

Chapter 5

Capacitance, Capacitors, and Dielectrics

First Edition

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Preface

This chapter introduces capacitance as the first device-level concept in electrostatics. In the previous chapters students learned how charges produce electric fields, how symmetric charge distributions can be treated with Gauss’s law, and how potential difference describes energy per unit charge. A capacitor turns those ideas into a concrete component: two conductors arranged so that equal and opposite charges can be stored at a controlled potential difference.

This chapter is written for Technical Physics, engineering, computer science, applied science, space science and related programmes. It combines rigorous electrostatic derivations with the engineering language of devices, tolerances, dielectric breakdown, sensors, energy storage and circuit modelling. The central lesson is that capacitance is not “the amount of charge” in a capacitor. It is a property of the conductor geometry and the material between the conductors, defined by the ratio of charge magnitude to potential difference.

Capacitors are ubiquitous. They appear in filters, timing circuits, power supplies, pulsed lasers, detector electronics, microelectromechanical sensors, touch screens, memory cells, medical instruments and high-voltage laboratories. Yet the ideal capacitor is also one of the cleanest bridges between field theory and circuits. The same object can be described by charge on conductors, potential difference between terminals, electric field in space, stored energy, energy density and dielectric polarization. Learning to move between those descriptions is one of the main goals of this chapter.

The Guideline

Treat a capacitor problem as a constraint problem before treating it as an algebra problem. First decide what is fixed: the free charge on the plates, the potential difference imposed by a battery, the geometry, the dielectric material, or a network connection. Then choose the correct relation among Q , ΔV , C , E and U . Most wrong capacitor solutions come from silently changing the constraint halfway through the calculation.

Pedagogical boundary

This chapter uses electrostatic fields and potentials, but it does not yet develop full time-dependent circuit theory. Capacitor charging through a resistor, impedance, displacement current and electromagnetic radiation are postponed to later chapters. Here the main focus is electrostatic capacitance, energy stored in electric fields, dielectric response and the algebra of ideal capacitor networks.

Conventions and notation

Vectors are written in bold notation using the macro `\vect{}`. The electric field is $\mathbf{E}$, the electric displacement field is $\mathbf{D}$, and surface elements are written as $d\mathbf{A}$. The symbol Q denotes the magnitude of the free charge on either plate of an ideal two-conductor capacitor. The algebraic charges on the two plates are then $+Q$ and $-Q$. The symbol ΔV denotes the potential difference between the conductors, usually the positive potential minus the negative potential, so that $\Delta V > 0$ for an ordinary charged capacitor.

The capacitance is denoted by C and is measured in farads:

$$1 \text{ F} = 1 \text{ C V}^{-1} = 1 \text{ C}^2 \text{ J}^{-1}.$$

Because one farad is a very large capacitance for many laboratory components, practical values are often expressed in microfarads, nanofarads and picofarads:

$$1 \mu\text{F} = 10^{-6} \text{ F}, \quad 1 \text{ nF} = 10^{-9} \text{ F}, \quad 1 \text{ pF} = 10^{-12} \text{ F}.$$

The vacuum permittivity is ε_0 , a material permittivity is ε , and a relative permittivity or dielectric constant is $\kappa = \varepsilon/\varepsilon_0$ for a linear homogeneous isotropic dielectric.

Symbol	Meaning	SI unit or dimension
C	capacitance of a two-terminal capacitor or equivalent capacitance of a network	$\text{F} = \text{C V}^{-1}$
Q	magnitude of free charge on either conductor in a two-conductor capacitor	C
q	algebraic charge, often used for a variable charge during charging	C
ΔV	potential difference between capacitor terminals	V
A	plate area in an ideal parallel-plate approximation	m^2
d	separation between parallel plates	m
ε_0	vacuum permittivity	F m^{-1}
ε	absolute permittivity of a linear dielectric	F m^{-1}
κ	relative permittivity or dielectric constant, $\varepsilon/\varepsilon_0$	dimensionless
σ_f	free surface charge density on a conductor plate	C m^{-2}
σ_b	bound surface charge density on a dielectric surface	C m^{-2}
U	electrostatic energy stored in a capacitor or field	J
u_E	electric-field energy density	J m^{-3}
E	magnitude of electric field	V m^{-1} or N C^{-1}
$\mathbf{D}$	electric displacement field, $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$	C m^{-2}

Throughout the chapter, ΔV denotes a terminal potential difference. For an isolated conductor, self-capacitance uses potential zero at infinity; nearby conductors change that boundary-value problem. The symbol C in units means coulomb, whereas italic C denotes capacitance.

Chapter 5

Capacitance, Capacitors, and Dielectrics

5.1 Motivation and roadmap

A capacitor is a deliberately arranged system of conductors that can store separated charge and electrostatic energy. The simplest mental picture is two conducting plates facing one another, with charge $+Q$ on one plate and $-Q$ on the other. The field between the plates stores energy; the potential difference between the plates measures how much work per unit charge is required to move charge from one plate to the other; and the capacitance tells how much charge is stored for a given potential difference.

The word “store” must be used carefully. A capacitor does not create charge. In an ideal two-conductor capacitor the net charge of the whole device is zero. The device stores separated charge: one conductor has $+Q$ and the other has $-Q$. It also stores energy in the electric field produced by that separation. This distinction is important in electronics, in high-voltage safety and in the field theory that will later lead to Maxwell’s equations.

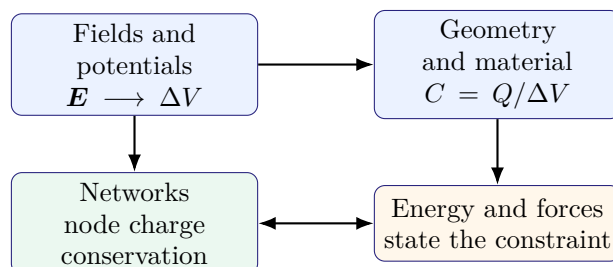


Figure 5.1: The chapter connects field calculations to circuit relations and then to energy, dielectric response and measurement. The advanced section develops the same connections with more general mathematics.

Figure 5.1 organizes these descriptions into a single sequence.

Learning goals

After studying this chapter, the student should be able to:

- define capacitance for a two-conductor system and interpret the meaning of $C = Q/\Delta V$;
- derive the ideal parallel-plate capacitance from Gauss's law and the relation between field and potential;
- derive and use the capacitance of cylindrical, spherical and isolated-sphere geometries;
- reduce ideal series and parallel capacitor networks and explain which quantity is common in each connection;
- compute the energy stored in a charged capacitor and relate it to electric-field energy density;
- explain what changes and what remains fixed when a dielectric is inserted under fixed- Q or fixed- ΔV constraints;
- distinguish free charge on conductors from bound charge in a dielectric;
- estimate breakdown limits, energy density and engineering safety margins;
- build and check simple computational models for capacitor networks;
- solve conceptual, analytical and engineering-style capacitor problems with explicit assumptions.

Key message

The central bridge of the chapter is

$$C = \frac{Q}{\Delta V}. \quad (5.1)$$

The numerator is the magnitude of the charge on either conductor, not the net charge of the whole capacitor. The denominator is the potential difference between the conductors, not an absolute potential.

5.2 Prerequisite ideas from electrostatics

Capacitance cannot be understood as a new isolated formula. It rests on four electrostatic ideas.

First, a conductor in electrostatic equilibrium is an equipotential object. Charges can move in the conductor; if there were an electric field inside the conducting material, free charges would continue to move. In equilibrium the electric field inside the conducting material is zero and the conductor has a single potential.

Second, electric field and potential difference are related by a line integral:

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\boldsymbol{\ell}. \quad (5.2)$$

For a uniform field of magnitude E between two parallel plates separated by distance d , this reduces to

$$\Delta V = Ed, \quad (5.3)$$

provided ΔV is taken as the higher potential minus the lower potential.

Third, Gauss's law gives a powerful way to find electric fields when the symmetry is high:

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\epsilon_0}. \quad (5.4)$$

In this chapter it will be used for parallel plates, coaxial cylinders and concentric spheres.

Fourth, electrostatic potential energy is related to work. A capacitor is charged by moving small amounts of charge from one conductor to another against a growing potential difference. The work done in that process becomes stored electrostatic energy.

Method

When deriving a capacitance from electrostatics, use the same four-step pattern:

1. assign charges $+Q$ and $-Q$ to the conductors;
2. find $\mathbf{E}$ using symmetry and Gauss's law;
3. compute ΔV from a line integral of $\mathbf{E}$;
4. form $C = Q/\Delta V$ and simplify.

This method is safer than memorising unrelated formulas, because it also tells you the assumptions behind each formula.

Guided checks

Check 1. If both conductors of a capacitor are shifted by the same absolute potential, does C change? *Answer:* No. Capacitance depends on potential difference, geometry and material, not on an arbitrary additive constant in potential.

Check 2. Why is a conductor surface an equipotential in electrostatic equilibrium? *Answer:* A tangential electric field would drive surface charge motion, so equilibrium requires no tangential field.

Check 3. Can Gauss's law always find the electric field of a capacitor? *Answer:* Gauss's law is always true, but it is directly useful only when symmetry makes the field simple enough to take outside the flux integral.

5.3 Definition of capacitance

An ideal two-conductor capacitor consists of two conductors carrying equal and opposite charges $+Q$ and $-Q$. The conductors are called plates even when their shapes are not literally flat. The capacitance is defined by

$$C = \frac{Q}{\Delta V}, \quad (5.5)$$

where Q is the magnitude of the charge on either conductor and ΔV is the magnitude of the potential difference between them.

For many ideal systems C is independent of Q and ΔV . Doubling the charge doubles the potential difference, so their ratio remains the same. Such a capacitor is called linear. Real components can deviate from this ideal behaviour if the dielectric is nonlinear, if the voltage is near breakdown, if temperature changes substantially, or if the geometry changes under electrostatic force.

Definition 5.1: Capacitance of a two-conductor system

For a two-conductor electrostatic system with charges $+Q$ and $-Q$ and potential difference ΔV between the conductors, the capacitance is

$$C = \frac{Q}{\Delta V}. \quad (5.6)$$

For a linear capacitor, C is a property of the geometry and the intervening material, not of the particular value of Q used to measure it.

The farad is a large unit because a capacitance of 1 F stores 1 C of charge at only 1 V. In ordinary electronics, capacitances from picofarads to millifarads are more common. Large supercapacitors can have capacitances of farads or thousands of farads, but their voltage ratings are usually low and their behaviour is not simply that of an ideal parallel-plate capacitor.

Caution

A capacitor with plates $+Q$ and $-Q$ has total charge zero, but its capacitance is not zero. Capacitance measures the ability to store separated charge at a potential difference. Confusing total charge of the device with charge magnitude on one plate is one of the most common mistakes in this topic.

Example 5.1: Interpreting a measured charge-voltage slope

A laboratory capacitor is charged to several potential differences. The measured charge magnitudes are $Q = 2.4 \text{ nC}$ at $\Delta V = 20 \text{ V}$ and $Q = 7.2 \text{ nC}$ at $\Delta V = 60 \text{ V}$. Estimate its capacitance and decide whether these two measurements are consistent with a linear capacitor.

Solution. From the first measurement,

$$C_1 = \frac{2.4 \times 10^{-9} \text{ C}}{20 \text{ V}} = 1.2 \times 10^{-10} \text{ F} = 120 \text{ pF}. \quad (5.7)$$

From the second measurement,

$$C_2 = \frac{7.2 \times 10^{-9} \text{ C}}{60 \text{ V}} = 1.2 \times 10^{-10} \text{ F} = 120 \text{ pF}. \quad (5.8)$$

The two estimates agree, so the data are consistent with a linear capacitance of approximately 120 pF. Agreement at two voltages does not prove perfect linearity, but it is a good first check.

Reasonableness check. A capacitance of 120 pF is typical of a small laboratory or cable-scale capacitance, not of a large energy-storage capacitor.

5.4 Ideal parallel-plate capacitor

The parallel-plate capacitor is the central model because it exposes the physical meaning of capacitance with minimal geometry. Consider two large conducting plates of area A , separated by a distance d , with d much smaller than the linear size of the plates. The ideal approximation neglects fringing fields near the edges. The field between the plates is then nearly uniform, and the field outside is nearly zero.

Let the plates carry charges $+Q$ and $-Q$. The free surface charge density magnitude is

$$\sigma_f = \frac{Q}{A}. \quad (5.9)$$

For ideal large plates in vacuum, Gauss's law gives the field magnitude between them:

$$E = \frac{\sigma_f}{\epsilon_0} = \frac{Q}{\epsilon_0 A}. \quad (5.10)$$

The potential difference is

$$\Delta V = Ed = \frac{Qd}{\epsilon_0 A}. \quad (5.11)$$

Therefore

$$C = \epsilon_0 \frac{A}{d}. \quad (5.12)$$

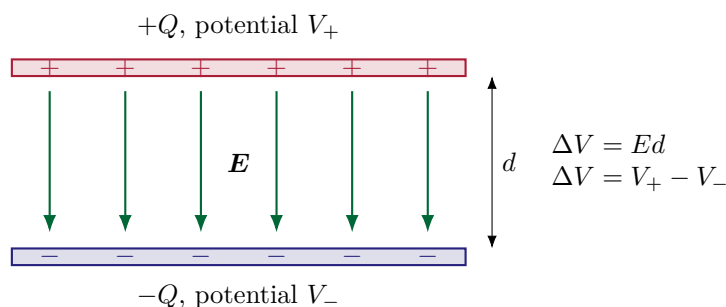


Figure 5.2: Parallel plates viewed in cross-section; each facing surface has area A . The field points from higher to lower potential. Edge fields are neglected in this model.

Figure 5.2 fixes the field and voltage directions.

Equation (5.12) carries a lot of physics. Increasing area gives more room for charge at the same field and potential difference. Increasing separation raises the potential difference for the same charge, so the capacitance decreases. Replacing vacuum by a dielectric changes the permittivity and is treated later.

Note

The parallel-plate formula is not exact for finite plates. It becomes accurate when the plate separation is much smaller than the plate dimensions and when one is interested mainly in the central region. The neglected field outside and near the edges is called fringing field.

5.4.1 Why the field has no extra factor of two

An isolated infinite sheet of charge produces a field $\sigma_f/(2\varepsilon_0)$ on each side. Two oppositely charged sheets produce fields in the same direction between them, so the contributions add to σ_f/ε_0 ; outside, they cancel. Equivalently, a small Gaussian pillbox straddling a conductor surface has zero field on its metal side and flux EA on its gap side. Thus $\varepsilon_0 EA = \sigma_f A$. These are two consistent arguments for the same result. One must not add a second factor of two to the conductor-surface result.

The cancellation is an ideal large-plate statement. Real finite plates have fields outside and a nonuniform charge density near their rims. Consequently, Q/A is an average surface density, and using it as the local density is part of the uniform-field approximation. Reducing d relative to both plate dimensions improves the approximation, but the plate leads and surrounding conductors still matter in precision measurements.

The potential is particularly simple in the gap. With the lower plate at $z = 0$ and $V_- = 0$, and the upper plate at $z = d$ and $V_+ = \Delta V$,

$$V(z) = \frac{\Delta V}{d}z, \quad \mathbf{E} = -\frac{\Delta V}{d}\hat{\mathbf{z}}. \quad (5.13)$$

The sign refers to the vector component; the positive magnitude is $E = \Delta V/d$. A potential that increases with height therefore corresponds to a downward field, as in Fig. 5.2.

Example 5.2: Design estimate for an air-filled parallel-plate capacitor

Two square conducting plates of side length 10.0 cm are separated by 1.00 mm in air. Estimate the capacitance, treating air as vacuum.

Solution. The area is

$$A = (0.100 \text{ m})^2 = 1.00 \times 10^{-2} \text{ m}^2. \quad (5.14)$$

Using $C = \epsilon_0 A/d$,

$$C = (8.85 \times 10^{-12} \text{ F m}^{-1}) \frac{1.00 \times 10^{-2} \text{ m}^2}{1.00 \times 10^{-3} \text{ m}} = 8.85 \times 10^{-11} \text{ F}. \quad (5.15)$$

Thus

$$C \simeq 88.5 \text{ pF}. \quad (5.16)$$

Interpretation. Even plates that are fairly large on a laboratory table produce only a picofarad-scale capacitance when the separation is millimetric. This explains why large capacitances require thin dielectrics, large effective area, or electrochemical mechanisms.

Guided checks

Check 1. If the plate area is doubled while d is fixed, what happens to C ? *Answer:* It doubles.

Check 2. If the separation is doubled while A is fixed, what happens to C ? *Answer:* It is halved in the ideal model.

Check 3. Does the ideal formula include edge effects? *Answer:* No. Edge effects are neglected by the large-plate, small-separation approximation.

5.5 Cylindrical capacitors

A coaxial cable is an important practical example of a cylindrical capacitor. It has an inner conductor, an insulating dielectric layer and an outer conducting shield. The capacitance per unit length controls signal propagation, impedance and stored electric energy.

Consider two long coaxial conducting cylinders of length L , inner radius a and outer radius b , with $L \gg b$ so that end effects can be neglected. Let the inner cylinder carry charge $+Q$ and the outer cylinder carry $-Q$. A cylindrical Gaussian surface of radius r and length L with $a < r < b$ encloses charge Q . By symmetry the field is radial and has constant magnitude on the curved Gaussian surface:

$$E(2\pi rL) = \frac{Q}{\epsilon_0}, \quad E(r) = \frac{Q}{2\pi\epsilon_0 L} \frac{1}{r}. \quad (5.17)$$

The potential difference between the cylinders is

$$\Delta V = \int_a^b E(r) dr = \frac{Q}{2\pi\epsilon_0 L} \ln \frac{b}{a}. \quad (5.18)$$

Thus the capacitance is

$$C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}. \quad (5.19)$$

The capacitance per unit length is

$$C' = \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln(b/a)}. \quad (5.20)$$

The radii and Gaussian surface are identified in Fig. 5.3.

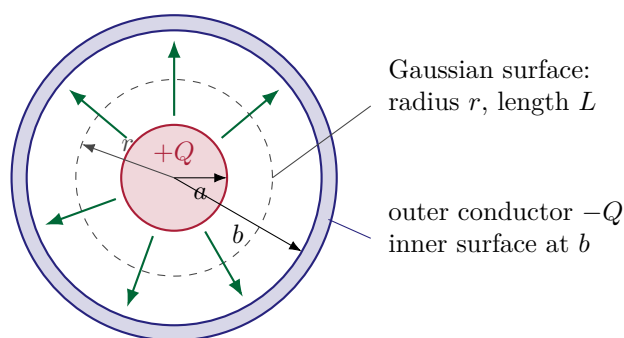


Figure 5.3: Cross-section of a long coaxial capacitor. The outer conductor is an annulus surrounding the inner conductor. The dashed circle is the cross-section of the cylindrical Gaussian surface. Radii refer to the facing conductor surfaces.

Example 5.3: Capacitance per unit length of a coaxial geometry

A coaxial cable has inner conductor radius $a = 0.50$ mm and inner radius of the outer conductor $b = 2.5$ mm. Estimate the capacitance per unit length in air.

Solution. The radius ratio is $b/a = 5.0$, so

$$C' = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi(8.85 \times 10^{-12} \text{ F m}^{-1})}{\ln 5.0} = 3.46 \times 10^{-11} \text{ F m}^{-1}. \quad (5.21)$$

Thus

$$C' \simeq 34.6 \text{ pF m}^{-1}. \quad (5.22)$$

A real coaxial cable often uses a dielectric with $\kappa > 1$, which increases this value.

Engineering interpretation. Capacitance per unit length is not a small parasitic detail in signal cables. Together with inductance per unit length it determines the characteristic impedance and wave speed.

Caution

The cylindrical formula assumes $L \gg b$ and neglects end effects. It is most useful for long coaxial structures and for capacitance per unit length. It should not be applied blindly to a short ring-shaped object.

