

Chapter 7

Direct-Current Circuits

First Edition

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Preface

This chapter develops direct-current circuits as the practical language that connects electric charge, current, resistance, potential difference, capacitance and energy. A circuit diagram is not merely a drawing of wires. It is a compact physical model: it tells us which elements are treated as ideal, where charge can accumulate, where energy is supplied or dissipated, and which measured quantities are expected to be the same in different parts of a network.

The chapter is placed after electric potential, capacitance, current and resistance because circuit analysis depends on all of them. Charge conservation becomes Kirchhoff's junction rule. Energy conservation becomes Kirchhoff's loop rule. Ohm's law becomes the element relation for ideal resistors. The capacitor relation $q = CV$ becomes the starting point for the time-dependent charging and discharging processes that introduce exponential transients. These transients will reappear in later physics and engineering contexts, including thermal relaxation, radioactive decay, mechanical damping, detector response, signal filtering and alternating-current circuits.

The emphasis is not only on calculation. Students must learn what the symbols in a circuit equation mean physically. The emf of a source is not literally a mechanical force. The terminal voltage of a real battery is not always equal to its emf. A capacitor in a DC circuit does not allow steady conduction through its dielectric, yet it can carry transient current while its plates charge. An ammeter is not connected the way a voltmeter is connected. Currents in Kirchhoff equations are assumptions until the signs of the solved currents are interpreted.

The chapter is written for Technical Physics, engineering, computer science, applied-science and space-science students. It therefore treats DC circuits both as physics and as engineering modelling. The same laws describe a textbook loop, a loaded battery, a voltage divider feeding a sensor input, a discharge curve measured on an oscilloscope, and the first step of a numerical circuit simulation.

The exposition and diagrams are developed for this book, with comparative use of the supplied university physics texts listed in the references. The core follows the course syllabus; the clearly separated enrichment develops network methods, sensor readout and first-order dynamics further.

The Guideline

Treat every circuit problem as a conservation-law problem before treating it as algebra. Identify nodes and loops, mark assumed current directions, assign potential changes consistently, write enough independent equations, solve them, and then interpret the signs of the answers. A negative current is not a failed solution; it is information that the actual current is opposite to the direction first assumed.

Pedagogical boundary

This chapter treats direct-current circuits in the lumped-element approximation. We neglect electromagnetic wave propagation along wires, spatially distributed capacitance and inductance unless they are explicitly introduced in an enrichment discussion. Inductors, alternating current, phasors, resonance, transformers and full Maxwell-equation propagation are postponed to later chapters.

Conventions and notation

The symbol I denotes electric current. The sign of I is defined by a chosen reference direction in a branch. A positive answer means that the actual conventional current is in the chosen direction; a negative answer means that the actual conventional current is opposite to it. Conventional current is the direction in which positive charge would move. In a metal, the mobile microscopic carriers are usually electrons, but circuit equations normally use conventional current.

Potential differences are written as ΔV or $V_a - V_b$. The symbol V may denote a potential difference across a circuit element when the reference points are clear. The emf of a source is denoted by $\mathcal{E}$ and is measured in volts. Resistance is denoted by R , internal resistance by r , capacitance by C , charge on a capacitor plate by q , and time constant by τ .

The following sign convention is used throughout unless stated otherwise.

- Across an ideal resistor traversed in the direction of the chosen current, the potential change is $-IR$.
- Across an ideal resistor traversed opposite to the chosen current, the potential change is $+IR$.
- Across an ideal source from its negative terminal to its positive terminal, the potential change is $+\mathcal{E}$.
- Across an ideal source from its positive terminal to its negative terminal, the potential change is $-\mathcal{E}$.
- Across a capacitor from the negative plate to the positive plate, the potential change is $+q/C$; from the positive plate to the negative plate it is $-q/C$.

Symbol	Meaning	SI unit or dimension
q	charge on a conductor or capacitor plate	C
I	current in a branch, with sign set by chosen direction	A = C s ⁻¹
J	current density, used only in reminders from resistance theory	A m ⁻²
V	potential difference, voltage	V = J C ⁻¹
$\mathcal{E}$	emf of a source, work per unit charge supplied internally	V
r	internal resistance of a real source	Ω
R	resistance of an ideal resistor	Ω
R_{eq}	equivalent resistance	Ω
P	power transferred electrically or dissipated thermally	W
C	capacitance	F = C V ⁻¹
τ	RC time constant, $\tau = RC$	s
V_C	capacitor potential difference	V
V_R	resistor potential difference	V
R_A	ammeter resistance	Ω
R_V	voltmeter resistance	Ω
$V_{\text{Th}}, R_{\text{Th}}$	Thevenin equivalent voltage and resistance	V, Ω

A volt is a joule per coulomb. A loop equation tracks energy per charge; power checks use the actual signed current of each element. Keep reference arrows and traversal directions unchanged while solving, then interpret negative results.

Chapter 7

Direct-Current Circuits

7.1 From fields to circuit models

A steady direct current has constant magnitude and direction. This chapter studies networks driven by constant sources and the transient responses produced when those sources or connections are switched. During a transient, branch currents may change in magnitude and can reverse if the initial capacitor voltage or switching arrangement requires it. Reference arrows are bookkeeping choices; they do not define whether a current is DC or AC.

The power of circuit theory is idealisation. A real wire has small resistance, small inductance, small capacitance to nearby objects and finite signal propagation speed. In a DC lumped circuit we usually replace the wire by an ideal conductor with negligible potential difference along it. A real battery has chemistry, internal resistance, capacity, temperature dependence and voltage sag. We often begin with an ideal emf source and then add internal resistance when needed. A real resistor may heat, drift and have parasitic capacitance. We begin with Ohm's law and then discuss limitations.

Learning goals

After studying this chapter, the student should be able to:

- explain emf as work per unit charge supplied by a source and distinguish it from an electrostatic potential difference;
- compute terminal voltage for a real source with internal resistance under load;
- combine resistors in series and parallel and state the assumptions behind equivalent resistance;
- apply Kirchhoff's junction rule as charge conservation at a node;
- apply Kirchhoff's loop rule as energy conservation around a closed circuit path;
- set up and solve systematic equations for one-loop and multi-loop DC circuits;
- interpret negative solved currents correctly;
- compute electric power supplied or dissipated by circuit elements;
- explain how ammeters and voltmeters must be connected and why their ideal resistances differ;
- derive and use the charging and discharging equations for a series RC circuit;
- interpret the time constant $\tau = RC$ physically and graphically;
- connect DC-circuit theory to instrumentation, sensors, laboratory measurements and numerical simulation.

Key message

Kirchhoff's rules are not arbitrary circuit tricks. The junction rule is charge conservation in a lumped circuit. The loop rule is energy conservation per unit charge around a closed path.

7.1.1 Required prior knowledge

This chapter assumes that students already know the definitions of current, resistance, potential difference and capacitance. The central previous relations are

$$I = \frac{dq}{dt}, \quad V = \frac{\Delta U}{q}, \quad R = \frac{V}{I} \quad \text{for an ohmic resistor,} \quad q = CV. \quad (7.1)$$

The chapter also relies on Physics 1 conservation laws. At a circuit junction, charge cannot disappear. Around a closed loop, the net energy gained per unit charge must equal the net energy lost per unit charge for a steady idealised circuit. For RC transients, the mathematics is that of a first-order linear differential equation, the same type of exponential relaxation encountered in many physical systems.

7.2 Circuit variables and the lumped-element model

A circuit diagram represents a physical model rather than a photograph. A line in a circuit diagram usually means an ideal conductor. Every point on an ideal conductor connected without intervening circuit elements is treated as being at the same electric potential. A resistor symbol represents an element for which the potential difference and current are related by Ohm's law in the operating range considered. A capacitor symbol represents two conductors separated by an insulator, with charge stored on plates and electric-field energy stored between them. A source symbol represents a device that supplies energy per unit charge.

Definition 7.1: Lumped-element approximation

In the lumped-element approximation, circuit elements are treated as localised objects connected by ideal wires. Voltages and currents are assumed to be well-defined at each instant across or through each element. Signal-propagation delay, radiation and spatially distributed electromagnetic fields are neglected.

The approximation is extremely useful, but it has limits. If a circuit is physically large compared with the electromagnetic wavelength associated with the signal variation, wires behave as transmission lines rather than ideal equipotential connectors. If current changes rapidly, inductive effects may become important. If elements are very close, parasitic capacitance can matter. For DC laboratory circuits with slowly varying transients and modest dimensions, the lumped approximation is usually the correct first model.

Note

Circuit theory does not abolish electric fields. It reorganises them. A potential difference across a resistor corresponds to an electric field inside it that drives charge carriers. An emf source uses non-electrostatic processes, such as chemical forces in a battery, to maintain charge separation.

7.2.1 Nodes, branches and loops

A *node* is a conducting connection point at which two or more element terminals meet. An *essential node* is a node connected to three or more branches. A *branch* is a path containing one or more elements connected between two nodes and carrying a single branch current. A *loop* is any closed path through the circuit. A *mesh* is a loop that contains no smaller loop inside it in a planar circuit.

These definitions matter because a circuit with many components may have more possible loops than independent equations. Kirchhoff's rules are always true in the ideal model, but writing redundant

equations wastes time and can obscure the physics. Later in the chapter we shall use systematic loop and node methods to avoid that problem.

Guided checks

Check 1. Does an ideal wire have zero resistance or infinite resistance?

Answer. In the usual model it has zero resistance, so there is no potential drop along it.

Check 2. Does a circuit diagram show the real physical size of components?

Answer. Usually no. It shows topology and idealised element relations.

Check 3. What conservation law is suggested by studying currents at a node?

Answer. Conservation of charge.

7.2.2 Reference potential, ideal connections and physical limits

A node is an entire ideal conducting connection, not merely a dot on a drawing. Two separated dots connected by an uninterrupted ideal wire represent one node. Conversely, lines crossing without a junction dot need not be connected. The positions of symbols may change when a diagram is redrawn; its connectivity must not change.

Choose any one node as the reference $V = 0$. Changing every potential by the same constant V_0 changes no branch voltage: $(V_a + V_0) - (V_b + V_0) = V_a - V_b$. It therefore changes no current. In circuit equations, a ground symbol often specifies this reference; it does not by itself imply a connection to Earth or a protective conductor. Joining two previously separate nodes with a wire is a physical alteration, whereas choosing a different zero of potential is not.

An open branch has zero current but may have a finite voltage. An ideal short has zero voltage but may carry a finite current determined by the rest of the circuit. Ohm's law does not justify the undefined expression $0/0$ for the current through a zero-resistance wire. That current is found using the connected network. An ideal nonzero voltage source directly shorted by an ideal wire imposes inconsistent constraints; internal and wire resistance are needed to obtain a finite prediction.

For a circuit of characteristic size ℓ and signal speed v_{sig} , a useful first check is

$$\frac{\ell}{v_{\text{sig}}} \ll T_{\text{variation}}. \quad (7.2)$$

The relevant time is the fastest change that matters, such as a switch rise time, not merely the repetition period. Stray inductance and capacitance must also be negligible for the intended accuracy. These are model checks, not extra terms to insert indiscriminately into every DC calculation.

Example 7.1: One circuit, two voltage references

A 12 V ideal source drives $2.0 \text{ k}\Omega$ and $4.0 \text{ k}\Omega$ in series. Take the negative source terminal as 0 V. The current is $12/(6.0 \text{ k}\Omega) = 2.0 \text{ mA}$, so the midpoint is 8.0 V and the positive source terminal is 12 V. If the midpoint is chosen as the new zero, these three node potentials become -8.0 , 0, and $+4.0 \text{ V}$. The resistor voltages remain 4.0 and 8.0 V and the current is unchanged. A negative node potential is therefore not evidence of a reversed source or an error.

7.3 Emf, sources and terminal voltage

A steady current through a resistor requires a maintained potential difference. If a charged capacitor is connected across a resistor, the capacitor initially drives a current, but the current decays as the capacitor discharges. A sustained DC current requires a device that continually supplies energy to charge carriers. Such a device is called an emf source.

Definition 7.2: Electromotive force

The emf $\mathcal{E}$ of a source is the work per unit charge supplied internally by the source as it moves charge from lower potential to higher potential inside the source:

$$\mathcal{E} = \frac{W_{\text{source}}}{q}. \quad (7.3)$$

Despite the historical name, emf is not a force. Its SI unit is the volt.

In an ideal source, the terminal potential difference equals the emf regardless of the current. A real battery, power supply or cell has internal resistance and other limitations. The simplest real-battery model is an ideal emf $\mathcal{E}$ in series with an internal resistance r . If the battery is delivering current I to an external circuit, the terminal voltage is

$$V_{\text{terminal}} = \mathcal{E} - Ir. \quad (7.4)$$

If a rechargeable battery is driven by an external source so that a current of positive magnitude I enters its positive terminal, the simple model gives

$$V_{\text{terminal}} = \mathcal{E} + Ir \quad (7.5)$$

for the same simple internal-resistance model. Alternatively, retain one signed current I_{out} defined as leaving the positive terminal: then $V_{\text{terminal}} = \mathcal{E} - rI_{\text{out}}$ covers both delivery ($I_{\text{out}} > 0$) and absorption ($I_{\text{out}} < 0$). These equations describe an electrical model; actual charging also depends on the cell chemistry and its specified operating conditions.

Figure 7.1 fixes the source boundary and terminal polarity.

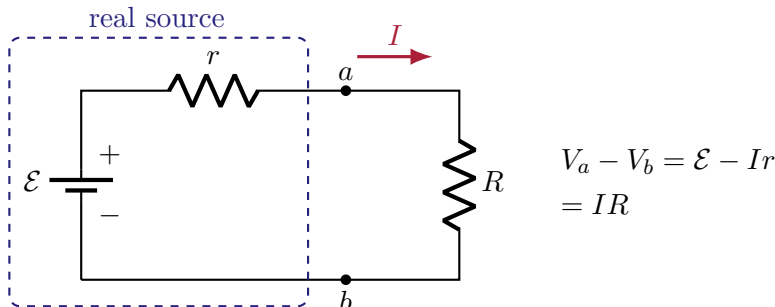


Figure 7.1: A real source model with explicitly identified external terminals a and b . Conventional current leaves a during delivery. The dashed boundary includes the ideal emf and internal resistance, not the load.

Caution

An ideal battery with fixed voltage and zero internal resistance is a model, not a physical object with unlimited power. Real sources have maximum current, internal resistance, heating, finite stored energy and safety limits.

Example 7.2: Terminal voltage of a loaded battery

A battery has emf $\mathcal{E} = 12.0 \text{ V}$ and internal resistance $r = 0.20 \Omega$. It is connected to a load that draws $I = 3.0 \text{ A}$. Find the terminal voltage and the power dissipated inside the battery.

Solution. Since the battery is delivering current,

$$V_{\text{terminal}} = \mathcal{E} - Ir = 12.0 - (3.0)(0.20) = 11.4 \text{ V}. \quad (7.6)$$

The internal thermal power is

$$P_r = I^2 r = (3.0)^2(0.20) = 1.8 \text{ W}. \quad (7.7)$$

The chemical source supplies power $I\mathcal{E} = 36 \text{ W}$, while the external load receives $IV_{\text{terminal}} = 34.2 \text{ W}$. The difference is the internal heating.

Reasonableness check. The internal voltage drop $Ir = 0.60 \text{ V}$ is small compared with 12 V , so the battery is not severely loaded in this model.

7.3.1 Measuring the source model rather than assuming it

The linear model predicts a straight line when terminal voltage is plotted against delivered current:

$$V = \mathcal{E} - rI. \quad (7.8)$$

Its intercept estimates emf and its negative slope estimates internal resistance over the tested operating range. The open-circuit condition means negligible external current, not that the battery contains no mobile charge. Use a range of finite loads for the fit; a short circuit is neither necessary nor a reliable way to characterize a real source.

Example 7.3: Inferring emf and internal resistance from load tests

Two illustrative measurements give $(I, V) = (0.20 \text{ A}, 5.80 \text{ V})$ and $(0.80 \text{ A}, 5.20 \text{ V})$. Assume a constant-emf, constant- r model over this range.

Strategy. Subtract the two equations to remove the unknown emf:

$$r = -\frac{5.20 - 5.80}{0.80 - 0.20} = 1.00 \Omega, \quad \mathcal{E} = 5.80 + (0.20)(1.00) = 6.00 \text{ V}. \quad (7.9)$$

For a 9.0Ω load, the model predicts $I = 6.00/(9.0 + 1.00) = 0.600 \text{ A}$ and $V = 5.40 \text{ V}$. This prediction lies between the tested currents, where interpolation is more defensible than extrapolation to very large current.

Uncertainty and model check. If each voltage has independent standard uncertainty 0.02 V and current uncertainty is negligible, the slope uncertainty from these two points is

$$\sigma_r = \frac{\sqrt{(0.02)^2 + (0.02)^2}}{0.60} = 0.047 \Omega. \quad (7.10)$$

A fit to several loads can reveal curvature or drift. Heating and state of charge can make r and $\mathcal{E}$ change during the measurements; such changes are model limitations, not failures of Kirchhoff's rules.

7.4 Single-loop circuits

A single-loop circuit contains only one independent current. Suppose an emf source with internal resistance r is connected to an external resistor R . The same current flows through both resistances because there is no junction at which current can split. Applying the loop rule gives

$$+\mathcal{E} - Ir - IR = 0. \quad (7.11)$$

Thus

$$I = \frac{\mathcal{E}}{R + r}. \quad (7.12)$$

For an ideal source and a finite positive load resistance, $r = 0$, so $I = \mathcal{E}/R$.

The simple expression (7.12) contains several important physical messages. The internal and external resistances are in series, so both limit the current. The terminal voltage is the part of the emf left after the internal drop:

$$V_{\text{terminal}} = IR = \mathcal{E} - Ir. \quad (7.13)$$

The same result was obtained by the real-battery model.

Method

Solving a single-loop DC circuit.

1. Choose a current direction. Any direction is acceptable.
2. Choose a loop traversal direction. It may be the same as the assumed current direction or not.
3. Record each potential rise or drop using the sign convention.
4. Set the algebraic sum of potential changes to zero.
5. Solve for the current and interpret its sign.
6. Check power: total power supplied should equal total power dissipated in steady state.

Example 7.4: A source, internal resistance and load

A 9.0 V battery with internal resistance 0.50Ω is connected to a 5.5Ω resistor. Find the current, the terminal voltage and the power dissipated in the external resistor.

Solution. The total series resistance is

$$R_{\text{tot}} = R + r = 5.5 + 0.50 = 6.0 \Omega. \quad (7.14)$$

The current is

$$I = \frac{\mathcal{E}}{R + r} = \frac{9.0}{6.0} = 1.5 \text{ A}. \quad (7.15)$$

The terminal voltage is

$$V_{\text{terminal}} = IR = (1.5)(5.5) = 8.25 \text{ V}, \quad (7.16)$$

which is also $\mathcal{E} - Ir = 9.0 - (1.5)(0.50) = 8.25 \text{ V}$. The external resistor receives

$$P_R = I^2 R = (1.5)^2 (5.5) = 12.4 \text{ W}. \quad (7.17)$$

Interpretation. The source is not ideal: 0.75 V of its emf is lost across the internal resistance.

Guided checks

Check 1. In a single-loop circuit, why is the current the same through every element?

Answer. There is no junction where charge can split or accumulate in steady state; charge conservation requires one branch current.

Check 2. If the solved current is negative, should the magnitude be discarded?

Answer. No. The negative sign means the actual current is opposite to the assumed direction.

Check 3. Why is terminal voltage smaller than emf when a battery delivers current through internal resistance?

Answer. Part of the energy per unit charge is dissipated inside the source as Ir .

7.5 Resistors in series and in parallel

Equivalent resistance is a modelling tool. A group of resistors can be replaced by a single resistor if the replacement has the same current-voltage relation as the original group at its terminals. The replacement is external-terminal equivalent; it does not mean that the internal currents and powers in the group are the same as in the single resistor.

7.5.1 Series combination

Resistors are in series when the same current must pass through each resistor. For n resistors in series, the total potential drop is

$$V = IR_1 + IR_2 + \cdots + IR_n = I \sum_{j=1}^n R_j. \quad (7.18)$$

Therefore the equivalent resistance is

$$R_{\text{eq}} = \sum_{j=1}^n R_j. \quad (7.19)$$

Series resistances add because the same charge carriers lose energy per unit charge across each element sequentially.

7.5.2 Parallel combination

Resistors are in parallel when the same potential difference appears across each resistor. For n resistors in parallel, the total current is the sum of branch currents:

$$I = \frac{V}{R_1} + \frac{V}{R_2} + \cdots + \frac{V}{R_n} = V \sum_{j=1}^n \frac{1}{R_j}. \quad (7.20)$$

Since $I = V/R_{\text{eq}}$, the equivalent resistance satisfies

$$\frac{1}{R_{\text{eq}}} = \sum_{j=1}^n \frac{1}{R_j}. \quad (7.21)$$

For two resistors,

$$R_{\text{eq}} = \frac{R_1 R_2}{R_1 + R_2}. \quad (7.22)$$

A parallel combination has an equivalent resistance smaller than the smallest finite positive branch resistance, because it provides additional paths for charge flow. Figure 7.2 compares the node connections of series and parallel groups.

