

Chapter 6

Electric Current, Resistance, and Power

First Edition

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Preface

This chapter introduces steady electric current, resistance, resistivity, Ohm's law and electric power as the first systematic step from electrostatics toward circuit physics. The earlier electrostatic chapters described charge at rest, electric field, electric potential, capacitance and stored electric-field energy. Real devices, however, are not built only from static charge distributions. They operate by controlling charge transport through conductors, semiconductors, resistive elements, contacts, sensors and power supplies.

The central shift in this chapter is from the question “where is the charge?” to the question “how fast is charge crossing a chosen surface, and how much energy is transferred while it does so?” This shift is subtle. A current is not a substance flowing like water through a pipe; it is a rate of charge transport. A resistor does not consume current; it converts electromagnetic energy into internal thermal energy. A battery does not always supply a fixed current; it helps maintain a potential difference, and the rest of the circuit determines the current. These distinctions are essential for the circuit chapters that follow.

The chapter is written for Technical Physics, engineering, computer-science, space-science and applied-science students. It therefore combines microscopic reasoning about charge carriers with macroscopic modelling in terms of current, current density, potential difference, resistance and power. The microscopic picture explains why drift speeds are usually small compared with thermal speeds. The macroscopic picture explains why a small current can deliver large energy over time, why wire gauge matters, why heating elements glow, why fuses protect circuits, and why the same component can be safe in one configuration and dangerous in another.

The core develops a calculus-based introduction to charge transport. The separate advanced-enrichment part develops field methods, electrothermal feedback, measurement and computation. The distinction between these layers allows the chapter to support both a first engineering course and deeper independent study. Suggested textbook sections are identified in the further reading.

The Guideline

Treat every current problem first as a charge-transport and energy-transfer problem. Identify the surface through which charge is counted, state the sign convention for current, decide whether the current density is uniform, check whether the material is being treated as ohmic, and only then select an equation. The equations $I = dq/dt$, $I = \int \mathbf{J} \cdot d\mathbf{A}$, $R = \rho L/A$, $\Delta V = IR$ and $P = I\Delta V$ answer different physical questions.

Pedagogical boundary

This chapter develops steady charge transport and resistive energy dissipation. It does not yet solve complete multi-loop circuits, does not introduce Kirchhoff's loop and junction rules in full, and does not treat capacitive or inductive transients as dynamical circuit problems. Those topics belong to the next circuit chapters. Here the goal is to understand the local and material meaning of current and resistance so that circuit rules later have physical content rather than becoming memorized algebra.

Conventions and notation

Vectors are written in bold notation. The electric field is $\mathbf{E}$, the current density is $\mathbf{J}$, and a directed surface element is $d\mathbf{A} = \hat{\mathbf{n}} dA$. The scalar current through a chosen oriented surface is denoted by I . A positive value of I means that positive charge is transported in the direction of the chosen surface normal, or equivalently that negative charge is transported in the opposite direction.

For a two-terminal element, choose a reference current from terminal a into the element and out through b , and define $\Delta V = V_a - V_b$. This is the *passive sign convention*: $P = I\Delta V$ is the signed electrical power absorbed. A positive value means absorption; a negative value means delivery. For an ordinary passive resistor with positive reference current, $V_a > V_b$. The terminal voltage drop $V_a - V_b$ must not be confused with the potential change along the path, $V_b - V_a$.

Symbol	Meaning	SI unit or dimension
q, Q	electric charge; total charge when capitalized	C
e	elementary charge, $1.602\,176\,634 \times 10^{-19}$ C	C
I	electric current through an oriented surface	A = C s ⁻¹
$\mathbf{J}$	current density vector	A m ⁻²
$d\mathbf{A}$	directed surface element	m ²
n	number density of mobile carriers	m ⁻³
q_c	charge of one mobile carrier; often $-e$ for electrons	C
$\mathbf{v}_d$	drift velocity of charge carriers	m s ⁻¹
R	resistance between two terminals	$\Omega = \text{V A}^{-1}$
ρ	electrical resistivity	$\Omega \text{ m}$
σ	electrical conductivity, $1/\rho$ in an isotropic medium	S m ⁻¹
L	length of uniform conducting sample	m
A	cross-sectional area, not to be confused with ampere	m ²
ΔV	potential difference or voltage drop, depending on convention	V
P	power or rate of energy transfer	W
α	temperature coefficient of resistivity in a linear approximation	K ⁻¹
T	temperature	K or °C, with differences in K
τ	effective momentum-relaxation time in the Drude model	s
μ	carrier mobility, $ v_d / E $ in the linear regime	m ² V ⁻¹ s ⁻¹

Typography. Italic A denotes area; upright A is the ampere. Likewise, italic V denotes potential and upright V the volt. Terminals and current directions fix voltage signs. Ohm's law describes a material response; it is distinct from the charge and energy conservation statements used in circuit analysis.

Chapter 6

Electric Current, Resistance, and Power

6.1 Charge transport and energy transfer

Electrical technology is possible because charges can be made to move in controlled ways. A laptop charger, a temperature sensor, an accelerator magnet power supply, an LED display, a superconducting coil, a neural probe and a household kettle all involve charge transport. The microscopic carriers and the engineering details differ, but the first modelling questions are common: what charge crosses a surface per unit time, what field drives it, what material relation connects the field to the motion, and how much energy is transferred?

Current first describes how much charge crosses a surface in a given time. Current density then makes that description local, and the drift model connects it to microscopic carriers. Material response and geometry determine resistance; energy accounting determines power. Measurements finally test which parts of this model are adequate for a real sample.

Learning goals

After studying this chapter, the student should be able to:

- define electric current and explain the sign convention for conventional current;
- relate current to charge transported during a time interval for constant and time-dependent currents;
- define current density and compute current from $I = \int \mathbf{J} \cdot d\mathbf{A}$;
- connect current density to carrier density, carrier charge and drift velocity;
- distinguish drift speed from microscopic thermal speed and signal propagation;
- derive $R = \rho L/A$ for a uniform ohmic conductor from $\mathbf{J} = \sigma \mathbf{E}$;
- distinguish resistance, resistivity, conductivity and conductance;
- state clearly that Ohm's law is a model for a class of materials and operating regimes, not a universal law;
- compute electrical power and Joule heating with the correct assumptions;
- interpret I - V graphs and identify ohmic and non-ohmic behaviour;
- discuss temperature dependence, semiconductors and superconductors at an introductory level;
- connect the theory to wire sizing, fuses, sensors, measurement uncertainty and data fitting.

Key message

In steady state, the same current enters and leaves a resistor. Electrical energy is transferred into internal energy at rate $P = I\Delta V$; charge is conserved.

6.2 From electrostatics to steady charge transport

In electrostatics, charges are at rest and the electric field in a conductor is zero in static equilibrium. A current-carrying wire is not in electrostatic equilibrium. Mobile carriers drift through the material, and a nonzero electric field is usually present inside a resistive conductor. This field is not large in typical metal wires, but it is essential: without it there would be no sustained drift and no continuous energy transfer.

A steady current means that macroscopic quantities do not change with time at a fixed point. It does not mean that every microscopic electron moves smoothly or that every electron travels from battery to lamp in a simple straight path. In a metal, conduction electrons undergo rapid microscopic motion and frequent scattering. The steady current is described by a small average drift superposed on much larger random motion.

The word “steady” also does not mean “static”. In a steady current, charge does not accumulate in the bulk of a uniform wire. The amount of charge entering a small segment during a short time equals the amount leaving it, so the segment’s net charge remains approximately constant. Surface charge distributions and electric fields arrange themselves so that this steady transport can occur. The full field description of how circuits establish their steady state is richer than the algebra of ideal circuits, but the local bookkeeping begins here.

Pedagogical boundary

We shall often use idealized steady currents. This approximation ignores the short transient during which charges redistribute after a circuit is connected. It also ignores high-frequency effects, wave propagation along conductors and microscopic fluctuations. Those effects matter in fast electronics and transmission lines, but the steady approximation is the correct first model for ordinary DC circuit elements.

Guided checks

Check 1. If a lamp glows steadily, is charge accumulating inside the filament?

Answer. Not in the steady approximation. Charge enters and leaves at the same rate. Energy is transferred to the filament and then to the surroundings.

Check 2. Can there be an electric field inside a current-carrying metal?

Answer. Yes. In a resistive conductor a small internal electric field is needed to maintain drift against scattering.

Check 3. Is a steady current the same as electrostatic equilibrium?

Answer. No. Electrostatic equilibrium in a conductor has no current and no internal electric field. A steady current has time-independent macroscopic transport.

6.3 Electric current as a rate of charge transfer

Choose an oriented surface. The electric current through that surface is the rate at which charge crosses it:

$$I = \frac{dq}{dt}. \quad (6.1)$$

Here dq is the net charge transported through the surface in time dt , counted with the chosen orientation. If the current is constant during a finite time interval Δt , then

$$\Delta q = I \Delta t. \quad (6.2)$$

If the current depends on time, the transported charge is obtained by integration:

$$\Delta q = \int_{t_1}^{t_2} I(t) dt. \quad (6.3)$$

The SI unit of current is the ampere:

$$1 \text{ A} = 1 \text{ C s}^{-1}. \quad (6.4)$$

Figure 6.1 fixes the sign convention. Conventional current is defined as the direction in which positive charge would move. In a metal the mobile carriers are electrons, so their drift velocity is opposite to the conventional current direction. This convention is not a mistake. It is a consistent historical and physical bookkeeping convention. For equal crossing rates, negative charge moving to the left contributes the same conventional current as positive charge of equal magnitude moving to the right.

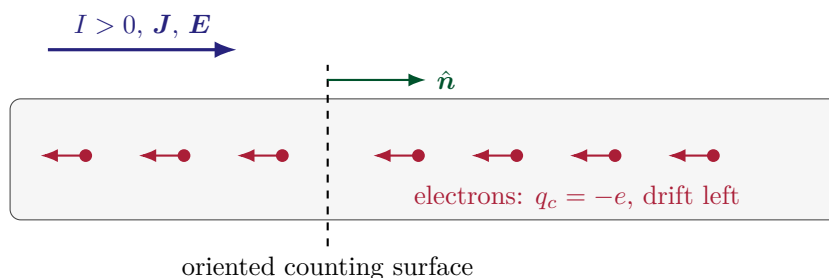


Figure 6.1: Sign conventions in a resistive metal. The reference normal, conventional current, current density and driving field point right. Electrons drift left, so $n(-e)\mathbf{v}_d$ points right. The dots are a schematic carrier representation.

Definition 6.1: Electric current

Electric current through an oriented surface is the net charge per unit time crossing that surface. The sign of I is fixed by the orientation convention. Positive current corresponds to positive charge moving with the surface normal or negative charge moving opposite to it.

Caution

Do not say that current is “the number of electrons per second” unless you also include the electron charge and the sign convention. Current is charge per unit time. A particle count per unit time becomes a current only after multiplication by the carrier charge.

Example 6.1: Charge transported by a time-dependent current

A sensor draws a current

$$I(t) = I_0 \left(1 - \frac{t}{T}\right) \quad (6.5)$$

for $0 \leq t \leq T$, where $I_0 = 40 \text{ mA}$ and $T = 5.0 \text{ s}$. Find the total charge transported.

Solution. Because the current is not constant, use Eq. (6.3):

$$\Delta q = \int_0^T I_0 \left(1 - \frac{t}{T}\right) dt = I_0 \left[t - \frac{t^2}{2T} \right]_0^T = \frac{I_0 T}{2}. \quad (6.6)$$

Substituting the numbers,

$$\Delta q = \frac{(40 \times 10^{-3} \text{ A})(5.0 \text{ s})}{2} = 0.10 \text{ C}. \quad (6.7)$$

Interpretation. The average current over the interval is $I_0/2$, so the result also equals $I_{\text{av}}T$. The triangular area under the I - t graph is the transported charge.

Example 6.2: Conventional current from electron transport

Electrons cross a plane in a wire from right to left at a rate of 3.0×10^{18} electrons per second. What is the conventional current through the plane if the positive surface normal points to the right?

Solution. Each electron has charge $-e$. Electron motion from right to left is opposite to the chosen positive normal. Negative charge moving opposite to the normal corresponds to positive conventional current. The magnitude is

$$I = Ne = (3.0 \times 10^{18} \text{ s}^{-1})(1.602 \times 10^{-19} \text{ C}) = 0.48 \text{ A.} \quad (6.8)$$

The conventional current is $+0.48 \text{ A}$ with respect to the rightward normal.

Reasonableness check. A current of half an ampere corresponds to roughly 3×10^{18} elementary charges per second, so the number is reasonable for an ordinary wire.

Guided checks

Check 1. A positive ion beam moves to the right. What is the conventional current direction?

Answer. To the right.

Check 2. An electron beam moves to the right. What is the conventional current direction?

Answer. To the left.

Check 3. If $I(t)$ is plotted versus t , what is the physical meaning of the area under the curve?

Answer. The net transported charge through the chosen surface during the interval.

6.3.1 Net transport, backflow and energy

The charge q in $I = dq/dt$ is a cumulative *signed transport* through the selected surface. It is not the net charge stored in the piece of wire. A wire may remain nearly neutral while a large cumulative charge passes through it. Random electrons crossing in both directions make contributions of opposite signs; a current is the small imbalance between them.

Figure 6.2 gives a useful distinction between charge and energy. The signed area under the current curve can vanish while a resistor still heats. Counting both pulse lobes with positive signs would give an absolute transported-charge measure, not the net transport.

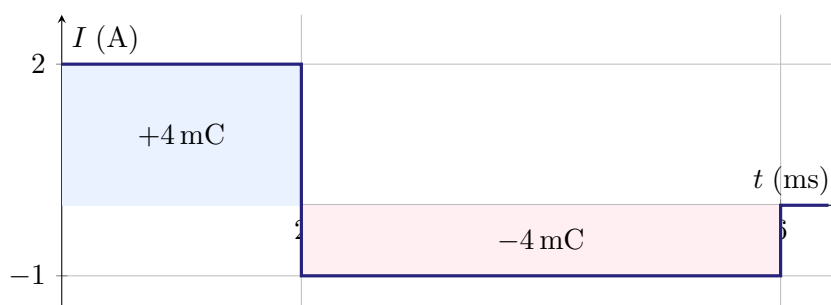


Figure 6.2: An idealized bipolar pulse: equal positive and negative signed areas imply zero net transported charge. Resistive heating depends on I^2 and adds during both intervals. Abrupt edges idealize finite switching times.

Example 6.3: Zero net transported charge with nonzero heating

The pulse in Fig. 6.2 passes through a 5.0Ω resistor whose temperature remains approximately constant. Find net transported charge and dissipated energy.

Charge accounting. With seconds used in the products,

$$\Delta q = (2.0)(2.0 \times 10^{-3}) + (-1.0)(4.0 \times 10^{-3}) = 0 \text{ C.}$$

The separate signed amounts are $+4.0 \text{ mC}$ and -4.0 mC .

Energy accounting. Using the resistive power relation derived in Section 6.9,

$$W = \int I^2 R dt = 5.0[2.0^2(0.002) + (-1.0)^2(0.004)] = 0.060 \text{ J.}$$

Both intervals heat the resistor because current and voltage reverse together. The units are $\text{A}^2 \Omega \text{s} = \text{J}$. Net charge transport alone cannot determine the energy dissipated.

6.4 Current density and local charge flow

Current is a global quantity associated with a chosen surface (Fig. 6.3). To describe how charge flow is distributed across space, we introduce the current density vector $\mathbf{J}$. Its direction is the direction of conventional current flow at a point, and its magnitude is current per unit area when the flow is perpendicular to the surface. The general relation is

$$I = \int_S \mathbf{J} \cdot d\mathbf{A}, \quad (6.9)$$

where $d\mathbf{A} = \hat{\mathbf{n}} dA$ is the directed area element.

If $\mathbf{J}$ is uniform over a flat cross-section and perpendicular to it, Eq. (6.9) reduces to

$$I = JA. \quad (6.10)$$

If $\mathbf{J}$ makes an angle θ with the surface normal, the contribution is $JA \cos \theta$. Only the component normal to the surface transports charge through that surface.

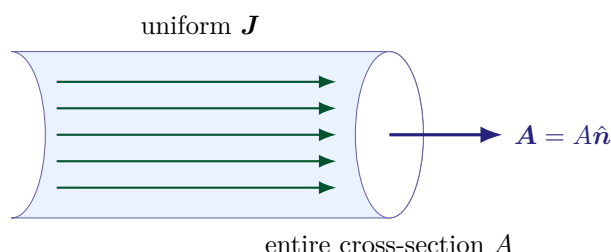


Figure 6.3: For uniform flow normal to a cross-section, $I = JA$. The arrow represents the area vector of the entire end face. A local patch would instead have the differential vector $d\mathbf{A}$.

Equation (6.9) resembles the flux integrals used for electric field. The interpretation is different: electric flux is a geometrical measure of field through a surface, while current is an actual rate of charge transport. The mathematical similarity is useful because both involve a vector field and an oriented surface.

Definition 6.2: Current density

The current density $\mathbf{J}$ at a point is a vector whose flux through a surface gives the current through that surface. In a simple uniform conductor carrying current I perpendicular to area A , its magnitude is $J = I/A$.

Example 6.4: Current density in a round wire

A copper wire of radius $r = 0.80$ mm carries a steady current of $I = 2.0$ A uniformly over its cross-section. Find the current density magnitude.

Solution. The cross-sectional area is

$$A = \pi r^2 = \pi(0.80 \times 10^{-3} \text{ m})^2 = 2.01 \times 10^{-6} \text{ m}^2. \quad (6.11)$$

For uniform current density,

$$J = \frac{I}{A} = \frac{2.0 \text{ A}}{2.01 \times 10^{-6} \text{ m}^2} = 9.9 \times 10^5 \text{ A m}^{-2}. \quad (6.12)$$

Interpretation. Current density is large in SI units because one square metre is an enormous cross-section for an ordinary wire.

Example 6.5: Nonuniform current density across a strip

A conducting strip has width w and thickness b . Treat the following current-density profile as prescribed; for example, conductivity may vary across the width in a uniform axial field. Across its width coordinate $x \in [0, w]$, the current density is

$$J(x) = J_0 \left(1 + \frac{x}{w} \right) \quad (6.13)$$

and is everywhere perpendicular to the cross-section. Find the total current.

Solution. The area element is $dA = b dx$, so

$$I = \int_0^w J(x) b dx = b J_0 \int_0^w \left(1 + \frac{x}{w} \right) dx \quad (6.14)$$

$$= b J_0 \left[w + \frac{w}{2} \right] = \frac{3}{2} J_0 b w. \quad (6.15)$$

Interpretation. The answer is the area times the average current density. Since J rises linearly from J_0 to $2J_0$, the average is $3J_0/2$.

Method

To compute current from current density:

1. draw the oriented surface and its normal;
2. write $d\mathbf{A} = \hat{\mathbf{n}} dA$;
3. take the dot product $\mathbf{J} \cdot d\mathbf{A}$;
4. integrate over the actual surface area;
5. check the sign against the chosen orientation.

Guided checks

Check 1. If $\mathbf{J}$ is tangent to a surface, what current crosses that surface?

Answer. Zero, because $\mathbf{J} \cdot d\mathbf{A} = 0$.

Check 2. If the same current passes through a narrower section of a wire, what happens to J ?

Answer. It increases if the current is uniformly distributed, because $J = I/A$.

Check 3. Is J a scalar or a vector?

Answer. A vector. Its direction carries physical information about the direction of conventional charge transport.

6.4.1 Changing a surface does not necessarily change the current

For a fixed flat patch of area A , tilting its normal through angle θ reduces the current to $JA \cos \theta$ in a uniform field. A different question is to tilt a cut that must still intercept an *entire wire*. Its area increases to $A_{\text{cut}} = A_{\perp} / \cos \theta$, where $A_{\perp}$ is the perpendicular cross-section. Therefore

$$I = JA_{\text{cut}} \cos \theta = JA_{\perp}. \quad (6.16)$$

The same steady current crosses both complete cuts; the factor of $\cos \theta$ is canceled by the enlarged cut area. This is shown geometrically in Fig. 6.4. Reversing the normal reverses the signed current, while leaving its magnitude unchanged.

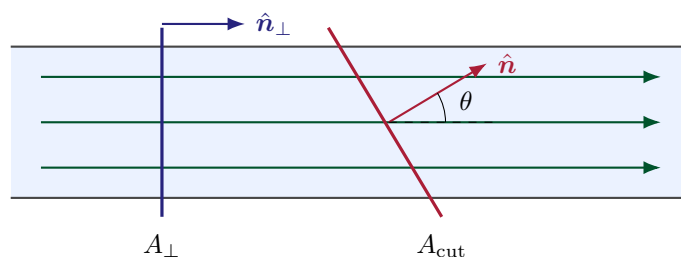


Figure 6.4: Side view of two complete cuts through a uniform wire. The tilted surface has a larger area. Its normal makes angle θ with the current density, so both surfaces carry the same flux I .

6.4.2 Circular geometry: choose rings, not strips

When the magnitude depends only on radial distance r , all points on a thin ring have the same $J(r)$. Its area is $\pi[(r + dr)^2 - r^2] = 2\pi r dr$ to first order. Thus

$$I = 2\pi \int_0^a J(r) r dr. \quad (6.17)$$

The factor r expresses how much area belongs to each radial interval. Ignoring it gives both the wrong physics and the wrong dimensions. Figure 6.5 makes this weighting explicit.

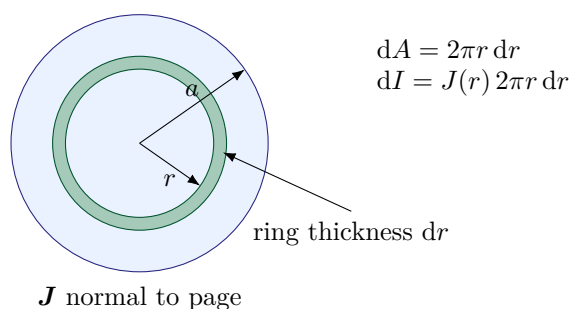


Figure 6.5: Annular integration element for an axially directed, radially varying current density. The finite-width colored ring illustrates an infinitesimal element; no radial current is implied.

