

Chapter 8

Magnetic Fields and Magnetic Forces

First Edition

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Preface

This chapter develops the action of a prescribed magnetic field on moving charges, currents and current loops. It belongs to *Physics for Technical and Engineering Students- Volume II* and continues the transition from electric fields and circuits to electromagnetism. The central new feature is a force that depends on velocity and is perpendicular to that velocity. Learning to interpret this geometry is as important as learning to calculate its magnitude.

The core assumes elementary vectors, differential and integral calculus, Newton's laws, circular motion, torque and electric potential. These tools are recalled where they enter a derivation. Circular and helical particle motion lead naturally to velocity selection, mass analysis and the Hall effect. Summing the forces on moving carriers then explains forces on wires and torques on loops. The engineering and laboratory sections connect these results to measurable voltages, forces, deflections and uncertainties.

A separate enrichment section develops exact trajectory solutions, magnetic rigidity, curvature measurements, a transport model of the Hall effect, numerical particle tracking and the limits of magnetic confinement. It is intended for students who want a deeper mathematical treatment after mastering the core. The problem set distinguishes core, applied and advanced work, and selected detailed solutions illustrate how to choose assumptions and check a result.

The discussion treats fields as given. Chapter 9 develops fields produced by currents; induction and magnetic field energy follow in Chapter 10, and relativistic mechanics is developed systematically in Chapter 17. Limited previews here are identified explicitly. The numerical examples use illustrative, rounded parameters. Data created for calibration and computation exercises are labelled as synthetic rather than presented as experimental observations.

The Guideline

Identify the object on which the force acts, specify the coordinate system, and write the vector law before calculating. Determine charge sign and current direction explicitly. Finish by checking dimensions, direction, limiting cases and the assumptions that make the chosen model applicable.

Conventions and notation

Vectors are bold, with Cartesian unit vectors $\hat{i}, \hat{j}, \hat{k}$ along a right-handed x, y, z system. In a page diagram, x normally points right, y up and z toward the reader. A dot in a circle ($\odot$) denotes a vector out of the page; a cross in a circle ($\otimes$) denotes a vector into the page. Every diagram that uses a different view states its axes.

The particle charge q includes its sign; $|q|$ is its magnitude. The field magnitude $B = |\mathbf{B}|$ is non-negative, whereas a component such as B_z can have either sign. Unless specified otherwise, fields are external, particles are nonrelativistic, and collisions, radiation reaction and space-charge effects are neglected. The SI Lorentz force is not a nonrelativistic approximation; replacing momentum by $m\mathbf{v}$ is.

Symbol	Meaning	SI unit
$\mathbf{B}$	magnetic field (also called magnetic flux density)	T
$\mathbf{E}$	electric field	V/m
q, q_c	signed particle charge; signed carrier charge	C
m, m_e, m_p	particle, electron and proton invariant masses	kg
$v_\perp, v_\parallel$	perpendicular speed and signed parallel velocity component	m/s
r, r_L	radius of transverse gyromotion	m
h	signed axial advance in one gyroperiod (helix pitch)	m
ω_c, Ω	positive cyclotron angular frequency; signed qB/m	s ⁻¹
T_c	cyclotron period (also written T in calculations)	s
$\mathbf{p}, p$	momentum vector and its magnitude	kg m/s
$I, \mathbf{J}$	conventional current; current density	A; A/m ²
$\mathbf{L}, d\ell$	directed length along conventional current	m
A, A_c	loop area; wire cross-sectional area	m ²
$\boldsymbol{\mu} = NIA\hat{n}$	loop magnetic dipole moment	A m ²
κ, I_{rot}	torsion constant; rotational inertia	N m/rad; kg m ²
n, t, w	carrier density; Hall thickness; Hall width	m ⁻³ ; m; m
R_H	Hall coefficient defined by $E_y = R_H J_x B_z$	m ³ /C

We use h for helix pitch and reserve p for momentum. A negative pitch means motion opposite to the chosen positive field direction; the distance advanced per turn is $|h|$. In Hall formulas, t denotes thickness, while in trajectory equations it denotes time; the local context is always stated.

For numerical exercises, use $e = 1.602 \times 10^{-19}$ C, $m_e = 9.11 \times 10^{-31}$ kg, $m_p = 1.67 \times 10^{-27}$ kg, $c = 3.00 \times 10^8$ m/s and $g = 9.81$ m/s² unless values are supplied. These are rounded teaching constants, not precision metrology values. One electronvolt is the energy 1 eV = 1.602×10^{-19} J at this precision. The unified atomic mass unit is written $u \simeq 1.6605 \times 10^{-27}$ kg.

Chapter 8

Magnetic Fields and Magnetic Forces

8.1 Magnetic fields: observations and operational meaning

The first chapters of electromagnetism developed electric charge, electric field, electric potential, capacitance, current and circuits. Those ideas already explain many static and circuit phenomena, but they are not the whole story. A wire carrying current can be pushed sideways by a magnet. A beam of electrons can curve in a region where no electric field is intentionally applied. A current loop can rotate in a magnetic field and become the core of an electric motor. A strip of semiconductor can develop a transverse Hall voltage that reveals the sign and density of its charge carriers.

These phenomena require a field whose effect depends on motion and direction. That field is the magnetic field $\mathbf{B}$. The central microscopic force law is

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}. \quad (8.1)$$

It says three things at once: the force is proportional to charge, proportional to speed perpendicular to the field, and perpendicular to both velocity and magnetic field. A magnetic field therefore bends a moving charged particle but, by itself, does not change the particle's kinetic energy.

Learning goals

By the end of the core chapter, you should be able to:

- determine magnetic-force directions with diagrams and component algebra;
- explain and use the zero-work property of the magnetic Lorentz force;
- derive radii, periods and pitches for uniform-field particle motion;
- analyse velocity selection, magnetic mass analysis and Hall polarity;
- derive forces on wires and torque and energy for current loops;
- connect a measurement or device model to its assumptions and uncertainties;
- test a numerical trajectory against analytic predictions.

8.1.1 From electric interactions to magnetic interactions

An electric field exerts the force $\mathbf{F}_E = q\mathbf{E}$ on a charge. The direction of this force is parallel to $\mathbf{E}$ for a positive charge and antiparallel to $\mathbf{E}$ for a negative charge. A stationary charge can therefore feel an electric force.

A magnetic field is different. A charge at rest in a magnetic field feels no magnetic force. A moving charge feels a force only from the part of its velocity perpendicular to the magnetic field. This difference is not a small correction to electrostatics; it is a new vector structure. It also explains why magnetic devices often involve moving charges, currents, rotating coils or particle beams.

The transition from electric to magnetic interactions may be summarized as

$$\text{electric field:} \quad \mathbf{F}_E = q\mathbf{E}, \quad (8.2)$$

$$\text{magnetic field:} \quad \mathbf{F}_B = q\mathbf{v} \times \mathbf{B}. \quad (8.3)$$

The electric force is controlled by a vector field and a scalar charge. The magnetic force is controlled by a vector field, a velocity vector and a scalar charge. This is why magnetic-force problems require three-dimensional vector thinking even when the drawing is two-dimensional.

Definition 8.1: Magnetic field through force on a moving charge

The magnetic field $\mathbf{B}$ at a point is defined operationally by the magnetic part of the force on a charge q moving with velocity $\mathbf{v}$ through that point:

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}. \quad (8.4)$$

This definition is used for a test particle whose presence does not significantly disturb the sources of the magnetic field.

Pedagogical boundary

At this point $\mathbf{B}$ is introduced by what it does. The question of what produces $\mathbf{B}$ is postponed to the next chapter, where currents and the Biot–Savart and Ampere laws appear. For the present chapter, the field is assumed to be known or measured.

Guided checks

Check 1. A charge is stationary in a uniform magnetic field. Is there a magnetic force?

Answer. No. The magnetic force contains $\mathbf{v}$, so $\mathbf{v} = \mathbf{0}$ gives $\mathbf{F}_B = \mathbf{0}$.

Check 2. A positive charge moves parallel to $\mathbf{B}$. Is there a magnetic force?

Answer. No. The cross product vanishes when the vectors are parallel or antiparallel.

Check 3. A negative charge has the same velocity as a positive charge in the same magnetic field. How are the magnetic-force directions related?

Answer. They are opposite, because the sign of q is reversed.

8.1.2 What an operational definition can determine

A single force measurement does not determine all three components of $\mathbf{B}$. For example, if $\mathbf{v} = v\hat{\mathbf{i}}$, then

$$\mathbf{F}_B = qv(B_y\hat{\mathbf{k}} - B_z\hat{\mathbf{j}}). \quad (8.5)$$

The component B_x is invisible in this measurement. It can be determined using another nonparallel test velocity. Moreover, a force measurement must separate the electric contribution: a stationary test charge probes $\mathbf{E}$, and reversing a velocity reverses the magnetic force while leaving the electric force unchanged in fixed fields. For equal and opposite velocities at the same position,

$$\frac{\mathbf{F}(\mathbf{v}) + \mathbf{F}(-\mathbf{v})}{2} = q\mathbf{E}, \quad \frac{\mathbf{F}(\mathbf{v}) - \mathbf{F}(-\mathbf{v})}{2} = q\mathbf{v} \times \mathbf{B}. \quad (8.6)$$

This is a useful example of a reversal measurement: changing one controllable quantity separates two contributions that would otherwise be mixed.

8.1.3 Fields, compasses and field lines

A small freely turning compass approximately aligns its magnetic dipole moment with the local field. This provides a directional probe complementary to the force on a moving charge. A bar magnet and a current-carrying coil can both create a magnetic field; their microscopic descriptions differ. Intrinsic particle magnetic moments require quantum physics and should not be imagined as literal spinning charged balls.

A field line is a curve whose tangent follows $\mathbf{B}$; it is a drawing convention, not a wire or a particle path. Equal spacings of parallel arrows represent an approximately uniform field. In magnetostatics the standard no-monopole description has no sources or sinks of magnetic flux. Familiar bar-magnet lines run from north to south outside the magnet and continue through its interior. Field-line source laws are developed later; here the essential distinction is between *field direction* and *force direction*.

Figure 8.1 makes that distinction explicit. A charged particle with velocity parallel to the arrows can move straight along them. A particle with perpendicular velocity curves across the field, and one with both components follows a helix. The particle's initial conditions are needed in addition to the field drawing.

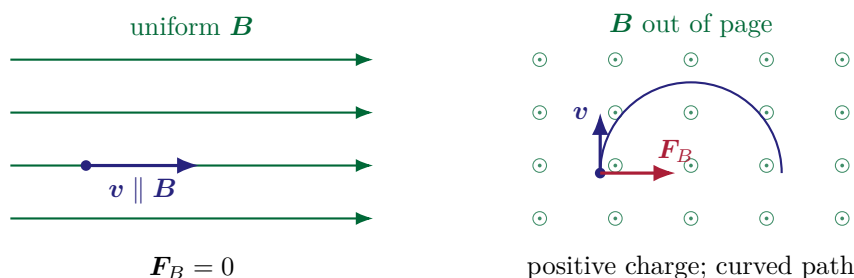


Figure 8.1: A field specifies possible magnetic forces, not a unique trajectory. Left: parallel motion has zero magnetic force. Right: with $\mathbf{B}$ toward the reader and initial velocity upward, a positive charge curves to the right. The blue arc is the particle path, not a field line.

8.2 The vector product and the right-hand rule

For two vectors $\mathbf{a}$ and $\mathbf{b}$, the cross product $\mathbf{a} \times \mathbf{b}$ has magnitude

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \theta, \quad (8.7)$$

where θ is the smaller angle between the vectors. Its direction is perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ and is fixed by the right-hand rule. For the magnetic force,

$$F_B = |q|vB \sin \theta = |q|v_{\perp}B. \quad (8.8)$$

Here $v_{\perp} = v \sin \theta$ is the component of the velocity perpendicular to $\mathbf{B}$.

For a positive charge, point the fingers of the right hand along $\mathbf{v}$ and curl them toward $\mathbf{B}$. The thumb points in the direction of $\mathbf{v} \times \mathbf{B}$ and hence in the direction of $\mathbf{F}_B$. For a negative charge, the force direction is opposite to the thumb direction.

Figure 8.2 uses explicit out-of-page symbols so that no projected arrow can be mistaken for a vector in the plane.

Method

Reliable direction method for $q\mathbf{v} \times \mathbf{B}$.

1. Draw $\mathbf{v}$ and $\mathbf{B}$ at the particle position.
2. Find $\mathbf{v} \times \mathbf{B}$ using the right hand, ignoring the sign of q temporarily.
3. If $q > 0$, keep that direction. If $q < 0$, reverse it.
4. Attach the force vector to the particle, not to the field source.
5. Check whether the force is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$.

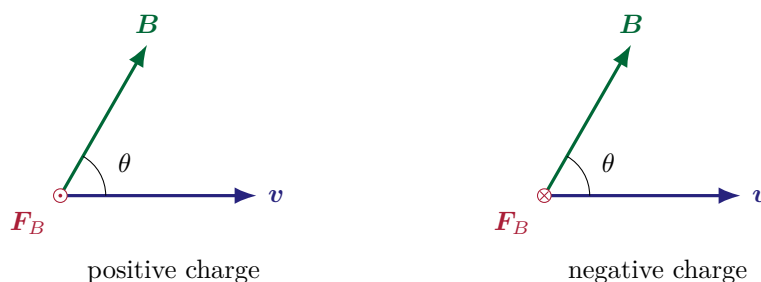


Figure 8.2: For velocity to the right and a field in the upper half of the page, $\mathbf{v} \times \mathbf{B}$ points out of the page. A positive charge has force out of the page (dot); a negative charge has force into it (cross). In both panels $+z$ points toward the reader.

Caution

Do not use the right hand to point along $\mathbf{B}$ and curl toward $\mathbf{v}$ unless you intentionally want $\mathbf{B} \times \mathbf{v}$. The order matters: $\mathbf{v} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{v})$.

Example 8.1: Direction of the magnetic force

A proton moves in the $+\hat{i}$ direction through a region where $\mathbf{B} = B\hat{k}$. Determine the direction of the magnetic force. Repeat for an electron with the same velocity.

Solution. For the proton,

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B} = e(v\hat{i}) \times (B\hat{k}) = evB(\hat{i} \times \hat{k}). \quad (8.9)$$

Using $\hat{i} \times \hat{j} = \hat{k}$, cyclic order gives $\hat{k} \times \hat{i} = \hat{j}$, and therefore $\hat{i} \times \hat{k} = -\hat{j}$. The proton force is in the $-\hat{j}$ direction.

For an electron, $q = -e$ while $\mathbf{v}$ and $\mathbf{B}$ are unchanged. The force is therefore in the opposite direction, $+\hat{j}$.

Reasonableness check. In both cases the force is perpendicular to the velocity and perpendicular to the magnetic field, as required.

8.2.1 Magnitude, units and dimensional checks

The magnitude of the magnetic force on a point charge is

$$F_B = |q|vB \sin \theta. \quad (8.10)$$

If $\mathbf{v} \perp \mathbf{B}$, then $F_B = |q|vB$. If $\mathbf{v} \parallel \mathbf{B}$, then $F_B = 0$.

The SI unit of B is the tesla. From $F = |q|vB$,

$$1 \text{ T} = 1 \frac{\text{N}}{\text{C m s}^{-1}} = 1 \frac{\text{N s}}{\text{C m}}. \quad (8.11)$$

Since $1 \text{ A} = 1 \text{ C s}^{-1}$, this is also

$$1 \text{ T} = 1 \frac{\text{N}}{\text{A m}}. \quad (8.12)$$

The latter form is especially useful for the force on a current-carrying wire.

Note

The tesla is a large field unit on everyday scales. It is useful to compare $50 \mu\text{T}$, 5 mT and 0.5 T as illustrative field scales; these differ by factors of one hundred. Also $1 \text{ G} = 10^{-4} \text{ T}$, where G denotes the gauss. The equations of this chapter use tesla throughout.

Example 8.2: Force magnitude on a moving ion

A singly ionized positive ion has charge $q = +e$ and speed $v = 2.4 \times 10^5 \text{ m/s}$. It moves perpendicular to a uniform magnetic field of magnitude $B = 0.18 \text{ T}$. Find the magnetic-force magnitude.

Solution. Since the motion is perpendicular to $\mathbf{B}$,

$$F_B = |q|vB \quad (8.13)$$

$$= (1.602 \times 10^{-19} \text{ C})(2.4 \times 10^5 \text{ m/s})(0.18 \text{ T}) \quad (8.14)$$

$$= 6.9 \times 10^{-15} \text{ N}. \quad (8.15)$$

Interpretation. This force is tiny on a macroscopic scale, but it may be dynamically important for an ion because the ion's mass is also extremely small. Particle beams are controlled by forces of this kind.

8.2.2 The full Lorentz force

When both electric and magnetic fields are present, the total electromagnetic force on a point charge is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (8.16)$$

This is the Lorentz three-force law in SI units. The law itself also holds relativistically as $d\mathbf{p}/dt = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$; the approximation in most of this chapter is $\mathbf{p} = m\mathbf{v}$ and hence $\mathbf{F} = m\mathbf{a}$. The electric part can change the speed and kinetic energy of the particle. The magnetic part changes the direction of motion but does no work on a point charge by itself.

A key point is that the two contributions may be perpendicular, parallel or antiparallel depending on the geometry. In a crossed-field device, one often arranges $\mathbf{E}$ and $\mathbf{B}$ so that

$$q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = \mathbf{0} \quad (8.17)$$

enables particles of one selected speed to pass undeflected.

Definition 8.2: Lorentz force

For a point charge q moving with velocity $\mathbf{v}$ in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, the electromagnetic force is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \quad (8.18)$$

The equation assumes the particle can be treated as a point charge and that the fields are evaluated at the instantaneous position of the particle.

Guided checks

Check 1. If $\mathbf{E} = \mathbf{0}$ and $\mathbf{v}$ is perpendicular to $\mathbf{B}$, can the magnetic field change the particle speed?

Answer. No. It can change the velocity direction, but the force remains perpendicular to velocity.

Check 2. If $\mathbf{B} = \mathbf{0}$, does the Lorentz force reduce to a familiar expression?

Answer. Yes. It becomes $\mathbf{F} = q\mathbf{E}$.

Check 3. Can the electric and magnetic forces cancel for all speeds?

Answer. No. In a simple crossed-field selector, cancellation occurs only for a particular speed and direction.

8.2.3 A component method for arbitrary directions

Right-hand rules are efficient, but component algebra is the safest method for complicated geometry. If

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}, \quad \mathbf{B} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}}, \quad (8.19)$$

then

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix}. \quad (8.20)$$

Expanding gives

$$\mathbf{v} \times \mathbf{B} = (v_y B_z - v_z B_y) \hat{\mathbf{i}} + (v_z B_x - v_x B_z) \hat{\mathbf{j}} + (v_x B_y - v_y B_x) \hat{\mathbf{k}}. \quad (8.21)$$

This expression is not meant to replace physical thinking. It is a check on physical thinking. In a well-written solution, a component result should be interpreted afterwards in words.

Example 8.3: Component cross product in a three-dimensional geometry

A particle of charge $q = +2e$ has velocity $\mathbf{v} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$ in units of 10^5 m/s. The magnetic field is $\mathbf{B} = 0.20\hat{\mathbf{j}} + 0.10\hat{\mathbf{k}}$ in tesla. Find the force vector.

Solution. First write the velocity in SI units:

$$\mathbf{v} = (3.0 \times 10^5) \hat{\mathbf{i}} + (4.0 \times 10^5) \hat{\mathbf{j}}. \quad (8.22)$$

The cross product is

$$\mathbf{v} \times \mathbf{B} = (3.0 \times 10^5 \hat{\mathbf{i}} + 4.0 \times 10^5 \hat{\mathbf{j}}) \times (0.20 \hat{\mathbf{j}} + 0.10 \hat{\mathbf{k}}) \quad (8.23)$$

$$= (6.0 \times 10^4) \hat{\mathbf{k}} - (3.0 \times 10^4) \hat{\mathbf{j}} + (4.0 \times 10^4) \hat{\mathbf{i}}. \quad (8.24)$$

Multiplying by $q = 2e = 3.204 \times 10^{-19}$ C,

$$\mathbf{F}_B = (1.28 \times 10^{-14} \hat{\mathbf{i}} - 9.61 \times 10^{-15} \hat{\mathbf{j}} + 1.92 \times 10^{-14} \hat{\mathbf{k}}) \text{ N}. \quad (8.25)$$

Check. The force should be perpendicular to $\mathbf{v}$. The dot product of the unrounded $\mathbf{v} \times \mathbf{B}$ with $\mathbf{v}$ is zero, as required.

8.2.4 Reconstructing a field from controlled force measurements

The forward problem takes $\mathbf{v}$ and $\mathbf{B}$ as inputs and predicts a force. In an inverse problem the force is measured and the field is unknown. The missing component along a single test velocity is a genuine information limit, not a defect of the algebra.

Suppose equal and opposite velocities can be prepared at the same point in fixed fields. For a known nonzero charge, define

$$\mathbf{D}(\mathbf{v}) = \frac{\mathbf{F}(\mathbf{v}) - \mathbf{F}(-\mathbf{v})}{2q} = \mathbf{v} \times \mathbf{B}. \quad (8.26)$$

The half-sum determines the electric field, while the half-difference removes it. Taking a second cross product gives

$$\mathbf{D} \times \mathbf{v} = (\mathbf{v} \times \mathbf{B}) \times \mathbf{v} = v^2 \mathbf{B} - \mathbf{v}(\mathbf{v} \cdot \mathbf{B}). \quad (8.27)$$

Therefore the component perpendicular to the probe velocity is

$$\mathbf{B}_\perp = \frac{\mathbf{D} \times \mathbf{v}}{v^2}. \quad (8.28)$$

Any field $\mathbf{B} = \mathbf{B}_\perp + B_\parallel \hat{\mathbf{v}}$ gives the same measurement. A second nonparallel velocity supplies the missing information. Nearly parallel probe directions are a poor choice because a small force error can then produce a large uncertainty in the weakly constrained component.

Example 8.4: Two probe directions and one consistency test

A known positive charge is probed at equal speed v along x and y . After subtracting the electric contribution by velocity reversal, the force data give

$$\frac{\mathbf{D}(v\hat{\mathbf{i}})}{v} = (-0.030\hat{\mathbf{j}} + 0.020\hat{\mathbf{k}}) \text{ T}, \quad \frac{\mathbf{D}(v\hat{\mathbf{j}})}{v} = (0.030\hat{\mathbf{i}} - 0.010\hat{\mathbf{k}}) \text{ T}.$$

Find $\mathbf{B}$ and identify a redundant measured component.

Strategy. Compare each cross product with its component expansion:

$$\hat{\mathbf{i}} \times \mathbf{B} = B_y \hat{\mathbf{k}} - B_z \hat{\mathbf{j}}, \quad \hat{\mathbf{j}} \times \mathbf{B} = B_z \hat{\mathbf{i}} - B_x \hat{\mathbf{k}}.$$

The first measurement gives $B_y = 0.020 \text{ T}$ and $B_z = 0.030 \text{ T}$. The second gives $B_x = 0.010 \text{ T}$ and independently $B_z = 0.030 \text{ T}$. Thus

$$\mathbf{B} = (0.010\hat{\mathbf{i}} + 0.020\hat{\mathbf{j}} + 0.030\hat{\mathbf{k}}) \text{ T}, \quad B = 0.0374 \text{ T}.$$

The two determinations of B_z agree. If they disagreed beyond measurement uncertainty, no single uniform local field could explain the data under the assumed velocities. Possible causes include a changed probe position, an incorrect velocity direction or a field that changed between measurements.