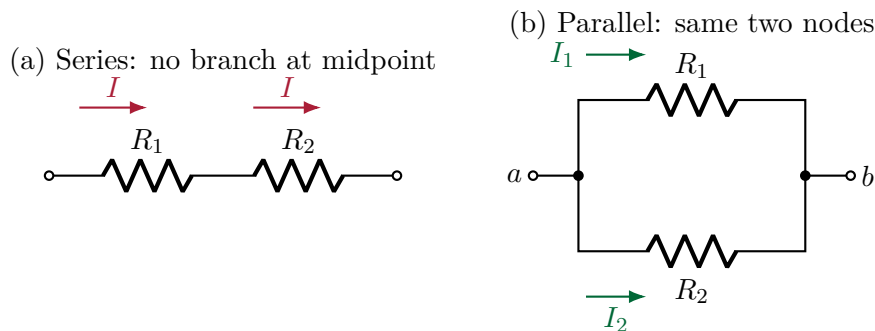


Figure 7.2: Connectivity determines series and parallel combinations. In (a), the current is common. In (b), both resistors connect to the same nodes a and b , so their voltage $V_a - V_b$ is common.

Key message

In series, current is common and voltage is divided. In parallel, voltage is common and current is divided.

Example 7.5: Equivalent resistance and power distribution

A 12 V ideal source is connected to $R_1 = 6.0 \Omega$ in series with a parallel combination of $R_2 = 12 \Omega$ and $R_3 = 4.0 \Omega$. Find the source current and the power in each resistor.

Solution. The parallel part has

$$R_{23} = \frac{R_2 R_3}{R_2 + R_3} = \frac{(12)(4)}{12 + 4} = 3.0 \Omega. \quad (7.23)$$

The total equivalent resistance is

$$R_{\text{eq}} = R_1 + R_{23} = 6.0 + 3.0 = 9.0 \Omega. \quad (7.24)$$

The source current is

$$I = \frac{12}{9.0} = 1.33 \text{ A}. \quad (7.25)$$

The voltage drop across R_1 is $IR_1 = (1.33)(6.0) = 8.0 \text{ V}$, so the parallel branch has 4.0 V across it. Hence

$$I_2 = \frac{4.0}{12} = 0.333 \text{ A}, \quad I_3 = \frac{4.0}{4.0} = 1.00 \text{ A}. \quad (7.26)$$

The powers are

$$P_1 = I^2 R_1 = (1.33)^2 (6.0) = 10.7 \text{ W}, \quad (7.27)$$

$$P_2 = \frac{(4.0)^2}{12} = 1.33 \text{ W}, \quad (7.28)$$

$$P_3 = \frac{(4.0)^2}{4.0} = 4.0 \text{ W}. \quad (7.29)$$

The total is about 16.0 W, matching $P_{\text{source}} = VI = (12)(1.33) = 16.0 \text{ W}$ within rounding.

Caution

Do not identify series or parallel merely by visual appearance. Two elements are in series only if the same current must pass through both. They are in parallel only if both terminals of one element connect to the same two nodes as the corresponding terminals of the other.

7.5.3 Voltage division, current division and a changed load

For series resistors across a fixed terminal voltage V , the same current $I = V / \sum_j R_j$ gives

$$V_j = V \frac{R_j}{\sum_k R_k}. \quad (7.30)$$

For parallel resistors carrying a total current I , the common voltage and the conductances $G_j = 1/R_j$ give

$$I_j = I \frac{G_j}{\sum_k G_k}. \quad (7.31)$$

For two parallel branches, $I_1 = IR_2/(R_1 + R_2)$: the resistance in the numerator belongs to the *other* branch. The larger resistance carries less current because the voltage is common. These are

consequences of Ohm's law and conservation, not additional independent laws.

Adding a parallel branch to an *ideal voltage source* increases the total source current without changing the voltage or current in the original branch. With a real source, the increased total current increases the internal drop, so the original branch voltage can fall. Students should ask which quantity the source holds fixed before predicting how a network changes.

Example 7.6: A second load changes the first load's current

A source with $\mathcal{E} = 12\text{ V}$ and $r = 1.0\ \Omega$ initially drives one $11\ \Omega$ resistor. The current is 1.0 A and the terminal voltage is 11 V . An identical second resistor is now connected in parallel. The external resistance becomes $5.5\ \Omega$, so the source current becomes $12/6.5 = 1.846\text{ A}$. The terminal voltage is $12 - 1.846 = 10.154\text{ V}$, and each load carries $10.154/11 = 0.923\text{ A}$. The original branch current has decreased by 7.69% , although total source current has increased. In the limiting case $r \rightarrow 0$, each load would retain $12/11\text{ A}$ and only the total current would change.

7.5.4 Equivalent resistance as a terminal property

Equivalent resistance is often taught as a set of formula patterns. The deeper idea is terminal behaviour. A two-terminal network has an equivalent resistance R_{eq} if applying a voltage V across its terminals produces a total current I satisfying $V = IR_{\text{eq}}$. The equivalent resistor reproduces the same relation between the external terminal voltage and external terminal current. It does not reproduce all internal details.

For example, two parallel resistors dissipate power in two physical locations. A single equivalent resistor dissipates the same total power for the same terminal voltage, but it does not tell us how hot each original resistor becomes. Similarly, replacing a network by R_{eq} can be useful for finding total current but insufficient for component selection, thermal design or failure analysis.

This distinction matters in engineering. Suppose a circuit draws 1 A from a 12 V supply, so its input equivalent resistance is $12\ \Omega$. That single number does not tell whether one component dissipates 11 W and another 1 W , or whether twelve components each dissipate 1 W . The input equivalent resistance is useful, but it is not the full design.

Example 7.7: Same equivalent resistance, different internal powers

Network A is a single $12\ \Omega$ resistor across 12 V . Network B consists of two $24\ \Omega$ resistors in parallel across 12 V . Compare the input current and internal power distribution.

Solution. Network A has current

$$I_A = \frac{12}{12} = 1.0\text{ A} \quad (7.32)$$

and dissipates

$$P_A = VI = 12\text{ W} \quad (7.33)$$

in one resistor. Network B has equivalent resistance

$$R_B = 24 \parallel 24 = 12\ \Omega, \quad (7.34)$$

so it draws the same total current 1.0 A and dissipates the same total power 12 W . However, each branch current is $12/24 = 0.50\text{ A}$, and each resistor dissipates

$$P = (0.50)^2(24) = 6.0\text{ W}. \quad (7.35)$$

Interpretation. The two networks are equivalent at their terminals, but their thermal design requirements differ.

7.6 Power and energy in DC circuits

Electric power is the rate at which energy is transferred. If charge dq moves through a potential difference V , the energy change is

$$dU = V dq. \quad (7.36)$$

Dividing by dt gives

$$P = IV. \quad (7.37)$$

For an ohmic resistor, $V = IR$, so the dissipated thermal power can be written in equivalent forms:

$$P_R = I^2 R = \frac{V^2}{R} = IV. \quad (7.38)$$

Define $v = V_a - V_b$ and let the signed current i enter terminal a , the terminal marked $+$ for the voltage reference. The *passive sign convention* is $p_{\text{abs}} = vi$. Positive p_{abs} means absorption; negative p_{abs} means delivery. The reference $+$ need not be the higher actual potential if the solved voltage is negative. For a resistor, $v = Ri$ and $p_{\text{abs}} = Ri^2 \geq 0$. A source can either supply or absorb energy, depending on its current. This signed convention makes a network-wide check possible: the sum of all absorbed powers is zero in steady state.

Theoretical point 7.1: Power balance from the loop rule

For a single loop containing an ideal source and resistors,

$$\mathcal{E} - IR_1 - IR_2 - \dots = 0. \quad (7.39)$$

Multiplying by I gives

$$I\mathcal{E} = I^2 R_1 + I^2 R_2 + \dots. \quad (7.40)$$

Thus the rate at which the source supplies energy equals the rate at which the resistors dissipate energy.

Example 7.8: Energy cost of a DC load

A laboratory heater is modelled as a $24\ \Omega$ resistor connected to a $12\ \text{V}$ DC supply for 30 min. Find the current, power and energy delivered.

Solution. The current is

$$I = \frac{V}{R} = \frac{12}{24} = 0.50\ \text{A}. \quad (7.41)$$

The power is

$$P = IV = (0.50)(12) = 6.0\ \text{W}. \quad (7.42)$$

The operating time is $30\ \text{min} = 1800\ \text{s}$, so

$$E = Pt = (6.0)(1800) = 1.08 \times 10^4\ \text{J}. \quad (7.43)$$

Interpretation. The calculation assumes constant resistance. If the heater warms substantially, its resistance may change and the constant-power estimate becomes an approximation.

7.6.1 Resistor ratings and thermal margins

A resistor value alone is not enough for design. Every physical resistor also has a power rating, tolerance, temperature coefficient, voltage rating and package-dependent thermal resistance. In introductory circuit theory we often compute power as $P = I^2 R$ or $P = V^2/R$. In a real design, the computed power should be safely below the component rating.

A common engineering practice is derating: using a component at a fraction of its nominal maximum rating to improve reliability and reduce temperature rise. For example, a 0.25 W resistor dissipating 0.24 W is mathematically within rating but may run very hot in a compact enclosure. A conservative design might choose a larger package or a higher power rating.

Example 7.9: Choosing a resistor power rating

A 1.0 k Ω resistor is to be connected continuously across a 15 V DC supply. Find the dissipated power and recommend whether a 0.25 W resistor is adequate if the design rule is to use no more than half the rated power.

Solution. The power is

$$P = \frac{V^2}{R} = \frac{15^2}{1000} = 0.225 \text{ W}. \quad (7.44)$$

Although 0.225 W is below 0.25 W, it exceeds half of 0.25 W, which is 0.125 W. Under the stated design rule, the 0.25 W resistor is not adequate. A 0.5 W or larger resistor would be more appropriate.

7.6.2 Battery capacity and load current

Battery capacity is often quoted in ampere-hours. This unit is a charge unit because

$$1 \text{ A h} = 3600 \text{ C}. \quad (7.45)$$

A capacity rating does not by itself specify energy unless the voltage is known and approximately constant. A rough energy estimate is

$$E \approx VQ = V(I\Delta t). \quad (7.46)$$

Real batteries have discharge curves, rate dependence, temperature dependence and cutoff voltages. The simple estimate is useful for first design checks but not for final battery engineering.

Example 7.10: Ideal runtime estimate

A battery pack is labelled 12 V, 2.0 A h. Estimate the runtime for a device drawing 0.40 A, assuming the ideal capacity model.

Solution. The runtime is

$$\Delta t = \frac{2.0 \text{ A h}}{0.40 \text{ A}} = 5.0 \text{ h}. \quad (7.47)$$

The approximate stored energy is

$$E \approx (12 \text{ V})(2.0 \text{ A h}) = 24 \text{ W h}. \quad (7.48)$$

In joules, this is $24 \times 3600 = 8.64 \times 10^4 \text{ J}$.

7.7 Kirchhoff's junction rule

At a circuit node, charge cannot build up indefinitely in steady-state DC operation. Therefore the total current entering the node must equal the total current leaving the node:

$$\sum I_{\text{in}} = \sum I_{\text{out}}. \quad (7.49)$$

Equivalently, if currents directed into the node are positive and currents directed out are negative, then

$$\sum_k I_k = 0. \quad (7.50)$$

This is Kirchhoff's junction rule.

The rule can be derived directly from charge conservation. Let a small region enclose the junction. If Q_{node} is the charge inside that region, then

$$\frac{dQ_{\text{node}}}{dt} = \sum I_{\text{in}} - \sum I_{\text{out}}. \quad (7.51)$$

In a steady lumped circuit, $dQ_{\text{node}}/dt = 0$, giving (7.49). During some transients, small surface-charge rearrangements occur, but the lumped-circuit version still applies to currents in the ideal node when capacitive storage at the node is neglected.

Method

Using the junction rule.

1. Draw a small closed boundary around the node.
2. Mark each current arrow as entering or leaving that boundary.
3. Write one equation: total entering equals total leaving.
4. If using signed currents, define a sign convention and keep it consistently.
5. Avoid writing both $I_1 = I_2 + I_3$ and $I_2 + I_3 = I_1$ as if they were two independent equations.

Example 7.11: Current splitting at a junction

At a node, currents $I_1 = 2.0 \text{ A}$ and $I_2 = 0.70 \text{ A}$ enter. A current $I_3 = 1.1 \text{ A}$ leaves, and another current I_4 is unknown. If I_4 is defined as leaving, find I_4 .

Solution. Charge conservation gives

$$I_1 + I_2 = I_3 + I_4. \quad (7.52)$$

Therefore

$$I_4 = I_1 + I_2 - I_3 = 2.0 + 0.70 - 1.1 = 1.6 \text{ A}. \quad (7.53)$$

The positive result means that the assumed leaving direction is correct.

7.8 Kirchhoff's loop rule

Kirchhoff's loop rule states that the algebraic sum of potential changes around any closed loop is zero:

$$\sum_{\text{closed loop}} \Delta V_i = 0. \quad (7.54)$$

The physical basis is conservation of energy per unit charge. If a charge returns to its starting point after a complete circuit traversal, its electric potential energy per unit charge must return to its starting value. Energy supplied by sources is balanced by energy dissipated or stored in elements.

The loop rule is often where sign errors enter. Figure 7.3 illustrates the standard rules. The loop traversal direction is a bookkeeping choice; it need not match the actual current direction. The potential change across a resistor depends on whether the traversal is with or against the chosen current direction. The potential change across an emf device depends on whether the traversal goes from negative to positive terminal or from positive to negative terminal.

Method

Standard loop-sign convention.

- Resistor: with current $\Rightarrow -IR$; against current $\Rightarrow +IR$.

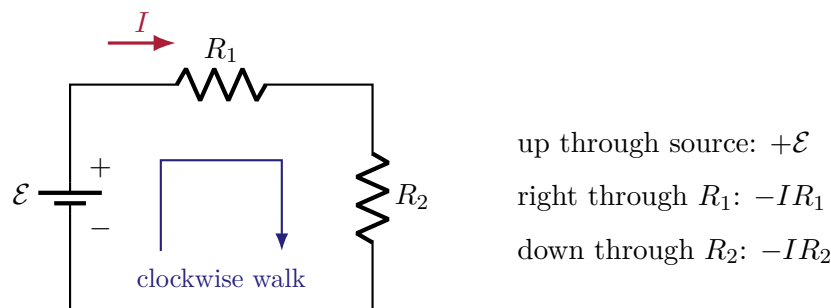


Figure 7.3: The assumed current and the chosen loop traversal are separate choices. This clockwise walk gives $\mathcal{E} - IR_1 - IR_2 = 0$. Reversing the walk reverses every term.

- Ideal source: from $-$ to $+$ terminal $\Rightarrow +\mathcal{E}$; from $+$ to $-$ terminal $\Rightarrow -\mathcal{E}$.
 - Capacitor: from $-$ plate to $+$ plate $\Rightarrow +q/C$; from $+$ plate to $-$ plate $\Rightarrow -q/C$.
- Once you choose directions, do not repair signs by intuition. Let the algebra decide.

Example 7.12: Loop equation with an initially unknown current direction

A loop contains an ideal 10 V source opposing an ideal 4 V source and a total resistance of $3.0\,\Omega$. The assumed current direction passes from negative to positive through the 4 V source and from positive to negative through the 10 V source. Find the current.

Solution. Traversing in the assumed current direction gives

$$+4.0 - 10.0 - I(3.0) = 0. \quad (7.55)$$

Therefore

$$I = -2.0\,\text{A}. \quad (7.56)$$

The actual current has magnitude 2.0 A and flows opposite to the assumed direction. The larger source drives the current.

Reasonableness check. The net emf is 6 V and the resistance is $3\,\Omega$, so the current magnitude should be 2 A.

Guided checks

Check 1. Is the loop rule a statement about charge conservation or energy conservation?

Answer. Energy conservation per unit charge.

Check 2. If you traverse a resistor opposite to the assumed current, is the potential change $+IR$ or $-IR$?

Answer. $+IR$.

Check 3. If a loop equation gives $I = -0.4\,\text{A}$, what does that mean?

Answer. The current is 0.4 A opposite to the chosen reference direction.

7.8.1 What the loop rule assumes

A scalar potential assigns a unique voltage difference to two nodes in the electroquasistatic approximation. A closed walk then telescopes to zero: $(V_b - V_a) + (V_c - V_b) + (V_a - V_c) = 0$. Source emf supplies non-electrostatic work inside the source, while the terminal voltages still enter a consistent voltage balance. This explains why a battery may maintain a potential rise without violating conservation of energy.

If a changing magnetic flux threads the physical loop, the circulation of the electric field need not vanish. An induced emf must then be included with a consistent orientation. The simple resistor–battery equations used here neglect that effect; Faraday’s law and inductance are developed later. It is therefore misleading to apply the present loop equation to an arbitrary rapidly changing field without first checking the model.

7.8.2 A disciplined voltage-walk method

A voltage walk is a careful traversal of a circuit path in which each element contributes a signed potential change. Students often try to guess whether a term is positive or negative from the final physical direction of current. That is not necessary and often leads to mistakes. A voltage walk is a bookkeeping procedure: the signs are determined by the chosen traversal direction, the assumed branch-current directions and the labelled polarities of sources or capacitors.

Consider a path from node a to node b through several elements. If the potential changes along that path are $\Delta V_1, \Delta V_2, \dots, \Delta V_n$, then

$$V_b - V_a = \sum_{j=1}^n \Delta V_j. \quad (7.57)$$

For a closed loop, $b = a$, so the left-hand side is zero. The loop rule is therefore just the special case of a voltage walk whose endpoint equals its starting point.

This viewpoint is useful because it prevents a hidden assumption: not every path in a circuit is a closed loop. Sometimes we want the voltage between two nodes, not just a current. We can find that voltage by walking from one node to the other and adding potential changes. The same sign rules apply.

Method

Voltage-walk method.

1. Mark a start node and an end node.
2. Choose a path between them.
3. For each element, write the potential change encountered along the path.
4. Sum the changes to obtain $V_{\text{end}} - V_{\text{start}}$.
5. If the path is closed, set the sum equal to zero.

Figure 7.4 makes the accumulated potential changes visible without confusing path position with time.

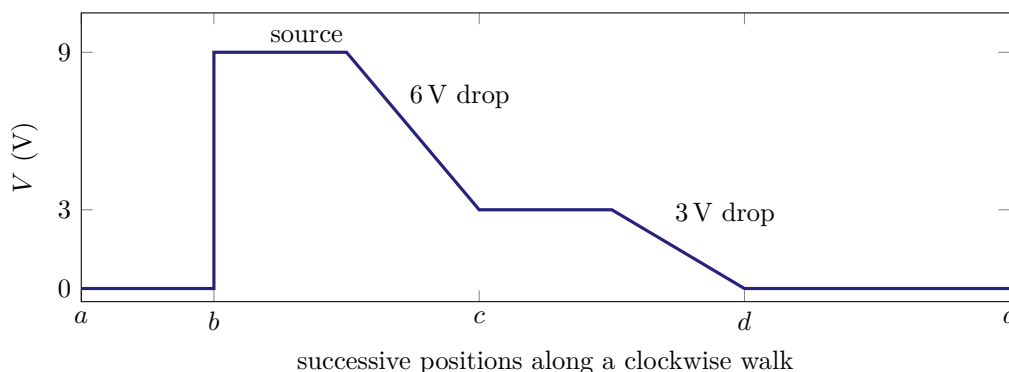


Figure 7.4: A voltage walk around a 9 V source and series resistors 6 k Ω and 3 k Ω carrying 1 mA. The horizontal coordinate orders elements; it is not a spatial scale. Ideal wires are level segments. The walk returns to the initial potential.

Example 7.13: Voltage walk between two points

A branch from node a to node b contains a 4.0 V ideal source traversed from $-$ to $+$, followed by a 3.0Ω resistor carrying 0.50 A in the direction from a to b . Find $V_b - V_a$.

Solution. The source contributes $+4.0\text{ V}$. The resistor is traversed in the direction of current, so it contributes

$$-IR = -(0.50)(3.0) = -1.5\text{ V}. \quad (7.58)$$

Therefore

$$V_b - V_a = 4.0 - 1.5 = 2.5\text{ V}. \quad (7.59)$$

Node b is at a higher potential than node a by 2.5 V .

Guided checks

Check 1. In a voltage walk, must the path be the actual direction of current?

Answer. No. It is a chosen mathematical traversal path.

Check 2. If a path starts and ends at the same node, what is the total potential change?

Answer. Zero.

Check 3. Why is a resistor rise possible in a voltage walk?

Answer. If the path is traversed opposite to the current, the potential change across the resistor is $+IR$.

