ bead of mass 2.0 mg is to be held at rest against gravity in a uniform vertical field. Find the field magnitude and direction, and the required potential difference between an upper and lower plate separated by 3.0 cm .
- P29.** Two widely separated spheres of radii 1.0 cm and 3.0 cm share total charge 8.0 nC after connection by a thin wire. Neglect the wire's charge and mutual-potential corrections. Find the charge on each, their common potential and their surface-field ratio. State the approximation that fails when the spheres approach each other.
- P30.** For a uniformly charged solid insulating sphere, derive the centre potential using the surface value and an interior field integral. Find the work required to move a small positive test charge from the centre to the surface while the prescribed source density stays fixed.
- P31.** An ideal vacuum gap has electrodes at 0 and 500 V separated by 5.0 mm . Find the average field component along an axis from the 0 to the 500 V electrode. Find the electrical work on an electron crossing in that direction. Explain why a silicon carrier crossing the same voltage need not emerge with that kinetic-energy increase.
- P32.** Compare fields for 12 V across 3.0 mm and 1.2 V across 6.0 nm in uniform-gap models. Can these calculations alone determine breakdown or electrical safety? State the physical information missing from such a conclusion.

Measurement and computation

- P33.** The readings at $x \pm h$ are 3.20 V and 2.80 V , respectively, with $h = 1.0\text{ mm}$. Each has independent standard uncertainty 0.010 V . Estimate E_x and its reading uncertainty. How does a common additive calibration offset affect the result?
- P34.** Reproduce the least-squares field estimate from the five synthetic readings in Section 4.13. Inspect the residuals. Explain why a straight-line fit over a wider region can give a biased local field even when its statistical uncertainty decreases.

- P35.** For $V(x) = k_e Q/x$, $x > h > 0$, derive the central-difference field exactly and compare it with $k_e Q/x^2$. Find the relative error for $h/x = 0.10$ and 0.05 .
- P36.** Compute the potential of a dipole on a square grid, masking disks around both charges. Compare central-difference and analytic fields only where every stencil point is unmasked. Report how the maximum absolute error changes when the spacing is halved over a fixed observation region away from the sources.
- P37.** Explain why a steady-current conductive sheet of uniform conductivity can reproduce a charge-free potential problem. What changes if the sheet thickness is nonuniform? Which boundary condition describes an insulating sheet edge?
- P38.** Numerically integrate the potential of the nonuniform rod in **PP14.** with the midpoint rule. Compare with the analytic answer for $a = 2L$ at $N = 10, 20, 40, 80$ subdivisions. Keep the source density and observation point fixed while refining the discretization.
- P39.** A measured map is represented locally by $V = V_0 + ax + by$. Describe the minimum number and arrangement of noiseless measurement points needed to recover a and b . Why do three collinear points fail? Explain what extra measurements accomplish with noisy data.
- P40.** A potential map is shifted by a constant, rescaled by a positive factor, or reflected as $V(x, y) \rightarrow V(-x, y)$. Predict the effect of each operation on field components, contour shapes and the work for fixed charges and endpoints. Distinguish a reference change from a physical change of the voltage scale.

4.17 Selected detailed solutions

The problem numbers below refer to the preceding set. These solutions emphasize modelling, not only numerical answers.

PP3.: projected displacement in a uniform field

Equation (4.16) gives $\Delta V = -450\Delta y$ with units carried as (V/m)m. Thus the three voltage changes are -90 V, $+90$ V and zero. A displacement along x is tangent to an equipotential plane. An arbitrary curve between the same endpoints has the same net Δy and therefore the same integral. Multiplying field magnitude by the curved path length would lose the projection factor and would incorrectly predict path dependence.

PP10.: the zero-potential sphere

The distance to $-3Q$ must be three times the distance to $+Q$. In Cartesian coordinates this gives

$$(x - d)^2 + y^2 + z^2 = 9(x^2 + y^2 + z^2). \quad (4.129)$$

Rearranging and completing the square yields

$$\left(x + \frac{d}{8}\right)^2 + y^2 + z^2 = \left(\frac{3d}{8}\right)^2. \quad (4.130)$$

The centre is $(-d/8, 0, 0)$ and the radius $3d/8$. The axial intersections are $x = -d/2$ and $x = d/4$. Neither source lies on the surface. Substitution into the original unsquared equation checks that no false solutions were introduced: the distances are positive and in the required ratio. The surface is a zero of V with the specified infinity reference, not a locus of zero force.

PP11.: same potential, different fields

Each corner is at $r = a/\sqrt{2}$, so in either arrangement the algebraic charge sum is zero and $V = 0$. In the alternating arrangement, each charge has an equal charge diagonally opposite it. Their fields at the centre cancel pairwise, giving $\mathbf{E} = 0$.

In the second arrangement, the horizontal components cancel by left–right symmetry. All vertical components point downward: away from the upper positive charges and toward the lower negative charges. Each has magnitude

$$E_y^{\text{magnitude}} = \frac{k_e q}{a^2/2} \frac{1}{\sqrt{2}} = \frac{\sqrt{2}k_e q}{a^2}. \quad (4.131)$$

Adding four contributions gives $\mathbf{E} = -4\sqrt{2}k_e q \hat{\mathbf{j}}/a^2$. This comparison isolates the difference between scalar addition and vector addition without changing the total charge or any source-to-centre distance.

PP14.: potential of a nonuniform rod

The total charge is

$$Q = \frac{\lambda_0}{L} \int_0^L x' dx' = \frac{\lambda_0 L}{2}. \quad (4.132)$$

Since $a > L$, the source-to-point distance is $a - x' > 0$. Decompose $x'/(a - x') = a/(a - x') - 1$. Then

$$V(a) = \frac{k_e \lambda_0}{L} \int_0^L \frac{x'}{a - x'} dx' \quad (4.133)$$

$$= \frac{k_e \lambda_0}{L} \left[a \ln \left(\frac{a}{a - L} \right) - L \right]. \quad (4.134)$$

The bracket has units of length, so the result has units $k_e \lambda_0 = \text{V}$. For $L/a \ll 1$, $\ln[a/(a - L)] = L/a + L^2/(2a^2) + \dots$. The first term cancels the $-L$, leaving $V \simeq k_e \lambda_0 L/(2a) = k_e Q/a$. The cancellation also warns against evaluating the unrearranged expression carelessly at extremely large a/L with limited numerical precision.