Example 6.6: A prescribed nonuniform profile

Suppose $J(r) = J_0(1 - r^2/a^2)$ along the axis of a circular sample. Find its total current and the fraction carried within radius $a/2$.

Model. Treat this as a prescribed profile, for example arising from radially varying conductivity under a uniform axial field. It is not the DC solution for a homogeneous, isothermal metal wire with full-face contacts.

$$I = 2\pi J_0 \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right]_0^a = \frac{\pi J_0 a^2}{2},$$

$$I(r < a/2) = 2\pi J_0 a^2 \left(\frac{1}{8} - \frac{1}{64} \right) = \frac{7\pi J_0 a^2}{32}.$$

The fraction is 7/16, which exceeds the inner disk's area fraction 1/4 because the current is concentrated near the center. The average current density is $J_0/2$, not J_0 .

6.5 Drift velocity and the microscopic meaning of current

Consider a conductor with mobile charge carriers of number density n , carrier charge q_c and average drift velocity $\mathbf{v}_d$. To count the net transported charge, replace the microscopic velocity distribution by its mean drift. During a time Δt , the effective carrier population in a length $v_d \Delta t$ crosses a cross-section, assuming the drift is perpendicular to it. Random crossings in opposite directions cancel in the net current; the construction does not assert that individual electrons all follow straight paths. The volume that contributes is $A v_d \Delta t$, the number of carriers is $n A v_d \Delta t$, and the transported charge magnitude is $n |q_c| A v_d \Delta t$. Therefore,

$$J = n |q_c| v_d \quad (6.18)$$

for the magnitude in a one-carrier uniform model. In vector form, with conventional-current direction included through the sign of q_c ,

$$\mathbf{J} = n q_c \mathbf{v}_d. \quad (6.19)$$

For electrons, $q_c = -e$, so $\mathbf{J}$ is opposite to $\mathbf{v}_d$.

The drift speed is usually very small. This is not a contradiction with the observation that a lamp lights almost immediately after a switch is closed. Changes of the electromagnetic field propagate through the circuit and its surroundings at finite speed. In ordinary wiring the propagation time is far shorter than an electron drift transit time; the actual speed and settling time depend on geometry, dielectrics and circuit response. The distinction between carrier drift and signal propagation is one of the most important conceptual points in this chapter.

Example 6.7: Drift speed in a household wire

A copper wire carries $I = 5.0 \text{ A}$. Its cross-sectional area is $A = 2.5 \text{ mm}^2$, and the density of conduction electrons is approximately $n = 8.5 \times 10^{28} \text{ m}^{-3}$. Estimate the electron drift speed.

Solution. Use $I = JA$ and $J = nev_d$ in magnitude:

$$v_d = \frac{I}{neA}. \quad (6.20)$$

Substitution gives

$$v_d = \frac{5.0 \text{ A}}{(8.5 \times 10^{28} \text{ m}^{-3})(1.602 \times 10^{-19} \text{ C})(2.5 \times 10^{-6} \text{ m}^2)} \quad (6.21)$$

$$= 1.5 \times 10^{-4} \text{ m s}^{-1}. \quad (6.22)$$

This is about 0.15 mm s^{-1} .

Interpretation. The drift is slow. The rapid response of the circuit is not caused by one electron racing from the switch to the lamp.

Caution

A small drift speed does not mean a small current. A macroscopic wire contains an enormous number of mobile electrons per cubic metre. The product $neAv_d$ can be amperes even when v_d is a fraction of a millimetre per second.

Note

Thermal speeds of conduction electrons are not described accurately by a simple classical room-temperature gas picture, because electrons in metals are quantum particles in a Fermi sea. For this chapter, the essential introductory point is only that microscopic random motion is much faster than the small average drift associated with ordinary currents.

Guided checks

Check 1. In a metal, are the electrons drifting in the same direction as conventional current?

Answer. No. They drift opposite to conventional current.

Check 2. If the same current flows through a wire with twice the area, what happens to the drift speed in the same material?

Answer. It is halved, because $v_d = I/(neA)$.

Check 3. Does a lamp turn on quickly because individual electrons move through the whole circuit at near light speed?

Answer. No. The field/signal establishes throughout the circuit much faster than the drift of individual carriers.

6.5.1 Where does the carrier density come from?

The number density n is a model input. For a metal approximated as having z conduction electrons per atom, the atomic count gives

$$n = z \frac{\rho_{\text{mass}}}{M} N_A, \quad (6.23)$$

where ρ_{mass} is mass density, M molar mass and N_A Avogadro's constant. The ratio ρ_{mass}/M has units of moles per cubic metre. It must not be confused with electrical resistivity ρ . For copper, using illustrative rounded values $z = 1$, $\rho_{\text{mass}} = 8.96 \times 10^3 \text{ kg m}^{-3}$, $M = 63.55 \times 10^{-3} \text{ kg mol}^{-1}$ and

$N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$ gives $n \simeq 8.49 \times 10^{28} \text{ m}^{-3}$, the estimate used above. The effective number of conducting carriers in a more complicated solid is not always inferred from a simple integer valence.

For several carrier species the signed contributions must be added:

$$\mathbf{J} = \sum_s n_s q_s \mathbf{v}_{d,s}. \quad (6.24)$$

Equal positive and negative charge densities moving with the *same* velocity make canceling contributions. Oppositely charged species drifting in opposite directions can instead reinforce the same conventional current. Neutrality is a statement about charge density; zero current is a statement about charge-weighted velocities. Neither condition implies the other.

6.6 Resistance, resistivity and conductance

Resistance is a two-terminal property of a device or piece of material under specified conditions. If a potential difference ΔV is maintained across a device and a current I passes through it, the resistance is defined by

$$R = \frac{\Delta V}{I} \quad (6.25)$$

for an operating point with $I \neq 0$. At zero current, the ratio is undefined; the resistance of a linear resistor is obtained by its slope or the appropriate limit. The SI unit is the ohm:

$$1 \Omega = 1 \text{ V A}^{-1}. \quad (6.26)$$

Resistivity ρ is a material property, not merely a device property. For a uniform wire of length L , cross-sectional area A and resistivity ρ , the resistance is

$$R = \rho \frac{L}{A}. \quad (6.27)$$

The conductivity is

$$\sigma = \frac{1}{\rho}. \quad (6.28)$$

Conductance is the reciprocal of resistance:

$$G = \frac{1}{R}. \quad (6.29)$$

The unit of conductance is the siemens, $\text{S} = \Omega^{-1}$.

The distinction between R and ρ is analogous to the distinction between a mechanical object's mass and the density of the material from which it is made. A long thin copper wire has larger resistance than a short thick copper bar, even though the material resistivity is the same at the same temperature and purity.

Definition 6.3: Resistance and resistivity

Resistance R characterizes the current-voltage response of a particular device or sample between two terminals. Resistivity ρ characterizes how strongly a material opposes current density for a given electric field in an isotropic linear regime. Geometry converts material resistivity into sample resistance.

Example 6.8: Resistance of a copper wire

A copper wire has length $L = 12.0$ m, cross-sectional area $A = 1.50$ mm² and resistivity $\rho = 1.68 \times 10^{-8}$ Ω m at room temperature. Estimate its resistance.

Solution. Convert area to SI units: $A = 1.50 \times 10^{-6}$ m². Then

$$R = \rho \frac{L}{A} = (1.68 \times 10^{-8} \Omega \text{ m}) \frac{12.0 \text{ m}}{1.50 \times 10^{-6} \text{ m}^2} = 0.134 \Omega. \quad (6.30)$$

Interpretation. The resistance is much smaller than a typical laboratory resistor, but not zero. At several amperes this wire can still dissipate noticeable power.

Example 6.9: Voltage drop in a supply-and-return cable

A device is 20 m from a DC supply and draws 8.0 A. Two identical copper conductors provide the outgoing and return paths. Require their *combined* voltage drop to be at most 0.50 V. Use $\rho = 1.7 \times 10^{-8}$ Ω m and neglect contact resistance and heating initially.

Strategy. Current traverses both conductors. The resistive path length is $2L = 40$ m, so $R_{\text{pair}} = 2\rho L/A$. The maximum pair resistance is

$$R_{\text{max}} = \frac{0.50}{8.0} \Omega = 0.0625 \Omega.$$

Consequently each conductor must have area

$$A \geq \frac{2\rho L}{R_{\text{max}}} = \frac{2(1.7 \times 10^{-8})(20)}{0.0625} \text{ m}^2 = 10.9 \text{ mm}^2 \simeq 11 \text{ mm}^2.$$

At the limiting area the pair dissipates $P = I\Delta V = 4.0$ W. Doubling area halves both voltage drop and power loss at the same specified current.

Check and limitation. Counting only one conductor would underestimate the required area by a factor of two. This is a voltage-drop estimate; it does not establish a thermal current rating. A real design must also account for operating resistivity, contacts and heat removal.

Method

For a uniform wire resistance calculation, use the chain

$$\text{material} \rightarrow \rho, \quad \text{geometry} \rightarrow L, A, \quad R = \rho L/A.$$

Do not insert current into $R = \rho L/A$. Current determines voltage drop and power once R is known.

Guided checks

Check 1. If the length of a uniform wire is doubled, what happens to its resistance?

Answer. It doubles.

Check 2. If the cross-sectional area is doubled, what happens to the resistance?

Answer. It is halved.

Check 3. Is resistivity a property of a sample's shape?

Answer. No. Resistivity is a material property under specified conditions; resistance also depends on geometry.

6.6.1 Geometry and contact placement are part of the model

A quoted resistance is always between specified terminals. Full-face contacts at the ends of a long uniform bar are approximately equipotential and establish nearly parallel current paths. Small contacts at arbitrary positions produce spreading and crowding. In that case assigning the object's longest dimension to L and an arbitrary area to A need not give the measured resistance.

The dimensions of resistivity follow from $\rho = RA/L$:

$$[\rho] = \Omega \text{ m}^2/\text{m} = \Omega \text{ m}.$$

This dimensional check distinguishes resistivity from resistance per length, R/L , whose units are Ω/m . The latter still depends on the wire area.

The following retained example shows why a drawing of the contacted faces is useful. In Fig. 6.6, rotating the contacts changes both path length and transverse area, even though the material is unchanged.

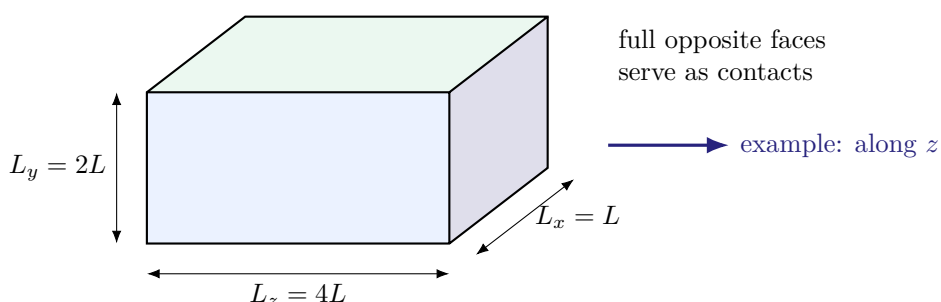


Figure 6.6: A rectangular bus bar with dimensions L , $2L$ and $4L$, drawn schematically. Each resistance calculation uses the dimension along the current as path length and the product of the other two as cross-section.

Example 6.10: A rectangular bus bar: resistance along different directions

A rectangular copper block has dimensions $L_x = L$, $L_y = 2L$ and $L_z = 4L$. The resistivity is ρ . Large conducting plates are attached to two opposite faces so that current flows uniformly along one coordinate direction at a time. Find the resistance for current along x , y and z .

Model. The contacts cover entire opposite faces, so the current density is treated as uniform and one-dimensional. The length is the distance between contacts, and the area is the area of the contacted face.

Solution. For current along x , the length is L and the cross-sectional area is $(2L)(4L) = 8L^2$, so

$$R_x = \rho \frac{L}{8L^2} = \frac{\rho}{8L}. \quad (6.31)$$

For current along y , the length is $2L$ and the area is $(L)(4L) = 4L^2$, so

$$R_y = \rho \frac{2L}{4L^2} = \frac{\rho}{2L}. \quad (6.32)$$

For current along z , the length is $4L$ and the area is $(L)(2L) = 2L^2$, so

$$R_z = \rho \frac{4L}{2L^2} = \frac{2\rho}{L}. \quad (6.33)$$

Interpretation. The largest resistance occurs when current must travel along the longest dimension through the smallest cross-section. Geometry matters as much as material.

6.6.2 Thin films and resistance per square

A conducting film of thickness t , length L and width w has transverse area wt . For uniform in-plane transport,

$$R = \frac{\rho L}{t w} = R_{\square} \frac{L}{w}, \quad R_{\square} = \frac{\rho}{t}. \quad (6.34)$$

The sheet resistance $R_{\square}$ has units of ohms. The phrase “ohms per square” emphasizes that L/w is a dimensionless number of squares in series, not a new SI unit or resistance divided by an area. A square film with full contacts on opposite edges has resistance $R_{\square}$ regardless of its lateral size, provided the thickness and material response remain the same.

This is useful for patterned resistors and conductive coatings. A meander lengthens the current path within a compact footprint. The simple count of squares neglects resistance changes at turns and contact spreading; narrow corners may produce crowding. For very thin films the resistivity itself can depend on thickness and microstructure, so bulk ρ should not automatically be inserted.

Example 6.11: A film resistor and a geometry check

A film has measured sheet resistance $R_{\square} = 12 \Omega$ and a straight rectangular active region $L = 8.0 \text{ mm}$, $w = 0.50 \text{ mm}$ with full-width contacts. Find its resistance and power at 50 mA. What happens if both L and w are doubled at fixed thickness?

$$R = 12 \frac{8.0}{0.50} = 192 \Omega, \quad P = I^2 R = (0.050)^2 (192) = 0.480 \text{ W}.$$

Doubling both lateral dimensions leaves L/w , resistance and total power unchanged at the same current. The active volume and plan area increase by four, so average volumetric power decreases by four. Temperature will also depend on the substrate and heat flow; unchanged electrical resistance does not imply unchanged thermal behavior.

6.7 Ohm’s law as a linear-response model

Ohm’s law appears in two closely related forms. At the material level, an isotropic ohmic conductor in the linear regime satisfies

$$\mathbf{J} = \sigma \mathbf{E}, \quad (6.35)$$

where σ is the conductivity. At the two-terminal device level, an ohmic resistor with constant resistance satisfies

$$\Delta V = IR. \quad (6.36)$$

These equations are not definitions of all electrical behaviour. They describe a class of materials and devices over a specified range of temperature, field strength, current, frequency and physical condition.

For a uniform wire of length L and area A , Eq. (6.35) leads to Eq. (6.27). If the electric field is approximately uniform along the wire,

$$E \simeq \frac{\Delta V}{L}, \quad (6.37)$$

and if $J = I/A$, then

$$\frac{I}{A} = \sigma \frac{\Delta V}{L}. \quad (6.38)$$

Rearranging gives

$$\Delta V = I \frac{L}{\sigma A} = I \rho \frac{L}{A}, \quad (6.39)$$

so the resistance is $R = \rho L/A$.

Theoretical point 6.1: Status of Ohm's law

Ohm's law is not a fundamental conservation law. It is a constitutive relation: a relation describing how a material responds to an applied electric field under specified conditions. Charge conservation and energy conservation are deeper constraints. Ohm's law is an accurate model for many metals at moderate fields and fixed temperature. Diodes and gas discharges are generally nonlinear; a self-heated lamp can be nonlinear even though its metal is locally ohmic at fixed temperature. A superconductor requires additional dynamical and magnetic physics beyond a finite positive conductivity model.

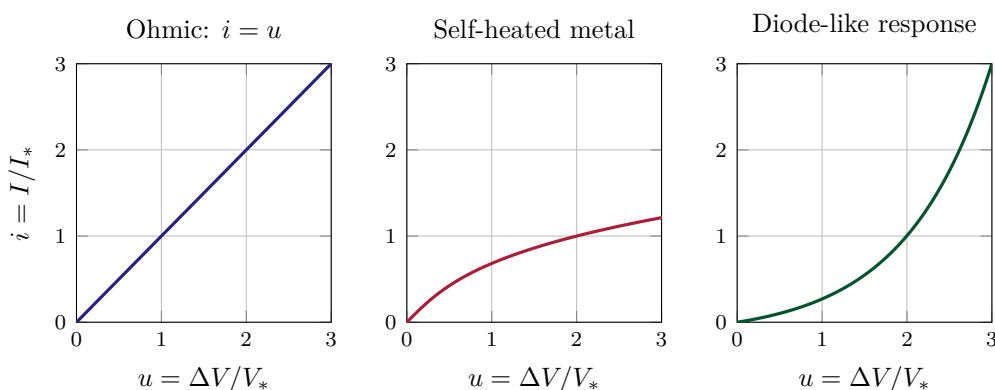


Figure 6.7: Dimensionless illustrative models, not measured device data. The middle curve is parametrized by $u = i(1 + i^2)$, a phenomenological low-heating approximation with increasing static resistance. Its I – V slope decreases as the sample heats. The right curve is $i = 3(\exp u - 1)/(\exp 3 - 1)$ and illustrates increasing forward conductance; it is not a calibration of a particular diode. Each panel may use different positive scales I_* , V_* .

Example 6.12: Resistance from an I – V graph

A resistor is tested with a DC supply. The illustrative data points are $(\Delta V, I) = (1.0 \text{ V}, 0.050 \text{ A})$, $(2.0 \text{ V}, 0.101 \text{ A})$, $(3.0 \text{ V}, 0.149 \text{ A})$ and $(4.0 \text{ V}, 0.201 \text{ A})$. Estimate the resistance.

Solution. The points lie close to a straight line through the origin. The slope of the I – V graph is approximately

$$\frac{I}{\Delta V} \simeq \frac{0.201 \text{ A}}{4.0 \text{ V}} = 0.0503 \text{ A V}^{-1}. \quad (6.40)$$

Thus

$$R = \frac{1}{\text{slope}} \simeq 19.9 \Omega. \quad (6.41)$$

Interpretation. The ratios cluster near 20Ω . This is a useful estimate, but a statistical claim of agreement requires measurement uncertainties and residuals; Section 6.12 develops that test.

Example 6.13: Identifying non-ohmic behaviour from data

A device has the following illustrative data, with uncertainties assumed small compared with the changes shown: $(\Delta V, I) = (1.0 \text{ V}, 0.10 \text{ A})$, $(2.0 \text{ V}, 0.17 \text{ A})$, $(3.0 \text{ V}, 0.22 \text{ A})$ and $(4.0 \text{ V}, 0.26 \text{ A})$. Does it have a constant resistance over this range?

Solution. Compute the operating-point resistance $R = \Delta V/I$:

$$R_1 = 10 \Omega, \quad R_2 = 11.8 \Omega, \quad R_3 = 13.6 \Omega, \quad R_4 = 15.4 \Omega. \quad (6.42)$$

The resistance changes significantly with voltage. The device is non-ohmic over this range.

Interpretation. A single resistance value may still approximate the device near one operating point, but not over the full range.

Caution

The equation $R = \Delta V/I$ can define an operating-point resistance even for a non-ohmic device. Ohm's law requires more: R must be independent of ΔV over the relevant operating range. Do not conclude that every device obeys Ohm's law just because $\Delta V/I$ can be calculated.

Guided checks

Check 1. Does a diode obey Ohm's law over a wide voltage range?

Answer. No. Its I - V curve is strongly nonlinear.

Check 2. Can a material be approximately ohmic in one temperature range but not another?

Answer. Yes. Material response depends on temperature and operating conditions.

Check 3. What does the slope of an I - V graph represent for an ohmic resistor?

Answer. The conductance $1/R$.

6.7.1 From material response to a terminal law: the full argument

The relation $R = \rho L/A$ is often presented as a rule about longer and thinner wires. A more complete derivation makes clear which field problem has been solved. Consider a homogeneous bar of length L , uniform area A , scalar conductivity σ , insulating sidewalls and ideal end contacts covering the entire faces. The left face is held at potential V_a and the right at V_b , with $V_a > V_b$.

A candidate solution is

$$V(x) = V_a - \frac{V_a - V_b}{L}x. \quad (6.43)$$

It has the prescribed end values. The electric field is $\mathbf{E} = -(\mathrm{d}V/\mathrm{d}x)\hat{\mathbf{x}} = [(V_a - V_b)/L]\hat{\mathbf{x}}$, which is uniform and parallel to the sides. Thus no current crosses the sidewalls, and $\mathbf{J} = \sigma\mathbf{E}$ is uniform. Its divergence is zero, consistent with steady charge conservation. Integrating over a transverse face gives

$$I = \sigma \frac{V_a - V_b}{L} A, \quad \frac{V_a - V_b}{I} = \frac{L}{\sigma A} = \rho \frac{L}{A}. \quad (6.44)$$

The result follows from material response plus boundary conditions. It does not follow from dimensional reasoning alone. Dimensional reasoning can suggest $\rho L/A$ but cannot decide whether the actual field is uniform.

Figure 6.8 ties the potential slope, electric-field direction and current together. Positive conventional current goes from higher to lower potential. Electrons drift the other way, but their force and drift are both opposite to $\mathbf{E}$, so the electric work rate on the electron ensemble is positive. Confusing electron direction with the sign of power is a common consequence of omitting the carrier charge from $\mathbf{J} = nq_c\mathbf{v}_d$.