5.6 Spherical capacitors and isolated conductors

A spherical capacitor consists of two concentric conducting spheres of radii a and b , with $b > a$. The inner sphere carries $+Q$ and the outer sphere carries $-Q$. A spherical Gaussian surface of radius r with $a < r < b$ encloses charge Q , so

$$E(r)4\pi r^2 = \frac{Q}{\epsilon_0}, \quad E(r) = \frac{Q}{4\pi\epsilon_0 r^2}. \quad (5.23)$$

The potential difference is

$$\Delta V = \int_a^b E(r) dr = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right). \quad (5.24)$$

Hence

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}. \quad (5.25)$$

If the outer conductor is moved infinitely far away, the isolated sphere capacitance relative to infinity is

$$C = 4\pi\epsilon_0 R. \quad (5.26)$$

This result is useful for estimating why everyday objects can reach large voltages with surprisingly small charges.

Example 5.4: Capacitance and voltage of an isolated conducting sphere

A conducting sphere of radius $R = 5.0\text{ cm}$ carries charge $Q = 10\text{ nC}$ and is far from other conductors. Estimate its capacitance and potential relative to infinity.

Solution. The isolated-sphere capacitance is

$$C = 4\pi\epsilon_0 R = 4\pi(8.85 \times 10^{-12}\text{ F m}^{-1})(0.050\text{ m}) = 5.56 \times 10^{-12}\text{ F}. \quad (5.27)$$

Thus $C = 5.56\text{ pF}$. The potential is

$$V = \frac{Q}{C} = \frac{10 \times 10^{-9}\text{ C}}{5.56 \times 10^{-12}\text{ F}} = 1.80 \times 10^3\text{ V}. \quad (5.28)$$

Interpretation. A charge of only 10 nC can put a small isolated conductor at kilovolt potential. High voltage does not necessarily mean large charge or large energy.

Guided checks

Check 1. Why does the isolated-sphere capacitance increase with radius? *Answer:* A larger sphere spreads the same charge over a larger surface, producing a smaller surface potential for the same Q , and therefore a larger Q/V .

Check 2. Is an isolated sphere really a two-conductor capacitor? *Answer:* The second conductor is effectively a reference conductor at infinity. This is an idealization that works when other conductors are far away.

Check 3. Does a kilovolt potential imply dangerous stored energy? *Answer:* Not by itself. Energy depends on $U = Q^2/(2C) = CV^2/2$, so both voltage and capacitance matter.

5.6.1 Curvature, limiting cases and geometric scaling

Figure 5.4 identifies the two surfaces that enter the spherical calculation. The outer radius b means the inner surface of the outer shell, not its exterior radius. In an ideal concentric system, charge $-Q$ resides on that inner surface and the exterior field vanishes when the net charge is zero.

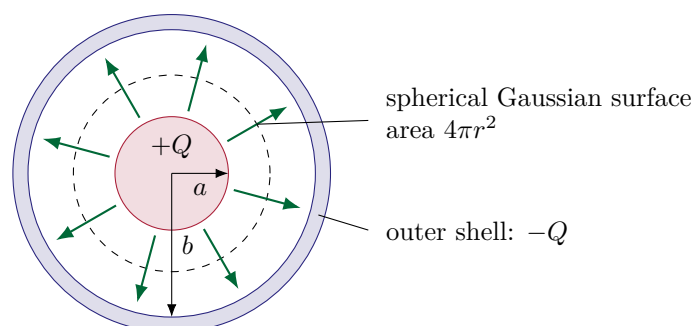


Figure 5.4: Central section of a spherical capacitor. The circular drawing is a section through spheres, whereas Fig. 5.3 is a section through cylinders. The distinction changes the Gaussian area and hence the radial dependence of the field.

The cylindrical and spherical formulas should both recover a plane gap when the curvature is small. Put $b = a + d$ with $d \ll a$. For the cylinder,

$$\ln \frac{b}{a} = \ln(1 + d/a) \simeq \frac{d}{a}, \quad C \simeq \frac{\varepsilon_0(2\pi aL)}{d}. \quad (5.29)$$

For the spheres,

$$C = \frac{4\pi\varepsilon_0 a(a+d)}{d} = \frac{\varepsilon_0(4\pi a^2)}{d} \left(1 + \frac{d}{a}\right). \quad (5.30)$$

The leading terms have precisely the plane-capacitor form $\varepsilon_0 A/d$, using the inner facing surface area. The correction d/a in the spherical result quantifies the first curvature effect for that choice of area.

A different operation is to enlarge *every* length by a common factor s . In that case $A \rightarrow s^2 A$ and $d \rightarrow sd$, so capacitance scales as $C \rightarrow sC$. The coaxial logarithm remains unchanged, while its length becomes sL ; the isolated-sphere radius also scales by s . Capacitance therefore scales like a length for geometrically similar three-dimensional systems with unchanged permittivity. Enlarging just the plate area at fixed separation is a different operation.

Guided checks

A sphere of radius R surrounded by a grounded concentric shell of radius $2R$ has $C = 8\pi\varepsilon_0 R$, twice its isolated self-capacitance. The nearby return conductor lowers the voltage required for the same charge. This is a concrete example of why the electrical environment belongs in a capacitance model.

5.7 Capacitors in parallel

Capacitors are in parallel when their corresponding terminals are connected to the same two nodes. In an ideal parallel connection, every capacitor has the same potential difference ΔV . The charges add:

$$Q_{\text{total}} = Q_1 + Q_2 + \cdots + Q_n = C_1\Delta V + C_2\Delta V + \cdots + C_n\Delta V. \quad (5.31)$$

Defining the equivalent capacitance by $Q_{\text{total}} = C_{\text{eq}}\Delta V$ gives

$$C_{\text{eq}} = \sum_{i=1}^n C_i \quad (\text{parallel}). \quad (5.32)$$

The equivalent capacitance is larger than any individual capacitance because the network provides more effective area for charge storage at the same potential difference.

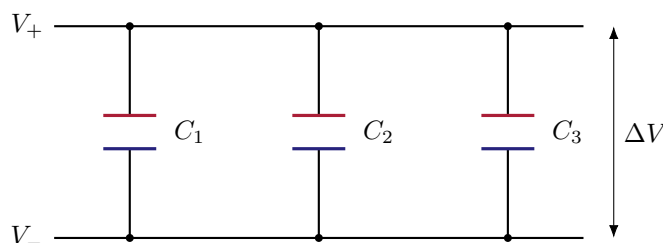


Figure 5.5: Parallel connection: every upper plate belongs to one node and every lower plate to the other. The node potentials, rather than the appearance of the drawing, determine the common voltage.

The two common nodes are visible in Fig. 5.5.

Method

For parallel capacitors, write first what is common:

$$\Delta V_1 = \Delta V_2 = \cdots = \Delta V. \quad (5.33)$$

Then compute charges from $Q_i = C_i \Delta V$ and add them. Do not force the charges to be equal; they are equal only if the capacitances happen to be equal.

Example 5.5: Parallel network and charge distribution

Three capacitors, $C_1 = 2.0 \mu\text{F}$, $C_2 = 5.0 \mu\text{F}$ and $C_3 = 8.0 \mu\text{F}$, are connected in parallel across $\Delta V = 12 \text{ V}$. Find the equivalent capacitance and the charge on each capacitor.

Solution. The equivalent capacitance is

$$C_{\text{eq}} = C_1 + C_2 + C_3 = 15.0 \mu\text{F}. \quad (5.34)$$

The voltage across each capacitor is 12 V , so

$$Q_1 = (2.0 \mu\text{F})(12 \text{ V}) = 24 \mu\text{C}, \quad (5.35)$$

$$Q_2 = (5.0 \mu\text{F})(12 \text{ V}) = 60 \mu\text{C}, \quad (5.36)$$

$$Q_3 = (8.0 \mu\text{F})(12 \text{ V}) = 96 \mu\text{C}. \quad (5.37)$$

The total charge supplied by the source to the positive node is $180 \mu\text{C}$, which also equals $C_{\text{eq}} \Delta V$.

Reasonableness check. The largest capacitor carries the largest charge because all three share the same potential difference.

5.8 Capacitors in series

A simple series chain connects capacitors end to end, with each internal node connected only to the two adjacent capacitor plates. If those internal conductors are initially neutral and remain isolated, the charge magnitudes on the capacitors are equal. To see why, write the charges on an internal island as $-Q_1$ and $+Q_2$: conservation gives $-Q_1 + Q_2 = 0$. No conduction charge crosses either dielectric. Equal charge is thus a consequence of the initial node charge and connectivity, not a universal property of components that happen to be drawn in a row. We use neutral internal nodes throughout the elementary series examples. The potential differences add:

$$\Delta V = \Delta V_1 + \Delta V_2 + \cdots + \Delta V_n. \quad (5.38)$$

Since $\Delta V_i = Q/C_i$, we have

$$\Delta V = Q \left(\frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n} \right). \quad (5.39)$$

With $\Delta V = Q/C_{\text{eq}}$,

$$\boxed{\frac{1}{C_{\text{eq}}} = \sum_{i=1}^n \frac{1}{C_i}} \quad (\text{series}). \quad (5.40)$$

The equivalent capacitance is smaller than the smallest individual capacitance. This is not a mysterious rule; it reflects the fact that the same charge must produce a larger total potential difference across several gaps.

Figure 5.6 shows the neutral island that makes the equal-charge argument possible.

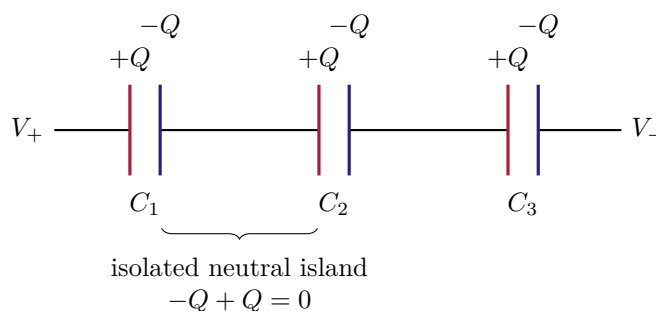


Figure 5.6: Series connection of three capacitors. Each internal island consists of a wire and the two plates it joins. Initially neutral isolated islands enforce equal charge magnitudes. The plate gaps are the only breaks in conduction paths.

Caution

Students often remember that resistors add in series and then incorrectly apply the same rule to capacitors. Capacitors in series add by reciprocals because equal charge magnitudes, established by conservation on neutral internal nodes, produce a sum of potential drops. Conduction charge does not pass through the dielectric gaps.

Example 5.6: Two capacitors in series

A $3.0\ \mu\text{F}$ capacitor and a $6.0\ \mu\text{F}$ capacitor are connected in series across $18\ \text{V}$. Find the equivalent capacitance, the charge on each capacitor and the potential difference across each capacitor.

Solution. The equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{3.0\ \mu\text{F}} + \frac{1}{6.0\ \mu\text{F}} = \frac{1}{2.0\ \mu\text{F}}, \quad (5.41)$$

so $C_{\text{eq}} = 2.0\ \mu\text{F}$. The common charge is

$$Q = C_{\text{eq}}\Delta V = (2.0\ \mu\text{F})(18\ \text{V}) = 36\ \mu\text{C}. \quad (5.42)$$

Therefore

$$\Delta V_1 = \frac{Q}{C_1} = \frac{36\ \mu\text{C}}{3.0\ \mu\text{F}} = 12\ \text{V}, \quad \Delta V_2 = \frac{Q}{C_2} = \frac{36\ \mu\text{C}}{6.0\ \mu\text{F}} = 6\ \text{V}. \quad (5.43)$$

The potential differences add to $18\ \text{V}$.

Interpretation. The smaller capacitor has the larger voltage across it in a series connection. This is important for voltage ratings in capacitor stacks.

Guided checks

Check 1. In a series connection, do all capacitors have the same voltage? *Answer:* No. They have the same charge magnitude under the neutral-internal-node assumption.

Check 2. Which capacitor in a series pair gets the larger voltage? *Answer:* The smaller capacitance, because $\Delta V = Q/C$.

Check 3. Why is C_{eq} smaller than the smallest series capacitance? *Answer:* The same charge produces a sum of voltage drops, so the total voltage is larger than the voltage across any single capacitor.

5.9 Mixed capacitor networks

Realistic circuit diagrams often contain combinations that must be reduced step by step. The physical rule is not to search for a memorised pattern, but to identify nodes. Components that share both nodes are in parallel. Components connected through an isolated internal node that has no other branch form a series pair; the usual equal-charge reduction assumes that node is initially neutral.

For ideal networks, the reduction process has three layers:

1. reduce obvious series or parallel groups to an equivalent capacitance;
2. compute the total charge or total voltage using the equivalent capacitor;
3. expand the network backward, using common-voltage rules for parallel groups and common-charge rules for series groups.

Method

A reliable network workflow is: reduce forward, solve the equivalent problem, expand backward. During the backward expansion, write on the diagram whether a subnetwork shares a common voltage or a common charge. This prevents the most common error: using a series rule during numerical reduction and then treating the same group as if it were parallel during charge calculation.

Figure 5.7 supplies the topology for the following example. The junction between C_1 and C_2 is a neutral floating island; the endpoints of the two branches are the same nodes.

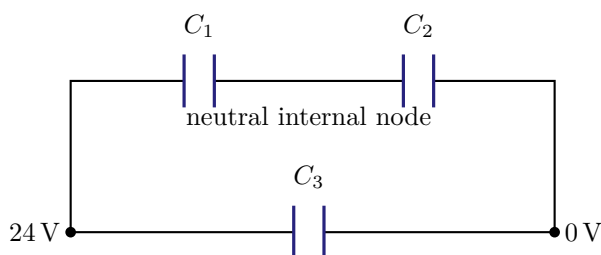


Figure 5.7: The mixed network used in Example 5.7: C_1 and C_2 form one series branch, in parallel with C_3 . Wires meeting at the marked nodes are electrically connected.

Example 5.7: A mixed network reduced step by step

Capacitors $C_1 = 4.0 \mu\text{F}$ and $C_2 = 12.0 \mu\text{F}$ are connected in series. That series pair is connected in parallel with $C_3 = 6.0 \mu\text{F}$. The whole network is connected across 24 V . Find the equivalent capacitance and the charge on each capacitor.

Solution. First reduce the series pair:

$$\frac{1}{C_{12}} = \frac{1}{4.0} + \frac{1}{12.0} \quad (\mu\text{F})^{-1}, \quad C_{12} = 3.0 \mu\text{F}. \quad (5.44)$$

This equivalent capacitor is in parallel with C_3 , so

$$C_{\text{eq}} = C_{12} + C_3 = 9.0 \mu\text{F}. \quad (5.45)$$

The total charge supplied to the network is

$$Q_{\text{total}} = C_{\text{eq}} \Delta V = (9.0 \mu\text{F})(24 \text{ V}) = 216 \mu\text{C}. \quad (5.46)$$

The parallel branches each have 24 V across them. Thus

$$Q_3 = C_3 \Delta V = (6.0 \mu\text{F})(24 \text{ V}) = 144 \mu\text{C}. \quad (5.47)$$

The series pair has charge

$$Q_{12} = C_{12} \Delta V = (3.0 \mu\text{F})(24 \text{ V}) = 72 \mu\text{C}. \quad (5.48)$$

Because C_1 and C_2 are in series, each has charge magnitude

$$Q_1 = Q_2 = 72 \mu\text{C}. \quad (5.49)$$

The individual voltages are

$$\Delta V_1 = \frac{72 \mu\text{C}}{4.0 \mu\text{F}} = 18 \text{ V}, \quad \Delta V_2 = \frac{72 \mu\text{C}}{12.0 \mu\text{F}} = 6 \text{ V}. \quad (5.50)$$

They add to the 24 V across the series branch.

Caution

Equivalent capacitance is a tool for replacing a network as seen from two terminals. It does not mean that every original capacitor has the same charge as the equivalent capacitor. After reduction, always expand backward to recover individual charges and voltages.

5.9.1 When the internal node is not neutral

Charge conservation is more general than the equal-charge shortcut. Let C_1 join a fixed potential V_0 to an island X , and let C_2 join X to ground. If that island has a trapped net charge q_X , then

$$q_X = C_1(V_X - V_0) + C_2 V_X, \quad V_X = \frac{C_1 V_0 + q_X}{C_1 + C_2}. \quad (5.51)$$

The relation is signed: the charge on the X plate of C_1 is $C_1(V_X - V_0)$, which is negative when $V_X < V_0$. A nonzero q_X changes the absolute charges and voltages, even though the incremental two-terminal capacitance at fixed q_X remains $C_1 C_2 / (C_1 + C_2)$.

Example 5.8: A precharged internal island

Take $C_1 = 2.0 \mu\text{F}$, $C_2 = 4.0 \mu\text{F}$, $V_0 = 12 \text{ V}$ and $q_X = +6.0 \mu\text{C}$. Find the island potential and the plate charges attached to it.

Strategy. Conserve the specified island charge; equal capacitor charges would impose the wrong initial condition. Equation (5.51) gives

$$V_X = \frac{(2.0 \mu\text{F})(12 \text{ V}) + 6.0 \mu\text{C}}{6.0 \mu\text{F}} = 5.0 \text{ V}.$$

The island plate of C_1 carries $(2.0 \mu\text{F})(5.0 - 12) \text{ V} = -14 \mu\text{C}$; that of C_2 carries $(4.0 \mu\text{F})(5.0 \text{ V}) = +20 \mu\text{C}$. Their sum is the required $+6.0 \mu\text{C}$.

Check. The voltage drops are 7.0 V and 5.0 V , summing to 12 V . Setting $q_X = 0$ restores $V_X = 4.0 \text{ V}$ and equal charge magnitudes of $16 \mu\text{C}$.

5.10 Work required to charge a capacitor

The energy stored in a capacitor is not $Q\Delta V$ for a capacitor charged gradually from zero. During charging, the potential difference is not constant. Suppose a capacitor has capacitance C and at some intermediate stage carries charge q . Its potential difference is

$$v(q) = \frac{q}{C}. \quad (5.52)$$

To move an additional small charge dq onto the capacitor requires work

$$dW = v(q) dq = \frac{q}{C} dq. \quad (5.53)$$

The total work required to charge from 0 to Q is

$$U = W = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}. \quad (5.54)$$

Using $Q = C\Delta V$, this can be written in three equivalent forms:

$$U = \frac{Q^2}{2C} = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}Q\Delta V. \quad (5.55)$$

These forms are equivalent for a linear capacitor.

Key message

The factor $1/2$ appears because the capacitor voltage grows from zero to its final value during charging. The average voltage during the ideal charging process is $\Delta V/2$, so the stored energy is $(\Delta V/2)Q$.

5.10.1 Stored energy and energy delivered by a source

The integral $\int_0^Q v(q) dq$ gives the increase in stored energy at fixed geometry. It can be realized as reversible work by raising the applied voltage slowly so that it remains almost equal to the instantaneous capacitor voltage. It is not the total work done by a battery of constant voltage during ordinary charging through a resistance.

A fixed source V_s moving charge $Q = CV_s$ supplies $W_s = V_s Q = CV_s^2$, while the final capacitor stores only $\frac{1}{2}CV_s^2$. The remaining half is transferred to other degrees of freedom during the transient. For

the elementary resistance-limited charging process it becomes heat. No time-dependent calculation is needed to establish this energy balance; the RC chapter will describe the time dependence.

Figure 5.8 displays the work integral. Notice that the appropriate area is under voltage as a function of charge. The complementary area under charge as a function of voltage is an electrical *coenergy*; the two areas are equal for a linear capacitor, but not generally for a nonlinear component.

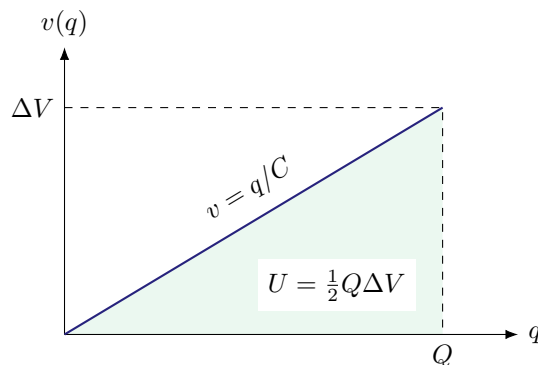


Figure 5.8: Charging work for a linear capacitor is the triangular area $\int_0^Q v dq$. The rectangle $Q\Delta V$ would count every charge increment at the final voltage. Axes are normalized schematic coordinates; the relation itself is exact for constant C .