PP16.: the annulus and its hole

Subtracting an inner disk from an outer disk cancels the $|z|$ terms:

$$V(z) = \frac{\sigma}{2\varepsilon_0} \left(\sqrt{R_2^2 + z^2} - \sqrt{R_1^2 + z^2} \right). \quad (4.135)$$

For $R_1 > 0$ this is differentiable at the origin. Its axial field is

$$E_z = \frac{\sigma z}{2\varepsilon_0} \left(\frac{1}{\sqrt{R_1^2 + z^2}} - \frac{1}{\sqrt{R_2^2 + z^2}} \right). \quad (4.136)$$

The field approaches zero continuously from either side of the hole, because there is no surface charge at the axial centre. For $z > 0$ and $\sigma > 0$, it is positive. At large $|z|$, the potential approaches that of the charge $Q = \pi\sigma(R_2^2 - R_1^2)$. If R_1 is taken to zero first, the hole disappears and the disk's one-sided surface-field limits are recovered. Limits at a singular idealized surface must be handled in their stated order.

PP20.: equilibrium and charge sign

Differentiation with dimensional coefficients gives

$$E_x = (2.0 \text{ V/m}) - (15.0 \text{ V/m}^3)x^2. \quad (4.137)$$

The two zeros are $x_{\pm} = \pm\sqrt{2/15} \text{ m} \simeq \pm 0.365 \text{ m}$. The curvature is $V'' = (30.0 \text{ V/m}^3)x$, positive at x_+ and negative at x_- . For positive q , $U = qV$ has a local minimum at x_+ and a local maximum at x_- . For negative q the classifications reverse. These statements concern the constrained coordinate. The potential along one axis contains no information about stability against transverse displacement.

PP22.: field from a piecewise-linear graph

The slope in the first interval is $-4 \text{ V/(2 mm)} = -2000 \text{ V/m}$, so $E_x = +2000 \text{ V/m}$. The middle interval has zero slope and zero field component. The last has slope $-6 \text{ V/(2 mm)} = -3000 \text{ V/m}$, giving $E_x = +3000 \text{ V/m}$.

The total potential change is -10 V , independent of this subdivision. Thus the field does $W_{\text{elec}} = -q\Delta V = +20 \text{ nJ}$ of work. The corners of the potential graph correspond to discontinuities in the field. In a plane-symmetric idealization they represent surface charge sheets, with $\sigma = \epsilon_0(E_{x,+} - E_{x,-})$. A jump in slope is consistent with continuous potential; a finite jump in potential would require a different singular model.

PP23.: assembly and removal are different changes

The initial configuration contains one positive pair contribution and two negative ones:

$$U_i = \frac{k_e}{a}(q^2 - q^2 - q^2) = -\frac{k_e q^2}{a}. \quad (4.138)$$

After the negative charge is removed to infinity, only the repulsive pair remains, so $U_f = +k_e q^2/a$. The change is $\Delta U = 2k_e q^2/a > 0$. An external agent moving quasistatically must supply this positive work. Equivalently, the potential of the two fixed positive charges at the negative charge's initial location is $2k_e q/a$ and at infinity is zero. Then $(-q)[0 - 2k_e q/a]$ gives the same result. Using the initial total energy alone as the removal work would omit the final energy of the remaining pair.

PP25.: electron energy and speed

Here $\Delta V = 80 \text{ V}$ and $q = -e$. Therefore

$$K_f = -q\Delta V = 80 \text{ eV} = 1.2816 \times 10^{-17} \text{ J}. \quad (4.139)$$

Using $v = \sqrt{2K_f/m_e}$ gives $v = 5.30 \times 10^6 \text{ m/s}$. The ratio $K_f/(m_e c^2) \simeq 1.57 \times 10^{-4}$ is small, so the nonrelativistic speed is adequate at the stated precision. The electron accelerates toward increasing potential, where its potential energy is lower. The initial value -20 V alone determines neither the acceleration nor the gain.

PP27.: locating the turning point

The initial kinetic energy is $K_i = \frac{1}{2}mv_i^2 = 6.4 \times 10^{-7} \text{ J}$. The potential-energy increase along the track is

$$q[V(x) - V_i] = (-4.0 \times 10^{-9} \text{ C})(-2.0 \times 10^3 \text{ V/m})x = (8.0 \times 10^{-6} \text{ J/m})x. \quad (4.140)$$

Set this equal to K_i to obtain $x_{\text{turn}} = 0.080 \text{ m}$. A force check gives $E_x = +2000 \text{ V/m}$, $F_x = -8.0 \mu\text{N}$ and acceleration -4.0 m/s^2 . The kinematic stopping distance $v_i^2/(2|a|) = 0.080 \text{ m}$ agrees. The electric force is opposite the initial velocity, so the bead initially moves uphill in potential energy while slowing down.

PP29.: connected spheres

Within the stated approximation, equal surface potentials require $Q_1/R_1 = Q_2/R_2$, so $Q_2 = 3Q_1$. Charge conservation gives $Q_1 = 2.0 \text{ nC}$ and $Q_2 = 6.0 \text{ nC}$. The common potential is

$$V_s \simeq \frac{k_e Q_1}{R_1} = 1.80 \times 10^3 \text{ V}. \quad (4.141)$$

The surface fields are $E_1 \simeq 1.80 \times 10^5 \text{ V/m}$ and $E_2 \simeq 5.99 \times 10^4 \text{ V/m}$, giving $E_1/E_2 = 3$. The smaller sphere carries less total charge but has higher surface charge density and field. At smaller separations, each sphere contributes significantly to the potential of the other and polarizes its charge distribution. The isolated-sphere approximation must then be replaced by a joint boundary-value calculation.

PP33.: uncertainty in a voltage derivative

The positive-side reading is 3.20 V and the negative-side reading is 2.80 V . Thus

$$E_x = -\frac{0.40 \text{ V}}{2.0 \times 10^{-3} \text{ m}} = -200 \text{ V/m}, \quad u(E_x) = \frac{0.010 \text{ V}}{\sqrt{2}(10^{-3} \text{ m})} = 7.1 \text{ V/m}. \quad (4.142)$$

The field points toward negative x , because potential increases with x . A common additive offset cancels exactly in the difference. This cancellation is why treating a fully correlated offset as two independent uncertainties would overstate its effect. A multiplicative calibration error, finite-spacing bias and position uncertainty require separate treatment.

PP35.: exact central-difference error

Substitute the Coulomb potential directly:

$$E_x^{(h)} = -\frac{k_e Q}{2h} \left(\frac{1}{x+h} - \frac{1}{x-h} \right) = \frac{k_e Q}{x^2 - h^2}, \quad (4.143)$$

$$\frac{E_x^{(h)} - E_x}{E_x} = \frac{h^2}{x^2 - h^2}. \quad (4.144)$$

The errors are 1.0101% and 0.2506% for the two specified spacings. Their ratio is close to four, consistent with second-order convergence. As h approaches x , one sample approaches the charge and the error diverges. Refinement tests must therefore keep the observation point and its distance from the singularity fixed. Near a true zero field, use an absolute-error measure instead of dividing by zero.