Check. Each magnetic force is perpendicular to its probe velocity. A measured component along that velocity should be investigated before solving for the field. It might be residual electric force, drag or a measurement error; it cannot be explained by changing $\mathbf{B}$ in the point-charge law.

The same reasoning is useful even when a direct particle-force probe is impractical. It teaches how to distinguish an observable component from a quantity inferred only by an assumption. A Hall probe, for example, supplies one calibrated field component rather than all three components at once.

8.3 Magnetic force, work and kinetic energy

The instantaneous power delivered by a force is

$$P = \mathbf{F} \cdot \mathbf{v}. \quad (8.29)$$

For the magnetic force,

$$P_B = \mathbf{F}_B \cdot \mathbf{v} \quad (8.30)$$

$$= q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}. \quad (8.31)$$

The vector $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$, so the scalar product is zero:

$$P_B = 0. \quad (8.32)$$

Thus a magnetic field alone does no work on a point charge. It can bend the trajectory, but it cannot directly change the particle's kinetic energy.

Theoretical point 8.1: Magnetic force and work

For a point charge acted on only by a magnetic force,

$$\frac{dK}{dt} = \mathbf{F}_B \cdot \mathbf{v} = 0. \quad (8.33)$$

The speed is constant, although the velocity vector may change direction. This statement assumes the magnetic field is the only force doing work on the particle and that radiation losses or collisions are neglected.

This result often surprises students because a magnetic field can visibly deflect a beam. Deflection does not imply work. A centripetal force in uniform circular motion changes the direction of velocity without changing the speed. Magnetic force is a natural example of such a perpendicular force.

Caution

A magnetic field can be part of a device that changes particle energy, but then the energy change is caused by electric forces (including induced electric fields), collisions, radiation, or external mechanical work. The force $q\mathbf{v} \times \mathbf{B}$ itself is perpendicular to $\mathbf{v}$ and does no instantaneous work on a point charge.

Example 8.5: Showing explicitly that the magnetic power is zero

Let $\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}$ and $\mathbf{B} = B \hat{\mathbf{k}}$. Compute $\mathbf{F}_B$ and verify that $\mathbf{F}_B \cdot \mathbf{v} = 0$.

Solution. The cross product is

$$\mathbf{v} \times \mathbf{B} = (v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}) \times B \hat{\mathbf{k}} \quad (8.34)$$

$$= B(v_x \hat{\mathbf{i}} \times \hat{\mathbf{k}} + v_y \hat{\mathbf{j}} \times \hat{\mathbf{k}}) \quad (8.35)$$

$$= B(-v_x \hat{\mathbf{j}} + v_y \hat{\mathbf{i}}). \quad (8.36)$$

Therefore

$$\mathbf{F}_B = qB(v_y \hat{\mathbf{i}} - v_x \hat{\mathbf{j}}). \quad (8.37)$$

The power is

$$\mathbf{F}_B \cdot \mathbf{v} = qB(v_y \hat{\mathbf{i}} - v_x \hat{\mathbf{j}}) \cdot (v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}}) \quad (8.38)$$

$$= qB(v_x v_y - v_x v_y) = 0. \quad (8.39)$$

Interpretation. The components rotate the velocity direction but leave $v_x^2 + v_y^2$ unchanged.

8.3.1 What the zero-work statement does and does not require

The scalar triple product is zero at every position and instant. The magnetic contribution to power therefore vanishes even in a spatially nonuniform field. Uniformity is required for the simple circle and helix solutions, not for $P_B = 0$. A time-dependent magnetic field is generally accompanied by

an induced electric field; then the *electric* term can change kinetic energy. For the full Lorentz force,

$$\frac{dK}{dt} = q\mathbf{E} \cdot \mathbf{v}. \quad (8.40)$$

For prescribed static fields, this result also follows by integrating the work along a trajectory. Since $d\mathbf{r} = \mathbf{v}dt$, the magnetic integrand $q(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$ is identically zero. Neither a long path nor a large deflection changes that conclusion.

A motor does not contradict this statement. Its moving conductor contains charges with both a bulk mechanical velocity $\mathbf{u}$ and a drift velocity $\mathbf{v}_d$ relative to the conductor. The magnetic power associated with the bulk motion can be written as part of the decomposition

$$q[(\mathbf{u} + \mathbf{v}_d) \times \mathbf{B}] \cdot (\mathbf{u} + \mathbf{v}_d) = 0. \quad (8.41)$$

The two mixed terms obey

$$q(\mathbf{v}_d \times \mathbf{B}) \cdot \mathbf{u} = -q(\mathbf{u} \times \mathbf{B}) \cdot \mathbf{v}_d. \quad (8.42)$$

The mechanical force on the conductor and the electrical response of the carriers belong to different parts of the energy account. Maintaining current requires a source; constraints and electric forces transfer energy between the carriers, lattice and source. The macroscopic mechanical power $\mathbf{F}_{\text{wire}} \cdot \mathbf{u}$ can be nonzero although the magnetic force on each charge has zero total instantaneous power. The induced voltage needed for a complete moving-circuit account is developed with induction.

8.4 Circular motion in a uniform magnetic field

Suppose a particle of mass m and charge q moves with speed v perpendicular to a uniform magnetic field of magnitude B . The magnetic force has magnitude $|q|vB$ and is always perpendicular to the velocity. It acts as a centripetal force:

$$|q|vB = \frac{mv^2}{r}. \quad (8.43)$$

Solving gives the radius of the circular path,

$$r = \frac{mv}{|q|B}. \quad (8.44)$$

The angular frequency is

$$\omega = \frac{v}{r} = \frac{|q|B}{m}. \quad (8.45)$$

This nonrelativistic frequency is independent of speed. That fact is central to the ideal cyclotron concept. At high speeds the momentum becomes $\mathbf{p} = \gamma m\mathbf{v}$, with m the invariant mass, and the frequency is $|q|B/(\gamma m)$. Section 8.13.3 develops this correction.

The geometry in Figure 8.3 shows that the magnetic force supplies the centripetal acceleration; it is not an extra force to add to a separate “centripetal force.”

Example 8.6: Radius of an electron path in a uniform magnetic field

An electron with speed $v = 3.0 \times 10^6$ m/s enters a region of uniform magnetic field $B = 0.020$ T perpendicular to its velocity. Estimate the radius of its path. Use $m_e = 9.11 \times 10^{-31}$ kg and $e = 1.602 \times 10^{-19}$ C.

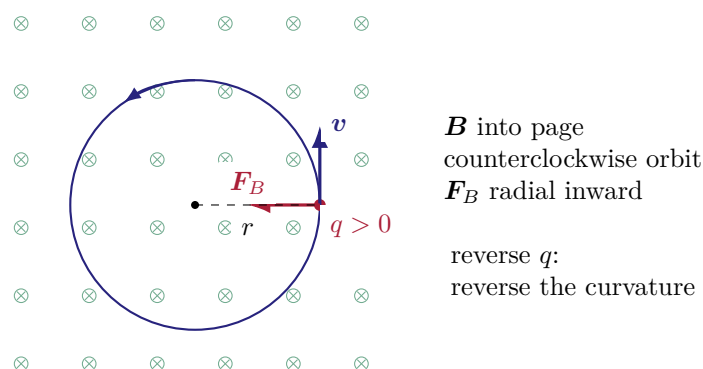


Figure 8.3: Uniform circular motion viewed perpendicular to the magnetic field. The velocity is tangent and the magnetic force points toward the centre at every point. Charge sign determines the sense of rotation.

Solution. The radius depends on the magnitude of the charge:

$$r = \frac{m_e v}{eB} \quad (8.46)$$

$$= \frac{(9.11 \times 10^{-31} \text{ kg})(3.0 \times 10^6 \text{ m/s})}{(1.602 \times 10^{-19} \text{ C})(0.020 \text{ T})} \quad (8.47)$$

$$= 8.5 \times 10^{-4} \text{ m}. \quad (8.48)$$

Thus $r \approx 0.85 \text{ mm}$.

Direction. The electron curves opposite to the direction a positive charge would curve. The radius calculation uses $|q|$, so it does not determine clockwise versus counterclockwise motion by itself.

Reasonableness check. A millimetre-scale radius is plausible because electrons have very small mass; modest laboratory magnetic fields can strongly bend electron beams.

Method

Circular-motion recipe in a uniform B .

1. Check that $\mathbf{v} \perp \mathbf{B}$, or use only $v_{\perp}$.
2. Set magnetic-force magnitude equal to centripetal-force magnitude.
3. Use $r = mv_{\perp}/(|q|B)$ and $\omega = |q|B/m$ in the nonrelativistic limit.
4. Determine the turning direction separately from the sign of q and the right-hand rule.
5. Check whether relativistic corrections are negligible if v is a significant fraction of c .

8.4.1 Radius, period and scaling

The distance travelled in one orbit is $2\pi r$. Thus

$$T_c = \frac{2\pi r}{v} = \frac{2\pi m}{|q|B}, \quad f_c = \frac{1}{T_c} = \frac{|q|B}{2\pi m}, \quad \omega_c = 2\pi f_c. \quad (8.49)$$

A larger speed increases both the circumference and the distance travelled per second by the same factor, so the period is unchanged. This is a property of the nonrelativistic, uniform-field model. It is not a statement that every charged particle circles at the same frequency: the ratio $|q|/m$ matters.

The dimensions provide independent checks. Because $|q|B$ has units kg/s , $mv/(|q|B)$ has units m and $|q|B/m$ has units s^{-1} . The radius becomes large as $B \rightarrow 0$, corresponding to almost straight motion. At exactly $B = 0$ there is no circular orbit, so the divergent radius is a limiting description rather than a measurable circle.

Comparisons depend on what is held fixed. At fixed speed, $r \propto m/|q|$. At fixed nonrelativistic kinetic energy,

$$K = \frac{1}{2}mv^2 \implies r = \frac{\sqrt{2mK}}{|q|B}. \quad (8.50)$$

At fixed momentum, $r = p/(|q|B)$ is independent of mass. A radius alone therefore cannot identify a particle species without additional information about speed, energy or momentum and charge state.

Example 8.7: Equal speed is different from equal energy

Compare a proton with an alpha particle, taking $m_\alpha \simeq 4m_p$ and $q_\alpha = +2e$, for perpendicular motion in the same field.

Strategy. Determine which quantity is equal before using the radius formula. At equal speed,

$$\frac{r_\alpha}{r_p} = \frac{4m_p/(2e)}{m_p/e} = 2.$$

At equal kinetic energy, however,

$$\frac{r_\alpha}{r_p} = \frac{\sqrt{4m_p}/(2e)}{\sqrt{m_p}/e} = 1.$$

Their nonrelativistic periods have ratio $T_\alpha/T_p = 2$ in both comparisons.

Interpretation. Equal curvature does not imply equal mass. The second result combines a larger mass with a larger charge and a smaller speed. Since both charges are positive, they curve in the same sense for equal initial directions. This comparison neglects the small departure of the alpha mass from $4m_p$.

8.4.2 Helical motion

If the velocity has both perpendicular and parallel components relative to a uniform magnetic field, the perpendicular component produces circular motion while the parallel component remains unchanged. The result is a helix around the magnetic-field direction.

Let

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel, \quad (8.51)$$

where $\mathbf{v}_\parallel$ is parallel to $\mathbf{B}$. Since $\mathbf{v}_\parallel \times \mathbf{B} = \mathbf{0}$, the magnetic force is controlled only by $\mathbf{v}_\perp$:

$$\mathbf{F}_B = q\mathbf{v}_\perp \times \mathbf{B}. \quad (8.52)$$

The radius and angular frequency are therefore

$$r = \frac{mv_\perp}{|q|B}, \quad \omega = \frac{|q|B}{m}. \quad (8.53)$$

The pitch, meaning the distance advanced parallel to the field in one revolution, is

$$h = v_\parallel T = v_\parallel \frac{2\pi}{\omega} = \frac{2\pi m v_\parallel}{|q|B}. \quad (8.54)$$

Figure 8.4 shows a perspective view together with the transverse projection. The latter is a circle even when the three-dimensional path is not.

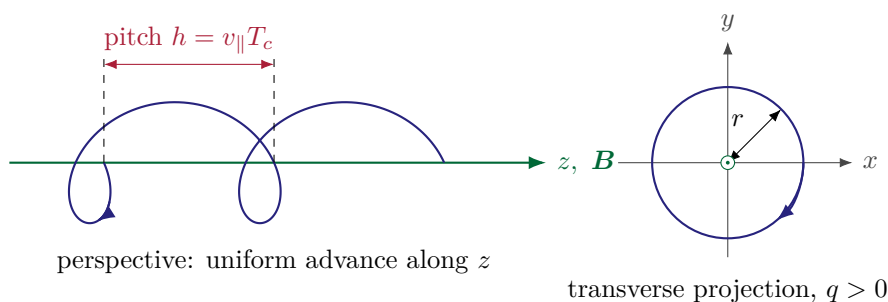


Figure 8.4: Helical motion for a positive charge with $v_{\parallel} > 0$. The perspective drawing is a projection of a circular helix with equal axial advances at equal phase. Viewed from $+z$ toward the origin, the transverse rotation is clockwise for $\mathbf{B} = B\hat{\mathbf{k}}$.

Example 8.8: Helical motion and pitch

A positive ion of mass $m = 3.3 \times 10^{-26}$ kg and charge $q = +e$ moves in a uniform field $B = 0.10$ T. Its velocity components are $v_{\perp} = 1.5 \times 10^4$ m/s and $v_{\parallel} = 2.0 \times 10^4$ m/s. Find the radius and pitch.

Solution. The radius is

$$r = \frac{mv_{\perp}}{eB} \quad (8.55)$$

$$= \frac{(3.3 \times 10^{-26})(1.5 \times 10^4)}{(1.602 \times 10^{-19})(0.10)} \text{ m} \quad (8.56)$$

$$= 3.1 \times 10^{-2} \text{ m}. \quad (8.57)$$

The angular frequency is

$$\omega = \frac{eB}{m} = \frac{(1.602 \times 10^{-19})(0.10)}{3.3 \times 10^{-26}} = 4.85 \times 10^5 \text{ s}^{-1}. \quad (8.58)$$

Thus

$$h = v_{\parallel} \frac{2\pi}{\omega} \quad (8.59)$$

$$= (2.0 \times 10^4) \frac{2\pi}{4.85 \times 10^5} \text{ m} \quad (8.60)$$

$$= 0.26 \text{ m}. \quad (8.61)$$

Interpretation. The ion advances 0.26 m along the field in each turn. Increasing $v_{\parallel}$ increases the pitch but does not change the radius.

Pitch angle and the geometry of a measured track

Define the pitch angle α relative to $+\mathbf{B}$. For $0 < \alpha < \pi$,

$$v_{\perp} = v \sin \alpha, \quad v_{\parallel} = v \cos \alpha, \quad \frac{h}{r} = 2\pi \cot \alpha. \quad (8.62)$$

This ratio is geometric: during one turn the transverse distance is $2\pi r$, while the axial displacement is h . At $\alpha = 90^\circ$, the pitch vanishes. At $\alpha \rightarrow 0$, the radius tends to zero and the path tends to a straight line along the field. In the exactly parallel case there is no actual gyration, although the formal period can still be defined from m, q, B .

For $0 < \alpha < 90^\circ$, a measured helix gives $\alpha = \tan^{-1}(2\pi r/h)$. If the axial motion is opposite to $\mathbf{B}$,

use the signs of the velocity components to choose the angle in the correct quadrant. The charge sign sets the handedness; it does not change r or T_c at fixed mass and velocity components.

The radius here is the radius of the *transverse projection*, not the local three-dimensional radius of curvature of the helix. A particle with large $v_{\parallel}$ can travel a long distance per turn without changing its transverse radius. This distinction becomes especially important when interpreting detector tracks.

Reconstructing motion from a three-dimensional track

A projected circle alone determines $mv_{\perp}/|q|$, but a helix with timing information contains more. Suppose a track in a known field gives transverse radius r_L , signed pitch h and measured gyroperiod T_c . Then

$$v_{\perp} = \frac{2\pi r_L}{T_c}, \quad v_{\parallel} = \frac{h}{T_c}, \quad v = \frac{\sqrt{(2\pi r_L)^2 + h^2}}{T_c}. \quad (8.63)$$

These first relations are geometric. The dynamical law supplies

$$\frac{m}{|q|} = \frac{BT_c}{2\pi} \quad (8.64)$$

in the nonrelativistic regime. Without the timing, the ratio h/r_L determines the pitch angle but leaves an overall speed scale unknown. This is another example of an inverse problem with an identifiable ratio and an undetermined scale.

Example 8.9: A track and a transit-time measurement

A particle makes five turns while travelling 0.500 m along $+\mathbf{B}$. Its transverse radius is 20.0 mm, the transit time is 4.00 μs , and $B = 0.200$ T. Assume the track is a uniform helix and the transit measurement spans exactly those five turns. Find the velocity components and mass-to-charge ratio.

Extract the geometric quantities. One turn takes $T_c = 4.00/5 = 0.800 \mu\text{s}$ and advances $h = 0.500/5 = 0.100$ m. Hence

$$\begin{aligned} v_{\perp} &= \frac{2\pi(0.0200)}{0.800 \times 10^{-6}} = 1.57 \times 10^5 \text{ m/s}, \\ v_{\parallel} &= \frac{0.100}{0.800 \times 10^{-6}} = 1.25 \times 10^5 \text{ m/s}, \\ v &= 2.01 \times 10^5 \text{ m/s}, \quad \alpha = \tan^{-1}(v_{\perp}/v_{\parallel}) = 51.5^\circ. \end{aligned}$$

The speed is small compared with c , so the nonrelativistic inference is consistent:

$$\frac{m}{|q|} = \frac{(0.200)(0.800 \times 10^{-6})}{2\pi} = 2.55 \times 10^{-8} \text{ kg/C}.$$

If $|q| = e$, the mass would be 4.08×10^{-27} kg; the data alone do not establish that charge state or identify a species. The temporal sense of the transverse rotation, together with the field direction, establishes charge sign.

Avoid a path-length error. The actual distance travelled is five times $\sqrt{(2\pi r_L)^2 + h^2}$, about 0.803 m, not the axial distance 0.500 m. Dividing the axial distance by the transit time gives $v_{\parallel}$, not the total speed.

The local radius of curvature of the spatial track is also different from r_L . Since the acceleration magnitude is $a = |q|Bv_{\perp}/m$ and is perpendicular to the total velocity, the geometric curvature radius is

$$\rho_{\text{curve}} = \frac{v^2}{a} = \frac{r_L}{\sin^2 \alpha}. \quad (8.65)$$

This result reduces to r_L for a circle and grows as the helix becomes more nearly straight. A three-dimensional curve-fitting routine and a transverse circle fit can therefore report different radii while both are correct. The fitted quantity must match the quantity in the physical formula.

8.4.3 A finite magnetic region: exit displacement and angle

A real bending magnet occupies a finite region. A particle may leave it long before completing a circle. Consider a positive charge entering at $(0, 0)$ with velocity along $+x$ into a uniform field $\mathbf{B} = -B\hat{\mathbf{k}}$ that occupies $0 < x < \ell$. The trajectory initially curves upward. Ignoring fringe fields, its circle has centre $(0, r)$ and satisfies

$$x^2 + (y - r)^2 = r^2. \quad (8.66)$$

On the first forward-going part of the orbit,

$$y(x) = r - \sqrt{r^2 - x^2}. \quad (8.67)$$

If $\ell < r$, the particle exits the far boundary with

$$\sin \phi = \frac{\ell}{r}, \quad y_{\text{exit}} = r - \sqrt{r^2 - \ell^2}, \quad \tan \phi = \frac{\ell}{\sqrt{r^2 - \ell^2}}. \quad (8.68)$$

After travelling a further horizontal distance D in a field-free region, the displacement is

$$y_{\text{screen}} = y_{\text{exit}} + D \tan \phi. \quad (8.69)$$

The second term can dominate. A detector downstream measures both the displacement accumulated within the field and the effect of the exit angle.

For weak bending, $\ell/r \ll 1$, expansion of the square root gives

$$y_{\text{screen}} \simeq \frac{\ell^2}{2r} + \frac{D\ell}{r}. \quad (8.70)$$

This is a controlled small-angle approximation. If $\ell > r$, the particle turns back before reaching the plane $x = \ell$; assuming a traversal time ℓ/v would then be impossible.

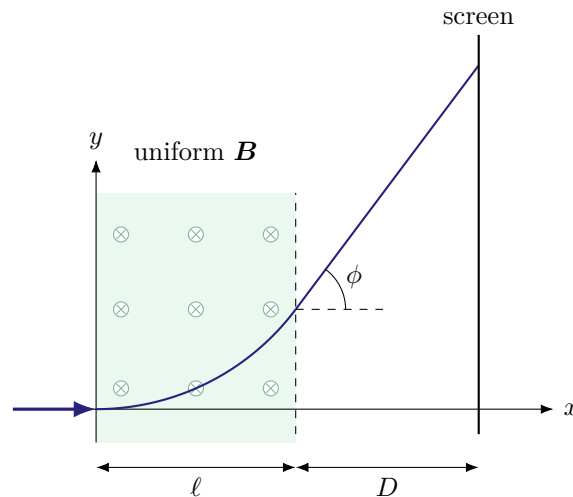


Figure 8.5: A circular arc inside a finite magnet followed by a straight tangent outside it. The drawing exaggerates the bending for clarity; the exact circular relation, rather than a small-angle sketch, determines the exit quantities.

Figure 8.5 is also a reminder that field boundaries and detector locations are part of a problem's model.

Example 8.10: Deflection at a downstream screen

A beam has bending radius $r = 0.50$ m in a magnet of horizontal length $\ell = 0.10$ m. A screen is $D = 0.30$ m beyond the exit. Find its displacement for the positive-charge geometry above. **Solution.** The exit angle is $\phi = \sin^{-1}(0.20) = 11.5^\circ$. The exit height is $0.50 - \sqrt{0.50^2 - 0.10^2} = 0.0101$ m. Thus

$$y_{\text{screen}} = 0.0101 + 0.30 \frac{0.10}{\sqrt{0.50^2 - 0.10^2}} = 0.0713 \text{ m.}$$

The small-angle result is 0.0700 m, about 1.8% lower. Merely reporting the exit height would underestimate the observed displacement by a factor of about seven.

Checks. The field changes direction, not kinetic energy. As $D \rightarrow 0$ the screen displacement approaches the exit height, and as $B \rightarrow 0$ the radius tends to infinity and both displacement terms vanish.

8.5 Crossed electric and magnetic fields

Crossed fields are arrangements in which $\mathbf{E}$ and $\mathbf{B}$ are perpendicular. They are important because the electric and magnetic forces can oppose one another for a selected particle velocity. Suppose a particle moves with velocity $\mathbf{v}$ through a region where $\mathbf{E}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$, and where $\mathbf{E}$ and $\mathbf{v} \times \mathbf{B}$ point in opposite directions. The condition for no deflection is

$$q\mathbf{E} + q\mathbf{v} \times \mathbf{B} = \mathbf{0}. \quad (8.71)$$

For the usual perpendicular geometry, magnitudes give

$$|q|E = |q|vB, \quad (8.72)$$

so the selected speed is

$$v = \frac{E}{B}. \quad (8.73)$$

For a nonzero charge this result is independent of charge sign, charge magnitude and mass, provided the velocity direction is the specified one.