7.9 Multi-loop circuits and systematic sign conventions

Many practical circuits cannot be reduced by series and parallel combinations alone. Kirchhoff's rules then become a system of linear equations. The safest approach is systematic:

1. label essential nodes;
2. assign branch currents;
3. write independent junction equations;
4. choose independent loops;
5. write loop equations using the sign rules;
6. solve the simultaneous equations;
7. interpret signs and check power.

For the circuit in Figure 7.5, suppose I_1 and I_2 enter the top junction and I_3 leaves downward. The junction rule gives

$$I_1 + I_2 = I_3. \quad (7.60)$$

Loop equations must account for the shared resistor using the chosen branch current through that resistor. If the central resistor current is I_3 downward, then traversing it downward gives $-I_3R_3$ and traversing it upward gives $+I_3R_3$.

Example 7.14: A complete two-loop calculation

In Figure 7.5, let $\mathcal{E}_1 = 12\text{ V}$, $\mathcal{E}_2 = 6\text{ V}$, $R_1 = 4\Omega$, $R_2 = 3\Omega$ and $R_3 = 6\Omega$. Use the indicated reference directions to find all three currents and check energy conservation.

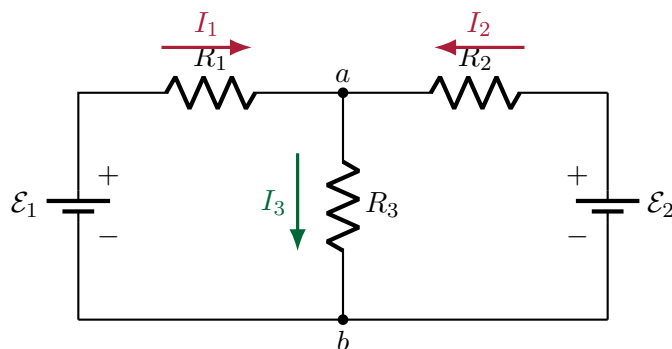


Figure 7.5: Three branch-current references in a two-loop network: I_1 and I_2 enter the top node, and I_3 leaves downward. Each source has its positive terminal above its negative terminal. These arrows are retained even if a solved current is negative.

Model and strategy. Both sources are ideal, all resistors are ohmic and the circuit is in steady state. The bottom node b is the zero of potential. Define I_1 and I_2 toward node a and I_3 downward from a to b . First solve the branch equations; then calculate the same node voltage independently from each branch.

The junction equation is

$$I_1 + I_2 = I_3. \quad (7.61)$$

The left loop, traversed upward through the source and downward through R_3 , gives

$$12 - 4I_1 - 6I_3 = 0. \quad (7.62)$$

For the right loop, walk from a to the right across R_2 , down through $\mathcal{E}_2$, and up through R_3 . The traversal is *opposite* to I_2 and I_3 , so both resistor terms are positive:

$$+3I_2 - 6 + 6I_3 = 0. \quad (7.63)$$

Equivalently, $6 - 3I_2 - 6I_3 = 0$ follows by reversing that entire walk. Eliminating I_3 yields

$$10I_1 + 6I_2 = 12, \quad (7.64)$$

$$6I_1 + 9I_2 = 6. \quad (7.65)$$

With coefficients in ohms and right-hand sides in volts, the solution is

$$I_1 = \frac{4}{3} \text{ A}, \quad I_2 = -\frac{2}{9} \text{ A}, \quad I_3 = \frac{10}{9} \text{ A}. \quad (7.66)$$

Thus current actually flows *out of* node a toward the 6 V source in the right branch. A negative current is the physical result of a consistent reference convention.

Voltage check. Three independent walks give

$$V_a = 12 - 4I_1 = 6 - 3I_2 = 6I_3 = \frac{20}{3} \text{ V}. \quad (7.67)$$

The node is above 6 V, so current must flow from it through R_2 toward the lower-potential source terminal.

Power check. The 12 V source delivers $12(4/3) = 16$ W. Current enters the positive terminal of the 6 V source, so it absorbs $6(2/9) = 4/3$ W. Resistor heating is

$$P_R = 4 \left(\frac{4}{3} \right)^2 + 3 \left(\frac{2}{9} \right)^2 + 6 \left(\frac{10}{9} \right)^2 = \frac{44}{3} \text{ W}. \quad (7.68)$$

Hence $16 = 44/3 + 4/3$ in watts. The smaller source absorbs energy; an emf device need not always deliver it.

Caution

Do not write one loop equation for every visually possible loop and assume all are independent. In a two-mesh planar circuit, two independent loop equations plus appropriate junction information are enough. Extra loop equations may be true but redundant.

7.9.1 Linear algebra viewpoint for circuit equations

When several unknown currents appear, Kirchhoff equations form a linear system. A compact representation helps students avoid random manipulation. For example,

$$\begin{pmatrix} 10 & 6 \\ 6 & 9 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} \quad (7.69)$$

represents the two equations obtained after eliminating I_3 in Example 7.14. The solution can be obtained by substitution, elimination, matrix inversion or numerical linear algebra.

For large networks, manual loop equations become inefficient. Nodal analysis, mesh-current analysis and computer-aided circuit simulation are more systematic. The physics, however, remains the same: conservation of charge plus element current-voltage relations plus energy consistency around loops.

Method**Linear-system checklist.**

1. Count unknown branch currents or node voltages.
2. Write independent equations equal in number to the unknowns.
3. Put the equations in standard algebraic form.
4. Check units of every coefficient: resistance coefficients multiply currents to give volts.
5. Solve and substitute back into the original equations.
6. Interpret negative values physically.

Example 7.15: Solving a small circuit by matrices

A circuit analysis gives

$$5I_1 - 2I_2 = 9, \quad (7.70)$$

$$-2I_1 + 6I_2 = 4, \quad (7.71)$$

where currents are in amperes when voltages are in volts and resistances in ohms. Solve for I_1 and I_2 .

Solution. The determinant is

$$D = (5)(6) - (-2)(-2) = 30 - 4 = 26. \quad (7.72)$$

Cramer's rule gives

$$I_1 = \frac{(9)(6) - (-2)(4)}{26} = \frac{62}{26} = 2.38 \text{ A}, \quad (7.73)$$

$$I_2 = \frac{(5)(4) - (9)(-2)}{26} = \frac{38}{26} = 1.46 \text{ A}. \quad (7.74)$$

Both signs are positive, so both currents flow in their assumed directions.

7.10 Meters and measurement loading

Measurement is part of circuit physics. Connecting a meter changes the circuit unless the meter is ideal or its loading effect is negligible.

An ammeter measures current. To measure the current through a component, the ammeter must be placed in series with that component so the same current passes through the meter. An ideal ammeter has zero resistance. A real ammeter has small but nonzero resistance R_A . If R_A is not small compared with the circuit resistance, inserting the ammeter changes the current being measured.

A voltmeter measures potential difference. To measure the potential difference across a component, the voltmeter is connected in parallel with that component. An ideal voltmeter has infinite resistance. A real voltmeter has large but finite resistance R_V . The loading must be judged against the output resistance of the network driving the meter; if it is not negligible, the meter changes the voltage being measured. Figure 7.6 shows both meter connections and identifies the current actually measured by the ammeter.

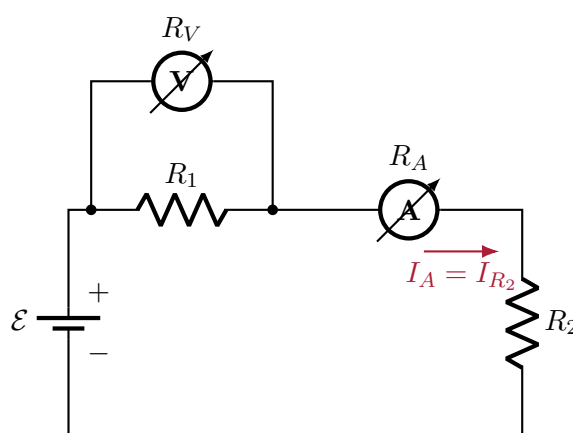


Figure 7.6: The voltmeter bridges the terminals of R_1 ; its branch contains the meter, with no bypass wire. The ammeter is in series with R_2 and measures I_{R_2} . With finite voltmeter resistance, this current is the sum of the currents through R_1 and the voltmeter, not just I_{R_1} .

Key message

A good ammeter disturbs the circuit little by having very small resistance. A good voltmeter disturbs the circuit little by having very large resistance.

Example 7.16: Voltmeter loading of a voltage divider

A voltage divider consists of $R_1 = 100\text{ k}\Omega$ in series with $R_2 = 100\text{ k}\Omega$ across an ideal 10.0 V source. The intended output is across R_2 . First find the unloaded output. Then find the output when a voltmeter of resistance $R_V = 100\text{ k}\Omega$ is connected across R_2 .

Solution. Without the voltmeter, equal resistances divide the source equally:

$$V_{\text{out,unloaded}} = 10.0 \frac{R_2}{R_1 + R_2} = 5.0\text{ V}. \quad (7.75)$$

With the voltmeter connected, it is in parallel with R_2 . The effective lower resistance is

$$R_{2V} = \frac{R_2 R_V}{R_2 + R_V} = \frac{(100)(100)}{200}\text{ k}\Omega = 50\text{ k}\Omega. \quad (7.76)$$

Thus

$$V_{\text{out,loaded}} = 10.0 \frac{50}{100 + 50} = 3.33\text{ V}. \quad (7.77)$$

Interpretation. The meter resistance is not large compared with the divider resistance, so the measurement significantly changes the circuit. This is not a small experimental error; it is a modelling error if ignored.

Guided checks

Check 1. Why is an ammeter placed in series?

Answer. To ensure the current through the meter is the same current to be measured.

Check 2. Why should an ideal voltmeter have infinite resistance?

Answer. So that it draws no current and does not change the voltage across the measured element.

Check 3. What can go wrong if an ammeter is connected directly across a low-resistance source?

Answer. It can create a near short circuit with very large current, damaging the meter or source.

7.10.1 Quantifying how much a meter changes the answer

If a source with fixed voltage drives a total resistance R_0 , inserting an ammeter with resistance R_A changes the current from $I_0 = V/R_0$ to $I_A = V/(R_0 + R_A)$. The fractional reduction is

$$\frac{I_0 - I_A}{I_0} = \frac{R_A}{R_0 + R_A}. \quad (7.78)$$

The voltage $I_A R_A$ across the meter is its resistive burden in this model. A small numerical resistance is not enough: it must be small *relative to the circuit resistance*. A meter suitable for a kilohm branch can seriously disturb a low-resistance current path.

Example 7.17: Selecting an ammeter range by its burden

A 2.0 V source drives $R_0 = 10.0 \Omega$. The unmeasured current is 0.200 A. A current range represented by $R_A = 0.20 \Omega$ gives $I_A = 2.0/10.20 = 0.1961$ A, a 1.96% reduction. If the allowed disturbance is 0.5%, the necessary bound is

$$\frac{R_A}{10.0 + R_A} \leq 0.005 \implies R_A \leq 0.0503 \Omega. \quad (7.79)$$

The range must also accommodate the expected current. Resolution and loading are separate specifications: more displayed digits do not correct the physical disturbance caused by the meter.

A moving-coil meter illustrates how resistance controls measurement range. If its full-scale current and coil resistance are I_g and R_g , a shunt R_s carrying the remaining current extends it to range $I_{\max}$:

$$I_g R_g = (I_{\max} - I_g) R_s, \quad R_s = \frac{I_g R_g}{I_{\max} - I_g}. \quad (7.80)$$

A series resistance extends the voltage range to $V_{\max}$:

$$R_{\text{series}} = \frac{V_{\max}}{I_g} - R_g. \quad (7.81)$$

The magnetic mechanism of the moving coil belongs to the magnetism chapters; the range-extension equations require only circuit laws.

Resistance measurement introduces another circuit. A meter applies a test current or voltage and infers R from the response. The component should be unpowered and isolated sufficiently that parallel paths do not alter the result. In a two-wire measurement, lead resistances add to the resistance under test. For very small resistances, use separate current and voltage leads: an approximately ideal voltmeter draws negligible current through its sense leads, so their voltage drops are negligible. Figure 7.7 shows this four-terminal principle.

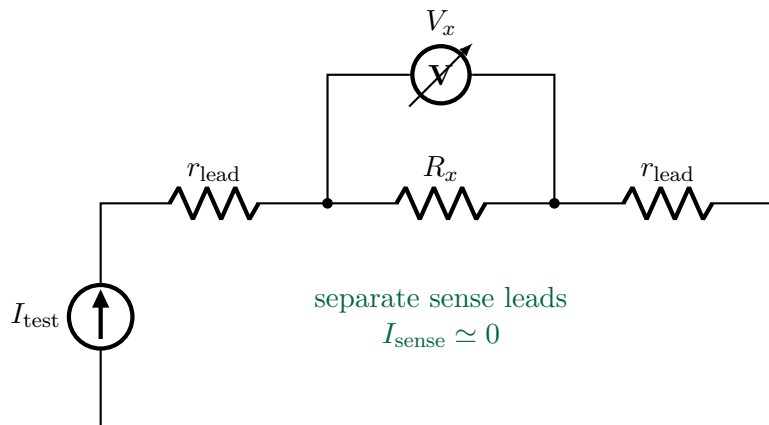


Figure 7.7: Four-terminal resistance measurement. The test current flows through the current leads and R_x ; separate sense leads contact R_x directly. A sufficiently high input resistance makes their voltage drops negligible, so $R_x = V_x/I_{\text{test}}$ excludes the current-lead drops.

7.11 Capacitors in DC circuits: steady state and transient state

A capacitor consists of two conductors separated by an insulating region. In a DC circuit, no steady conduction current passes through the dielectric of an ideal capacitor. However, when a capacitor is being charged or discharged, current flows in the wires connected to the plates while charge accumulates or leaves the plates.

This distinction is a common source of confusion. During charging, electrons are moved onto one plate and away from the other plate by the external circuit. The dielectric between plates is not a normal conducting path. The current in the wires is associated with changing charge on the plates:

$$I = \frac{dq}{dt}. \quad (7.82)$$

When the capacitor reaches its final DC voltage, q is constant, so $dq/dt = 0$ and the current in that branch becomes zero.

Caution

An ideal capacitor does not conduct steady DC through its dielectric. It can nevertheless be part of a time-dependent current path while its charge and voltage are changing.

7.12 RC charging

Consider an initially uncharged capacitor C connected in series with a resistor R and an ideal DC source $\mathcal{E}$ at time $t = 0$. Let $q(t)$ be the charge on the capacitor plate connected to the positive terminal of the source. The current is

$$I(t) = \frac{dq}{dt}. \quad (7.83)$$

Applying the loop rule during charging gives

$$\mathcal{E} - IR - \frac{q}{C} = 0. \quad (7.84)$$

Substituting $I = dq/dt$ gives the differential equation

$$R \frac{dq}{dt} + \frac{q}{C} = \mathcal{E}. \quad (7.85)$$

Rearrange:

$$\frac{dq}{dt} = \frac{C\mathcal{E} - q}{RC}. \quad (7.86)$$

The solution satisfying $q(0) = 0$ is

$$q(t) = C\mathcal{E} \left(1 - e^{-t/RC}\right). \quad (7.87)$$

Therefore

$$V_C(t) = \frac{q(t)}{C} = \mathcal{E} \left(1 - e^{-t/RC}\right), \quad (7.88)$$

$$I(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R} e^{-t/RC}. \quad (7.89)$$

Figure 7.8 connects the charging circuit to the quantitative curves. The time constant is

$$\tau = RC. \quad (7.90)$$

Dimensional check:

$$[RC] = \Omega \text{F} = \frac{\text{V}}{\text{A}} \frac{\text{C}}{\text{V}} = \frac{\text{C}}{\text{A}} = \text{s}. \quad (7.91)$$

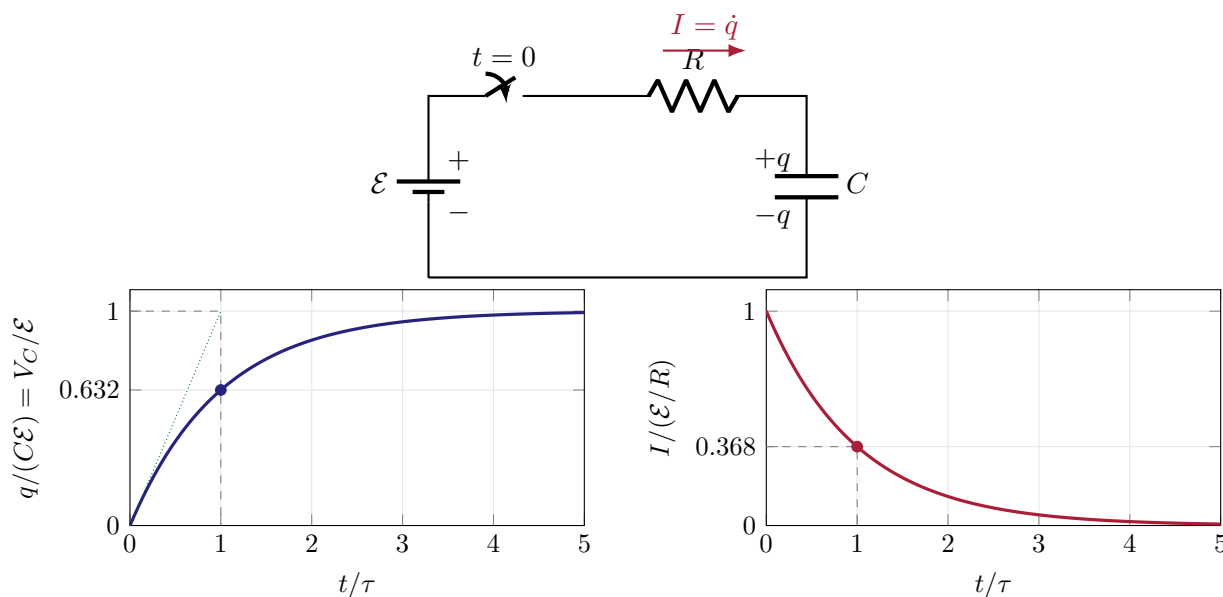


Figure 7.8: An initially uncharged capacitor begins charging when the switch closes. Separate quantitative axes show the normalized charge/voltage and current. The dotted initial tangent in the left panel reaches the final-value level at $t = \tau$, while the actual charge reaches only $1 - e^{-1}$.

7.12.1 Deriving the exponential and reading its graph

The differential equation does more than supply a formula: it says that the charging rate is proportional to the charge still needed to reach equilibrium. Separate variables, with an integration variable q' that is distinct from the upper limit:

$$\int_0^{q(t)} \frac{dq'}{C\mathcal{E} - q'} = \int_0^t \frac{dt'}{RC}. \quad (7.92)$$

The left integral is $-\ln[(C\mathcal{E} - q(t))/(C\mathcal{E})]$, so

$$\ln\left(1 - \frac{q(t)}{C\mathcal{E}}\right) = -\frac{t}{RC}. \quad (7.93)$$

Exponentiating and solving for q gives Eq. (7.87). Substitution back into the differential equation and evaluation at $t = 0$ check both the dynamics and the initial condition.

The initial slope is $\dot{V}_C(0) = \mathcal{E}/(RC)$. Its tangent reaches the final-voltage level at $t = \tau$, although the actual curve bends below that tangent. At any later time the slope is smaller because less voltage remains across the resistor. The charge and voltage have identical normalized curves because $q = CV_C$; the current has a different curve because it is proportional to the slope, not the charge.

The residual fraction after time t is $\epsilon = \exp(-t/\tau)$. Reaching a chosen accuracy therefore requires

$$t = -\tau \ln \epsilon. \quad (7.94)$$

For 1% residual, $t = 4.605\tau$; for 0.1%, $t = 6.908\tau$. Five time constants leave about 0.674% of the initial deviation. “Fully charged” in a measurement means within a stated tolerance, not that the mathematical exponential reaches its asymptote at a finite time.

Example 7.18: Charging to 63 percent

A $10\ \mu\text{F}$ capacitor is charged through a $200\ \text{k}\Omega$ resistor from an ideal $9.0\ \text{V}$ source. Find the time constant, the capacitor voltage at one time constant and the current at one time constant.

Solution. The time constant is

$$\tau = RC = (200 \times 10^3)(10 \times 10^{-6}) = 2.0\ \text{s}. \quad (7.95)$$

At $t = \tau$,

$$V_C(\tau) = \mathcal{E}(1 - e^{-1}) = 9.0(0.632) = 5.69\ \text{V}. \quad (7.96)$$

The initial current is

$$I_0 = \frac{\mathcal{E}}{R} = \frac{9.0}{200 \times 10^3} = 45\ \mu\text{A}. \quad (7.97)$$

At $t = \tau$,

$$I(\tau) = I_0 e^{-1} = 16.6\ \mu\text{A}. \quad (7.98)$$

Interpretation. One time constant is not the time to become fully charged. It is the time to reach about 63.2% of the final voltage in the ideal charging model.