4.18 Advanced Topics for Technical and Engineering Students

The core develops voltage from electrical work and then obtains potentials for prescribed sources. The enrichment asks a complementary question: how can the field be determined when the electrode voltages are known but the surface charges are not? This is the natural formulation for many instruments, sensors and field-shaping structures. The extra mathematics consists of partial derivatives, a volume integral identity, local series expansions and finite differences. Each is introduced where it enters.

Pedagogical boundary

The following sections extend Chapter 4; they do not constitute Chapter 5. The core outcomes do not require solving partial differential equations. Capacitance networks and dielectric response are reserved for the next chapter, while induction and relativistic dynamics are deferred to their later chapters. Vacuum electrostatics remains the default physical model here.

4.19 Poisson's equation and electrostatic boundary data

4.19.1 From flux to local curvature

Gauss's law relates electric flux out of a closed surface to its enclosed charge. Applied to an infinitesimal volume, it states

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \quad (4.145)$$

The divergence $\nabla \cdot \mathbf{E} = \partial E_x / \partial x + \partial E_y / \partial y + \partial E_z / \partial z$ measures local outward flux per unit volume. Substituting $\mathbf{E} = -\nabla V$ yields

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0}. \quad (4.146)$$

This is Poisson's equation. Where $\rho = 0$, it becomes Laplace's equation, $\nabla^2 V = 0$. A charge-free region can still contain a strong field produced by charges outside it. Laplace's equation constrains the *sum* of the directional curvatures, not each curvature separately.

Dimensionally, $\nabla^2 V$ has units V/m^2 . Since ε_0 has units $\text{C}/(\text{V m})$, the ratio ρ/ε_0 has the same units. The minus sign means that positive local charge gives a negative sum of curvatures. This sign is a useful debugging check when implementing a numerical solver.

4.19.2 Which boundary conditions are specified?

A differential equation alone has many solutions. Even in one dimension, $V'' = 0$ allows $V = A + Bx$ with arbitrary constants. Specifying two electrode values fixes those constants. In higher dimensions one needs compatible information along the boundary:

- Prescribed potential, called a Dirichlet condition, models an electrode maintained at a stated voltage.
- Prescribed normal derivative, called a Neumann condition, specifies normal field or flux. A zero normal derivative models an insulating boundary in the uniform conductive-sheet analogue or a symmetry plane with zero normal field.
- A floating conductor has an unknown but spatially constant potential. Its total charge provides an additional constraint. Specifying both its potential and an unrelated total charge generally overconstrains the problem.

A purely Neumann problem determines potential only up to a constant and requires a compatibility condition: the prescribed flux must agree with the total enclosed charge. A finite numerical outer boundary also needs a physical interpretation. Grounding it at a convenient distance can noticeably distort the solution for a device that is actually in open space.

4.19.3 Uniqueness explains why boundary methods work

Suppose V_1 and V_2 solve Poisson's equation with the same charge density in a connected volume Ω and the same potentials on its entire boundary. Their difference $w = V_1 - V_2$ satisfies $\nabla^2 w = 0$ and $w = 0$ on the boundary. The product rule gives

$$\nabla \cdot (w \nabla w) = |\nabla w|^2 + w \nabla^2 w. \quad (4.147)$$

Integrate over the volume and turn the divergence into a boundary flux:

$$\int_{\Omega} |\nabla w|^2 d\tau = \oint_{\partial\Omega} w \frac{\partial w}{\partial n} dA = 0. \quad (4.148)$$

The integrand is nonnegative, so $\nabla w = 0$ everywhere in a regular solution. Thus w is constant; its zero boundary value makes that constant zero. The solution is unique. The same proof applies to an exterior problem if the boundary condition at infinity makes the remote surface term vanish.

This result has practical force. If an analytic construction satisfies the equation and all boundary conditions, it is the physical solution; one need not independently reconstruct every source charge first. It also justifies the empty-cavity argument: constant boundary potential and no interior charge have the unique constant solution. A nonzero voltage difference is not required to specify a field-free region, but the entire boundary condition is.

Example 4.25: A uniform charge density between planar electrodes

An ideal laterally infinite vacuum slab occupies $0 < x < d$, has prescribed uniform volume charge density ρ_0 , and is bounded by electrodes at $V(0) = 0$ and $V(d) = V_b$. Find $V(x)$ and $E_x(x)$.

With no lateral variation, Poisson's equation is $V'' = -\rho_0/\varepsilon_0$. Two integrations give $V = -\rho_0 x^2/(2\varepsilon_0) + Ax + B$. The first boundary fixes $B = 0$ and the second fixes $A = V_b/d + \rho_0 d/(2\varepsilon_0)$. Therefore

$$V(x) = \frac{V_b x}{d} + \frac{\rho_0}{2\varepsilon_0} x(d - x), \quad (4.149)$$

$$E_x(x) = -\frac{V_b}{d} + \frac{\rho_0}{\varepsilon_0} \left(x - \frac{d}{2} \right). \quad (4.150)$$

The voltage sets the average field $-V_b/d$, while charge density gives a linear field variation. Integrating E_x from 0 to d returns $-V_b$; differentiating gives ρ_0/ε_0 , as Gauss's law requires. For the illustrative values $d = 1.0$ cm, $V_b = 100$ V and $\rho_0 = 1.0 \times 10^{-5}$ C/m³, the endpoint fields are approximately -1.565×10^4 and -4.35×10^3 V/m. Their average remains -1.00×10^4 V/m. The midpoint potential is about 64.1 V, rather than the charge-free value 50.0 V. These are prescribed-source model values; whether a real medium can maintain that charge distribution is a separate physical question.

4.19.4 Bias voltage is not a complete field specification

The slab example shows why a detector bias divided by thickness can be a useful average yet fail as a local field model. In a depleted semiconductor, fixed ionized dopants and dielectric permittivity modify Poisson's equation; mobile carriers and contacts add further constraints. With

a homogeneous permittivity the same one-dimensional integration can be adapted by replacing ε_0 with the appropriate material permittivity and specifying a justified charge density. This is a model extension, not a claim that every detector has uniform charge or a uniform field.

For positive ρ_0 and positive V_b , keeping E_x negative throughout the ideal slab requires $V_b > \rho_0 d^2 / (2\varepsilon_0)$. Equality makes the field vanish at the right boundary; a smaller voltage allows an interior field reversal. This criterion is a consequence of the assumed static model and helps distinguish the work set by terminal voltage from the transport affected by local field minima.