Example 5.9: Energy stored in a camera-flash capacitor

A flash unit contains a capacitor of capacitance $C = 120 \mu\text{F}$ charged to $\Delta V = 300 \text{ V}$. Estimate the stored energy.

Solution. Use

$$U = \frac{1}{2} C (\Delta V)^2. \quad (5.56)$$

Substitution gives

$$U = \frac{1}{2} (120 \times 10^{-6} \text{ F}) (300 \text{ V})^2 = 5.4 \text{ J}. \quad (5.57)$$

Interpretation. A few joules delivered in a very short time can produce a bright flash. The same energy delivered slowly would be much less dramatic. Power depends on the time scale of energy transfer, not only on the stored energy.

Guided checks

Check 1. If the voltage on a fixed capacitor is doubled, what happens to stored energy? *Answer:* It increases by a factor of four.

Check 2. If the charge on an isolated fixed capacitor is doubled, what happens to stored energy? *Answer:* It increases by a factor of four, since $U = Q^2/(2C)$.

Check 3. Why is $U \neq Q\Delta V$ for gradual charging from zero? *Answer:* The full voltage exists only at the end; earlier charge increments are moved through smaller potential differences.

5.11 Energy density in the electric field

For an ideal parallel-plate capacitor in vacuum,

$$C = \epsilon_0 \frac{A}{d}, \quad \Delta V = Ed. \quad (5.58)$$

The stored energy is

$$U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}\left(\varepsilon_0 \frac{A}{d}\right)(Ed)^2 = \frac{1}{2}\varepsilon_0 E^2(Ad). \quad (5.59)$$

The volume between the plates is Ad , so the energy per unit volume is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 \quad (\text{vacuum}). \quad (5.60)$$

In a linear dielectric, the corresponding expression is

$$u_E = \frac{1}{2}\varepsilon E^2 = \frac{1}{2}\kappa\varepsilon_0 E^2, \quad (5.61)$$

for a reversible, linear, isotropic dielectric at fixed material state. It holds locally with position-dependent ε as well. In matter, this macroscopic stored energy includes the reversible polarization response; $\frac{1}{2}\varepsilon_0 E^2$ alone is the vacuum-field contribution. Dissipative or strongly dispersive media require additional care.

This result is more than a convenient formula. It marks a shift in interpretation: energy can be regarded as stored in the electric field itself. Later, magnetic fields will have their own energy density, and electromagnetic waves will transport energy through space.

Example 5.10: Energy density between capacitor plates

An air-filled parallel-plate capacitor has plate separation $d = 0.80$ mm and is connected to a 400 V source. Estimate the electric field and the energy density between the plates.

Solution. The ideal field is

$$E = \frac{\Delta V}{d} = \frac{400 \text{ V}}{0.80 \times 10^{-3} \text{ m}} = 5.0 \times 10^5 \text{ V m}^{-1}. \quad (5.62)$$

The vacuum energy density is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}(8.85 \times 10^{-12})(5.0 \times 10^5)^2 \text{ J m}^{-3} = 1.11 \text{ J m}^{-3}. \quad (5.63)$$

Interpretation. This energy density is modest compared with chemical fuels or batteries, which is why ordinary electrostatic capacitors are not usually high-energy-density storage devices. Their advantage is rapid charging and discharging, not high energy per unit mass.

Note

The energy-density formula is local. For nonuniform fields, the total field energy is obtained by integration:

$$U = \int u_E d\tau. \quad (5.64)$$

For ideal parallel plates the field is approximately uniform, so the integral reduces to $u_E Ad$.

5.11.1 Changing a plate separation: who supplies the energy?

A common misconception is that disconnecting a capacitor prevents its voltage or energy from changing. Isolation fixes free charge, not geometry. Moving a plate changes the field-filled volume and the potential difference, and mechanical work can change the energy even without an electrical connection.

For an air-filled capacitor initially at gap d_i , define $r = d_f/d_i$. At fixed area,

$$C_f = C_i/r. \quad (5.65)$$

With fixed Q , the ideal field $E = Q/(\epsilon_0 A)$ is unchanged, but it occupies a larger volume if $r > 1$. Therefore $\Delta V_f = r\Delta V_i$ and $U_f = rU_i$. A quasistatic external agent separating the attracting plates must supply $W_{\text{ext}} = U_f - U_i = (r - 1)U_i$.

With a source maintaining ΔV , the field instead falls to $E_f = E_i/r$ as the gap grows. The charge and stored energy both decrease by $1/r$. It would be wrong to infer that separating the plates produces energy for free. Charge returns through the source, which receives energy. Taking source work as positive when delivered to the capacitor,

$$W_s = \Delta V(Q_f - Q_i) = (\Delta V)^2(C_f - C_i), \quad (5.66)$$

$$W_{\text{ext}} = \Delta U - W_s = -\frac{1}{2}(\Delta V)^2(C_f - C_i). \quad (5.67)$$

For separation, $C_f < C_i$, so $W_s < 0$ while $W_{\text{ext}} > 0$. The source gains both the decrease in capacitor energy and the mechanical work supplied from outside.

Example 5.11: Doubling the gap in two different experiments

A capacitor of initial capacitance 100 pF is at 100 V. Its gap is doubled slowly. Compare (a) disconnecting it before the motion and (b) keeping an ideal 100 V source attached. Neglect fringing and all dissipation.

Initial state. $Q_i = 10 \text{ nC}$ and $U_i = 0.500 \mu\text{J}$. Both experiments end at $C_f = 50 \text{ pF}$.

(a) Isolated. $Q_f = 10 \text{ nC}$, $\Delta V_f = 200 \text{ V}$, and $U_f = 1.00 \mu\text{J}$. The external mechanical work is $+0.500 \mu\text{J}$, all stored in the capacitor.

(b) Connected. $Q_f = 5.0 \text{ nC}$ and $U_f = 0.250 \mu\text{J}$. The source work is $(100 \text{ V})(-5.0 \text{ nC}) = -0.500 \mu\text{J}$. Consequently the external agent supplies $(-0.250) - (-0.500) = +0.250 \mu\text{J}$. The source receives $0.500 \mu\text{J}$: half comes from the field and half from the agent.

Checks. Increasing separation lowers capacitance in both cases. It does not imply the same voltage, field or energy change. Positive external work is required in both experiments because the plates attract.

Figure 5.9 summarizes these two paths from the same initial state. The curves are exact within the plane model, and their different slopes show why an energy formula must be accompanied by a controlled-variable statement.

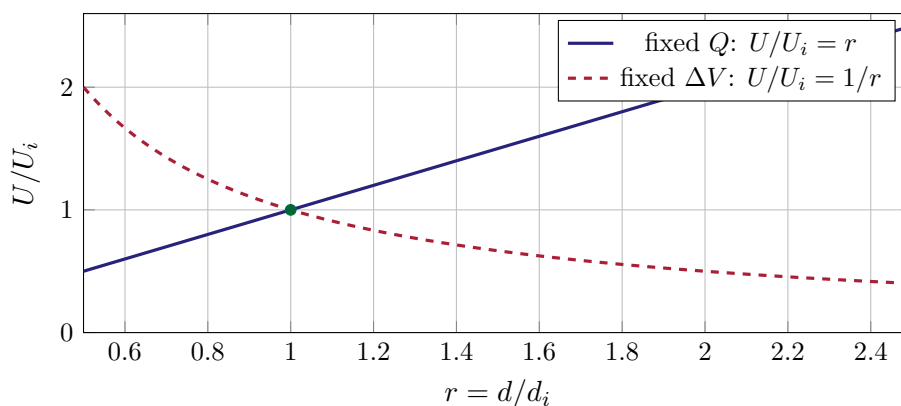


Figure 5.9: Stored energy as the gap changes from a common initial state. The source is part of the energy accounting for the dashed curve. The graph does not imply that force follows the negative slope of capacitor energy along both paths.

5.12 Dielectrics and relative permittivity

A dielectric is an insulating material that responds to an electric field by polarisation. Charges in the material are not free to move throughout the sample as they would in a metal, but positive and negative charge distributions can shift slightly relative to one another. This microscopic rearrangement produces bound charges on dielectric surfaces and changes the electric field inside the capacitor.

If a dielectric fully fills the space between the plates of an ideal capacitor, the capacitance becomes

$$C = \kappa C_0, \quad (5.68)$$

where C_0 is the capacitance without the dielectric and κ is the dielectric constant or relative permittivity. For a parallel-plate capacitor fully filled by a linear dielectric,

$$C = \kappa \epsilon_0 \frac{A}{d} = \epsilon \frac{A}{d}. \quad (5.69)$$

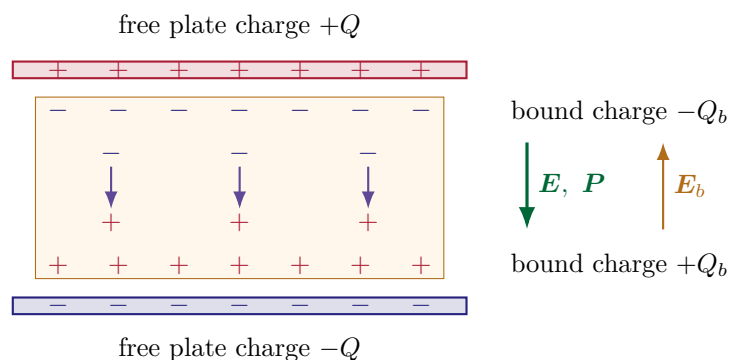


Figure 5.10: Polarization for a positively charged upper plate. Dipole moments point from negative to positive charge, downward along $\mathbf{E}$. The negative upper bound charge and positive lower bound charge produce an opposing field $\mathbf{E}_b$. Small clearances separate the schematic labels; the full-fill formula neglects air gaps.

Figure 5.10 separates the applied-field direction from the opposing contribution of the bound charges.

Definition 5.2: Free and bound charge

Free charge is charge placed on conductors or supplied by external circuits. Bound charge is the effective charge associated with polarization inside a dielectric. Bound charge is not charge that can freely flow through the dielectric; it represents microscopic displacement or alignment of charges within molecules or atoms.

Key message

A dielectric does not supply free charge to the capacitor plates. It changes the relation between free charge and electric field by polarization. For a fully filled ideal capacitor, this increases the capacitance by the factor κ .

Guided checks

Check 1. Does inserting a dielectric into an isolated capacitor change the free charge on the plates? *Answer:* No. If the capacitor is disconnected, free charge has no path to enter or leave.

Check 2. Does inserting a dielectric into a battery-connected capacitor change the free charge? *Answer:* Yes. The battery maintains ΔV , and additional free charge can flow to the plates.

Check 3. Are bound charges in a dielectric the same as mobile conduction charges? *Answer:* No. Bound charge arises from polarization; it is not free to travel through the material as conduction charge.

5.12.1 From molecular dipoles to free-charge Gauss's law

Polarization may result from displacement of positive and negative charge within a nonpolar molecule, or from partial alignment of permanent dipoles. Thermal motion prevents perfect alignment in ordinary polar dielectrics. The vector $\mathbf{P}$ is the electric dipole moment per unit volume; it points, by convention, from negative to positive bound charge and has units C m^{-2} .

For a uniform slab polarized perpendicular to its faces, the internal dipole charges cancel in the bulk, leaving surface charge. If $\hat{\mathbf{n}}$ points outward from the dielectric,

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}. \quad (5.70)$$

In Fig. 5.10, $\mathbf{P}$ points downward. The upper outward normal points upward, hence the upper dielectric face has negative bound charge. The lower face is positive. Those surface charges produce a field opposing the plate field.

For a reversible linear isotropic dielectric, introduce its susceptibility χ_e through

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} \equiv \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}, \quad \kappa = 1 + \chi_e. \quad (5.71)$$

Here $\mathbf{E}$ is the total macroscopic field, not the field of the free plates alone. The definition $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ is general; the constitutive equation $\mathbf{D} = \varepsilon \mathbf{E}$ is the additional linear isotropic model.

Ordinary Gauss's law has not changed: $\varepsilon_0 \oint \mathbf{E} \cdot d\mathbf{A}$ equals *all* enclosed charge, free plus bound. By including polarization on the field side, the same law can be written

$$\oint \mathbf{D} \cdot d\mathbf{A} = Q_{f,\text{enc}}. \quad (5.72)$$

For plane symmetry, a pillbox gives $D = \sigma_f = Q/A$ in the gap. Consequently,

$$E = \frac{\sigma_f}{\varepsilon}, \quad P = \frac{\kappa - 1}{\kappa} \sigma_f, \quad Q_b = \left(1 - \frac{1}{\kappa}\right) Q. \quad (5.73)$$

Q_b is the magnitude on either dielectric face. It is smaller than the *simultaneous* free plate charge Q for a finite static $\kappa > 1$. These equations derive the increase of capacitance from polarization rather than treating $\varepsilon_0 \rightarrow \varepsilon$ as an unexplained substitution.

Example 5.12: Counting free and bound charge

An ideal fully filled capacitor has $A = 20 \text{ cm}^2$, $Q = 30 \text{ nC}$ and $\kappa = 3$. Determine D , E and the bound surface charge magnitude.

Strategy. The free charge fixes D by Gauss's law; the material relation then fixes E . Convert $A = 2.0 \times 10^{-3} \text{ m}^2$:

$$\begin{aligned} D &= Q/A = 1.5 \times 10^{-5} \text{ C m}^{-2}, \\ E &= D/(3\epsilon_0) = 5.65 \times 10^5 \text{ V m}^{-1}, \\ P &= (2/3)D = 1.0 \times 10^{-5} \text{ C m}^{-2}, \quad Q_b = PA = 20 \text{ nC}. \end{aligned}$$

The negative bound face is adjacent to the positive conductor. Checking with total charge gives $E = (Q - Q_b)/(\epsilon_0 A)$, the same result. No charge has crossed the dielectric or been supplied by it to the plates.

5.13 Fixed charge versus fixed voltage

Dielectric insertion is the point in the chapter where constraints become unavoidable. The same physical action, inserting a dielectric, has different consequences depending on whether the capacitor is isolated or connected to a voltage source.

5.13.1 Isolated capacitor: fixed Q

If the capacitor is charged and disconnected from any circuit, the free charge Q remains fixed. A dielectric increases the capacitance from C_0 to κC_0 . The new potential difference is

$$\Delta V_f = \frac{Q}{\kappa C_0} = \frac{\Delta V_i}{\kappa}. \quad (5.74)$$

The stored energy changes from $U_i = Q^2/(2C_0)$ to

$$U_f = \frac{Q^2}{2\kappa C_0} = \frac{U_i}{\kappa}. \quad (5.75)$$

The field energy decreases. In an ideal quasistatic insertion, the capacitor pulls the dielectric inward and does mechanical work.

5.13.2 Battery-connected capacitor: fixed ΔV

If the capacitor remains connected to a battery, the potential difference is fixed. The capacitance increases to κC_0 , so the free charge becomes

$$Q_f = (\kappa C_0)\Delta V = \kappa Q_i. \quad (5.76)$$

The stored energy becomes

$$U_f = \frac{1}{2}(\kappa C_0)(\Delta V)^2 = \kappa U_i. \quad (5.77)$$

The energy stored in the capacitor increases, but the full energy balance also includes work done by the battery and mechanical work associated with dielectric insertion.

For fixed geometry and complete filling by a homogeneous dielectric, the ideal plane field at fixed voltage is $E = \Delta V/d$, exactly unchanged within that model. Its unchanged value does not mean that polarization is absent: the bound charge is present, and the source supplies enough additional free charge to compensate its opposing field.

Table 5.1: Comparison of dielectric insertion in an ideal capacitor fully filled by a dielectric of constant κ .

Quantity	Isolated capacitor	Battery-connected capacitor
Fixed constraint	Q fixed	ΔV fixed
Capacitance	$C_f = \kappa C_0$	$C_f = \kappa C_0$
Free charge	$Q_f = Q_i$	$Q_f = \kappa Q_i$
Potential difference	$\Delta V_f = \Delta V_i / \kappa$	$\Delta V_f = \Delta V_i$
Electric field	$E_f = E_i / \kappa$	$E_f = E_i$ (plane model)
Stored energy	$U_f = U_i / \kappa$	$U_f = \kappa U_i$

Example 5.13: Dielectric insertion under fixed Q and fixed ΔV

An air-filled capacitor has capacitance $C_0 = 200$ pF and is initially at $\Delta V_i = 100$ V. A dielectric with $\kappa = 4.0$ fully fills the capacitor. Compare the final charge, voltage and stored energy for (a) an isolated capacitor and (b) a capacitor kept connected to a 100 V source. Assume quasistatic, reversible insertion with the plate geometry fixed.

Initial values. The initial charge is

$$Q_i = C_0 \Delta V_i = (200 \times 10^{-12} \text{ F})(100 \text{ V}) = 20.0 \text{ nC}. \quad (5.78)$$

The initial energy is

$$U_i = \frac{1}{2} C_0 \Delta V_i^2 = 1.00 \times 10^{-6} \text{ J} = 1.00 \mu\text{J}. \quad (5.79)$$

(a) Fixed Q . The final capacitance is $C_f = 800$ pF. The free charge remains 20.0 nC. The final voltage is

$$\Delta V_f = \frac{Q_i}{C_f} = 25.0 \text{ V}. \quad (5.80)$$

The final energy is

$$U_f = \frac{U_i}{\kappa} = 0.250 \mu\text{J}. \quad (5.81)$$

(b) Fixed ΔV . The voltage remains 100 V. The final charge is

$$Q_f = C_f \Delta V = (800 \text{ pF})(100 \text{ V}) = 80.0 \text{ nC}. \quad (5.82)$$

The final stored energy is

$$U_f = \kappa U_i = 4.00 \mu\text{J}. \quad (5.83)$$

Interpretation. Same dielectric, same final capacitance, different physical constraints. This is why a problem statement must say whether the battery is connected or disconnected.

Caution

Never say simply “the energy decreases when a dielectric is inserted” or “the energy increases when a dielectric is inserted” without specifying the constraint. Under fixed Q it decreases; under fixed ΔV the capacitor’s stored energy increases.

5.13.3 Closing the energy balance

In Example 5.13, fixed-charge insertion releases $1.00 - 0.250 = 0.750 \mu\text{J}$ of stored energy. A restraining mechanism can receive this as mechanical work during a reversible insertion. At fixed voltage, the source delivers

$$W_s = \Delta V(Q_f - Q_i) = (100 \text{ V})(60.0 \text{ nC}) = 6.00 \mu\text{J}. \quad (5.84)$$

The increase of stored energy is $3.00 \mu\text{J}$; the remaining $3.00 \mu\text{J}$ is mechanical work done by the field on the slab in the reversible process. Thus the slab is pulled inward under either constraint. For rapid unrestrained insertion, mechanical kinetic energy and dissipation must also be included; the initial and final capacitor states do not specify how the released work is partitioned.

5.14 Partial dielectrics and equivalent models

A dielectric does not always fill the whole capacitor. Some arrangements can be treated by dividing the capacitor into series or parallel sub-capacitors. The rule is geometric.

If two dielectric regions occupy different parts of the plate area side by side, and each region spans the full plate separation, they share the same potential difference. The arrangement behaves like capacitors in parallel. If two dielectric layers are stacked along the field direction, so that the field passes through one layer and then the other, the normal electric displacement is the same in both layers, provided there is no free charge at their interface. Their voltage drops add, giving the same algebra as series capacitors. The physical interface contains bound polarization charge; it is not an extra metal plate carrying free charge.

Figure 5.11 distinguishes the two arrangements. In the lateral case the field is tangential to the dielectric interface and is approximately the same on both sides. In the stacked case it is normal to the interface, so D_n is continuous while E_n changes. A series or parallel analogy is useful only after the field geometry has been identified.