Guided checks

Check 1. What is the initial current in an ideal RC charging circuit?

Answer. $I(0) = \mathcal{E}/R$, because the initially uncharged capacitor has zero voltage.

Check 2. What is the final current after a long time?

Answer. Zero, because the capacitor voltage equals the source emf and no steady current flows.

Check 3. What is the physical meaning of $\tau = RC$?

Answer. It is the characteristic time scale over which the charge, voltage and current change by factors involving e^{-1} .

7.13 RC discharging

Now suppose a capacitor initially has charge Q_0 and is connected across a resistor R with no source. Choose the positive current direction to correspond to decreasing positive charge on the initially

positive plate. Then

$$I = -\frac{dq}{dt} \quad (7.99)$$

if $q(t)$ denotes the remaining positive charge on that plate. A loop equation gives

$$IR - \frac{q}{C} = 0, \quad (7.100)$$

or equivalently

$$\frac{dq}{dt} = -\frac{q}{RC}. \quad (7.101)$$

The solution is

$$q(t) = Q_0 e^{-t/RC}. \quad (7.102)$$

Therefore

$$V_C(t) = \frac{Q_0}{C} e^{-t/RC} = V_0 e^{-t/RC}, \quad (7.103)$$

$$\frac{dq}{dt} = -\frac{Q_0}{RC} e^{-t/RC}. \quad (7.104)$$

Figure 7.9 marks the physical discharge direction and compares voltage with energy decay. If the reference current is defined as $I_q = dq/dt$, then it is negative. If the reference current is defined in the physical direction of discharge, its magnitude is

$$I_{\text{discharge}}(t) = \frac{Q_0}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC}. \quad (7.105)$$

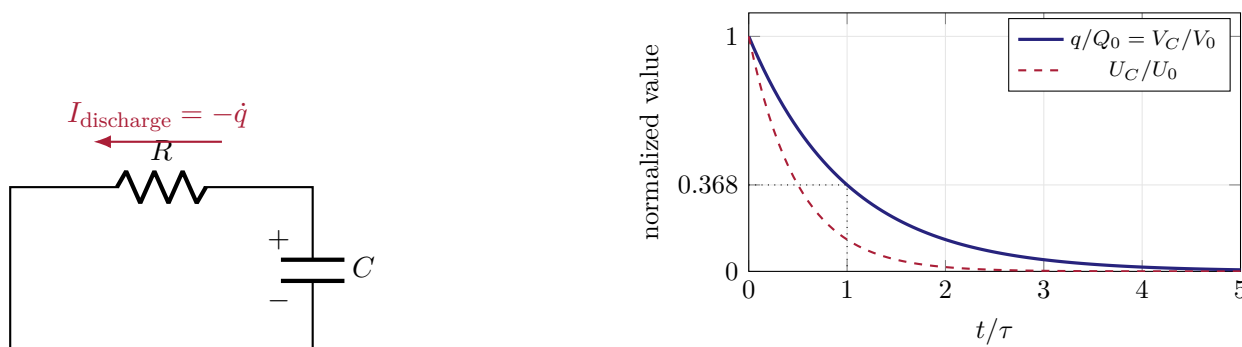


Figure 7.9: A charged capacitor discharges through R . The current arrow follows the physical discharge direction for $q > 0$. Voltage and charge decay as $e^{-t/\tau}$, but stored energy decays faster, as $e^{-2t/\tau}$.

Example 7.19: Discharging time from a voltage threshold

A capacitor initially at 5.0 V discharges through a resistor with time constant $\tau = 0.80$ s. How long does it take for the voltage to fall to 1.0 V?

Solution. The discharge law is

$$V_C(t) = V_0 e^{-t/\tau}. \quad (7.106)$$

Thus

$$\frac{1.0}{5.0} = e^{-t/0.80}. \quad (7.107)$$

Taking natural logarithms,

$$t = -0.80 \ln(0.20) = 1.29 \text{ s}. \quad (7.108)$$

Reasonableness check. After one time constant the voltage is $0.368V_0 = 1.84$ V. Reaching 1.0 V should take more than one time constant, and it does.

7.14 Energy in RC charging and discharging

When a capacitor is charged by an ideal battery through a resistor, the final energy stored in the capacitor is

$$U_C = \frac{1}{2}C\mathcal{E}^2. \quad (7.109)$$

The energy delivered by the ideal battery during the charging process is

$$W_{\text{battery}} = \int \mathcal{E} dq = \mathcal{E}(C\mathcal{E}) = C\mathcal{E}^2. \quad (7.110)$$

Thus half of the supplied energy is stored in the capacitor and half is dissipated as thermal energy in the resistor:

$$W_R = \frac{1}{2}C\mathcal{E}^2. \quad (7.111)$$

This result is independent of the value of R in the ideal model. The resistance controls how fast the process occurs, not the total energy partition for charging from an ideal voltage source.

During discharge, the initial capacitor energy

$$U_0 = \frac{Q_0^2}{2C} = \frac{1}{2}CV_0^2 \quad (7.112)$$

is dissipated in the resistor. The instantaneous resistor power is

$$P_R(t) = I^2(t)R = \frac{V_0^2}{R}e^{-2t/RC}. \quad (7.113)$$

Integrating from $t = 0$ to infinity gives

$$\int_0^\infty P_R(t) dt = \frac{V_0^2}{R} \int_0^\infty e^{-2t/RC} dt = \frac{V_0^2}{R} \frac{RC}{2} = \frac{1}{2}CV_0^2. \quad (7.114)$$

Example 7.20: Energy supplied and stored during charging

A $100 \mu\text{F}$ capacitor is charged from 0 to 12 V through a resistor. Find the energy supplied by the ideal source, the final capacitor energy and the energy dissipated in the resistor.

Solution. The source supplies

$$W_{\text{source}} = C\mathcal{E}^2 = (100 \times 10^{-6})(12)^2 = 1.44 \times 10^{-2} \text{ J}. \quad (7.115)$$

The final capacitor energy is

$$U_C = \frac{1}{2}C\mathcal{E}^2 = 7.2 \times 10^{-3} \text{ J}. \quad (7.116)$$

The resistor dissipates the difference,

$$W_R = 7.2 \times 10^{-3} \text{ J}. \quad (7.117)$$

Interpretation. A larger resistor would make charging slower and reduce peak current, but in this ideal model it would not change the total heat produced during a full charge from zero to $\mathcal{E}$.

Caution

The statement that half the energy is dissipated during ideal RC charging applies to charging from an ideal fixed-voltage source through resistance. It is not a universal law for every charging protocol. A carefully controlled current source or stepwise charging process can change the dissipated energy.

For arbitrary initial voltage V_i and final source voltage V_f across a finite positive series resistance, the same reasoning gives

$$W_s = CV_f(V_f - V_i), \quad (7.118)$$

$$\Delta U_C = \frac{1}{2}C(V_f^2 - V_i^2), \quad (7.119)$$

$$W_R = W_s - \Delta U_C = \frac{1}{2}C(V_f - V_i)^2 \geq 0. \quad (7.120)$$

The source work can be negative: a capacitor initially above the applied source voltage may return energy to an ideal source that can absorb it. The heat is always nonnegative. The familiar half-energy statement is specifically the case $V_i = 0$.

Taking R toward zero leaves this integrated heat finite while the current pulse becomes narrower and larger. Setting $R = 0$ *at the outset* is a different, singular idealization: an ideal source imposing a voltage jump on an ideal capacitor requires an impulsive current. Stray inductance, radiation and source limitations then matter. The finite- R result must not be interpreted as a prediction of ordinary heating in a literally zero-resistance element.

7.15 Steady-state versus transient reasoning

A common mistake is to apply steady-state intuition to a transient circuit or transient intuition to a final DC state. In a resistor-only DC circuit, currents reach steady values extremely quickly compared with ordinary measurement times. In an RC circuit, the capacitor voltage changes over a time scale RC .

At $t = 0^+$, immediately after switching, an uncharged capacitor behaves like a short circuit in the sense that its voltage is initially zero. A fully charged capacitor in a long-time DC steady state behaves like an open circuit in the sense that no steady current flows through the capacitor branch. These phrases are useful shortcuts but must be interpreted carefully: the capacitor is not literally replaced by a wire or a broken component for all quantities at all times.

Method**Transient-state checklist for RC circuits.**

1. Identify the capacitor voltage just before switching. With finite current, capacitor voltage is continuous: $V_C(0^+) = V_C(0^-)$. An imposed ideal voltage jump requires an impulse of current and lies outside the finite-resistance switching model used here.
2. Determine the initial current using the initial capacitor voltage.
3. Determine the final steady-state capacitor voltage by treating the capacitor branch as open for DC.
4. Find the resistance seen by the capacitor for the time constant, after independent ideal voltage sources are replaced by short circuits and ideal current sources by open circuits.
5. Write the response as final value plus initial deviation times $e^{-t/\tau}$.

Theoretical point 7.2: General first-order RC form

For a linear circuit containing one independent capacitor state, constant sources after switching and a finite positive resistance seen by the capacitor, the capacitor voltage can be written as

$$V_C(t) = V_C(\infty) + [V_C(0^+) - V_C(\infty)] e^{-t/\tau}, \quad (7.121)$$

where $\tau = R_{\text{seen}}C$ and R_{seen} is the Thevenin resistance seen by the capacitor terminals with independent sources deactivated. This compact form is an engineering bridge to the Thevenin discussion in the advanced section.

7.15.1 A capacitor connected to a loaded divider

The general response is most useful when the final voltage is not the source emf. Figure 7.10 shows a capacitor connected at $t = 0$ across the lower arm of a divider. The resistor network remains connected while the capacitor charges; the upper and lower resistors do not generally carry the same transient current.

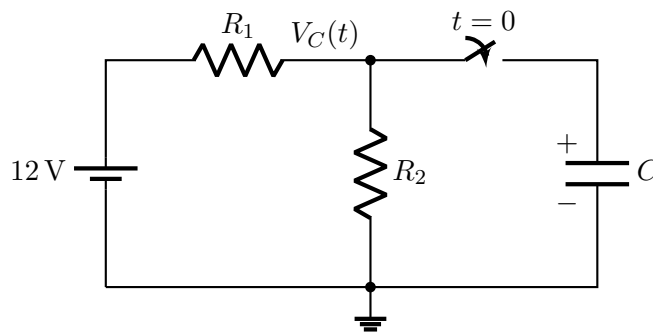


Figure 7.10: After the switch closes, the capacitor is across R_2 . The initial capacitor voltage is specified by its prior preparation. Both R_1 and R_2 influence the relaxation time; the nearest resistor alone does not determine it.

Example 7.21: Finding initial current, final voltage and time constant

An ideal 12 V supply feeds $R_1 = 3.0 \text{ k}\Omega$ and $R_2 = 6.0 \text{ k}\Omega$ in series. At $t = 0$, a $100 \mu\text{F}$ capacitor with initial voltage 2.0 V is connected across R_2 , with its positive voltage reference at the divider node.

Initial state. With finite resistance, $V_C(0^+) = 2.0 \text{ V}$. Immediately after connection,

$$I_1(0^+) = \frac{12 - 2}{3.0 \text{ k}\Omega} = 3.333 \text{ mA}, \quad (7.122)$$

$$I_2(0^+) = \frac{2}{6.0 \text{ k}\Omega} = 0.333 \text{ mA}, \quad (7.123)$$

$$I_C(0^+) = I_1 - I_2 = 3.000 \text{ mA}. \quad (7.124)$$

The difference charges the capacitor. Assuming equal resistor currents at this instant would violate charge conservation at the node.

Final state and dynamics. At long times $I_C = 0$, so $V_f = 12(6/9) = 8.0 \text{ V}$. Setting the ideal source voltage to zero connects both resistors between the capacitor terminals; hence $R_{\text{seen}} = 3 \parallel 6 = 2.0 \text{ k}\Omega$ and $\tau = 0.200 \text{ s}$. Therefore

$$V_C(t) = 8.0 - 6.0e^{-t/(0.200 \text{ s})} \text{ V}. \quad (7.125)$$

At 0.300 s, $V_C = 6.661 \text{ V}$. The capacitor current is $C\dot{V}_C = 3.000e^{-t/\tau} \text{ mA}$, agreeing with the initial junction calculation.

Direct check without equivalent circuits. The node equation

$$C\dot{V}_C = \frac{12 - V_C}{R_1} - \frac{V_C}{R_2} \quad (7.126)$$

reduces to $R_{\text{seen}}C\dot{V}_C + V_C = 8.0 \text{ V}$. It independently confirms the final value and time constant. The full Thevenin method is developed in the enrichment section.

7.16 Laboratory connection: measuring an RC time constant

A simple laboratory experiment can make the exponential law visible. Connect a resistor and capacitor in series with a square-wave source or a switchable DC source. Measure the capacitor voltage with an oscilloscope or data-acquisition interface. During charging, fit

$$V_C(t) = V_f - (V_f - V_i)e^{-t/\tau}. \quad (7.127)$$

During discharge, fit

$$V_C(t) = V_i e^{-t/\tau} \quad (7.128)$$

if the final voltage is zero.

A practical measurement should record uncertainties in R , C , source voltage, oscilloscope time base and voltage readings. A powerful linearisation for discharge is

$$\ln\left(\frac{V_C(t)}{V_0}\right) = -\frac{t}{\tau}. \quad (7.129)$$

A plot of $\ln(V_C/V_0)$ versus t should be a straight line with slope $-1/\tau$ in the ideal model. Deviations can reveal leakage, meter loading, capacitor tolerance, switch bounce or parasitic resistance.

Example 7.22: Extracting τ from two voltage readings

During discharge, a capacitor voltage is 4.0 V at $t = 0$ and 1.5 V at $t = 3.0 \text{ s}$. Estimate the time constant.

Solution. Use

$$\frac{V(t)}{V_0} = e^{-t/\tau}. \quad (7.130)$$

Thus

$$\frac{1.5}{4.0} = e^{-3.0/\tau}. \quad (7.131)$$

Taking logs,

$$\tau = \frac{-3.0}{\ln(1.5/4.0)} = 3.06 \text{ s}. \quad (7.132)$$

Experimental note. A two-point estimate is sensitive to measurement noise. A fit to many data points is more reliable.

7.16.1 Offsets, loading and a reproducible data analysis

A discharge trace rarely comes with a perfectly known zero. A useful measurement model is

$$V_{\text{meas}}(t) = V_{\text{off}} + Ae^{-t/\tau}. \quad (7.133)$$

Here A is the initial signal above the final offset, not necessarily the initial raw reading. Subtract the offset before taking a logarithm. A plot of $\ln[(V_{\text{meas}} - V_{\text{off}})/A]$ is linear only where the offset model is appropriate and the signal remains above the noise floor.

For additive voltage noise of approximately constant standard deviation σ_V , the logarithmic uncertainty grows as $\sigma_{\ln V} \simeq \sigma_V/(V - V_{\text{off}})$. Giving all log-transformed points equal weight can therefore overemphasize noisy late-time points. When uncertainties matter, fit the original voltage model with the appropriate weights, or propagate uncertainties into a weighted logarithmic fit. Residuals should be inspected for systematic structure rather than judged only by an attractive curve.

t (s)	0	1	2	3	4	5
V_{meas} (V)	5.10	3.13	1.94	1.22	0.777	0.510

Table 7.1: Synthetic teaching data obtained by rounding $0.100 + 5.000e^{-t/(2.000\text{ s})}$ V. These are model-generated values, not experimental observations.

Example 7.23: A fitted time constant depends on the zero level

Use the illustrative data in Table 7.1, with independently known $V_{\text{off}} = 0.100$ V. Write $y_i = \ln[(V_i - V_{\text{off}})/(1\text{ V})]$ so the logarithm has a dimensionless argument. An equal-weight straight-line fit $y = c + mt$ is sufficient here to illustrate the method because the deviations arise only from rounding. Its slope is

$$m = \frac{\sum_i (t_i - \bar{t})(y_i - \bar{y})}{\sum_i (t_i - \bar{t})^2} = -0.49993\text{ s}^{-1}. \quad (7.134)$$

Thus $\tau = -1/m = 2.0003\text{ s}$ and $A = (1\text{ V})e^c \simeq 5.001\text{ V}$. The extra digits diagnose agreement with the generating model; they are not a claim about laboratory precision.

If the offset is ignored and $\ln(V_i/1\text{ V})$ is fitted instead, the slope is -0.46162 s^{-1} and the inferred time constant is 2.1663 s , about 8.3% too large. The positive offset makes the late-time trace appear to decay more slowly. A plausible fitted number can therefore be wrong because the measurement model is incomplete.

Standard-library Python reproduces the illustrative fit.

```
from math import log, exp
```

```
t = [0, 1, 2, 3, 4, 5] # seconds
```

```
v = [5.10, 3.13, 1.94, 1.22, .777, .510] # volts
```

```
for offset in (.100, 0.0):
```

```
    y = [log(vi-offset) for vi in v] # numeric volt values
```

```
    tm, ym = sum(t)/len(t), sum(y)/len(y)
```

```
    m = sum((ti-tm)*(yi-ym) for ti, yi in zip(t,y))
```

```
    m /= sum((ti-tm)**2 for ti in t)
```

```
    print("offset, tau, amplitude:", offset, -1/m,
          exp(ym-m*tm))
```

If a meter resistance R_M and probe capacitance C_M are connected across a capacitor discharging through R , the model predicts $\tau_{\text{meas}} = (R \parallel R_M)(C + C_M)$. Resistive loading shortens the decay, while added capacitance lengthens it. A near-correct measured time constant can result from partial cancellation and does not prove that both effects are negligible. For a charging measurement, include source output resistance and determine the loaded final voltage as well.

7.16.2 Checking Kirchhoff's rules with real instruments

Before energizing a low-voltage teaching circuit, record component values and the intended node connections. Measure node voltages with respect to a common reference and infer resistor currents using measured resistances. This avoids inserting an ammeter successively into different branches and thereby creating a different loaded circuit for each reading. Include the voltmeter input resistance

when its current is significant.

For a node with three measured branch currents, form the residual $\delta I = I_1 + I_2 - I_3$. If uncertainties are independent, $\sigma_{\delta I} = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}$. For currents inferred from shared voltage measurements, their errors may be correlated and the covariance must be retained. Report the residual together with its uncertainty; an exact numerical zero is not expected from finite-resolution measurements.

Loop residuals obtained simply by subtracting the same set of node voltages telescope to zero automatically. They check bookkeeping but are not independent evidence for the resistor model. A stronger check compares measured voltage drops with independently inferred IR values, then checks source and resistor power within the uncertainty budget. The aim is to test a physically specified model, not merely to demonstrate that numbers can be made to add.

7.17 Computational connection: numerical RC simulation

The charging equation

$$\frac{dq}{dt} = \frac{C\mathcal{E} - q}{RC} \quad (7.135)$$

can be integrated numerically. The simplest explicit Euler method with time step Δt is

$$q_{n+1} = q_n + \Delta t \frac{C\mathcal{E} - q_n}{RC}. \quad (7.136)$$

This method is easy to implement but has numerical limitations. If Δt is too large compared with RC , the numerical solution may be inaccurate or even qualitatively wrong.

Note

A numerical simulation should be checked against the analytic solution whenever an analytic solution is available. The RC circuit is an ideal first test problem for a circuit-simulation code because the exact exponential response is known.

```
# Pseudocode: RC charging by explicit Euler
R = 200e3
C = 10e-6
E = 9.0
dt = 0.01
t = 0.0
q = 0.0
while t <= 10.0:
    print(t, q, q/C)
    dqdt = (C*E - q)/(R*C)
    q = q + dt*dqdt
    t = t + dt
```

The same approach extends to more complicated circuits. In professional circuit simulation, element equations and Kirchhoff laws are assembled into systems of algebraic and differential equations. SPICE-like tools then solve those systems using numerical methods. Even when software does the algebra, the user must understand the modelling assumptions.

Example 7.24: One Euler step for charging

Let $R = 1.0 \text{ k}\Omega$, $C = 100 \mu\text{F}$ and $\mathcal{E} = 5.0 \text{ V}$. Starting from $q_0 = 0$, use explicit Euler with $\Delta t = 0.010 \text{ s}$ to estimate q_1 after one time step.

Solution. The time constant is

$$RC = (1000)(100 \times 10^{-6}) = 0.100 \text{ s.} \quad (7.137)$$

At $q_0 = 0$,

$$\left. \frac{dq}{dt} \right|_0 = \frac{C\mathcal{E}}{RC} = \frac{\mathcal{E}}{R} = \frac{5.0}{1000} = 5.0 \times 10^{-3} \text{ C/s.} \quad (7.138)$$

Thus

$$q_1 = 0 + (0.010)(5.0 \times 10^{-3}) = 5.0 \times 10^{-5} \text{ C.} \quad (7.139)$$

The corresponding capacitor voltage estimate is

$$V_{C,1} = \frac{q_1}{C} = \frac{5.0 \times 10^{-5}}{100 \times 10^{-6}} = 0.50 \text{ V.} \quad (7.140)$$

Check. The exact value after 0.010 s is $5(1 - e^{-0.1}) = 0.476 \text{ V}$, so the Euler estimate is close but already slightly high.