4.20 Saddles, stability and the limits of static trapping

4.20.1 Two like charges provide a concrete saddle

Place two equal positive charges at $(\pm a, 0, 0)$. Symmetry makes the field zero at the origin, but the potential there is $2k_e Q/a$. Expand the exact potential for $|x|, |y|, |z| \ll a$:

$$V(x, y, z) = \frac{2k_e Q}{a} + \frac{k_e Q}{a^3}(2x^2 - y^2 - z^2) + \mathcal{O}(r^4/a^5) k_e Q. \quad (4.151)$$

Along the charge axis the origin is a local minimum of V ; in either transverse direction it is a local maximum. For a positive test charge,

$$F_x \simeq -\frac{4k_e Qq}{a^3}x, \quad F_y \simeq +\frac{2k_e Qq}{a^3}y, \quad F_z \simeq +\frac{2k_e Qq}{a^3}z. \quad (4.152)$$

The axial force is restoring and the transverse forces destabilizing. A negative charge reverses all three signs, exchanging which directions are stable but never making every direction stable.

Figure 4.14 shows two exact cuts, not a guessed landscape. A one-dimensional scan along the charge axis would suggest stability for a positive test charge, but the transverse cut disproves unconstrained trapping.

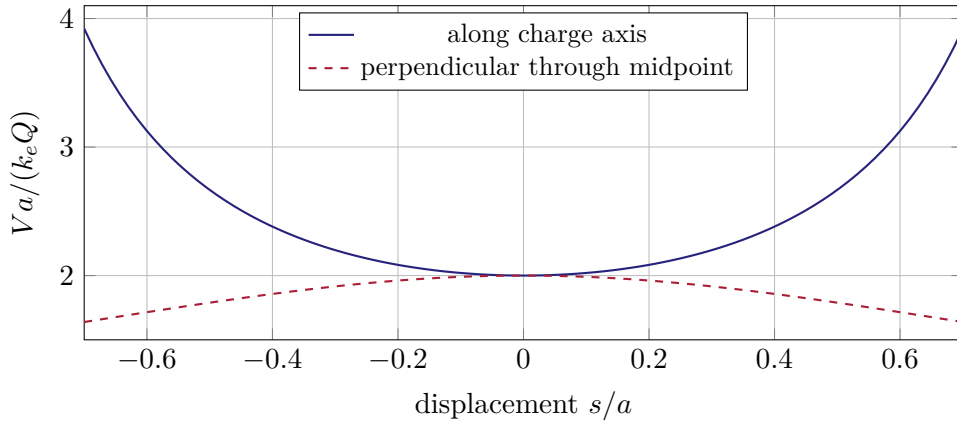


Figure 4.14: Exact potential cuts for two equal positive charges at $x = \pm a$. The origin is a minimum along x and a maximum along a transverse line: a saddle of the full potential.

4.20.2 Why a charge-free harmonic potential has no strict interior minimum

At a smooth local minimum, the curvatures in all principal directions are nonnegative. Laplace's equation forces their sum to vanish. This already rules out a quadratic minimum with a positive curvature in every direction. The stronger statement, including flat higher-order possibilities, follows from the mean-value property: the value of a harmonic potential at the centre of any sufficiently small sphere equals its average over that sphere. A strict local minimum would make all surrounding

values larger and contradict that average. A harmonic function attaining an interior maximum or minimum must be constant on the connected region under the usual regularity assumptions.

Consequently, a fixed nonzero point charge cannot have a stable isolated equilibrium in a charge-free region using only static external electric forces. For a negative charge, stability would require a local maximum of V and encounters the same obstruction. This is the relevant form of Earnshaw's theorem. It does not forbid constraints, time-dependent fields, magnetic forces, feedback, or other forces. Nor does it assert that every possible form of matter must behave as a fixed point charge.

For a uniformly charged ring, the near-centre axial curvature in Eq. (4.53) is negative for $Q > 0$. Rotational symmetry gives equal transverse curvatures; Laplace's equation fixes each to $+k_e Q/(2R^3)$. The full quadratic potential is therefore

$$V \simeq \frac{k_e Q}{R} + \frac{k_e Q}{4R^3}(x^2 + y^2) - \frac{k_e Q}{2R^3}z^2. \quad (4.153)$$

A negative charge is stable along the ring axis only if constrained there. The transverse instability is hidden by the axial formula alone.

Example 4.26: A constrained oscillation near the centre of a ring

A bead with negative charge q and mass m is constrained to the axis of a fixed positive uniform ring. For $|z| \ll R$, find its angular frequency of small oscillations. Ignore gravity and friction. The axial field is $E_z \simeq k_e Qz/R^3$, so

$$m\ddot{z} = qE_z = -\frac{k_e Q|q|}{R^3}z, \quad \omega = \sqrt{\frac{k_e Q|q|}{mR^3}}. \quad (4.154)$$

The coefficient inside the square root has units s^{-2} . For $Q = 5.0 \text{ nC}$, $|q| = 1.0 \text{ nC}$, $R = 0.050 \text{ m}$ and $m = 1.0 \text{ mg}$, $\omega \simeq 19.0 \text{ s}^{-1}$ and the period is 0.331 s . These values assume the bead is a nonperturbing test charge, the support supplies any transverse force, and its displacement remains small. Removing the constraint invalidates the stable-trap interpretation, even though the axial calculation remains correct.

4.21 Multipoles and distant-source approximations

4.21.1 How a source reveals its structure with distance

For a localized charge distribution of size a_s observed at $r \gg a_s$, expand the inverse distance in the source coordinate $\mathbf{r}'$:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r^2} + \frac{3(\hat{\mathbf{r}} \cdot \mathbf{r}')^2 - r'^2}{2r^3} + \dots \quad (4.155)$$

This follows by factoring r from the square root and expanding in r'/r . Substituting into the source integral gives

$$V(\mathbf{r}) = k_e \left[\frac{Q}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{2r^3} \int \rho(\mathbf{r}') [3(\hat{\mathbf{r}} \cdot \mathbf{r}')^2 - r'^2] d\tau' + \dots \right], \quad (4.156)$$

where $Q = \int \rho d\tau'$ and $\mathbf{p} = \int \rho \mathbf{r}' d\tau'$. The three displayed terms are monopole, dipole and quadrupole contributions. Their dimensions are all volts. Differentiation adds an inverse power of distance to the corresponding field scales.