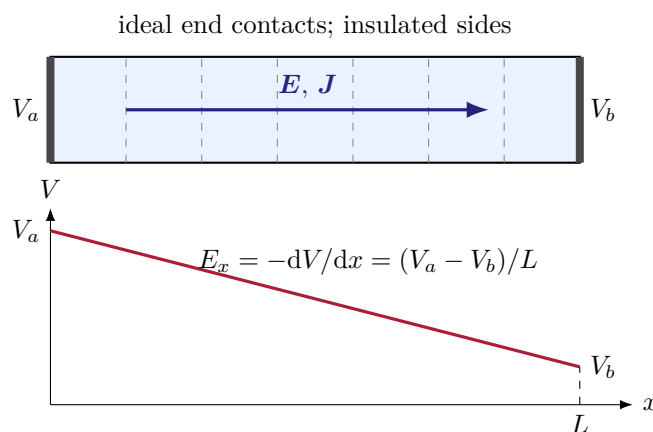


Figure 6.8: The uniform-bar solution. Dashed vertical lines in the bar represent equally spaced equipotentials. The potential decreases linearly, while $\mathbf{E}$ and $\mathbf{J}$ point right. The lower graph allows an arbitrary potential reference; only the terminal difference matters.

Example 6.14: A microscopic and macroscopic consistency check

A homogeneous copper bar has $L = 0.50$ m, $A = 2.0$ mm² and $\rho = 1.7 \times 10^{-8}$ Ω m. A small terminal voltage drop 17 mV is maintained. Use $n = 8.5 \times 10^{28}$ m⁻³ and $e = 1.602 \times 10^{-19}$ C. Find current and drift speed, and check the power through two descriptions.

Field route. The uniform field is $E = 0.017/0.50 = 0.034$ V m⁻¹. Thus $J = E/\rho = 2.0 \times 10^6$ A m⁻², $I = JA = 4.0$ A and

$$v_d = \frac{J}{ne} = 1.47 \times 10^{-4} \text{ m s}^{-1}.$$

The electron drift is opposite to the current. The field need not be large to generate an ampere-scale current in a good metal.

Terminal route. The resistance is $\rho L/A = 4.25$ m Ω , so $I = \Delta V/R = 4.0$ A again. The terminal power is $I\Delta V = 0.068$ W. The local density is $JE = 6.8 \times 10^4$ W m⁻³, which gives 0.068 W when multiplied by the bar volume $AL = 1.0 \times 10^{-6}$ m³.

Interpretation. Small field, slow drift and appreciable total current are mutually consistent because the mobile carrier density is enormous. The agreement of the two power calculations checks the model across microscopic, local and terminal descriptions.

6.7.2 What changes when a material is locally ohmic?

A useful model-testing question is to double the imposed terminal voltage while keeping geometry and temperature fixed. In the uniform-bar model, E , J , drift speed and total current all double; resistance remains unchanged and power increases by four. If the device is then allowed to warm, conductivity can change and these simple scalings no longer describe the final state. This separates an isothermal material test from a steady operating test that includes heat transfer.

The supplied I - V models in Fig. 6.7 illustrate the same distinction. A heated metallic element can show decreasing incremental conductance as voltage increases. A diode can show increasing forward conductance because of its junction physics, even at approximately fixed temperature. Both are nonlinear terminal responses, but the mechanisms and suitable models differ. The observation of curvature alone does not identify its cause.

6.8 Microscopic view: scattering, mobility and the Drude estimate

A simple microscopic model helps explain why a metal can have a linear relation between $\mathbf{J}$ and $\mathbf{E}$. Suppose an electron of mass m and charge $-e$ experiences an electric force $-e\mathbf{E}$. Between scattering events it accelerates, but scattering relaxes the directed momentum. If τ is an effective momentum-relaxation time, the average drift velocity magnitude is roughly

$$v_d \simeq \frac{eE\tau}{m}. \quad (6.45)$$

The current density magnitude is therefore

$$J = nev_d \simeq \frac{ne^2\tau}{m}E. \quad (6.46)$$

Thus the conductivity estimate is

$$\sigma \simeq \frac{ne^2\tau}{m}, \quad \rho \simeq \frac{m}{ne^2\tau}. \quad (6.47)$$

This is the Drude model estimate. It is classical and incomplete, but it gives useful intuition: more carriers and longer mean scattering time increase conductivity, while stronger scattering increases resistivity.

Note

The Drude model is not a complete theory of metals. A realistic description uses quantum mechanics, band structure, the Pauli exclusion principle and electron-phonon scattering. Nevertheless, the Drude estimate is pedagogically valuable because it connects macroscopic resistivity to carrier density and scattering.

Example 6.15: Estimating a scattering time from resistivity

Use the Drude estimate to find the effective scattering time for copper with $\rho = 1.68 \times 10^{-8} \Omega \text{ m}$ and $n = 8.5 \times 10^{28} \text{ m}^{-3}$. Use $m = 9.11 \times 10^{-31} \text{ kg}$ and $e = 1.602 \times 10^{-19} \text{ C}$.

Solution. From Eq. (6.47),

$$\tau = \frac{m}{ne^2\rho}. \quad (6.48)$$

Substituting,

$$\tau = \frac{9.11 \times 10^{-31}}{(8.5 \times 10^{28})(1.602 \times 10^{-19})^2(1.68 \times 10^{-8})} \text{ s} = 2.5 \times 10^{-14} \text{ s}. \quad (6.49)$$

Interpretation. This very short time is an effective scattering time. It explains why a small electric field produces only a small average drift despite the continuous electric force.

Definition 6.4: Mobility

The mobility μ of a carrier is defined in the linear regime by

$$|\mathbf{v}_d| = \mu|\mathbf{E}|. \quad (6.50)$$

For one carrier type, $\sigma = n|q_c|\mu$. Mobility is especially useful in semiconductor physics, where carrier density and mobility can vary strongly with doping, temperature and material quality.

Guided checks

Check 1. In the Drude model, what happens to resistivity if scattering becomes more frequent?

Answer. The mean time τ decreases, so resistivity increases.

Check 2. Why does a larger carrier density usually increase conductivity?

Answer. More mobile charges contribute to the current for the same drift velocity.

Check 3. Does the Drude model replace quantum solid-state physics?

Answer. No. It is an introductory model that captures useful trends but not the full microscopic theory.

6.8.1 Why a steady drift does not imply zero force

A free electron accelerated by a constant electric field would gain directed speed indefinitely. A resistive conductor is different: directed momentum is continually transferred to other degrees of freedom. The steady drift is an average maintained by a balance of drive and relaxation. It is analogous to a terminal velocity in a dissipative mechanical model, but the relaxation term describes an ensemble of quantum carriers rather than a literal viscous force on an isolated electron.

Writing the averaged equation explicitly removes ambiguity about factors of two:

$$m \frac{d\mathbf{v}_d}{dt} = q_c \mathbf{E} - \frac{m\mathbf{v}_d}{\tau}. \quad (6.51)$$

For a steady field and a settled state, the derivative vanishes and $\mathbf{v}_d = q_c \tau \mathbf{E} / m$. Multiplication by nq_c gives $\mathbf{J} = (nq_c^2 \tau / m) \mathbf{E}$. The conductivity is positive for either sign of carrier charge. The transient solution and the assumptions behind this relaxation model are developed in the advanced enrichment.

The time τ refers to loss of *directed momentum*. It need not equal every microscopic collision time. A perfect periodic lattice does not act as a collection of independent classical obstacles; phonons, disorder, impurities and other momentum-relaxing processes determine the effective response. This is why the estimate is useful without being a full theory of resistivity.

6.9 Electric power and Joule heating

Choose current I entering terminal a and leaving terminal b , and define $\Delta V = V_a - V_b$. During a short interval, the signed transported charge is $dq = I dt$. The electrical work delivered to a two-terminal element in the lumped, quasistatic description is

$$dW = (V_a - V_b) dq.$$

Therefore the signed absorbed power is

$$P = \frac{dW}{dt} = I \Delta V. \quad (6.52)$$

This sign convention does not assume that the element absorbs energy: if current exits the terminal labeled positive, $I < 0$ and the element delivers electrical power. A motor, a charging cell and a resistor may all absorb positive electrical power, but the forms of energy produced differ.

For a resistor represented at its operating state by $\Delta V = IR$,

$$P = I^2 R, \quad (6.53)$$

$$P = \frac{(\Delta V)^2}{R}. \quad (6.54)$$

For a linear resistor, R is constant over the stated range. A temperature-dependent or nonlinear resistive element can also satisfy these algebraic forms *at a specified operating point* when $R = R_{\text{static}} = \Delta V/I$ is used. This does not imply Ohm's law over a range, and substituting the differential resistance into $I^2 R$ generally gives the wrong total power.

In a resistor the absorbed power becomes internal energy and ultimately heat transfer or radiation to the surroundings. “Dissipation” does not mean energy destruction: it describes conversion to less readily recoverable forms. Figure 6.9 distinguishes conserved charge flow from this energy transfer.

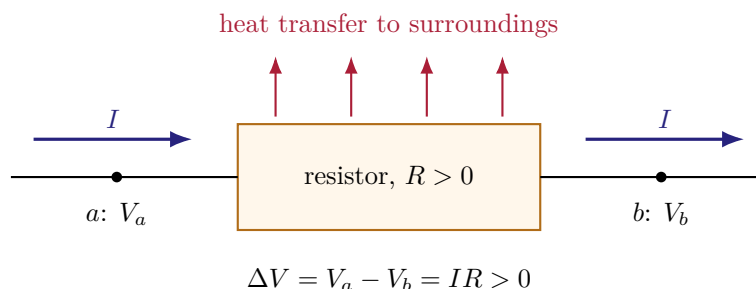


Figure 6.9: Passive sign convention for a resistor. Equal currents enter and leave, while absorbed electrical power is $I(V_a - V_b)$. At steady temperature the heat loss equals that power; during warming some energy increases the resistor's internal energy.

Example 6.16: Power dissipated in a heating element

A heating element has resistance $R = 26.5 \Omega$ when operating and is connected across a constant DC voltage $\Delta V = 230 \text{ V}$. Estimate the current and power.

Solution. Assuming the element is ohmic at its operating temperature,

$$I = \frac{\Delta V}{R} = \frac{230 \text{ V}}{26.5 \Omega} = 8.68 \text{ A.} \quad (6.55)$$

The power is

$$P = I\Delta V = (8.68 \text{ A})(230 \text{ V}) = 2.00 \times 10^3 \text{ W.} \quad (6.56)$$

Interpretation. This is a 2.0 kW heater. In one hour it transfers about 2.0 kWh of energy.

Example 6.17: Energy use from electrical power

A device consumes 75 W for 6.0 h each day. How much energy does it use in 30 days, in kilowatt-hours?

Solution. The operating time is 180 h and the power is 0.075 kW, so

$$E = P\Delta t = (0.075 \text{ kW})(180 \text{ h}) = 13.5 \text{ kWh.} \quad (6.57)$$

Interpretation. The kilowatt-hour is an energy unit. It is not a unit of power.

6.9.1 Energy, average power and the thermal destination

For varying current or voltage the correct energy expression is

$$W = \int_{t_1}^{t_2} I(t)\Delta V(t) dt. \quad (6.58)$$

Power and energy must be distinguished in both language and units: $1 \text{ kWh} = 10^3 \text{ J s}^{-1} \times 3600 \text{ s} = 3.6 \times 10^6 \text{ J}$. Over a time interval, average power is $W/\Delta t$. In general the average of a product is not

the product of the averages. For a fixed resistor, $\langle P \rangle = R\langle I^2 \rangle$, which need not equal $R\langle I \rangle^2$. The bipolar pulse already illustrates this distinction without requiring AC circuit theory.

Temperature is determined by an energy balance, not by electrical power alone. With effective heat capacity C_{th} and heat loss $\dot{Q}_{\text{loss}}$,

$$C_{\text{th}} \frac{dT}{dt} = P_{\text{el}} - \dot{Q}_{\text{loss}}. \quad (6.59)$$

A body can absorb electrical power continuously at constant temperature when its heat loss is equal to that input. Conversely, the same power can produce very different temperatures in an insulated component and a well-cooled component.

Example 6.18: Electrical heating and a calorimetric check

A 24.0 V DC heater raises 0.300 kg of water through 15.0 K in 180 s. Take $c = 4.18 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1}$. A calibrated fraction $\eta = 0.80$ of the electrical input reaches the water. Find the heater's operating power, current and resistance.

Model. Use a uniform water temperature, constant specific heat, negligible evaporation and a constant effective efficiency over the interval. The water receives

$$Q = mc\Delta T = 0.300(4180)(15.0) = 1.881 \times 10^4 \text{ J}.$$

Since $Q = \eta P \Delta t$,

$$P = \frac{1.881 \times 10^4}{0.80(180)} = 131 \text{ W}, \quad I = \frac{P}{24.0} = 5.44 \text{ A}, \quad R_{\text{hot}} = \frac{24.0^2}{P} = 4.41 \Omega.$$

Check. The electrical input is $2.35 \times 10^4 \text{ J}$, exceeding the water's gain by the assumed losses. Taking $\eta = 1$ would give only 105 W, underestimating the required input. The derived resistance is an operating value, not a prediction of the cold resistance.

Example 6.19: Electrical input is not all winding heat

A small DC motor takes 0.80 A at 12.0 V while lifting a 1.50 kg mass vertically at 0.40 m s^{-1} . With $g = 9.81 \text{ m s}^{-2}$, find electrical input, useful mechanical power and the remaining loss.

Solution. At constant speed,

$$P_{\text{in}} = IV = 9.60 \text{ W}, \quad P_{\text{mech}} = mgv = 5.89 \text{ W}, \quad P_{\text{loss}} = 3.71 \text{ W}.$$

The useful efficiency is $5.89/9.60 = 0.613$. The ratio $V/I = 15.0 \Omega$ is not the winding resistance. If *all* loss were winding Joule heat, one would infer $R_w = P_{\text{loss}}/I^2 = 5.80 \Omega$; friction and magnetic losses would make the actual winding resistance smaller than this inferred value. The terminal power and its conversion must be separated before applying a resistive model.

Caution

The general terminal relation is $P = I\Delta V$ with paired reference directions. The formulas $I^2 R$ and $(\Delta V)^2/R$ describe resistive absorption with the relevant resistance. Neither the differential resistance of a nonlinear device nor the terminal ratio V/I of an energy-converting machine may automatically be identified with a physical Joule-heating resistance.

6.10 Temperature dependence of resistivity

Resistivity depends on temperature; Fig. 6.10 compares two specified models. For many metals over a moderate temperature range near a reference temperature T_0 , a useful linear approximation is

$$\rho(T) = \rho_0 [1 + \alpha(T - T_0)], \quad (6.60)$$

where $\rho_0 = \rho(T_0)$ and α is the temperature coefficient of resistivity. For many pure metals near room temperature, α is positive: resistivity increases with temperature because lattice vibrations increase scattering. For semiconductors, the trend can be very different because carrier density may increase strongly with temperature.

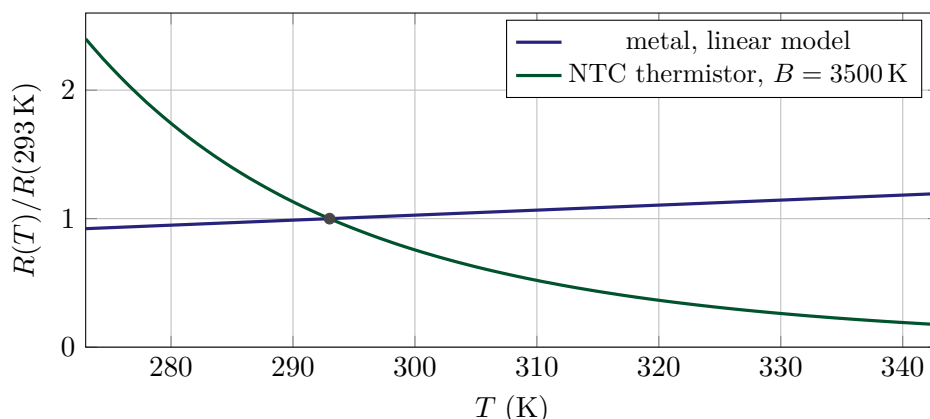


Figure 6.10: Calculated normalized resistance curves at fixed geometry. The metal model uses $\alpha = 3.9 \times 10^{-3} \text{ K}^{-1}$ and the illustrative thermistor model uses $R/R_0 = \exp[B(1/T - 1/T_0)]$ with $T_0 = 293 \text{ K}$. These specified models replace universal-looking material curves; semiconductor temperature trends depend on carrier density and mobility.

Example 6.20: Temperature correction for a copper wire

A copper wire has resistance $R_0 = 10.0 \Omega$ at $T_0 = 20^\circ\text{C}$. Estimate its resistance at $T = 70^\circ\text{C}$ using $\alpha = 3.9 \times 10^{-3} \text{ K}^{-1}$.

Solution. For fixed geometry, resistance scales like resistivity:

$$R(T) = R_0 [1 + \alpha(T - T_0)]. \quad (6.61)$$

The temperature change is 50 K, so

$$R = (10.0 \Omega)[1 + (3.9 \times 10^{-3} \text{ K}^{-1})(50 \text{ K})] = 11.95 \Omega. \quad (6.62)$$

Interpretation. The resistance increases by about 20%. Temperature effects are not small in precision measurements or high-power devices.

Method

Temperature differences in degrees Celsius and kelvin have the same numerical size. In Eq. (6.60), $T - T_0$ may be inserted as 50 K for a change from 20°C to 70°C . Do not insert absolute Celsius values where absolute thermodynamic temperature is required in other contexts.

Guided checks

Check 1. Why does a metal's resistance often increase when heated?

Answer. Increased lattice vibrations increase scattering of mobile electrons.

Check 2. Does every material have positive α ?

Answer. No. Semiconductors and other materials can behave differently.

Check 3. Is the linear law exact for all temperatures?

Answer. No. It is a local approximation over a limited range.

6.10.1 Temperature coefficients are local derivatives

The precise local coefficient at a reference temperature is

$$\alpha_0 = \frac{1}{\rho(T_0)} \left. \frac{d\rho}{dT} \right|_{T_0}. \quad (6.63)$$

The linear formula is a first-order Taylor approximation. Even within an exactly linear model the normalized coefficient referred to a different temperature changes:

$$\alpha(T_1) = \frac{\alpha_0}{1 + \alpha_0(T_1 - T_0)}.$$

Thus a tabulated coefficient should be used with its reference temperature. Extrapolating a room-temperature straight line until it predicts negative resistivity is a failure of the approximation, not a physical prediction.

The approximation $R(T)/R_0 = \rho(T)/\rho_0$ also neglects thermal expansion. If a freely expanding isotropic specimen has linear expansion coefficient β , then to first order $L/L_0 \simeq 1 + \beta\Delta T$ and $A/A_0 \simeq 1 + 2\beta\Delta T$. Hence

$$\frac{R(T)}{R_0} \simeq 1 + (\alpha_0 - \beta)\Delta T. \quad (6.64)$$

For many metals near room temperature the geometric correction is much smaller than the resistivity change, but it can matter in precision measurements. A bonded strain gauge may instead be mechanically constrained; its calibration then includes the actual mounting conditions.

A negative-temperature-coefficient (NTC) thermistor can be modeled over a stated interval by

$$R(T) = R_0 \exp \left[B \left(\frac{1}{T} - \frac{1}{T_0} \right) \right], \quad T, T_0 > 0, \quad (6.65)$$

with absolute temperatures in kelvin and $B > 0$ in kelvin. Differentiating gives $R^{-1}dR/dT = -B/T^2$. This stronger, nonlinear temperature sensitivity is useful for sensing; it does not mean that every semiconductor has the same temperature dependence. Calibration and self-heating are treated below and in the advanced enrichment.

6.11 Semiconductors, superconductors and non-ohmic devices

A semiconductor has electrical behaviour controlled by carrier density, mobility, doping, temperature, illumination and device structure. In an intrinsic semiconductor the number of mobile carriers can be small at low temperature, but thermal excitation or doping can create electrons and holes that carry current. Doping with donor or acceptor atoms changes the carrier population and is the foundation of diodes, transistors, sensors, photodetectors and integrated circuits.

A superconductor can sustain a nondissipative DC transport state within a material- and geometry-dependent range of temperature, magnetic field and current. These limits are interdependent, and merely cooling below a critical temperature does not guarantee loss-free operation at arbitrary field

or current. Zero DC resistance is accompanied by distinct magnetic and quantum properties; it is not a complete theory of the superconducting state. In particular, type-II materials can admit magnetic flux in vortices, and moving vortices can produce dissipation. Chapter 21 develops the solid-state context; magnetic energy and magnet protection connect to Chapters 9–10.

At fixed temperature and sufficiently small measuring field, a homogeneous semiconductor can be ohmic. Temperature dependence of resistance is therefore distinct from intrinsic nonlinearity of an isothermal I – V curve.

Non-ohmic devices include diodes, filament lamps over a large temperature range, gas discharge tubes, thermistors, varistors, light-dependent resistors and many biological or electrochemical interfaces. Their I – V curves are not straight lines through the origin. They may still be operated in a small-signal regime where a differential resistance is meaningful.

Definition 6.5: Static and differential resistance

For a two-terminal device at an operating point, the static resistance is

$$R_{\text{static}} = \frac{\Delta V}{I}. \quad (6.66)$$

The differential resistance is

$$r_{\text{diff}} = \frac{d(\Delta V)}{dI} \quad (6.67)$$

at that operating point. For a perfectly ohmic resistor these are the same constant. For a nonlinear device they are generally different.

For electron and hole drift in a homogeneous semiconductor, let n be electron density and p_h hole density; the subscript avoids confusion with power density later. With positive mobilities μ_n and μ_p , electrons drift as $\mathbf{v}_{d,n} = -\mu_n \mathbf{E}$ and holes as $\mathbf{v}_{d,p} = \mu_p \mathbf{E}$. Equation (6.24) then gives

$$\mathbf{J} = e(n\mu_n + p_h\mu_p)\mathbf{E}, \quad \sigma = e(n\mu_n + p_h\mu_p). \quad (6.68)$$

A hole represents an unoccupied electronic state with an effective positive carrier response; it is not a proton moving through the crystal. Density gradients can also produce diffusion currents, so the drift formula alone is not a complete junction model.