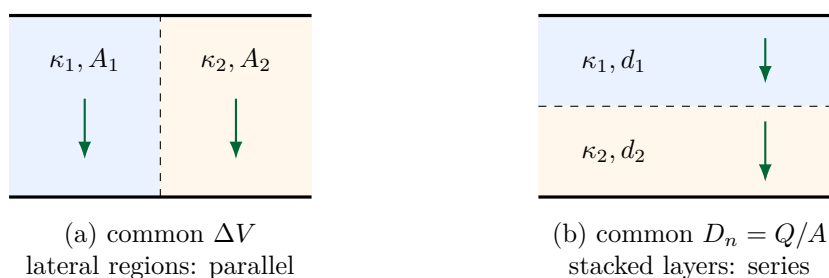


Figure 5.11: Partial dielectric filling in the plane-field approximation. The internal dashed lines mark dielectric interfaces, not conductors. Panel (b) illustrates differing field magnitudes; the lengths of those arrows are qualitative.

Example 5.14: Two dielectrics side by side

An ideal parallel-plate capacitor of area A and separation d is half filled by a dielectric κ_1 and half by a dielectric κ_2 , with the materials side by side across the area. Find the capacitance.

Solution. Each half has area $A/2$ and separation d . The two halves share the same voltage, so they are in parallel:

$$C_1 = \kappa_1 \epsilon_0 \frac{A/2}{d}, \quad C_2 = \kappa_2 \epsilon_0 \frac{A/2}{d}. \quad (5.85)$$

Therefore

$$C = C_1 + C_2 = \frac{\epsilon_0 A}{2d} (\kappa_1 + \kappa_2). \quad (5.86)$$

Interpretation. Side-by-side dielectrics add like parallel capacitors because each region sees the same potential difference between the same plates.

Example 5.15: Two dielectric layers stacked along the field

An ideal parallel-plate capacitor of area A is filled by two dielectric slabs stacked along the field direction. Slab 1 has thickness d_1 and dielectric constant κ_1 ; slab 2 has thickness d_2 and dielectric constant κ_2 . Find the capacitance.

Solution. The free charge Q is on the outer conducting plates. With no free interfacial charge, $D_n = Q/A$ is common to the layers. Their effective capacitances are

$$C_1 = \kappa_1 \epsilon_0 \frac{A}{d_1}, \quad C_2 = \kappa_2 \epsilon_0 \frac{A}{d_2}. \quad (5.87)$$

Thus

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d_1}{\kappa_1 \epsilon_0 A} + \frac{d_2}{\kappa_2 \epsilon_0 A}. \quad (5.88)$$

The capacitance is

$$C = \frac{\epsilon_0 A}{d_1/\kappa_1 + d_2/\kappa_2}. \quad (5.89)$$

Interpretation. A thin low-permittivity layer can dominate the total voltage drop. This is important in insulation design, coatings and biological membranes.

Guided checks

Check 1. A dielectric fills the left half of the plate area, and air fills the right half. Series or parallel model? *Answer:* Parallel.

Check 2. A dielectric slab fills half the plate separation, with air filling the other half along the field direction. Series or parallel model? *Answer:* Series.

Check 3. Why can a thin air gap strongly reduce capacitance even if most of the volume contains a high- κ material? *Answer:* In a series stack, the low-permittivity gap can carry a large fraction of the voltage drop.

5.14.1 Air gaps and a floating metal slab

A dielectric sheet of thickness $t < d$ leaves air gaps with total thickness $g = d - t$. Their placement above or below the sheet does not matter in the ideal one-dimensional model. The voltage is the sum of air and dielectric drops, so

$$C = \frac{\epsilon_0 A}{g + t/\kappa}, \quad E_{\text{air}} = \frac{\Delta V}{g + t/\kappa}, \quad E_{\text{diel}} = \frac{E_{\text{air}}}{\kappa}. \quad (5.90)$$

Even a small air gap can dominate the electrical thickness $g + t/\kappa$. Inserting a high-permittivity sheet without controlling the air gaps may therefore yield a capacitance far below κC_0 .

Example 5.16: A small void controls the voltage distribution

A capacitor has area 10 cm^2 and separation 1.0 mm . A sheet of thickness 0.90 mm and $\kappa = 9$ leaves a total air gap of 0.10 mm . Find C and the two fields at $\Delta V = 100 \text{ V}$.

Solution. The effective electrical thickness is

$$g + t/\kappa = 0.10 \text{ mm} + 0.90 \text{ mm}/9 = 0.20 \text{ mm}.$$

Thus $C = 44.3 \text{ pF}$, $E_{\text{air}} = 5.0 \times 10^5 \text{ V m}^{-1}$ and $E_{\text{diel}} = 5.56 \times 10^4 \text{ V m}^{-1}$. Each region takes 50 V , despite the dielectric being nine times thicker. A full 1.0 mm of the same dielectric would give 79.7 pF .

Interpretation. The air gap occupies only one tenth of the physical separation but half of the electrical thickness. The appropriate breakdown check uses each local field and its own specified material limit, not the average $\Delta V/d$.

A floating uncharged conducting slab provides a useful limiting comparison. If its thickness is t and it touches neither plate, its interior field is zero, so it contributes no voltage drop. The two remaining vacuum gaps have total thickness $d - t$, giving

$$C = \frac{\epsilon_0 A}{d - t}. \quad (5.91)$$

For a slab of negligible thickness the original capacitance is unchanged; for finite thickness it increases. Moving a plane slab within the gap does not change this result as long as it remains isolated, parallel and fully covers the active area. Contact with a plate changes the electrical connectivity. This conductor problem should not be described as a finite- κ dielectric acquiring free charge from a battery.

5.15 Breakdown field and real capacitor limitations

Ideal electrostatics allows any voltage in the equations. Real materials do not. If the electric field is too large, a dielectric can break down: it becomes conducting, often irreversibly or destructively. The breakdown field depends on material, thickness, defects, temperature, humidity, geometry, time dependence and manufacturing quality. Therefore it should be treated as an engineering limit, not as a universal constant.

For an ideal parallel-plate capacitor, the approximate field is $E = \Delta V/d$. If the maximum safe field is E_{max} , the voltage must satisfy

$$\Delta V \lesssim E_{\text{max}} d. \quad (5.92)$$

The maximum energy density in a linear dielectric before breakdown is roughly

$$u_{\text{max}} \simeq \frac{1}{2} \epsilon E_{\text{max}}^2. \quad (5.93)$$

This expression explains why high-energy-density capacitors require both large permittivity and high breakdown strength. A large κ alone is not enough.

Example 5.17: Voltage rating from breakdown estimate

A polymer dielectric layer of thickness $25\ \mu\text{m}$ is used in a film capacitor. For this illustrative design exercise, an allowable field of $80\ \text{MV m}^{-1}$ is specified; it is not a universal property of polymers. Estimate the maximum working voltage for one layer.

Solution. Use $\Delta V_{\text{max}} = E_{\text{max}}d$:

$$\Delta V_{\text{max}} = (80 \times 10^6\ \text{V m}^{-1})(25 \times 10^{-6}\ \text{m}) = 2.0 \times 10^3\ \text{V}. \quad (5.94)$$

The estimated working voltage is 2.0 kV for this idealized layer.

Engineering comment. Real designs include safety margins, edge grading, impurity control, temperature limits and aging. The simple calculation is a first estimate, not a complete qualification test.

Caution

A capacitor's voltage rating is not determined by capacitance alone. Two capacitors with the same capacitance may have very different voltage ratings and stored energies. Always check both C and the allowed ΔV when estimating safety or energy storage.

Note

Real capacitors also have leakage resistance, equivalent series resistance, equivalent series inductance, dielectric absorption and temperature dependence. These effects are essential in electronics but are beyond the ideal electrostatic model of the main chapter. They are introduced qualitatively in the advanced enrichment section.

5.16 Laboratory and measurement connection

A useful laboratory exercise is to measure capacitance as a function of plate separation. Two metal plates are mounted parallel to each other, and a capacitance meter measures C while d is varied. The ideal model predicts

$$C(d) = \varepsilon_0 \frac{A}{d}. \quad (5.95)$$

A plot of C versus $1/d$ should be approximately linear. Deviations can reveal fringing fields, imperfect parallelism, stray capacitance from cables and the finite input capacitance of the measuring instrument.

Another experiment uses a known capacitor charged to a known voltage, then disconnected and connected in parallel with an unknown capacitor initially uncharged. Charge conservation gives a final voltage from which the unknown capacitance can be inferred. This experiment is conceptually powerful because it distinguishes charge conservation from voltage conservation.

Method

For a capacitance measurement, list all parasitic capacitances before interpreting small discrepancies. Cable capacitance, input capacitance of instruments, humidity on insulating supports and nearby grounded objects can be comparable to the capacitance of small plate systems.

Example 5.18: Estimating an unknown capacitance by charge sharing

A known capacitor $C_1 = 1.00 \mu\text{F}$ is charged to $V_i = 12.0 \text{ V}$ and disconnected from the source. It is then connected in parallel with an initially uncharged capacitor C_2 . The final common voltage is $V_f = 4.00 \text{ V}$. Find C_2 , assuming negligible charge leakage and negligible meter loading after the redistribution has settled.

Solution. Initial charge on the known capacitor is

$$Q_i = C_1 V_i = (1.00 \mu\text{F})(12.0 \text{ V}) = 12.0 \mu\text{C}. \quad (5.96)$$

After connection, the total capacitance is $C_1 + C_2$ and the final charge is

$$Q_f = (C_1 + C_2)V_f. \quad (5.97)$$

Charge conservation gives

$$C_1 V_i = (C_1 + C_2)V_f. \quad (5.98)$$

Thus

$$C_2 = C_1 \left(\frac{V_i}{V_f} - 1 \right) = (1.00 \mu\text{F})(3.00 - 1) = 2.00 \mu\text{F}. \quad (5.99)$$

Energy comment. The initial energy is $72 \mu\text{J}$, while the final energy is $\frac{1}{2}(3.00 \mu\text{F})(4.00 \text{ V})^2 = 24 \mu\text{J}$. The decrease of $48 \mu\text{J}$ is transferred to heat and possibly radiation during settling. This is not a loss of charge. An exactly lossless model with finite inductance would oscillate and would not settle to a unique static voltage; an instantaneous connection through a zero-impedance wire is a singular idealization.

Guided checks

Check 1. In the charge-sharing experiment, is voltage conserved? *Answer:* No. Charge is conserved for the isolated connected system; the final voltage is determined by the total capacitance.

Check 2. Why can energy decrease even when charge is conserved? *Answer:* Energy is dissipated during the transient redistribution of charge. Conservation of charge does not imply conservation of electrostatic field energy alone.

Check 3. Why are cable capacitances important in picofarad experiments? *Answer:* A short cable can have capacitance comparable to or larger than the small capacitor being measured.

5.16.1 Charge sharing with arbitrary initial voltages

The conservation law applies separately to each conductor formed by the connection, not merely to the net charge of the complete two-capacitor apparatus (which may be zero). Define signed initial voltages V_1 and V_2 relative to the terminals that will be joined. With no source or appreciable leakage,

$$V_f = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2}. \quad (5.100)$$

Opposite initial polarities are handled by a negative V_2 . Substituting this voltage in the initial and final energy expressions and collecting terms gives

$$\begin{aligned} U_i - U_f &= \frac{1}{2}C_1 V_1^2 + \frac{1}{2}C_2 V_2^2 - \frac{(C_1 V_1 + C_2 V_2)^2}{2(C_1 + C_2)} \\ &= \boxed{\frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} (V_1 - V_2)^2}. \end{aligned} \quad (5.101)$$

This quantity is nonnegative and vanishes only when the initial terminal voltages agree. Charge sharing never increases stored electrostatic energy in this isolated passive connection. What appears as “missing energy” from an electrostatic calculation is transferred during the transient; electrostatics specifies its amount, but not its detailed destination or time scale.

Example 5.19: Opposite-polarity connection

Two capacitors of $2.0\ \mu\text{F}$ and $1.0\ \mu\text{F}$ initially have terminal voltages $+12\ \text{V}$ and $-6.0\ \text{V}$ relative to the terminals about to be joined. Find the settled voltage and released energy.

Strategy. Conserve signed charge on the joined upper conductor. Its initial charge is $24 - 6 = 18\ \mu\text{C}$. Therefore $V_f = 18/3 = 6.0\ \text{V}$. The smaller capacitor reverses its polarization during redistribution.

$$U_i = \frac{1}{2}(2.0\ \mu\text{F})(12\ \text{V})^2 + \frac{1}{2}(1.0\ \mu\text{F})(6.0\ \text{V})^2 = 162\ \mu\text{J},$$

whereas $U_f = \frac{1}{2}(3.0\ \mu\text{F})(6.0\ \text{V})^2 = 54\ \mu\text{J}$. The released energy is $108\ \mu\text{J}$, in agreement with Eq. (5.101), since $V_1 - V_2 = 18\ \text{V}$.

Check. The final charges on the joined upper plates are 12 and $6\ \mu\text{C}$, whose sum remains $18\ \mu\text{C}$. Using unsigned initial charge magnitudes would give the wrong final voltage.

5.16.2 A measurement model with stray capacitance

The measurement is not always the ideal active capacitance. A useful first model is

$$C_{\text{meas}}(d) = \frac{\varepsilon_0 A}{d} + C_s, \quad (5.102)$$

where C_s represents an approximately separation-independent contribution from cables and the instrument. Plotting C_{meas} against $1/d$ gives slope $\varepsilon_0 A$ and intercept C_s . Subtracting an arbitrary constant until a graph appears plausible is not calibration: C_s must be fitted from a justified model or independently measured. Fringing can depend on d , so a constant offset cannot absorb all departures from the plane-field model.

Example 5.20: Separating geometry from an instrument offset

The following numbers are *synthetic*, generated to illustrate a calibration calculation; they are not laboratory data. A plate fixture has area $100\ \text{cm}^2$ and gives:

d (mm)	$1/d$ (mm^{-1})	C_{meas} (pF)
0.50	2.00	189.0
1.00	1.00	100.5
2.00	0.50	56.25

Using the first two points, the slope is $(189.0 - 100.5)/(2.00 - 1.00) = 88.5\ \text{pF mm}$. Since $1\ \text{pF mm} = 10^{-15}\ \text{F m}$, the inferred area is

$$A = \frac{88.5 \times 10^{-15}\ \text{F m}}{8.85 \times 10^{-12}\ \text{F m}^{-1}} = 0.0100\ \text{m}^2.$$

The intercept is $100.5 - 88.5 = 12.0\ \text{pF}$. The third point checks the same line: $88.5(0.50) + 12.0 = 56.25\ \text{pF}$.

Interpretation. At $d = 2\ \text{mm}$, ignoring the offset would overestimate the active capacitance by $12/44.25 = 27.1\%$. More points and realistic uncertainties would be needed to test the model experimentally; exact agreement here is built into the synthetic data.

5.16.3 Plate tilt as a systematic geometry error

Suppose rectangular plates have width w , length L , and a gap that varies slowly along the length as $d(x) = d_0 + \alpha x$, with $0 \leq x \leq L$. Here d_0 is the gap at one end, and the slope α is dimensionless. When $|\alpha| \ll 1$ and the local gap is small relative to the plate dimensions, narrow strips of area $w dx$ can be treated as capacitors in parallel. Thus

$$C \simeq \varepsilon w \int_0^L \frac{dx}{d_0 + \alpha x} = \frac{\varepsilon w}{\alpha} \ln \left(1 + \frac{\alpha L}{d_0} \right). \quad (5.103)$$

This is a local plane-gap approximation, not an exact solution of a wedge-shaped boundary-value problem. It neglects the small transverse field and the outer fringing field. The gap must remain positive everywhere.

Let $\eta = \alpha L/d_0$. Dividing by $C_0 = \varepsilon w L/d_0$ gives $C/C_0 = \ln(1 + \eta)/\eta$. For $|\eta| \ll 1$,

$$\frac{C}{C_0} = 1 - \frac{\eta}{2} + \frac{\eta^2}{3} + \dots \quad (5.104)$$

The first-order correction depends on defining d_0 at an *end*. If the measured quantity is instead the mean gap $\bar{d} = d_0 + \alpha L/2$, comparison with $C_{\text{mean}} = \varepsilon w L/\bar{d}$ gives

$$\frac{C}{C_{\text{mean}}} = 1 + \frac{1}{12} \left(\frac{\alpha L}{\bar{d}} \right)^2 + \dots \quad (5.105)$$

The capacitance is slightly larger than a uniform-gap estimate using the mean, because the narrow region contributes more strongly than the wider region suppresses it. Stating where the gap was measured is therefore part of the physics, not merely laboratory paperwork.

Example 5.21: Tilt bias versus random measurement uncertainty

An air-gap fixture has $w = L = 0.100$ m, with end gaps 1.00 mm and 1.10 mm. Estimate its capacitance using the slowly varying gap model and compare two uniform-gap estimates.

Solution. The gap slope is $\alpha = 0.10 \text{ mm}/100 \text{ mm} = 0.0010$. Equation (5.103) gives

$$C = \frac{(8.85 \times 10^{-12})(0.100)}{0.0010} \ln(1.10) \text{ F} = 84.35 \text{ pF}.$$

Using the small end gap alone predicts 88.50 pF, about 4.92% too high. Using the mean gap 1.05 mm predicts 84.29 pF, only about 0.076% low within this model.

Interpretation. Repeating a precise gap measurement at the same end cannot remove the bias caused by tilt. The remedy is to measure or constrain the geometry, not simply to average more capacitance readings. The remaining fringe correction is a separate systematic effect.

5.16.4 Measuring relative permittivity and its uncertainty

Compare an empty and filled fixture at the same separation and with the same leads. Let M_0 and M_1 denote the two measured capacitances, and C_s the common stray term. Full filling gives

$$\kappa = \frac{M_1 - C_s}{M_0 - C_s}. \quad (5.106)$$

The cancellation of A/d is useful, but it requires unchanged geometry. Inserting a sheet that bends the plates, leaves air gaps or changes the lead position invalidates that cancellation. Record the meter frequency and voltage amplitude: an inferred permittivity belongs to those measurement conditions.

For small independent standard uncertainties $u(M_0)$, $u(M_1)$ and $u(C_s)$, first-order propagation gives

$$u(\kappa)^2 = \frac{u(M_1)^2 + \kappa^2 u(M_0)^2 + (\kappa - 1)^2 u(C_s)^2}{(M_0 - C_s)^2}. \quad (5.107)$$

The same C_s enters numerator and denominator, so its derivative is $(\kappa - 1)/(M_0 - C_s)$. Treating the two stray corrections as independent would be wrong. If the meter readings themselves have common calibration errors, their covariance must also be included rather than assuming independence.

Example 5.22: A corrected dielectric-constant estimate

Suppose $M_0 = 112$ pF, $M_1 = 412$ pF and $C_s = 12$ pF. Independent standard uncertainties for this illustrative calculation are 1, 2 and 1 pF, respectively. The corrected result is $\kappa = 400/100 = 4.00$. Equation (5.107) gives

$$u(\kappa) = \frac{\sqrt{2^2 + 4^2(1)^2 + 3^2(1)^2}}{100} = 0.054.$$

Thus report $\kappa = 4.00 \pm 0.05$ (one standard uncertainty under the stated model). The uncorrected ratio $412/112 = 3.68$ has a systematic bias far larger than this uncertainty.

Model limitation. This uncertainty budget excludes unknown air gaps. It is not a guarantee of accuracy if the full-fill model is false. Measurement repeatability and model validity answer different questions.

For a geometry-based prediction $C = \epsilon_0 L^2/d$, independent small uncertainties give

$$\left(\frac{u(C)}{C}\right)^2 = \left(2\frac{u(L)}{L}\right)^2 + \left(\frac{u(d)}{d}\right)^2. \quad (5.108)$$

With $L = (100.0 \pm 0.2)$ mm and $d = (1.00 \pm 0.03)$ mm, interpreted as standard uncertainties, the relative uncertainty is $\sqrt{0.004^2 + 0.030^2} = 3.03\%$. The nominal value 88.5 pF therefore has $u(C) \simeq 2.7$ pF before fringe and stray effects. Precision in plate area cannot compensate for poor knowledge of a very small gap.