If the total charge is nonzero, the leading potential is generally monopolar. If $Q = 0$ but $\mathbf{p} \neq 0$, the dipole dominates. If both vanish, a quadrupole or higher term can lead. Being neutral does not mean being electrostatically invisible. Conversely, the leading term may vanish along particular observation directions, so a general distance scaling must be interpreted with its angular dependence.

4.21.2 The origin matters unless the net charge vanishes

Move the coordinate origin by $\mathbf{a}$, so the source coordinates become $\mathbf{r}'' = \mathbf{r}' - \mathbf{a}$. The new dipole moment is

$$\mathbf{p}' = \int \rho(\mathbf{r}')(\mathbf{r}' - \mathbf{a}) d\tau' = \mathbf{p} - Q\mathbf{a}. \quad (4.157)$$

For a neutral source, the dipole moment is origin independent. For a nonneutral source, the individual multipole coefficients depend on origin, though the exact potential does not. Changing the expansion origin changes both the source moments and the observation coordinates. Mixing coefficients from one origin with distances from another creates an inconsistent approximation.

Example 4.27: A neutral source whose dipole also vanishes

Place charges $+q$ at $z = +a$ and $z = -a$, and $-2q$ at the origin. Find the leading distant potential and verify it on the positive axis.

The total charge and dipole moment both vanish. With θ measured from the z axis, each outer charge contributes $qa^2(3\cos^2\theta - 1)$ to the quadrupole integral. Thus

$$V(r, \theta) \simeq \frac{k_e qa^2}{r^3} (3\cos^2\theta - 1). \quad (4.158)$$

For $z > a$, the exact axial expression is

$$V(z) = k_e q \left(\frac{1}{z-a} + \frac{1}{z+a} - \frac{2}{z} \right) = \frac{2k_e qa^2}{z(z^2 - a^2)}. \quad (4.159)$$

Expanding at large z gives $2k_e qa^2/z^3$, as the angular formula predicts for $\theta = 0$. In the equatorial plane the leading potential is negative. Its nodal cones have $\cos^2\theta = 1/3$ at leading order. The field falls as $1/r^4$ away from directions with additional cancellation. This example turns the multipole hierarchy into a concrete symmetry test for numerical calculations.

4.22 A grounded plane and the method of images

4.22.1 Constructing the potential without guessing surface charge

A point charge q is held at $(0, 0, a)$ above an infinite grounded conducting plane $z = 0$. The physical vacuum region is $z > 0$. Replace the conductor, mathematically, by an image charge $-q$ at $(0, 0, -a)$. In the upper half-space define

$$V(x, y, z) = k_e q \left[\frac{1}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+a)^2}} \right]. \quad (4.160)$$

The two distances are equal on $z = 0$, so $V = 0$ there. The expression has the correct real-source singularity, solves Laplace's equation elsewhere in $z > 0$, and tends to zero far away. The uniqueness theorem therefore identifies it as the physical potential in that region.

The image charge is not a particle hidden inside the conductor. Equation (4.160) must not be used as the physical potential in the lower half-space; the grounded metal is at zero potential. Figure 4.15 shows the domain distinction.

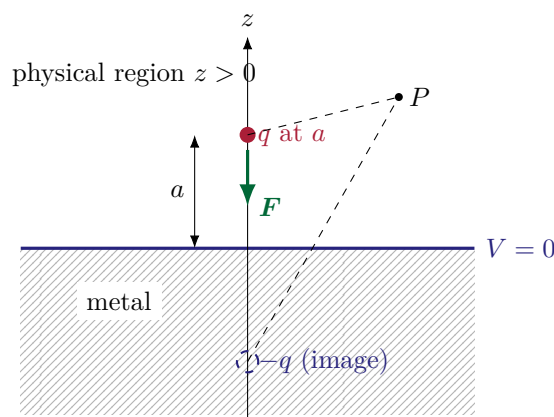


Figure 4.15: Image construction for a point charge above an infinite grounded plane. The image reproduces the upper-half-space field; it is not a real source in the metal. The force on the real charge points toward the plane for either sign of q .

4.22.2 Induced surface charge and attractive force

Let $s = \sqrt{x^2 + y^2}$ be the radial distance in the plane. Differentiating the potential just above it gives

$$E_z(s, 0^+) = -\frac{2k_e q a}{(s^2 + a^2)^{3/2}}, \quad \sigma(s) = -\frac{q a}{2\pi(s^2 + a^2)^{3/2}}. \quad (4.161)$$

The second relation uses $\sigma = \epsilon_0 E_z$ with outward normal $+\hat{z}$. Integrating over the plane provides a nontrivial normalization check:

$$Q_{\text{ind}} = 2\pi \int_0^\infty \sigma(s) s \, ds = -q. \quad (4.162)$$

The opposite total charge is drawn from the grounding reservoir. The result depends on the infinite grounded-plane model and should not be transferred unchanged to a finite isolated metal plate.

The force on the real charge is the force from the induced charges, represented by the image field at distance $2a$:

$$\mathbf{F} = -\frac{k_e q^2}{4a^2} \hat{z}. \quad (4.163)$$

The square of q makes the force attractive for both charge signs. The source's own field must be excluded when evaluating its force.

4.22.3 Why the interaction energy is not the real–image pair energy

When the real charge moves, the induced surface charge distribution changes. The image position follows it, so treating the image as a second fixed real source miscounts work. With zero interaction energy at $a \rightarrow \infty$, integrate the actual force:

$$U(a) = -\int_\infty^a F_z(a') \, da' = -\frac{k_e q^2}{4a}. \quad (4.164)$$

The induced potential evaluated at the charge is $V_{\text{ind}} = -k_e q/(2a)$, so $U = \frac{1}{2} q V_{\text{ind}}$. The naive real–image pair energy $-k_e q^2/(2a)$ is twice too negative. A force check, $-dU/da = -k_e q^2/(4a^2)$, confirms the correct expression. This is a special case where the source response must be included explicitly; qV remains correct for a genuinely fixed external potential.

Example 4.28: Work to lift a charge away from a grounded plane

A charge $q = 1.0 \text{ nC}$ is moved quasistatically from $a_1 = 1.0 \text{ cm}$ to $a_2 = 3.0 \text{ cm}$ above the ideal plane. Calculate the required external work and the initial force magnitude.

$$W_{\text{app}} = U(a_2) - U(a_1) = \frac{k_e q^2}{4} \left(\frac{1}{a_1} - \frac{1}{a_2} \right) = 1.50 \times 10^{-7} \text{ J}, \quad (4.165)$$

$$|F(a_1)| = \frac{k_e q^2}{4a_1^2} = 2.25 \times 10^{-5} \text{ N}. \quad (4.166)$$

The work is positive because the charge is lifted against attraction. It is smaller than $|F(a_1)|(a_2 - a_1) = 4.50 \times 10^{-7} \text{ J}$ because the force decreases strongly during the lift. This comparison independently checks the sign and scale. Gravity and any support energy are excluded from this electrical calculation.