Figure 8.6 fixes the coordinate convention used for the selector: $\mathbf{E} = -E\hat{\mathbf{j}}$, $\mathbf{B} = -B\hat{\mathbf{k}}$ and the incident velocity is along $+x$.

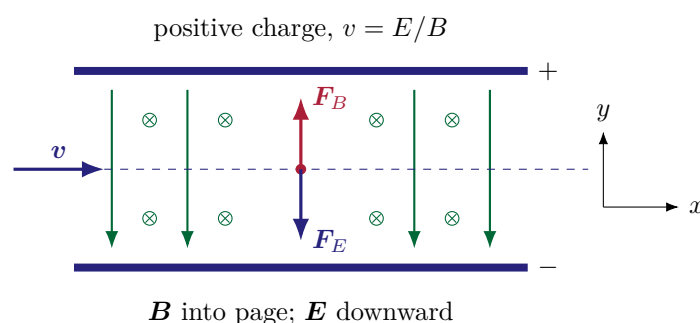


Figure 8.6: For a positive particle moving right, the magnetic force points up and the electric force down. Both arrows are attached to the particle. Reversing charge reverses both forces, so the same velocity remains undeflected.

Example 8.11: Velocity selector

A beam of singly charged positive ions passes through crossed fields with $E = 4.0 \times 10^4 \text{ V/m}$ and $B = 0.20 \text{ T}$. What speed passes through undeflected?

Solution. For the selector geometry,

$$v = \frac{E}{B} = \frac{4.0 \times 10^4 \text{ V/m}}{0.20 \text{ T}} = 2.0 \times 10^5 \text{ m/s.} \quad (8.74)$$

Unit check. Since $1 \text{ V/m} = 1 \text{ N/C}$ and $1 \text{ T} = 1 \text{ N s/(C m)}$, the ratio $(\text{N/C})/(\text{N s/(C m)})$ is m/s .

Interpretation. Faster particles have a larger magnetic force than needed for cancellation and are deflected one way; slower particles are deflected the other way.

Guided checks

Check 1. In a velocity selector, why does $v = E/B$ not contain q ?

Answer. For $q \neq 0$, both electric and magnetic forces are proportional to q , so q cancels in the no-deflection condition.

Check 2. Does every particle in the beam pass through undeflected?

Answer. No. Only particles with the selected velocity and direction do.

Check 3. If the magnetic field is reversed while the electric field is not, does the same selector condition hold?

Answer. The magnitude condition $v = E/B$ remains possible, but the forces no longer oppose for the same beam direction; the geometry must be reconsidered.

8.5.1 What the selector actually selects

For the geometry above the initial vertical force is $F_y = q(Bv - E)$. A positive particle that is faster than $v_0 = E/B$ initially bends upward; one that is slower bends downward. A negative charge reverses those deflections. There is no force restoring a particle to a preferred speed: the instrument rejects unwanted trajectories with apertures.

The qualification “for the specified beam direction” matters. If a velocity component along $\mathbf{B}$ is allowed, it produces no magnetic force and need not vanish for force cancellation. The general force-free velocity in uniform perpendicular fields is

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + v_{\parallel} \hat{\mathbf{b}}, \quad \hat{\mathbf{b}} = \mathbf{B}/B. \quad (8.75)$$

A collimated beam with $v_{\parallel} = 0$ is therefore essential when describing E/B as selection of the total speed. Neutral particles have zero Lorentz force for every speed in this point-particle model and are not speed-selected by these fields.

The selected speed need not correspond to a unique kinetic energy. Ions with masses m_1 and m_2 but the same selected speed have energy ratio $K_1/K_2 = m_1/m_2$. A mass analyser downstream measures the remaining mass-to-charge dependence. Finite slit widths, velocity spread and fringe fields broaden an ideal equality into a transmission band; Section 8.13.11 quantifies that distinction.

8.5.2 Mass spectrometry and charge-to-mass measurements

Magnetic bending is a measurement tool. If particles of known speed enter a uniform magnetic field perpendicular to their velocity, their path radius is

$$r = \frac{mv}{|q|B}. \quad (8.76)$$

Rearranging gives the mass-to-charge ratio

$$\frac{m}{|q|} = \frac{rB}{v}. \quad (8.77)$$

A velocity selector followed by a magnetic-bending region can therefore separate ions by mass-to-charge ratio. In a simple nonrelativistic design, the selector gives $v = E/B_s$, where B_s is the selector field, and the analyser magnet gives

$$\frac{m}{|q|} = \frac{rB_a B_s}{E}, \quad (8.78)$$

where B_a is the analyser field.

Example 8.12: Simple mass-spectrometer estimate

Singly charged positive ions pass through a velocity selector with $E = 3.0 \times 10^4$ V/m and $B_s = 0.15$ T, then enter an analyser field $B_a = 0.30$ T. The observed circular radius is $r = 0.21$ m. Estimate the ion mass.

Solution. The selected speed is

$$v = \frac{E}{B_s} = \frac{3.0 \times 10^4}{0.15} \text{ m/s} = 2.0 \times 10^5 \text{ m/s}. \quad (8.79)$$

The mass is

$$m = \frac{|q|B_a r}{v} \quad (8.80)$$

$$= \frac{(1.602 \times 10^{-19} \text{ C})(0.30 \text{ T})(0.21 \text{ m})}{2.0 \times 10^5 \text{ m/s}} \quad (8.81)$$

$$= 5.0 \times 10^{-26} \text{ kg}. \quad (8.82)$$

Interpretation. Dividing by the atomic mass unit 1.66×10^{-27} kg gives about 30 atomic mass units. The result is reasonable for a light molecular ion or medium-mass atomic ion.

Method

Magnetic analyser modelling steps.

1. Identify how the speed is known: direct acceleration, velocity selector, time of flight or other measurement.
2. Use the analyser field only for the region where the trajectory is circular.
3. Use $r = mv/(|q|B)$ in the nonrelativistic limit.
4. Remember that magnetic bending measures $m/|q|$, not m alone unless the charge state is known.
5. Check whether the detected radius is large compared with beam size and aperture uncertainty.

The detector geometry must be related to the circular radius. In a 180° analyser the displacement between entrance and impact points is the diameter $2r$, not the radius (Figure 8.7).

8.5.3 Acceleration through a voltage and the charge-to-mass ratio

Another way to determine speed is electric acceleration from negligible initial kinetic energy. Define $V_{\text{acc}} > 0$ as the magnitude of an accelerating potential difference, chosen so that $-q\Delta V = |q|V_{\text{acc}}$. Then

$$\frac{1}{2}mv^2 = |q|V_{\text{acc}}, \quad r = \frac{mv}{|q|B}. \quad (8.83)$$

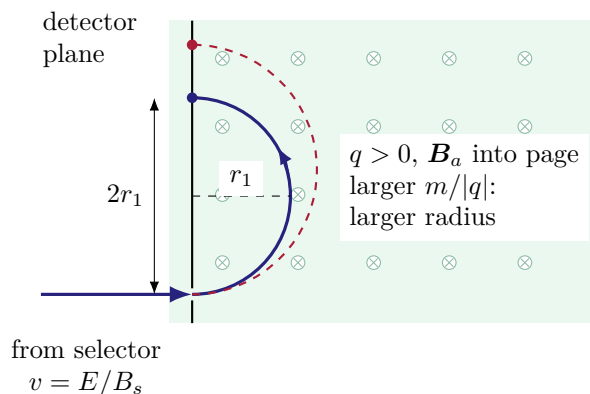


Figure 8.7: A schematic semicircular mass analyser. Two positive ions enter at the same point and speed. Their impact separation along the detector plane is $2(r_2 - r_1)$. The regions outside the analysing magnet, including the selector, are represented only schematically.

Eliminating v gives

$$\frac{|q|}{m} = \frac{2V_{\text{acc}}}{B^2 r^2}, \quad \frac{m}{|q|} = \frac{B^2 r^2}{2V_{\text{acc}}}. \quad (8.84)$$

These expressions assume nonrelativistic motion, negligible energy loss, a known acceleration voltage and perpendicular entry into a uniform analysing field. An unknown initial kinetic energy would enter as an additional term in the energy equation. The magnetic region does not supply the kinetic energy used in this calculation.

Example 8.13: A consistent electron charge-to-mass measurement

Electrons are accelerated through 250 V and bend in a field of 1.50 mT. A circular track has measured radius 3.56 cm. Estimate e/m_e and assess the nonrelativistic approximation.

Model and calculation. The acceleration and bending regions are separate. With $r = 0.0356$ m,

$$\frac{e}{m_e} = \frac{500}{(1.50 \times 10^{-3})^2 (0.0356)^2} = 1.75 \times 10^{11} \text{ C/kg}.$$

The rounded constants used in this chapter give $e/m_e \simeq 1.76 \times 10^{11} \text{ C/kg}$, consistent with these illustrative data. The implied speed is

$$v = \sqrt{2(e/m_e)V_{\text{acc}}} = 9.36 \times 10^6 \text{ m/s} \simeq 0.0312c.$$

The small ratio v/c supports the nonrelativistic treatment at this precision.

Diagnostic. If the same apparatus instead reported a radius of 7.1 cm, the inferred e/m_e would be about four times smaller. Such a discrepancy would call for a check of radius versus diameter, field calibration and the model. It cannot be justified merely by calling a measurement “coarse.”

At fixed acceleration voltage, the radius scales as $\sqrt{m/|q|}$; at fixed selected speed, it scales as $m/|q|$. These different scalings should not be mixed. For two close masses with the same charge, $\Delta r/r \simeq \frac{1}{2}\Delta m/m$ at fixed voltage, but $\Delta r/r = \Delta m/m$ at fixed speed.

8.6 The Hall effect: force balance and voltage polarity

The Hall effect turns a sideways magnetic force on moving carriers into a measurable transverse voltage. A conductor or semiconductor carries a steady current along its length. When a magnetic field is applied perpendicular to the plate, carriers initially deflect toward one edge. The resulting

charge separation produces a transverse electric field. In an open transverse circuit, the steady Hall field prevents a continuing transverse current.

The current along the bar still requires a longitudinal electric field. The statement that electric and magnetic forces balance refers to the *transverse* component. Scattering and the longitudinal driving field are part of the resistive current model introduced in Chapter 6. They should not be discarded merely because the transverse balance is simple.

8.6.1 One geometry, one signed convention

Let the conventional current be $I > 0$ along $+x$, the magnetic field be $\mathbf{B} = B_z \hat{\mathbf{k}}$, and the Hall width be w along y . The thickness t is along z . Label the lower edge a at $y = -w/2$ and the upper edge b at $y = +w/2$. Define the measured voltage by connecting the positive meter terminal to a :

$$V_H \equiv V_a - V_b = E_y w. \quad (8.85)$$

The equality follows from $\mathbf{E} = -\nabla V$: a positive E_y means that potential decreases as y increases. Connecting the meter leads in the opposite order reverses the displayed voltage.

For one mobile carrier type of charge q_c , density n and drift velocity v_{dx} ,

$$J_x = nq_c v_{dx} = \frac{I}{wt}. \quad (8.86)$$

The transverse magnetic-force component is $-q_c v_{dx} B_z$. The steady balance gives

$$q_c(E_y - v_{dx} B_z) = 0 \implies E_y = v_{dx} B_z = \frac{J_x B_z}{nq_c}. \quad (8.87)$$

Define the Hall coefficient by $E_y = R_H J_x B_z$. Then

$$\boxed{R_H = \frac{1}{nq_c}, \quad V_H = \frac{IB_z}{nq_c t}}. \quad (8.88)$$

Its magnitude is $|V_H| = |IB_z|/(n|q_c|t)$. The width cancels because increasing width reduces drift speed at fixed current in exactly the proportion that it increases the distance over which the Hall field is integrated. This cancellation assumes uniform current density and an ideal rectangular bar.

Figure 8.8 compares positive and negative carriers at the same conventional current and field. In both cases the initial magnetic force on the carriers points toward the lower edge. What differs is the *sign of the charge* that accumulates there.

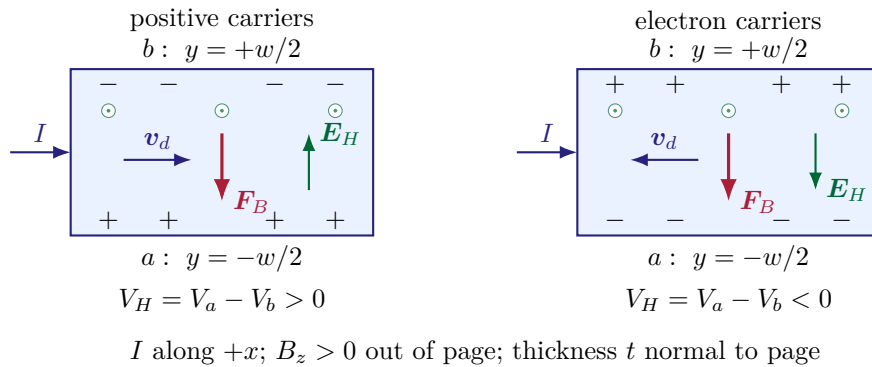


Figure 8.8: Hall polarity for the explicitly defined voltage $V_a - V_b$. Charge accumulation is shown schematically. The red arrow is the initial magnetic force; the Hall electric force eventually opposes it. The positive and negative carrier cases have opposite Hall fields and opposite meter readings.

Example 8.14: Hall density, polarity and uncertainty

A bar with $t = 0.50$ mm carries $I = 20$ mA along $+x$ in $B_z = +0.80$ T. With the terminal convention above, $V_a - V_b = -12$ mV. Assume a single carrier type with $|q_c| = e$.

Sign first. Since I and B_z are positive, the negative voltage implies $R_H < 0$ and therefore $q_c = -e$ in this model. Reversing the voltage leads would change the reading without changing the material.

Magnitude. Rearranging the Hall relation gives

$$n = \frac{IB_z}{et|V_H|} \quad (8.89)$$

$$= \frac{(20 \times 10^{-3})(0.80)}{(1.602 \times 10^{-19})(0.50 \times 10^{-3})(12 \times 10^{-3})} = 1.7 \times 10^{22} \text{ m}^{-3}. \quad (8.90)$$

The coefficient is $R_H = V_H t / (IB_z) = -3.75 \times 10^{-4} \text{ m}^3/\text{C}$.

Uncertainty. If the independent relative standard uncertainties in $I, B_z, t, |V_H|$ are 1%, 1%, 3%, 2%, then first-order propagation gives

$$\frac{\sigma_n}{n} = \sqrt{0.01^2 + 0.01^2 + 0.03^2 + 0.02^2} = 3.9\%.$$

Thickness is the largest contribution. This uncertainty describes measurement precision *within the model*; it does not quantify failure of the single-carrier assumption.

8.6.2 What a Hall measurement can establish

A calibrated Hall plate can measure a magnetic-field component; if the field and geometry are known, it can instead estimate a carrier density. In either use, the formula contains assumptions: approximately homogeneous material, a known current path, negligible transverse current through the meter, and a regime where one effective carrier type is adequate.

A raw voltage may contain a longitudinal contribution because the two contacts are not exactly opposite one another. Offset voltages and heating can also matter. Reversing B_z separates a component odd in field from a field-even background:

$$V_H^{\text{odd}}(B_z) = \frac{V_{\text{meas}}(B_z) - V_{\text{meas}}(-B_z)}{2}. \quad (8.91)$$

This procedure requires stable conditions between the two measurements. An odd instrumental artefact would not be removed. Multiple mobile carrier populations can also alter R_H ; the enrichment section derives a simple case in which the Hall sign is governed by mobilities as well as densities.

Guided checks

If the current and field are both reversed, what happens to the Hall voltage? In the linear single-carrier model it is unchanged because it is proportional to IB_z . If only one is reversed, it changes sign. If only the voltage leads are interchanged, the displayed number changes sign but the charge distribution does not.

8.7 Magnetic force on a current-carrying wire

A current is moving charge. Therefore a current-carrying wire in a magnetic field can experience a magnetic force. For a straight wire segment of vector length $\mathbf{L}$ carrying conventional current I in a uniform field $\mathbf{B}$,

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}. \quad (8.92)$$

The magnitude is

$$F = ILB \sin \theta, \quad (8.93)$$

where θ is the angle between the current direction and the magnetic field.

8.7.1 From carrier motion to the force on a wire

Consider a straight wire segment of length L and cross-sectional area A_c . Suppose mobile carriers have charge q_c , number density n and drift velocity $\mathbf{v}_d$ along the wire. The number of carriers in the segment is nA_cL . The magnetic force on one carrier is $q_c\mathbf{v}_d \times \mathbf{B}$, so the total force is

$$\mathbf{F} = nA_cLq_c\mathbf{v}_d \times \mathbf{B}. \quad (8.94)$$

The current density is $\mathbf{J} = nq_c\mathbf{v}_d$, and the current is $I = JA_c$ in magnitude. Therefore

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B}, \quad (8.95)$$

where $\mathbf{L}$ points in the direction of conventional current. The sign of the mobile carrier is already included in the conventional current direction.

Figure 8.9 shows the force on the selected segment. Return leads are excluded from this force calculation; they must be included when the entire circuit is the object of interest.

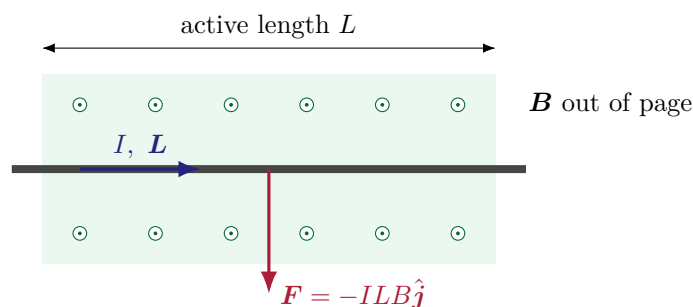


Figure 8.9: A straight segment carrying conventional current rightward in a field out of the page feels a downward force. Only the length inside the stated field region contributes to the displayed ILB estimate.

Example 8.15: Force on a straight current-carrying wire

A straight wire segment of length $L = 0.45$ m carries current $I = 3.0$ A in the $+\hat{i}$ direction. The wire lies in a uniform magnetic field $\mathbf{B} = 0.20\hat{k}$ T. Find the magnetic force on the segment.

Solution. The vector length is $\mathbf{L} = 0.45\hat{i}$ m. Thus

$$\mathbf{F} = I\mathbf{L} \times \mathbf{B} \quad (8.96)$$

$$= (3.0)(0.45\hat{i}) \times (0.20\hat{k}) \text{ A m T} \quad (8.97)$$

$$= 0.27(\hat{i} \times \hat{k}) \text{ N} \quad (8.98)$$

$$= -0.27\hat{j} \text{ N}. \quad (8.99)$$

Interpretation. The force is downward in the $-\hat{j}$ direction for the chosen axes. If the current is reversed, the force reverses.

Guided checks

Check 1. A straight wire is parallel to a uniform magnetic field. What is the magnetic force?

Answer. Zero, because $\sin \theta = 0$.

Check 2. If the current is doubled, what happens to the force magnitude?

Answer. It doubles, provided L , B and the angle are unchanged.

Check 3. Does $\mathbf{L}$ point with electron drift in a metal?

Answer. No. It points with conventional current.

8.7.2 Finite wires and differential force

For a wire of arbitrary shape in a magnetic field, the force on a differential segment is

$$d\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B}. \quad (8.100)$$

The total force is

$$\mathbf{F} = I \int d\boldsymbol{\ell} \times \mathbf{B}. \quad (8.101)$$

If $\mathbf{B}$ is uniform, it can be taken outside the integral:

$$\mathbf{F} = I \left(\int d\boldsymbol{\ell} \right) \times \mathbf{B}. \quad (8.102)$$

For a closed loop, $\int d\boldsymbol{\ell} = \mathbf{0}$, so the net force on a closed current loop in a uniform magnetic field is zero. The torque may still be nonzero.

Example 8.16: Net force on a semicircular wire in a uniform field

A wire consists of a semicircular arc of radius a carrying current from the left end to the right end along the upper arc. It lies in the xy plane. A uniform field $\mathbf{B} = B\hat{\mathbf{k}}$ is applied. Find the net force on the arc.

Solution. Since $\mathbf{B}$ is uniform,

$$\mathbf{F} = I \left(\int d\boldsymbol{\ell} \right) \times \mathbf{B}. \quad (8.103)$$

The integral of $d\boldsymbol{\ell}$ over the arc is simply the displacement from the starting point to the ending point, $2a\hat{\mathbf{i}}$. Thus

$$\mathbf{F} = I(2a\hat{\mathbf{i}}) \times (B\hat{\mathbf{k}}) = -2IaB\hat{\mathbf{j}}. \quad (8.104)$$

Interpretation. The net force is the same as it would be on a straight wire connecting the same endpoints in the same uniform field. This result is special to uniform $\mathbf{B}$.

8.7.3 How the force reaches the conductor

The drift velocity is an average over carrier motion. Random thermal velocities do not constitute a steady current and average to zero in the elementary isotropic model. The collective magnetic force density is consequently

$$\mathbf{f}_B = \mathbf{J} \times \mathbf{B}. \quad (8.105)$$

It has units force per volume. The conductor as a whole can remain electrically neutral while having nonzero $\mathbf{J}$ and hence a magnetic force. Transverse charge rearrangement, electric forces on the lattice and carrier-lattice interactions transmit this force to the solid. A zero transverse acceleration of the carriers after Hall equilibrium does not imply zero force on the entire supported wire.

The field in $I d\boldsymbol{\ell} \times \mathbf{B}$ is the specified *external* field. One should not insert the wire's singular idealized self-field into the same formula to infer a net force of the whole isolated wire on itself. Internal magnetic stresses require a more careful extended-current description.

For a nonuniform external field the integral is essential. If a straight wire lies along x , current flows in $+x$, and $\mathbf{B} = B_0(1 + x/L)\hat{\mathbf{k}}$ for $0 < x < L$, then

$$\mathbf{F} = -I\hat{\mathbf{j}} \int_0^L B_0(1 + x/L)dx = -\frac{3}{2}IB_0L\hat{\mathbf{j}}. \quad (8.106)$$

The effective field is the average along the active segment. Replacing it by the endpoint value would overestimate the force by one third.

Example 8.17: Suspending a wire: force and current direction

A horizontal active wire segment has length 0.60 m and mass 12 g. It lies perpendicular to a horizontal field of magnitude 0.40 T. Neglect the weight and forces on its flexible leads. What current makes the magnetic force balance its weight?

Model. The two forces on the segment are gravity downward and magnetic force upward. Take the wire direction as $+x$, vertical as $+y$ and the field along $-z$. Since $\hat{\mathbf{i}} \times (-\hat{\mathbf{k}}) = +\hat{\mathbf{j}}$, conventional current along $+x$ gives the required force.

$$I = \frac{mg}{LB} = \frac{(0.012)(9.81)}{(0.60)(0.40)} = 0.49 \text{ A}.$$

Reversing the current would add a downward magnetic force rather than support the wire. The equation establishes force balance at the chosen position; it does not establish stability against every possible displacement or rotation.

8.8 Torque on a current loop

A uniform field exerts zero net force on a closed current loop, yet it can produce a torque. Equal and opposite forces at separated positions form a couple: they cancel in the translational equation but add in the rotational equation. This is the basis of a simple motor and of moving-coil instruments.