Method

A useful low-voltage laboratory study records capacitance at several separations, fits an explicit offset model, inspects residuals, and repeats readings after reversing the order of separations. For a dielectric measurement, measure thickness and repeat empty/filled readings without moving the leads. Compare the influence of plate pressure, cable position and nearby objects. A guard electrode held at the active electrode potential can reduce edge contributions to the *measured active-electrode charge*, provided the guard current is separately excluded; simply connecting a larger plate to the measured terminal changes the measured area.

5.17 Computational connection: capacitor networks as linear systems

For simple series-parallel networks, hand reduction is efficient. For general networks with many nodes, a systematic computational method is better. Capacitors connected between nodes behave, in electrostatic equilibrium, like charge-voltage relations. If a capacitor C_{ij} connects node i to node j , the charge contribution at node i is

$$q_i^{(ij)} = C_{ij}(V_i - V_j). \quad (5.109)$$

Adding contributions from all capacitors connected to node i gives

$$q_i = \sum_j C_{ij}(V_i - V_j), \quad (5.110)$$

where the sum is over neighbouring nodes. This is a linear system for the node potentials when node charges and boundary voltages are specified.

Method

A computational network model should preserve charge conservation at every floating node. For an isolated internal conductor, the algebraic sum of charges on all connected capacitor plates must remain fixed. This condition is the capacitor-network analogue of Kirchhoff's current law in DC circuits.

Example 5.23: Two-node floating conductor calculation

A capacitor $C_1 = 2.0 \mu\text{F}$ connects a fixed 12 V node to a floating node X . A capacitor $C_2 = 4.0 \mu\text{F}$ connects node X to ground. The floating conductor at X has zero net charge initially. Find V_X .

Solution. The net charge on node X is the sum of the charges on the plates connected to X :

$$q_X = C_1(V_X - 12 \text{ V}) + C_2(V_X - 0) = 0. \quad (5.111)$$

Therefore

$$(C_1 + C_2)V_X = C_1(12 \text{ V}). \quad (5.112)$$

Thus

$$V_X = \frac{C_1}{C_1 + C_2} 12 \text{ V} = \frac{2.0}{6.0} 12 \text{ V} = 4.0 \text{ V}. \quad (5.113)$$

Interpretation. This is the capacitive voltage-divider formula. The floating node is closer to ground because the capacitance to ground is larger.

Note

This nodal formulation is the natural bridge to circuit simulation. In time-dependent circuits, capacitor current is $I = C d(\Delta V)/dt$, and the same nodal structure becomes a system of differential equations. That extension belongs to later circuit chapters.

5.17.1 A bridge that cannot be reduced by elementary rules

Figure 5.12 contains five capacitors and two floating nodes. Neither floating node has only two connections, so the central branch prevents a direct series reduction. Let every branch value be a positive physical capacitance, measured here in μF , and take both floating nodes initially neutral.

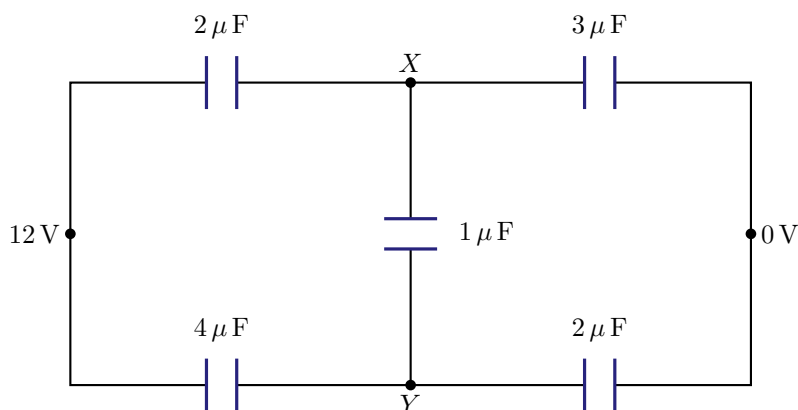


Figure 5.12: A five-capacitor bridge. The central $1\mu\text{F}$ capacitor connects two floating nodes and makes their charge equations coupled. All branch values are shown; the outside terminals are fixed.

Example 5.24: Solving a two-floating-node bridge

For Fig. 5.12, use charge conservation to determine V_X , V_Y and the terminal capacitance.

Equations. Expressing capacitances in μF and voltages in volts, the neutral-node conditions are

$$\begin{aligned} 2(V_X - 12) + 3V_X + (V_X - V_Y) &= 0, \\ 4(V_Y - 12) + 2V_Y + (V_Y - V_X) &= 0. \end{aligned}$$

Therefore $6V_X - V_Y = 24$ and $-V_X + 7V_Y = 48$. Solving,

$$V_X = \frac{216}{41} \text{ V} = 5.268 \text{ V}, \quad V_Y = \frac{312}{41} \text{ V} = 7.610 \text{ V}.$$

The central capacitor has $V_Y - V_X = 96/41 \text{ V}$; its Y plate is positive. The source-terminal charge is

$$Q_A = 2(12 - V_X) + 4(12 - V_Y) = \frac{1272}{41} \mu\text{C}. \quad (5.114)$$

Consequently $C_{\text{eq}} = Q_A/(12 \text{ V}) = 106/41 \mu\text{F} = 2.585 \mu\text{F}$.

Independent check. Summing $\frac{1}{2}C_{ij}(V_i - V_j)^2$ over all five branches gives $U = 186.15 \mu\text{J}$, equal to $\frac{1}{2}C_{\text{eq}}(12 \text{ V})^2$. Both node potentials lie between the imposed boundary values, as expected for this passive neutral network.

A minimal numerical implementation is shown below. Capacitances are in farads, node charges in coulombs and potentials in volts. The right-hand side changes if an island has specified nonzero charge; the matrix of capacitances does not.

```
import numpy as np
M = 1e-6 * np.array([[6.0, -1.0], [-1.0, 7.0]])
b = 1e-6 * np.array([24.0, 48.0])
VX, VY = np.linalg.solve(M, b)
QA = 2e-6*(12-VX) + 4e-6*(12-VY)
Ceq = QA/12
print(VX, VY, Ceq)
assert np.max(np.abs(M @ np.array([VX, VY]) - b)) < 1e-15
```

The residual is a charge imbalance, so its tolerance has units of coulombs. A small residual tests the algebraic solve, while the separate energy and topology checks test different possible mistakes.

This is more informative than merely obtaining a number from a matrix routine.

5.18 Engineering applications and physical interpretation

Applications can be organized by the quantity a capacitor controls: energy, a temporary charge supply, or a change in geometry or material. Each use starts from the same electrostatic relation, but the relevant constraint differs.

5.18.1 Charge reserve and energy reserve are different specifications

A circuit may require a known charge δQ while allowing only a small voltage decrease δV . For fixed capacitance,

$$C \geq \frac{\delta Q}{\delta V}. \quad (5.115)$$

If a load instead needs energy W while the capacitor falls from V_i to V_f , the requirement is

$$C \geq \frac{2W}{V_i^2 - V_f^2}. \quad (5.116)$$

Using $W = \frac{1}{2}CV_i^2$ would incorrectly count the energy still present at the allowed final voltage. Both estimates also ignore resistance, inductance and conversion loss, which determine whether the stored energy can be delivered on the required time scale.

Example 5.25: Two ways of specifying a supply capacitor

(a) A digital load needs 30 nC of extra charge and permits 50 mV of droop. (b) An ideal energy buffer must deliver 0.20 J while its voltage falls from 24 V to 20 V. Find the minimum capacitance for each independent specification.

Solution. In (a), $C_{\min} = 30 \text{ nC}/0.050 \text{ V} = 0.60 \mu\text{F}$. In (b),

$$C_{\min} = \frac{0.40 \text{ J}}{(24^2 - 20^2) \text{ V}^2} = 2.27 \text{ mF}.$$

At the latter capacitance the initial energy is 0.655 J and the final energy is 0.455 J, whose difference is the required 0.20 J.

Interpretation. The two answers address different loads and tolerances. Capacitance alone does not specify response time or allowable current, and the values are ideal lower bounds rather than qualified component selections.

5.18.2 Displacement sensing and linearization

For a separation sensor $C(d) = \varepsilon A/d$, define a signed displacement δd about d_0 . A first-order Taylor expansion gives

$$C(d_0 + \delta d) \simeq C_0 \left(1 - \frac{\delta d}{d_0}\right), \quad C_0 = \frac{\varepsilon A}{d_0}. \quad (5.117)$$

The next relative term is $(\delta d/d_0)^2$. This estimates the error of the linearization without requiring an experiment. A sensor that changes the overlap $A = wx$ at fixed gap instead has $C(x) = \varepsilon wx/d$ and constant ideal sensitivity $dC/dx = \varepsilon w/d$. In both cases the useful change may be much smaller than the stray capacitance.

The measurement circuit determines the observed output. With fixed free charge, a plane separation sensor has $\Delta V = Qd/(\varepsilon A)$, which is linear in d . With fixed applied voltage, it produces a charge change instead. Neither description may be substituted for the other merely because the mechanical motion is the same.

Example 5.26: Sensor sensitivity and approximation error

An air-gap sensor has $A = 25 \text{ mm}^2$ and $d_0 = 50 \mu\text{m}$. Estimate its response to a $+1 \mu\text{m}$ increase in gap.

Solution. $C_0 = 4.425 \text{ pF}$ and $dC/dd = -C_0/d_0 = -0.0885 \text{ pF } \mu\text{m}^{-1}$. The linear estimate is $C \simeq 4.3365 \text{ pF}$. The exact plane model gives $C = C_0(50/51) = 4.3382 \text{ pF}$. Their difference is 0.0017 pF , approximately 0.04% of the baseline, consistent with $(1/50)^2$.

Interpretation. The approximation error is small here, but an uncorrected 1 pF stray contribution is much larger than the entire displacement signal. Good sensor design must consider both the mathematical approximation and the measurement environment.

5.18.3 Small charge signals and membranes

A packet Q_s deposited on an electrically floating sensing node changes its potential by $\delta V = Q_s/C_{\text{node}}$ if leakage and feedback are negligible during the observation. For example, 16000 elementary charges correspond to 2.56 fC ; on 5.0 pF they produce a 0.512 mV step. This illustrates why input capacitance matters for voltage sensing. A charge-sensitive amplifier with feedback is a different circuit: its output conversion is primarily set by its feedback capacitance, and must not automatically be equated to Q_s/C_{input} .

A biological membrane provides another useful scale estimate. Modelling it as a 5 nm thick insulating slab with assumed $\kappa = 5$ gives

$$\frac{C}{A} = \frac{\kappa\epsilon_0}{d} = 8.85 \times 10^{-3} \text{ F m}^{-2} = 0.885 \mu\text{F cm}^{-2}. \quad (5.118)$$

A 20 pF patch acquiring 2.0 pC changes voltage by 0.10 V . These are illustrative passive-model estimates; ion channels, ionic screening and pumps require additional physics.

5.18.4 Material choice and voltage sharing

At a specified permissible local field, the ideal energy density is $u_{\text{max}} = \frac{1}{2}\kappa\epsilon_0 E_{\text{max}}^2$. Therefore a larger dielectric constant is not enough to establish a larger usable energy density. Compare the product κE_{max}^2 , then consider loss, temperature, leakage, electrode volume and reliability. The dielectric-only field volume is not the volume of the complete packaged device.

For series capacitors with initially neutral nodes, the largest permissible common charge is

$$Q_{\text{max}} = \min_i(C_i V_{i,\text{rated}}), \quad V_{\text{stack,max}} = Q_{\text{max}} \sum_i \frac{1}{C_i}. \quad (5.119)$$

Simply adding ratings is valid only if the resulting division keeps every element within its limit. In a long-time DC connection, leakage can control the final voltage sharing instead of the initial capacitance ratios. Balancing circuits are therefore a separate requirement from the ideal electrostatic calculation.

Example 5.27: A series stack limited by one component

Two initially uncharged capacitors, $2.0 \mu\text{F}$ and $4.0 \mu\text{F}$, are each rated at 100 V . What is the ideal maximum stack voltage under capacitance-controlled sharing?

Solution. The common charge cannot exceed $\min(200, 400) \mu\text{C} = 200 \mu\text{C}$. At this charge the voltages are 100 V and 50 V , so the stack limit is 150 V , not 200 V . Its equivalent capacitance is $4/3 \mu\text{F}$, and its energy at the limit is 0.015 J .

Check. The individual energies are 0.010 J and 0.0050 J . At a proposed 200 V stack voltage the smaller capacitance would have 133 V and exceed its rating. Component tolerances and leakage require further margin in an actual design.

Pedagogical boundary

Filters, RC timing and the frequency response of real capacitors require current, resistance and inductance. They are developed in later chapters. The present energy and charge calculations determine possible states and ideal bounds; they do not determine the entire transient waveform or certify a real device.

5.19 Common mistakes and how to avoid them

1. **Calling capacitance stored charge.** Capacitance is $Q/\Delta V$, not Q . It is measured in farads, not coulombs.
2. **Using total charge of a two-plate capacitor.** The net charge of an ideal capacitor is zero, but Q in $C = Q/\Delta V$ is the magnitude on one plate.
3. **Confusing series and parallel rules.** Parallel capacitors share voltage and add directly. Series capacitors with initially neutral floating nodes have equal charge magnitudes and add by reciprocals.
4. **Forgetting the battery.** If a battery remains connected, it fixes ΔV and can supply charge. If the capacitor is disconnected, Q is fixed.
5. **Treating dielectric bound charge as free charge.** Bound charge arises from polarization. It does not flow through the external circuit.
6. **Ignoring units.** Microfarads, nanofarads and picofarads differ by factors of 10^3 . A unit error can change energy by orders of magnitude.
7. **Overusing ideal formulas.** The formula $C = \epsilon_0 A/d$ assumes negligible fringing, parallel plates and a uniform dielectric.
8. **Forgetting voltage ratings.** A large capacitance is not useful if the dielectric breaks down at the intended voltage.

Method

Before finalising a capacitor calculation, perform three checks: dimensional consistency, constraint consistency and limiting behaviour. A correct expression for capacitance should have units of farads, should respect the stated fixed quantity, and should behave sensibly when A becomes larger, d becomes smaller or κ increases.

5.20 Main-chapter synthesis

The main chapter can be summarised as a chain of interpretations. A pair of conductors can store separated charge. Separated charge creates an electric field. The field gives a potential difference. The ratio of charge magnitude to potential difference is capacitance. Work is required to build the separated-charge state, and that work is stored as electric-field energy. A dielectric changes the field-charge relation by polarization and therefore changes the capacitance.

Summary

- Capacitance is defined by $C = Q/\Delta V$ for a two-conductor system.
- For ideal parallel plates in vacuum, $C = \varepsilon_0 A/d$.
- For a long cylindrical capacitor, $C = 2\pi\varepsilon_0 L / \ln(b/a)$.
- For concentric spheres, $C = 4\pi\varepsilon_0 ab/(b-a)$; for an isolated sphere, $C = 4\pi\varepsilon_0 R$.
- Parallel capacitors share voltage and their capacitances add directly.
- A simple series chain with initially neutral floating nodes has equal charge magnitudes; its inverse capacitances add.
- The energy stored in a linear capacitor is $U = Q^2/(2C) = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}Q\Delta V$.
- Electric-field energy density is $u_E = \frac{1}{2}\varepsilon E^2$ in a linear medium.
- A dielectric fully filling an ideal capacitor increases capacitance by κ .
- Under fixed Q , dielectric insertion reduces voltage and stored energy; under fixed ΔV , it increases charge and stored energy.

Formula summary

$$C = \frac{Q}{\Delta V}, \quad (5.120)$$

$$C_{\text{pp}} = \varepsilon_0 \frac{A}{d} \quad (\text{parallel plates, vacuum}), \quad (5.121)$$

$$C_{\text{cyl}} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}, \quad (5.122)$$

$$C_{\text{sph}} = 4\pi\varepsilon_0 \frac{ab}{b-a}, \quad (5.123)$$

$$C_{\text{isolated sphere}} = 4\pi\varepsilon_0 R, \quad (5.124)$$

$$C_{\parallel} = \sum_i C_i, \quad (5.125)$$

$$\frac{1}{C_{\text{series}}} = \sum_i \frac{1}{C_i}, \quad (5.126)$$

$$U = \frac{Q^2}{2C} = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}Q\Delta V, \quad (5.127)$$

$$u_E = \frac{1}{2}\varepsilon E^2, \quad (5.128)$$

$$C_{\text{dielectric}} = \kappa C_0 \quad (\text{full filling, ideal linear dielectric}). \quad (5.129)$$

5.21 Advanced Topics for Technical and Engineering Students

The elementary models assume a simple geometry, a linear material and a specified constraint. This section relaxes those restrictions one at a time. The purpose is to understand where a capacitance comes from, how it produces mechanical forces, and how to calculate it when a single series/parallel reduction is impossible. The main prerequisites are derivatives, line integrals and basic linear algebra; the additional vector calculus is introduced where needed.

5.21.1 Polarization charge and dielectric boundary conditions

A spatially uniform polarization has cancelling dipoles in the bulk, but a nonuniform polarization need not. Imagine a small closed volume in which more dipole moment points outward through one face than enters through the opposite face. The resulting imbalance is described by the bound volume charge density

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}. \quad (5.130)$$

The volume term has units C m^{-3} and the surface term has units C m^{-2} . Applying the divergence theorem inside a finite dielectric body gives $\int \rho_b d\tau + \oint \sigma_b dA = 0$ when all of its polarization charge is included. Polarization separates charge rather than adding net charge to an initially neutral dielectric.

Starting from $\epsilon_0 \nabla \cdot \mathbf{E} = \rho_f + \rho_b$ and substituting Eq. (5.130),

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f, \quad \nabla \cdot \mathbf{D} = \rho_f. \quad (5.131)$$

This derivation explains why Gauss's law for $\mathbf{D}$ contains only free charge without asserting that the bound charge has disappeared.

At a flat boundary, take the unit normal $\hat{\mathbf{n}}$ from medium 1 into medium 2. A vanishingly thin pillbox gives

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_{f,\text{interface}}. \quad (5.132)$$

A narrow rectangular loop straddling the interface gives a second condition, because the electrostatic field is conservative:

$$\mathbf{E}_{2t} = \mathbf{E}_{1t}. \quad (5.133)$$

Here the subscript t means the component tangent to the interface. With no free surface charge, normal D is continuous; tangential E is continuous. These statements do *not* say that the complete field vector is continuous.

For isotropic media and field angles θ_i measured from the normal, the two conditions imply

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}, \quad \frac{\tan \theta_2}{\tan \theta_1} = \frac{\epsilon_2}{\epsilon_1}. \quad (5.134)$$

This is a geometric relation for static field components, not an optical refraction law. At a conductor boundary the tangential field vanishes and the exterior normal displacement equals the free surface charge density, with the normal pointing out of the metal.

For a nonuniform linear dielectric, combining $\mathbf{E} = -\nabla V$ and $\mathbf{D} = \epsilon(\mathbf{r})\mathbf{E}$ yields

$$\nabla \cdot [\epsilon(\mathbf{r})\nabla V] = -\rho_f. \quad (5.135)$$

Only in a charge-free region of *constant* permittivity does this reduce to $\nabla^2 V = 0$. Applying the ordinary equal-neighbor Laplace update across a permittivity jump without modification can violate displacement continuity and produce a wrong capacitance.

Example 5.28: Checking the bound charge at an interface

Two plane layers carry the same downward displacement D and have $\kappa_1 = 2$ above and $\kappa_2 = 6$ below, with no free interfacial charge. Find their fields and the net bound charge density at the contact surface.

Solution. Let the downward normal be $\hat{n}$. Then $E_1 = D/(2\epsilon_0)$ and $E_2 = D/(6\epsilon_0)$, so $E_1 = 3E_2$. The polarizations are $P_1 = D/2$ and $P_2 = 5D/6$. The bottom face of layer 1 contributes $+P_1$, while the top face of layer 2 contributes $-P_2$. Thus

$$\sigma_{b,\text{net}} = P_1 - P_2 = -D/3.$$

As a check, the jump in electric field gives $\epsilon_0(E_{2n} - E_{1n}) = -D/3$, exactly the same total interfacial charge. The displacement remains continuous because that charge is bound, not free.