4.23 Energy stored in the electrostatic field

4.23.1 From assembly work to a spatial energy density

For a smooth localized charge distribution, imagine building the charge density by scaling it from zero to its final value. At fraction η of the final charge, linearity makes the potential ηV . Adding $d\eta \rho d\tau$ requires work $(\eta V)(d\eta \rho d\tau)$. Integrating η from zero to one gives

$$U = \frac{1}{2} \int \rho V d\tau. \quad (4.167)$$

The factor one half again expresses progressive assembly, not a reduction of the force on any individual charge. The derivation assumes the final geometry and the shape of the density distribution are fixed while their amplitudes scale.

Using $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$ and the identity

$$\nabla \cdot (V \mathbf{E}) = \nabla V \cdot \mathbf{E} + V \nabla \cdot \mathbf{E} = -E^2 + V \nabla \cdot \mathbf{E}, \quad (4.168)$$

we obtain for a volume Ω

$$\frac{1}{2} \int_{\Omega} \rho V d\tau = \frac{\epsilon_0}{2} \int_{\Omega} E^2 d\tau + \frac{\epsilon_0}{2} \oint_{\partial\Omega} V \mathbf{E} \cdot d\mathbf{A}. \quad (4.169)$$

For an isolated localized system integrated over all space with $V(\infty) = 0$, the boundary term vanishes: at large radius $V \sim 1/r$, $E \sim 1/r^2$ and area grows as r^2 , leaving a term that vanishes as $1/r$. Thus

$$U = \int \frac{1}{2} \epsilon_0 E^2 d\tau, \quad u_E = \frac{1}{2} \epsilon_0 E^2. \quad (4.170)$$

A finite integration volume or maintained external boundaries cannot be substituted without checking the surface term. When voltage sources exchange charge, their energy accounting also matters. The next chapter develops that issue for capacitors.

4.23.2 Positive field energy and negative pair energy

The density u_E is nonnegative. How can the pair energy of opposite charges be negative? The two expressions use different levels of idealization. The all-space field energy includes each source's self-energy. For mathematical point charges those self-energies diverge. The pair sum in the core explicitly excludes them and retains only the finite interaction energy.

For two extended distributions, write $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$. The square contains two positive self terms and a cross term:

$$U = U_1 + U_2 + \varepsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau. \quad (4.171)$$

The cross term can be negative, representing a reduction of energy when the distributions are brought together. Subtracting unchanged self-energies before taking an ideal point-source limit leaves the finite pair interaction $k_e q_1 q_2 / r_{12}$. There is no contradiction between positive total field-energy density and negative interaction energy relative to separated sources.

Example 4.29: Charging an isolated conducting sphere

Build charge Q on an isolated conducting sphere of fixed radius R . At an intermediate charge Q' , its potential is $k_e Q' / R$. The charging work is

$$U = \int_0^Q \frac{k_e Q'}{R} dQ' = \frac{k_e Q^2}{2R}. \quad (4.172)$$

The same answer follows by integrating the exterior field energy:

$$U = \frac{\varepsilon_0}{2} \int_R^\infty \left(\frac{k_e Q}{r^2} \right)^2 4\pi r^2 dr = \frac{k_e Q^2}{2R}. \quad (4.173)$$

There is no interior contribution because the field in the metal is zero. For $Q = 2.0 \text{ nC}$ and $R = 5.0 \text{ cm}$, $U = 3.60 \times 10^{-7} \text{ J}$. This is also $QV_s/2$, not QV_s : the sphere was not at its final potential during the entire charging process. At fixed Q , shrinking R makes the energy grow as $1/R$, displaying the classical self-energy divergence as the source approaches a point.

4.24 Numerical potentials: approximation, solution and validation

4.24.1 A softened Coulomb potential changes the physical source

Suppose a visualization uses

$$V_a(r) = \frac{k_e q}{\sqrt{r^2 + a^2}}, \quad a > 0. \quad (4.174)$$

This removes the point singularity, but the implied field is

$$E_r = \frac{k_e q r}{(r^2 + a^2)^{3/2}}. \quad (4.175)$$

For a spherically symmetric function, the divergence is $r^{-2} d(r^2 E_r) / dr$. Applying Gauss's law gives

$$\rho_a(r) = \frac{3qa^2}{4\pi(r^2 + a^2)^{5/2}}. \quad (4.176)$$

The enclosed charge is $Q(< r) = qr^3 / (r^2 + a^2)^{3/2}$, which approaches q at infinity. Thus softening corresponds to an extended distribution with tails, not to an unchanged point charge with a harmless numerical repair. At $r \gg a$, its potential approaches Coulomb's; at $r = 0$, its field vanishes. Both the resolution scale and the source model must be reported.

For a true point-charge benchmark it is often better to mask a small excluded region and compare only where the exact potential is smooth. For a finite charged object, an explicit density model is more informative than an unexplained softening parameter. Refining the grid while keeping a fixed converges to the softened model, not to the point-charge model.

4.24.2 Finite differences turn Laplace's equation into neighbour averages

In a two-dimensional charge-free region with equal grid spacings h , central differences give

$$\frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{h^2} + \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{h^2} \simeq 0. \quad (4.177)$$

Solving for the central value yields

$$V_{i,j} \simeq \frac{1}{4}(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}). \quad (4.178)$$

In the Jacobi method, all interior values are updated from the previous iteration while boundary values remain fixed. Gauss–Seidel uses newly updated neighbours immediately. Both solve the same discrete system when converged, but their iteration histories differ. The averaging rule assumes equal spacings and no charge; nonzero density adds $+h^2\rho_{i,j}/(4\epsilon_0)$ to the update. Unequal spacings require weighted neighbours.

A small update is not the same as an accurate physical answer. Three errors must be separated: incomplete iteration, finite spatial resolution, and an inaccurate physical model or boundary placement. A residual tests whether the discrete equation is solved. A grid-refinement study tests spatial error. Comparison with analytic solutions or independent measurements tests the whole setup.