7.18 Common mistakes and how to avoid them

Summary

1. **Random loop signs.** Avoid this by writing the sign rules next to the circuit and using them mechanically.
2. **Treating negative current as impossible.** A negative current means the actual direction is opposite to the assumed direction.
3. **Confusing emf with force.** Emf is energy supplied per unit charge, measured in volts.
4. **Putting an ammeter in parallel.** This can create a near short circuit. Ammeters go in series.
5. **Putting a voltmeter in series.** A high-resistance voltmeter in series can nearly stop the current and measure the wrong thing.
6. **Assuming all visually adjacent resistors are series.** Check whether the same current must pass through them.
7. **Assuming all side-by-side resistors are parallel.** Check whether they share both nodes.
8. **Saying a capacitor conducts steady DC.** An ideal capacitor carries transient current while charging or discharging but blocks steady DC.
9. **Forgetting meter loading.** Real meters change circuits unless their resistance is appropriately small or large.
10. **Using $\tau = RC$ with the wrong resistance.** In more complicated circuits, R is the resistance seen by the capacitor, not necessarily the nearest labelled resistor.

7.19 Chapter summary

Summary

Direct-current circuit analysis is an application of conservation laws to idealised networks. Current represents charge flow, potential difference represents energy per unit charge, and emf represents energy per unit charge supplied by a source. Real sources can be modelled by an ideal emf in series with internal resistance, giving $V_{\text{terminal}} = \mathcal{E} - Ir$ when delivering current. Resistors in series share current and have $R_{\text{eq}} = \sum R_j$. Resistors in parallel share voltage and have $1/R_{\text{eq}} = \sum 1/R_j$. Kirchhoff's junction rule expresses charge conservation at a node, while the loop rule expresses energy conservation around a closed path. Ammeters must be connected in series and have small resistance; voltmeters must be connected in parallel and have large resistance. A series RC circuit obeys a first-order differential equation. During charging, $q = C\mathcal{E}(1 - e^{-t/RC})$ and $I = (\mathcal{E}/R)e^{-t/RC}$. During discharging, $q = Q_0e^{-t/RC}$. The time constant $\tau = RC$ sets the scale for exponential relaxation.

7.20 Formula summary

Formula summary

$$I = \frac{dq}{dt}, \quad (7.141)$$

$$V_{\text{terminal}} = \mathcal{E} - Ir \quad (\text{real source delivering current}), \quad (7.142)$$

$$R_{\text{eq}} = \sum_{j=1}^n R_j \quad (\text{series resistors}), \quad (7.143)$$

$$\frac{1}{R_{\text{eq}}} = \sum_{j=1}^n \frac{1}{R_j} \quad (\text{parallel resistors}), \quad (7.144)$$

$$\sum I_{\text{in}} = \sum I_{\text{out}} \quad (\text{junction rule}), \quad (7.145)$$

$$\sum_{\text{closed loop}} \Delta V_i = 0 \quad (\text{loop rule}), \quad (7.146)$$

$$P = IV = I^2 R = \frac{V^2}{R} \quad (\text{resistor}), \quad (7.147)$$

$$\tau = RC, \quad (7.148)$$

$$q_{\text{charge}}(t) = C\mathcal{E} \left(1 - e^{-t/RC}\right), \quad (7.149)$$

$$I_{\text{charge}}(t) = \frac{\mathcal{E}}{R} e^{-t/RC}, \quad (7.150)$$

$$q_{\text{discharge}}(t) = Q_0 e^{-t/RC}, \quad (7.151)$$

$$V_C(t) = V_C(\infty) + \left[V_C(0^+) - V_C(\infty)\right] e^{-t/\tau}. \quad (7.152)$$

7.21 Advanced Topics for Technical and Engineering Students

The core chapter supplies the circuit laws and single-capacitor response required by the course. The following modules deepen those same ideas for strong undergraduate and early graduate students. They connect terminal equivalents to measurement design, node equations to linear algebra, and exponential relaxation to numerical and multi-state models. Inductance, sinusoidal phasors and transmission lines remain topics for later chapters.

7.21.1 Thevenin equivalent circuits

Many networks are used only through two output terminals. From the viewpoint of a load connected to those terminals, a complicated linear resistor-source network can often be replaced by a Thevenin equivalent: an ideal voltage source V_{Th} in series with a resistance R_{Th} .

Definition 7.3: Thevenin equivalent

For a linear two-terminal network of resistors and independent sources with a well-defined finite voltage-versus-current port relation, the Thevenin equivalent is an ideal voltage source V_{Th} in series with a resistance R_{Th} that produces the same terminal voltage-current relation as the original network for any external load.

Figure 7.11 defines the port and load-current reference. The open-circuit terminal voltage is

$$V_{Th} = V_{open}. \quad (7.153)$$

The Thevenin resistance can be found by deactivating independent sources: ideal voltage sources are replaced by short circuits, and ideal current sources by open circuits. The resistance seen into the terminals is then R_{Th} . Alternatively, when a finite nonzero short-circuit current I_{sc} exists,

$$R_{Th} = \frac{V_{Th}}{I_{sc}}. \quad (7.154)$$

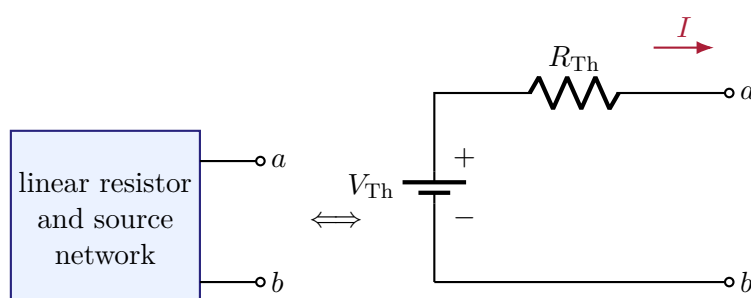


Figure 7.11: Thevenin equivalence preserves the terminal relation $V_a - V_b = V_{Th} - IR_{Th}$, with I positive out of terminal a . The open port is shown explicitly. Internal currents and dissipation need not be reproduced by the equivalent network.

Example 7.25: Thevenin equivalent of a voltage divider

A 12 V ideal source feeds a divider with $R_1 = 3.0\text{ k}\Omega$ from the source to the output node and $R_2 = 6.0\text{ k}\Omega$ from the output node to ground. Find the Thevenin equivalent seen from the output node to ground.

Solution. The open-circuit output voltage is

$$V_{Th} = 12 \frac{R_2}{R_1 + R_2} = 12 \frac{6.0}{9.0} = 8.0\text{ V}. \quad (7.155)$$

To find R_{Th} , deactivate the ideal voltage source by replacing it with a short circuit. Then R_1 and R_2 are both connected from the output node to ground, so

$$R_{Th} = R_1 \parallel R_2 = \frac{(3.0)(6.0)}{3.0 + 6.0} \text{ k}\Omega = 2.0 \text{ k}\Omega. \quad (7.156)$$

The divider therefore behaves at its output like an 8.0 V ideal source in series with 2.0 k Ω .

Why the equivalent works and when to use a test source

Let $V = V_a - V_b$ and define the port current I as leaving terminal a . In a linear resistor–source network with a well-defined port relation, linearity gives an affine response $V = A - BI$. Setting $I = 0$ identifies $A = V_{open} = V_{Th}$. Setting all independent sources to zero removes the constant term; a test source then measures the slope. If a test current I_t enters terminal a , it gives $R_{Th} = V_t/I_t$. The sign change between outward load current and inward test current explains the minus sign in $V = V_{Th} - IR_{Th}$.

Source deactivation means setting the specified source value to zero. A zero-voltage ideal source is a short; a zero-current ideal source is an open. Internal resistances remain in the network. A dependent source, if included in a more advanced model, must remain active while a test source is used; it is not set to zero merely because it is drawn as a source.

The shortcut $R_{Th} = V_{Th}/I_{sc}$ is usable when a finite nonzero short-circuit current and a meaningful ratio exist. It fails for a source-free resistive port because both numerator and denominator are zero. A test source still determines the resistance. A port containing an ideal voltage source can have $R_{Th} = 0$; an ideal current-source port does not admit a finite ordinary Thevenin representation. Conflicting ideal-source constraints or floating unconstrained nodes also need separate attention.

Example 7.26: A loading-error requirement becomes a design inequality

A sensor has $V_{Th} = 2.5 \text{ V}$ and $R_{Th} = 20 \text{ k}\Omega$. Choose a minimum meter input resistance so the measured DC voltage is within 1% of its open-circuit value.

Derivation. With R_M as the load,

$$V_M = V_{Th} \frac{R_M}{R_{Th} + R_M}, \quad \epsilon = \frac{V_{Th} - V_M}{V_{Th}} = \frac{R_{Th}}{R_{Th} + R_M}. \quad (7.157)$$

Solving the inequality before inserting numbers gives

$$R_M \geq \frac{1 - \epsilon_{max}}{\epsilon_{max}} R_{Th} = 99 R_{Th} = 1.98 \text{ M}\Omega. \quad (7.158)$$

A 2.0 M Ω input gives $V_M = 2.475 \text{ V}$ and 0.990% loading. A 10 M Ω input gives 2.495 V and 0.200% loading.

Interpretation. The often-used condition $R_M \gg R_{Th}$ becomes meaningful only after the acceptable error is specified. This is a bound on loading alone; calibration uncertainty, offset and noise require their own error budget.

7.21.2 Source transformations and Norton equivalents

A Thevenin source V_{Th} with finite positive series resistance R_{Th} has an equivalent Norton form: an ideal current source I_N in parallel with the same resistance $R_N = R_{Th}$, where

$$I_N = \frac{V_{Th}}{R_{Th}}. \quad (7.159)$$

The two forms have the same terminal current-voltage relation. For a load current I leaving the positive Thevenin terminal, the terminal voltage is

$$V = V_{\text{Th}} - IR_{\text{Th}}. \quad (7.160)$$

For the Norton form, current division gives

$$I = I_N - \frac{V}{R_N}, \quad (7.161)$$

which rearranges to

$$V = I_N R_N - IR_N. \quad (7.162)$$

The forms are equivalent if $I_N R_N = V_{\text{Th}}$ and $R_N = R_{\text{Th}}$.

Example 7.27: Thevenin-to-Norton transformation

Convert a Thevenin source with $V_{\text{Th}} = 9.0 \text{ V}$ and $R_{\text{Th}} = 3.0 \text{ k}\Omega$ to a Norton equivalent.

Solution. The Norton current is

$$I_N = \frac{V_{\text{Th}}}{R_{\text{Th}}} = \frac{9.0}{3.0 \times 10^3} = 3.0 \text{ mA}. \quad (7.163)$$

The Norton resistance is

$$R_N = R_{\text{Th}} = 3.0 \text{ k}\Omega. \quad (7.164)$$

Thus the equivalent is a 3.0 mA ideal current source in parallel with $3.0 \text{ k}\Omega$.

7.21.3 Superposition in linear DC circuits

In a linear circuit containing multiple independent sources and resistors, the response can be found by superposition. The current or voltage of interest equals the sum of the responses produced by each independent source acting alone, with all other independent sources deactivated. Deactivating an ideal voltage source means replacing it by a short circuit. Deactivating an ideal current source means replacing it by an open circuit.

Superposition applies to voltages and currents in linear circuits. It does not apply directly to power because power is quadratic in voltage or current. If two sources produce currents I_1 and I_2 in a resistor, the total power is $R(I_1 + I_2)^2$, not $RI_1^2 + RI_2^2$ in general.

Example 7.28: Superposition warning for power

Two sources acting separately would produce currents 0.30 A and 0.20 A in the same direction through a 10Ω resistor. Find the actual power and compare with the sum of separate powers.

Solution. The total current is

$$I = 0.30 + 0.20 = 0.50 \text{ A}. \quad (7.165)$$

The actual power is

$$P = I^2 R = (0.50)^2 (10) = 2.5 \text{ W}. \quad (7.166)$$

The separate powers would be

$$(0.30)^2 (10) + (0.20)^2 (10) = 0.90 + 0.40 = 1.30 \text{ W}. \quad (7.167)$$

These are not equal. Superposition applies to current, not directly to power.

7.21.4 Maximum power transfer

A Thevenin source connected to a load R_L delivers load power

$$P_L = I^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L. \quad (7.168)$$

To find the load resistance that maximises P_L , differentiate with respect to R_L :

$$P_L = V_{Th}^2 \frac{R_L}{(R_{Th} + R_L)^2}. \quad (7.169)$$

Then

$$\frac{dP_L}{dR_L} = V_{Th}^2 \frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4}. \quad (7.170)$$

Setting the numerator to zero gives

$$(R_{Th} + R_L) - 2R_L = 0, \quad (7.171)$$

so

$$R_L = R_{Th}. \quad (7.172)$$

This is the maximum-power-transfer condition for a purely resistive DC Thevenin source.

Caution

Maximum power transfer is not the same as maximum efficiency. For the simple Thevenin source and series resistance model, $R_L = R_{Th}$ gives equal power in the load and the equivalent resistance. This does not determine every internal loss of the original network. Power-transfer matching is useful in some signal and instrumentation contexts but is not how high-efficiency power distribution is normally designed.

Example 7.29: Maximum power from a sensor source

A sensor output is modelled by $V_{Th} = 2.0 \text{ V}$ and $R_{Th} = 500 \Omega$. What load resistance gives maximum power, and what is that power?

Solution. Maximum power occurs for

$$R_L = R_{Th} = 500 \Omega. \quad (7.173)$$

The current is

$$I = \frac{2.0}{500 + 500} = 2.0 \text{ mA}. \quad (7.174)$$

The load power is

$$P_L = I^2 R_L = (2.0 \times 10^{-3})^2 (500) = 2.0 \text{ mW}. \quad (7.175)$$

The maximum is meaningful for fixed V_{Th} and finite $R_{Th} > 0$. If $R_{Th} = 0$, the ideal formula has no finite maximizing load: decreasing R_L increases power without bound, revealing the ideal source's missing current limit.

Using $x = R_L/R_{Th}$,

$$\frac{P_L}{P_{\max}} = \frac{4x}{(1+x)^2}, \quad P_{\max} = \frac{V_{Th}^2}{4R_{Th}}, \quad \eta_{\text{eq}} = \frac{R_L}{R_{Th} + R_L} = \frac{x}{1+x}. \quad (7.176)$$

Figure 7.12 separates maximum power from efficient transfer in the equivalent series-source model. This efficiency is not automatically the efficiency of the original network. For example, a powered divider dissipates energy even with no external load, whereas its open-circuit Thevenin equivalent has no current and no resistor heating. Terminal equivalence preserves externally delivered power, not the internal power budget of the original circuit.

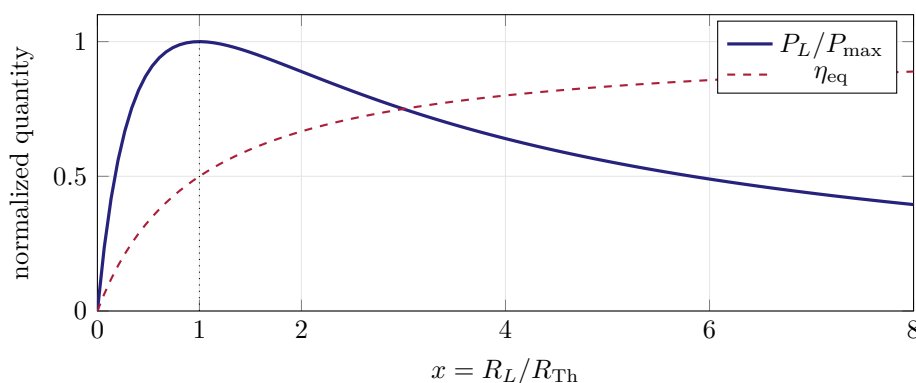


Figure 7.12: For $R_{Th} > 0$, maximum load power occurs at $R_L = R_{Th}$. The efficiency η_{eq} refers to the simple equivalent source and series resistance. It rises toward unity as the load resistance grows, while the absolute load power eventually falls toward zero.

7.21.5 Nodal analysis

Nodal analysis uses node voltages rather than branch currents as unknowns. Choose one node as ground, with potential 0. For each remaining essential node, apply the junction rule using element currents expressed in terms of node voltages. For a resistor between nodes a and b , the current from a to b is

$$I_{a \rightarrow b} = \frac{V_a - V_b}{R}. \quad (7.177)$$

Example 7.30: One-node nodal analysis

A node is connected to a 12 V source through $R_1 = 2.0 \text{ k}\Omega$, to ground through $R_2 = 3.0 \text{ k}\Omega$, and to ground through $R_3 = 6.0 \text{ k}\Omega$. Find the node voltage V .

Solution. Apply the junction rule: current entering from the source branch equals currents leaving to ground:

$$\frac{12 - V}{R_1} = \frac{V}{R_2} + \frac{V}{R_3}. \quad (7.178)$$

Substitute values in $\text{k}\Omega$ units:

$$\frac{12 - V}{2} = \frac{V}{3} + \frac{V}{6} = \frac{V}{2}. \quad (7.179)$$

Thus $12 - V = V$, so

$$V = 6.0 \text{ V}. \quad (7.180)$$

Check. The two ground resistors in parallel equal $2.0 \text{ k}\Omega$, so the circuit is a divider of two equal resistances. The result 6 V is expected.

Conductance-weighted voltages and a unique solution

For a node connected through positive resistances R_j to fixed voltages V_j , the outgoing-current convention gives

$$\sum_j \frac{V - V_j}{R_j} = 0 \quad \Rightarrow \quad V = \frac{\sum_j G_j V_j}{\sum_j G_j}, \quad G_j = \frac{1}{R_j}. \quad (7.181)$$

The node voltage is a weighted average, so it lies between the smallest and largest applied voltages. A computed value outside this range indicates an error or an additional element, such as a current source, that invalidates this restricted model. The weights are conductances because a smaller resistance constrains the node more strongly.

For several unknown nodes, KCL gives $\mathbf{G}\mathbf{v} = \mathbf{b}$. Each diagonal entry of $\mathbf{G}$ is the sum of conductances

leaving that node; the off-diagonal entry G_{jk} is minus the conductance directly joining nodes j and k . Source contributions enter $\mathbf{b}$. In a passive resistor network whose unknown-node components each connect to a fixed-potential reference, the matrix is symmetric positive definite. To see the physical reason, set all fixed potentials to zero and consider arbitrary trial node voltages x_j :

$$\mathbf{x}^T \mathbf{G} \mathbf{x} = \sum_{\text{internal edges } jk} g_{jk} (x_j - x_k)^2 + \sum_{\text{edges to reference } j} g_{j0} x_j^2. \quad (7.182)$$

Every term is nonnegative and their sum vanishes only for the zero vector under the stated connectivity condition. The equations therefore have a unique solution. A floating component with no voltage reference leaves an arbitrary common potential and makes the reduced problem singular; adding more decimal places cannot repair the missing reference.

An ideal voltage source between unknown nodes

A resistor current follows from a voltage difference, but an ideal voltage source's current is not given by V/R . If such a source joins two unknown nodes, combine them into a *supernode*. KCL for a boundary enclosing both nodes omits the internal source current, while a separate equation supplies the source voltage constraint.

Example 7.31: Solving a supernode

Node a connects to ground through $2.0 \text{ k}\Omega$ and node b through $4.0 \text{ k}\Omega$. An ideal source imposes $V_a - V_b = 6.0 \text{ V}$, with its positive terminal at a . There are no other branches. The supernode has no net current injection, so

$$\frac{V_a}{2.0 \text{ k}\Omega} + \frac{V_b}{4.0 \text{ k}\Omega} = 0, \quad V_a - V_b = 6.0 \text{ V}. \quad (7.183)$$

Thus $2V_a + V_b = 0$, giving $V_a = 2.0 \text{ V}$ and $V_b = -4.0 \text{ V}$. The resistor currents, referenced from each node to ground, are $+1.0 \text{ mA}$ and -1.0 mA . The source delivers 6.0 mW ; the resistors absorb $2.0 + 4.0 = 6.0 \text{ mW}$. The negative node voltage is essential to the physical loop current and is fully consistent with the chosen ground.

7.21.6 Mesh-current analysis

Mesh-current analysis assigns a circulating current to each mesh of a planar circuit. Shared resistors carry the algebraic difference of adjacent mesh currents. This method reduces the number of unknowns in many planar circuits.

For two meshes with clockwise mesh currents I_A and I_B and a shared resistor R_s , the current through the shared resistor in the direction of mesh A is $I_A - I_B$. The voltage drop across the shared resistor in mesh A is therefore $R_s(I_A - I_B)$; in mesh B it is $R_s(I_B - I_A)$. This is a common source of signs, but it follows from the same resistor rule.

Example 7.32: Two-mesh equations with a shared resistor

Two clockwise meshes share a resistor $R_s = 4 \Omega$. The left mesh has source 12 V and outer resistance 2Ω . The right mesh has source 6 V opposing the clockwise direction and outer resistance 3Ω . Write the mesh equations.