5.21.2 Breakdown in curved geometries

The approximation $E = \Delta V/d$ is specific to a uniform plane gap. A coaxial capacitor has

$$E(r) = \frac{\Delta V}{r \ln(b/a)}, \quad E_{\text{max}} = E(a) = \frac{\Delta V}{a \ln(b/a)}. \quad (5.136)$$

If a uniform dielectric has a specified allowable field E_{allow} in the intended model, the voltage constraint is

$$\Delta V \leq E_{\text{allow}} a \ln(b/a). \quad (5.137)$$

The largest field occurs at the inner conductor, not at the average radial position. This is why a very thin wire can be a poor high-voltage electrode even if the outer conductor is far away.

Now hold the outer radius b fixed and ask which inner radius maximizes the ideal allowable voltage. Differentiate $f(a) = a \ln(b/a)$:

$$f'(a) = \ln(b/a) - 1, \quad f''(a) = -1/a < 0. \quad (5.138)$$

The optimum is therefore $a = b/e$ and $\Delta V_{\text{max}} = E_{\text{allow}} b/e$. Making the inner wire either much thinner or nearly as large as the outer bore reduces the withstand voltage for fixed b . This is a concrete example in which maximizing capacitance and maximizing voltage rating are different design goals.

Example 5.29: Choosing a coaxial radius ratio

A long coaxial fixture has fixed outer-conductor inner radius $b = 5.0$ mm. An allowable dielectric field of 10 MV m^{-1} is specified for this idealized calculation. Find the inner radius that maximizes allowable voltage and compare with $a = 0.50$ mm.

Solution. The optimum is $a = b/e = 1.84$ mm and the ideal voltage limit is 18.4 kV. With the thinner inner conductor,

$$\Delta V_{\text{max}} = (10^7)(0.50 \times 10^{-3}) \ln(10) \text{ V} = 11.5 \text{ kV}.$$

The average-gap estimate $E_{\text{allow}}(b - a) = 45 \text{ kV}$ for that thinner wire is much too large because it ignores the radial field enhancement.

Check. At the optimum the logarithm is one, so $C' = 2\pi\epsilon$. The optimum voltage depends on geometry and the specified field limit; the capacitance additionally depends on permittivity. End terminations, roughness and defects are excluded and would need separate analysis.

For a spherical capacitor, the corresponding relations are

$$E(a) = \frac{Q}{4\pi\epsilon a^2} = \frac{\Delta V b}{a(b-a)}, \quad \Delta V_{\max} = E_{\text{allow}} \frac{a(b-a)}{b}. \quad (5.139)$$

At fixed b , this ideal limit is maximized at $a = b/2$, giving $E_{\text{allow}}b/4$. As $b \rightarrow \infty$, it becomes the isolated-sphere estimate $\Delta V_{\max} = E_{\text{allow}}a$. These results are field-based estimates under stated assumptions, not empirical universal breakdown voltages.

5.21.3 Continuously graded dielectrics

Layered formulas extend naturally to a continuously varying permittivity. Let z run from the lower to the upper plate of a plane capacitor, with $\epsilon(z) > 0$, no free charge in the gap, and negligible fringing. The magnitude $D = Q/A$ is constant, so

$$\Delta V = \frac{Q}{A} \int_0^d \frac{dz}{\epsilon(z)}, \quad \boxed{C = \frac{A}{\int_0^d dz/\epsilon(z)}}. \quad (5.140)$$

The effective permittivity is a harmonic, not arithmetic, average along the field. A narrow region of low permittivity can therefore control the result. Lateral regions spanning the entire gap instead contribute in parallel and use an area-weighted arithmetic average in the ideal plane geometry.

For $\epsilon(z) = \epsilon_0\kappa_0(1 + \alpha z/d)$ with $\alpha > -1$,

$$C = \frac{\epsilon_0\kappa_0 A}{d} \frac{\alpha}{\ln(1 + \alpha)}. \quad (5.141)$$

The apparent $0/0$ at $\alpha = 0$ is removable: $\ln(1 + \alpha) = \alpha - \alpha^2/2 + \dots$, so $C \rightarrow \epsilon_0\kappa_0 A/d$. If the permittivity increases with z , the field is strongest at the lower-permittivity end even though D is constant.

The same method works radially. For a long coaxial capacitor with $\epsilon = \epsilon(r)$,

$$D_r = \frac{Q}{2\pi L r}, \quad C = \frac{2\pi L}{\int_a^b \frac{dr}{\epsilon(r)r}}. \quad (5.142)$$

A homogeneous medium returns the logarithmic formula. This is useful in graded insulation, but one must still check local fields and material limits rather than inferring robustness from a larger capacitance alone.

5.21.4 Field energy beyond uniform plates

For a reversible linear material at fixed geometry, build the displacement field from zero while keeping the constitutive response linear. The incremental work density is $du = \mathbf{E} \cdot d\mathbf{D}$, hence

$$u = \frac{1}{2} \mathbf{E} \cdot \mathbf{D} = \frac{1}{2} \epsilon E^2, \quad U = \frac{1}{2} \int \epsilon |\nabla V|^2 d\tau. \quad (5.143)$$

For linear anisotropic media, $\mathbf{D} = \epsilon \mathbf{E}$ and the dot-product form is retained; the scalar formula requires isotropy. For a nonlinear reversible material the integral must be evaluated using the full constitutive relation. Hysteresis generally prevents a single energy function depending only on the instantaneous field.

A useful check is to recover a known capacitance from its nonuniform field. For a coaxial capacitor in a homogeneous dielectric,

$$\begin{aligned} U &= \int_a^b \frac{1}{2} \epsilon \left(\frac{Q}{2\pi \epsilon L r} \right)^2 (2\pi r L dr) \\ &= \frac{Q^2}{4\pi \epsilon L} \ln \frac{b}{a} = \frac{Q^2}{2C}. \end{aligned} \quad (5.144)$$

The factor $2\pi rL$ belongs to the cylindrical volume element; replacing it by A independent of r would be incorrect. For a spherical capacitor the corresponding shell volume is $4\pi r^2 dr$, giving

$$U = \frac{Q^2}{8\pi\epsilon} \left(\frac{1}{a} - \frac{1}{b} \right). \quad (5.145)$$

These integrals agree with the charging-work result even though the energy density is far from uniform.

For an isolated sphere, $b \rightarrow \infty$. Integrating only from R to a radius r shows that the fraction of its energy inside that radius is $1 - R/r$. Half of the energy lies between R and $2R$, and half lies farther out. The capacitance of an isolated conductor therefore depends on a field that extends well beyond the metal itself.

Theoretical point 5.1: Capacitance from a field solution

For a balanced two-terminal excitation with voltage difference ΔV , compute the entire energy associated with that excitation and use

$$C = \frac{2U}{(\Delta V)^2}.$$

In a translationally invariant two-dimensional model, the integral gives energy *per unit length*, U' , and the result is $C' = 2U'/(\Delta V)^2$ in F m^{-1} . It is not a total three-dimensional capacitance. Multiplying by a length is justified only when end effects are negligible.

5.21.5 Nonlinear charge-voltage relations

A measured capacitor may have a smooth nonlinear law $Q = Q(v)$, where v is a signed terminal voltage and $Q(0) = 0$. Two different capacitances can be defined:

$$C_{\text{sec}}(v) = \frac{Q(v)}{v}, \quad C_{\text{diff}}(v) = \frac{dQ}{dv}. \quad (5.146)$$

The secant capacitance relates the complete state to the origin. The differential capacitance relates a small extra charge to a small voltage perturbation about a chosen bias: $\delta Q \simeq C_{\text{diff}}(v_0)\delta v$. They coincide only when the characteristic is linear through the origin.

For reversible charging at fixed geometry, the work is

$$U(v) = \int_0^{Q(v)} v(q) dq = \int_0^v v' \frac{dQ(v')}{dv'} dv'. \quad (5.147)$$

Integration by parts gives $U(v) = vQ(v) - \int_0^v Q(v') dv'$. The second integral is coenergy, denoted U^* . Thus $U + U^* = vQ$; for a linear capacitor both equal $\frac{1}{2}Cv^2$, but this equality must not be assumed for nonlinear devices.

Example 5.30: Differential capacitance is not an energy formula

A reversible model has $Q = av + bv^3$, where $a = 20 \text{ pF}$ and $b = 0.010 \text{ pF V}^{-2}$. Find both capacitances and the stored energy at $v = 10 \text{ V}$.

Solution. $C_{\text{sec}} = a + bv^2 = 21 \text{ pF}$ and $C_{\text{diff}} = a + 3bv^2 = 23 \text{ pF}$. The stored energy is

$$U = \int_0^v v'(a + 3bv'^2) dv' = \frac{1}{2}av^2 + \frac{3}{4}bv^4 = 1.075 \text{ nJ}. \quad (5.148)$$

Neither $\frac{1}{2}C_{\text{sec}}v^2 = 1.050 \text{ nJ}$ nor $\frac{1}{2}C_{\text{diff}}v^2 = 1.150 \text{ nJ}$ is correct. The coenergy is $\frac{1}{2}av^2 + \frac{1}{4}bv^4 = 1.025 \text{ nJ}$, and its sum with U is $2.100 \text{ nJ} = vQ$.

Validity. This polynomial is an assumed local model, not a universal dielectric law. Its coefficients have the required units, and $C_{\text{diff}} > 0$ in the specified range. Hysteresis, leakage and mechanical motion are excluded.

5.21.6 Capacitance matrices: branches and physical conductors

Two matrices occur in this subject, and their entries should not be confused. In a lumped network, a branch capacitance $C_{ij} = C_{ji} \geq 0$ connects nodes i and j . Its contribution to node charge is $C_{ij}(V_i - V_j)$. The resulting nodal matrix has

$$M_{ii} = \sum_{j \neq i} C_{ij}, \quad M_{ij} = -C_{ij} \quad (i \neq j), \quad (5.149)$$

including connections to fixed boundary nodes in the diagonal. Separating floating nodes f from fixed nodes b gives

$$M_{ff}\mathbf{V}_f = \mathbf{q}_f - M_{fb}\mathbf{V}_b. \quad (5.150)$$

This is the system solved in Example 5.24. If every floating connected component reaches at least one fixed-potential node, M_{ff} is positive definite. If a complete component has no reference connection, a common shift of all its potentials is undetermined; the matrix is singular until a reference is supplied.

For any voltage vector, the branch energy is

$$U = \frac{1}{2} \sum_{i < j} C_{ij}(V_i - V_j)^2 \geq 0. \quad (5.151)$$

This expression explains both the sign pattern of M and why a common potential shift changes no energy when all reference nodes are shifted too. It also provides a strong numerical check.

A physical arrangement of conductors is more generally described by *Maxwell capacitance coefficients*, which we denote C_{ij}^M :

$$Q_i = \sum_j C_{ij}^M V_j, \quad U = \frac{1}{2} \sum_{i,j} C_{ij}^M V_i V_j. \quad (5.152)$$

The potentials must be defined relative to a specified reference boundary, often infinity. For passive linear reciprocal dielectrics the matrix is symmetric, diagonal entries are positive, and off-diagonal entries are nonpositive. These entries are coefficients of a boundary-value problem; a positive physical branch capacitance is not the same as an off-diagonal Maxwell coefficient.

Symmetry follows from electrostatic reciprocity. Let ϕ_i be the potential solution with conductor i at unit voltage and all other conductors and the reference at zero. In a scalar dielectric,

$$C_{ij}^M = \int \epsilon \nabla \phi_i \cdot \nabla \phi_j d\tau. \quad (5.153)$$

The integrand is symmetric under exchanging i, j . The energy expression is nonnegative because it is the integral of the squared total field. This is a physical consistency condition on a computed coefficient matrix, not an arbitrary numerical convention.

Setting conductor 1 at V and conductor 2 at zero does *not* by itself ensure $Q_1 = -Q_2$ in an open geometry with a third reference at infinity. Charge may also be associated with the exterior field. For a truly isolated pair constrained to $+Q, -Q$, write $P = (C^M)^{-1}$ for the two-conductor matrix relative to infinity. Then

$$V_1 - V_2 = (P_{11} + P_{22} - 2P_{12})Q, \quad C_{\text{pair}} = \frac{1}{P_{11} + P_{22} - 2P_{12}}. \quad (5.154)$$

A surrounding grounded enclosure is a third conductor and changes the coefficients. This distinction matters for sensors, detector strips and capacitance-to-ground measurements.

Guided checks

An additive change of potential zero shifts every potential, including the reference, and changes no field. Raising only two electrodes relative to a nearby fixed grounded shield is a physical common-mode excitation; it can change their charges and exterior fields. Gauge freedom and changing a boundary voltage are different operations.

5.21.7 Eliminating internal nodes by minimizing energy

There is a useful second route to the network equations. Hold all boundary potentials fixed and vary the potential of a neutral floating node. Differentiating the branch energy in Eq. (5.151) gives

$$\frac{\partial U}{\partial V_i} = \sum_j C_{ij}(V_i - V_j) = q_i. \quad (5.155)$$

Thus a neutral node settles at a stationary point of this quadratic function. Its Hessian is M_{ff} ; when every floating connected component reaches a reference node, the stationary point is a unique minimum. A node carrying specified charge $q_i^{(0)}$ instead makes $U - \sum_i q_i^{(0)} V_i$ stationary. This is a variational statement about solving potentials, distinct from minimizing mechanical energy while moving electrodes at fixed voltage.

For neutral floating nodes, eliminate $\mathbf{V}_f = -M_{ff}^{-1} M_{fb} \mathbf{V}_b$ from the boundary-charge equations. The reduced relation is

$$\mathbf{q}_b = M_{\text{eff}} \mathbf{V}_b, \quad M_{\text{eff}} = M_{bb} - M_{bf} M_{ff}^{-1} M_{fb}. \quad (5.156)$$

This operation is called a Schur complement. It provides a systematic equivalent network even when no local series/parallel simplification is visible. In numerical work, solve systems with M_{ff} rather than explicitly forming its inverse.

The five-capacitor bridge gives a revealing limiting example. If the central branch has capacitance $c = 0$, the upper and lower branches are independent series pairs. Their equivalent capacitance is

$$C_{\text{eq}}(0) = \frac{2 \cdot 3}{2 + 3} + \frac{4 \cdot 2}{4 + 2} = \frac{38}{15} \mu\text{F} = 2.533 \mu\text{F}. \quad (5.157)$$

As $c \rightarrow \infty$, any finite imbalance of V_X and V_Y would require unbounded branch charge or energy; the node equations force $V_X - V_Y \rightarrow 0$. The two left capacitors then act in parallel, as do the two right capacitors, giving

$$C_{\text{eq}}(\infty) = \frac{(2 + 4)(3 + 2)}{2 + 4 + 3 + 2} \mu\text{F} = \frac{30}{11} \mu\text{F} = 2.727 \mu\text{F}. \quad (5.158)$$

The $c = 1 \mu\text{F}$ result, $2.585 \mu\text{F}$, lies between these limits. The infinite-capacitance limit makes the two node potentials equal for this static response; it should not be confused with a statement that a real capacitor conducts DC through its dielectric.

One can go further without solving the entire family of bridges. Normalize the terminal voltage to V_0 and let $U_{\min}$ be the energy after the neutral floating potentials have been determined. Increasing the central branch by dc changes the minimized energy by

$$\frac{dU_{\min}}{dc} = \frac{1}{2}(V_X - V_Y)^2, \quad \frac{dC_{\text{eq}}}{dc} = \left(\frac{V_X - V_Y}{V_0} \right)^2 \geq 0. \quad (5.159)$$

The implicit changes in V_X, V_Y contribute no first-order term because the energy is stationary with respect to them. A branch joining equal-potential nodes therefore makes no first-order contribution to terminal capacitance. This result is useful for both sensitivity analysis and recognizing balanced bridges.

5.21.8 Electrostatic forces and the controlled variable

Let a mechanical coordinate x change the capacitance $C(x)$. Define F_x as the force exerted by the electric system in the positive x direction. For fixed free charge,

$$F_x = - \left(\frac{\partial U}{\partial x} \right)_Q = \frac{Q^2}{2C(x)^2} \frac{dC}{dx} = \frac{1}{2} [\Delta V(x)]^2 \frac{dC}{dx}. \quad (5.160)$$

Thus a reversible linear capacitor tends to move toward increasing capacitance. The voltage in the last expression is the instantaneous $Q/C(x)$; it is not a fixed constant throughout that motion.

At fixed voltage, the source performs work $\Delta V dQ = (\Delta V)^2 dC$. Of this, $dU = \frac{1}{2}(\Delta V)^2 dC$ increases the stored energy. The remaining work is $F_x dx$, giving

$$F_x = + \frac{1}{2} (\Delta V)^2 \frac{dC}{dx} \quad (\Delta V \text{ fixed}). \quad (5.161)$$

The algebraic force formula at a specified state is the same, but the force along a path differs because different variables are held fixed. Equivalently, the mechanical potential at fixed voltage is $U - \Delta V Q = -\frac{1}{2} C (\Delta V)^2$, which includes the source's energy change. Differentiating only the positive capacitor energy at fixed voltage gives the wrong force sign.

For plane plates, use $x = d$ as the separation. Then $dC/dd = -\epsilon A/d^2$, so

$$F_d = - \frac{Q^2}{2\epsilon A} \text{quad}(Q \text{ fixed}), \quad F_d = - \frac{1}{2} \epsilon A \frac{(\Delta V)^2}{d^2} \text{quad}(\Delta V \text{ fixed}). \quad (5.162)$$

Both components are negative: the force reduces the separation. The positive *magnitude* is $|F_d|$, and the pressure is $p = |F_d|/A = \frac{1}{2} \epsilon E^2$. For a vacuum gap this can also be obtained from $F = QE_{\text{other}}$: the opposite plate produces only $E/2$, since a plate must not exert a net translational force on itself. For a deformable dielectric, material stresses and changes of permittivity may require more than this fixed-material ideal model.

Example 5.31: Electrostatic force in a MEMS-style parallel-plate actuator

A plate of area $A = 1.0 \text{ mm}^2$ has an air gap $d = 2.0 \mu\text{m}$ and applied voltage 20 V. Estimate the instantaneous attractive force in the ideal plane model.

Solution. The field is $E = 10^7 \text{ V m}^{-1}$. The pressure and force are

$$p = \frac{1}{2} (8.85 \times 10^{-12}) (10^7)^2 \text{ Pa} = 442.5 \text{ Pa},$$

$$|F_d| = pA = (442.5 \text{ Pa}) (10^{-6} \text{ m}^2) = 4.43 \times 10^{-4} \text{ N} = 0.443 \text{ mN}.$$

Independent check. Here $C = 4.425 \text{ pF}$ and $U = 0.885 \text{ nJ}$; $U/d = 4.43 \times 10^{-4} \text{ N}$ agrees. The force points toward smaller d .

Limits. This is a force estimate at a specified geometry, not proof of a stable operating point or a qualified voltage. At micrometre gaps, surface condition and small-gap breakdown physics matter; a macroscopic air-breakdown rule must not be applied indiscriminately.

5.21.9 Force on a partially inserted dielectric

Consider a slab filling the gap over an insertion distance x within rectangular plates of width w and length L , with $0 < x < L$. Neglecting edge corrections to the total capacitance,

$$C(x) = \frac{\varepsilon_0 w}{d} [L + (\kappa - 1)x], \quad C'(x) = \frac{\varepsilon_0 w}{d} (\kappa - 1). \quad (5.163)$$

Figure 5.13 fixes the coordinate: increasing x means inserting more dielectric. Equations (5.160) and (5.161) then give

$$F_x|_{\Delta V} = \frac{\varepsilon_0 w (\kappa - 1)}{2d} (\Delta V)^2, \quad F_x|_Q = \frac{Q^2 \varepsilon_0 w (\kappa - 1)}{2d C(x)^2}. \quad (5.164)$$

For $\kappa > 1$ both are positive. The fixed-voltage force is approximately constant away from the entrance and exit; the fixed-charge force decreases as the slab enters and the voltage falls.