4.24.3 An analytic boundary problem for testing a solver

On a square $0 \leq x, y \leq L$, set the bottom and side potentials to zero and prescribe the top boundary as $V(x, L) = V_0 \sin(\pi x/L)$. The top is a prescribed distributed boundary, which can be approximated by segmented electrodes; it is not one equipotential metal strip. A direct separation-of-variables solution is

$$V(x, y) = V_0 \sin\left(\frac{\pi x}{L}\right) \frac{\sinh(\pi y/L)}{\sinh \pi}. \quad (4.179)$$

The second x derivative is $-(\pi/L)^2 V$, while the second y derivative is $+(\pi/L)^2 V$, so their sum vanishes. The boundary conditions are satisfied at all four sides, including the corners. Uniqueness makes this the exact solution.

The corresponding field is

$$E_x = -\frac{\pi V_0}{L} \cos(\pi x/L) \frac{\sinh(\pi y/L)}{\sinh \pi}, \quad (4.180)$$

$$E_y = -\frac{\pi V_0}{L} \sin(\pi x/L) \frac{\cosh(\pi y/L)}{\sinh \pi}. \quad (4.181)$$

At $x = L/2$, reflection symmetry gives $E_x = 0$. The field points down from the positive top boundary. No interior value should exceed the largest boundary value or fall below zero; the discrete neighbour average provides an elementary version of the maximum principle.

4.24.4 A reproducible calculation

The following NumPy program solves for the dimensionless potential $u = V/V_0$ on coordinates $X = x/L$, $Y = y/L$. Array index zero labels Y , and index one labels X . This convention matters when taking derivatives. The boundary row at $Y = 1$ is fixed to $\sin(\pi X)$. The stopping criterion uses the discrete Laplacian of u , rather than merely the difference between successive arrays.

```
import numpy as np

def solve_square(N, tolerance=1e-7):
    h = 1.0 / N
    x = np.linspace(0.0, 1.0, N + 1)
```

```

X, Y = np.meshgrid(x, x, indexing="xy")
exact = np.sin(np.pi*X)*np.sinh(np.pi*Y)
exact /= np.sinh(np.pi)
u = np.zeros_like(exact)
u[-1, :] = np.sin(np.pi*x)
for step in range(1, 200001):
    new = u.copy()
    new[1:-1, 1:-1] = 0.25*(
        u[2:, 1:-1] + u[:-2, 1:-1]
        + u[1:-1, 2:] + u[1:-1, :-2])
    u = new
    residual = (u[2:, 1:-1] + u[:-2, 1:-1]
        + u[1:-1, 2:] + u[1:-1, :-2]
        - 4*u[1:-1, 1:-1]) / h**2
    if np.max(np.abs(residual)) < tolerance:
        break
else:
    raise RuntimeError("Iteration did not converge")
error = np.max(np.abs(u - exact))
return u, exact, step, error

for N in (16, 32, 64):
    u, exact, steps, error = solve_square(N)
    print(N, steps, error)

```

The computed maximum potential errors, normalized to V_0 , are shown below. They measure the difference from the analytic solution at the same grid nodes, not an experimental discrepancy.

Intervals per side N	Iterations	$\max V_{\text{num}} - V_{\text{exact}} /V_0$
16	926	0.0011088
32	3720	0.00027796
64	14898	$6.9623e-05$

The approximately fourfold reduction on doubling N is consistent with the second-order spatial approximation. Increasing the number of iterations on one grid cannot remove its discretization error. Jacobi convergence also becomes slower on finer grids; more efficient solvers change the computational cost, not the underlying electrostatic boundary problem.

For a physical square with $L = 1.0$ cm and $V_0 = 10$ V, multiply u by 10 V and its dimensionless derivatives by $V_0/L = 1000$ V/m. At the centre the exact values are $V = V_0 \sinh(\pi/2)/\sinh \pi \simeq 1.99$ V and $E_y \simeq -683$ V/m. Reporting a dimensionless gradient as volts per metre without this scale conversion is a substantial error.

4.24.5 Turning a numerical map into a defensible model

First verify boundary values and the range of the interior potential. Then reconstruct field components on interior nodes with central differences and compare with the analytic field. For this smooth benchmark, a refinement test over a fixed region away from the outer boundary should show convergence of both potential and field. Boundary derivatives require one-sided formulas or a separate treatment.

An electrode charge can be estimated from the outward normal derivative through $\sigma = -\epsilon_0 \partial_n V$. This derivative is often more sensitive to discretization than the potential itself, especially near corners. A potential map that appears smooth on screen may therefore give an inaccurate peak field or surface charge. In real geometries, edge rounding and material interfaces need physical length scales rather than a mesh-dependent singular peak.

The computed dipole contours used earlier in this chapter follow the exact point-source potential with source neighbourhoods excluded. The electrode benchmark instead solves a two-dimensional boundary problem. Their methods are complementary, but their geometries must not be mixed: a planar slice of a three-dimensional point field does not obey the same source-free two-dimensional Laplace equation everywhere in that plane.

4.25 How electrode detail fades away from a boundary

4.25.1 A sinusoidal boundary produces an exponential spatial decay

Potential methods can quantify how small geometric or voltage structure near an electrode affects a distant measurement region. Consider the charge-free half-space $y > 0$, with no z dependence, and a prescribed boundary variation

$$V(x, 0) = V_1 \cos(kx), \quad k > 0, \quad (4.182)$$

that vanishes at large y . The symbol k here is a spatial wave number, distinct from the Coulomb constant k_e . Seek a separated solution $V = \cos(kx)f(y)$. Laplace's equation gives

$$f''(y) - k^2 f(y) = 0. \quad (4.183)$$

The two possible dependences are $\exp(ky)$ and $\exp(-ky)$. The growing term violates the distant boundary condition, and the surface value fixes the remaining coefficient:

$$V(x, y) = V_1 \cos(kx)e^{-ky}. \quad (4.184)$$

The field components are $E_x = kV_1 \sin(kx)e^{-ky}$ and $E_y = kV_1 \cos(kx)e^{-ky}$. Direct differentiation checks that the field is conservative and divergence free. The exponential is a spatial attenuation, not energy dissipation or time decay.

The characteristic decay distance is $1/k = \ell/(2\pi)$, where ℓ is the period along the boundary. Shorter-wavelength voltage variations decay over shorter distances. Their near-boundary fields can nevertheless be large, because differentiation multiplies the amplitude by k . A boundary may therefore have strong local gradients while its fine structure has little effect far into the observation region.

Example 4.30: Suppressing a periodic boundary ripple

An ideal prescribed boundary has a sinusoidal potential ripple of amplitude $V_1 = 0.20$ V and period $\ell = 4.0$ mm. At what height is its potential amplitude reduced to 1% of the boundary amplitude? What is the ripple field magnitude there?