Solution. Left mesh:

$$12 - 2I_A - 4(I_A - I_B) = 0. \quad (7.184)$$

Right mesh, with the source a drop in the chosen traversal:

$$-6 - 3I_B - 4(I_B - I_A) = 0. \quad (7.185)$$

Equivalently,

$$6I_A - 4I_B = 12, \quad (7.186)$$

$$4I_A - 7I_B = 6. \quad (7.187)$$

Solving gives $I_A = 30/13$ A and $I_B = 6/13$ A. The shared current is $I_A - I_B = 24/13$ A in mesh A 's direction. Substitution gives $12 - 2(30/13) - 4(24/13) = 0$ in the left loop and $-6 - 3(6/13) + 4(24/13) = 0$ in the right. The 12 V source delivers $360/13$ W, the 6 V source absorbs $36/13$ W, and the three resistors dissipate $324/13$ W in total. The power balance therefore closes. Mesh currents are algebraic loop variables; a shared branch current must be reconstructed from their difference.

7.21.7 Bridge balance, unbalance and finite detector loading

A bridge compares two voltage dividers. Figure 7.13 gives its topology: R_1, R_2 form the left arm, R_3, R_4 the right arm, and R_5 connects the midpoints. When midpoint voltages differ, current in R_5 changes both divider currents. Treating each arm as an unloaded divider is then incorrect.

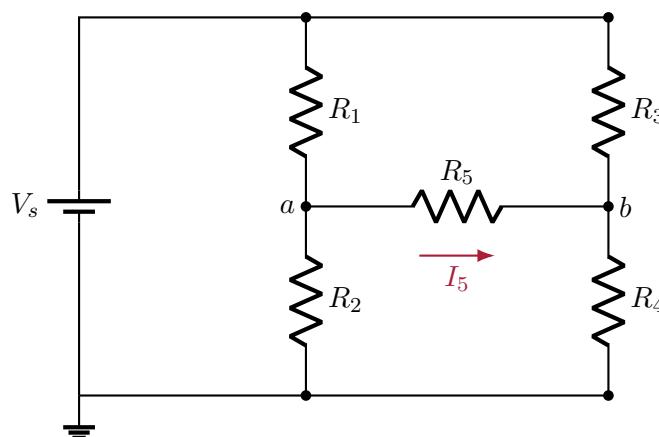


Figure 7.13: A bridge with a finite central resistance. Except at balance, the central branch changes both midpoint voltages. Balance requires $R_1/R_2 = R_3/R_4$ and gives $I_5 = 0$, independent of finite R_5 .

Example 7.33: A balanced bridge as an exact diagnostic

Let $V_s = 10$ V, $R_1 = 2$ k Ω , $R_2 = 3$ k Ω , $R_3 = 4$ k Ω , $R_4 = 6$ k Ω and $R_5 = 5$ k Ω . Before doing algebra, observe that $R_1/R_2 = R_3/R_4 = 2/3$. The unloaded midpoints are both 6 V, so connecting any finite positive R_5 produces no bridge current.

Algebraic check. With resistances in kilohms and currents in milliamperes, KCL gives

$$\frac{V_a - 10}{2} + \frac{V_a}{3} + \frac{V_a - V_b}{5} = 0, \quad (7.188)$$

$$\frac{V_b - 10}{4} + \frac{V_b}{6} + \frac{V_b - V_a}{5} = 0. \quad (7.189)$$

Equivalently, $31V_a - 6V_b = 150$ and $-12V_a + 37V_b = 150$. Substitution verifies the exact solution $V_a = V_b = 6$ V and $I_5 = 0$. A nonzero bridge current for these exact inputs is an algebraic error, not a rounding effect. In a real bridge, component mismatch or measurement uncertainty can create a small unbalance; that is a different physical model.

Example 7.34: A genuinely unbalanced bridge

Keep the same circuit but change R_4 to $3\text{ k}\Omega$. Compute the midpoint voltages and the current I_5 from a to b .

Strategy. The ratios are now unequal, so solve the two node equations with the central branch included. The first remains $31V_a - 6V_b = 150$. The second becomes

$$\frac{V_b - 10}{4} + \frac{V_b}{3} + \frac{V_b - V_a}{5} = 0 \implies -12V_a + 47V_b = 150. \quad (7.190)$$

Solving yields

$$V_a = \frac{1590}{277} \text{ V} = 5.740 \text{ V}, \quad V_b = \frac{1290}{277} \text{ V} = 4.657 \text{ V}, \quad I_5 = \frac{60}{277} \text{ mA} = 0.2166 \text{ mA}. \quad (7.191)$$

The current is positive, from the higher midpoint potential to the lower one.

Independent terminal check. Remove R_5 . The open midpoint difference is $6 - 30/7 = 12/7 \text{ V}$. With the source deactivated, the resistance between midpoints is

$$R_{ab} = (2 \parallel 3) + (4 \parallel 3) = \frac{6}{5} + \frac{12}{7} = \frac{102}{35} \text{ k}\Omega. \quad (7.192)$$

Reconnecting R_5 gives $I_5 = (12/7)/(102/35 + 5) = 60/277 \text{ mA}$, exactly matching nodal analysis. This second method tests the topology as well as the arithmetic.

7.21.8 Balanced Wheatstone bridge

A Wheatstone bridge is an important measurement circuit. Four resistive arms form a diamond, and a detector is connected across the middle. When the detector current is zero, the two middle nodes have the same potential. That condition can determine an unknown resistance with high precision because the measurement is based on a null result rather than an absolute meter calibration.

Let the left branch contain R_1 above R_2 and the right branch contain R_3 above R_x . A source is connected between the top and bottom nodes. The bridge detector connects the two middle nodes. If the detector current is zero, each side behaves as an independent voltage divider. Balance requires

$$V_s \frac{R_2}{R_1 + R_2} = V_s \frac{R_x}{R_3 + R_x}. \quad (7.193)$$

Canceling V_s and cross-multiplying gives

$$R_2(R_3 + R_x) = R_x(R_1 + R_2). \quad (7.194)$$

The R_2R_x terms cancel, leaving

$$R_x = \frac{R_2R_3}{R_1}. \quad (7.195)$$

Example 7.35: Unknown resistance from a bridge balance

A bridge is balanced when $R_1 = 100 \Omega$, $R_2 = 250 \Omega$ and $R_3 = 80 \Omega$. Find the unknown resistance R_x .

Solution. The balance condition gives

$$R_x = \frac{R_2R_3}{R_1} = \frac{(250)(80)}{100} = 200 \Omega. \quad (7.196)$$

No detector current flows at balance, so the detector's internal resistance does not affect the ideal balance condition.

7.21.9 Voltage dividers as sensor readout circuits

A voltage divider converts a resistance change into a voltage change. Suppose a sensor resistance R_s is connected in series with a fixed resistor R_f across an input voltage V_{in} , and the output is measured across R_s :

$$V_{\text{out}} = V_{\text{in}} \frac{R_s}{R_f + R_s}. \quad (7.197)$$

The sensitivity to resistance change is

$$\frac{dV_{\text{out}}}{dR_s} = V_{\text{in}} \frac{R_f}{(R_f + R_s)^2}. \quad (7.198)$$

The optimum depends on which parameters are fixed. For a specified sensor, sensitivity must be balanced against loading, self-heating and supply-current requirements.

Example 7.36: Thermistor-style divider sensitivity

A resistive sensor has nominal resistance $R_s = 10 \text{ k}\Omega$ and is used with $R_f = 10 \text{ k}\Omega$ across $V_{\text{in}} = 5.0 \text{ V}$. Find the nominal output and the local sensitivity dV_{out}/dR_s .

Solution. The nominal output is

$$V_{\text{out}} = 5.0 \frac{10}{10 + 10} = 2.5 \text{ V}. \quad (7.199)$$

The sensitivity is

$$\frac{dV_{\text{out}}}{dR_s} = 5.0 \frac{10 \times 10^3}{(20 \times 10^3)^2} = 1.25 \times 10^{-4} \text{ V}/\Omega. \quad (7.200)$$

Thus a 100Ω small-signal change produces approximately 12.5 mV change in output near this operating point.

Choosing a fixed resistor for a specified sensor

The phrase “maximum sensitivity” requires stating what is varied. For fixed nominal R_s and supply voltage, differentiating the sensitivity with respect to R_f gives

$$\frac{\partial}{\partial R_f} \left(\frac{dV_{\text{out}}}{dR_s} \right) = V_{\text{in}} \frac{R_s - R_f}{(R_s + R_f)^3}. \quad (7.201)$$

Thus the absolute sensitivity to a small resistance change is largest at $R_f = R_s$. If both resistances are scaled together while a *fractional* sensor change is fixed, the voltage response is unchanged; reducing both mainly increases current and heating. The normalized sensitivity makes this clear:

$$\frac{R_s}{V_{\text{out}}} \frac{dV_{\text{out}}}{dR_s} = \frac{R_f}{R_f + R_s}. \quad (7.202)$$

If R_s depends on a physical variable x , the measured sensitivity is

$$\frac{dV_{\text{out}}}{dx} = \frac{V_{\text{in}} R_f}{(R_f + R_s)^2} \frac{dR_s}{dx}. \quad (7.203)$$

The sign is determined by the sensor law. For example, resistance decreasing with temperature makes the output across the sensor decrease with temperature. A linear approximation to $R_s(x)$ must be restricted to its stated range; the circuit cannot make an inaccurate material model correct.

Sensor heating follows from the same circuit:

$$P_s = \frac{V_{\text{in}}^2 R_s}{(R_f + R_s)^2}. \quad (7.204)$$

If a supplied thermal resistance Θ relates temperature rise to power, the approximate self-heating error is $\Delta T \simeq \Theta P_s$ in steady state. This model assumes a fixed thermal environment and sufficiently small temperature change. Reducing excitation can improve the measurement even though it reduces voltage sensitivity.

7.21.10 Sensor-readout mini-project

A useful mini-project is to model a resistive sensor readout from first principles. The system consists of a source voltage V_{in} , a fixed resistor R_f , a sensor resistor $R_s(x)$ that depends on a physical variable x , and a measuring device with finite input resistance R_M . The goal is to predict the measured output voltage and decide whether the meter loading is acceptable.

The unloaded output across the sensor is

$$V_0(x) = V_{\text{in}} \frac{R_s(x)}{R_f + R_s(x)}. \quad (7.205)$$

With the meter connected, the lower resistance is

$$R_{\text{low}}(x) = R_s(x) \parallel R_M, \quad (7.206)$$

so the measured output is

$$V_M(x) = V_{\text{in}} \frac{R_{\text{low}}(x)}{R_f + R_{\text{low}}(x)}. \quad (7.207)$$

A good design should compare V_M and V_0 over the full expected range of x , not only at one nominal point.

Method

Suggested project tasks.

1. Choose a plausible sensor law, for example $R_s(x) = R_0(1 + \alpha x)$ over a small range.
2. Plot or tabulate $V_0(x)$ and $V_M(x)$ for several R_M values.
3. Compute the relative loading error $(V_M - V_0)/V_0$.
4. Estimate power dissipated in the sensor.
5. Discuss the tradeoff between sensitivity and power consumption.

7.21.11 Uncertainty propagation in resistor networks

Suppose a voltage divider output is

$$V_{\text{out}} = V_{\text{in}} \frac{R_2}{R_1 + R_2}. \quad (7.208)$$

If R_1 and R_2 have tolerances, the output also has uncertainty. For small independent uncertainties,

$$\sigma_V^2 = \left(\frac{\partial V_{\text{out}}}{\partial R_1} \sigma_{R_1} \right)^2 + \left(\frac{\partial V_{\text{out}}}{\partial R_2} \sigma_{R_2} \right)^2 + \left(\frac{\partial V_{\text{out}}}{\partial V_{\text{in}}} \sigma_{V_{\text{in}}} \right)^2. \quad (7.209)$$

The partial derivatives are

$$\frac{\partial V_{\text{out}}}{\partial R_1} = -V_{\text{in}} \frac{R_2}{(R_1 + R_2)^2}, \quad (7.210)$$

$$\frac{\partial V_{\text{out}}}{\partial R_2} = V_{\text{in}} \frac{R_1}{(R_1 + R_2)^2}, \quad (7.211)$$

$$\frac{\partial V_{\text{out}}}{\partial V_{\text{in}}} = \frac{R_2}{R_1 + R_2}. \quad (7.212)$$

This calculation connects circuit theory to laboratory uncertainty analysis.

Component tolerance is a bound supplied for a component or batch; it is not automatically a standard deviation. If a bound $\pm a$ is modeled by a uniform distribution, its standard uncertainty is $a/\sqrt{3}$. If the distribution is unknown, retain a worst-case bound rather than inventing a probability model.

For correlated uncertainties, add covariance terms $2(\partial V/\partial x_i)(\partial V/\partial x_j) \text{Cov}(x_i, x_j)$. In an equal-resistor divider, a common fractional drift of both resistors cancels to first order (indeed exactly for equal scaling), while differential mismatch changes the ratio. This is why matched resistor ratios can matter more than individual absolute accuracy.

Example 7.37: Divider tolerance and a common-mode change

Take $V_{\text{in}} = 5.0 \text{ V}$ and nominal $R_1 = R_2 = 10 \text{ k}\Omega$, each with a $\pm 1\%$ bound. The exact worst-case output is

$$V_{\min} = 5 \frac{9.9}{10.1 + 9.9} = 2.475 \text{ V}, \quad (7.213)$$

$$V_{\max} = 5 \frac{10.1}{9.9 + 10.1} = 2.525 \text{ V}. \quad (7.214)$$

The $\pm 25 \text{ mV}$ range is a worst-case bound. If the resistor deviations are independent and uniform over their bounds, each has $\sigma_R = 100/\sqrt{3} \Omega$. At nominal values, the derivative magnitudes are $1.25 \times 10^{-4} \text{ V}/\Omega$, so $\sigma_V = 10.2 \text{ mV}$ with an exact supply. These two uncertainty statements answer different questions.

If both resistances increase by exactly 1% together, their ratio stays one and the output stays 2.5 V . Treating that perfectly correlated change as two independent errors would overestimate the ratio error.

7.21.12 Capacitor timing and comparator thresholds

RC charging is often used to create a time delay. Suppose a capacitor charges toward $\mathcal{E}$ and a comparator switches when $V_C(t)$ reaches a threshold V_T . From

$$V_T = \mathcal{E} \left(1 - e^{-t/RC} \right), \quad (7.215)$$

we obtain

$$t = -RC \ln \left(1 - \frac{V_T}{\mathcal{E}} \right). \quad (7.216)$$

This equation is meaningful only for $0 < V_T < \mathcal{E}$ in the ideal charging model.

Example 7.38: Designing a one-second RC delay

A comparator threshold is $V_T = 3.0 \text{ V}$, the supply is $\mathcal{E} = 5.0 \text{ V}$, and a delay of 1.0 s is desired. If $C = 10 \mu\text{F}$, choose the ideal resistance R .

Solution. Use

$$t = -RC \ln \left(1 - \frac{V_T}{\mathcal{E}} \right). \quad (7.217)$$

Thus

$$R = \frac{t}{-C \ln(1 - V_T/\mathcal{E})} = \frac{1.0}{-(10 \times 10^{-6}) \ln(1 - 3/5)}. \quad (7.218)$$

Since $\ln(0.4) = -0.916$,

$$R = 1.09 \times 10^5 \Omega = 109 \text{ k}\Omega. \quad (7.219)$$

A standard resistor near $110 \text{ k}\Omega$ would be chosen, subject to tolerance and leakage considerations.

7.21.13 Nonideal capacitors: leakage and equivalent series resistance

Real capacitors are not ideal. A leakage resistance in parallel allows slow discharge even when no external path is intended. Equivalent series resistance, abbreviated ESR, contributes heating and modifies fast transients. Dielectric absorption can cause a capacitor that was discharged briefly to recover some voltage. Breakdown voltage limits safe operation.

A simple leakage model places a resistance R_L in parallel with the capacitance. If a capacitor initially at voltage V_0 is left open except for leakage, then

$$V(t) = V_0 e^{-t/(R_L C)}. \quad (7.220)$$

Large R_L gives a long self-discharge time. In precision timing circuits, leakage can dominate the error budget.

Caution

Large capacitance values do not automatically produce accurate long delays. Electrolytic capacitors may have broad tolerances and leakage. For precision timing, component tolerances, temperature dependence and input bias currents may be more important than the ideal formula.

Leakage changes both the final value and the time constant

A leakage resistor R_ℓ in parallel with C , driven from V_s through R , has node equation

$$C\dot{V}_C = \frac{V_s - V_C}{R} - \frac{V_C}{R_\ell}. \quad (7.221)$$

Consequently,

$$V_f = V_s \frac{R_\ell}{R + R_\ell}, \quad \tau = (R \parallel R_\ell)C. \quad (7.222)$$

An input resistance across the capacitor has the same effect and combines in parallel with leakage. With $V_s = 5\text{ V}$, $R = 1\text{ M}\Omega$, $R_\ell = 4\text{ M}\Omega$ and $C = 10\text{ }\mu\text{F}$, the final voltage is 4 V and the time constant is 8 s , rather than 5 V and 10 s . A 4.5 V threshold is never reached. Simply adjusting RC to compensate the delay cannot repair an unreachable final voltage.

For a simple series ESR model, let u denote the voltage of the ideal capacitance inside the component. When it is charged from V_s through R and ESR r_C , $(R + r_C)C\dot{u} + u = V_s$. The externally measured component voltage is $u + r_C I$ in the charging direction. The internal state u remains continuous with finite current, but the terminal voltage can change abruptly by a resistive contribution when the current changes. This distinction prevents misuse of ideal-capacitor continuity for a whole nonideal component.

7.21.14 Square-wave drive of an RC circuit

If a square wave alternates between 0 and V_s , an RC circuit may not reach its final value during each half-period. The response depends on the ratio of the half-period to the time constant. If the half-period is much longer than τ , the capacitor nearly fully charges and discharges each cycle. If the half-period is much shorter than τ , the capacitor voltage changes only slightly during each half-cycle and behaves like a smoothed version of the input.

This is the beginning of filter physics. A resistor-capacitor network can act as a low-pass or high-pass filter depending on where the output is taken. The full sinusoidal steady-state treatment belongs to the AC chapter, but the time-domain intuition begins here.

Example 7.39: Partial charging during a pulse

An initially uncharged capacitor in an RC circuit with $\tau = 10$ ms is connected to 5.0 V for 5.0 ms. What voltage does it reach by the end of the pulse?

Solution. The charging formula gives

$$V_C = 5.0 \left(1 - e^{-5.0/10} \right) = 5.0(1 - e^{-0.5}). \quad (7.223)$$

Since $e^{-0.5} = 0.607$,

$$V_C = 1.97 \text{ V}. \quad (7.224)$$

The capacitor does not come close to 5 V because the pulse duration is only half a time constant.

Deriving the periodic steady state

A single pulse starting from zero and a repeated pulse train are different initial-value problems. Let the source alternate between V_H for duration T_H and V_L for duration T_L , through the same R to C . Define $a = e^{-T_H/\tau}$ and $b = e^{-T_L/\tau}$. In the periodic steady state, the endpoint voltages satisfy

$$V_{\max} = V_H + (V_{\min} - V_H)a, \quad (7.225)$$

$$V_{\min} = V_L + (V_{\max} - V_L)b. \quad (7.226)$$

Solving these simultaneous equations gives

$$V_{\max} = \frac{V_H(1 - a) + aV_L(1 - b)}{1 - ab}, \quad (7.227)$$

$$V_{\min} = \frac{V_L(1 - b) + bV_H(1 - a)}{1 - ab}. \quad (7.228)$$

The deviation from this repeating waveform is multiplied by $ab = e^{-(T_H+T_L)/\tau}$ each cycle, so the stable periodic state is approached exponentially.

For equal half-periods, $V_L = 0$ and $V_H = V_s$, $a = b = \alpha$ and

$$V_{\max} = \frac{V_s}{1 + \alpha}, \quad V_{\min} = \frac{\alpha V_s}{1 + \alpha}, \quad \Delta V_{pp} = V_s \frac{1 - \alpha}{1 + \alpha}. \quad (7.229)$$

When $T_H \gg \tau$, $\alpha \rightarrow 0$ and the capacitor almost reaches each source level. When $T_H \ll \tau$, $\alpha \simeq 1 - T_H/\tau$ and the ripple is approximately $V_s T_H/(2\tau)$. Integrating $\tau \dot{V}_C + V_C = V_s(t)$ over a complete settled cycle gives $\langle V_C \rangle = \langle V_s \rangle$ because the capacitor returns to its starting state.