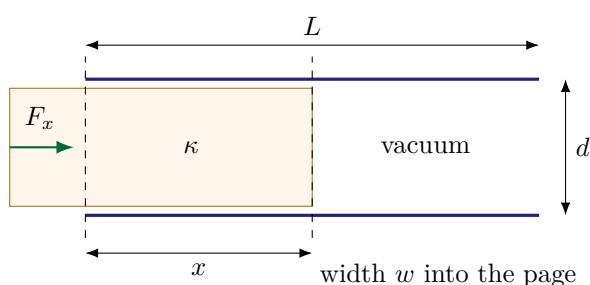


Figure 5.13: Lateral dielectric insertion with fixed plate gap. The filled and unfilled areas form a parallel decomposition. The slab force points toward increasing insertion x . The drawing is a cross-section; w is perpendicular to the page.

A subtle point is that the lateral force is produced in the nonuniform field near the slab entrance. A perfectly uniform infinite field cannot pull a neutral dipole sideways. The leading capacitance approximation nevertheless gives a useful total force from energy: it compares the energies of neighbouring configurations, whose changing edge region mediates the work. It does not resolve the local stress there. The constant-force result fails close to complete entry or exit, where the neglected fringe corrections are no longer a small relative correction to the derivative.

Example 5.32: Comparing two insertion protocols

Let $L = 50$ mm, $w = 20$ mm, $d = 1.0$ mm and $\kappa = 4$. The initial empty capacitor is charged to 100 V. Compare forces halfway through insertion for a connected and a disconnected source.

Solution. $C_0 = \varepsilon_0 w L / d = 8.85$ pF. At $x = L/2$, $C = 2.5C_0 = 22.125$ pF, and $C' = 5.31 \times 10^{-10} \text{ F m}^{-1}$. The connected-source force is

$$F_V = \frac{1}{2} (100 \text{ V})^2 C' = 2.66 \mu\text{N}.$$

For the disconnected capacitor $Q = C_0(100 \text{ V}) = 0.885$ nC and the halfway voltage is 40 V. Its force is $F_Q = \frac{1}{2} (40 \text{ V})^2 C' = 0.425 \mu\text{N}$, smaller by $(40/100)^2 = 0.16$.

Check. At the initial empty state the two voltages and forces coincide. The subsequent difference arises from the source constraint, not from a different local force law.

5.21.10 A reversible variable-capacitance cycle

A complete cycle is a useful test of the claim that a capacitor can deliver unlimited energy just by changing geometry. Let a movable device vary reversibly between C_a and $C_b > C_a$. Choose two

voltages $V_H > V_L \geq 0$. Consider four quasistatic legs: increase C from C_a to C_b at V_H ; lower the voltage from V_H to V_L at fixed C_b ; decrease C back to C_a at V_L ; then raise the voltage to V_H at fixed C_a .

During a geometry-changing leg, Eq. (5.161) gives mechanical work done by the electric system as $dW_{\text{mech}} = \frac{1}{2}V^2 dC$. During a fixed-geometry leg that work is zero. The net mechanical work per cycle is therefore

$$W_{\text{mech}} = \frac{1}{2}(C_b - C_a)(V_H^2 - V_L^2) > 0. \quad (5.165)$$

The capacitor returns to the same initial state, so its net stored-energy change is zero. The positive mechanical output must be supplied by net electrical work from the sources.

To verify this explicitly, use $dW_s = V dQ$. At fixed voltage, $dQ = V dC$; at fixed capacitance, $dQ = C dV$. The four source-work contributions are

$$\begin{aligned} W_{s,1} &= V_H^2(C_b - C_a), & W_{s,2} &= \frac{1}{2}C_b(V_L^2 - V_H^2), \\ W_{s,3} &= V_L^2(C_a - C_b), & W_{s,4} &= \frac{1}{2}C_a(V_H^2 - V_L^2). \end{aligned}$$

Their sum equals Eq. (5.165). Negative source work on a leg means that energy is returned to a reversible source. A real switch that directly connects a charged capacitor to a mismatched fixed voltage does not implement this quasistatic reversible leg; it introduces a transient and generally dissipation.

Table 5.2: Four states of the reversible variable-capacitance cycle. The charge is the signed charge on the positive-voltage terminal.

State	Capacitance	Voltage	Charge
A	C_a	V_H	$C_a V_H$
B	C_b	V_H	$C_b V_H$
C	C_b	V_L	$C_b V_L$
D	C_a	V_L	$C_a V_L$

Example 5.33: Energy accounting over a complete cycle

Take $C_a = 100 \text{ pF}$, $C_b = 300 \text{ pF}$, $V_H = 100 \text{ V}$ and $V_L = 40 \text{ V}$. The four state charges in Table 5.2 are 10, 30, 12 and 4 nC, respectively.

Mechanical output. The high-voltage motion produces $1.00 \mu\text{J}$, while the return motion requires $0.160 \mu\text{J}$. The net output is $0.840 \mu\text{J}$.

Source work. The four electrical contributions are $+2.00$, -1.26 , -0.320 and $+0.420 \mu\text{J}$, summing to $+0.840 \mu\text{J}$. The capacitor begins and ends with $0.500 \mu\text{J}$ stored, so its net change is zero.

Interpretation. The device converts electrical work into mechanical work by arranging different electrical constraints on the outward and return paths. Reversing the cycle makes it an ideal mechanical-to-electrical converter. Neither direction creates energy.

The electrical work $\oint V dQ$ is the oriented area enclosed by the cycle in a charge–voltage plot. This geometric interpretation is another reason to distinguish stored energy (a state function for the reversible model) from source work (which depends on the path). A plot and a list of endpoint capacitor energies answer different questions.

5.21.11 Spring-supported actuators and pull-in

A realistic movable electrode must also satisfy mechanical equilibrium. Let x be displacement toward a fixed plate, the initial gap be d_0 , and the spring restoring force be kx . With $d = d_0 - x$

and constant applied voltage V , the electrostatic attraction is

$$F_e(x) = \frac{\varepsilon_0 A V^2}{2(d_0 - x)^2}. \quad (5.166)$$

Equilibrium requires $kx = F_e(x)$. A stable equilibrium must also have a restoring response to a small perturbation: $k > dF_e/dx$. Combining the equality at marginal stability with the equilibrium condition gives

$$x_{\text{PI}} = \frac{d_0}{3}, \quad V_{\text{PI}} = \sqrt{\frac{8kd_0^3}{27\varepsilon_0 A}}. \quad (5.167)$$

To see the algebra, $dF_e/dx = \varepsilon_0 A V^2 / (d_0 - x)^3 = 2kx / (d_0 - x)$ at equilibrium. Setting this equal to k gives $d_0 - x = 2x$. The voltage source supplies more charge as the gap shrinks, reinforcing the attraction until the spring can no longer stabilize the plate.

Introduce $y = x/d_0$ and $\lambda = \varepsilon_0 A V^2 / (2kd_0^3)$. The equilibrium curve is $\lambda = y(1 - y)^2$, whose maximum is $4/27$ at $y = 1/3$. Figure 5.14 is generated from this equation. It distinguishes existence of an equilibrium from its stability.

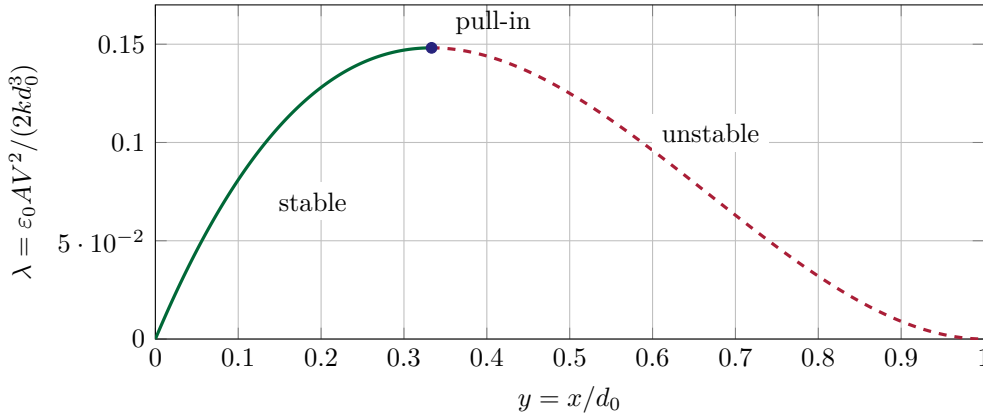


Figure 5.14: Equilibria of the ideal voltage-driven plane actuator. The solid branch is stable, the dashed branch unstable. Above $\lambda = 4/27$, there is no noncontact equilibrium in the model. Surface forces, finite plate rigidity and stops are omitted.

For $k = 1.0 \text{ N m}^{-1}$, $A = 1.0 \text{ mm}^2$ and $d_0 = 20 \mu\text{m}$, the model predicts $V_{\text{PI}} = 16.4 \text{ V}$ and $x_{\text{PI}} = 6.67 \mu\text{m}$. The dimensional check is $kd_0^3/(\varepsilon_0 A) \sim \text{V}^2$. Under ideal fixed-charge control the plane-model force $Q^2/(2\varepsilon_0 A)$ is independent of gap, so this particular feedback instability is absent before contact. That conclusion does not exclude instabilities from other mechanical or electrical effects.

5.21.12 Numerical electrostatics and fringing fields

When no useful symmetry remains, specifying the conductor boundaries is still enough to define an electrostatic problem. In a homogeneous charge-free region,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \quad (5.168)$$

On a square mesh of spacing h , replacing each second derivative by a central difference gives

$$4V_{i,j} - V_{i+1,j} - V_{i-1,j} - V_{i,j+1} - V_{i,j-1} = 0. \quad (5.169)$$

Each interior value equals the average of its four neighbors. Conductor nodes remain fixed. One can solve all equations as a sparse linear system or iteratively replace free nodes by neighbor averages. Convergence of the linear solve does not imply convergence with respect to grid spacing or the artificial outer boundary.

For a concrete dimensionless model, take strip electrodes $-\ell \leq x \leq \ell$ at $y = \pm\ell/4$, extending indefinitely perpendicular to the drawing. Hold the upper strip at $+\Delta V/2$ and the lower at $-\Delta V/2$. Place them inside a grounded square boundary $-B \leq x, y \leq B$, with uniform permittivity ε . Reflection symmetry makes the two electrode charges equal and opposite. The enclosure is part of the specified model; moving it farther away tests approach to an open exterior.

The elementary plane estimate is $C'_{pp} = \varepsilon W/d = 4\varepsilon$, with units F m^{-1} . Here the width-to-gap ratio is only four, so fringing is expected to be appreciable. Figure 5.15 shows equipotentials calculated from Eq. (5.169). A discrete energy gives a useful estimator without taking noisy derivatives at a

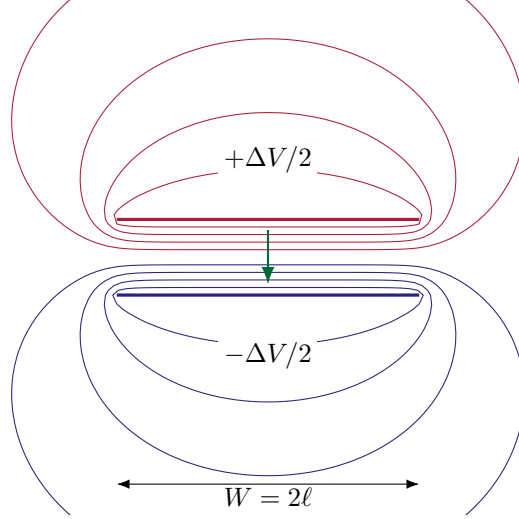


Figure 5.15: Computed equipotentials for two thin strip electrodes of width $W = 2\ell$, gap $d = 0.5\ell$, and potentials $\pm\Delta V/2$. Contours are at $V/\Delta V = \pm 0.1, \pm 0.2, \pm 0.3, \pm 0.4$. They are derived from the finite-difference solution on a square enclosure of half-width $B = 3\ell$ with grounded outer boundary and grid spacing $h = \ell/32$. Only the central region is shown. Contour bending near the tips exposes the fringing omitted by the plane model; the field is perpendicular to these curves.

sharp edge. Count each horizontal or vertical grid link once. In a two-dimensional square mesh,

$$U'_h = \frac{\varepsilon}{2} \sum_{\langle p,q \rangle} (V_p - V_q)^2, \quad C'_h = \frac{2U'_h}{(\Delta V)^2}. \quad (5.170)$$

The factors of h from the cell area cancel the $1/h^2$ in the squared gradient. The independent electrode-flux estimate is $Q'_+ = \varepsilon \sum (V_+ - V_q)$ over links leaving the upper electrode. For this symmetric unit-voltage excitation, $C'_h = Q'_+/\Delta V$. If the same discrete equations are used consistently, the two estimates agree to solver precision; this is a conservation check, not evidence that the continuum error is negligible.

Table 5.3: Computed strip-capacitor results for the stated finite-difference model. The flux and energy estimates agree to better than 10^{-11} in C'/ε . The displayed changes with grid and enclosure size are much larger than that algebraic residual.

B/ℓ	h/ℓ	Grid nodes	C'_h/ε
3	1/8	2401	5.75330
3	1/16	9409	5.64601
3	1/32	37249	5.59475
4	1/32	66049	5.54439
6	1/32	148225	5.50868

Table 5.3 separates two errors. At fixed enclosure $B = 3\ell$, halving h from $\ell/16$ to $\ell/32$ changes the result by about 0.91%. Keeping that finer mesh and enlarging B from 3ℓ to 6ℓ changes it by about

1.54%. Both effects are systematic. The final row still has unresolved grid and boundary error and is not an exact open-space capacitance. It nevertheless demonstrates why the ideal value 4ϵ misses an appreciable edge contribution for this width-to-gap ratio.

A reproducible sparse calculation

The following Python 3 implementation uses NumPy and SciPy. It produces every row of Table 5.3; no input files are needed. The supplied spacings align the electrode surfaces and endpoints with grid nodes. Other spacings require a separate boundary representation rather than silently moving the electrodes to nearby nodes.

The four-neighbour stencil places 4 on each unknown's diagonal, -1 in columns for unknown neighbours, and known neighbour potentials on the right-hand side. After solving, the code evaluates both upper-electrode flux and the sum of squared voltage differences across unique grid links. The returned numbers are dimensionless C'/ϵ , not a three-dimensional capacitance in farads.

```
import numpy as np
from scipy.sparse import lil_matrix
from scipy.sparse.linalg import spsolve

def strip_capacitance(B, h):
    # Coordinates in units of ell; potentials in units of Delta V.
    x = np.arange(-B, B + h/2, h)
    X, Y = np.meshgrid(x, x)
    n = len(x)
    fixed = np.zeros((n, n), dtype=bool)
    fixed[[0, -1], :] = True
    fixed[:, [0, -1]] = True
    upper = (abs(Y - .25) < h/5) & (abs(X) <= 1 + h/5)
    lower = (abs(Y + .25) < h/5) & (abs(X) <= 1 + h/5)
    fixed |= upper | lower
    V = np.zeros((n, n))
    V[upper], V[lower] = .5, -.5
    index = -np.ones((n, n), dtype=int)
    index[~fixed] = np.arange(np.count_nonzero(~fixed))
    rows, cols = np.where(~fixed)
    N = len(rows)
    M, rhs = lil_matrix((N, N)), np.zeros(N)
    steps = [(1, 0), (-1, 0), (0, 1), (0, -1)]
    for k, (i, j) in enumerate(zip(rows, cols)):
        M[k, k] = 4
        for di, dj in steps:
            ii, jj = i + di, j + dj
            if fixed[ii, jj]:
                rhs[k] += V[ii, jj]
            else:
                M[k, index[ii, jj]] = -1
    V[~fixed] = spsolve(M.tocsr(), rhs)
    # Each grid link is counted once in the energy sum.
    C_energy = sum(np.sum(np.diff(V, axis=a)**2) for a in (0, 1))
    # Outward flux from all upper-electrode grid nodes.
    C_flux = 0.0
    for i, j in zip(*np.where(upper)):
        for di, dj in steps:
            C_flux += V[i, j] - V[i + di, j + dj]
    return C_flux, C_energy

for B, h in [(3, 1/8), (3, 1/16), (3, 1/32),
             (4, 1/32), (6, 1/32)]:
    cf, ce = strip_capacitance(B, h)
    print(f"B={B} h={h:.5f} C/eps={cf:.5f} diff={cf-ce:.2e}")
```

Agreement of these two estimates checks the assembled discrete model, including charge accounting.

It does not show that the grid or enclosure is sufficiently refined. The last two printed columns answer different questions: the capacitance changes expose discretization and boundary effects; the small flux–energy difference checks algebraic consistency. Multiplying by a physical permittivity gives capacitance per length in F m^{-1} . Multiplying further by a finite electrode length is an additional approximation that neglects end effects in the third dimension.

The strip endpoints create strongly varying fields. Although central differences are second-order accurate for a smooth solution away from boundaries, it is unsafe to assume a globally second-order capacitance error near ideal zero-thickness edges. Rounding electrodes, resolving their thickness or fitting several refinement levels changes the numerical problem and should be stated explicitly. For a piecewise dielectric, discretize Eq. (5.135) in flux form with face permittivities; the appropriate normal link coefficient for two half-cells is their harmonic mean. This preserves the series-layer physics already derived analytically.

5.21.13 Real dielectrics and the boundary of the electrostatic model

A real material responds on finite time scales. A static dielectric constant is sufficient only when the polarization follows the applied field and losses are negligible. In a sinusoidal steady state one often writes, with the explicit convention $\mathbf{E}(t) = \text{Re}[\tilde{\mathbf{E}} e^{i\omega t}]$,

$$\tilde{\mathbf{D}} = (\epsilon' - i\epsilon'')\tilde{\mathbf{E}}, \quad \tan \delta = \frac{\epsilon''}{\epsilon'}. \quad (5.171)$$

For a passive dielectric under this convention, $\epsilon'' \geq 0$. The displacement-current amplitude is $i\omega\tilde{\mathbf{D}}$; its in-phase component is $\omega\epsilon''\tilde{\mathbf{E}}$. Therefore the average dissipated power density is

$$\langle p \rangle = \omega\epsilon'' E_{\text{rms}}^2 = \frac{1}{2}\omega\epsilon'' |\tilde{\mathbf{E}}|^2. \quad (5.172)$$

The factor of two depends on whether amplitude or rms field is used. Reversing the time convention reverses the sign used for the imaginary part; a sign without its convention is ambiguous. In a strongly dispersive medium, stored energy is not obtained simply by replacing ϵ with ϵ' in the static formula.

Leakage is often represented by a large parallel resistance, while electrode resistance is represented by an equivalent series resistance. Lead and internal geometry add inductance. These are different mechanisms and must not be combined into one unexplained “loss.” A simple series *RLC* approximation has impedance

$$Z(\omega) = R + i \left(\omega L - \frac{1}{\omega C} \right), \quad (5.173)$$

with resonance at $\omega = 1/\sqrt{LC}$ and heating $I_{\text{rms}}^2 R$. This is only a preview of the later AC chapter. Real assemblies can have several resonances, and parallel combinations can introduce antiresonances; adding more capacitor values is not by itself a guarantee of improved high-frequency behavior.

Supercapacitors illustrate the limits of extrapolating a simple geometric formula. Large accessible electrode area and very small charge-separation scales explain the possibility of large capacitance, but electrochemical double layers, ion transport and sometimes charge-transfer reactions determine the detailed response. The appropriate stored energy is still obtained from the measured reversible charging relation where that description applies; it must not be assumed to be $\frac{1}{2}CV^2$ with one constant C over every operating range.

Summary

The advanced results extend the same principles used in the core chapter. Free charge fixes displacement flux; material response converts displacement to electric field; boundary conditions determine potential; work determines energy; and the controlled variable determines the appropriate mechanical energy derivative. Numerical and circuit models are useful only when their geometry, reference nodes, material assumptions and error checks are explicit.