The ratio is $\exp(-2\pi y/\ell)$. Setting it to 0.01 gives

$$y = \frac{\ell}{2\pi} \ln 100 = 2.93 \text{ mm}. \quad (4.185)$$

The two field components combine to a magnitude $E = k|V_1|e^{-ky}$, independent of x for this single spatial mode. At the stated height,

$$E = \frac{2\pi}{4.0 \times 10^{-3} \text{ m}} (0.20 \text{ V})(0.01) = 3.14 \text{ V/m}. \quad (4.186)$$

At the boundary the corresponding magnitude is 314 V/m. This calculation models an imposed periodic boundary pattern, such as a mathematical approximation to multiple biased strips; one perfect connected conductor could not maintain a varying surface potential in electrostatic equilibrium.

4.25.2 Superposition connects simple modes to structured electrodes

If a periodic boundary function is decomposed into sine and cosine terms, linearity allows each term to be propagated into the half-space independently. The mode with wave number nk decays as $\exp(-nky)$. For example,

$$V(x, 0) = V_1 \cos(kx) + V_2 \cos(2kx) \quad (4.187)$$

produces

$$V(x, y) = V_1 \cos(kx)e^{-ky} + V_2 \cos(2kx)e^{-2ky}. \quad (4.188)$$

Farther from the boundary, the larger-scale first term becomes relatively more important. This gives a physical reason that coarse boundary structure often controls a distant field region while fine details dominate very near an edge. A constant boundary mode needs separate treatment: it cannot both remain nonzero and decay to zero at infinity in this translationally invariant half-space model.

The square benchmark in Section 4.24 uses the same separation principle with a second boundary at finite distance. Hyperbolic sine replaces a single decaying exponential because both the lower and upper conditions must be satisfied. Analytic modes are useful not only as exact special cases but also as independent checks on a numerical solver's treatment of length scales and boundary values.

4.26 When the electrostatic description must be extended

The identity $\mathbf{E} = -\nabla V$ is sufficient for a static conservative electric field. In general electromagnetism the field may also contain an induction contribution, commonly written $\mathbf{E} = -\nabla V - \partial \mathbf{A}/\partial t$. A changing magnetic flux can make the electric-field integral around a closed loop nonzero. Then a single electrostatic potential difference cannot represent every field integral independently of path.

A slowly varying device can often be approximated by a sequence of electrostatic solutions, provided induction effects and propagation delays are negligible on its operating scales. The dimensionless comparison $L/(cT) \ll 1$, for size L and variation time T , is one necessary scale check for negligible propagation delay, but does not by itself guarantee negligible induction in every geometry. The physical approximation must be assessed from the actual fields and circuit arrangement.

For a particle crossing a time-dependent accelerator gap, the fundamental energy transfer is $q \int \mathbf{E}(\mathbf{r}(t), t) \cdot \mathbf{v}(t) dt$. A static voltage estimate is justified only when the field is effectively unchanged during transit and the relevant path can be treated consistently. This boundary explains why the chapter's energy language remains valuable while the later induction and wave chapters add new structure.

4.27 Advanced problems and solution checks

These tasks use the enrichment directly. The checks provide intermediate or final constraints without repeating the full derivations.

- A1.** Starting from Eq. (4.150), determine when an interior zero of E_x occurs. Find its position and classify the extremum of V for positive ρ_0 . *Check:* $x_* = d/2 + \varepsilon_0 V_b/(\rho_0 d)$, valid only if $0 < x_* < d$; $V'' < 0$.
- A2.** Use the local expansion for two like charges to find the small-oscillation frequency along the constrained charge axis and the exponential growth rate for a small transverse displacement of a positive test charge. *Check:* $\omega_x^2 = 4k_e Qq/(ma^3)$ and $\gamma_y^2 = 2k_e Qq/(ma^3)$.
- A3.** For the neutral three-charge quadrupole, determine the leading axial and equatorial radial fields. *Check:* $E_r = 3k_e qa^2(3 \cos^2 \theta - 1)/r^4$; along these symmetry directions the angular field component vanishes.

- A4.** Integrate the induced surface density of a grounded plane out to radius S . What fraction of the total induced charge lies within $S = a$? *Check:* $Q(< S) = -q[1 - a/\sqrt{S^2 + a^2}]$; the fraction is $1 - 1/\sqrt{2}$.
- A5.** For a charge above the grounded plane, derive the work needed to take it from height a to infinity. Compare with the work for an image treated incorrectly as a fixed source. *Check:* the physical work is $k_e q^2/(4a)$; the false pair-energy procedure gives twice that.
- A6.** Compute the field energy of a uniformly charged solid insulating sphere, including both interior and exterior contributions. *Check:* $U_{\text{in}} = k_e Q^2/(10R)$, $U_{\text{out}} = k_e Q^2/(2R)$ and $U = 3k_e Q^2/(5R)$.
- A7.** For the softened charge density, find the radius containing half the total charge and determine the large- r leading correction to its potential. *Check:* $r_{1/2} = a/\sqrt{2^{2/3} - 1}$ and $V_a = (k_e q/r)[1 - a^2/(2r^2) + \dots]$.
- A8.** Modify the numerical benchmark to use unequal spacings h_x, h_y . Derive the update weights before writing code. *Check:* the central value is $[(V_E + V_W)/h_x^2 + (V_N + V_S)/h_y^2]/[2/h_x^2 + 2/h_y^2]$ in a charge-free region.
- A9.** Verify the numerical benchmark's centre field on successively refined grids. Compare maximum potential error, centre-field error and discrete residual. Explain why they are different diagnostics and why none alone validates an unrelated real electrode geometry.
- A10.** Derive the pure-Neumann compatibility condition by integrating Poisson's equation. Explain the remaining reference freedom. *Check:* $\oint \partial_n V \, dA = -Q_{\text{enclosed}}/\epsilon_0$, with the normal outward from the solution volume.

Take-home message

Potential methods connect three descriptions of one static physical system: work and energy, local field gradients, and source or electrode boundary data. A successful calculation states which data are prescribed, respects the source model, and checks the resulting field as well as the voltage. The value of a scalar potential lies in making those connections easier to calculate and to interpret.

Bibliography

- [1] D. Halliday, R. Resnick and J. Walker, *Fundamentals of Physics*, 12th edition, Volume 2, Wiley, 2022, Chapter 24: Electric Potential.
- [2] R. A. Serway and J. W. Jewett, Jr., *Physics for Scientists and Engineers with Modern Physics*, 9th edition, Brooks/Cole, Cengage Learning, 2014, Chapter 25: Electric Potential.
- [3] S. J. Ling, W. Moebs and J. Sanny, *University Physics*, Volume 2, OpenStax, 2016, Chapter 7: Electric Potential.