Example 6.21: Static and differential resistance of a simple nonlinear model

A device has current-voltage relation

$$I = a(\Delta V)^2 \quad (6.69)$$

for positive voltage, with $a = 0.020 \text{ A V}^{-2}$. Find the static and differential resistance at $\Delta V = 5.0 \text{ V}$.

Solution. The current is

$$I = (0.020)(5.0)^2 \text{ A} = 0.50 \text{ A}. \quad (6.70)$$

The static resistance is

$$R_{\text{static}} = \frac{5.0 \text{ V}}{0.50 \text{ A}} = 10 \Omega. \quad (6.71)$$

Because $I = aV^2$, $dI/dV = 2aV$, so

$$r_{\text{diff}} = \frac{dV}{dI} = \frac{1}{2aV} = \frac{1}{2(0.020)(5.0)} \Omega = 5.0 \Omega. \quad (6.72)$$

Interpretation. The two resistance concepts differ because the device is nonlinear.

Example 6.22: Current density with two carrier types

A semiconductor has electron density $n = 2.0 \times 10^{21} \text{ m}^{-3}$ and hole density $p_h = 5.0 \times 10^{20} \text{ m}^{-3}$. Mobilities are $\mu_n = 0.12 \text{ m}^2 \text{ V}^{-1} \text{ s}^{-1}$ and $\mu_p = 0.045 \text{ m}^2 \text{ V}^{-1} \text{ s}^{-1}$. Estimate conductivity.

Solution. In the simple two-carrier drift model,

$$\sigma = e(n\mu_n + p_h\mu_p). \quad (6.73)$$

Substitution gives

$$\sigma = (1.602 \times 10^{-19})[(2.0 \times 10^{21})(0.12) + (5.0 \times 10^{20})(0.045)] \quad (6.74)$$

$$= 38.5 \text{ S m}^{-1} + 3.60 \text{ S m}^{-1} = 42.1 \text{ S m}^{-1}. \quad (6.75)$$

Interpretation. Electrons dominate here because both their density and mobility are larger.

Caution

A superconducting loop can carry persistent current without a voltage drop in the ideal DC state, but it is not a perfect ordinary wire in every situation. Critical currents, magnetic fields, material defects, AC losses and transitions to the normal state matter in real systems.

Guided checks

Check 1. Is a semiconductor simply a poor metal?

Answer. No. Its carrier density and response can be engineered strongly by doping, fields, illumination and temperature.

Check 2. Does zero resistance mean infinite current in any superconducting device?

Answer. No. Real superconductors have critical currents and stability limits.

Check 3. Why can differential resistance matter?

Answer. Small signal variations around a nonlinear operating point are governed by the local slope of the I - V curve, not necessarily by V/I .

6.12 Laboratory and data analysis: what resistance is measured?

A resistance measurement joins a physical device model to an instrument model. The desired quantities are the current through the specimen and the voltage between specified specimen terminals. A supply display need not give that terminal voltage, because leads, contacts and meters may have appreciable drops. Nor is a single ratio V/I enough to show that the device is ohmic over a range.

6.12.1 Separate current delivery from voltage sensing

In the simplest two-wire method the same conductors carry the excitation current and connect the voltage measurement. If each lead and contact combination has resistance R_ℓ , the inferred value is approximately

$$R_{\text{two}} = R_s + 2R_\ell, \quad (6.76)$$

where R_s is the sample resistance. This error is small for a kilohm specimen with milliohm leads, but can dominate a milliohm specimen.

The four-wire (Kelvin) arrangement in Fig. 6.11 uses separate sense connections at the boundaries of the region of interest. A high-input-resistance voltmeter draws very little current through its leads, making their voltage drops small. Contact drops outside the sense points are excluded. Any contact or bulk resistance inside the sense boundaries remains part of the measured quantity. Four-wire

sensing is therefore a precise definition of *where* the voltage is measured, not a claim that all contact effects disappear.

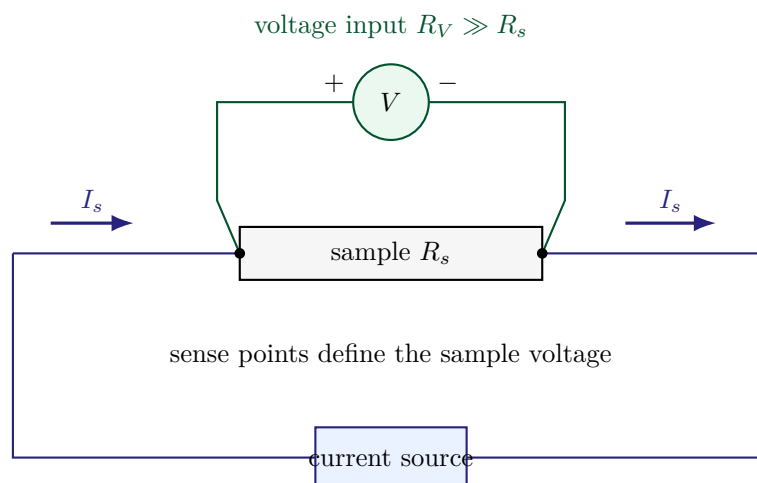


Figure 6.11: Four-wire measurement with a complete excitation loop and a voltmeter inserted between separate sense leads. There is no wire directly joining the sense nodes. Equality of source and sample currents is the approximation that voltmeter loading is negligible.

If the voltmeter input resistance is finite, it provides another current path. Ignoring sense-lead drops for this estimate, $I_s = I_R + I_V$, with $I_V = V/R_V$. Dividing the measured voltage by the *source* current gives

$$\frac{V}{I_s} = \frac{R_s R_V}{R_s + R_V} \simeq R_s \left(1 - \frac{R_s}{R_V}\right), \quad R_s \ll R_V.$$

For a low-resistance sample the loading correction is usually tiny. For very high resistance it can dominate, even though a four-wire connection removes series lead drops. Detailed meter placement in general circuits is developed in Chapter 7.

Example 6.23: Lead error and Kelvin sensing

A $20.0 \text{ m}\Omega$ specimen is connected by two leads, each with $15.0 \text{ m}\Omega$ resistance including its current contact. A current source supplies 1.00 A .

Two-wire reading. The voltage including both leads is $1.00(0.0200 + 0.0300) = 0.0500 \text{ V}$, giving $50.0 \text{ m}\Omega$. The relative error compared with the specimen is 150%.

Four-wire reading. Sense leads at the specimen terminals measure approximately 0.0200 V . With $R_V = 10 \text{ M}\Omega$, the sense current is about 2.00 nA , so loading is negligible at this scale. The specimen dissipates $I^2 R_s = 20.0 \text{ mW}$ and the current leads together dissipate 30.0 mW ; four-wire sensing removes a voltage error, not these heat sources.

6.12.2 Offsets, current reversal and self-heating

Thermoelectric junctions and instrument zero errors can add an approximately current-independent offset b . If the specimen is stable and ohmic, measurements at equal opposite currents obey

$$V(+I) = RI + b, \quad V(-I) = -RI + b.$$

Adding and subtracting gives

$$R = \frac{V(+I) - V(-I)}{2I}, \quad b = \frac{V(+I) + V(-I)}{2}. \quad (6.77)$$

This cancels a constant offset. It does not remove self-heating, since I^2R is unchanged on reversal. The two readings must be made closely enough in time for the offset and sample state to remain stable, yet after the relevant electrical transient has settled.

Example 6.24: Canceling an offset in a small-voltage measurement

At currents $+0.500\text{ A}$ and -0.500 A , a Kelvin measurement gives $+10.35\text{ mV}$ and -9.65 mV . Find resistance and offset.

$$R = \frac{10.35 - (-9.65)}{2(0.500)}\text{ m}\Omega = 20.00\text{ m}\Omega, \quad b = 0.350\text{ mV}.$$

A single positive-current ratio would give $20.70\text{ m}\Omega$. The difference is an offset error; it is not evidence of a new material resistivity. Repeat at a second current magnitude to test whether the specimen resistance changes with heating.

6.12.3 Fit a model and inspect what it fails to explain

A straight-line fit $V = RI + b$ allows both a slope and an offset. For observations I_i, V_i , define residuals $r_i = V_i - (RI_i + b)$. A small fitted intercept and residuals compatible with the stated uncertainties support an ohmic model over the tested range. They do not establish linearity at arbitrarily high current or at another temperature.

Table 6.1 contains *synthetic teaching data*, not measurements. Data set A was constructed from $V = 47.2I + 0.0003\text{ V}$ plus small specified perturbations. Set B adds cI^3 with $c = 1.20 \times 10^4\text{ V A}^{-3}$. The cubic term models increasing resistance at higher current and is not a universal heating law. Treat each voltage as having independent standard uncertainty $u_V = 0.8\text{ mV}$ and neglect current uncertainty for this exercise.

Table 6.1: Synthetic data for linearity and offset tests. The signed currents span the origin, which helps distinguish offset from slope.

$I\text{ (mA)}$	$V_A\text{ (V)}$	$V_B\text{ (V)}$
-20	-0.9432	-1.0392
-15	-0.7081	-0.7486
-10	-0.4715	-0.4835
-5	-0.2360	-0.2375
0	0.0004	0.0004
5	0.2367	0.2382
10	0.4721	0.4841
15	0.7086	0.7491
20	0.9437	1.0397

Example 6.25: A plausible slope can hide the wrong model

Fit a line with an intercept to each column of Table 6.1. Use the equal-uncertainty formulas derived in Section 6.22.

Set A. The fit gives

$$R_A = (47.192 \pm 0.021) \Omega, \quad b_A = (0.300 \pm 0.267) \text{ mV}.$$

The quoted uncertainties are standard uncertainties obtained from the specified u_V , not from rescaling to the observed scatter. The residuals range from -0.52 to $+0.44$ mV and show no large systematic curvature (Fig. 6.12). The reduced chi-square is 0.246, unusually small rather than evidence of exceptionally exact physics: the chosen illustrative perturbations are small relative to u_V .

Set B. The full-range line has slope 50.732Ω and essentially the same offset, but residuals reach approximately 25 mV and form a systematic pattern. Its reduced chi-square is about 699. A formal slope uncertainty of 0.021Ω under an invalid linear model is not an adequate description of this data set. Fit a physically motivated nonlinear model or report a restricted operating range instead.

Interpretation. The display of a straight fitted line is insufficient. Model adequacy is a comparison between residual structure, uncertainty and the physical assumptions.

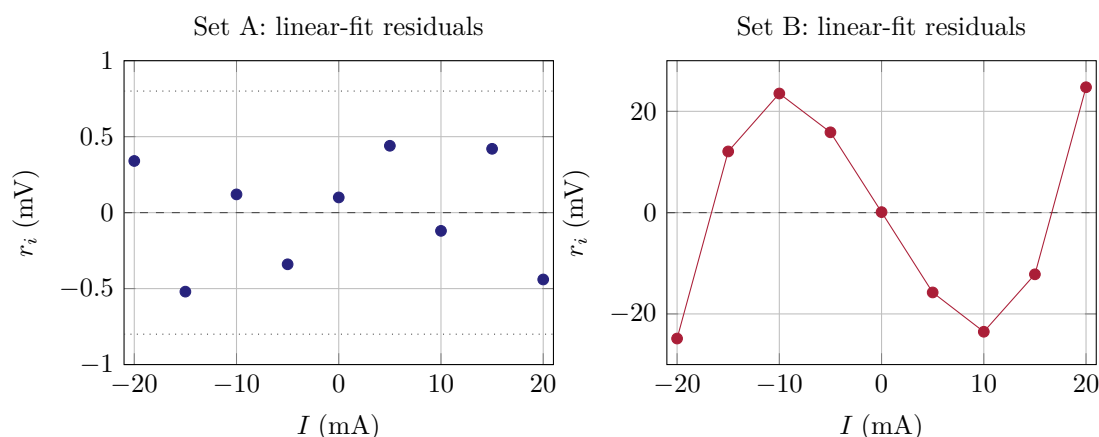


Figure 6.12: Residuals computed from the embedded data and ordinary least-squares fits. Dotted lines in the left panel mark $\pm u_V$. The right panel uses a much larger vertical scale; connecting lines guide the eye. The cubic response cannot be represented by a single resistance over this range.

6.12.4 A focused laboratory investigation

Use a current-limited low-voltage supply, suitable meters and a sample within its specified ratings. Record the voltage at the sample terminals and the actual current. A useful investigation has three stages: measure a nominal resistor at low current with both polarities; increase the current range while allowing temperature to settle; then compare the results with a lamp or thermistor whose state is allowed to change. Plot the raw data before fitting and retain the individual observations.

At each stage record instrument ranges, uncertainty estimates, measurement time and temperature if available. Distinguish reproducible curvature from scatter and from hysteresis between increasing and decreasing sweeps. A dependence on sweep rate suggests that the temperature has not equilibrated. A change of current-meter range may introduce a discontinuity that should not be attributed to the sample.

The report should state a measured resistance with its uncertainty, the tested range and the

conditions under which the model worked. For a nonlinear specimen, report the curve and one operating-point ratio together with a local differential estimate. Do not infer a universal I – V law from a few uncalibrated points.

6.12.5 Choosing a measurement current quantitatively

Two common error sources favor opposite choices. If the voltage readout has approximately current-independent standard uncertainty u_V , then the relative resistance uncertainty at small current is roughly $u_V/(IR_0)$, which decreases as current increases. On the other hand, self-heating of a positive-coefficient sensor gives a fractional resistance bias approximately

$$\frac{R - R_0}{R_0} \simeq \alpha \Theta_{\text{th}} I^2 R_0, \quad (6.78)$$

where $\Theta_{\text{th}} = 1/H$ and the rise is small. This grows as current squared. Averaging additional readings can reduce some random noise, but it does not remove this systematic bias.

Example 6.26: A feasible current window for resistance thermometry

A sensor has $R_0 = 100 \, \Omega$, $\alpha = 0.0040 \, \text{K}^{-1}$ and $\Theta_{\text{th}} = 200 \, \text{K W}^{-1}$. The readout has $u_V = 10 \, \mu\text{V}$. Require the voltage-induced relative standard uncertainty to be at most 0.10% and the self-heating resistance bias to be at most 0.020%.

Readout lower bound. A signal IR_0 must be at least $u_V/0.001 = 10 \, \text{mV}$, so $I \geq 0.100 \, \text{mA}$.

Heating upper bound. Equation (6.78) requires

$$I \leq \sqrt{\frac{0.00020}{(0.0040)(200)(100)}} = 1.58 \, \text{mA}.$$

Thus a feasible model window exists between 0.100 and 1.58 mA. At the upper bound the temperature rise is $\Theta_{\text{th}} I^2 R_0 = 0.050 \, \text{K}$, consistent with the small-heating approximation.

Interpretation. The two requirements describe different quantities, a standard uncertainty and a bias; they should not be added as though they were identical random errors. A practical choice within the interval also checks offset cancellation, current calibration, sensor thermal contact and meter range. If the interval were empty, changing the readout resolution or thermal coupling would be necessary.

A direct way to investigate heating is to repeat the measurement at several current magnitudes. To leading order, a plot of the offset-corrected R against I^2 has intercept R_0 and slope $\alpha \Theta_{\text{th}} R_0^2$. An extrapolation to zero current estimates the unheated resistance only if that leading-order model fits and the ambient state remains stable. The intercept uncertainty should include both measurement scatter and the sensitivity to the chosen fit range.

6.13 Engineering applications: losses, sensors and model limits

The same dissipation may be useful in a heater and undesirable in a cable. Engineering interpretation therefore begins with the system boundary: identify the useful output, where loss occurs, and the path by which heat can escape.

6.13.1 Power delivery and the cost of current

For a specified load power P_L delivered at load voltage V_L , the current is $I = P_L/V_L$. A supply-and-return pair with total resistance R_c loses

$$P_c = I^2 R_c = \frac{P_L^2 R_c}{V_L^2}, \quad \frac{P_c}{P_L} = \frac{P_L R_c}{V_L^2}. \quad (6.79)$$

This scaling explains why raising delivery voltage can reduce conductor losses for the same delivered power and cable resistance. It does not determine the overall best voltage: insulation, conversion, protection and load requirements also matter. Transformer and AC-system theory belong to later chapters.

Example 6.27: Comparing two DC distribution voltages

A load receives 96 W through a cable pair with total resistance $0.20\ \Omega$. Compare operation at actual load voltages 12 V and 48 V.

At 12 V. The current is 8.0 A, the cable drop is 1.6 V and the loss is 12.8 W. The source must supply 13.6 V at 8.0 A, or 108.8 W.

At 48 V. The current is 2.0 A, the cable drop is 0.40 V and the loss is 0.80 W. The source supplies 48.4 V at 2.0 A, or 96.8 W.

Check. Multiplying the load voltage by four divides cable loss by sixteen, as Eq. (6.79) predicts. The comparison specifies *load* voltages; assuming those values at the source without allowing for the cable drop would change the problem.

6.13.2 Current shunts and the signal–heating compromise

A current shunt deliberately creates a measurable voltage $V_s = IR_s$. Making R_s larger increases the signal but also increases burden voltage and power. Four-wire sensing is useful because voltage errors from current contacts can be comparable to the intended signal.

Example 6.28: A shunt resistor as a current sensor

A current sensor uses a shunt resistor $R_s = 5.0\ \text{m}\Omega$. Find the voltage across the shunt and its power dissipation at $I = 20\ \text{A}$.

Solution.

$$V_s = IR_s = (20)(5.0 \times 10^{-3})\ \text{V} = 0.10\ \text{V}. \quad (6.80)$$

The power is

$$P_s = I^2 R_s = (20)^2 (5.0 \times 10^{-3})\ \text{W} = 2.0\ \text{W}. \quad (6.81)$$

Interpretation. The shunt voltage is small enough not to disturb the circuit too much, but the shunt must dissipate real heat.

A shunt rating is conditional on cooling. Its resistance may change as it warms, causing a current estimate $I = V_s/R_s$ based on the cold calibration to become biased. The thermal model in the advanced enrichment quantifies this coupling.

6.13.3 Resistance thermometry and strain gauges

A thermometer infers ambient temperature from resistance only if the sensor temperature follows the environment closely. The measurement current itself dissipates power. Using thermal resistance Θ_{th} in K W^{-1} , a first estimate is $\Delta T_{\text{self}} = \Theta_{\text{th}} I^2 R$, assuming a small perturbation about the ambient state.

Example 6.29: Thermal self-heating of a resistance thermometer

A resistance thermometer of $100\ \Omega$ is read with current $1.0\ \text{mA}$. The thermal resistance to the environment is $200\ \text{K W}^{-1}$. Estimate the temperature rise due to self-heating.

Solution. The electrical power is

$$P = I^2 R = (1.0 \times 10^{-3})^2 (100)\ \text{W} = 1.0 \times 10^{-4}\ \text{W}. \quad (6.82)$$

The temperature rise is

$$\Delta T = (200\ \text{K W}^{-1})(1.0 \times 10^{-4}\ \text{W}) = 0.020\ \text{K}. \quad (6.83)$$

Interpretation. The self-heating is small here, but increasing measurement current by a factor of ten would increase power and temperature rise by a factor of one hundred.

Reducing current improves this systematic error quadratically but reduces the sensed voltage linearly. Good sensor design balances readout resolution, electrical noise and self-heating; simply choosing the smallest possible current may make the reading unusably noisy.

Mechanical deformation changes $R = \rho L/A$. Logarithmic differentiation gives

$$\frac{\delta R}{R} = \frac{\delta \rho}{\rho} + \frac{\delta L}{L} - \frac{\delta A}{A}. \quad (6.84)$$

For small uniaxial strain $\epsilon = \delta L/L$, an isotropic material with Poisson ratio ν has each transverse strain approximately $-\nu\epsilon$, so $\delta A/A \simeq -2\nu\epsilon$. The gauge factor is therefore

$$K_g = \frac{\delta R/R}{\epsilon} = 1 + 2\nu + \frac{\delta \rho/\rho}{\epsilon}. \quad (6.85)$$

Even when resistivity is constant, stretching raises resistance through both lengthening and narrowing. The piezoresistive contribution is material dependent and is absorbed into a calibrated K_g .

Example 6.30: Reading a small strain electrically

A calibrated gauge has $R = 120\ \Omega$, $K_g = 2.0$, and tensile strain $\epsilon = 500 \times 10^{-6}$. Then

$$\delta R = R K_g \epsilon = 120(2.0)(500 \times 10^{-6}) = 0.12\ \Omega.$$

With an approximately constant measuring current of $1.0\ \text{mA}$, the voltage change is only $0.12\ \text{mV}$. The baseline dissipated power is $0.12\ \text{mW}$. The small fractional signal motivates bridge readout in Chapter 7, while the baseline heating motivates a thermal check.

6.13.4 Localized heating and protection

In a uniform resistive segment $P/L = I^2 \rho/A$. At the same current, a narrow segment heats more per length; its heating per volume rises even faster, as $\rho I^2/A^2$. Contacts and constrictions can therefore be critical even when their contribution to the total resistance seems modest. Temperature also depends on local heat removal.

A fuse responds to heating over time, not just to an instantaneous current value. In a short-pulse, approximately adiabatic model with constant fuse resistance R_f , mass m_f , specific heat c_f , melting temperature T_m and latent heat per mass ℓ_f ,

$$R_f \int I^2 dt \simeq m_f [c_f (T_m - T_0) + \ell_f]. \quad (6.86)$$

This explains the usefulness of an I^2t scale. It is not a replacement for a component's measured time-current behavior: resistance changes during heating, heat can escape, and real failure involves material and geometry details. No installation or human-exposure limits are inferred from this idealized model.