For a planar loop define its oriented area $A\hat{\mathbf{n}}$. Curl the fingers of the right hand in the direction of conventional current; the thumb gives $\hat{\mathbf{n}}$. The magnetic dipole moment of N identical turns is

$$\boldsymbol{\mu} = NIA\hat{\mathbf{n}}. \quad (8.107)$$

It is perpendicular to the loop's plane, not parallel to any side of the wire. Figure 8.10 gives complementary views of the geometry.

8.8.1 Deriving the torque from the four sides

Take two sides of length b parallel to y , separated by the vector $a\hat{\mathbf{u}}$, where

$$\hat{\mathbf{u}} = \sin \theta \hat{\mathbf{i}} - \cos \theta \hat{\mathbf{k}}, \quad \mathbf{B} = B\hat{\mathbf{i}}.$$

Let current on the side at $+(a/2)\hat{\mathbf{u}}$ flow along $+y$. The current on the opposite side flows along $-y$. The normal is

$$\hat{\mathbf{n}} = \hat{\mathbf{u}} \times \hat{\mathbf{j}} = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{k}},$$

so θ is indeed the angle between the normal and the field. On these two sides the forces are $-IbB\hat{\mathbf{k}}$ and $+IbB\hat{\mathbf{k}}$. Their magnitudes are independent of θ , but their moment arms are not.

The other two sides have forces parallel to y . They are equal and opposite and have zero net torque about the loop centre. The active-side couple gives

$$\boldsymbol{\tau} = \frac{a}{2}\hat{\mathbf{u}} \times (-IbB\hat{\mathbf{k}}) + \left(-\frac{a}{2}\hat{\mathbf{u}}\right) \times (IbB\hat{\mathbf{k}}) \quad (8.108)$$

$$= IabB \sin \theta \hat{\mathbf{j}}. \quad (8.109)$$

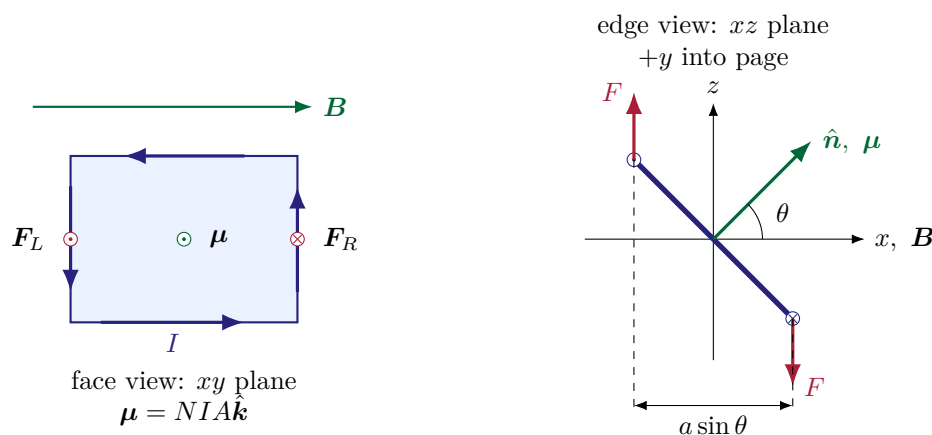


Figure 8.10: Left: a loop in the xy plane with counterclockwise current has moment out of the page. A rightward field gives forces out of the page on the left side and into it on the right. Right: a tilted loop viewed along $+y$ has two active sides perpendicular to the view; their force lines are separated by $a \sin \theta$. The two panels show different loop orientations.

With $A = ab$, the vector result is

$$\boxed{\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}, \quad \tau = NIAB \sin \theta.} \quad (8.110)$$

The factor of two from the two forces cancels the factor of one half in each lever arm. Multiplying the force by the *full* separation between their lines of action is an equivalent way to compute the couple.

At $\theta = 0$, the moment is parallel to the field and the torque is zero. At $\theta = 90^\circ$, the field lies in the loop's plane and the torque is maximal. If an angle is instead given between the *plane* and the field, its cosine enters the magnitude. Confusing these two angles is a common source of otherwise correct-looking errors.

The torque law extends to planar loops of arbitrary shape in a uniform field. One can subdivide a planar area into small rectangular loops: currents on internal shared edges cancel, leaving the original boundary, while the area moments add. In a nonuniform field, a small loop admits a dipole approximation, but an extended loop requires integrating the actual force and torque distribution.

Example 8.18: Torque on a rectangular coil

A rectangular coil has $N = 80$ turns, area $A = 2.5 \times 10^{-3} \text{ m}^2$ and current $I = 0.60 \text{ A}$. It is placed in a uniform magnetic field $B = 0.40 \text{ T}$. The angle between $\boldsymbol{\mu}$ and $\mathbf{B}$ is 60° . Find the torque magnitude.

Solution. The dipole-moment magnitude is

$$\mu = NIA = (80)(0.60)(2.5 \times 10^{-3}) = 0.120 \text{ A m}^2. \quad (8.111)$$

The torque magnitude is

$$\tau = \mu B \sin 60^\circ \quad (8.112)$$

$$= (0.120)(0.40)(0.866) \text{ N m} \quad (8.113)$$

$$= 4.2 \times 10^{-2} \text{ N m}. \quad (8.114)$$

Interpretation. The torque vanishes when the dipole moment is parallel or antiparallel to the field and is maximum when the two vectors are perpendicular.

8.9 Magnetic dipole energy and alignment

The torque can be expressed in terms of an orientation-dependent potential energy when the moment magnitude and external field are held fixed. Choose θ to increase away from alignment, and let τ_θ denote the generalized torque conjugate to this angle. For $0 < \theta < \pi$, the field tends to decrease θ , so

$$\tau_\theta = -\mu B \sin \theta = -\frac{dU}{d\theta}. \quad (8.115)$$

Integrating and setting the zero at $\theta = \pi/2$ gives

$$U(\theta) = -\mu B \cos \theta = -\boldsymbol{\mu} \cdot \mathbf{B}. \quad (8.116)$$

This is an effective mechanical potential for a fixed dipole moment in a prescribed external field. For a powered coil, it is not the total electromagnetic energy of all coils, sources and fields. A complete energy balance also includes the devices that maintain the currents.

Figure 8.11 connects torque and stability. Parallel alignment is an energy minimum; antiparallel alignment is a maximum. Both have zero torque, so zero torque alone is not a stability test.

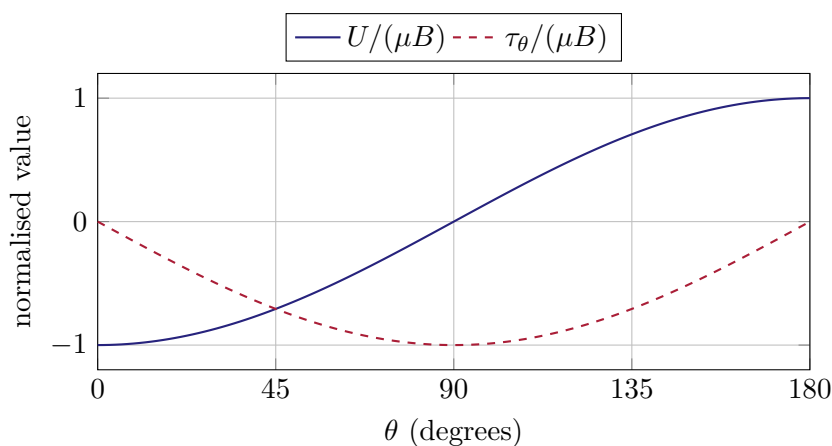


Figure 8.11: Calculated energy and signed angular torque for a fixed magnetic moment. Alignment at 0° is stable; opposition at 180° is unstable. The energy rises as the dipole is turned away from alignment, while the torque opposes that change.

A freely moving, undamped loop does not simply stop at the aligned orientation. It accelerates toward it, can overshoot and oscillate. Dissipation is needed for settling. If the loop is constrained to rotate about one axis and released from rest at θ_0 , conservation of mechanical energy gives

$$\frac{1}{2} I_{\text{rot}} \dot{\theta}^2 - \mu B \cos \theta = -\mu B \cos \theta_0. \quad (8.117)$$

The notation I_{rot} distinguishes rotational inertia from electric current I .

Example 8.19: Work required to turn a coil

A coil with fixed $\mu = 0.050 \text{ A m}^2$ is slowly turned from alignment to opposition in a field $B = 0.80 \text{ T}$. Find the external work in a quasistatic, lossless rotation.

Solution. The external agent balances the field torque at every angle, so its work equals the increase of the effective potential:

$$W_{\text{ext}} = U(\pi) - U(0) = 2\mu B = 0.080 \text{ J}.$$

The field torque performs the opposite work during this imposed rotation. If the coil is released

close to opposition, the effective potential decreases as it turns away; the exact opposed state has zero torque but is unstable.

Check. The units are $(\text{A m}^2)(\text{N}/(\text{A m})) = \text{J}$. The result concerns mechanical rotation at fixed μ ; energy exchanged with the current-maintaining circuit must be included if that circuit is part of the system being studied.

8.10 Motors, loudspeakers and moving-coil instruments

8.10.1 Continuous rotation requires control of torque sign

A single loop with fixed current in a uniform field tends toward alignment and, without suitable switching, cannot sustain rotation against a dissipative load through successive turns. A commutator reverses current at selected angular positions. Since μ reverses with the current, switching changes the torque sign at the appropriate stage of the mechanical cycle. Electronic commutation serves the same purpose in other motor designs.

The single-loop scale $NIAB$ is useful, but it is not a complete motor model. Multiple windings reduce angular intervals of weak torque, and the field geometry determines where conductor forces add. The electrical source supplies the energy delivered to a load and lost as heat. In steady operation,

$$P_{\text{in}} = P_{\text{mechanical}} + P_{\text{losses}}, \quad P_{\text{mechanical}} = \tau_L \omega_{\text{mech}}. \quad (8.118)$$

Induction adds a back emf to the circuit equation once the coil moves. The present force law determines torque for a given current; the later circuit and induction analysis determines that current self-consistently.

Example 8.20: Coil current with a thermal constraint

A coil has $N = 100$, $A = 8.0 \times 10^{-4} \text{ m}^2$ and operates in $B = 0.25 \text{ T}$. Its resistance is $R = 4.0 \Omega$. What current gives maximum torque 0.020 N m , and is it consistent with an assumed continuous Joule-heating allowance of 3.0 W ?

Electromagnetic model. Maximum torque requires the normal perpendicular to the field:

$$I = \frac{0.020}{(100)(8.0 \times 10^{-4})(0.25)} = 1.0 \text{ A}.$$

At that current $I^2 R = 4.0 \text{ W}$, which exceeds the specified allowance. The thermal current limit is $I_{\text{max}} = \sqrt{3.0/4.0} = 0.866 \text{ A}$, giving at most

$$\tau_{\text{max}} = NI_{\text{max}}AB = 0.0173 \text{ N m}.$$

Design interpretation. With these assumptions the target cannot be sustained continuously. A change in area, field, winding resistance or cooling is needed. Increasing turns at fixed current increases torque but usually also changes resistance and thermal loading; it is not an independent improvement at fixed electrical power.

8.10.2 Why a voice coil can have a net force

A loudspeaker's circular voice coil lies in an approximately radial field in an annular gap. Each local current element is tangential while the field is radial. Their cross product is axial, with the same sense around the circumference. The field vectors vary in direction around the loop, so the theorem of zero net force on a closed loop in a *uniform vector field* does not apply.

If N turns of radius a lie fully in a gap with nearly constant radial magnitude B , the active length is $L_{\text{active}} = N2\pi a$. The axial force magnitude is

$$F \simeq BL_{\text{active}}I \equiv k_F I. \quad (8.119)$$

The force factor k_F has units N/A. When the coil leaves the uniform part of the gap, k_F becomes position dependent; distortion can then occur even if the current is sinusoidal.

Example 8.21: Voice-coil force and active length

A voice coil has $k_F = 6.5 \text{ N/A}$ and carries a current of amplitude 0.40 A . The gap field is approximated as radial with magnitude 0.50 T . Find the force amplitude and effective active length.

Solution. $F_0 = k_F I_0 = 2.6 \text{ N}$ and $L_{\text{active}} = k_F/B = 13 \text{ m}$. This is the sum of the circumferences of all active turns, not the coil's axial height.

Interpretation. The force drives a mass, suspension and acoustic load. For a simple linear mechanical model,

$$M\ddot{x} + b\dot{x} + kx = k_F I(t).$$

This connects magnetic force to the driven-oscillator physics of Physics I. The force amplitude alone does not determine sound pressure or efficiency.

8.10.3 A meter with a radial field

A torsion spring provides a restoring torque $-\kappa\phi$ for a moving-coil meter. In a properly shaped radial field, the effective angle between the coil's normal and the local field stays close to 90° throughout its useful range. The magnetic torque is then approximately $NIAB$, and equilibrium gives

$$\phi = \frac{NAB}{\kappa} I. \quad (8.120)$$

The approximately linear scale follows from the radial field geometry. A coil in a uniform fixed-direction field generally has a torque containing an angle-dependent sine or cosine and does not automatically have a linear response over a large range. Calibration should distinguish this geometry from the fixed-field torque derivation.

8.11 Laboratory connections: reversal, calibration and inference

A formula becomes an experimental model only after specifying the measured quantity, geometry and nuisance effects. The following three investigations use the chapter's force laws while teaching complementary methods: extracting a slope, removing an offset by reversal, and propagating uncertainty through a nonlinear relation.

8.11.1 Current balance: measuring force rather than inferring its direction

Mount a straight conductor on a force sensor, or place the magnet on a balance while supporting the conductor independently. These are different mechanical systems. A sensor attached to the wire measures its magnetic force; a balance supporting the magnet records the corresponding reaction. State explicitly which object is weighed and assign the sign of the reading accordingly.

For an active segment perpendicular to an approximately uniform field, fit a signed component to

$$F_{\text{meas}} = F_0 + sI, \quad |s| = LB. \quad (8.121)$$

The offset F_0 includes the tare and other forces that remain unchanged when the current reverses. The half-difference

$$F_{\text{odd}}(I) = \frac{F_{\text{meas}}(I) - F_{\text{meas}}(-I)}{2} = sI \quad (8.122)$$

eliminates a constant offset. If a balance displays mass instead of force, multiply its signed mass change by g before comparing with ILB . A reading difference between $+I$ and $-I$ corresponds to *twice* the one-direction magnetic force.

I (A)	-2.0	-1.5	-1.0	-0.5	0	0.5	1.0	1.5	2.0
F (mN)	-22.7	-16.9	-10.7	-4.9	1.1	7.3	13.1	19.3	25.2

Table 8.1: Synthetic teaching data for a wire-force calibration with $L = 0.150$ m. These are model-generated values with deliberately added sub-millinewton deviations, not laboratory observations.

For the synthetic values in Table 8.1, an unweighted least-squares fit gives $s = 0.0120$ N/A and $F_0 = 0.00120$ N. Thus $B = |s|/L = 0.0800$ T in the stated model. A fit forced through the origin would ignore a real nuisance parameter. In this symmetric current set the offset happens not to bias the fitted slope, but that protection disappears for an asymmetric current set.

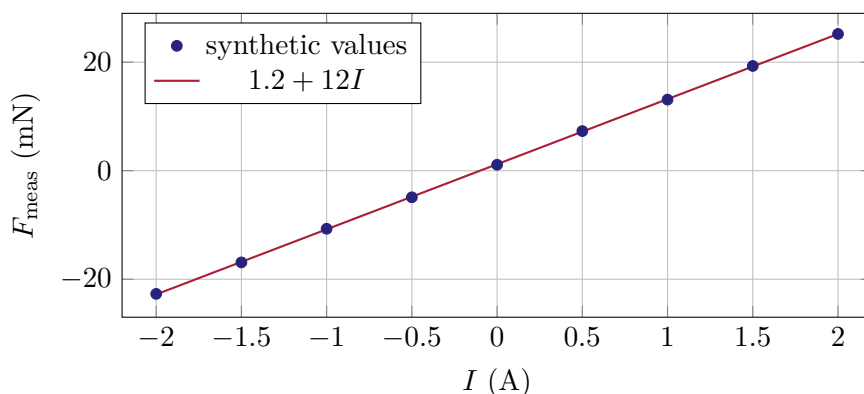


Figure 8.12: Synthetic calibration data and their fitted linear model. The slope measures LB ; the intercept measures an offset. Deviations are much smaller than the plotted full force range, so a residual plot would be needed to inspect them clearly.

Figure 8.12 tests proportionality to current over the chosen range. It does not by itself prove field uniformity. A nonuniform field with fixed geometry can also give a linear relation whose slope is the appropriate line integral of the field.

If independent standard uncertainties are σ_s and σ_L , then

$$\left(\frac{\sigma_B}{B}\right)^2 \simeq \left(\frac{\sigma_s}{s}\right)^2 + \left(\frac{\sigma_L}{L}\right)^2. \quad (8.123)$$

A slope uncertainty obtained from repeated measurements captures random scatter, while an incorrectly chosen active length causes a systematic bias. Near perpendicular alignment, writing the true angle as $90^\circ - \epsilon$ gives $\sin \theta = \cos \epsilon \simeq 1 - \epsilon^2/2$ with ϵ in radians. Small alignment errors then enter at second order. That fact does not justify ignoring large or poorly known angles.

Example 8.22: The factor of two in a reversal measurement

The balance reading changes by 0.96 g when current reverses from $+2.0$ A to -2.0 A. The active wire length is 0.30 m. Estimate the field magnitude, assuming the change is entirely magnetic.

Solution. Reversal changes $+F$ into $-F$, so the change is $2F$:

$$F = \frac{0.00096 \times 9.81}{2} = 4.71 \times 10^{-3} \text{ N}, \quad B = \frac{F}{IL} = 7.9 \times 10^{-3} \text{ T}.$$

Failing to halve the reading difference would overestimate B by a factor of two. The magnitude does not specify the sign of the force on the wire; that requires the mechanical mounting and the field-current directions.

Reproducing the calibration fit and its residuals

The fitted slope should be reproducible from the data, not accepted as an unexplained number. Let n_d be the number of readings, let $\bar{I}$ and $\bar{F}$ be their arithmetic means, and minimize

$$S(F_0, s) = \sum_{i=1}^{n_d} (F_i - F_0 - sI_i)^2.$$

Setting the two derivatives to zero gives

$$s = \frac{\sum_i (I_i - \bar{I})(F_i - \bar{F})}{\sum_i (I_i - \bar{I})^2}, \quad F_0 = \bar{F} - s\bar{I}. \quad (8.124)$$

For the synthetic table, $n_d = 9$, $\bar{I} = 0$ and $\bar{F} = 1.20 \text{ mN}$. Also $\sum I_i^2 = 15.0 \text{ A}^2$ and $\sum I_i F_i = 180 \text{ A mN}$, giving $s = 12.0 \text{ mN/A}$. This is the value used in the field inference above.

The residuals $\epsilon_i = F_i - (F_0 + sI_i)$ are shown in Figure 8.13. Residuals reveal deviations that are hidden by the full vertical range of the calibration plot. A systematic curve could indicate heating, geometry changes or a nonlinear sensor response; random scatter alone does not identify its physical cause.

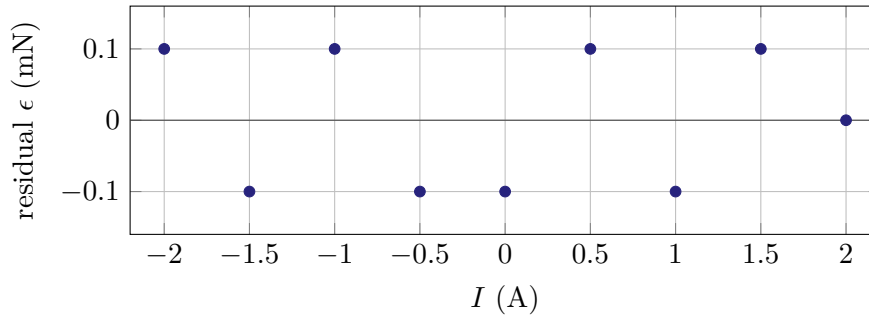


Figure 8.13: Residuals of the synthetic wire-force calibration. These small, deliberately chosen deviations are visible on their own scale. The plot illustrates a diagnostic technique; it is not evidence of the noise distribution of a physical instrument.

Under a model of independent equal-variance force errors and negligible current errors, the residual estimate of force scatter is

$$u_F = \sqrt{\frac{\sum_i \epsilon_i^2}{n_d - 2}} = \sqrt{\frac{0.0800}{7}} \text{ mN} = 0.107 \text{ mN}. \quad (8.125)$$

Two fitted parameters account for the subtraction of two degrees of freedom. The associated slope standard uncertainty is

$$\sigma_s = \frac{u_F}{\sqrt{\sum_i (I_i - \bar{I})^2}} = 0.0276 \text{ mN/A}, \quad (8.126)$$

or 0.230% relative to the slope. If the length has independent uncertainty $\sigma_L = 0.0005$ m at $L = 0.150$ m, then $\sigma_B/B = 0.405\%$, giving an illustrative $B = (0.08000 \pm 0.00032)$ T. This uncertainty calculation is conditional on the statistical model; deliberately created synthetic data cannot validate that model experimentally.

A useful counterexample uses only the five nonnegative-current readings and constrains $F_0 = 0$. The resulting slope is $\sum I_i F_i / \sum I_i^2 = 12.813$ mN/A, about 6.8% too large. The regression has absorbed part of the offset into its slope. An attractive straight line is therefore insufficient: the fit model must include plausible nuisance parameters and its assumptions must be checked.

8.11.2 Hall probe: field component, offset and angular response

A simple calibrated probe has output

$$V_{\text{out}} = V_0 + SB_n, \quad B_n = \mathbf{B} \cdot \hat{\mathbf{n}}_{\text{probe}}. \quad (8.127)$$

Here S is a signed sensitivity fixed by the wiring and orientation. Rotate a probe in a constant field through an angle β between its normal and the field. The expected response is $V_0 + SB \cos \beta$. A zero output after offset subtraction can therefore mean a zero normal component rather than a zero field.

A useful calibration uses equal positive and negative reference components:

$$S = \frac{V(+B_{\text{ref}}) - V(-B_{\text{ref}})}{2B_{\text{ref}}}, \quad V_0 = \frac{V(+B_{\text{ref}}) + V(-B_{\text{ref}})}{2}. \quad (8.128)$$

For example, synthetic outputs of 1.024 V and 0.976 V at ± 0.20 T give $V_0 = 1.000$ V and $S = 0.120$ V/T. An output of 1.006 V then corresponds to $B_n = 0.050$ T. Treating the raw output as a Hall voltage without subtracting the offset would be meaningless.

A field map can test whether the uniform-field assumption is appropriate over a particle orbit or wire segment. Record position and orientation as well as output. Report the variation over the region of interest, for example $(B_{\text{max}} - B_{\text{min}})/|\overline{B}|$, and connect that variation to the accuracy required in the intended force or radius calculation. If the mean is close to zero this fractional measure is ill-conditioned; use an absolute tolerance instead.

8.11.3 Electron-beam curvature: a nonlinear measurement

For acceleration from rest followed by perpendicular bending, the inferred ratio is $\eta = e/m_e = 2V_{\text{acc}}/(B^2 r^2)$. Logarithmic differentiation gives

$$\frac{d\eta}{\eta} = \frac{dV_{\text{acc}}}{V_{\text{acc}}} - 2\frac{dB}{B} - 2\frac{dr}{r}. \quad (8.129)$$

This signed relation describes how systematic biases propagate. A radius overestimate lowers the inferred ratio. For independent small random uncertainties,

$$\left(\frac{\sigma_\eta}{\eta}\right)^2 \simeq \left(\frac{\sigma_V}{V_{\text{acc}}}\right)^2 + 4\left(\frac{\sigma_B}{B}\right)^2 + 4\left(\frac{\sigma_r}{r}\right)^2. \quad (8.130)$$

The factors of four arise by squaring the exponents, not because the measurements are repeated four times. If errors are correlated, covariance terms must be included.