Example 7.40: Ripple after many pulses

A source alternates between 0 and 5.0 V with 5.0 ms in each state. For $\tau = 10$ ms, $\alpha = e^{-0.5} = 0.60653$. The settled endpoints are

$$V_{\max} = 3.1123 \text{ V}, \quad V_{\min} = 1.8877 \text{ V}, \quad \Delta V_{pp} = 1.2246 \text{ V}. \quad (7.230)$$

The first pulse from an uncharged state reaches only 1.9673 V; subsequent pulses begin above zero and build up to the periodic response in Figure 7.14. Using the first-pulse formula for every cycle would discard the capacitor's memory.

7.21.15 Stability of explicit Euler integration

For the discharge equation

$$\frac{dq}{dt} = -\frac{q}{\tau}, \quad (7.231)$$

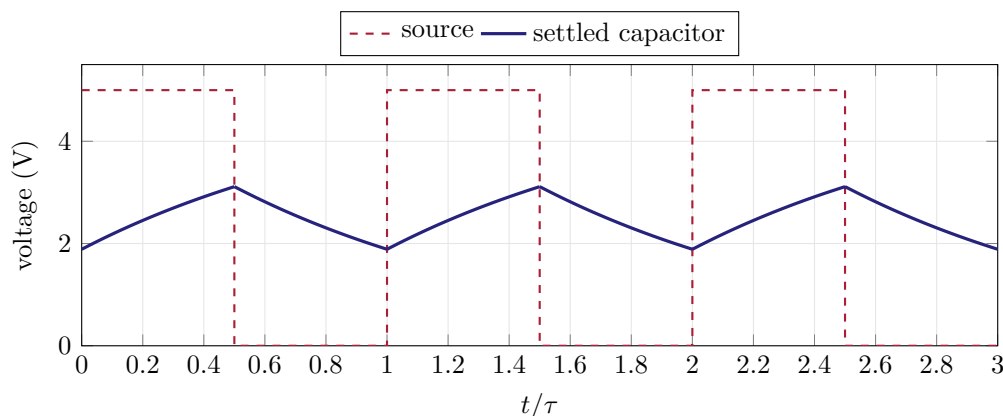


Figure 7.14: Periodic steady state for a 0–5 V square-wave input with half-period 0.5τ . The capacitor voltage is continuous at every transition and varies between 1.888 and 3.112 V. The plotted time origin starts a high interval after startup transients have decayed.

explicit Euler gives

$$q_{n+1} = q_n \left(1 - \frac{\Delta t}{\tau} \right). \quad (7.232)$$

If $\Delta t < \tau$, the numerical solution decays monotonically. If $\tau < \Delta t < 2\tau$, the numerical solution alternates sign while decaying, which is unphysical for capacitor charge in this simple discharge problem. If $\Delta t > 2\tau$, the magnitude grows, producing numerical instability. Thus a method can fail even when the differential equation is physically stable.

Example 7.41: Euler sign-alternation demonstration

Let the normalized initial charge be $q_0/Q_0 = 1$ and let $\tau = 1$ s. Use explicit Euler with $\Delta t = 1.5$ s for two discharge steps.

Solution. The update factor is

$$1 - \frac{\Delta t}{\tau} = 1 - 1.5 = -0.5. \quad (7.233)$$

Thus

$$q_1/Q_0 = -0.5, \quad q_2/Q_0 = (-0.5)(-0.5) = 0.25. \quad (7.234)$$

The exact discharge never changes sign, so the oscillation in sign is a numerical artifact. A smaller time step or a more appropriate integration method is needed.

Accuracy, monotonicity and energy are different tests

Let $x = V_C - V_f$ and $h = \Delta t/\tau$. The exact update is $x_{n+1} = e^{-h}x_n$. Explicit Euler uses $x_{n+1} = (1 - h)x_n$, which is stable in magnitude for $0 < h < 2$ but monotone with the correct sign only for $0 < h < 1$. At $h = 1$ it reaches zero in one step, a stable but inaccurate result. At $h = 2$ it alternates forever; for $h > 2$ it grows. Stability alone therefore does not ensure a physically faithful trajectory.

Backward Euler evaluates the derivative at the new state:

$$\frac{x_{n+1} - x_n}{\Delta t} = -\frac{x_{n+1}}{\tau} \implies x_{n+1} = \frac{x_n}{1 + h}. \quad (7.235)$$

It is stable and monotone for any positive h , but large steps still distort the time scale. At $t = \tau$ with ten steps, the normalized discharge predictions are $0.9^{10} = 0.34868$ (explicit), $(1/1.1)^{10} = 0.38554$

(backward), and $e^{-1} = 0.36788$ (exact). Halving the step and comparing errors is an accuracy test that an absence of numerical blow-up cannot replace.

```
# Reproducible standard-library Python: normalized discharge
from math import exp
for n in (10, 20, 40, 80):
    h = 1.0/n
    explicit = (1.0-h)**n
    backward = (1.0+h)**(-n)
    exact = exp(-1.0)
    print(n, explicit, backward,
          abs(explicit-exact), abs(backward-exact))
```

For a passive discharge, also inspect stored energy $U_n = CV_n^2/2$. Explicit Euler multiplies it by $(1 - h)^2$, while backward Euler multiplies it by $(1 + h)^{-2}$. An energy that decays numerically is a useful check, but does not detect the unphysical sign alternation for $1 < h < 2$. A trustworthy computation needs both the state-variable behavior and energy behavior to agree with the physical problem.

7.21.16 Oscilloscope input as a circuit element

An oscilloscope input is not an invisible observer. A common model is a resistance, such as $1\text{ M}\Omega$, in parallel with a capacitance, such as tens of picofarads. At DC, the resistance dominates loading. For fast transients, the input capacitance can affect rise times and measured waveforms. A probe can reduce this loading by increasing effective input resistance and reducing effective capacitance, but probe compensation must be correct.

In this chapter we only need the DC lesson: measurement devices are circuit elements. If the resistance being measured is comparable to the meter input resistance, the act of measurement changes the circuit. In later AC and signal chapters, the capacitive loading becomes equally important.

Example 7.42: Oscilloscope input resistance loading

A sensor can be modelled as a Thevenin source with $V_{\text{Th}} = 1.0\text{ V}$ and $R_{\text{Th}} = 2.0\text{ M}\Omega$. It is measured by an oscilloscope input resistance of $1.0\text{ M}\Omega$. What voltage is measured in the DC model?

Solution. The oscilloscope input is the load. Therefore

$$V_{\text{meas}} = V_{\text{Th}} \frac{R_L}{R_{\text{Th}} + R_L} = 1.0 \frac{1.0}{2.0 + 1.0} = 0.333\text{ V}. \quad (7.236)$$

The measurement is far below the open-circuit sensor voltage because the input resistance is not large compared with the source resistance.

7.21.17 Pull-up and pull-down resistors

Digital input pins should not be left floating. A pull-up resistor connects a signal line weakly to a positive supply so that the default logic level is high when no switch or transistor pulls it low. A pull-down resistor connects the line weakly to ground so that the default logic level is low. The resistor must be small enough to define the input against leakage and noise, but large enough to avoid wasting power when the line is pulled to the opposite state.

If a pull-up resistor R_P connects a line to V_s and a switch closes the line to ground, the current

through the resistor is approximately

$$I = \frac{V_s}{R_P}. \quad (7.237)$$

The static power wasted while the switch is closed is

$$P = \frac{V_s^2}{R_P}. \quad (7.238)$$

Example 7.43: Choosing a pull-up current

A 3.3 V digital input uses a 10 k Ω pull-up resistor. Find the current and resistor power when the input is pulled to ground.

Solution. The current is

$$I = \frac{3.3}{10 \times 10^3} = 0.33 \text{ mA}. \quad (7.239)$$

The power is

$$P = \frac{3.3^2}{10 \times 10^3} = 1.09 \text{ mW}. \quad (7.240)$$

This is small for many applications, but in ultra-low-power systems even such currents may matter.

A pull-up also charges the input and wiring capacitance. If the input capacitance is C_{in} and leakage is negligible, release of the grounding switch gives $V(t) = V_s(1 - e^{-t/(R_P C_{\text{in}})})$. The 10–90% rise time is $\ln 9 R_P C_{\text{in}} \simeq 2.197 R_P C_{\text{in}}$. Thus a large resistor reduces low-state power but slows the transition. A constant leakage current I_ℓ toward ground also lowers the final voltage to $V_s - I_\ell R_P$. Problem 7.43 combines these three design constraints quantitatively.

7.21.18 Beyond one time constant: two coupled RC stages

A circuit with two independent capacitor voltages generally has two relaxation modes. It cannot always be described by finding a single “total RC .” Consider Figure 7.15: a source V_s feeds R_1 to node v_1 , then R_2 to node v_2 ; capacitors C_1, C_2 connect the corresponding nodes to ground. No buffer separates the stages, so the second stage loads the first.

KCL gives

$$C_1 \dot{v}_1 = \frac{V_s - v_1}{R_1} - \frac{v_1 - v_2}{R_2}, \quad (7.241)$$

$$C_2 \dot{v}_2 = \frac{v_1 - v_2}{R_2}. \quad (7.242)$$

For positive finite resistances and capacitances, the final DC state is $v_1 = v_2 = V_s$. Subtracting that state gives a homogeneous matrix equation for the deviations. For $R_1 = R_2 = R$, $C_1 = C_2 = C$, define $u_j = v_j - V_s$ and $s = t/(RC)$. Then

$$\frac{d}{ds} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (7.243)$$

A mode proportional to $e^{-\lambda s}$ exists if

$$(2 - \lambda)(1 - \lambda) - 1 = 0, \quad \lambda_{\pm} = \frac{3 \pm \sqrt{5}}{2}. \quad (7.244)$$

The slow and fast time constants are $RC/\lambda_- = 2.618RC$ and $RC/\lambda_+ = 0.382RC$. The slow response is longer than the time constant of either isolated stage because the stages interact.

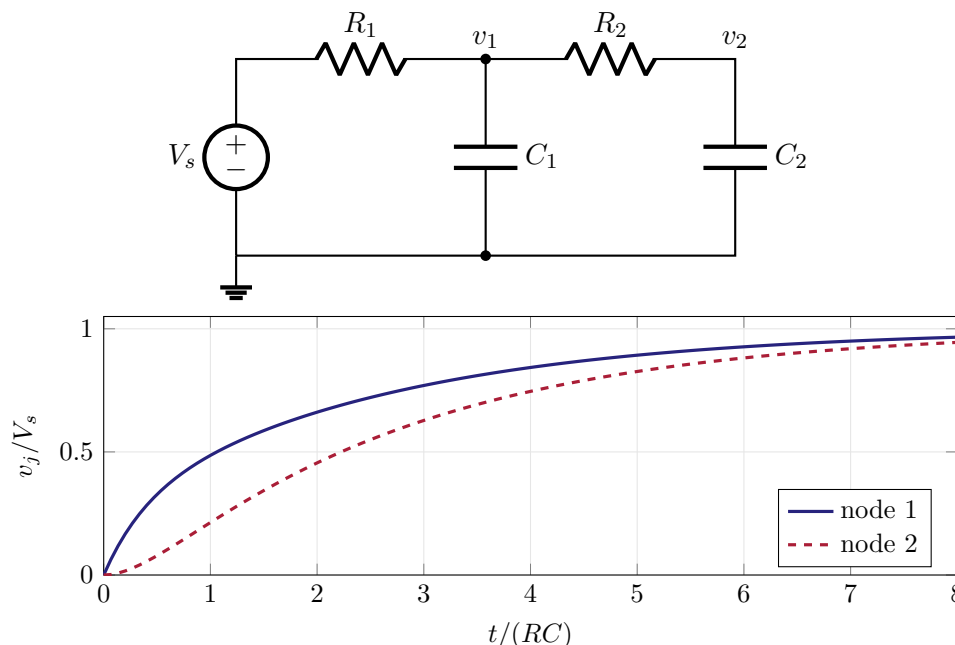


Figure 7.15: Two coupled stages with $R_1 = R_2 = R$ and $C_1 = C_2 = C$, initially uncharged, respond to a positive source step. Node 2 initially has zero slope and approaches the final voltage through two relaxation modes. These are calculated curves from the coupled node equations.

For a step from zero to constant V_s with initially uncharged capacitors, write $a = \lambda_-$ and $b = \lambda_+$. Solving the initial conditions yields

$$\frac{v_2}{V_s} = 1 - \frac{be^{-as} - ae^{-bs}}{b - a}, \quad (7.245)$$

$$\frac{v_1}{V_s} = 1 - \frac{(b-1)e^{-as} + (1-a)e^{-bs}}{b - a}. \quad (7.246)$$

At $t = 0^+$, both capacitor voltages are zero. Consequently $\dot{v}_2(0^+) = 0$, while $\dot{v}_1(0^+) = V_s/(RC)$. The delayed initial response at node 2 is a strong check on the formulas and on a numerical implementation. At long times both exponentials vanish and both voltages approach V_s .

With the source set to zero, the stored energy satisfies

$$\frac{d}{dt} \left(\frac{1}{2} C_1 v_1^2 + \frac{1}{2} C_2 v_2^2 \right) = -\frac{v_1^2}{R_1} - \frac{(v_1 - v_2)^2}{R_2} \leq 0. \quad (7.247)$$

This gives a physical passivity test for a simulation: energy cannot spontaneously grow in a source-free resistor–capacitor network. The matrix description extends the conservation-law approach; it does not replace it.

Method

Mini-project: when is a single exponential adequate? Use $R = 10 \text{ k}\Omega$ and $C = 10 \mu\text{F}$, so $RC = 0.100 \text{ s}$. Integrate the two node equations from $v_1 = v_2 = 0$ after a 5 V step. Compare with the two-mode formulas at 0.1, 0.3 and 0.6 s. Fit only the late-time residual to one exponential and compare its fitted time constant with 0.2618 s. Explain why extending that late-time fit back to $t = 0$ cannot reproduce the zero initial slope at node 2.

7.21.19 Limits and connections to later chapters

Large currents can overheat resistors and conductors, and a charged capacitor retains energy after its supply is disconnected. The energy $CV^2/2$ and initial discharge power V^2/R answer different

questions and must both be considered in a laboratory model. The examples here support supervised low-voltage teaching; they do not specify component approval or installation rules.

Inductors introduce current as another energy-storage state, and LC/RLC networks combine magnetic and electric energy. Frequency-domain methods later express the same linear dynamics through impedance and phase. At still shorter time scales, distributed electromagnetic fields replace ideal equipotential wires. The transferable method remains the one used here: identify the state and reference variables, apply conservation with the correct element laws, solve, and test the physical limits of the result.

7.22 Practice problems

Use ideal wires, ohmic resistors and constant source values unless stated otherwise. Numerical values specify teaching models, not measurements or product specifications. Problems marked $\star$ use the enrichment section. Define reference directions before solving. Selected solutions follow the complete set; attempt the problems before reading them.

A. Concepts, topology and interpretation

- 7.1. Explain how a battery maintains current even though an electrostatic field cannot do nonzero net work around a closed path. Distinguish emf, terminal voltage and energy supplied per second.
- 7.2. A wire connects nodes a and c directly. One resistor connects a to b and another connects b to c . Are the resistors in series or parallel as viewed between a and b ? Explain using node identities rather than the drawing's shape.
- 7.3. Node potentials are $V_a = 12\text{ V}$, $V_b = 5\text{ V}$ and $V_c = 0$. Re-express them with node b as zero. What happens to the current through a resistor between a and c ?
- 7.4. A branch current defined into the positive terminal of a 6 V ideal source is -0.20 A . State its actual direction and whether the source supplies or absorbs power. Give the signed absorbed power.
- 7.5. An open switch separates two parts of a powered circuit. Is its current necessarily zero? Is its voltage necessarily zero? Contrast with an ideal closed switch and identify the missing information needed to determine its current.
- 7.6. Explain why $I = C\dot{V}_C$ is compatible with an insulating capacitor dielectric. Give a switching situation in which calling a capacitor an “initial short circuit” would be wrong.
- 7.7. Two resistors R and $4R$ are first connected in series and then in parallel across the same ideal voltage source. Rank their powers in each arrangement and explain why the ranking reverses.
- 7.8. A large circuit is driven by pulses with a slow repetition rate but very fast edges. Which time scale belongs in the lumped-model check? Explain why slow repetition alone is insufficient.

B. Sources, networks and power

- 7.9. A source with $\mathcal{E} = 9.0\text{ V}$ and $r = 1.0\ \Omega$ drives $R = 8.0\ \Omega$. Find the current, terminal voltage, internal heating and load power. Check the energy balance over 60 s .
- 7.10. A source gives 11.8 V at 0.50 A and 11.2 V at 2.00 A . Infer $\mathcal{E}$ and r . Predict the current and terminal voltage with a $5.6\ \Omega$ load, and discuss one limitation of extrapolating this model.
- 7.11. Two ideal sources, 15 V and 9 V , oppose each other in a loop containing $3\ \Omega$. Define current positive in the direction driven by the smaller source. Find the signed current and the power supplied or absorbed by each source.
- 7.12. A 12 V ideal source drives $3\ \Omega$ in series with $6\ \Omega$ and $12\ \Omega$ in parallel. Find all currents, voltage drops and powers. Sketch the reduction and then reconstruct the branch quantities.
- 7.13. A 12.0 V ideal supply feeds a $5.0\ \Omega$ load through two leads, each $0.25\ \Omega$. Find the load voltage, cable heating and fraction of supplied power reaching the load. What changes if both lead resistances are halved?

- 7.14.** A $1.0\text{ k}\Omega$ resistor with tolerance $\pm 5\%$ is connected to a supply bounded by $15.0\text{ V} \pm 2\%$. Find the worst-case power. Under a stated design rule allowing at most 50% of rated power, is a 0.50 W part adequate? Do not interpret this rule as a universal component specification.
- 7.15.** A source with $\mathcal{E} = 6.0\text{ V}$ and $r = 0.50\ \Omega$ initially drives one $5.5\ \Omega$ load. A second identical load is added in parallel. Find the change in the first load's current and in total source current.
- 7.16.** A hypothetical pack delivers 0.40 A for 5.0 h with approximately constant terminal voltage 11.5 V . Find delivered charge in ampere-hours and coulombs and energy in watt-hours and joules. Why would a capacity label alone not determine the energy precisely?

C. Kirchhoff equations and measurement

- 7.17.** A student chooses a clockwise current I through two series resistors $2\ \Omega$ and $4\ \Omega$ and a source that is a 6 V drop during the clockwise walk. The student writes $-6 + 2I + 4I = 0$. Diagnose the inconsistency and find the physically correct signed current.
- 7.18.** In Figure 7.5, use $\mathcal{E}_1 = 10\text{ V}$, $\mathcal{E}_2 = 4\text{ V}$, $R_1 = 2\ \Omega$, $R_2 = 4\ \Omega$ and $R_3 = 4\ \Omega$. Determine all signed currents, $V_a - V_b$, and the complete power balance.
- 7.19.** For the same topology with positive resistances and $\mathcal{E}_1 > 0$, derive the value of $\mathcal{E}_2$ that makes $I_2 = 0$. Does zero current imply that the right branch can have no source voltage?
- 7.20.** Starting from $V_a = 0$, a path crosses a 9 V source from minus to plus, then a $2\text{ k}\Omega$ resistor in the direction of a 1 mA current, then a $3\text{ k}\Omega$ resistor opposite to a 0.50 mA current. Find the final potential and explain why this path need not be a complete loop.
- 7.21.** An ammeter with $R_A = 0.20\ \Omega$ is inserted into a circuit whose total original resistance is $10.0\ \Omega$ and whose source voltage is fixed. Find the fractional current disturbance. Derive the general upper bound on R_A for a permitted disturbance ϵ .
- 7.22.** A divider consists of two $10\text{ k}\Omega$ resistors across 5.0 V . A $10\text{ k}\Omega$ meter is connected across the lower resistor. Find the reading, fractional loading error, source current and meter current.
- 7.23.** A meter movement has $I_g = 100\ \mu\text{A}$ and $R_g = 1.0\text{ k}\Omega$. Find the parallel shunt for a 100 mA full-scale ammeter and the series resistance for a 10 V full-scale voltmeter. State which total input resistance is low and which is high.
- 7.24.** A $0.100\ \Omega$ resistor is measured using 0.500 A . Each current lead has $0.050\ \Omega$. Compare the voltage measured at the supply-side terminals with the voltage measured directly across the resistor using separate ideal sense leads. What resistances would be inferred?

D. Transients, energy and data

- 7.25.** A $10\ \mu\text{F}$ capacitor initially uncharged is connected to 5.0 V through $100\text{ k}\Omega$. Find τ , $V_C(1.0\text{ s})$, $I(0^+)$ and $I(1.0\text{ s})$. Check the loop voltage at 1.0 s .
- 7.26.** Derive the time to reach 90% of the final voltage during charging and the time to fall to 10% during discharge. Are they equal for the same τ ? Find the time for the *energy* of a discharging capacitor to fall to 10%.
- 7.27.** A normalized discharge graph has value 0.60 at $t = 1.0\text{ s}$. Infer τ , the initial normalized slope and the value at 3.0 s . Explain which graphical feature distinguishes a true exponential from a straight-line discharge.