6.14 Common mistakes and how to avoid them

Mistake	Why it is wrong	Better habit
Saying current is consumed in a resistor	Charge is conserved in steady flow; energy is transferred, not charge	Track charge flow and power separately
Assuming electrons move at near light speed in a wire	Drift speed is usually very small; the signal/field propagation is fast	Distinguish drift velocity from electromagnetic signal propagation
Using $P = V^2/R$ for any device	This form assumes a resistive element with appropriate R	Use $P = I\Delta V$ first; then decide whether Ohm's law applies
Treating Ohm's law as universal	Many devices and materials are non-ohmic	Inspect the I - V curve and operating range
Confusing resistance and resistivity	Resistance depends on geometry; resistivity is material response	Use $R = \rho L/A$ only after identifying L and A
Forgetting area conversion from mm^2 to m^2	SI units require $1 \text{ mm}^2 = 10^{-6} \text{ m}^2$	Convert before substitution
Ignoring temperature changes	Resistive heating can change R	State constant-temperature assumptions
Confusing conventional current with electron drift	Electron charge is negative	Draw arrows for both when learning
Averaging V/I ratios blindly in nonlinear data	Nonlinear devices have changing operating-point resistance	Use graphs, slopes and model fits
Forgetting the dot product in current density flux	Only normal component crosses a surface	Write $I = \int \mathbf{J} \cdot d\mathbf{A}$ before simplifying

Summary

The chapter's main chain of ideas is: current counts charge crossing a surface per unit time; current density describes that transport locally; drift velocity connects the macroscopic current to microscopic carriers; resistance combines material response and geometry; Ohm's law is a linear-response model; and electric power measures the rate of energy transfer. The formulas are simple, but their assumptions are not optional.

6.15 Core synthesis and formula guide

The core rests on three distinct ingredients. Charge conservation constrains how currents can enter and leave a region. A constitutive relation describes the response of a particular material or device. Energy accounting determines absorbed or delivered power. These ingredients remain distinct even when a familiar algebraic formula combines them.

Relation	Meaning and conditions
$I = dq/dt, \Delta q = \int I dt$	Signed transported charge through a specified oriented surface.
$I = \int_S \mathbf{J} \cdot d\mathbf{A}$	Local-to-global current relation; $I = JA$ only for uniform normal flow.
$\mathbf{J} = \sum_s n_s q_s \mathbf{v}_{d,s}$	Sum signed carrier contributions.
$\mathbf{J} = \sigma \mathbf{E}, \rho = 1/\sigma$	Isotropic, local, linear material response.
$R = \rho L/A$	Uniform material and cross-section, full end contacts, effectively one-dimensional transport.
$R_{\text{static}} = \Delta V/I$	Operating-point ratio for $I \neq 0$. Does not establish linearity.
$r_{\text{diff}} = d(\Delta V)/dI$	Small-signal derivative at a specified operating point and state.
$P = I\Delta V$	Signed absorbed terminal power for passive reference directions.
$P = I^2 R = (\Delta V)^2/R$	Resistive absorption using the relevant static or linear resistance, not r_{diff} in general.
$W = \int P dt$	Energy; $1 \text{ kWh} = 3.6 \times 10^6 \text{ J}$.
$\rho \simeq \rho_0[1 + \alpha_0(T - T_0)]$	Local temperature approximation; geometry fixed for the corresponding R law.
$\sigma = ne^2\tau/m$	Drude relaxation model, one carrier species.
$R = [V(+I) - V(-I)]/(2I)$	Offset-canceling measurement for a stable, linear specimen.

6.16 Advanced Topics for Technical and Engineering Students

The following material is a connected extension of Chapter 6; all numbering remains within that chapter. The core treated current, resistance and power as measurable quantities. Here we ask how those quantities emerge from local conservation, spatial geometry, relaxation and coupled thermal response. The additional mathematics consists of divergence, first-order differential equations, linearization and elementary least squares. Each is introduced when needed.

6.17 Charge conservation, interfaces and steady fields

6.17.1 Continuity from a finite-volume balance

Let $\rho_q(\mathbf{r}, t)$ be charge density, distinct from resistivity ρ . For a fixed volume $\mathcal{V}$ with outward-oriented boundary $\partial\mathcal{V}$, charge conservation gives

$$\frac{d}{dt} \int_{\mathcal{V}} \rho_q d\mathcal{V} = - \oint_{\partial\mathcal{V}} \mathbf{J} \cdot d\mathbf{A}. \quad (6.87)$$

The sign is fixed physically: net outward conventional current decreases the enclosed charge. The divergence theorem converts outward flux to the volume integral of $\nabla \cdot \mathbf{J}$. Since the volume can be chosen arbitrarily,

$$\frac{\partial \rho_q}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (6.88)$$

The divergence is a local net-outflow density: in Cartesian coordinates it is $\partial J_x/\partial x + \partial J_y/\partial y + \partial J_z/\partial z$. Its units are A m^{-3} , equal to the units of $\partial \rho_q/\partial t$. In a steady state $\nabla \cdot \mathbf{J} = 0$. This does not require $\mathbf{J}$ to be spatially constant, and it does not require the local charge density itself to vanish.

Choose a short unbranched conductor segment whose side wall carries no current. The boundary flux is $I_{\text{out}} - I_{\text{in}}$, so

$$\frac{dQ_{\text{segment}}}{dt} = I_{\text{in}} - I_{\text{out}}.$$

In steady state the two currents are equal, even if area, material and drift velocity differ between the ends. This argument uses conservation alone; it also applies to a non-ohmic conductor.

6.17.2 A constriction changes fields, not the conserved current

In two long sections of the same isotropic material, $J_i = I/A_i$ and $E_i = \rho I/A_i$. If section 2 has half the area, both J and E double there. The potential gradient $-dV/dx = E_x$ becomes steeper. Figure 6.13 shows this one-dimensional approximation away from the junction. Near an abrupt step, current crowds and the three-dimensional solution can have an additional spreading resistance.

The same distinction appears at an interface between two materials with equal area. Steady normal current satisfies $J_{1n} = J_{2n}$ if surface charge is not changing. The fields can differ, because $E_{in} = \rho_i J_n$. A thin accumulation of interface charge supports the normal field discontinuity. For total microscopic-plus-bound charge represented by a macroscopic total surface density, Gauss's law gives $\sigma_{\text{surf, total}} = \epsilon_0(E_{2n} - E_{1n})$. If a dielectric displacement description is used instead, $D_{2n} - D_{1n}$ is the *free* surface charge. The distinction matters in materials with appreciable polarization.

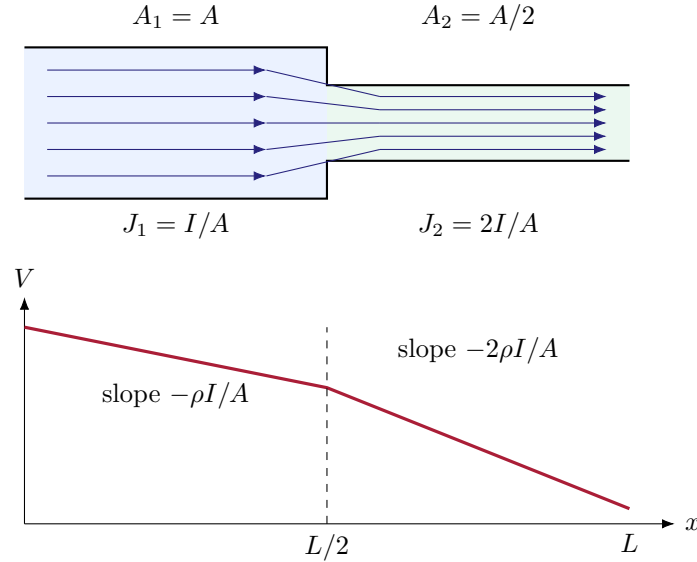


Figure 6.13: Steady current through two equal-length sections. Top: schematic current paths, not an exact field solution near the step. Bottom: the piecewise-linear potential in the one-dimensional model. Potential is continuous, while the bulk slope is larger in the narrow region.

Example 6.31: A two-material interface

Two long slabs of equal cross-section carry $J_x = 2.0 \times 10^6 \text{ A m}^{-2}$. Their resistivities are $\rho_1 = 2.0 \times 10^{-8}$ and $\rho_2 = 6.0 \times 10^{-8} \Omega \text{ m}$. Find the bulk fields and the total interface charge density in an ideal planar model, with normal from material 1 to 2.

$$E_{1x} = 0.040 \text{ V m}^{-1}, \quad E_{2x} = 0.120 \text{ V m}^{-1},$$

$$\sigma_{\text{surf, total}} = \epsilon_0(0.120 - 0.040) \simeq 7.1 \times 10^{-13} \text{ C m}^{-2}.$$

The sign is positive for the chosen normal. A very small net charge can sustain a field difference while a large current flows. The result assumes no extra interfacial voltage drop and a macroscopic planar interface; real contact layers can add resistance.

6.17.3 The boundary-value formulation

In the DC limit with a conservative electric field, $\mathbf{E} = -\nabla V$. Combining local Ohm response with steady conservation yields

$$\nabla \cdot [\sigma(\mathbf{r}) \nabla V] = 0. \quad (6.89)$$

Terminal potentials provide boundary conditions; insulated surfaces obey $\hat{\mathbf{n}} \cdot \sigma \nabla V = 0$. Across an ideal interface, potential and normal current are continuous. For uniform scalar σ , Eq. (6.89) reduces to Laplace's equation. It is the field-level problem hidden by a lumped resistance.

For anisotropic media replace the scalar conductivity by a tensor: $J_i = \sum_j \sigma_{ij} E_j$. Then $\mathbf{J}$ need not be parallel to $\mathbf{E}$. A passive resistive medium must have nonnegative dissipative power $\mathbf{E} \cdot \sigma \mathbf{E}$; equivalently its symmetric conductivity part must be positive semidefinite. The Hall response and magnetic-field effects are deferred to later chapters.

6.18 Resistance from nonuniform geometry

6.18.1 Axial transport in a slowly varying conductor

Suppose current crosses each slice of length dx through area $A(x)$, with resistivity $\rho(x)$. If each slice can be approximated as equipotential across its area and current is effectively axial, then

$$dV = I \frac{\rho(x)}{A(x)} dx.$$

Here dV denotes a positive *drop*; the actual potential differential along positive current is its negative. Adding the drops gives

$$R = \int_0^L \frac{\rho(x)}{A(x)} dx. \quad (6.90)$$

This is exact within the one-dimensional model. A gradual taper helps justify the model; abrupt contacts, rapid area changes or lateral inhomogeneity may require Eq. (6.89). The harmonic rather than arithmetic averaging of area weights narrow regions strongly.

Example 6.32: Resistance of a tapered conductor

A conductor of length L has resistivity ρ and cross-sectional area increasing linearly from A_0 at $x = 0$ to $2A_0$ at $x = L$:

$$A(x) = A_0 \left(1 + \frac{x}{L}\right). \quad (6.91)$$

Find its resistance in a one-dimensional approximation.

Solution. Use Eq. (6.90):

$$R = \int_0^L \frac{\rho}{A_0(1 + x/L)} dx = \frac{\rho L}{A_0} \int_0^1 \frac{du}{1 + u} \quad (6.92)$$

$$= \frac{\rho L}{A_0} \ln 2. \quad (6.93)$$

Interpretation. The resistance is smaller than $\rho L/A_0$ because much of the conductor is wider than A_0 , but larger than $\rho L/(2A_0)$ because the entire conductor is not at the maximum area.

For a fixed total material volume $\mathcal{V} = \int_0^L A(x) dx$, constant ρ , fixed length and the same one-dimensional approximation, the Cauchy–Schwarz inequality gives

$$\left(\int_0^L A dx \right) \left(\int_0^L \frac{dx}{A} \right) \geq L^2.$$

Consequently $R \geq \rho L^2/\mathcal{V}$, with equality for constant area. The result explains why a constriction increases resistance even when extra material is added elsewhere to preserve volume. It is not a general optimal-design theorem once cooling, stress or three-dimensional current paths are included.

6.18.2 Radial leakage is a different current path

The desired current in a coaxial cable usually flows *along* its metal conductors. Leakage through the separating material flows *radially*. For coaxial electrodes with radii $a < b$ and active length ℓ , a cylindrical shell of thickness dr has crossing area $2\pi r\ell$ (Fig. 6.14). If the material between them is homogeneous and approximately ohmic,

$$R_{\text{radial}} = \int_a^b \frac{\rho}{2\pi r\ell} dr = \frac{\rho}{2\pi\ell} \ln \frac{b}{a}. \quad (6.94)$$

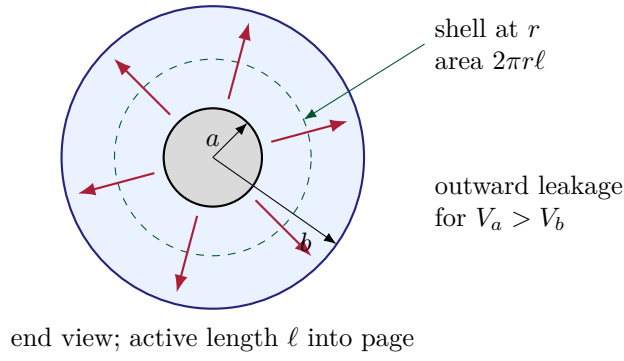


Figure 6.14: Radial conduction between coaxial cylindrical electrodes. The dashed circle represents a thin cylindrical shell, not an annular axial-current element. The direction of current determines the area used in the resistance integral.

The result decreases with length because a longer cable provides more radial paths in parallel. This reverses the length dependence of axial resistance. End leakage and surface leakage are excluded.

The current density is $J_r = I/(2\pi r\ell)$, and $E_r = \rho I/(2\pi r\ell)$. Taking $V_b = 0$ and $V_a = \Delta V$ yields

$$V(r) = \Delta V \frac{\ln(b/r)}{\ln(b/a)}, \quad E_r(r) = \frac{\Delta V}{r \ln(b/a)}. \quad (6.95)$$

Differentiation verifies $E_r = -dV/dr$. The greatest field and local heating density occur near the inner electrode, even though the same total leakage current crosses every cylindrical shell.

Example 6.33: Leakage, field and power between coaxial electrodes

An illustrative homogeneous leakage layer has $\rho = 2.0 \times 10^{10} \Omega \text{ m}$ between radii $a = 1.0 \text{ mm}$ and $b = 3.0 \text{ mm}$ over length $\ell = 0.50 \text{ m}$. Apply 100 V between the electrodes. Assume DC equilibrium, no surface leakage and no field dependence of ρ .

$$\begin{aligned} R &= \frac{2.0 \times 10^{10}}{2\pi(0.50)} \ln 3 = 6.99 \times 10^9 \Omega, \\ I &= 100/R = 1.43 \times 10^{-8} \text{ A} = 14.3 \text{ nA}, \\ P &= \Delta V I = 1.43 \mu\text{W}, \\ E_r(a) &= \frac{100}{10^{-3} \ln 3} = 9.10 \times 10^4 \text{ V m}^{-1}. \end{aligned}$$

Doubling active length doubles leakage current while leaving $E_r(r)$ unchanged for the same terminal voltage. The estimate is not an insulation qualification: real surface contamination, humidity and breakdown are outside the constant-resistivity model.

6.19 Local energy transfer and relaxation

6.19.1 Deriving the power density

For carrier species s , the electric work rate on one carrier is $q_s \mathbf{E} \cdot \mathbf{v}_s$. Multiplying its ensemble mean by number density and summing gives

$$p = \sum_s n_s q_s \mathbf{v}_{d,s} \cdot \mathbf{E} = \mathbf{J} \cdot \mathbf{E}. \quad (6.96)$$

This is local electrical power transferred to matter per volume, with units $\text{A m}^{-2}\text{V m}^{-1} = \text{W m}^{-3}$. In an ohmic resistive medium the transferred energy becomes heat through relaxation, giving

$$p = \sigma E^2 = \rho J^2 \geq 0. \quad (6.97)$$

In an active or energy-converting medium, $\mathbf{J} \cdot \mathbf{E}$ should first be interpreted as field-to-matter energy transfer, not automatically as heat. For a uniform resistor,

$$\int p \, dV = \rho(I/A)^2 AL = I^2 \rho L/A = I^2 R.$$

For a tapered sample, integrating $\rho I^2/A(x)$ over length reproduces $I^2 R$ from Eq. (6.90). Thus local and terminal accounting agree.

Example 6.34: Local heating in a narrowed section

Two sections of the same metal carry the same current I . Section 1 has area A , section 2 has area $A/2$. Compare their Joule heating per unit length, assuming uniform current density in each.

Solution. Power per unit length is

$$\frac{P}{L} = pA = \rho J^2 A = \rho \left(\frac{I}{A}\right)^2 A = \rho \frac{I^2}{A}. \quad (6.98)$$

For area $A/2$,

$$\left(\frac{P}{L}\right)_2 = \rho \frac{I^2}{A/2} = 2\rho \frac{I^2}{A} = 2 \left(\frac{P}{L}\right)_1. \quad (6.99)$$

Interpretation. The narrower section heats more per unit length. Current crowding and reduced area are common causes of hotspots.

In that example the heating *per unit volume* rises by a factor of four, whereas heating *per unit length* rises by two because the narrow section has half the volume per length. Neither ratio alone specifies the temperature; thermal conduction and the exposed surface also change.

6.19.2 The transient Drude model

Using the relaxation equation (6.51) with a constant applied field for $t \geq 0$ and initially zero drift gives

$$\mathbf{v}_d(t) = \frac{q_c \tau}{m} \mathbf{E}(1 - e^{-t/\tau}), \quad \mathbf{J}(t) = \frac{nq_c^2 \tau}{m} \mathbf{E}(1 - e^{-t/\tau}). \quad (6.100)$$

One may verify the result by differentiating and substituting, and by checking both $t = 0$ and $t \rightarrow \infty$. At very short times $\mathbf{v}_d \simeq (q_c \mathbf{E}/m)t$; at long times drift saturates. When the field is removed, the directed motion decays exponentially in this model.

The derivation treats the field as a specified local input. It is not a propagation calculation for a newly connected circuit. The fast microscopic relaxation time must not be substituted for the travel time of an electromagnetic signal or for an RC or thermal settling time.

The typical distance traveled between momentum-relaxing events is estimated from the microscopic speed, $\ell_{\text{tr}} \sim v_{\text{micro}} \tau$, not the drift speed. For the earlier copper estimate $\tau \simeq 2.5 \times 10^{-14} \text{ s}$, a stipulated effective microscopic speed of $1.6 \times 10^6 \text{ m s}^{-1}$ gives about 40 nm. Using $v_d \sim 10^{-4} \text{ m s}^{-1}$ would instead give a meaningless microscopic path estimate: drift measures a tiny directional bias, not the distance actually traveled.

6.19.3 Energy balance in the relaxation model

Dot Eq. (6.51) with nv_d . For constant n ,

$$\frac{d}{dt} \left(\frac{1}{2} n m v_d^2 \right) = \mathbf{J} \cdot \mathbf{E} - \frac{n m v_d^2}{\tau}. \quad (6.101)$$

This tracks the directed kinetic energy density of the ensemble. In the stationary solution the relaxation loss equals $\mathbf{J} \cdot \mathbf{E}$. During a transient some power initially changes the directed energy. The model does not imply that the total microscopic kinetic energy of the electrons equals $n m v_d^2 / 2$; the much larger equilibrium electronic energy is not represented by the drift energy alone.

A formal $\tau \rightarrow \infty$ limit removes the stationary resistive solution and permits continued acceleration under a fixed field. It is not a derivation of superconductivity. The latter has quantum coherence and magnetic response absent from this relaxation model.

6.20 Electrothermal feedback and stability

6.20.1 A coupled electrical–thermal model

Let the ambient temperature be T_0 and write $\theta = T - T_0$. Assume a spatially uniform sample temperature, constant heat capacity $C_{th} > 0$, thermal conductance $H > 0$ in W K^{-1} , and

$$R(\theta) = R_0(1 + \alpha\theta), \quad 1 + \alpha\theta > 0.$$

The dynamic energy balance is

$$C_{th} \dot{\theta} = P_{el}(\theta) - H\theta. \quad (6.102)$$

Here H is a whole-device thermal conductance, not a convective heat-transfer coefficient per unit area. Linear heat loss is an approximation over a limited temperature range. Radiation, changing geometry and distributed temperatures can require a richer model.

An equilibrium θ_* satisfies $P_{el}(\theta_*) = H\theta_*$. Set $\theta = \theta_* + \delta\theta$ and keep first-order terms:

$$C_{th} \delta \dot{\theta} = [P'_{el}(\theta_*) - H] \delta\theta. \quad (6.103)$$

The equilibrium is stable if $P'_{el}(\theta_*) < H$. Then small disturbances decay with time constant $C_{th}/[H - P'_{el}(\theta_*)]$. Existence of an algebraic root alone is not enough; its stability and the model's validity must also be checked.

6.20.2 Fixed current with positive temperature coefficient

At fixed I ,

$$C_{th} \dot{\theta} = I^2 R_0 + (\alpha I^2 R_0 - H)\theta.$$

Define the dimensionless feedback parameter $\gamma = \alpha I^2 R_0 / H$. For $\alpha > 0$ and $\gamma < 1$,

$$\theta_* = \frac{I^2 R_0}{H - \alpha I^2 R_0}, \quad \tau_{th,eff} = \frac{C_{th}}{H - \alpha I^2 R_0}. \quad (6.104)$$

From an initially ambient sample,

$$\theta(t) = \theta_* [1 - \exp(-t/\tau_{th,eff})].$$

As γ approaches one from below, both equilibrium rise and relaxation time increase. At $\gamma = 1$ this linear model heats at the constant rate $I^2 R_0 / C_{th}$; for $\gamma > 1$ it predicts runaway growth. Actual materials and cooling laws will change before temperature becomes unlimited. The threshold is a property of this model, not a component rating.

6.20.3 Fixed voltage with positive temperature coefficient

At fixed terminal voltage V ,

$$\frac{V^2}{R_0(1 + \alpha\theta_*)} = H\theta_* \quad (6.105)$$

For $\alpha > 0$ the physical root is

$$\theta_* = \frac{\sqrt{1 + 4\alpha V^2/(HR_0)} - 1}{2\alpha}. \quad (6.106)$$

Since $P'_{\text{el}} = -\alpha V^2/[R_0(1 + \alpha\theta)^2] < 0$, the root is stable within this model. The $\alpha \rightarrow 0$ limit is $V^2/(HR_0)$; multiplying numerator and denominator by the conjugate verifies this limit without a numerical cancellation error.