Track thickness, perspective and parallax can bias a radius. A diameter measured across the visible track must be divided by two. Beam alignment also matters: an oblique beam forms a helix and its transverse radius depends on $v_\perp$, whereas the accelerating voltage sets total speed. Using that radius as though all kinetic energy were perpendicular gives a biased result. The laboratory model must therefore include both the optical measurement and the particle geometry.

8.12 Computational connection: trajectory simulation

The Lorentz force naturally leads to numerical simulation. For a particle in prescribed fields,

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad \frac{d\mathbf{r}}{dt} = \mathbf{v}. \quad (8.131)$$

A simple numerical model updates position and velocity in small time steps. However, magnetic motion is a useful warning: a naive Euler method can slowly change the speed even when the exact magnetic force does no work. A good algorithm should respect the near-constant speed in a static magnetic field.

For introductory work, students can still learn much from a small-step simulation: circular motion, helix pitch, effect of charge sign, crossed-field deflection and radius fitting. The advanced section discusses the Boris pusher, a standard particle-tracking idea that handles magnetic rotation more robustly than naive Euler stepping.

Example 8.23: Numerical time-step scale for magnetic motion

An electron moves in a uniform magnetic field $B = 5.0$ mT. Estimate the cyclotron period and suggest a time step that resolves the motion with about 100 steps per revolution.

Solution. The angular frequency is

$$\omega = \frac{eB}{m_e} = \frac{(1.602 \times 10^{-19})(5.0 \times 10^{-3})}{9.11 \times 10^{-31}} = 8.79 \times 10^8 \text{ s}^{-1}. \quad (8.132)$$

The period is

$$T = \frac{2\pi}{\omega} = 7.15 \times 10^{-9} \text{ s}. \quad (8.133)$$

For 100 steps per revolution,

$$\Delta t \approx 7 \times 10^{-11} \text{ s}. \quad (8.134)$$

Interpretation. Magnetic particle tracking can require very small time steps for light particles in even modest fields. This is one reason specialized integration algorithms are important.

8.12.1 Choose diagnostics before choosing a method

A trajectory plot alone can conceal a wrong simulation. A slowly expanding spiral may look circular over a short interval, and a reflected plot can hide a sign error. Before integrating, write down the expected observables: constant speed, constant $v_{\parallel}$, radius $mv_{\perp}/(|q|B)$, period $2\pi m/(|q|B)$, and the initial direction of acceleration.

For uniform $\mathbf{B} = B\hat{\mathbf{k}}$ and $q > 0$, define dimensionless time $\tau = \omega_c t$, velocity $\mathbf{u} = \mathbf{v}/v_0$, and position $\mathbf{X} = \mathbf{r}/r_0$, where $r_0 = v_0/\omega_c$. The planar equations become

$$\frac{du_x}{d\tau} = u_y, \quad \frac{du_y}{d\tau} = -u_x, \quad \frac{dX}{d\tau} = u_x, \quad \frac{dY}{d\tau} = u_y. \quad (8.135)$$

This form avoids extremely small electron time scales in the first code and isolates the numerical problem from unit conversions. Dimensional results are restored only after validating the dimensionless model.

For $\mathbf{u}(0) = (1, 0)$ and $\mathbf{X}(0) = (0, 0)$, the exact result is $X = \sin \tau$, $Y = \cos \tau - 1$. Compare the simulated position with this solution at matched times, not merely with any circle of similar size. Repeat with negative charge, halved time step and a nonzero parallel component. In crossed fields, a launch with the vector cancellation condition should remain straight; an off-selected launch should agree with the drift-plus-gyration solution developed below.

A practical report should distinguish three questions: Does the algorithm preserve the appropriate invariants? Does it accumulate the correct orbital phase? Does it converge as the time step decreases? The enrichment section supplies an executable algorithm and calculated comparisons showing why these are separate tests.

8.13 Advanced Topics for Technical and Engineering Students

The following modules deepen the same force models rather than introducing a separate chapter. They require simple coupled differential equations, matrix algebra and first-order uncertainty propagation. The topics are linked by an experimental question: how can a predicted trajectory or force be used reliably to infer a particle property, a material parameter or a device response?

8.13.1 Coordinate solution of motion in a uniform magnetic field

Let $\mathbf{B} = B\hat{\mathbf{k}}$ and $\mathbf{E} = \mathbf{0}$. The Lorentz equation becomes

$$m \frac{d\mathbf{v}}{dt} = q\mathbf{v} \times B\hat{\mathbf{k}}. \quad (8.136)$$

Writing $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ gives

$$\dot{v}_x = \Omega v_y, \quad (8.137)$$

$$\dot{v}_y = -\Omega v_x, \quad (8.138)$$

$$\dot{v}_z = 0, \quad (8.139)$$

where

$$\Omega = \frac{qB}{m} \quad (8.140)$$

is the signed cyclotron frequency. Differentiating the first equation gives

$$\ddot{v}_x = \Omega \dot{v}_y = -\Omega^2 v_x. \quad (8.141)$$

Thus v_x and v_y oscillate sinusoidally, while v_z remains constant. The sign of Ω determines the sense of rotation.

With initial conditions $v_x(0) = v_{x0}$ and $v_y(0) = v_{y0}$,

$$v_x(t) = v_{x0} \cos(\Omega t) + v_{y0} \sin(\Omega t), \quad (8.142)$$

$$v_y(t) = v_{y0} \cos(\Omega t) - v_{x0} \sin(\Omega t), \quad (8.143)$$

$$v_z(t) = v_{z0}. \quad (8.144)$$

Integrating once more gives the trajectory. This solution makes the conservation of $v_x^2 + v_y^2$ explicit.

Example 8.24: Analytic trajectory for a positive charge

A positive particle starts at the origin with $\mathbf{v}(0) = v_0\hat{\mathbf{i}}$ in a uniform field $\mathbf{B} = B\hat{\mathbf{k}}$. Find $x(t)$ and $y(t)$.

Solution. Here $\Omega = qB/m > 0$, $v_{x0} = v_0$ and $v_{y0} = 0$. The velocity components are

$$v_x(t) = v_0 \cos \Omega t, \quad (8.145)$$

$$v_y(t) = -v_0 \sin \Omega t. \quad (8.146)$$

Integrating from zero initial position,

$$x(t) = \frac{v_0}{\Omega} \sin \Omega t, \quad (8.147)$$

$$y(t) = \frac{v_0}{\Omega} (\cos \Omega t - 1). \quad (8.148)$$

This is a circle of radius $r = v_0/\Omega = mv_0/(qB)$ centred at $(0, -r)$.

Interpretation. The negative y curvature agrees with the right-hand rule: $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$ for a positive charge.

General positions and the guiding centre

For $\Omega = qB/m \neq 0$, integration of the component solution gives

$$x(t) = x_0 + \frac{v_{x0}}{\Omega} \sin \Omega t + \frac{v_{y0}}{\Omega} (1 - \cos \Omega t), \quad (8.149)$$

$$y(t) = y_0 + \frac{v_{y0}}{\Omega} \sin \Omega t - \frac{v_{x0}}{\Omega} (1 - \cos \Omega t), \quad (8.150)$$

$$z(t) = z_0 + v_{z0}t. \quad (8.151)$$

The apparently singular factors $1/\Omega$ have regular limits as $B \rightarrow 0$. Using $\sin \Omega t \simeq \Omega t$ and $1 - \cos \Omega t \simeq \Omega^2 t^2/2$ recovers straight-line motion at fixed finite time.

The transverse centre is constant:

$$X_c = x + \frac{v_y}{\Omega} = x_0 + \frac{v_{y0}}{\Omega}, \quad Y_c = y - \frac{v_x}{\Omega} = y_0 - \frac{v_{x0}}{\Omega}. \quad (8.152)$$

Differentiating confirms $\dot{X}_c = \dot{Y}_c = 0$. Also

$$(x - X_c)^2 + (y - Y_c)^2 = \frac{v_x^2 + v_y^2}{\Omega^2} = \left(\frac{v_\perp}{|\Omega|} \right)^2. \quad (8.153)$$

This proves the circular transverse orbit without guessing its centre from a sketch. In a nonuniform field these simple centre coordinates are generally not constant.

The velocity equation can be written $\dot{\mathbf{v}} = \mathbf{A}\mathbf{v}$ with

$$\mathbf{A} = \begin{pmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{A}^\top = -\mathbf{A}. \quad (8.154)$$

Antisymmetry implies $d(\mathbf{v}^\top \mathbf{v})/dt = 0$. The exact matrix exponential is a rotation in the transverse plane. This observation provides both a compact proof of energy conservation and a design principle for a numerical method.

8.13.2 Crossed fields as drift plus gyration

The undeflected velocity is also the mean transverse velocity of a more general trajectory in uniform crossed fields. Let the fields be constant with $\mathbf{E} \cdot \mathbf{B} = 0$, and define

$$\mathbf{v}_D = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \quad (8.155)$$

The vector identity $(\mathbf{E} \times \mathbf{B}) \times \mathbf{B} = \mathbf{B}(\mathbf{E} \cdot \mathbf{B}) - \mathbf{E}B^2$ gives $\mathbf{v}_D \times \mathbf{B} = -\mathbf{E}$. Substitute $\mathbf{v} = \mathbf{v}_D + \mathbf{u}$ into the nonrelativistic equation of motion:

$$m\dot{\mathbf{u}} = q[\mathbf{E} + (\mathbf{v}_D + \mathbf{u}) \times \mathbf{B}] = q\mathbf{u} \times \mathbf{B}. \quad (8.156)$$

Because $\mathbf{v}_D$ is constant, the remaining velocity is a uniform magnetic rotation. Its mean transverse component over a gyroperiod is zero. The guiding centre therefore moves with $\mathbf{v}_D$, independently of charge sign and mass. The gyration radius and period still depend on both.

This is a change of variables in the nonrelativistic equations, not a derivation of relativistic field transformations. It requires velocities small compared with c . If E/B is comparable to or greater than c , the nonrelativistic drift interpretation cannot be used as a physical particle velocity.

An explicit trajectory and its energy

Take $\mathbf{E} = E\hat{\mathbf{j}}$, $\mathbf{B} = B\hat{\mathbf{k}}$ and $v_D = E/B$. A positive particle released from rest has $u_x(0) = -v_D$, $u_y(0) = 0$, so

$$v_x(t) = v_D(1 - \cos \Omega t), \quad v_y(t) = v_D \sin \Omega t, \quad (8.157)$$

$$x(t) = v_D \left(t - \frac{\sin \Omega t}{\Omega} \right), \quad y(t) = \frac{v_D}{\Omega} (1 - \cos \Omega t). \quad (8.158)$$

This is a cycloid. The laboratory speed is not constant:

$$K(t) = mv_D^2(1 - \cos \Omega t) = qEy(t). \quad (8.159)$$

The electric field performs work as the particle moves in y . Over a complete gyroperiod, y and the kinetic energy return to their original values even though the guiding centre has moved in x by $v_D T_c$. There is no conflict between a nonzero instantaneous electric power and zero net work over the full period.

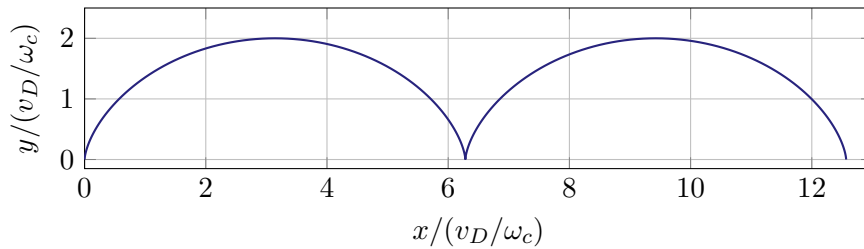


Figure 8.14: Calculated cycloid for a positive charge released from rest in $\mathbf{E} = E\hat{\mathbf{j}}$ and $\mathbf{B} = B\hat{\mathbf{k}}$. The drift is along $+x$. Cusps occur when the instantaneous laboratory velocity vanishes, once per gyroperiod.

Figure 8.14 distinguishes instantaneous motion from the guiding-centre motion. A negative charge has the same $+x$ drift but opposite transverse excursion for this release condition. A particle launched exactly with $\mathbf{v} = \mathbf{v}_D$ has $\mathbf{u} = 0$ and follows a straight line instead of a cycloid.

Example 8.25: Drift speed and gyration are separate scales

Let $E = 600 \text{ V/m}$ and $B = 0.020 \text{ T}$. Find the drift speed of a proton and an electron, and the maximum transverse displacement of each if released from rest.

Solution. Both have $v_D = E/B = 3.0 \times 10^4 \text{ m/s}$. Equation (8.158) gives excursion magnitude $2v_D/\omega_c = 2mv_D/(eB)$:

$$y_{\max,p} = 3.1 \times 10^{-2} \text{ m}, \quad |y_{\min,e}| = 1.7 \times 10^{-5} \text{ m}.$$

The proton moves initially toward $+y$, the electron toward $-y$. The different excursion scales arise from their masses, while the identical mean drift follows from cancellation of q in the force-free velocity.

If an electric field has a component along $\mathbf{B}$, no magnetic force can cancel it. Uniform $E_{\parallel}$ gives $\dot{v}_{\parallel} = qE_{\parallel}/m$, while the perpendicular part can still be decomposed as above. Collisions, field gradients and time-dependent fields introduce additional physics; the simple drift formula should not be applied to all such situations without analysis.

8.13.3 Magnetic rigidity and relativistic bending

Momentum is a better quantity than speed for comparing magnetic bending across energy regimes. For a circular orbit with momentum perpendicular to a uniform field, the momentum vector rotates

through angle $d\phi = vdt/r$. Its change has magnitude $dp_{\text{vector}} = p d\phi$, so

$$\left| \frac{d\mathbf{p}}{dt} \right| = \frac{pv}{r} = |q|vB \implies p = |q|Br. \quad (8.160)$$

The *magnetic rigidity* is

$$B\rho = \frac{p}{|q|}. \quad (8.161)$$

Here ρ denotes the bending radius for motion perpendicular to $\mathbf{B}$. For a helix, the transverse projection gives $p_{\perp} = |q|Br_L$; it does not directly give the total momentum. If the pitch angle is known, $p = p_{\perp}/\sin \alpha$.

Relativistically, $p = \gamma mv$ with $\gamma = (1 - v^2/c^2)^{-1/2}$. A purely magnetic force conserves the particle energy and hence γ . The gyrofrequency is therefore

$$\omega_c = \frac{|q|B}{\gamma m}, \quad T_c = \frac{2\pi\gamma m}{|q|B}. \quad (8.162)$$

The invariant mass m has not changed. It is the relationship between momentum and velocity that differs from the nonrelativistic approximation.

When kinetic energy K is specified, the relativistic identity

$$(K + mc^2)^2 = p^2c^2 + m^2c^4 \quad (8.163)$$

gives

$$pc = \sqrt{K(K + 2mc^2)}. \quad (8.164)$$

For $K \ll mc^2$ this reduces to $p \simeq \sqrt{2mK}$. The form avoids using the nonrelativistic energy formula at energies where it is inaccurate.

A convenient engineering conversion, with $q = ze$, is

$$p[\text{GeV}/c] \simeq 0.2998 |z| B[\text{T}] \rho[\text{m}]. \quad (8.165)$$

To obtain the factor, use $1 \text{ GeV}/c = (10^9 \times 1.602 \times 10^{-19} \text{ J})/c \simeq 5.34 \times 10^{-19} \text{ kg m/s}$ and substitute $p = |z|eB\rho$. The bracketed units indicate numerical values in those units; they are not additional factors in an SI equation.

Example 8.26: A relativistic electron track

An electron has kinetic energy $K = 1.00 \text{ MeV}$ and travels perpendicular to a uniform field of 0.10 T . Use $m_e c^2 = 0.511 \text{ MeV}$ to find its radius and compare with a nonrelativistic estimate.

Solution. Equation (8.164) gives $pc = \sqrt{1.00(1.00 + 1.022)} = 1.422 \text{ MeV}$. Thus

$$r = \frac{0.001422}{0.2998(0.10)} = 0.0474 \text{ m}.$$

The nonrelativistic expression would give $pc = \sqrt{2(0.511)(1.00)} = 1.011 \text{ MeV}$ and $r = 0.0337 \text{ m}$, about 29% too small. Here $\gamma = 1 + 1.00/0.511 = 2.96$, so a nonrelativistic treatment is not justified.

Interpretation. The bending equation $p = eBr$ remains valid; the failed step is replacing relativistic momentum by $\sqrt{2mK}$. Charge sign determines the sense of curvature separately.

8.13.4 The cyclotron: a timing argument

In an ideal cyclotron, a uniform magnetic field bends a charged particle through semicircles inside two conducting dees. An alternating voltage accelerates it in the gap. The magnetic field sets the half-turn time $T_c/2$; the gap field supplies the energy gain. Neglecting the finite gap transit time, the voltage must reverse polarity after every half-turn so that the electric force accelerates the particle at each crossing.

For nonrelativistic motion, $T_c = 2\pi m/(|q|B)$ does not depend on radius. Increasing kinetic energy therefore increases the radius without changing the required radio-frequency period. With gap voltage magnitude V_g , a particle crossing at the accelerating phase gains approximately $|q|V_g$ per crossing, or $2|q|V_g$ per revolution. The energy gain occurs in short intervals at the gap, not continuously along a magnetic arc.

At a maximum orbit radius R , the ideal nonrelativistic extraction energy is

$$K_{\max} = \frac{q^2 B^2 R^2}{2m}. \quad (8.166)$$

This expression must be checked against $K_{\max} \ll mc^2$. At larger energies T_c grows with γ , producing phase slip in a fixed-frequency, uniform-field cyclotron. Frequency modulation or a suitably varying field can compensate within a more advanced design. In a synchrotron, approximately fixed orbit geometry instead requires the bending field to track momentum, $B\rho = p/|q|$, while the accelerating system stays synchronized with the revolution frequency. These are timing and field-control principles, not claims about particular current machines.

Example 8.27: Cyclotron radius and the size of a correction

A proton cyclotron has $B = 0.80$ T and outer orbit radius $R = 0.25$ m. Estimate its nonrelativistic frequency and extraction energy.

$$f_c = \frac{eB}{2\pi m_p} = 1.22 \times 10^7 \text{ Hz}, \quad K_{\max} = \frac{e^2 B^2 R^2}{2m_p} = 3.07 \times 10^{-13} \text{ J} = 1.92 \text{ MeV}.$$

The implied $v = eBR/m_p = 1.92 \times 10^7$ m/s $= 0.0640c$. The fractional period correction is approximately $\gamma - 1 \simeq v^2/(2c^2) = 0.00205$. This is small for a first estimate, but accumulated timing error over many turns can still matter. “Nonrelativistic enough” depends on the observable and required precision.

8.13.5 Curvature, sagitta and measurement resolution

A tracker usually sees an arc, not a full orbit. The sagitta s is the greatest distance between a circular arc and its chord of length L . Geometry gives

$$(R - s)^2 + (L/2)^2 = R^2 \implies R = \frac{L^2}{8s} + \frac{s}{2}. \quad (8.167)$$

The familiar $R \simeq L^2/(8s)$ assumes $s \ll L$, or equivalently weak bending. The neglected term has fractional size approximately $4s^2/L^2$ relative to the leading term. Equation (8.167) is exact for a circular arc; detector tracks that lose energy or traverse an inhomogeneous field need a richer model.

In Figure 8.15, a nearly horizontal chord gives the estimator $s = y_2 - (y_1 + y_3)/2$. For three independent equal vertical position uncertainties σ_x ,

$$\sigma_s = \sqrt{\sigma_x^2 + \frac{1}{4}\sigma_x^2 + \frac{1}{4}\sigma_x^2} = \sqrt{\frac{3}{2}} \sigma_x. \quad (8.168)$$

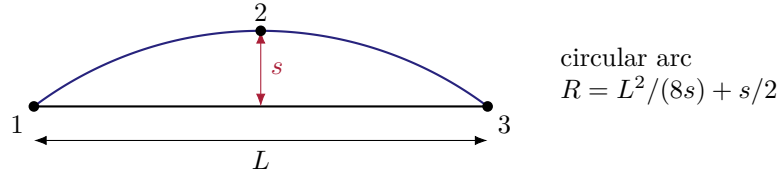


Figure 8.15: Three position samples define a sagitta for a symmetric arc. The curve is drawn from a circle, with its bending enlarged for readability. Small errors in the midpoint relative to the two endpoints become important when the sagitta is small.

The sign of s can encode charge sign once the field and direction of motion are known. The radius formula uses the magnitude for a positive radius.

For perpendicular motion and small sagitta, $p \simeq |q|BL^2/(8s)$. For independent small uncertainties and a known charge state,

$$\left(\frac{\sigma_p}{p}\right)^2 \simeq \left(\frac{\sigma_B}{B}\right)^2 + 4\left(\frac{\sigma_L}{L}\right)^2 + \left(\frac{\sigma_s}{s}\right)^2. \quad (8.169)$$

At fixed position resolution the fractional momentum uncertainty grows as the sagitta shrinks. A longer lever arm helps quadratically in this simplified model. Multiple scattering, alignment, energy loss and field-map errors can dominate in a real detector and are not represented by this equation.

Example 8.28: From a measured sagitta to momentum precision

A singly charged track in $B = 1.0\text{ T}$ has $L = 0.40\text{ m}$ and $s = 2.0\text{ mm}$. Each of three symmetric position measurements has independent uncertainty $50\text{ }\mu\text{m}$. Assume the field and chord are known precisely.

Geometry. The exact radius is $R = 0.40^2/[8(0.0020)] + 0.0020/2 = 10.001\text{ m}$. The small-sagitta approximation gives 10.000 m , a negligible change here.

Momentum and uncertainty. $p = eBR = 1.60 \times 10^{-18}\text{ kg m/s} \simeq 3.00\text{ GeV}/c$. Also $\sigma_s = 61\text{ }\mu\text{m}$, so $\sigma_p/p \simeq 0.0306$, or 3.1% .

Interpretation. The approximation error in the radius is far smaller than the measurement uncertainty. Improving an already adequate geometric approximation would bring little benefit; improving position resolution or lever arm would.

For more than three points one can fit a circle. Expanding $(x - a)^2 + (y - b)^2 = R^2$ yields a linear algebraic form $x^2 + y^2 = 2ax + 2by + c$, with $R^2 = a^2 + b^2 + c$. An ordinary least-squares fit of this form is a useful initial estimate, but errors in both coordinates mean that it is not generally an unbiased or optimal fit. A more faithful model minimizes radial residuals $\sqrt{(x_i - a)^2 + (y_i - b)^2} - R$, weighted according to the measurement covariance. Very short arcs are poorly conditioned: many large circles fit them almost equally well. Report this limitation rather than interpreting a small algebraic residual as proof of an accurate momentum.