- 7.28.** A $20\ \mu\text{F}$ capacitor starts at $8.0\ \text{V}$ and is connected with the same polarity to an ideal $3.0\ \text{V}$ source through $50\ \text{k}\Omega$. Find $V_C(t)$ and the signed current into its positive plate. Find the total resistor heat, capacitor energy change and signed source work.
- 7.29.** A $10\ \text{V}$ source feeds $R_1 = 2\ \text{k}\Omega$ and $R_2 = 3\ \text{k}\Omega$ in series. At $t = 0$ a $100\ \mu\text{F}$ capacitor initially at $1\ \text{V}$ is connected across R_2 . Find V_f , τ , the initial capacitor current and $V_C(0.24\ \text{s})$.
- 7.30.** A $220\ \mu\text{F}$ capacitor charges from zero to $9.0\ \text{V}$ through a resistor. Find the source work, final stored energy and resistor heat. At what time, in units of τ , is half the final stored energy present?
- 7.31.** A discharge gives $4.0\ \text{V}$ at $t = 0$ and $1.5\ \text{V}$ at $t = 3.0\ \text{s}$. Estimate τ . Derive its first-order standard uncertainty if each voltage has independent uncertainty $0.020\ \text{V}$ and timing uncertainty is negligible.
- 7.32.** Reproduce the fits to Table 7.1 with and without subtracting $0.100\ \text{V}$. Explain the bias and why a real late-time point may deserve less weight in a logarithmic fit.
- 7.33.** A $100\ \text{k}\Omega$ resistor discharges $100\ \text{nF}$. A probe modeled as $1.0\ \text{M}\Omega$ in parallel with $10\ \text{nF}$ is attached across the capacitor. Calculate the apparent time constant and compare it with the undisturbed value. Does agreement demonstrate negligible probe loading?
- 7.34.** For normalized discharge $x' = -x/\tau$, use explicit Euler with $\Delta t = 0.5\tau$, 1.5τ and 2.5τ . Compute the first four values from $x_0 = 1$. Distinguish accuracy, monotonicity and stability, and compare with backward Euler.

E. Engineering and advanced challenges

- 7.35.** ★ Find the Thevenin and Norton equivalents of a $10\ \text{V}$ divider with $R_1 = 2\ \text{k}\Omega$ and $R_2 = 3\ \text{k}\Omega$. With a $1.8\ \text{k}\Omega$ load, find terminal voltage and load power by either equivalent.
- 7.36.** ★ For $V_{\text{Th}} = 6\ \text{V}$ and $R_{\text{Th}} = 50\ \Omega$, find the load and power at maximum power transfer. Find the smallest load resistance that gives 90% efficiency in the equivalent series-source model. Calculate its power and explain why a general original network may have a different internal efficiency.
- 7.37.** ★ A sensor varies from 8 to $12\ \text{k}\Omega$ and forms the lower arm of a divider with $R_f = 10\ \text{k}\Omega$ and $V_s = 5\ \text{V}$. Find the unloaded voltage range. Determine the minimum meter resistance that keeps loading below 1% throughout the range.
- 7.38.** ★ In Figure 7.13, set $R_1 = R_2 = R_3 = R$, $R_4 = R(1 + \delta)$ and leave the central branch open. Derive $V_a - V_b$ exactly and to first order in small δ . What is the balance condition? Find the output resistance at $\delta = 0$.
- 7.39.** ★ Solve the unbalanced bridge of Example 7.34 with R_5 removed, then with $R_5 = 5\ \text{k}\Omega$. Explain why $V_a - V_b$ decreases under detector loading, and check the result using a Thevenin equivalent between the midpoints.
- 7.40.** ★ Repeat the supernode example with $V_a - V_b = 9\ \text{V}$, $R_a = 3\ \text{k}\Omega$ to ground and $R_b = 6\ \text{k}\Omega$ to ground. Find node voltages, branch current and source power.
- 7.41.** ★ A $5\ \text{V}$ source charges $10\ \mu\text{F}$ through $1\ \text{M}\Omega$. Leakage of $4\ \text{M}\Omega$ is parallel to the capacitor, initially uncharged. Find V_f , τ and the time to reach $3.0\ \text{V}$. Can it reach $4.5\ \text{V}$?
- 7.42.** ★ A source alternates between 0 and $5\ \text{V}$ for equal $5\ \text{ms}$ intervals, with $\tau = 10\ \text{ms}$. Derive the settled endpoints and peak-to-peak ripple. How many complete cycles reduce a startup deviation to less than 1% of its initial value?

- 7.43.** ★ A 3.3 V input has total capacitance 100 pF and worst-case leakage $1\ \mu\text{A}$ toward ground. The required high voltage is at least 3.0 V, the low-state switch current must not exceed 1 mA and the 10–90% rise time relative to the final high value must be at most $2.0\ \mu\text{s}$. Determine the allowed range of pull-up resistance in the stated ideal model.
- 7.44.** ★ Reproduce the exact worst-case and standard-uncertainty calculations of Example 7.37. Then explain the result for perfectly correlated equal fractional changes of the two resistors.
- 7.45.** ★ Derive the two eigenvalues for the equal-component network in Figure 7.15. Check the initial slopes and the final values of both step-response formulas. Why is fitting the whole response by one exponential inadequate?
- 7.46.** ★ For an arbitrary resistor network with fixed source terminal potentials, multiply each KCL equation by its node potential and sum over nodes. Show how paired resistor terms become $G_{ab}(V_a - V_b)^2$ and derive total source power equals total resistor heating. State the reference convention for source power.
- 7.47.** ★ A $10\ \text{k}\Omega$ sensor is used with a $10\ \text{k}\Omega$ fixed resistor across 5 V. Assume its thermal resistance to the surroundings is 400 K/W. Estimate self-heating using the nominal resistance. How does it change if both resistances increase by a factor of ten at the same excitation voltage? Discuss what remains unchanged in the response to the same fractional resistance change.
- 7.48.** ★ Design a low-voltage measurement of an unknown capacitance near $10\ \mu\text{F}$ using a known resistor, a switchable source and a voltage recorder. Choose a time constant near 1 s, state the fit model, account for finite input resistance, identify independent measurements and propose a residual check that would reject the single-exponential model.

7.23 Selected detailed solutions

These solutions emphasize model choice, reference conventions and checks. Intermediate digits are retained to make the checks reproducible; final precision should reflect the stated inputs.

Problem 7.10: source characterization

Use the same signed current in both measurements, positive out of the source. The equations are $11.8 = \mathcal{E} - 0.50r$ and $11.2 = \mathcal{E} - 2.00r$, with voltages in volts, currents in amperes and r in ohms. Subtracting removes the intercept:

$$r = \frac{11.8 - 11.2}{2.00 - 0.50} = 0.400 \Omega, \quad \mathcal{E} = 11.8 + 0.50(0.400) = 12.0 \text{ V}. \quad (7.248)$$

The 5.6Ω load is in series with this internal resistance, so

$$I = \frac{12.0}{5.6 + 0.4} = 2.00 \text{ A}, \quad V_{\text{terminal}} = IR = 11.2 \text{ V}. \quad (7.249)$$

This agrees with one of the original load-test points. The source supplies 24.0 W , the load absorbs 22.4 W and internal heating is 1.6 W . The inferred resistance is an operating-range parameter; it need not remain constant under heating, large current or a different state of charge. There is no justification for treating the extrapolated short-circuit current as a safe test current.

Problem 7.11: a source absorbs energy

Choose the given current direction, driven by the smaller source, and traverse the loop in that direction. The smaller source is a rise, the larger source a drop, and the resistor a signed drop:

$$9 - 15 - 3I = 0 \implies I = -2.0 \text{ A}. \quad (7.250)$$

The actual current is 2.0 A in the opposite direction. It leaves the positive terminal of the 15 V source, which supplies 30 W . It enters the positive terminal of the 9 V source, which absorbs 18 W . The resistor absorbs $I^2 R = 12 \text{ W}$. The signed absorbed-power sum is $-30 + 18 + 12 = 0$. Reporting both sources as delivering positive power would violate this check even if the current magnitude were correct.

Problem 7.12: reduce, then reconstruct

The parallel group is $R_p = 6 \times 12 / (6 + 12) = 4 \Omega$, so the input resistance is $3 + 4 = 7 \Omega$. The source current is $12/7 \text{ A}$. The series-resistor drop is $36/7 \text{ V}$ and the common parallel voltage is $48/7 \text{ V}$. The branch currents are

$$I_6 = \frac{48/7}{6} = \frac{8}{7} \text{ A}, \quad I_{12} = \frac{48/7}{12} = \frac{4}{7} \text{ A}. \quad (7.251)$$

Their sum is $12/7 \text{ A}$, as required at the junction. Resistor powers are

$$P_3 = 3(12/7)^2 = 8.816 \text{ W}, \quad (7.252)$$

$$P_6 = 6(8/7)^2 = 7.837 \text{ W}, \quad (7.253)$$

$$P_{12} = 12(4/7)^2 = 3.918 \text{ W}. \quad (7.254)$$

The total 20.571 W equals source power $12(12/7) \text{ W}$. The equivalent resistance alone supplies the source current; the original topology is needed to assign individual powers. These dissipations also show why resistance values alone are insufficient to select actual component ratings.

Problem 7.14: worst-case component power

For fixed voltage, $P = V^2/R$ increases when voltage rises and resistance falls. The worst allowed combination is therefore $V_{\max} = 15.3 \text{ V}$ and $R_{\min} = 950 \Omega$, not two deviations with the same sign:

$$P_{\max} = \frac{15.3^2}{950} = 0.2464 \text{ W}. \quad (7.255)$$

The specified 50% rule requires a rating of at least $2P_{\max} = 0.4928 \text{ W}$. A 0.50 W part meets this particular mathematical bound by a small margin. The conclusion is conditional on the stated model and rule; actual temperature, mounting and voltage ratings would still be checked against the chosen component specification. Tolerance bounds have been combined as worst cases, without assuming a probability distribution.

Problem 7.17: diagnose before solving

The resistor contributions must be $-2I$ and $-4I$ when the traversal follows the *chosen* current direction. This remains true even if that current later turns out to be negative. The consistent equation is

$$-6 - 2I - 4I = 0 \implies I = -1.0 \text{ A}. \quad (7.256)$$

Thus the actual current is counterclockwise. The student mixed a clockwise source traversal with resistor signs appropriate to the opposite reference. Reversing the entire correct loop equation gives $+6 + 2I + 4I = 0$, the same solution. A traversal reversal changes every term, not only the resistor terms.

Problem 7.18: use one node voltage as a check

Let $V_b = 0$ and $V_a = V$. With the figure's current references,

$$I_1 = \frac{10 - V}{2}, \quad I_2 = \frac{4 - V}{4}, \quad I_3 = \frac{V}{4}. \quad (7.257)$$

KCL gives $(10 - V)/2 + (4 - V)/4 = V/4$, so $24 - 3V = V$ and $V = 6 \text{ V}$. Therefore

$$I_1 = 2.0 \text{ A}, \quad I_2 = -0.50 \text{ A}, \quad I_3 = 1.50 \text{ A}. \quad (7.258)$$

Check both source-to-ground walks: $10 - 2(2) - 4(1.5) = 0$ and $4 - 4(-0.5) - 4(1.5) = 0$. The 10 V source supplies 20 W; the 4 V source absorbs 2 W. The resistor powers are 8, 1 and 9 W, totaling 18 W. The balance $20 = 18 + 2$ verifies the result and identifies the smaller source as an energy receiver.

Problem 7.22: the meter is a branch

Before connection, $V_0 = 2.50 \text{ V}$. After connection, the lower resistor and meter give $10 \parallel 10 = 5 \text{ k}\Omega$. Hence

$$V_M = 5.0 \frac{5}{10 + 5} = \frac{5}{3} \text{ V} = 1.667 \text{ V}. \quad (7.259)$$

The fractional reduction is $(2.50 - 1.667)/2.50 = 1/3$. The source current is $5/(15 \text{ k}\Omega) = 0.3333 \text{ mA}$; half flows through the meter and half through the lower resistor, so $I_M = 0.1667 \text{ mA}$. The two branch currents sum to the source current. A finite reading error is expected from the changed topology; it cannot be eliminated merely by using a more precise voltage display with the same input resistance.

Problem 7.24: distinguish lead drops from sample voltage

The two current leads add $0.100\,\Omega$, equal to the sample resistance. Measuring across the whole current path gives

$$V_{\text{outer}} = 0.500(0.050 + 0.100 + 0.050) = 0.100\,\text{V}. \quad (7.260)$$

The inferred two-wire resistance is $0.100/0.500 = 0.200\,\Omega$, twice the sample value. Direct sense contacts at the sample give $V_x = 0.500(0.100) = 0.0500\,\text{V}$ and $R_x = 0.100\,\Omega$. The ideal sense leads carry no current and therefore develop no resistive voltage drop. Finite input resistance and contact placement set the limits of the real measurement; four-wire wiring does not automatically remove all systematic effects.

Problem 7.28: a capacitor returns energy to a source

The finite series resistance ensures $V_C(0^+) = 8.0\,\text{V}$. At long times the source fixes $V_f = 3.0\,\text{V}$, and $\tau = (50\,\text{k}\Omega)(20\,\mu\text{F}) = 1.0\,\text{s}$. Thus

$$V_C(t) = 3.0 + 5.0e^{-t/(1.0\,\text{s})}\,\text{V}, \quad I_C = C\dot{V}_C = -100e^{-t/(1.0\,\text{s})}\,\mu\text{A}. \quad (7.261)$$

The negative current enters the capacitor's negative-reference direction: it flows out of its positive plate and into the positive terminal of the ideal source.

The total charge delivered by the source is $C(V_f - V_i) = -100\,\mu\text{C}$, so its work is $W_s = 3.0(-100\,\mu\text{C}) = -300\,\mu\text{J}$. The capacitor energy change is

$$\Delta U_C = \frac{1}{2}(20\,\mu\text{F})(3^2 - 8^2) = -550\,\mu\text{J}. \quad (7.262)$$

The resistor heat is $\frac{1}{2}C(8 - 3)^2 = 250\,\mu\text{J}$. The capacitor loses $550\,\mu\text{J}$: $300\,\mu\text{J}$ is absorbed by the source and $250\,\mu\text{J}$ becomes heat. Both the negative source work and the positive heat are necessary for energy conservation.

Problem 7.29: the nearest resistor is not enough

The final capacitor branch is open, leaving a divider:

$$V_f = 10 \frac{3}{2 + 3} = 6.0\,\text{V}. \quad (7.263)$$

With the source set to zero, both resistors connect from the capacitor node to ground, so $R_{\text{seen}} = 2 \parallel 3 = 1.2\,\text{k}\Omega$ and $\tau = 0.120\,\text{s}$. The solution is

$$V_C(t) = 6.0 - 5.0e^{-t/(0.120\,\text{s})}\,\text{V}. \quad (7.264)$$

Initially the upper branch supplies $(10 - 1)/(2\,\text{k}\Omega) = 4.50\,\text{mA}$ while the lower resistor takes $1/(3\,\text{k}\Omega) = 0.333\,\text{mA}$. Their difference is $4.167\,\text{mA}$ into the capacitor. Differentiating the solution gives the same value: $C(5/\tau) = 4.167\,\text{mA}$. At $0.24\,\text{s} = 2\tau$, $V_C = 6 - 5e^{-2} = 5.3233\,\text{V}$.

Problem 7.31: uncertainty in an exponential estimate

For a zero-offset discharge, $V_2/V_1 = e^{-\Delta t/\tau}$, so

$$L = \ln(V_1/V_2) = \ln(4/1.5) = 0.98083, \quad \tau = \frac{3.0}{L} = 3.0586\,\text{s}. \quad (7.265)$$

Differentiate with respect to the two independently measured voltages:

$$\frac{\partial \tau}{\partial V_1} = -\frac{\Delta t}{L^2 V_1}, \quad \frac{\partial \tau}{\partial V_2} = +\frac{\Delta t}{L^2 V_2}. \quad (7.266)$$

The first-order uncertainty is therefore

$$\sigma_\tau = \frac{\Delta t}{L^2} \sqrt{\left(\frac{\sigma_1}{V_1}\right)^2 + \left(\frac{\sigma_2}{V_2}\right)^2} = 0.0444 \text{ s}. \quad (7.267)$$

Under the stated assumptions, report $\tau \simeq 3.059 \pm 0.044 \text{ s}$ as a standard uncertainty. An unknown offset, correlated calibration error or timing uncertainty would require extending this calculation. In particular, nearly equal voltage readings make L small and the estimate poorly conditioned.

Problem 7.35: two equivalent descriptions

The open output voltage is $V_{\text{Th}} = 10(3/5) = 6.0 \text{ V}$. Deactivating the ideal source gives $R_{\text{Th}} = 2 \parallel 3 = 1.2 \text{ k}\Omega$. The Norton source is $I_N = 6/(1.2 \text{ k}\Omega) = 5.0 \text{ mA}$ in parallel with $1.2 \text{ k}\Omega$.

A load $R_L = 1.8 \text{ k}\Omega$ gives

$$I_L = \frac{6}{1.2 + 1.8} \text{ mA} = 2.0 \text{ mA}, \quad V_L = I_L R_L = 3.6 \text{ V}, \quad P_L = 7.2 \text{ mW}. \quad (7.268)$$

In the Norton form, the resistance branch takes $3.6/(1.2 \text{ k}\Omega) = 3.0 \text{ mA}$, leaving 2.0 mA for the load. As a check on internal-power interpretation, the original divider draws $(10 - 3.6)/(2 \text{ k}\Omega) = 3.2 \text{ mA}$ from its 10 V source, which supplies 32 mW . Its overall efficiency to this load is $7.2/32 = 22.5\%$, whereas the equivalent series-source ratio is $R_L/(R_{\text{Th}} + R_L) = 60\%$. Both circuits deliver the same terminal power; they do not have the same internal dissipation.

Problem 7.38: a small imbalance becomes a voltage

With the central branch open, $V_a = V_s/2$ and

$$V_b = V_s \frac{1 + \delta}{2 + \delta}. \quad (7.269)$$

For positive physical R_4 , require $\delta > -1$. The exact differential output is

$$V_a - V_b = -\frac{V_s \delta}{2(2 + \delta)} = -\frac{V_s}{4} \delta + O(\delta^2). \quad (7.270)$$

A positive increase of the lower right resistance raises V_b , so the sign is sensible. Balance is $\delta = 0$. At balance, deactivate the source: each midpoint sees $R \parallel R = R/2$ to the common reference node. Between the midpoints these two resistances add, giving $R_{\text{out}} = R$. A finite detector then loads the differential voltage by a divider factor determined by its input resistance.

Problem 7.42: periodicity supplies the initial condition

For each half-cycle, $\alpha = e^{-0.5}$. At the end of the high interval, $V_{\text{max}} = 5 + (V_{\text{min}} - 5)\alpha$. At the end of the low interval, periodicity requires $V_{\text{min}} = \alpha V_{\text{max}}$. Substitute the latter into the former:

$$V_{\text{max}}(1 - \alpha^2) = 5(1 - \alpha), \quad V_{\text{max}} = \frac{5}{1 + \alpha} = 3.1123 \text{ V}. \quad (7.271)$$

Thus $V_{\text{min}} = 1.8877 \text{ V}$ and ripple is 1.2246 V . A deviation from the settled waveform is multiplied by $\alpha^2 = e^{-1}$ after each complete cycle. Requiring $e^{-N} < 0.01$ gives $N > \ln 100 = 4.605$, so five complete cycles suffice to reduce the startup deviation below 1%. The residual here is relative to the initial deviation, not relative to the instantaneous waveform voltage.

Problem 7.43: combine three independent constraints

The low-state switch current is approximately V_s/R_P , so the 1 mA bound requires $R_P \geq 3.3 \text{ k}\Omega$. Treat the specified worst-case leakage as a constant current toward ground. The final high voltage is $V_H = 3.3 - I_\ell R_P$, giving the upper bound

$$R_P \leq \frac{3.3 - 3.0}{1 \mu\text{A}} = 300 \text{ k}\Omega. \quad (7.272)$$

The charging response has $\tau = R_P C_{\text{in}}$ about this final value. The 10–90% rise time relative to the final excursion is

$$t_{10-90} = [-\ln(0.1) + \ln(0.9)]\tau = \ln 9 R_P C_{\text{in}}. \quad (7.273)$$

Thus the rise-time requirement gives

$$R_P \leq \frac{2.0 \mu\text{s}}{(\ln 9)(100 \text{ pF})} = 9.10 \text{ k}\Omega. \quad (7.274)$$

Combining the bounds yields $3.3 \text{ k}\Omega \leq R_P \leq 9.10 \text{ k}\Omega$. The rise-time condition is more restrictive than the leakage condition. For example, $R_P = 4.7 \text{ k}\Omega$ gives about 0.702 mA low-state current, 3.2953 V final high voltage and 1.03 μs rise time. A real design would insert the specified input thresholds, leakage law and switch voltage; the calculation here answers the explicitly stated model.

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