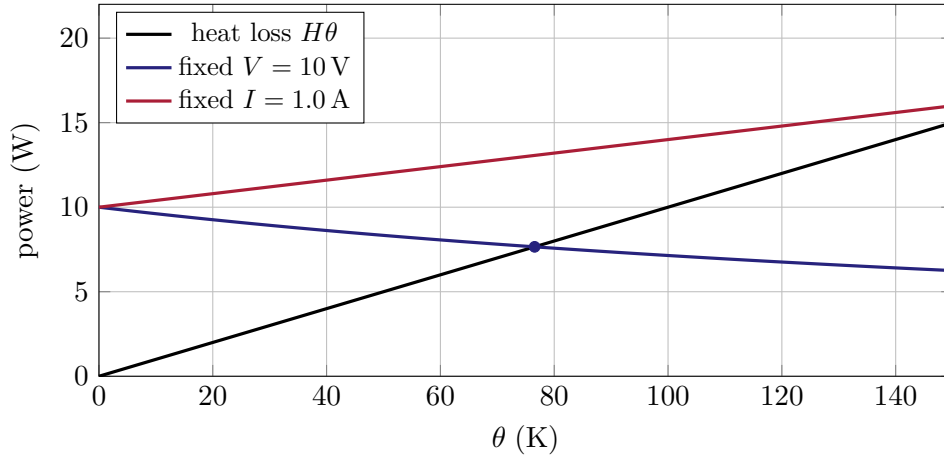


Figure 6.15: Calculated heat-generation and heat-loss curves for $R_0 = 10\ \Omega$, $\alpha = 0.004\ \text{K}^{-1}$ and $H = 0.10\ \text{W K}^{-1}$. Both sources supply 10 W at ambient temperature. Voltage control equilibrates at 76.6 K rise; current control equilibrates at 166.7 K rise, just beyond the plotted interval. The linear resistance and cooling laws are stipulated models, not a high-temperature material fit.

Example 6.35: Compare equilibria and response times

Use the model parameters in Fig. 6.15 with $C_{\text{th}} = 2.0\ \text{J K}^{-1}$.

Fixed current. With $I = 1.0\ \text{A}$, $\gamma = 0.40$, so $\theta_* = 10/(0.10 - 0.04) = 166.7\ \text{K}$ and $\tau_{\text{th,eff}} = 33.3\ \text{s}$. The model threshold is $I_{\text{crit}} = \sqrt{H/(\alpha R_0)} = 1.58\ \text{A}$.

Fixed voltage. At 10 V, Eq. (6.106) gives $\theta_* = 76.6\ \text{K}$, $R_* = 13.06\ \Omega$ and $P_* = 7.66\ \text{W}$. Here $P'_{\text{el}}(\theta_*) = -0.02345\ \text{W K}^{-1}$, so the local response time is $2.0/(0.10 + 0.02345) = 16.2\ \text{s}$.

Interpretation. Identical cold power does not imply identical hot power, equilibrium temperature or response time. Both computed rises are large enough that measured material and heat-loss data would be needed before treating the numbers as engineering predictions.

For an NTC sensor the electrical feedback reverses: current drive decreases power as resistance falls, while voltage drive increases it. Stability still follows Eq. (6.103); it should not be guessed solely from the word “thermistor.” A nonlinear loss law can create several roots, making a slope test at each root essential.

6.21 Differential response and sensor calibration

6.21.1 A tangent describes increments, not the whole device

Near an operating point (I_0, V_0) , a differentiable terminal law $V = f(I)$ has the expansion

$$V_0 + \delta V = f(I_0) + f'(I_0)\delta I + \frac{1}{2}f''(I_0)(\delta I)^2 + \dots \quad (6.107)$$

Thus $\delta V \simeq r_{\text{diff}}\delta I$ with $r_{\text{diff}} = f'(I_0)$. The small-signal model is valid when the neglected curvature term is small at the required accuracy. It describes a change from the bias state, not the bias voltage itself. Generally $V_0 \neq r_{\text{diff}}I_0$.

Figure 6.16 returns to the earlier model $I = aV^2$ for $V > 0$. At $V_0 = 5 \text{ V}$ and $a = 0.020 \text{ A V}^{-2}$, the line from the origin has slope $I_0/V_0 = 0.10 \text{ S}$ and the tangent has slope $dI/dV = 0.20 \text{ S}$. Their inverse slopes are the static and differential resistances, 10Ω and 5Ω .

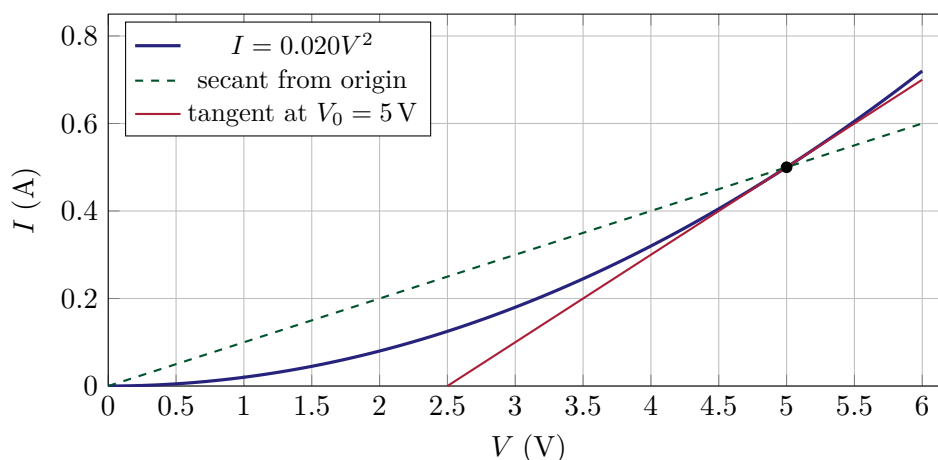


Figure 6.16: Calculated nonlinear response with its secant and tangent. On an I versus V graph the slopes are conductances; resistance is their inverse. The tangent's intercept need not be zero.

For this model the exact increment is $\delta I = 2aV_0\delta V + a(\delta V)^2$. The magnitude of the omitted quadratic term relative to the linear term is $|\delta V|/(2V_0)$. At $V_0 = 5 \text{ V}$, a 0.10 V increment gives a 1% correction to the predicted increment. This gives a quantitative meaning to “small signal.”

The thermal state must also be specified. A rapid perturbation compared with the thermal response may probe an approximately fixed-temperature resistance. A slow sweep permits self-heating to adjust and can have a different slope. The same specimen can therefore have different differential responses in different measurement protocols without violating any conservation law.

6.21.2 Calibration of an NTC thermistor

For Eq. (6.65), taking logarithms gives

$$\ln \frac{R}{R_0} = B \left(\frac{1}{T} - \frac{1}{T_0} \right), \quad \frac{1}{T} = \frac{1}{T_0} + \frac{1}{B} \ln \frac{R}{R_0}. \quad (6.108)$$

Two calibration points give

$$B = \frac{\ln(R_1/R_2)}{1/T_1 - 1/T_2}.$$

Both temperatures must be absolute. More points are needed to test whether a constant- B model is adequate. A line through only two points has zero residual by construction and supplies no test of model curvature.

Example 6.36: Thermistor readout and sensitivity

Use an illustrative calibrated model with $R_0 = 10.0 \text{ k}\Omega$ at $T_0 = 298.15 \text{ K}$ and $B = 3500 \text{ K}$. A measured resistance is $6.80 \text{ k}\Omega$.

$$T = \left[\frac{1}{298.15} + \frac{1}{3500} \ln \frac{6.80}{10.0} \right]^{-1} = 308.3 \text{ K} \simeq 35.1^\circ\text{C}.$$

For fixed calibration parameters, differentiating the inverse gives

$$\frac{dT}{dR} = -\frac{T^2}{BR}, \quad u_T \simeq \frac{T^2}{B} \frac{u_R}{R}.$$

A 0.5% relative standard uncertainty in resistance contributes approximately 0.136 K to temperature uncertainty. Calibration uncertainty in B and R_0 , bath uncertainty and self-heating must be added through their own derivatives or a joint fit; the number above is only the readout contribution.

6.21.3 Propagation of measurement uncertainty

For $R = V/I$ and small fluctuations,

$$\delta R \simeq \frac{\delta V}{I} - \frac{V}{I^2} \delta I.$$

Squaring and averaging gives the standard-uncertainty relation

$$u_R^2 \simeq \frac{u_V^2}{I^2} + \frac{V^2 u_I^2}{I^4} - \frac{2V}{I^3} \text{Cov}(V, I). \quad (6.109)$$

For independent measurements the covariance term vanishes, and at nonzero V, I this reduces to $(u_R/R)^2 \simeq (u_V/V)^2 + (u_I/I)^2$. Correlation can matter if both readings share a reference or if source drift affects them together. This linear propagation is unreliable when uncertainty is large compared with the denominator; measurements near zero current should not be interpreted through unstable individual ratios.

Example 6.37: Uncertainty in a single resistance measurement

A resistor has measured voltage $V = (5.00 \pm 0.02) \text{ V}$ and current $I = (0.250 \pm 0.002) \text{ A}$. Find R and its approximate uncertainty.

Solution. The resistance is

$$R = \frac{5.00}{0.250} \Omega = 20.0 \Omega. \quad (6.110)$$

The relative uncertainty is

$$\frac{\delta R}{R} \simeq \sqrt{(0.02/5.00)^2 + (0.002/0.250)^2} = 0.0089. \quad (6.111)$$

Thus

$$\delta R \simeq (20.0 \Omega)(0.0089) = 0.18 \Omega. \quad (6.112)$$

A sensible report is $R = (20.0 \pm 0.2) \Omega$.

The uncertainties in that example are interpreted as independent standard uncertainties. A stated meter accuracy limit is not automatically a standard deviation; its conversion depends on a justified uncertainty model.

6.22 Reproducible fitting and numerical integration

6.22.1 Deriving the equal-uncertainty line fit

For the model $V_i = RI_i + b + \epsilon_i$ with negligible current uncertainty and equal, independent voltage standard uncertainties, minimizing

$$S(R, b) = \sum_i [V_i - (RI_i + b)]^2$$

requires $\partial S/\partial b = 0$ and $\partial S/\partial R = 0$. The first equation gives $b = \bar{V} - R\bar{I}$. Substitution into the second yields

$$R = \frac{\sum_i (I_i - \bar{I})(V_i - \bar{V})}{S_{II}}, \quad b = \bar{V} - R\bar{I}, \quad S_{II} = \sum_i (I_i - \bar{I})^2. \quad (6.113)$$

If the known standard uncertainty of every voltage is u_V ,

$$u_R = \frac{u_V}{\sqrt{S_{II}}}, \quad u_b = u_V \sqrt{\frac{1}{N} + \frac{\bar{I}^2}{S_{II}}}, \quad \text{Cov}(R, b) = -\frac{\bar{I}u_V^2}{S_{II}}. \quad (6.114)$$

A wider spread of currents increases S_{II} and improves slope precision, but only while the same physical model remains valid. Currents centered around zero remove the slope–intercept covariance in this equal-weight setting. A forced fit through the origin is justified only if offsets have been independently excluded.

The normalized residual sum is

$$\chi_\nu^2 = \frac{1}{N-2} \sum_i \left(\frac{r_i}{u_V} \right)^2. \quad (6.115)$$

Its expected scale is near one under a correct linear model and independent Gaussian errors with correctly specified uncertainties, but small samples fluctuate. A small value may indicate conservative uncertainties or unusually small scatter; a large value may indicate underestimated uncertainty or model failure. Residual shape and physical evidence are essential alongside the number.

If only an unknown common noise scale is available, $s^2 = \sum r_i^2/(N-2)$ estimates it under the assumed linear model. Replacing u_V by s is then appropriate for parameter uncertainty. It is not legitimate to inflate uncertainties blindly until a visibly curved data set appears linear. Unequal known uncertainties require weighted least squares; appreciable current errors require a model that treats uncertainty in both coordinates.

6.22.2 An executable calculation for the embedded data

The following Python code uses only the standard library. Currents are converted to amperes before fitting; voltages are already in volts. It reproduces the numbers and residuals used in Fig. 6.12. All data needed to run it appear in the chapter.

```

from math import sqrt
I = [x/1000 for x in [-20,-15,-10,-5,0,5,10,15,20]]
VA = [-.9432,-.7081,-.4715,-.2360,.0004,
      .2367,.4721,.7086,.9437]
VB = [-1.0392,-.7486,-.4835,-.2375,.0004,
      .2382,.4841,.7491,1.0397]
uV = .0008

def fit_line(I, V, uV):
    N = len(I)
    xb, yb = sum(I)/N, sum(V)/N
    Sxx = sum((x-xb)**2 for x in I)
    R = sum((x-xb)*(y-yb) for x,y in zip(I,V))/Sxx
    b = yb-R*xb
    residual = [y-(R*x+b) for x,y in zip(I,V)]
    uR = uV/sqrt(Sxx)
    ub = uV*sqrt(1/N+xb**2/Sxx)
    chi2red = sum((r/uV)**2 for r in residual)/(N-2)
    return R, b, uR, ub, chi2red, residual

for name, V in [('A', VA), ('B', VB)]:
    R,b,uR,ub,chi,residual = fit_line(I,V,uV)
    print(name, 'R =', R, 'ohm; b =', 1000*b, 'mV')
    print('uR =', uR, 'ohm; ub =', 1000*ub, 'mV')
    print('reduced chi-square =', chi)
    print('residuals in mV:', [1000*r for r in residual])

```

6.22.3 Charge from sampled current: accuracy has a physical scale

For ordered samples (t_i, I_i) , the trapezoidal estimate is

$$Q_N = \sum_{i=0}^{N-1} \frac{I_i + I_{i+1}}{2} (t_{i+1} - t_i). \quad (6.116)$$

It integrates the piecewise-linear interpolant. It does not recover an unresolved narrow pulse between samples. If a baseline offset b_I persists throughout an interval T , its systematic charge error is $b_I T$, regardless of how finely the data are sampled.

Example 6.38: Convergence for a smooth current pulse

Take $I(t) = I_0 \exp(-t/\tau)$ with $I_0 = 2.0$ A, $\tau = 3.0$ ms, and integrate only to $T = 3\tau$. The exact finite-window charge is

$$Q = I_0 \tau (1 - e^{-3}) = 5.70128 \text{ mC}.$$

With N equal steps, $h = T/N$ and $z = h/\tau$, summing the geometric series in the trapezoidal rule gives

$$\frac{Q_N}{Q} = \frac{z}{2} \frac{1 + e^{-z}}{1 - e^{-z}}.$$

The ratios for $N = 3, 6, 12, 24$ are approximately 1.08198, 1.02075, 1.00520 and 1.00130. Halving the spacing reduces the leading smooth-function error by about four. The integral to infinity is 6.000 mC; the missing tail of 0.299 mC is a different error and does not disappear by refining samples within the same time window.

The corresponding code makes the integration range and convergence check explicit:

```

from math import exp
I0, tau = 2.0, .003
T = 3*tau
Qexact = I0*tau*(1-exp(-T/tau))
for N in [3,6,12,24]:
    h = T/N
    current = [I0*exp(-k*h/tau) for k in range(N+1)]
    Q = sum((current[k]+current[k+1])*h/2 for k in range(N))
    print(N, Q, 'C; ratio =', Q/Qexact)

```

6.23 A conservative numerical model of a nonuniform wire

6.23.1 Discretize the conservation law

Divide a wire into N cells with potentials $V_0, \dots, V_N$ at their boundaries. Approximate the resistance of cell k by

$$R_k \simeq \frac{\rho(x_{k+1/2})\Delta x}{A(x_{k+1/2})}, \quad I_k = \frac{V_k - V_{k+1}}{R_k}.$$

A steady internal node cannot accumulate charge, so $I_{k-1} = I_k$. Equivalently,

$$\frac{V_k - V_{k-1}}{R_{k-1}} + \frac{V_k - V_{k+1}}{R_k} = 0. \quad (6.117)$$

This is a tridiagonal linear system for the interior potentials when the end voltages are prescribed. In one dimension its solution can be obtained directly: the common current is $I = (V_0 - V_N) / \sum_k R_k$, and each potential drop is IR_k . This is a numerical realization of current conservation, not an assumption of a uniform field.

At a discontinuous material interface, a cell spanning two half-cells should use their series resistances. For constant area and equal half-cell lengths this gives a face conductivity $\sigma_f = 2\sigma_1\sigma_2/(\sigma_1 + \sigma_2)$, the harmonic mean. Using an arithmetic mean can overestimate transport through a poorly conducting layer.

6.23.2 A smooth benchmark with an exact answer

Choose, for $0 \leq x \leq L$,

$$\rho(x) = \rho_0(1 + x/L), \quad A(x) = A_0(1 + x/L)^2.$$

Then

$$R_{\text{exact}} = \frac{\rho_0 L}{A_0} \ln 2, \quad V(x) = V_0 - (V_0 - V_N) \frac{\ln(1 + x/L)}{\ln 2}. \quad (6.118)$$

Take $L = 0.20$ m, $\rho_0 = 1.0 \times 10^{-6} \Omega \text{ m}$, $A_0 = 1.0 \times 10^{-6} \text{ m}^2$, and $V_0 - V_N = 0.10$ V. The exact model resistance is 0.138629Ω , current 0.721348 A, and total electrical power 0.0721348 W. The small power is chosen to make the constant-temperature assumption more plausible; it is not a thermal calculation.

```

from math import log
rho0, L, A0 = 1e-6, .20, 1e-6
Vleft, Vright = .10, 0.0
Rexact = rho0*L/A0*log(2)
for N in [4,8,16,32]:
    dx = L/N
    cell_R = []
    for k in range(N):
        x = (k+.5)*dx
        rho = rho0*(1+x/L)
        area = A0*(1+x/L)**2
        cell_R.append(rho*dx/area)
    R = sum(cell_R)
    current = (Vleft-Vright)/R
    V = [Vleft]
    for r in cell_R:
        V.append(V[-1]-current*r)
    Pcells = sum(current**2*r for r in cell_R)
    Pterm = current*(Vleft-Vright)
    err = abs(R/Rexact-1)
    print(N, R, err, V[-1], Pcells-Pterm)

```

The final potential should agree with V_{right} to floating-point rounding, and the summed cell heating should agree with terminal power. These two checks test conservation. Comparing with R_{exact} tests spatial discretization. A result can conserve current perfectly and still have a large discretization error, so both checks are needed.

The relative resistance errors for $N = 4, 8, 16, 32$ are approximately 0.278%, 0.0702%, 0.0176% and 0.00440%. The smooth midpoint rule is second order, so refinement by two reduces error by about four. Abrupt interfaces require cells aligned to the interface or an explicit subcell resistance treatment.

6.23.3 From a benchmark to a useful mini-project

A useful extension is to prescribe a measured or synthetic temperature profile $T(x)$ and replace $\rho(x)$ by $\rho[T(x)]$. Ask where voltage drop and heating are largest. Then refine the grid until the answer changes by less than the intended accuracy. Report the material law, boundary conditions, current conservation, power balance and grid convergence.

For a more demanding project, solve Eq. (6.117) as a linear system rather than using the one-dimensional shortcut. This exposes the structure used in higher-dimensional conduction solvers. A two-dimensional sheet with a narrow contact is a natural later extension, but its contact geometry and boundary conditions must be specified before the computed “resistance” has meaning.

6.24 Spatial temperature profiles in a resistive wire

6.24.1 When a single temperature is insufficient

The lumped model assumed a uniform sample temperature. A long wire cooled strongly at its ends may instead have its highest temperature in the middle. A local power-density calculation should then be coupled to heat conduction. Consider a uniform wire of length L , area A , constant electrical resistivity ρ and constant thermal conductivity k_{th} in $\text{W m}^{-1}\text{K}^{-1}$. Both ends are maintained at T_0 . Neglect heat loss from the sidewall and assume steady current I .

Fourier’s conduction law gives the heat flow in the positive x direction,

$$\dot{Q}_x = -k_{\text{th}}A \frac{dT}{dx}.$$

Electrical generation per length is $g = I^2\rho/A$. In a short slice, outgoing heat minus incoming heat must equal generated heat: $d\dot{Q}_x/dx = g$. Therefore

$$k_{\text{th}}A \frac{d^2T}{dx^2} + \frac{I^2\rho}{A} = 0. \quad (6.119)$$

Twice integrating with $T(0) = T(L) = T_0$ gives

$$\theta(x) = T(x) - T_0 = \frac{I^2\rho}{2k_{\text{th}}A^2}x(L-x), \quad \theta_{\text{max}} = \frac{I^2\rho L^2}{8k_{\text{th}}A^2}. \quad (6.120)$$

The maximum occurs at $x = L/2$, where the derivative vanishes. Positive curvature would have implied a cold center and would contradict positive internal generation, so the negative second derivative is a useful sign check.

At the left end $\dot{Q}_x(0) = -I^2\rho L/(2A)$; at the right it is $+I^2\rho L/(2A)$. Heat leaves through each end at half the total power. The negative leftward value is not negative heat generation; it denotes energy moving against the positive x direction. Adding the two *outward* heat flows gives I^2R .