8.13.6 A transport model of the Hall coefficient

The transverse balance can be embedded in a simple dissipative transport model. Assume identical carriers of charge q_c , density n , effective mass m_* and a constant relaxation time τ_r . Their average drift obeys

$$m_* \dot{\mathbf{v}}_d = q_c(\mathbf{E} + \mathbf{v}_d \times \mathbf{B}) - \frac{m_*}{\tau_r} \mathbf{v}_d. \quad (8.170)$$

The friction term models momentum relaxation to the material. It is a phenomenological average, not the trajectory equation of one collision-free particle.

At steady state, substituting $\mathbf{J} = nq_c \mathbf{v}_d$ gives

$$\mathbf{E} = \rho_0 \mathbf{J} + R_H \mathbf{B} \times \mathbf{J}, \quad \rho_0 = \frac{m_*}{nq_c^2 \tau_r}, \quad R_H = \frac{1}{nq_c}. \quad (8.171)$$

For $\mathbf{B} = B_z \hat{\mathbf{k}}$, the in-plane resistivity relation is

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_0 & -R_H B_z \\ R_H B_z & \rho_0 \end{pmatrix} \begin{pmatrix} J_x \\ J_y \end{pmatrix}. \quad (8.172)$$

Open transverse contacts enforce $J_y = 0$, immediately reproducing $E_y = R_H J_x B_z$. The antisymmetric Hall part does not dissipate power:

$$\mathbf{J} \cdot \mathbf{E} = \rho_0 J^2 + R_H \mathbf{J} \cdot (\mathbf{B} \times \mathbf{J}) = \rho_0 J^2. \quad (8.173)$$

The longitudinal term is responsible for Joule heating. In this ideal single-carrier model $E_x/J_x = \rho_0$ at $J_y = 0$ is independent of magnetic field; observed magnetoresistance requires physics beyond this particular model.

Why two carrier types change the interpretation

Define a positive mobility $b_s = |q_s| \tau_s / m_s$ for each species; the letter b_s avoids confusion with magnetic moment μ . Consider electrons with density n_e and holes with density n_h , charges $-e, +e$, and mobilities b_e, b_h . In the weak-field limit $b_s |B_z| \ll 1$, the conductivity of each population is, to first order in field,

$$\begin{pmatrix} J_{xs} \\ J_{ys} \end{pmatrix} \simeq \sigma_s \begin{pmatrix} 1 & \beta_s \\ -\beta_s & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \quad \sigma_s = n_s e b_s, \quad \beta_s = \frac{q_s}{e} b_s B_z. \quad (8.174)$$

Conductivities add because both populations respond to the same electric field. Setting the total J_y to zero and retaining terms through first order in B_z gives

$$R_H \simeq \frac{n_h b_h^2 - n_e b_e^2}{e(n_e b_e + n_h b_h)^2}. \quad (8.175)$$

A carrier population is weighted by mobility squared in the numerator. The Hall sign therefore need not equal the sign of the more numerous carriers. Nor can one infer a total density simply by taking $1/(e|R_H|)$.

Example 8.29: A minority population can set the Hall sign

Take an illustrative two-carrier material with $n_e = 4n_0$, $n_h = n_0$, $b_e = b_0$ and $b_h = 3b_0$. Determine the low-field Hall sign and the density one would incorrectly infer with a single-carrier formula.

Solution. The numerator is $(9 - 4)n_0 b_0^2 > 0$, whereas the denominator is $e(4 + 3)^2 n_0^2 b_0^2$. Hence

$$R_H = \frac{5}{49en_0} > 0, \quad n_{\text{apparent}} = \frac{1}{eR_H} = \frac{49}{5}n_0 = 9.8n_0.$$

The actual total mobile density is $5n_0$ and electrons are more numerous. The Hall voltage nevertheless has the positive-carrier sign because holes have higher mobility in this model.

Validity. The result assumes independent, isotropic carrier populations, constant relaxation times and $3b_0|B_z| \ll 1$. It illustrates a model limitation; it is not a characterization of a particular material.

8.13.7 Numerical magnetic rotation: energy and phase accuracy

The exact uniform-field velocity update is a rotation. A forward Euler step instead adds a perpendicular increment:

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \frac{q\Delta t}{m} \mathbf{v}_n \times \mathbf{B}. \quad (8.176)$$

For perpendicular motion, with $\delta = \omega_c \Delta t$,

$$|\mathbf{v}_{n+1}|^2 = (1 + \delta^2)|\mathbf{v}_n|^2, \quad \frac{K_N}{K_0} = (1 + \delta^2)^N. \quad (8.177)$$

There is artificial energy growth for every nonzero time step. A smaller step reduces the error over a fixed finite interval but does not turn forward Euler into an energy-preserving method. The failure follows from the geometry of adding a perpendicular vector.

The magnetic midpoint step and its rotation form

Evaluate the magnetic force using the average of the old and new velocities:

$$\frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} = \frac{q}{m} \left(\frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2} \right) \times \mathbf{B}. \quad (8.178)$$

Dotting both sides with $\mathbf{v}_{n+1} + \mathbf{v}_n$ proves that $v_{n+1}^2 - v_n^2 = 0$. This is exact in arithmetic without rounding. The implicit equation can be implemented without a matrix inversion using the Boris rotation.

For fields evaluated at the time-step midpoint, split the velocity operation into an electric half-kick, magnetic rotation and electric half-kick:

$$\mathbf{v}^- = \mathbf{v}_n + \frac{q\Delta t}{2m} \mathbf{E}, \quad (8.179)$$

$$\mathbf{t}_B = \frac{q\Delta t}{2m} \mathbf{B}, \quad \mathbf{s}_B = \frac{2\mathbf{t}_B}{1 + |\mathbf{t}_B|^2}, \quad (8.180)$$

$$\mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \times \mathbf{t}_B, \quad (8.181)$$

$$\mathbf{v}^+ = \mathbf{v}^- + \mathbf{v}' \times \mathbf{s}_B, \quad (8.182)$$

$$\mathbf{v}_{n+1} = \mathbf{v}^+ + \frac{q\Delta t}{2m} \mathbf{E}. \quad (8.183)$$

The vector $\mathbf{t}_B$ is dimensionless and is unrelated to the time variable or Hall-bar thickness. Its sign includes q . The magnetic rotation preserves $|\mathbf{v}^+| = |\mathbf{v}^-|$ to floating-point precision; electric kicks can change the energy physically. A useful implementation reference is given in the further-reading list. The equations here are accompanied by an independent norm and phase check.

For $\mathbf{B} = B\hat{\mathbf{k}}$, the rotation angle is $-2\tan^{-1}(\Omega\Delta t/2)$ rather than exactly $-\Omega\Delta t$. Therefore the positive numerical angular frequency is

$$\omega_{\text{num}} = \frac{2}{\Delta t} \tan^{-1} \left(\frac{\omega_c \Delta t}{2} \right) = \omega_c \left[1 - \frac{(\omega_c \Delta t)^2}{12} + \dots \right]. \quad (8.184)$$

Energy conservation does not imply exact phase. A particle can stay on the correct circle while arriving at the wrong point at a specified time.

A reproducible constant-field implementation

The following Python code uses only the standard library. It implements a symmetric half-position drift, Boris velocity update and half-position drift. In constant fields this puts positions and velocities at the same integer times. For uniform pure B it also preserves the analytic guiding centre. This collocated position convention differs from the common staggered leapfrog convention, in which velocity is stored at half-integer times; do not mix their initial conditions or position formulas.

```

from math import pi, sqrt, atan

def add(a, b):
    return tuple(x+y for x, y in zip(a, b))

def scale(c, a):
    return tuple(c*x for x in a)

def dot(a, b):
    return sum(x*y for x, y in zip(a, b))

def cross(a, b):
    return (a[1]*b[2]-a[2]*b[1],
            a[2]*b[0]-a[0]*b[2],
            a[0]*b[1]-a[1]*b[0])

def step(r, v, dt, qm, E, B):
    rh = add(r, scale(0.5*dt, v))
    kick = scale(0.5*qm*dt, E)
    vm = add(v, kick)
    tb = scale(0.5*qm*dt, B)
    sb = scale(2/(1+dot(tb, tb)), tb)
    vp = add(vm, cross(vm, tb))
    vn = add(add(vm, cross(vp, sb)), kick)
    rn = add(rh, scale(0.5*dt, vn))
    return rn, vn

# Dimensionless: qm=1, Bz=1, v0=1, period=2*pi.
for per_turn in (100, 200, 400):
    dt = 2*pi/per_turn
    r, v = (0., 0., 0.), (1., 0., 0.)
    N = 10*per_turn
    for _ in range(N):
        r, v = step(r, v, dt, 1., (0.,0.,0.),
                    (0.,0.,1.))
    energy_ratio = dot(v, v)
    phase_lag = 2*pi*10 - 2*N*atan(dt/2)
    euler_energy = (1+dt*dt)**N
    print(per_turn, energy_ratio, phase_lag,
          euler_energy)

```

The calculated values are shown in Table 8.2. Its Euler column follows Equation (8.177), independently of any position update. The Boris phase lag is an angular lag accumulated over ten physical periods, not the error in just one step.

Steps per turn	Euler K_N/K_0	Boris K_N/K_0	Boris phase lag (rad)
100	51.42	1.000000	0.02066
200	7.192	1.000000	0.005167
400	2.683	1.000000	0.001292

Table 8.2: Calculated constant-field benchmark after ten gyroperiods, with $\omega_c \Delta t = 2\pi/N_{\text{turn}}$. Boris energy values are rounded to six decimal places. The phase error decreases by about four when the step is halved.

Figure 8.16 shows why a numerical method should be tested over the time interval relevant to the physical question. A step that appears acceptable for a fraction of a turn can be unusable after many turns.

For spatially varying fields, the half-drift position supplies an approximate midpoint for field evaluation. That extension still requires convergence tests, especially near abrupt field boundaries.

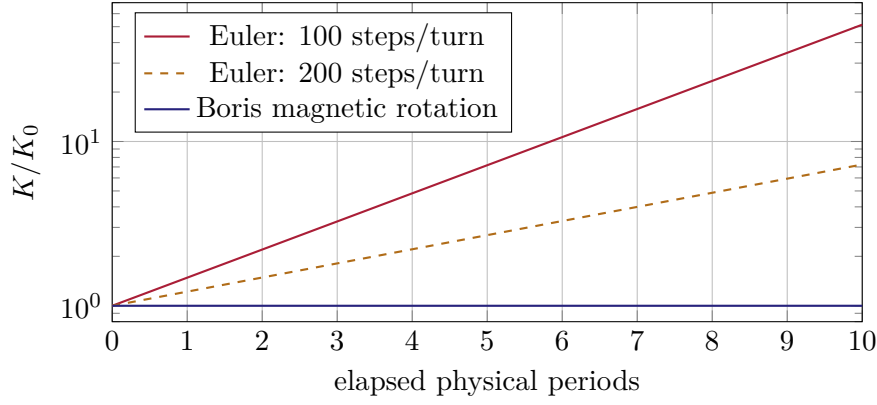


Figure 8.16: Energy behaviour computed from the discrete update laws. The logarithmic energy axis exposes Euler’s systematic growth. The horizontal Boris curve concerns pure magnetic motion; it is not an assertion of energy conservation for arbitrary electric fields.

A hard boundary crossed in one long step is not resolved by excellent speed conservation. Relativistic motion requires a momentum-based pusher; replacing m by an ad hoc value in this nonrelativistic code is not a reliable general prescription.

8.13.8 A small current loop in a field gradient

The result of zero net force on a closed loop depended on a uniform external field. A spatial gradient can produce a net force as well as a torque. For a small loop with fixed moment, the effective potential $U = -\boldsymbol{\mu} \cdot \mathbf{B}(\mathbf{r})$ predicts

$$\mathbf{F} = \nabla(\boldsymbol{\mu} \cdot \mathbf{B}), \quad (8.185)$$

where the derivative is taken with the moment held fixed. A loop whose moment changes with field requires extra care; the fixed-moment formula must not be combined blindly with a field-dependent moment.

It is instructive to recover a component of this force directly from the wire law. Consider a small circular loop of radius a centred on the z axis in an axisymmetric external field. Its current is counterclockwise as seen from $+z$, so $\boldsymbol{\mu} = I\pi a^2 \hat{\mathbf{k}}$. Near the axis, with field varying slowly over the loop,

$$\mathbf{B} \simeq B_z(z) \hat{\mathbf{k}} - \frac{r}{2} \frac{dB_z}{dz} \hat{\mathbf{r}}. \quad (8.186)$$

The radial component is required by $\nabla \cdot \mathbf{B} = 0$. On the loop, $d\boldsymbol{\ell} = a d\varphi \hat{\boldsymbol{\varphi}}$. Since $\hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = -\hat{\mathbf{k}}$, its axial force is

$$F_z = -I \int_0^{2\pi} a B_r d\varphi = I\pi a^2 \frac{dB_z}{dz} \quad (8.187)$$

$$= \mu \frac{dB_z}{dz}. \quad (8.188)$$

The forces from the dominant axial field point radially and cancel around the loop. The weaker radial field supplies axial forces with the same sign, so they add. A model that retains only a variable $B_z(z)$ while discarding the required radial component would incorrectly predict no axial force.

The aligned fixed moment is pulled toward stronger field because its effective potential decreases there. Reversing its moment reverses the force. The particle gyro-orbit in the next subsection has a moment antiparallel to the field for either sign of charge, so its averaged force points toward weaker field. This sign difference connects the coil calculation to magnetic reflection without confusing an externally maintained current with a freely gyrating particle.

Example 8.30: An aligned coil can translate without a torque

A small 50-turn loop of area $2.00 \times 10^{-4} \text{ m}^2$ carries 0.300 A. Its moment is along $+z$ in a field with $B_z > 0$ and local gradient $dB_z/dz = 4.00 \text{ T/m}$. Estimate the axial force and the torque when centred on the symmetry axis.

Calculation. The moment is $\mu = NIA = 3.00 \times 10^{-3} \text{ A m}^2$, so

$$F_z = \mu \frac{dB_z}{dz} = 1.20 \times 10^{-2} \text{ N}.$$

It points toward increasing B_z . The leading dipole torque is zero because $\boldsymbol{\mu} \parallel \mathbf{B}$ on the axis. Zero torque therefore does not imply zero net force.

Model check. The loop dimensions must be small compared with the field-variation length, and the current must stay fixed. The units $(\text{A m}^2)(\text{T/m}) = \text{N}$ agree with a force. The estimate says nothing about a stable equilibrium: a nonzero constant gradient continues to pull the loop unless another force opposes it.

8.13.9 Magnetisation, guiding centres and magnetic mirrors

A small Larmor radius is evidence that a particle's transverse motion is magnetised, not proof of confinement. A particle in a uniform field can leave a device through either end because its parallel velocity remains unchanged. In a varying field, drift, collisions and collective fields can cause additional losses. A useful local requirement for guiding-centre reasoning is

$$r_L \ll L_B, \quad L_B \sim \frac{B}{|\nabla B|}, \quad (8.189)$$

together with slow temporal changes compared with a gyroperiod and rare collisions over the motion being considered.

A magnetic mirror illustrates how a nonuniform field can exchange perpendicular and parallel kinetic energy while leaving total kinetic energy unchanged. A physical converging field has transverse components; the imagined field $\mathbf{B} = B(z)\hat{\mathbf{k}}$ with nonconstant $B(z)$ alone would violate the no-monopole condition. Near the axis of a slowly varying axisymmetric field, the radial component is approximately $B_r \simeq -(r/2)dB_z/dz$.

The radial component follows by inserting axisymmetry into the divergence equation:

$$\frac{1}{r} \frac{\partial(rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0.$$

Treat $\partial B_z/\partial z$ as constant across a small local orbit and integrate from the axis, requiring a finite field there. This gives $B_r \simeq -(r/2)dB_z/dz$.

To obtain the force sign explicitly, take the orbit centre on the axis and $B_z > 0$. The azimuthal velocity is $v_\varphi = -\text{sgn}(q)v_\perp$. Its force from B_r has axial component

$$F_z = -qv_\varphi B_r = |q|v_\perp B_r \simeq -\frac{|q|v_\perp r_L}{2} \frac{dB_z}{dz} = -\frac{mv_\perp^2}{2B} \frac{dB}{dz}.$$

The use of a local circular orbit is valid only to leading order in slow field variation. Gyro-averaging extends the result along the local guiding-centre path. The leading parallel force is

$$F_\parallel \simeq -\mathcal{M} \frac{dB}{ds}, \quad \mathcal{M} = \frac{mv_\perp^2}{2B} = \frac{K_\perp}{B}, \quad (8.190)$$

where s measures distance along the slowly varying field. The quantity $\mathcal{M}$ is the positive adiabatic magnetic moment magnitude of the gyration, not the vector moment $\boldsymbol{\mu}$ of an externally powered coil. The effective gyro-orbit moment is antiparallel to the field for either charge sign.

To see the energy consequence, use $dK_{\parallel}/dt = F_{\parallel}v_{\parallel}$ and $dB/dt \simeq v_{\parallel}dB/ds$. Since total kinetic energy is constant,

$$\frac{dK_{\perp}}{dt} \simeq \frac{K_{\perp}}{B} \frac{dB}{dt} \implies \frac{d}{dt} \left(\frac{K_{\perp}}{B} \right) \simeq 0. \quad (8.191)$$

Thus $\mathcal{M}$ is approximately conserved in this adiabatic model. This reasoning relies on gyro-averaging and slow variation; it is not an exact invariant in an arbitrary field.

Suppose at a field minimum B_0 the total speed is v and the pitch angle is α_0 . Then

$$K_{\perp}(B) = K \sin^2 \alpha_0 \frac{B}{B_0}, \quad K_{\parallel}(B) = K \left(1 - \sin^2 \alpha_0 \frac{B}{B_0} \right). \quad (8.192)$$

Reflection occurs where $K_{\parallel}$ reaches zero:

$$B_{\text{mirror}} = \frac{B_0}{\sin^2 \alpha_0}. \quad (8.193)$$

For a maximum field B_{max} , reflection requires $\sin^2 \alpha_0 \geq B_0/B_{\text{max}}$. Small pitch angles lie in a *loss cone*: their parallel energy is too large to be reversed by the available mirror ratio.

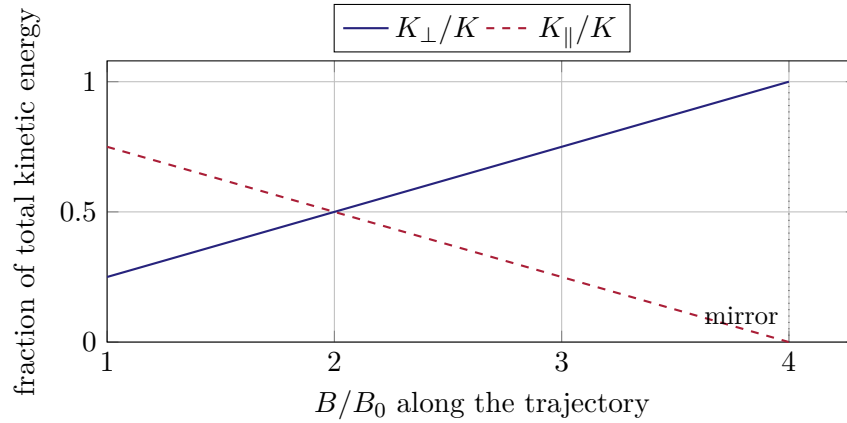


Figure 8.17: Adiabatic energy exchange for an initial pitch angle of 30° . The total kinetic energy stays constant. At $B = 4B_0$, parallel kinetic energy reaches zero and the particle turns back. Extending the descending line to negative kinetic energy would be unphysical.

Figure 8.17 shows how reflection can occur without loss of total kinetic energy. A field whose maximum is below the required mirror field cannot reflect that initial pitch angle in this model.

Example 8.31: Mirror reflection is pitch-angle dependent

A slowly varying magnetic bottle has $B_{\text{max}}/B_0 = 4$. Compare particles launched at the field minimum with pitch angles 20° and 45° , neglecting collisions and electric fields.

Solution. The loss-cone boundary is $\alpha_c = \sin^{-1}(1/2) = 30^\circ$ for motion toward one end. The 20° particle would require $B_{\text{mirror}}/B_0 = 1/\sin^2 20^\circ \simeq 8.55$, beyond the available ratio, and is not reflected. The 45° particle reflects when $B = 2B_0$.

Interpretation. Both can have small Larmor radii and still behave differently longitudinally. The mirror converts parallel kinetic energy into perpendicular kinetic energy; it does not slow the particle's total speed. Collisions that change pitch angle can scatter a trapped particle into the loss cone.

For a perpendicular thermal scale, $K_{\perp} \sim k_B T$ gives $r_L \sim \sqrt{2mk_B T}/(|q|B)$. In a classical isotropic thermal distribution the mean perpendicular energy contains two velocity degrees of freedom and equals $k_B T$. This estimate describes a characteristic gyroradius, not a complete plasma equilibrium or a guarantee of confinement. The full field geometry and the distribution of pitch angles remain essential.

8.13.10 Magnetic dipole energy, small oscillations and instruments

For a magnetic dipole in a uniform field,

$$U(\theta) = -\mu B \cos \theta. \quad (8.194)$$

Near the stable equilibrium $\theta = 0$, use $\cos \theta \simeq 1 - \theta^2/2$:

$$U(\theta) \simeq -\mu B + \frac{1}{2}\mu B \theta^2. \quad (8.195)$$

The restoring torque is approximately

$$\tau \simeq -\mu B \theta. \quad (8.196)$$

If a coil of rotational inertia I_{rot} is free to rotate with small angular displacement, the equation

$$I_{\text{rot}} \ddot{\theta} + \mu B \theta = 0 \quad (8.197)$$

has angular frequency

$$\omega_0 = \sqrt{\frac{\mu B}{I_{\text{rot}}}}. \quad (8.198)$$

This idealized model connects magnetic torque to mechanical oscillations and moving-coil instruments.

Damping and the actual meter geometry

The small-oscillation result assumes a fixed moment and a single rotational degree of freedom near alignment, without gyroscopic complications. Including linear rotational damping b_θ gives

$$I_{\text{rot}} \ddot{\theta} + b_\theta \dot{\theta} + \mu B \theta = 0. \quad (8.199)$$

Only with damping can the system settle at the energy minimum. For underdamped motion, the oscillation frequency is $\sqrt{\mu B/I_{\text{rot}} - (b_\theta/2I_{\text{rot}})^2}$. If a torsion spring is also relaxed at the aligned orientation, its stiffness adds to the magnetic stiffness, giving $\omega_0^2 = (\kappa + \mu B)/I_{\text{rot}}$ in the undamped limit.

A measuring coil is often arranged differently: at zero spring displacement its normal is perpendicular to a fixed-direction field. Let ϕ be the small deflection toward alignment. The static balance is then

$$\kappa \phi = NIAB \cos \phi. \quad (8.200)$$

Only for $|\phi| \ll 1$ does this become $\phi \simeq NIAB/\kappa$. The leading fractional correction to the magnetic torque is $\phi^2/2$, with ϕ in radians. At 10° it is about 1.5%. A radial-field meter avoids much of this angle dependence by changing the local field geometry, as discussed in the core.

Example 8.32: Frequency as an independent moment measurement

A rigid coil with fixed current has moment $\mu = 0.012 \text{ A m}^2$ and rotational inertia $I_{\text{rot}} = 2.0 \times 10^{-6} \text{ kg m}^2$. It undergoes small undamped oscillations about alignment in $B = 0.050 \text{ T}$, with no torsion spring. Find its period.

$$\omega_0 = \sqrt{\frac{\mu B}{I_{\text{rot}}}} = 17.3 \text{ s}^{-1}, \quad T_0 = \frac{2\pi}{\omega_0} = 0.363 \text{ s}.$$

The quantity under the square root has units s^{-2} . This period is a mechanical rotational period, not a charged-particle cyclotron period. Their different dependencies on mass, inertia and field follow from different equations of motion.