5.22 Exercises

Problems 1–10 emphasize concepts; 11–25 develop quantitative technique; 26–35 connect modelling, computation and measurement; 36–50 include extensions for stronger students. State the controlled variable and the approximation in every calculation. Unless specified otherwise, capacitors are linear and lossless in their static states, floating internal nodes are initially neutral, and $\varepsilon_0 = 8.85 \times 10^{-12} \text{ F m}^{-1}$ may be used. Charge-sharing questions assume a settled state with negligible charge leakage, not zero energy dissipation.

1. Define capacitance in words and explain why the net charge of an ideal two-plate capacitor can be zero while C is nonzero.
2. A capacitor is labelled 47 nF. What physical ratio does this number represent?
3. If every length of a capacitor is doubled, does its capacitance double or quadruple? Contrast this with doubling only the plate area at fixed separation.
4. Explain why the smaller capacitance in a simple series pair receives the larger voltage, and state the initial-charge condition needed for that argument.
5. State the difference between free charge and bound charge in a dielectric.
6. A capacitor is disconnected from a battery and then a dielectric is inserted. What stays fixed?
7. A capacitor remains connected to a battery while a dielectric is inserted. What stays fixed?
8. Why is the formula $C = \varepsilon_0 A/d$ not exact for small plates separated by a large distance?
9. Why does a high dielectric constant alone not guarantee a good high-energy capacitor?
10. In a charge-sharing experiment, why can electrostatic energy decrease even though charge is conserved?
11. An air-filled parallel-plate capacitor has $A = 25 \text{ cm}^2$ and $d = 0.50 \text{ mm}$. Find C .
12. A 220 pF capacitor is charged to 50 V. Find Q and U .
13. Two capacitors $3.0 \mu\text{F}$ and $9.0 \mu\text{F}$ are connected in parallel across 10 V. Find C_{eq} and the charge on each.
14. The same two capacitors are connected in series across 10 V. Find C_{eq} , the common charge and the voltage across each.
15. A cylindrical capacitor has $a = 1.0 \text{ mm}$, $b = 4.0 \text{ mm}$ and $L = 0.50 \text{ m}$. Find C in vacuum.
16. A spherical capacitor has $a = 2.0 \text{ cm}$ and $b = 6.0 \text{ cm}$. Find C in vacuum.
17. An isolated conducting sphere has radius 12 cm and potential 3.0 kV. Find its charge.

18. A capacitor $C_0 = 80 \text{ pF}$ is fully filled with a dielectric $\kappa = 3.5$. Find the new capacitance.
19. A $1.0 \mu\text{F}$ capacitor at 100 V is disconnected and then filled with $\kappa = 5$. Find final voltage and energy.
20. The same capacitor remains connected to the 100 V source while the dielectric is inserted. Find final charge and energy.
21. A parallel-plate capacitor with $A = 10 \text{ cm}^2$, $d = 1.0 \text{ mm}$ and $\kappa = 4$ is connected to 12 V . Find E , C , Q and U .
22. Two dielectric layers with $d_1 = 0.20 \text{ mm}$, $\kappa_1 = 2.0$ and $d_2 = 0.30 \text{ mm}$, $\kappa_2 = 6.0$ fill a capacitor of area 5.0 cm^2 . Find C .
23. A capacitor stores 0.50 J at 200 V . Find its capacitance.
24. A dielectric with safe field 50 MV m^{-1} and thickness $10 \mu\text{m}$ is used in a film capacitor. Estimate the voltage rating for one ideal layer.
25. Two capacitors, $10 \mu\text{F}$ and $20 \mu\text{F}$, initially have signed voltages $+12 \text{ V}$ and -3 V relative to the terminals to be joined. Find the settled parallel voltage, each final charge and the decrease in electrostatic energy.
26. The $2.0 \mu\text{F}$ – $4.0 \mu\text{F}$ series pair of Eq. (5.51) is connected between 12 V and ground, but its internal island carries $-6.0 \mu\text{C}$. Find the island voltage and both signed charges on the island. Explain why equal charge magnitudes fail.
27. A plate fixture produces $C_{\text{meas}} = 99 \text{ pF}$ at $d = 1.0 \text{ mm}$ and 54 pF at $d = 2.0 \text{ mm}$. Fit $C_{\text{meas}} = \varepsilon_0 A/d + C_s$ to these two points. Predict the capacitance at $d = 0.50 \text{ mm}$ and explain why a third measurement is valuable.
28. Plot $C = \varepsilon_0 A/d$ and $C_{\text{meas}} = C + 12 \text{ pF}$ against both d and $1/d$, for $A = 100 \text{ cm}^2$ and $0.2 \text{ mm} \leq d \leq 5 \text{ mm}$. At what separation does the offset equal 20% of the active capacitance?
29. For a floating node connected by C_1 to V_0 and by C_2 to ground, derive the voltage for specified charge q_X . Show that the incremental charge drawn from the V_0 terminal, at fixed q_X , has capacitance $C_1 C_2 / (C_1 + C_2)$.
30. Derive Eq. (5.169) from central differences. Explain why it is not directly valid across an interface where permittivity changes, and identify the quantity whose normal component must be conserved.
31. Design a numerical experiment to determine how a grounded enclosure changes a conductor capacitance. Specify conductor potentials, enclosure size, mesh-refinement tests and an analytic spherical benchmark. Distinguish a boundary-size error from a solver residual.
32. For a separation sensor $C = \varepsilon A/d$, obtain the first two correction terms in $\delta d/d_0$. Estimate the relative error of the linear approximation for $\delta d/d_0 = 0.05$. Compare with a sensor whose overlap area varies linearly.
33. Estimate the capacitance per unit area of a membrane with $d = 6 \text{ nm}$ and $\kappa = 4$.
34. A cable has capacitance 80 pF m^{-1} . Estimate the stored energy in a 2 m cable charged to 5 V .
35. Compare energy densities $u = \frac{1}{2} \varepsilon E^2$ for two dielectrics: $\kappa = 2$, $E_{\text{max}} = 200 \text{ MV m}^{-1}$ and $\kappa = 20$, $E_{\text{max}} = 30 \text{ MV m}^{-1}$.

36. Derive the capacitance per unit length of a coaxial cable filled with a dielectric κ and discuss the limit $b/a \rightarrow 1$.
37. A dielectric slab of relative permittivity κ fills the entire gap d but is inserted laterally a distance x into plates of length L and width w , with $0 < x < L$. Derive $C(x)$ and test the $x = 0$ and $x = L$ limits.
38. For the partially inserted slab in Problem 37, derive the force at fixed voltage and the source work during insertion from $x = 0$ to $x = L$. Explain where the local force acts.
39. For the same slab, derive the force and total mechanical work at fixed charge. Compare with fixed-voltage insertion starting from the same empty-capacitor state.
40. Show that the energy-density integral for a spherical capacitor reproduces $U = Q^2/(2C)$.
41. For the five-capacitor bridge in Fig. 5.12, replace the central branch capacitance by $c \geq 0$. Set up the two-node equations. Compute C_{eq} at $c = 0$ and in the limit $c \rightarrow \infty$, and interpret the limiting topologies.
42. Derive the capacitance of two concentric spherical shells in the limit $b = a + d$ with $d \ll a$ and compare with the parallel-plate formula.
43. Discuss qualitatively how sharp conductor edges affect breakdown risk.
44. Using Eq. (5.119), derive the maximum ideal series-string voltage for unequal capacitances and ratings. Explain why long-time leakage can invalidate the initial capacitance-controlled voltage division.
45. A capacitor bank contains ten identical capacitors each $100 \mu\text{F}$ rated at 100 V . Compare all-parallel and all-series connections in terms of equivalent capacitance, voltage rating and maximum stored energy under ideal voltage sharing.
46. For $\varepsilon(z) = \varepsilon_0 \kappa_0 (1 + \alpha z/d)$, derive Eq. (5.140) and the resulting capacitance. Expand the answer through first order in α and identify the location of the largest field when $\alpha > 0$.
47. A reversible nonlinear capacitor has $Q = av + bv^3$ with $a = 10 \text{ pF}$ and $b = 0.020 \text{ pF V}^{-2}$. At $v = 5 \text{ V}$, find Q , C_{sec} , C_{diff} , stored energy and coenergy. Check $U + U^* = vQ$.
48. A voltage-driven plate of area 0.50 mm^2 is supported by a spring of stiffness 2.0 N m^{-1} at initial gap $10 \mu\text{m}$. Derive and calculate its ideal pull-in displacement and voltage. Explain why an equilibrium below pull-in still requires a stability test.
49. Using $M_0 = 110 \text{ pF}$, $M_1 = 310 \text{ pF}$ and common stray correction $C_s = 10 \text{ pF}$, estimate κ . If independent standard uncertainties are $u(M_0) = 1 \text{ pF}$, $u(M_1) = 2 \text{ pF}$ and $u(C_s) = 2 \text{ pF}$, propagate the uncertainty with the shared correction treated correctly.
50. Reproduce at least two rows of Table 5.3. Compare charge and energy estimates, then change h and B separately. Explain why their mutual agreement does not by itself establish the exact open-space capacitance.

5.23 Selected solutions to representative exercises

Solution to Problem 11

Convert units: $A = 25 \text{ cm}^2 = 2.5 \times 10^{-3} \text{ m}^2$ and $d = 0.50 \text{ mm} = 5.0 \times 10^{-4} \text{ m}$. Then

$$C = \varepsilon_0 \frac{A}{d} = (8.85 \times 10^{-12}) \frac{2.5 \times 10^{-3}}{5.0 \times 10^{-4}} \text{ F} = 4.43 \times 10^{-11} \text{ F} = 44.3 \text{ pF}. \quad (5.174)$$

The result has units $\text{F m}^{-1} \text{m}^2 / \text{m} = \text{F}$. The plane model requires both plate dimensions to exceed the gap substantially; an area alone cannot establish that for an arbitrarily narrow plate.

Solution to Problem 14

For series capacitors,

$$C_{\text{eq}} = \frac{(3.0)(9.0)}{3.0 + 9.0} \mu\text{F} = 2.25 \mu\text{F}. \quad (5.175)$$

The common charge is $Q = C_{\text{eq}}V = 22.5 \mu\text{C}$. The voltages are 7.5 V across $3.0 \mu\text{F}$ and 2.5 V across $9.0 \mu\text{F}$.

The smaller capacitance takes three times the voltage. Its charge is nevertheless the same because the internal conductor is neutral. The equivalent capacitance $2.25 \mu\text{F}$ is below the smaller original value $3.0 \mu\text{F}$, as required.

Solution to Problem 19

For a disconnected capacitor, Q is fixed. The final capacitance is $5C_0$, so

$$\Delta V_f = \frac{\Delta V_i}{5} = 20 \text{ V}, \quad U_f = \frac{U_i}{5}. \quad (5.176)$$

The initial energy is $U_i = \frac{1}{2}(1.0 \mu\text{F})(100 \text{ V})^2 = 5.0 \text{ mJ}$, so $U_f = 1.0 \text{ mJ}$.

Solution to Problem 20

For a battery-connected capacitor, ΔV is fixed. The final capacitance is $5.0 \mu\text{F}$, so

$$Q_f = C_f V = (5.0 \mu\text{F})(100 \text{ V}) = 500 \mu\text{C}, \quad (5.177)$$

while

$$U_f = \frac{1}{2}C_f V^2 = 25 \text{ mJ}. \quad (5.178)$$

Here the initial charge is $100 \mu\text{C}$, so the source transfers $400 \mu\text{C}$ and supplies 40 mJ. The field-energy increase is 20 mJ, leaving 20 mJ for reversible mechanical work. This additional source term explains why an increase in stored energy is consistent with attraction of the dielectric.

Solution to Problem 22

There is no free charge at the dielectric interface, so D_n is common and the layer voltage drops add. Consequently

$$C = \frac{\varepsilon_0 A}{d_1/\kappa_1 + d_2/\kappa_2}. \quad (5.179)$$

With $A = 5.0 \times 10^{-4} \text{ m}^2$, $d_1 = 2.0 \times 10^{-4} \text{ m}$ and $d_2 = 3.0 \times 10^{-4} \text{ m}$,

$$C = \frac{(8.85 \times 10^{-12})(5.0 \times 10^{-4})}{(2.0 \times 10^{-4})/2 + (3.0 \times 10^{-4})/6} = 2.95 \times 10^{-11} \text{ F} = 29.5 \text{ pF}. \quad (5.180)$$

Solution to Problem 26

Use the signed node equation, with capacitances in μF and charges in μC :

$$2(V_X - 12) + 4V_X = -6, \quad V_X = 3.0 \text{ V}.$$

The island plate charges are $-18 \mu\text{C}$ and $+12 \mu\text{C}$, whose sum is $-6 \mu\text{C}$. The terminal voltage drops 9.0 V and 3.0 V add to 12 V. Equal magnitudes would contradict the specified island charge. The electrostatic laws remain valid; only the neutral-node shortcut fails.

Solution to Problem 27

Writing $C_{\text{meas}} = m/d + C_s$ with d in mm, the two equations are $99 = m + C_s$ and $54 = m/2 + C_s$ in picofarads. Hence $m = 90 \text{ pF mm}$ and $C_s = 9 \text{ pF}$. The area is

$$A = \frac{90 \times 10^{-15}}{8.85 \times 10^{-12}} \text{ m}^2 = 1.017 \times 10^{-2} \text{ m}^2 \simeq 102 \text{ cm}^2.$$

At $d = 0.50 \text{ mm}$ the prediction is 189 pF . Two points determine two parameters exactly and cannot test whether the offset model is appropriate; a third point can reveal curvature, changing stray coupling or a separation error.

Solution to Problem 35

For dielectric 1,

$$u_1 = \frac{1}{2}(2\varepsilon_0)(200 \times 10^6)^2 \simeq 3.54 \times 10^5 \text{ J m}^{-3}. \quad (5.181)$$

For dielectric 2,

$$u_2 = \frac{1}{2}(20\varepsilon_0)(30 \times 10^6)^2 \simeq 7.97 \times 10^4 \text{ J m}^{-3}. \quad (5.182)$$

The lower- κ material stores more energy density at its safe field because its breakdown field is much larger. This illustrates why breakdown strength matters.

Solution to Problem 37

If a dielectric fills a fraction x/L of the area, the filled and unfilled regions act in parallel:

$$C(x) = \varepsilon_0 \frac{w}{d} [\kappa x + (L - x)] = \varepsilon_0 \frac{w}{d} [L + (\kappa - 1)x]. \quad (5.183)$$

This linear dependence on insertion depth is the basis of simple capacitive displacement sensors.

Solutions to Problems 38 and 39

Let $C_0 = \varepsilon_0 wL/d$ and $\beta = \varepsilon_0 w(\kappa - 1)/d$, so $C(x) = C_0 + \beta x$. At fixed voltage,

$$F_x = \frac{1}{2}V_0^2\beta, \quad W_{\text{mech}} = \frac{1}{2}V_0^2(C_f - C_0) = \frac{1}{2}(\kappa - 1)C_0V_0^2.$$

The source supplies $W_s = V_0(Q_f - Q_i) = (\kappa - 1)C_0V_0^2$, twice the mechanical work; the other half increases stored energy. The leading plane-capacitance model gives the total force, while the local pulling stress resides in the nonuniform entrance field.

At fixed $Q_0 = C_0V_0$,

$$F_x = \frac{Q_0^2\beta}{2(C_0 + \beta x)^2}, \quad W_{\text{mech}} = \int_0^L F_x dx = \frac{Q_0^2}{2} \left(\frac{1}{C_0} - \frac{1}{\kappa C_0} \right).$$

Both forces point inward. For the same initial state, the isolated protocol yields $W_{\text{mech}} = \frac{1}{2}C_0V_0^2(1 - 1/\kappa)$, smaller than the voltage-controlled result by a factor $1/\kappa$. No source energy enters the isolated calculation.

Solution to Problem 40

For a spherical capacitor, the field between the shells is $E = Q/(4\pi\varepsilon_0 r^2)$. The energy integral is

$$U = \int_a^b \frac{1}{2}\varepsilon_0 E^2 (4\pi r^2 dr) = \frac{Q^2}{8\pi\varepsilon_0} \int_a^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right). \quad (5.184)$$

Using $C = 4\pi\varepsilon_0 ab/(b - a)$ gives exactly $U = Q^2/(2C)$.

Solution to Problem 45

All parallel: $C_{\text{eq}} = 1000 \mu\text{F}$ and the voltage rating remains 100 V, so the ideal maximum energy is $\frac{1}{2}(1000 \mu\text{F})(100)^2 = 5 \text{ J}$. All series: $C_{\text{eq}} = 10 \mu\text{F}$ and the ideal voltage rating is 1000 V if voltage sharing is perfect, so the maximum energy is $\frac{1}{2}(10 \mu\text{F})(1000)^2 = 5 \text{ J}$. The same total ideal energy appears because the same ten identical capacitors are each used at their rating. In practice, series strings need balancing components to ensure equal voltage sharing.

Solution to Problem 47

At $v = 5 \text{ V}$, the charge is $Q = 50 \text{ pC} + 2.5 \text{ pC} = 52.5 \text{ pC}$. Thus

$$C_{\text{sec}} = 10.5 \text{ pF}, \quad C_{\text{diff}} = a + 3bv^2 = 11.5 \text{ pF}.$$

Integrating $v \, dQ$ yields

$$U = \frac{1}{2}av^2 + \frac{3}{4}bv^4 = 125 \text{ pJ} + 9.375 \text{ pJ} = 134.375 \text{ pJ}.$$

The coenergy is $U^* = \frac{1}{2}av^2 + \frac{1}{4}bv^4 = 128.125 \text{ pJ}$. Their sum 262.5 pJ equals $(5 \text{ V})(52.5 \text{ pC})$. The units of the cubic coefficient are essential: bv^4 has units of farads times volts squared, hence joules.

Solution to Problem 48

At equilibrium, $kx = \varepsilon_0 AV^2/[2(d_0 - x)^2]$. Marginal stability additionally requires $k = \varepsilon_0 AV^2/(d_0 - x)^3$. Dividing the equations gives $x = (d_0 - x)/2$, hence $x_{\text{PI}} = d_0/3 = 3.33 \mu\text{m}$. Then

$$V_{\text{PI}} = \sqrt{\frac{8(2.0)(10^{-5})^3}{27(8.85 \times 10^{-12})(0.50 \times 10^{-6})}} \text{ V} = 11.6 \text{ V}.$$

Below this voltage, the ideal curve generally has two mathematical equilibria; only the smaller-displacement one has $k > dF_e/dx$ and is stable. A force balance alone cannot distinguish them.

Solution to Problem 49

Subtract the same stray contribution from both readings: $\kappa = (310 - 10)/(110 - 10) = 3.00$. The derivatives are $\partial\kappa/\partial M_1 = 1/100 \text{ pF}^{-1}$, $\partial\kappa/\partial M_0 = -3/100 \text{ pF}^{-1}$ and $\partial\kappa/\partial C_s = 2/100 \text{ pF}^{-1}$. Therefore

$$u(\kappa) = \frac{\sqrt{2^2 + 3^2(1)^2 + 2^2(2)^2}}{100} = 0.054.$$

The result is 3.00 ± 0.05 under the stated independence and full-fill assumptions. The correction's two appearances are correlated because they are the same measured quantity; assigning independent uncertainties to two copies of C_s would double-count it.

5.24 Bridge to the next chapter

Capacitance completes the electrostatic sequence from charge to field, field to potential and potential to a device relation. The next chapter introduces electric current, resistance and power. For an ideal capacitor of fixed capacitance, differentiating its signed terminal relation gives

$$I = \frac{dQ}{dt} = C \frac{d(\Delta V)}{dt}. \quad (5.185)$$

The current sign is defined as positive when positive charge enters the terminal whose voltage is used in ΔV . If geometry or material response changes with time, the additional term $\Delta V \, dC/dt$ must be included for a linear time-varying capacitor. The static chapter determines the charge and energy of each state; circuit theory determines how one state evolves into another.

Take-home message

State the geometry, material and electrical constraint before calculating. Use signed charge conservation on connected conductors, distinguish free from bound charge, and account for the voltage source when mechanical motion changes capacitance. Finish with units, limiting cases and a model-validity check.

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