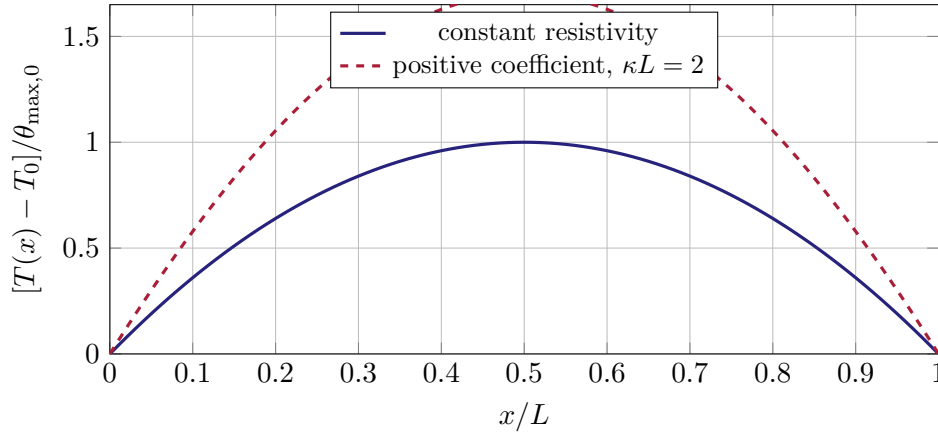


Figure 6.17: Calculated steady temperature profiles with both ends held at ambient. The normalization $\theta_{\text{max},0} = I^2\rho_0 L^2/(8k_{\text{th}}A^2)$ is the constant-resistivity maximum at the same current. The dashed curve includes the linear positive temperature coefficient through Eq. (6.122); heating is more concentrated than in the constant-resistivity approximation.

Example 6.39: A wire cooled at its ends

An illustrative alloy wire has $L = 0.10$ m, $A = 1.0$ mm², $\rho = 1.0 \times 10^{-6}$ Ω m and $k_{\text{th}} = 15$ W m⁻¹K⁻¹. It carries 0.20 A. Both ends are held at T_0 , and sidewall losses are neglected. **Electrical result.** The resistance is 0.10 Ω , the voltage drop is 0.020 V, and the total power is 0.0040 W.

Temperature result. Equation (6.120) gives

$$\theta_{\text{max}} = \frac{(0.20)^2(1.0 \times 10^{-6})(0.10)^2}{8(15)(1.0 \times 10^{-6})^2} = 3.33 \text{ K}.$$

The mean rise is

$$\bar{\theta} = \frac{1}{L} \int_0^L \theta(x) dx = \frac{I^2 \rho L^2}{12 k_{\text{th}} A^2} = 2.22 \text{ K}.$$

Each end removes 2.0 mW. The ratio $\theta_{\text{max}}/P = L/(8k_{\text{th}}A)$ is an effective thermal resistance for the hottest point under these boundary conditions. It differs from the mean-temperature thermal resistance $L/(12k_{\text{th}}A)$.

Limitation. Strong convection or substrate contact would alter the temperature profile. Finite thermal contact resistance at the ends would also alter the boundary temperatures. The model identifies the controlling geometry without claiming a universal wire temperature.

6.24.2 A spatial resistance correction

If $\rho(T) = \rho_0[1 + \alpha\theta(x)]$ and heating is weak, first use the constant- ρ_0 temperature profile to estimate the correction. The terminal resistance becomes

$$R \simeq \frac{\rho_0}{A} \int_0^L [1 + \alpha\theta(x)] dx = R_0[1 + \alpha\bar{\theta}]. \quad (6.121)$$

The appropriate first-order temperature is the length average for this uniform-area sample, not the hottest-point temperature. For the example with $\alpha = 4.0 \times 10^{-4}$ K⁻¹, the fractional resistance correction is 8.89×10^{-4} , or 0.0889%. The small product $\alpha\theta_{\text{max}} \simeq 0.00133$ supports the perturbative use of the uncorrected profile.

In a nonuniform wire, the resistance-weighted average is instead relevant: regions with large $\rho_0(x)/A(x)$ contribute more to R . A thermal camera reading at one visible surface point need not equal the temperature relevant to the measured resistance.

6.24.3 Positive feedback with distributed cooling

Keeping the linear temperature dependence within the heat equation rather than treating it perturbatively gives

$$\theta'' + \kappa^2 \theta = -\frac{I^2 \rho_0}{k_{\text{th}} A^2}, \quad \kappa^2 = \frac{\alpha I^2 \rho_0}{k_{\text{th}} A^2}, \quad \alpha > 0.$$

With both ends at T_0 , the symmetric physical branch for $\kappa L < \pi$ is

$$\theta(x) = \frac{1}{\alpha} \left[\frac{\cos[\kappa(x - L/2)]}{\cos(\kappa L/2)} - 1 \right]. \quad (6.122)$$

Expanding the cosines for small κL recovers the parabola. As $\kappa L \rightarrow \pi$ from below, the denominator approaches zero and the model predicts a loss of bounded steady operation. Figure 6.17 shows the enhancement before that threshold.

A stability check follows from the time-dependent heat equation. Let c_ℓ be heat capacity per length in J m⁻¹K⁻¹. For a temperature perturbation that vanishes at both ends,

$$c_\ell \frac{\partial \delta \theta}{\partial t} = k_{\text{th}} A \frac{\partial^2 \delta \theta}{\partial x^2} + \frac{\alpha I^2 \rho_0}{A} \delta \theta.$$

A mode $\sin(n\pi x/L) \exp(s_n t)$ has growth rate

$$s_n = \frac{1}{c_\ell} \left[-k_{\text{th}} A \left(\frac{n\pi}{L} \right)^2 + \frac{\alpha I^2 \rho_0}{A} \right].$$

All modes decay if the slowest mode $n = 1$ decays, giving exactly $\kappa L < \pi$. Thus the spatial model has a geometrical stability limit,

$$I < \frac{\pi A}{L} \sqrt{\frac{k_{\text{th}}}{\alpha \rho_0}}. \quad (6.123)$$

This is a model threshold with ideal end cooling and no side losses. At large temperature rises, changes of thermal conductivity, radiation, resistivity and material state must be included before making a physical prediction. The value of the derivation is the connection between heat-removal geometry and electrical feedback.

6.25 Field energy identities as checks on a conduction solution

6.25.1 Recovering terminal power from a field solution

Suppose a passive conducting region is in DC steady state, with potential fixed to V_a and V_b on two electrodes and zero normal current through all other boundaries. The bulk Joule power is

$$P = \int_{\mathcal{V}} \mathbf{J} \cdot \mathbf{E} \, d\mathcal{V} = - \int_{\mathcal{V}} \mathbf{J} \cdot \nabla V \, d\mathcal{V}.$$

Using $\nabla \cdot (V \mathbf{J}) = \mathbf{J} \cdot \nabla V + V \nabla \cdot \mathbf{J}$ and steady conservation,

$$P = - \oint_{\partial \mathcal{V}} V \mathbf{J} \cdot d\mathbf{A}. \quad (6.124)$$

On the input electrode, the outward current is $-I$; on the output electrode it is $+I$. The insulating boundary contributes zero. Hence

$$P = -[V_a(-I) + V_b I] = I(V_a - V_b).$$

This derivation works for nonuniform geometry. It explains why a single terminal power can be compared with the sum of local heating in a numerical solver, even when there is no uniform field or simple L/A description.

Potential has an arbitrary additive reference. Replacing V by $V + C$ in the boundary expression changes it by $-C \oint \mathbf{J} \cdot d\mathbf{A} = 0$ in steady state. The calculated power is therefore reference independent. A numerical code whose computed terminal power depends on adding the same constant to both electrode potentials contains a serious inconsistency.

6.25.2 A variational check, with its constraints stated

For a known positive scalar conductivity independent of potential and temperature during the solve, define

$$\mathcal{F}[V] = \int_{\mathcal{V}} \sigma |\nabla V|^2 \, d\mathcal{V}.$$

Let V_* solve the steady conduction problem, and let $V = V_* + \eta$ be a trial potential with $\eta = 0$ on the fixed-potential electrodes. Expanding,

$$\mathcal{F}[V] = \mathcal{F}[V_*] + 2 \int \sigma \nabla V_* \cdot \nabla \eta \, d\mathcal{V} + \int \sigma |\nabla \eta|^2 \, d\mathcal{V}.$$

Integration by parts makes the middle term vanish: the volume term contains $\nabla \cdot (\sigma \nabla V_*) = 0$, while the boundary term vanishes because $\eta = 0$ on the electrodes and the physical normal current is zero on insulating boundaries. Therefore

$$\mathcal{F}[V] - \mathcal{F}[V_*] = \int_{\mathcal{V}} \sigma |\nabla \eta|^2 d\mathcal{V} \geq 0. \quad (6.125)$$

The exact field minimizes this functional over potentials with the same electrode values. An arbitrary trial field need not satisfy current conservation, so its functional value should not be mistaken for a physical terminal-power measurement. This is a constrained mathematical check, not a claim that circuits always minimize their power when geometry or source conditions change.

The identity supports numerical methods: a trial potential that meets the electrode values but gives a much larger functional is not yet the correct field. It also proves uniqueness up to a constant when the boundary data leave the potential reference free. If conductivity depends on self-heating, the electrical and thermal problems must be coupled, and this fixed-conductivity proof cannot be applied unchanged to the whole nonlinear system.

Take-home message

An advanced calculation is valuable when it answers a physical question that the simpler model cannot. Conservation, constitutive response, geometry and heat transfer provide separate checks. Agreement with one check does not excuse failure of another.

6.26 Problems for independent study and tutorials

These problems progress from conceptual interpretation to quantitative modeling and advanced analysis. Unless a problem specifies otherwise, use $e = 1.602 \times 10^{-19} \text{ C}$, copper resistivity $\rho = 1.7 \times 10^{-8} \Omega \text{ m}$, copper carrier density $n = 8.5 \times 10^{28} \text{ m}^{-3}$, fixed temperature and negligible contact resistance. State any additional assumptions. Problems marked **A** use the advanced enrichment. A selected solution is identified by an explicit cross-reference, not by a separate repeated exercise bank.

Conceptual and graphical reasoning

1. A steady resistor absorbs 8 W. Explain why its output current equals its input current, yet its output terminal can be at lower potential. Identify which quantity is conserved locally and which is transferred to the surroundings.
2. A perpendicular cut through a wire has area $A_{\perp}$ and carries current I . A second complete cut has its normal at 60° to the wire axis. Find its area and signed current. Contrast this with tilting a fixed-area patch in an infinite uniform current density. See Solution S1.
3. A neutral fluid transports equal positive and negative charge densities with the same velocity. Does it carry electric current? Explain why the answer differs from a neutral metal in which electrons drift relative to the lattice.
4. In Fig. 6.2, replace the second lobe by -2 A lasting 2 ms. Compare net charge, absolute signed-lobe area, and resistor heating with the original pulse. Explain why equal net charge is insufficient to predict heating.
5. A perfectly straight measured V - I line has a nonzero intercept. Give two distinct physical or measurement interpretations. Which additional observation could distinguish an offset from intrinsic device behavior?
6. A positive resistor is replaced by another with twice its resistance. Compare power under fixed current and fixed voltage. Explain why neither comparison alone answers what happens in an unspecified circuit.
7. Electrons drift left and holes drift right in a semiconductor. Do their conventional current contributions add or cancel? Can the material remain nearly neutral? Distinguish the charge-density and current-density conditions.
8. A thermistor is linear in voltage at each externally fixed temperature, but its resistance changes strongly with temperature. Is it necessarily non-ohmic in an isothermal low-current experiment? Why can a slow high-current sweep be nonlinear?
9. Explain why setting $R = 0$ in an ordinary DC resistor formula does not predict the current of a superconducting loop or provide a theory of its magnetic response.
10. Two resistors dissipate equal power but have different temperatures. Give two physically distinct reasons. Which additional parameters enter a minimal thermal model?

Charge transport, material response and geometry

11. A current is $I(t) = (0.60 \text{ A s}^{-1})t$ for $0 \leq t \leq 4.0 \text{ s}$. Find transported charge, average current and the equivalent net number of elementary charges. See Solution S2.
12. Copper carries 3.0 A through area 1.0 mm^2 . Find drift speed and the drift time associated with a length of 1.0 m . Explain why this is not a circuit turn-on delay.

13. A round copper wire of radius 0.50 mm and length 2.0 m carries 1.5 A uniformly. Find J , E , R , voltage drop and power. Check both $P = I\Delta V$ and $P = \rho J^2 AL$. See Solution S3.
14. A block has dimensions L , $2L$, $3L$ and isotropic resistivity ρ . Find its resistance along each axis with full-face contacts. If all dimensions are doubled, how does each resistance change?
15. A wire is stretched uniformly to 1.50 times its original length at constant volume and unchanged resistivity. Find the resistance ratio. State why real deformation could invalidate the unchanged-resistivity assumption. See Solution S4.
16. A motor receives 24 V and 2.0 A and delivers 36 W of mechanical power. Find total loss and efficiency. If its measured winding resistance is $1.0\ \Omega$, separate winding heating from other loss. Why is $24/2\ \Omega$ not the winding resistance? See Solution S15.
17. A device uses 18 W continuously for seven days. Find energy in joules and kilowatt-hours. A second device uses twice the power for one-quarter of the time; compare energy use.
18. A metal sample has $R_0 = 50.0\ \Omega$ at 20°C and $\alpha_0 = 4.0 \times 10^{-3}\ \text{K}^{-1}$. Estimate R at 80°C for fixed geometry and compute the local normalized coefficient at that temperature within the same linear model.
19. A prescribed axial current density in a cylinder is $J(r) = Cr$, $0 \leq r \leq a$. Express C in terms of total current I , and find the fraction of current in $a/2 < r < a$. State the units of C . See Solution S5.
20. **A.** A slowly tapered circular wire has radius $r(x) = a(1 + x/L)$ and constant resistivity. Derive its resistance. Compare with a uniform wire having its initial area and with one having its arithmetic mean cross-sectional area. See Solution S6.
21. **A.** A conductor has constant area A and $\rho(x) = \rho_0(1 + x/L)$. Find total resistance and the potential $V(x)$ for boundary values $V(0) = V_0$, $V(L) = 0$. Check that the resulting current is independent of x .
22. A device obeys $I = aV + bV^3$, where $a = 0.020\ \text{A V}^{-1}$ and $b = 0.0010\ \text{A V}^{-3}$. At $V = 2.0\ \text{V}$, find I , static resistance, differential resistance and total absorbed power. Which resistance belongs in I^2R ? See Solution S16.

Engineering and measurement

23. A load 15 m from a source draws 10 A. Identical outgoing and return copper conductors must together drop at most 0.20 V. Find minimum area of each and pair loss at that area. Identify a missing input for a temperature-rise calculation. See Solution S7.
24. A shunt at 10 A must generate at least 40 mV but dissipate at most 0.60 W. Determine the allowed resistance interval and corresponding burden-voltage interval. Explain the design tradeoff. See Solution S17.
25. A 12 V heater warms 0.10 kg of water through 10 K in 120 s. Take $c = 4180\ \text{J kg}^{-1}\text{K}^{-1}$ and a stipulated water-heating efficiency 0.75. Find operating power and resistance. State which energy losses the efficiency summarizes.
26. A sample gives $V(+0.200\ \text{A}) = 8.50\ \text{mV}$ and $V(-0.200\ \text{A}) = -7.50\ \text{mV}$. Find resistance and offset, assuming a stable linear sample. Each voltage has independent standard uncertainty 0.04 mV; find u_R for exact currents. See Solution S8.

27. **A.** A resistance is inferred from $V = 5.00\text{ V}$ and $I = 0.250\text{ A}$, with $u_V = 0.02\text{ V}$ and $u_I = 0.002\text{ A}$. Compare u_R for correlation coefficients 0, $+0.8$ and -0.8 . Explain the sign of the covariance contribution. See Solution S18.
28. Reproduce the fit to data set A in Table 6.1. Then force $b = 0$ and compare the fitted resistance. Why is the symmetric current range special? Would agreement of slopes prove that the offset is absent? See Solution S19.
29. Fit the full and restricted range $|I| \leq 10\text{ mA}$ of data set B. Compare slope and residuals. Explain why restricting the range can improve linearity without giving a universal material resistance.
30. A $1.0\text{ M}\Omega$ sample is sensed by a $10.0\text{ M}\Omega$ voltmeter. An ideal source drives the parallel combination and its total current is used in V/I . Find the inferred resistance and relative bias. Does a four-wire connection alone remove this error?
31. A $100\text{ }\Omega$ sensor has thermal resistance 200 K W^{-1} . Limit self-heating to 0.010 K . Find the maximum measuring current in the small-heating approximation and its voltage signal. See Solution S9.
32. A gauge has $R = 350\text{ }\Omega$, $K_g = 2.1$ and strain 300×10^{-6} . Find its resistance change. For a hypothetical constant-resistivity material with $\nu = 0.30$, what would the purely geometric gauge factor be?

Advanced analysis and computation

33. **A.** Derive the differential continuity equation from integral conservation. Then consider $\mathbf{J} = kx\hat{x}$ in a small region. Find $\partial\rho_q/\partial t$ and interpret the sign for $k > 0$.
34. **A.** Two equal-area materials carry the same axial current. Their resistivities have ratio $\rho_2/\rho_1 = 4$. Compare J , E , drift speeds for equal carrier densities, and local heating densities. Which quantities change if carrier densities differ?
35. **A.** Derive radial resistance for coaxial electrodes of radii a, b and length ℓ . Find the thin-gap limit $b = a + d$, $d \ll a$, and interpret it using a slab of area $2\pi a\ell$. See Solution S10.
36. **A.** Homogeneous material of resistivity ρ fills the region between concentric spherical electrodes $a < r < b$. Derive radial resistance, current density and field for applied voltage ΔV . Compare the dimension and the $b \rightarrow a$ limit with a thin spherical shell. See Solution S20.
37. **A.** Two equal-length sections of the same material have areas A and $A/4$ and carry common current I . Compare heating per volume, per length, and total power. Why does the temperature ratio remain undetermined?
38. **A.** A sample obeys $R = R_0(1 + \alpha\theta)$ with $R_0 = 20\text{ }\Omega$, $\alpha = 0.004\text{ K}^{-1}$, $H = 0.20\text{ W K}^{-1}$, and $C_{\text{th}} = 4.0\text{ J K}^{-1}$. For $I = 1.0\text{ A}$, find equilibrium rise, stability and relaxation time. Compare with a fixed voltage of 20 V , giving the same cold power. See Solution S11.
39. **A.** For an NTC thermistor $R(T) = R_0 \exp[B(1/T - 1/T_0)]$ under fixed voltage, calculate dP_{el}/dT . Write the local stability inequality for linear heat loss $H(T - T_0)$. Do not assume that an equilibrium exists for arbitrary voltage. See Solution S21.
40. **A.** Solve the Drude relaxation equation for an initially settled current after the field is switched off at $t = 0$. Show how directed kinetic energy density decays and compare its decay time with the current decay time.

41. Reproduce the finite-window exponential-pulse integration in the text. Determine how many steps of the displayed sequence are needed for less than 0.2% discretization error. Separately compute the fraction of infinite-window charge omitted at $T = 3\tau$.
42. **A.** In the thermistor model of Example 6.36, find the resistance at 323.15 K and the contribution to temperature uncertainty from a 0.5% relative resistance uncertainty there. Explain why absolute kelvin is required. See Solution S12.
43. **A.** Reproduce the nonuniform-wire benchmark for $N = 4, 8, 16, 32$. Plot potential at cell boundaries and compare with Eq. (6.118). Check current conservation, total power, endpoint potential and resistance convergence separately.
44. **A.** A numerical cell consists of two equal-length subcells of area A and conductivities σ_1 and σ_2 . Derive its effective conductivity. Evaluate the ratio of arithmetic-mean to correct conductivity when $\sigma_2 = 100\sigma_1$. See Solution S13.
45. **A.** A material has $J_x = \sigma E_x$, $J_y = 3\sigma E_y$ and $\sigma > 0$. For $\mathbf{E} = E_0(\hat{x} + \hat{y})$, find the direction of $\mathbf{J}$, the angle between $\mathbf{J}$ and $\mathbf{E}$, and the power density. See Solution S22.
46. **A.** For constant resistivity, fixed length L and fixed volume $\mathcal{V}$, prove the minimum-resistance result in the one-dimensional model. State which physical assumptions prevent interpreting it as a universal wire-design prescription.
47. **A.** A stipulated short-pulse fuse model requires 0.50 J to reach melting and has constant $R_f = 0.020 \Omega$. Find the required $\int I^2 dt$ and the duration of a rectangular 10 A pulse. Explain why a pulse with the same transported charge but a different shape can have a different effect.
48. **A.** A low-resistance specimen gives different positive- and negative-current ratios V/I , while the difference-based resistance increases with $|I|$. Formulate a minimal model containing both a constant offset and leading self-heating curvature. Specify the measurements and residual tests needed to distinguish the two effects. See Solution S14.

6.27 Selected detailed solutions

The following solutions illustrate reasoning and validation. Solve the corresponding problems first; use these derivations to diagnose assumptions and methods, not only to compare final numbers.

Solution S1 to Problem 2: a complete tilted cut

For the same wire, the perpendicular projection of the tilted cut is fixed. Therefore $A_{\text{cut}} \cos 60^\circ = A_\perp$, so $A_{\text{cut}} = 2A_\perp$. With the normal having a positive axial component,

$$I_{\text{cut}} = J(2A_\perp) \cos 60^\circ = JA_\perp = I.$$

A fixed-area patch instead carries half its original flux at this angle, because its area does not change. The two questions differ in what is held fixed. Reversing the complete cut's normal gives $-I$. As the cut becomes nearly parallel to the wire, its required area grows; this geometric limit cannot be treated as a fixed finite patch.

Solution S2 to Problem 11: integrating a ramp

Write the coefficient with its time dimension explicitly:

$$\Delta q = \int_0^{4.0\text{s}} (0.60\text{ A s}^{-1})t\,dt = \frac{1}{2}(0.60)(4.0)^2\text{ A s} = 4.8\text{ C}.$$

The average current is $4.8/4.0 = 1.2\text{ A}$, half the final current 2.4 A . The equivalent net number of elementary charges is $4.8/e = 3.0 \times 10^{19}$. This is a net excess of crossings in the current-carrying sense; it is not the total number of microscopic crossings in both directions.