Designing the transient response of a radial-field meter

Static sensitivity tells where a pointer eventually settles, but a measuring instrument must also reach that reading in a useful time. For a radial-field meter driven by a prescribed current, write the torque constant as $k_\tau = NAB$. With rotational inertia I_{rot} , torsion stiffness κ and effective linear damping b_θ , the model is

$$I_{\text{rot}}\ddot{\phi} + b_\theta\dot{\phi} + \kappa\phi = k_\tau I(t). \quad (8.201)$$

This mechanical equation assumes that the current is known. If the instrument is voltage-driven, coil resistance, inductance and motional emf can change the current; those circuit dynamics belong in an extended model. Here the damping term represents a measured or independently modelled opposing torque proportional to angular speed.

Divide by I_{rot} and define

$$\omega_n = \sqrt{\kappa/I_{\text{rot}}}, \quad \zeta = \frac{b_\theta}{2\sqrt{\kappa I_{\text{rot}}}}, \quad \phi_\infty = \frac{k_\tau I_0}{\kappa}. \quad (8.202)$$

For a current step from zero to I_0 at $t = 0$, the displacement from the final value obeys a homogeneous equation. Trying $\phi - \phi_\infty \propto e^{\lambda t}$ gives

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0, \quad \lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \quad (8.203)$$

For $0 < \zeta < 1$, the pointer overshoots and oscillates while its envelope decays. For $\zeta > 1$, there are two negative real decay rates and no oscillation from a zero-state step. At critical damping, $\zeta = 1$, the two roots coincide and the solution includes a factor linear in time. Imposing $\phi(0) = \dot{\phi}(0) = 0$ yields

$$\frac{\phi(t)}{\phi_\infty} = 1 - (1 + \omega_n t)e^{-\omega_n t}. \quad (8.204)$$

The pointer approaches its final value monotonically. Increasing damping far beyond this value generally slows the final approach; suppressing oscillation is not the same as making the response arbitrarily fast.

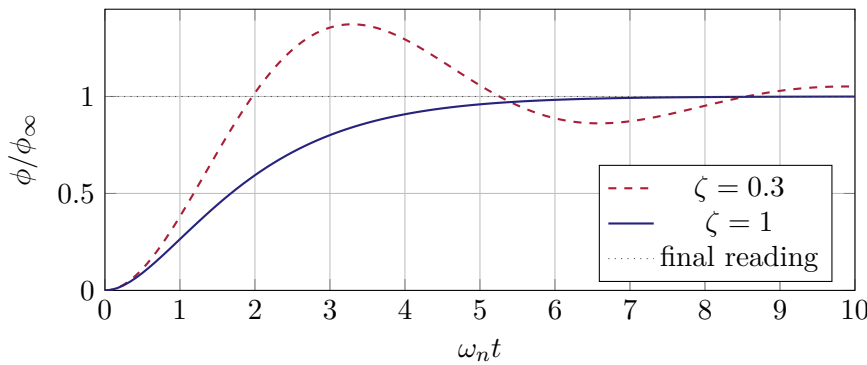


Figure 8.18: Calculated step responses of the same linear meter with two damping ratios. A lightly damped pointer can temporarily indicate a current greater than the actual final current. The model isolates the mechanical response to a prescribed current step.

Figure 8.18 separates static calibration from transient response. Both curves have the same limiting sensitivity k_τ/κ , but a reading taken too early can depend strongly on damping.

Example 8.33: Sensitivity and settling time are different specifications

A radial-field meter has $k_\tau = 3.60 \times 10^{-3} \text{ N m/A}$, $\kappa = 2.00 \times 10^{-5} \text{ N m/rad}$ and $I_{\text{rot}} = 2.00 \times 10^{-7} \text{ kg m}^2$. Find its undamped mechanical frequency, critical damping coefficient, final

deflection for a 1.00 mA step, and approximate time to reach within 2% of that final reading when critically damped.

Mechanical scales. $\omega_n = \sqrt{100} = 10.0 \text{ s}^{-1}$, corresponding to $f_n = 1.59 \text{ Hz}$. Critical damping requires

$$b_{\theta,c} = 2\sqrt{\kappa I_{\text{rot}}} = 4.00 \times 10^{-6} \text{ N m s/rad}.$$

The final reading is $\phi_\infty = k_\tau I_0 / \kappa = 0.180 \text{ rad} = 10.3^\circ$.

Settling criterion. Set $(1 + u)e^{-u} = 0.02$, where $u = \omega_n t$. Numerical substitution gives $u \simeq 5.83$, hence $t_{2\%} \simeq 0.583 \text{ s}$. The criterion uses the actual critically damped response, not just a guessed single exponential.

Interpretation. Reducing inertia can shorten the response without changing the ideal static sensitivity. Changing the spring stiffness changes both sensitivity and frequency. A complete instrument design therefore balances more than the maximum available magnetic torque.

8.13.11 Integrated design study: a selector and mass analyser

A useful design exercise combines force balance, finite flight time, orbital geometry and uncertainty. Consider a collimated positive-ion beam entering a selector along $+x$. Its electric field is $-E\hat{j}$ and its magnetic field is $-B_s\hat{k}$, with $E, B_s > 0$. A downstream magnetic analyser bends transmitted ions through a semicircle in a field of magnitude B_a . Both fields are treated as uniform within their respective regions; fringe fields and space charge are neglected.

The design goal is to separate singly charged ions with masses $20u$ and $22u$ at a nominal speed $v_0 = 2.00 \times 10^5 \text{ m/s}$. Choose $B_s = 0.0500 \text{ T}$ and $B_a = 0.200 \text{ T}$. The selector requires

$$E = v_0 B_s = 1.00 \times 10^4 \text{ V/m}.$$

For ideal parallel plates with gap $d = 10.0 \text{ mm}$, the required potential difference magnitude is $Ed = 100 \text{ V}$. That value establishes an ideal central field; finite plate geometry still has to be checked before assigning an actual field tolerance.

From a force-free speed to a transmission width

The equation $v_0 = E/B_s$ gives the central transmitted velocity, but it gives no bandwidth. Introduce a selector length ℓ_s and an exit aperture of half-width a . A particle enters at the centre with $v_x(0) = v_0 + \Delta v$, $v_y(0) = 0$. With $\Omega_s = qB_s/m > 0$, the exact velocity equations are

$$\dot{v}_x = -\Omega_s v_y, \quad \dot{v}_y = \Omega_s (v_x - v_0).$$

Subtracting the drift velocity gives a rotating vector. Integration with $x(0) = y(0) = 0$ yields

$$v_x = v_0 + \Delta v \cos \Omega_s t, \quad v_y = \Delta v \sin \Omega_s t, \quad (8.205)$$

$$x = v_0 t + \frac{\Delta v}{\Omega_s} \sin \Omega_s t, \quad y = \frac{\Delta v}{\Omega_s} (1 - \cos \Omega_s t). \quad (8.206)$$

The exit time must satisfy $x(t_e) = \ell_s$. It is not generally $\ell_s/v_x(0)$. For small $|\Delta v|/v_0$, use $t_0 = \ell_s/v_0$ in the already first-order expression for y :

$$y_e \simeq \frac{\Delta v}{\Omega_s} (1 - \cos \theta_s), \quad \theta_s = \frac{\Omega_s \ell_s}{v_0}. \quad (8.207)$$

Thus the approximate exit-position acceptance is

$$|\Delta v| \lesssim \frac{a \Omega_s}{1 - \cos \theta_s}. \quad (8.208)$$

If also $\theta_s \ll 1$, this reduces to

$$|\Delta v| \lesssim \frac{2amv_0^2}{qB_s\ell_s^2}. \quad (8.209)$$

This last estimate uses both small speed detuning and a short gyro-angle. Extending the selector improves speed discrimination quadratically only within that regime. It is not a universal scaling for arbitrarily long crossed-field devices.

Choose $\ell_s = 0.200$ m and $a = 0.100$ mm. For $20u$ ions, $\Omega_s = 2.41 \times 10^5 \text{ s}^{-1}$, $t_0 = 1.00 \mu\text{s}$ and $\theta_s = 0.241$. Equation (8.208) gives $|\Delta v| \lesssim 8.33 \times 10^2 \text{ m/s}$, or a fractional half-width of about 0.42%. The short-angle estimate differs by less than one percent. The $22u$ ions have a slightly wider acceptance because their transverse acceleration is smaller.

An aperture checks position, not velocity direction. The exiting beam also has

$$v_{y,e} \simeq \Delta v \sin \theta_s,$$

which produces an angular spread and additional displacement during any subsequent drift. Initial beam width and divergence further change the transmitted band. For example, an initial displacement y_0 adds directly to y_e . These effects belong in a detailed transport design. Near $\theta_s = 2\pi k$, the leading exit-position dispersion vanishes: a long selector can have a poor velocity discrimination at its chosen exit plane. Inspection of the complete trajectory and all apertures is then essential.

Separation in the analyser

For a common selected speed and charge e ,

$$r_{20} = \frac{20u v_0}{eB_a} = 0.207 \text{ m}, \quad r_{22} = 0.228 \text{ m}.$$

The detector distances from the entrance are $2r_{20}$ and $2r_{22}$. Their separation is therefore

$$\Delta x = \frac{2(2u)v_0}{eB_a} = 0.0415 \text{ m}.$$

The factor of two follows from the semicircle geometry. A radius difference alone is not the detector separation. A velocity half-width of 0.42% contributes a detector-position half-width of approximately $0.0042(2r_{20}) = 1.7 \text{ mm}$ for the lighter species. This estimate addresses speed spread only; it does not include angular spread, slit images, detector resolution or field aberrations. The nominal 41.5 mm separation is encouraging, but it is not by itself a complete resolution specification.

The analyser actually measures $m/|q|$. A doubly charged $40u$ ion has the same $m/|q|$ as a singly charged $20u$ ion and follows the same ideal selected-speed radius. Identifying an isotope therefore requires knowledge of charge state as well as a position measurement.

Calibration versus resolution

For a measured radius,

$$\frac{m}{|q|} = \frac{B_a r}{v_0} = \frac{B_a r B_s}{E}. \quad (8.210)$$

If B_a, r, B_s, E have independent small standard uncertainties, their relative variances add:

$$\left(\frac{\sigma_{m/q}}{m/|q|} \right)^2 \simeq \left(\frac{\sigma_{B_a}}{B_a} \right)^2 + \left(\frac{\sigma_r}{r} \right)^2 + \left(\frac{\sigma_{B_s}}{B_s} \right)^2 + \left(\frac{\sigma_E}{E} \right)^2.$$

With respective values 0.3%, 0.2%, 0.3%, 0.5%, the combined standard uncertainty is 0.69%. Shared field calibrations can introduce covariance, so independence is an assumption rather than a default truth.

Resolution concerns the width and separation of neighbouring distributions. Calibration concerns how accurately the position scale corresponds to mass-to-charge ratio. An instrument can distinguish two peaks while assigning both an incorrect absolute mass. Conversely, excellent absolute calibration does not make broad overlapping peaks narrow. This distinction recurs in many engineering measurements.

Method

A complete device calculation follows the chain from a physical law to an observable: field settings determine the central velocity; finite apertures and flight time determine acceptance; curvature and geometry determine detector positions; and calibration plus beam spread determine what can be inferred from those positions. Each stage introduces assumptions that must survive the final accuracy requirement.

8.14 Synthesis and a problem-solving checklist

The Lorentz force combines an electric part that can change speed and a magnetic part that changes the direction of momentum. A magnetic field is not a map of force directions: the force also depends on the charge and its velocity. That dependence explains circular and helical trajectories, and it is the starting point for every device in this chapter.

For a macroscopic current, the individual carrier forces add to a force on the conductor. In a uniform field, a closed current loop has zero net force but can experience a torque. Its magnetic moment summarizes the orientation and strength of that torque. A Hall voltage instead measures the transverse charge separation needed to balance the magnetic carrier force in a steady open transverse circuit.

Question	Relation	Conditions or caution
Force on a charge	$\dot{\mathbf{p}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	Use $\mathbf{p} = m\mathbf{v}$ only nonrelativistically.
Magnetic power	$P_B = \mathbf{F}_B \cdot \mathbf{v} = 0$	Also valid for nonuniform $\mathbf{B}$; induced electric fields can do work.
Orbit radius	$r_L = mv_{\perp}/(q B)$	Uniform field; nonrelativistic form.
Period and pitch	$T_c = 2\pi m/(q B)$; $h = v_{\parallel}T_c$	Pitch is a signed advance; $h/r_L = 2\pi \cot \alpha$.
Velocity selection	$\mathbf{v}_D = \mathbf{E} \times \mathbf{B}/B^2$	$\mathbf{E} \perp \mathbf{B}$; a parallel velocity is also force-free.
Hall voltage	$V_a - V_b = IB_z/(nqct)$	Single carrier; a below b , I along $+x$.
Wire force	$d\mathbf{F} = I d\mathbf{\ell} \times \mathbf{B}$	Integrate the local external field; $d\mathbf{\ell}$ follows conventional current.
Loop torque	$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$; $\boldsymbol{\mu} = NIA\hat{\mathbf{n}}$	Uniform field, planar loop; right-hand normal.
Orientational energy	$U = -\boldsymbol{\mu} \cdot \mathbf{B}$	Fixed moment in prescribed field; alignment requires damping to settle.
Rigidity	$p_{\perp} = q Br_L$	Relativistically valid; transverse momentum for a helix.

Question	Relation	Conditions or caution
Crossed-field drift	$\mathbf{v}_D = \mathbf{E} \times \mathbf{B} / B^2$	Uniform perpendicular fields in the nonrelativistic model used here.
Mirror condition	$\sin^2 \alpha_0 \geq B_0 / B_{\max}$	Adiabatic approximation, no parallel electric work.

Use the following sequence when approaching an unfamiliar problem.

1. **Specify the system.** Is the force on a particle, a straight segment, an entire circuit or the magnet supporting a balance? Specify which field is external to that system.
2. **Draw directions at one point.** Show velocity or conventional current, field and the relevant normal. Calculate a cross product for positive charge and then include the sign of q .
3. **Choose the dynamical variable.** Radius relates to perpendicular momentum, energy relates to total speed, and pitch carries the parallel component. Do not interchange them.
4. **Describe the measurement geometry.** Distinguish radius from diameter, a local displacement from a downstream displacement, and a voltage from its lead convention.
5. **Calculate and check.** Keep units in intermediate steps. Test zero field, zero current, parallel motion and sign reversals. Ask whether a predicted orbit fits inside the stated field region.
6. **Assess the model.** Check nonrelativistic speed, field uniformity, finite apertures, carrier assumptions and numerical convergence before interpreting precision as accuracy.

Caution

Common errors have a common source: using a scalar formula before establishing its geometry. Typical consequences are a reversed electron force, total speed substituted for $v_\perp$, an in-plane loop moment, a Hall sign with undefined terminals, a diameter used as a radius, or an orbit continued beyond the magnet. A carefully labelled sketch often prevents more errors than an extra line of algebra.

8.15 Problems

These 48 problems progress from interpretation to calculation, measurement and modelling. The first three groups use the core chapter; the final group uses the enrichment section. A dagger (†) marks a problem with a detailed solution in the next section. Use the rounded constants in the notation guide. Unless stated otherwise, assume nonrelativistic point particles, prescribed uniform fields and negligible collisions, radiation and fringe fields. Every numerical answer should include units and a check of direction or physical scale.

Conceptual questions: P1–P8

- P1.** A charged particle has zero magnetic force at one instant. List three different explanations consistent with the Lorentz force. Explain why a single zero-force measurement cannot in general establish that $\mathbf{B} = 0$. What changes if two nonparallel nonzero test velocities both give zero force at the same point?
- P2†.** A proton and an electron enter the same uniform magnetic field perpendicular to it. Compare their force directions, radii, periods and kinetic energies when they have (a) equal speeds and (b)

equal kinetic energies. Explain why “opposite charges follow the same circle in opposite senses” is incomplete.

- P3.** A particle moves through a static but spatially nonuniform magnetic field with no electric field. Does the field necessarily change its speed? Can it change the direction and magnitude of the acceleration? Discuss how the conclusion changes when an electric field is also present.
- P4.** A wire loop is released from rest with its moment at 60° to a uniform field. A student says it will rotate into alignment and stay there. Identify the missing physical mechanism. Compare undamped and damped motion, and explain the equilibrium at 180° .
- P5.** For the Hall convention of Section 8.6, predict the sign changes under reversal of (a) current only, (b) field only, (c) both and (d) meter leads only. Does a negative measured voltage always prove that the material has negative mobile carriers? State the extra information needed.
- P6.** A circular loop lies entirely in a uniform magnetic field. Explain how the net force can vanish while the net torque is nonzero. If only one short part of a complete circuit lies in the strong field of a magnet gap, does zero net force on the circuit still follow?
- P7.** A selector transmits a particle without deflection at speed E/B . Explain why this statement presumes a specified beam direction and nonzero charge. Can two transmitted species have different kinetic energies? Can an arbitrary velocity component parallel to $\mathbf{B}$ be detected by this force balance?
- P8.** A numerical orbit maintains constant speed to twelve decimal places. Give two reasons why its predicted detector position may nevertheless be wrong. Describe a convergence or consistency check for each reason.

Analytical and quantitative problems: P9–P22

- P9[†].** A particle with $q = -2.0 \text{ nC}$ has velocity $(3\hat{i} - 4\hat{j} + 2\hat{k}) \times 10^5 \text{ m/s}$ in $\mathbf{B} = (0.10\hat{i} + 0.20\hat{k}) \text{ T}$ and $\mathbf{E} = 50\hat{j} \text{ V/m}$. Calculate the magnetic force, total force and instantaneous electric power. Verify the magnetic power directly from the components.
- P10.** A positive test charge moving at $v\hat{i}$ experiences magnetic force $-F_0\hat{j}$. At the same point, the same charge moving at $v\hat{j}$ experiences $F_0\hat{i} - 2F_0\hat{k}$. Infer all components of $\mathbf{B}$ in terms of $F_0/(qv)$. Check consistency of the component determined by both measurements.
- P11[†].** A proton enters $\mathbf{B} = 0.400\hat{k} \text{ T}$ at speed $8.00 \times 10^5 \text{ m/s}$ and angle 60.0° to $+z$, with its transverse velocity initially along $+x$. Find the transverse radius, period, signed pitch and initial force direction. Determine its position relative to entry after one quarter-period.
- P12.** Singly charged ions of unknown mass move on a helix with radius 25.0 mm and pitch 0.100 m in a 0.300 T field. Their speed is $3.00 \times 10^5 \text{ m/s}$. Infer the pitch angle, mass and period. Explain which additional observation would determine charge sign.
- P13.** Derive the scalings of magnetic radius with mass and charge magnitude for particles (a) at equal speed, (b) at equal kinetic energy and (c) accelerated from rest through equal accelerating potential magnitudes. Which comparisons require knowledge of charge state?
- P14[†].** A positive ion enters a finite field $\mathbf{B} = -B\hat{k}$ along $+x$. Its radius is 0.250 m, and the field occupies $0 < x < 0.150 \text{ m}$. A screen lies a further horizontal distance 0.200 m beyond the field.

Find the exit angle, exit height and screen height. Quantify the error of the small-angle screen formula. Would the particle reach the far boundary if the magnet length were 0.300 m?

- P15.** A selector uses $\mathbf{B} = -0.0800\hat{\mathbf{k}}$ T and $\mathbf{E} = -1.60 \times 10^4 \hat{\mathbf{j}}$ V/m. Find its central transmitted speed along $+x$. Determine the initial deflection of positive and negative particles whose speeds are 10% larger. Write the full family of force-free velocities.
- P16.** Singly charged ions with masses $24u$ and $26u$ enter a semicircular analyser at the common speed 1.50×10^5 m/s in $B = 0.250$ T. Find the radii and detector separation. Repeat the radius ratio if the two species instead enter directly after acceleration through the same potential difference, with no velocity selector.
- P17[†].** A single-carrier Hall bar has $w = 4.00$ mm, $t = 0.200$ mm and $I = 50.0$ mA along $+x$. At $B_z = +0.500$ T, $V_a - V_b = -3.00$ mV. Assuming $|q_c| = e$, find the carrier sign, R_H , density and signed drift velocity. Calculate E_y and verify the transverse force balance on an individual carrier.
- P18.** A straight segment has directed length $\mathbf{L} = (0.20\hat{\mathbf{i}} + 0.10\hat{\mathbf{j}})$ m, carries conventional current 3.0 A, and lies in $\mathbf{B} = (0.40\hat{\mathbf{j}} + 0.20\hat{\mathbf{k}})$ T. Find its force vector and magnitude. Explain why replacing $\mathbf{L}$ by its magnitude would lose necessary information.
- P19[†].** A bent wire carries current I from $A = (0, 0, 0)$ to $C = (L, H, 0)$ in a uniform field $B\hat{\mathbf{k}}$. Compare the force when it follows (a) two straight segments through $(L, 0, 0)$ and (b) any smooth curve between the same endpoints. Close the circuit with a return wire from C to A in the same field. Find the total force and explain why the result says nothing by itself about the torque.
- P20.** A wire runs from $x = 0$ to $x = L$ with current along $+x$. The external field is $\mathbf{B}(x) = B_0(1+x/L)\hat{\mathbf{j}}$. Integrate its force. Compare with using the entry field, the midpoint field and the exit field in $I\mathbf{L} \times \mathbf{B}$. Explain why the midpoint happens to be exact here and need not be exact for a nonlinear field profile.
- P21.** A 40-turn circular coil of radius 30.0 mm carries 0.250 A. Its normal makes an angle 35.0° with a 0.200 T field. Find the magnetic moment, torque magnitude and orientational energy. How much work must an external agent do to rotate it quasistatically from 35.0° to 100° at fixed current?
- P22[†].** A loop with fixed $\mu = 0.0200$ A m² and rotational inertia $I_{\text{rot}} = 5.00 \times 10^{-6}$ kg m² moves in a 0.300 T field. Find the small-oscillation period about alignment. Released from rest at 30.0° without damping, what angular speed does it have at alignment? Explain why the energy result is more accurate than using the small-angle oscillator energy for this finite release angle.

Engineering, laboratory and computational problems: P23–P36

- P23.** A straight active wire of length 0.120 m must experience a magnetic force of 0.180 N. The maximum current is 2.50 A and the available field is 0.650 T. Determine whether perpendicular operation is adequate. Calculate the smallest allowed angle between wire and field at the maximum current. Identify a thermal constraint that the force calculation does not address.
- P24[†].** A magnet rests on a balance while an independently supported 0.0800 m wire crosses its gap perpendicularly. On reversing the current from $+3.00$ A to -3.00 A, the balance reading changes by 1.20 g. Find B . If the standard uncertainties in the reading difference, current and length are 0.04 g, 0.02 A and 0.0010 m, find the combined relative uncertainty in B .