Solution S3 to Problem 13: local and terminal checks

Convert the radius before squaring: $r = 5.0 \times 10^{-4}\text{ m}$, so $A = 7.854 \times 10^{-7}\text{ m}^2$. For uniform axial transport,

$$\begin{aligned} J &= I/A = 1.91 \times 10^6\text{ A m}^{-2}, \\ E &= \rho J = 0.0325\text{ V m}^{-1}, \\ R &= \rho L/A = 0.0433\,\Omega, \\ \Delta V &= EL = IR = 0.0649\text{ V}, \\ P &= I\Delta V = I^2 R = 0.0974\text{ W}. \end{aligned}$$

The local density is $p = \rho J^2 = 6.20 \times 10^4\text{ W m}^{-3}$. Multiplying by volume $AL = 1.571 \times 10^{-6}\text{ m}^3$ gives the same 0.0974 W . The large density in SI units is compatible with small total power because the volume is small. An estimate of temperature still requires thermal information.

Solution S4 to Problem 15: preserve volume

Let $L' = sL$, where $s = 1.50$. Conservation of material volume gives $A'L' = AL$, hence $A' = A/s$. With resistivity stipulated unchanged,

$$\frac{R'}{R} = \frac{\rho L'/A'}{\rho L/A} = s^2 = 2.25.$$

Using only the length change would miss the narrowing and predict 1.50 instead. Real strain can change resistivity and create defects or nonuniform necking, so the result is a geometry-only model rather than a universal tensile-test law.

Solution S5 to Problem 19: radial weighting

The current is axial, so the integration element is an annular disk element rather than the lateral surface of a cylinder:

$$I = \int_0^a Cr(2\pi r dr) = \frac{2\pi Ca^3}{3}, \quad C = \frac{3I}{2\pi a^3}.$$

Thus $[C] = \text{A m}^{-3}$, consistent with $J = Cr$ having units A m^{-2} . The outer-annulus fraction is

$$\frac{\int_{a/2}^a r^2 dr}{\int_0^a r^2 dr} = 1 - \frac{1}{8} = \frac{7}{8}.$$

This exceeds the annular area fraction $3/4$ because the stipulated profile increases toward the outer edge. Such a profile is not a default homogeneous DC-metal solution; a physical mechanism for it would be needed in an experiment.

Solution S6 to Problem 20: radius is not area

The radius varies linearly but the area varies quadratically:

$$A(x) = \pi a^2(1 + x/L)^2.$$

The one-dimensional resistance is

$$R = \frac{\rho L}{\pi a^2} \int_0^1 \frac{du}{(1+u)^2} = \frac{\rho L}{2\pi a^2}.$$

A uniform wire with the initial area has $R_{\text{initial}} = \rho L/(\pi a^2)$, twice as large. The mean area is

$$\bar{A} = \frac{1}{L} \int_0^L A(x) dx = \frac{7}{3} \pi a^2,$$

so a uniform wire of that area has $R_{\text{mean}} = 3\rho L/(7\pi a^2)$. Therefore $R/R_{\text{mean}} = 7/6$. Equal material volume does not make the tapered sample electrically equivalent to its mean-area wire: narrow regions have disproportionate resistance.

Solution S7 to Problem 23: include both conductors

The allowed combined resistance is $0.20/10 = 0.020 \Omega$. The complete resistive path is $2(15) = 30 \text{ m}$, giving

$$A_{\text{min}} = \frac{\rho(2L)}{R_{\text{max}}} = \frac{1.7 \times 10^{-8}(30)}{0.020} = 2.55 \times 10^{-5} \text{ m}^2 = 25.5 \text{ mm}^2.$$

Each conductor needs that area. At the boundary of the requirement, $P_c = I\Delta V = 2.0 \text{ W}$. A total power does not determine temperature without a heat-transfer model, including the installation geometry and environment. The result is a voltage-drop estimate under the stated resistivity assumption.

Solution S8 to Problem 26: an offset and its uncertainty

The difference and mean isolate the odd and even parts of the voltage:

$$R = \frac{8.50 - (-7.50)}{0.400} \text{ m}\Omega = 40.0 \text{ m}\Omega, \quad b = 0.50 \text{ mV}.$$

Since the two voltage uncertainties are independent, the standard uncertainty in their difference is $\sqrt{2}(0.04 \text{ mV})$. Exact current magnitude gives

$$u_R = \frac{\sqrt{2}(0.04 \text{ mV})}{0.400 \text{ A}} = 0.141 \text{ m}\Omega.$$

A sensible report is $(40.00 \pm 0.14) \text{ m}\Omega$ under the specified uncertainty model. If the offset drifts between reversals or current uncertainty is appreciable, additional terms are required.

Solution S9 to Problem 31: a measurement-current bound

Small self-heating permits use of the ambient resistance. The power limit is

$$P_{\max} = \frac{0.010 \text{ K}}{200 \text{ K W}^{-1}} = 5.0 \times 10^{-5} \text{ W}.$$

Therefore

$$I_{\max} = \sqrt{P_{\max}/R} = 0.707 \text{ mA}, \quad V = I_{\max} R = 70.7 \text{ mV}.$$

The units under the square root are $\text{W}/\Omega = \text{A}^2$. A lower current reduces the systematic heating error but also the voltage signal. The thermal resistance must describe the actual mounting and medium, not just the sensor material in isolation.

Solution S10 to Problem 35: the thin-gap limit

Every radial shell carries the same current through lateral area $2\pi r\ell$, so $dR = \rho dr/(2\pi r\ell)$. Integration gives

$$R = \frac{\rho}{2\pi\ell} \ln \frac{b}{a}.$$

For $b = a + d$ with $d/a \ll 1$, $\ln(1 + d/a) = d/a + \mathcal{O}((d/a)^2)$ and

$$R \simeq \frac{\rho d}{2\pi a\ell}.$$

This is the resistance of a flat layer with path length d and area $2\pi a\ell$, as expected when the curvature is small across the gap. The next term lowers the resistance relative to this inner-area slab approximation because shells farther out have larger areas.

Solution S11 to Problem 38: check the slope

For current drive, the cold power is 20 W and the feedback slope is $\alpha I^2 R_0 = 0.080 \text{ W K}^{-1}$. Since this is less than $H = 0.20 \text{ W K}^{-1}$, the equilibrium is stable:

$$\theta_* = \frac{20}{0.20 - 0.080} = 166.7 \text{ K}, \quad \tau_{\text{eff}} = \frac{4}{0.12} = 33.3 \text{ s}.$$

For fixed 20 V,

$$\theta_* = \frac{\sqrt{1 + 4(0.004)(20^2)/(0.20 \cdot 20)} - 1}{0.008} = 76.6 \text{ K}.$$

The equilibrium resistance is 26.12Ω and the power is 15.31 W , equal to $H\theta_*$. Here $P'_{\text{el}} = -0.04689 \text{ W K}^{-1}$, so $\tau_{\text{eff}} = 4/(0.20 + 0.04689) = 16.2 \text{ s}$. Both conclusions are conditional on the linear material and cooling laws, whose realism at large rises must be tested independently.

Solution S12 to Problem 42: thermistor sensitivity

With $T = 323.15 \text{ K}$,

$$R = 10.0 \text{ k}\Omega \exp \left[3500 \left(\frac{1}{323.15} - \frac{1}{298.15} \right) \right] \simeq 4.03 \text{ k}\Omega.$$

The uncertainty contribution from the resistance reading is

$$u_T = \frac{T^2}{B} \frac{u_R}{R} = \frac{323.15^2}{3500} (0.005) \text{ K} = 0.149 \text{ K}.$$

The reciprocal-temperature law is defined using thermodynamic temperature. Substituting Celsius values changes the model rather than performing a harmless unit conversion. Unlike a temperature difference, reciprocal absolute temperature is not invariant under a shift of the zero.

Solution S13 to Problem 44: preserve series resistance

For cell length d ,

$$R = \frac{d}{2A\sigma_1} + \frac{d}{2A\sigma_2} = \frac{d}{A\sigma_{\text{eff}}}.$$

Thus $\sigma_{\text{eff}} = 2\sigma_1\sigma_2/(\sigma_1 + \sigma_2)$. With $\sigma_2 = 100\sigma_1$, the correct result is $(200/101)\sigma_1 \simeq 1.9802\sigma_1$, whereas the arithmetic mean is $50.5\sigma_1$. Their ratio is 25.5025.

The poorly conducting half limits the shared current. Averaging conductivities arithmetically would correspond to side-by-side parallel channels, a different geometry. Numerical averaging therefore encodes physics and cannot be chosen solely for algebraic convenience.

Solution S14 to Problem 48: separate even and odd voltage parts

A useful minimal model over a limited range is

$$V(I) = b + R_0I + cI^3.$$

The constant term represents a stable offset; the odd cubic term is a leading phenomenological correction compatible with symmetric self-heating. Under weak current-driven heating, $\theta \simeq I^2 R_0/H$ and $V = IR_0(1 + \alpha\theta)$ give $c \simeq \alpha R_0^2/H > 0$ for a positive coefficient.

At each magnitude $I > 0$, form

$$V_{\text{even}} = \frac{V(+I) + V(-I)}{2} = b, \quad \frac{V_{\text{odd}}}{I} = \frac{V(+I) - V(-I)}{2I} = R_0 + cI^2.$$

Measure several magnitudes with both signs, recorded uncertainties and settling times. Plot the even voltage against I and the odd ratio against I^2 . A constant even part supports the offset model; an increasing odd ratio supports the stipulated curvature. Compare residuals with measurement uncertainty and repeat at different sweep rates or controlled bath temperatures to test the heating interpretation.

These observations do not prove that heating is the only possible source of curvature. A time-dependent offset, asymmetric contacts or intrinsic nonlinear response can fail the even/odd model and require a more specific explanation. The cubic approximation must be restricted to a range where omitted higher terms are negligible.

Solution S15 to Problem 16: distinguish conversion from winding loss

Choose the system. Take the entire motor as the system and assume steady operation: neither its rotational kinetic energy nor its internal energy is increasing. The electrical input and mechanical output are then

$$P_{\text{in}} = VI = (24)(2.0) = 48 \text{ W}, \quad P_{\text{mech}} = 36 \text{ W}.$$

Energy conservation gives a total loss of 12 W and efficiency

$$\eta = P_{\text{mech}}/P_{\text{in}} = 0.75 = 75\%.$$

Identify the resistive part. The stipulated winding resistance determines the winding heat rate, not the whole input power:

$$P_{\text{winding}} = I^2 R_w = (2.0)^2(1.0) = 4.0 \text{ W}.$$

Thus 8.0 W remains in other losses, such as mechanical friction or magnetic losses. The data do not determine how that remainder is divided among mechanisms.

The terminal ratio $V/I = 12 \Omega$ is an operating-point ratio for the running device. Inserting it into $I^2(V/I)$ gives 48 W, the *total electrical input*, because the expression is algebraically IV . It does not imply 48 W of winding heating. The motor exchanges energy with mechanical degrees of freedom, so its terminal voltage cannot be represented only by the measured winding resistance. This is the distinction that makes the power accounting meaningful.

Solution S16 to Problem 22: secant and tangent at one operating point

Evaluate the device law first. At $V = 2.0 \text{ V}$,

$$I = aV + bV^3 = (0.020)(2.0) + (0.0010)(2.0)^3 = 0.048 \text{ A}.$$

Both terms have units of amperes. The static resistance and incremental conductance are

$$R_{\text{static}} = \frac{2.0}{0.048} = 41.7 \Omega, \quad \frac{dI}{dV} = a + 3bV^2 = 0.032 \text{ S}.$$

Therefore $r_{\text{diff}} = 31.25 \Omega$. Graphically, $1/R_{\text{static}}$ is the slope of the line from the origin to the operating point, whereas $1/r_{\text{diff}}$ is the tangent slope there. The cubic term makes the tangent steeper.

The total absorbed power is

$$P = VI = 0.096 \text{ W} = I^2 R_{\text{static}}.$$

Using $I^2 r_{\text{diff}}$ would give 0.072 W , which contradicts the terminal energy-transfer rate. The differential resistance describes small changes, not the finite operating-point ratio.

Check the local approximation. A $+0.10 \text{ V}$ increment predicts $\delta I \simeq (0.032)(0.10) = 3.20 \text{ mA}$. Direct evaluation at 2.10 V gives $I = 51.261 \text{ mA}$, an actual increment of 3.261 mA . Their difference is 0.061 mA , precisely the omitted $3bV(\delta V)^2 + b(\delta V)^3$. This check states what “small signal” means quantitatively for this example.

Solution S17 to Problem 24: combine two competing inequalities

At the specified current, the minimum readable voltage imposes a lower resistance bound:

$$IR \geq 0.040 \text{ V} \implies R \geq 4.0 \text{ m}\Omega.$$

The heat limit imposes an upper bound:

$$I^2 R \leq 0.60 \text{ W} \implies R \leq 6.0 \text{ m}\Omega.$$

The allowed interval is therefore $4.0 \leq R/(\text{m}\Omega) \leq 6.0$, with burden voltage $40 \leq V/(\text{mV}) \leq 60$. The endpoint powers are 0.40 and 0.60 W .

Feasibility check. Before choosing a numerical resistance, one can combine the requirements as $IV_{\text{min}} \leq P_{\text{max}}$. Here $0.40 \text{ W} \leq 0.60 \text{ W}$, so an interval exists. If this inequality failed, no passive shunt value could satisfy both requirements at the stated current; changing only the resistance would not solve the design problem.

A higher resistance improves the voltage signal at the price of increased burden voltage and heat. The interval does not determine temperature, temperature-induced resistance error, or contact error. Those require the thermal resistance, temperature coefficient and sensing arrangement. A midpoint is not automatically an optimum; a design choice depends on which uncertainties and margins matter.

Solution S18 to Problem 27: a ratio can cancel a shared error

The nominal value is $R = 5.00/0.250 = 20.0 \Omega$. Write the correlation coefficient as c to distinguish it from electrical resistivity. First-order uncertainty propagation gives

$$\left(\frac{u_R}{R}\right)^2 = \left(\frac{u_V}{V}\right)^2 + \left(\frac{u_I}{I}\right)^2 - 2c \frac{u_V}{V} \frac{u_I}{I}.$$

The relative standard uncertainties are 0.004 and 0.008 . Thus

$$u_R = 20.0 \sqrt{80 \times 10^{-6} - 64c \times 10^{-6}} \Omega.$$

The results are

Correlation c	Relative standard uncertainty	u_R (Ω)
0	0.00894	0.179
+0.8	0.00537	0.107
-0.8	0.01145	0.229

Interpret the sign. To first order, $\delta R/R = \delta V/V - \delta I/I$. Errors that raise numerator and denominator together partly cancel. Errors that raise one while lowering the other reinforce. Correlation is a property of the measurement process; it must not be chosen to make the reported uncertainty smaller.

For a useful limiting check, suppose both readings share exactly the same multiplicative calibration factor. That shared factor cancels from their ratio. In the uncertainty formula this corresponds to equal relative contributions with $c = 1$, giving zero contribution from that particular shared source. Independent noise and other calibration terms would still remain.

Solution S19 to Problem 28: why symmetric sampling hides an offset in the slope

For a line constrained through the origin, minimize

$$S_0(R) = \sum_i (V_i - RI_i)^2.$$

Differentiation gives $R_{\text{zero}} = \sum I_i V_i / \sum I_i^2$. For an unconstrained fit, the slope is

$$R_{\text{free}} = \frac{\sum (I_i - \bar{I})(V_i - \bar{V})}{\sum (I_i - \bar{I})^2}.$$

The current values in Table 6.1 are symmetric about zero, hence $\bar{I} = 0$. Both expressions then reduce to the same ratio, so both fitted slopes are 47.192Ω . The unconstrained fit gives $b = 0.300 \text{ mV}$, whereas the forced fit has zero intercept by construction.

Compare the residuals. Let $e_i = V_i - (R_{\text{free}} I_i + b)$. The forced-fit residuals are $e_i + b$. In an ordinary equal-weight intercept fit, $\sum e_i = 0$, so

$$\sum (e_i + b)^2 = \sum e_i^2 + Nb^2.$$

Here $N = 9$, $\sum e_i^2 = 1.104 \text{ mV}^2$ and $Nb^2 = 0.810 \text{ mV}^2$. For $u_V = 0.8 \text{ mV}$, the reduced chi-squared values are $1.104/[7(0.8)^2] = 0.246$ for the two-parameter fit and $1.914/[8(0.8)^2] = 0.374$ for the one-parameter forced fit.

The slope agreement is an algebraic consequence of symmetric sampling; it is not evidence that the intercept is exactly zero. Equally, the fitted intercept alone does not establish a statistically resolved offset: its standard uncertainty is about 0.267 mV under the stated model, so the measured intercept is only about 1.1 such uncertainties from zero. A different current range, improved precision or repeated reversal measurements could make the offset test more informative.

Solution S20 to Problem 36: choose the surface that carries the conserved current

Assume full concentric spherical electrodes, homogeneous isotropic material, and $\Delta V = V_a - V_b > 0$. Symmetry makes current radial. The same outward current crosses every sphere, so

$$J_r(r) = \frac{I}{4\pi r^2}, \quad E_r(r) = \frac{\rho I}{4\pi r^2}.$$

The path element is dr and the current-carrying area is $4\pi r^2$. Therefore

$$R = \int_a^b \frac{\rho dr}{4\pi r^2} = \frac{\rho}{4\pi} \left(\frac{1}{a} - \frac{1}{b} \right), \quad I = \frac{4\pi \Delta V}{\rho(1/a - 1/b)}.$$

The product of ρ in $\Omega \text{ m}$ and the reciprocal-length bracket has units Ω . Substitution into the local expressions gives

$$E_r(r) = \frac{\Delta V}{(1/a - 1/b)r^2}, \quad J_r(r) = \frac{\Delta V}{\rho(1/a - 1/b)r^2}.$$

One compatible potential is

$$V(r) = V_b + \Delta V \frac{1/r - 1/b}{1/a - 1/b}.$$

It has the two required endpoint values, and $-dV/dr = E_r > 0$ is an independent sign check.

For a thin layer, $b = a + d$ and $d \ll a$ give

$$\frac{1}{a} - \frac{1}{a+d} = \frac{d}{a(a+d)} \simeq \frac{d}{a^2}, \quad R \simeq \frac{\rho d}{4\pi a^2}.$$

This is the slab result with thickness d and area $4\pi a^2$. At the other limit, $b \rightarrow \infty$, $R \rightarrow \rho/(4\pi a)$ remains finite because the spherical current-carrying area grows as r^2 . This mathematical limit assumes an unlimited homogeneous conducting medium with an asymptotic return boundary; it does not mean a physical return path is unnecessary.

Solution S21 to Problem 39: negative temperature coefficient under voltage drive

The reciprocal-temperature law gives

$$\frac{dR}{dT} = -\frac{B}{T^2}R.$$

For fixed voltage, $P_{\text{el}} = V^2/R$, and hence

$$\frac{dP_{\text{el}}}{dT} = -\frac{V^2}{R^2} \frac{dR}{dT} = \frac{BV^2}{T^2 R} = \frac{B}{T^2} P_{\text{el}} > 0.$$

The units are W K^{-1} , since B is in kelvin. Heating reduces the resistance, which raises power at fixed voltage: this is positive feedback.

If an equilibrium T_* exists, it satisfies $P_{\text{el}}(T_*) = H(T_* - T_0)$. A small perturbation obeys

$$C_{\text{th}} \frac{d\delta T}{dt} = \left[\frac{BP_{\text{el}}(T_*)}{T_*^2} - H \right] \delta T.$$

It decays only if

$$\frac{BP_{\text{el}}(T_*)}{T_*^2} < H, \quad \text{or equivalently} \quad \frac{B(T_* - T_0)}{T_*^2} < 1.$$

Equality is marginal in this linearized test and requires higher-order analysis. The condition classifies a specified equilibrium; it does not locate one or prove global stability for arbitrary disturbances. In particular, an extrapolated mathematical high-temperature root may lie far outside the calibration range or outside the solid state of the sensor.

For comparison, current drive gives $P_{\text{el}} = I^2 R$ and $P'_{\text{el}} = -BP_{\text{el}}/T^2 < 0$. With the same linear cooling law, a valid current-driven equilibrium is locally stable. The sign of the material coefficient alone therefore cannot determine feedback without also specifying what the source holds fixed.

Solution S22 to Problem 45: current need not follow the field direction

With $\mathbf{E} = E_0(\hat{\mathbf{x}} + \hat{\mathbf{y}})$, the stated component laws give

$$\mathbf{J} = \sigma E_0(\hat{\mathbf{x}} + 3\hat{\mathbf{y}}).$$

For $E_0 > 0$, the field makes 45° with $+x$, but the current makes $\tan^{-1} 3 = 71.565^\circ$. The angle between them is 26.565° . The dot-product check gives

$$\cos \phi = \frac{\mathbf{J} \cdot \mathbf{E}}{|\mathbf{J}| |\mathbf{E}|} = \frac{4}{\sqrt{10}\sqrt{2}} = \frac{2}{\sqrt{5}}.$$

The power density is

$$p = \mathbf{J} \cdot \mathbf{E} = \sigma E_x^2 + 3\sigma E_y^2 = 4\sigma E_0^2 > 0.$$

Its units are $(\text{A V}^{-1}\text{m}^{-1})(\text{V}^2\text{m}^{-2}) = \text{W m}^{-3}$. Positive dissipation does not require parallel field and current; it requires a positive projection in this passive model.

There is no single scalar conductivity that reproduces both component responses for arbitrary field directions. The magnitude ratio here, $|\mathbf{J}|/|\mathbf{E}| = \sqrt{5}\sigma$, characterizes only this chosen direction. Treating it as a universal conductivity would give the wrong response to a field along x or along y . The tensor description preserves the directional information that a scalar ratio loses.

Bridge to Chapter 7

Chapter 7 combines these ideas into DC circuit analysis. Junction equations follow from charge conservation, loop equations express potential and energy accounting within their stated circuit approximation, and device equations supply the material response. The distinction between emf and terminal voltage, source resistance, network reduction, meter placement and RC transients can then be developed without confusing current with energy or a constitutive law with a conservation law.

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