- P25.** Fit the synthetic data in Table 8.1 to $F = F_0 + sI$ using a spreadsheet or a short program. Report the fitted slope and intercept with units, plot residuals, and calculate B for the stated active length. Then use only the data with $I \geq 0$ and force the fit through the origin. Quantify and explain the resulting bias. Do not treat these synthetic residuals as evidence about an actual instrument.
- P26.** A Hall probe has synthetic outputs 2.460 V and 2.540 V at normal field components -0.400 T and $+0.400$ T. Infer its offset and sensitivity. In an unknown field the offset-corrected output varies sinusoidally with probe angle with amplitude 15.0 mV. Infer the field magnitude and explain what determines the direction modulo the probe's angular reference.
- P27.** A Hall measurement contains a field-even contact contribution. At fixed current, synthetic readings are $+7.6$ mV at $+B$ and -6.4 mV at $-B$. Separate the field-odd and field-even parts. Propose a four-reading combination using both current and field reversals that isolates a contribution odd in each variable. State one type of drift or artefact that would still invalidate that combination.
- P28[†].** In an electron-beam experiment, $V_{\text{acc}} = 300$ V, $B = 1.20$ mT and the observed circular diameter is 97.4 mm. Infer e/m_e and the electron speed. The independent standard uncertainties in voltage, field and diameter are 1.0%, 0.5% and 1.5%. Find the relative uncertainty in e/m_e . What result would be obtained if the diameter were mistakenly entered as the radius?
- P29.** A proposed electron-beam measurement assumes perpendicular entry, but the actual beam makes 80° with the field. The accelerating voltage sets the total kinetic energy, while an image supplies the transverse radius. Derive the ratio of the inferred e/m_e to its true value when perpendicular entry is assumed. Estimate the bias and compare it with a claimed 1% measurement uncertainty.
- P30.** A rectangular coil has $N = 100$, area $A = 4.0 \times 10^{-4}$ m², resistance $R = 8.0 \Omega$ and $B = 0.25$ T. Its maximum allowed resistive dissipation is 2.0 W. Find the largest continuous current and maximum torque. If the current is doubled for a short pulse, how do torque and instantaneous heating change? Do not infer a safe pulse duration without a thermal model.
- P31[†].** A moving-coil meter has a radial field $B = 0.180$ T, $N = 80$, coil area $A = 2.50 \times 10^{-4}$ m² and torsion constant $\kappa = 2.00 \times 10^{-5}$ N m/rad. Find its angular current sensitivity and current at 60° deflection. If the same coil instead operates in a fixed-direction uniform field, with the moment initially perpendicular to the field, find the current needed to hold 60° and explain the nonlinearity.
- P32.** A loudspeaker's active coil length is 8.0 m in a radial field of 0.40 T. Its moving mass is 12 g and suspension stiffness is 600 N/m. For a steady 0.20 A current, find force and static displacement. Find the undamped natural frequency. Explain why a sinusoidal current at that frequency cannot be analysed by static displacement alone.
- P33.** A semicircular analyser has radius readout uncertainty $\sigma_r = 0.40$ mm at $r = 0.160$ m. Its field and selected-speed uncertainties are 0.20% and 0.30%. Estimate the relative standard uncertainty of $m/|q|$, assuming independence. Distinguish this calibration uncertainty from peak width caused by beam divergence and slit width.
- P34[†].** Run the constant-field code in Section 8.13.7, then repeat it with 50 and 800 steps per turn for ten turns. Calculate Euler energy growth and Boris phase lag from the formulas as an independent check. How many steps per turn are needed approximately to keep the Boris

phase lag below 10^{-3} rad over ten turns? Verify the integer choice using the exact arctangent expression.

- P35.** Modify the numerical experiment to use $q/m = -1$, $\mathbf{B} = \hat{\mathbf{k}}$, $\mathbf{E} = 0$, $\mathbf{r}_0 = 0$ and $\mathbf{v}_0 = \hat{\mathbf{i}} + 0.5\hat{\mathbf{k}}$ in dimensionless units. Compare radius, pitch, sense of rotation and guiding centre with the analytic helix. Use at least three time steps and distinguish a sign error from a phase error.
- P36.** Design a numerical boundary check for the finite magnet of P14. Explain how a step that straddles $x = \ell$ should be handled or refined. Compare exit angle and downstream screen position with the exact solution. Report convergence with step size, not merely a plot of an apparently smooth orbit.

Advanced synthesis: P37–P48

- P37.** Derive the general uniform-field orbit in Equation (8.151) from the velocity equations. Show directly that the guiding-centre coordinates are constant. Evaluate the limit $B \rightarrow 0$ at fixed time without substituting $B = 0$ into a denominator.
- P38[†].** A positive ion is released from rest in $\mathbf{E} = E\hat{\mathbf{j}}$ and $\mathbf{B} = B\hat{\mathbf{k}}$, with $E, B > 0$. Derive $x(t), y(t), v_x(t), v_y(t)$. Determine the greatest speed and transverse excursion. Show that $K = qEy$ and find the displacement after one gyroperiod. Explain why a nonzero average drift is compatible with periodically vanishing speed.
- P39.** A particle in uniform perpendicular $\mathbf{E}, \mathbf{B}$ has an arbitrary initial velocity. Write it as drift plus gyromotion plus parallel motion. Determine conditions under which its projected path has cusps, loops, or no reversal of its component along the drift. Use the ratio of gyro-speed to drift speed and illustrate representative trajectories numerically.
- P40.** An electron of kinetic energy 2.00 MeV enters a 0.150 T field perpendicularly. Find its momentum, radius, speed as a fraction of c , and gyroperiod using $m_e c^2 = 0.511$ MeV. Compare the radius with the nonrelativistic result and identify which equation remains valid in both regimes.
- P41.** An ideal proton cyclotron has $B = 1.00$ T and final radius 0.300 m. Find the nonrelativistic frequency and extraction energy. With a 5.00 kV accelerating gap voltage at every crossing, estimate the number of complete revolutions from negligible injection energy. Estimate the final relativistic fractional period increase; explain why a small final correction can matter over many turns.
- P42[†].** A track in a 1.20 T field has transverse chord $L = 0.500$ m and sagitta $s = 3.00$ mm. Find the exact circle radius and transverse momentum for charge magnitude e . Each of three equally precise position measurements has independent uncertainty $\sigma_x = 40 \mu\text{m}$; the sagitta is the middle coordinate minus the mean of the endpoint coordinates. Estimate momentum uncertainty when field uncertainty is 0.20% and chord uncertainty is negligible. If the pitch angle is 45° , what is the total momentum?
- P43.** Starting from the one-carrier resistivity tensor, show that $\mathbf{J} \cdot \mathbf{E} = \rho_0(J_x^2 + J_y^2)$ in the transverse plane. Invert the tensor to obtain the conductivity tensor and explain why the off-diagonal terms change sign under field reversal. Does a Hall field produce Joule heating directly?
- P44.** In the low-field two-carrier model, let $n_e = 9n_0$, $n_h = n_0$, electron mobility $b_e = b_0$ and hole mobility $b_h = 4b_0$. Calculate R_H and the apparent single-carrier density $1/(e|R_H|)$. Determine

the mobility ratio at which the Hall sign changes. Explain why carrier majority alone does not establish the sign.

- P45.** Derive the norm conservation and rotation angle of the magnetic midpoint method. Compare its one-step phase error with the exact rotation. Explain why an energy-preserving method can still require small steps and why an abrupt field boundary poses a separate resolution problem.
- P46.** For the selector in the integrated design study, numerically solve $x(t_e) = \ell_s$ using Equation (8.206) for $\Delta v/v_0 = \pm 0.005$ and $20u$ ions. Compare exact exit heights with the first-order estimate. Repeat for twice the selector length. Identify which approximation improves as detuning decreases and which depends on gyro-angle.
- P47[†].** A particle enters an adiabatic magnetic mirror at $B_0 = 0.200$ T with pitch angle 30.0° and kinetic energy 4.00 keV. The greatest available field is 1.00 T. Find its perpendicular and parallel energies initially, its mirror field, and whether it reflects. Determine the loss-cone boundary angle. State the geometric and time-scale assumptions that must hold before these calculations describe actual confinement.
- P48.** A field is written near an axis as $\mathbf{B} = B_z(z)\hat{\mathbf{k}}$ with B_z increasing along z . Explain why that expression cannot be the complete local field in a source-free region. Derive the leading radial component $B_r \simeq -(r/2)dB_z/dz$ from $\nabla \cdot \mathbf{B} = 0$. Use a circular gyro-orbit to explain the sign of the averaged mirror force, and identify the small parameter controlling the approximation.

8.16 Selected detailed solutions

The following 14 solutions model the reasoning as well as the calculation. Attempt the problems first, including a diagram and an estimate. A different coordinate choice is equally valid if signs, directions and measured quantities are defined consistently.

P2: comparing species requires specifying what is equal

Let $M = m_p/m_e \simeq 1833$ using the rounded constants. Both particles have charge magnitude e , so at equal speeds their magnetic force magnitudes are the same, evB . Their directions are opposite because the charges have opposite signs. This does not make their accelerations equal: $a = evB/m$ is larger for the electron by M .

For equal perpendicular speeds,

$$\frac{r_p}{r_e} = \frac{m_p v / (eB)}{m_e v / (eB)} = M, \quad \frac{T_p}{T_e} = M, \quad \frac{K_p}{K_e} = M.$$

Thus the electron curves much more tightly and completes a turn much sooner. The circles cannot be the same circle merely because $|q|$ is equal.

At equal kinetic energies, $v = \sqrt{2K/m}$, so $v_e/v_p = \sqrt{M} \simeq 42.8$. Substitution in the radius formula gives

$$r = \frac{\sqrt{2mK}}{eB}, \quad \frac{r_p}{r_e} = \sqrt{M} \simeq 42.8.$$

The period ratio remains M because the nonrelativistic period is independent of speed. The kinetic energy ratio is now one by construction, while the electron force magnitude is larger by $\sqrt{M}$. At the entrance, where both velocity directions are the same, the force vectors are opposite. At later equal times their velocity directions generally differ, so one should not compare directions without specifying the point on each orbit.

The energy must be low enough for both species to remain nonrelativistic. The electron usually supplies the more restrictive check. This solution illustrates why a phrase such as “same beam conditions” should be replaced by an explicit statement about speed, energy or accelerating voltage.

P9: a component calculation and a power check

Keep the velocity scale visible and form the cross product before multiplying by charge:

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 \times 10^5 & -4 \times 10^5 & 2 \times 10^5 \\ 0.10 & 0 & 0.20 \end{vmatrix} = (-8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \times 10^4.$$

The units here are (m/s) T = N/C. Multiplication by the negative charge reverses every component:

$$\mathbf{F}_B = (1.60\hat{\mathbf{i}} + 0.80\hat{\mathbf{j}} - 0.80\hat{\mathbf{k}}) \times 10^{-4} \text{ N}.$$

The electric force is $q\mathbf{E} = -1.00 \times 10^{-7}\hat{\mathbf{j}}$ N. Therefore

$$\mathbf{F} = (1.600\hat{\mathbf{i}} + 0.799\hat{\mathbf{j}} - 0.800\hat{\mathbf{k}}) \times 10^{-4} \text{ N}.$$

The extra digits in this intermediate result expose the small electric contribution; they do not imply extra measurement precision.

The magnetic power is

$$\mathbf{F}_B \cdot \mathbf{v} = 48 - 32 - 16 = 0 \text{ W},$$

whereas the electric power is

$$P_E = q\mathbf{E} \cdot \mathbf{v} = (-2.0 \times 10^{-9})(50)(-4 \times 10^5) = 0.040 \text{ W}.$$

The kinetic energy is instantaneously increasing despite the electric force being much smaller than the magnetic force. Magnitude alone does not determine power; the component along velocity does. The negative charge and negative v_y make the electric work positive in this geometry.

P11: a helix from transverse and parallel motion

The field acts only on the transverse velocity. Decompose the given velocity first:

$$v_{\perp} = v \sin 60^\circ = 6.93 \times 10^5 \text{ m/s}, \quad v_{\parallel} = v \cos 60^\circ = 4.00 \times 10^5 \text{ m/s}.$$

Then

$$\begin{aligned} r_L &= \frac{m_p v_{\perp}}{eB} = 1.81 \times 10^{-2} \text{ m}, \\ T_c &= \frac{2\pi m_p}{eB} = 1.64 \times 10^{-7} \text{ s}, \\ h &= v_{\parallel} T_c = 6.55 \times 10^{-2} \text{ m}. \end{aligned}$$

The pitch is positive because the axial motion is along $+z$. A dimensional check on the radius gives (kg)(m/s)/(C T) = m.

At entry, $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$, so the proton initially curves toward $-y$. Its transverse circle is centred at $(0, -r_L)$ if entry is the origin. After a quarter-period its transverse angle has changed by -90° , and its position is

$$(x, y, z) = (r_L, -r_L, h/4) = (0.0181, -0.0181, 0.0164) \text{ m}.$$

The total speed is unchanged. The ratio $h/r_L = 2\pi \cot 60^\circ = 3.63$ provides an independent geometry check. Using v instead of $v_{\perp}$ would overestimate the radius; using $2\pi \tan 60^\circ$ would reverse the roles of the parallel and transverse components.

P14: the detector sees the exit angle as well as the exit height

The particle's circular arc has centre $(0, r)$ because the positive charge bends upward in a field into the page. At the exit,

$$\sin \phi = \frac{0.150}{0.250} = 0.600, \quad \phi = 36.9^\circ, \quad \sqrt{r^2 - \ell^2} = 0.200 \text{ m}.$$

Thus $y_e = 0.250 - 0.200 = 0.0500 \text{ m}$ and $\tan \phi = 0.750$. The field-free trajectory is a straight tangent, giving

$$y_{\text{screen}} = 0.0500 + (0.200)(0.750) = 0.200 \text{ m}.$$

The small-angle expression gives

$$\frac{\ell^2}{2r} + \frac{D\ell}{r} = 0.0450 + 0.120 = 0.165 \text{ m},$$

which is 17.5% below the exact result. Here $\ell/r = 0.60$ is not a small parameter, so the error is unsurprising. A large downstream lever arm amplifies the effect of an inaccurate exit angle.

For $\ell = 0.300 \text{ m} > r$, the particle never reaches the far boundary: its maximum x coordinate is r . In the ideal field region it turns back toward the entrance side. Substituting this longer length into a straight-flight-time formula would describe a nonexistent path. The basic force law is still valid; the assumed trajectory segment is what fails.

P17: signed Hall inference

The specified current and field are positive, so $V_H < 0$ implies $q_c = -e$ under the one-carrier assumption. The signed coefficient is

$$R_H = \frac{V_H t}{IB_z} = \frac{(-3.00 \times 10^{-3})(2.00 \times 10^{-4})}{(0.0500)(0.500)} = -2.40 \times 10^{-5} \text{ m}^3/\text{C}.$$

Consequently $n = 1/(e|R_H|) = 2.60 \times 10^{23} \text{ m}^{-3}$. The cross section is $wt = 8.00 \times 10^{-7} \text{ m}^2$, so

$$J_x = 6.25 \times 10^4 \text{ A/m}^2, \quad v_{dx} = \frac{J_x}{n(-e)} = -1.50 \text{ m/s}.$$

Electrons drift opposite to the conventional current. The measured Hall field is

$$E_y = \frac{V_H}{w} = -0.750 \text{ V/m} = v_{dx} B_z,$$

as required. The electric force on one electron is upward: $F_{E,y} = (-e)(-0.750) = +1.20 \times 10^{-19} \text{ N}$. The magnetic force is downward: $F_{B,y} = -q_c v_{dx} B_z = -1.20 \times 10^{-19} \text{ N}$. Their transverse sum is zero.

The small drift speed is compatible with a substantial current because many carriers cross a macroscopic area. It is not the typical microscopic thermal speed. Also, the measured voltage means $V_a < V_b$: a downward Hall field points toward increasing V in the opposite direction, consistent with $\mathbf{E} = -\nabla V$.

P19: endpoint dependence in a uniform field

For the first horizontal segment, $\mathbf{L}_1 = L\hat{\mathbf{i}}$, hence

$$\mathbf{F}_1 = ILB(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) = -ILB\hat{\mathbf{j}}.$$

For the second vertical segment, $\mathbf{L}_2 = H\hat{\mathbf{j}}$, hence $\mathbf{F}_2 = IHB\hat{\mathbf{i}}$. Their sum is

$$\mathbf{F} = IB(H\hat{\mathbf{i}} - L\hat{\mathbf{j}}).$$

For any smooth path between the endpoints, uniformity permits taking $\mathbf{B}$ outside the integral:

$$\mathbf{F} = I \int_A^C d\ell \times \mathbf{B} = I(\mathbf{r}_C - \mathbf{r}_A) \times \mathbf{B}.$$

This gives exactly the same result. The relevant length vector is the endpoint displacement, not the total wire length. The return wire has the opposite displacement and therefore the opposite force; the closed circuit's total force is zero.

Torque contains an additional position vector, $\int \mathbf{r} \times d\mathbf{F}$, so endpoint cancellation of force does not prove torque cancellation. A loop can have separated opposite forces and hence a couple. The argument also depends on a uniform field over the complete path. If the return wire is outside the magnet gap, it need not cancel the active segment's force.

P22: small oscillations and a finite-angle energy calculation

The restoring torque is $-\mu B \sin \theta$ about the rotation axis. Near alignment, $\sin \theta \simeq \theta$, so

$$I_{\text{rot}} \ddot{\theta} + \mu B \theta = 0, \quad \omega_0 = \sqrt{\frac{\mu B}{I_{\text{rot}}}} = \sqrt{1200} = 34.6 \text{ s}^{-1}.$$

The small-oscillation period is $T_0 = 2\pi/\omega_0 = 0.181 \text{ s}$. This is a rotational oscillator; the particle cyclotron period is unrelated.

For release at $\theta_0 = 30^\circ$, conserve rotational kinetic energy plus $U = -\mu B \cos \theta$. At alignment,

$$\frac{1}{2} I_{\text{rot}} \dot{\theta}^2 = \mu B (1 - \cos \theta_0),$$

so

$$|\dot{\theta}| = \sqrt{\frac{2(0.0200)(0.300)(1 - \cos 30^\circ)}{5.00 \times 10^{-6}}} = 17.9 \text{ rad/s}.$$

The loop passes through alignment with its greatest angular speed. It does not stop there without damping. The small-angle energy approximation would replace $1 - \cos \theta_0$ by $\theta_0^2/2$, giving 18.1 rad/s, about 1.2% high. The exact energy relation remains valid for this finite angle even though the simple sinusoidal time dependence is approximate.

P24: a reversal difference is twice the magnetic force

The balance measures the reaction on the magnet. Reversing the wire current reverses that reaction, so the magnitude of the reading change is $2F/g$. Therefore

$$B = \frac{g \Delta m}{2IL} = \frac{(9.81)(1.20 \times 10^{-3})}{2(3.00)(0.0800)} = 0.0245 \text{ T}.$$

The sign of a displayed mass change depends on the mounting and field direction; the supplied difference determines a magnitude. The factor of two is a consequence of subtraction between opposite forces, not a property of magnetic force itself.

Treating g as known to sufficient accuracy and propagating independent standard uncertainties gives

$$\begin{aligned} \frac{\sigma_B}{B} &= \sqrt{\left(\frac{0.04}{1.20}\right)^2 + \left(\frac{0.02}{3.00}\right)^2 + \left(\frac{0.0010}{0.0800}\right)^2} \\ &= 0.0362. \end{aligned}$$

Thus $B = (0.0245 \pm 0.0009) \text{ T}$ at one standard uncertainty within the stated model. The given 0.04 g is already the uncertainty of the difference; it should not be multiplied by $\sqrt{2}$ again. If individual readings had instead been assigned independent uncertainties, their variances would first need to be combined. Field nonuniformity or an incorrect active length is a separate modelling issue.

P28: radius, energy and an uncertainty budget

The measured diameter must first be halved: $r = 48.7 \text{ mm} = 0.0487 \text{ m}$. Combining $eV_{\text{acc}} = m_e v^2/2$ and $m_e v = eBr$ gives

$$\frac{e}{m_e} = \frac{2V_{\text{acc}}}{B^2 r^2} = \frac{600}{(1.20 \times 10^{-3})^2 (0.0487)^2} = 1.76 \times 10^{11} \text{ C/kg}.$$

The speed follows either from the energy or curvature equation:

$$v = \frac{e}{m_e} Br = 1.03 \times 10^7 \text{ m/s} \simeq 0.034c.$$

This is sufficiently nonrelativistic for the requested uncertainty scale. The relative radius uncertainty equals the relative diameter uncertainty because dividing by an exact factor of two changes value and absolute uncertainty together.

The propagated relative uncertainty is

$$\frac{\sigma_{e/m}}{e/m} = \sqrt{0.010^2 + 4(0.005)^2 + 4(0.015)^2} = 0.0332,$$

or 3.3%. The diameter measurement dominates, contributing 0.030 before quadrature. The corresponding absolute standard uncertainty is approximately $0.058 \times 10^{11} \text{ C/kg}$.

Using the diameter as the radius would multiply the denominator by four and yield $4.39 \times 10^{10} \text{ C/kg}$. This is a systematic geometry error, far larger than the uncertainty budget. Adding more repeated readings cannot repair that mistake; the measurement model must be corrected.

P31: radial and fixed-direction field geometries

In the ideal radial-field meter the active conductor forces maintain a torque magnitude $NIAB$ throughout the useful angular range. The equilibrium equation is

$$\kappa\phi = NIAB, \quad \frac{\phi}{I} = \frac{NAB}{\kappa} = 180 \text{ rad/A}.$$

At $\phi = 60^\circ = \pi/3 \text{ rad}$, $I = \phi/180 = 5.82 \text{ mA}$. Radians must be used with the supplied torsion constant.

In the fixed-direction field, the moment starts perpendicular to $\mathbf{B}$ and turns toward it. Its angle to the field is $90^\circ - \phi$, so the torque is $NIAB \cos \phi$. Equilibrium requires

$$I = \frac{\kappa\phi}{NAB \cos \phi}.$$

At 60° , $\cos \phi = 1/2$, giving $I = 11.6 \text{ mA}$. The restoring torque grows with angle while the magnetic torque per unit current shrinks. The scale therefore becomes nonlinear and the required current diverges as this ideal geometry approaches 90° . A radial-field design removes that angular factor over its designed range; it does not remove thermal or mechanical operating limits.

P34: an energy check and a phase tolerance

Let n be the number of steps per turn, $N = 10n$, and $\delta = 2\pi/n$. The exact discrete predictions are

$$\frac{K_N^{\text{Euler}}}{K_0} = (1 + \delta^2)^{10n}, \quad \Delta\phi = 20\pi - 20n \tan^{-1}(\pi/n).$$

For $n = 50$, these are approximately 2.52×10^3 and 0.0825 rad . For $n = 800$, they are 1.638 and $3.23 \times 10^{-4} \text{ rad}$. The pure-magnetic Boris update keeps $K_N/K_0 = 1$ to floating-point rounding in both cases. These checks concern different properties: norm preservation and arrival phase.

Expanding the arctangent for small π/n gives

$$\Delta\phi \simeq \frac{20\pi^3}{3n^2}.$$

Demanding a lag below 10^{-3} rad yields

$$n > \sqrt{\frac{20\pi^3}{3 \times 10^{-3}}} \simeq 454.65.$$

Choose $n = 455$. Substitution into the exact expression gives approximately 9.98×10^{-4} rad; $n = 454$ gives about 1.003×10^{-3} rad and misses the target. This criterion is for ten turns and constant fields. Doubling the observation time doubles the phase lag at fixed step, while more complicated fields can introduce other errors.

A program should be checked against these formulas before its plots are interpreted. An output close to unity for the Boris energy is expected; disagreement in phase or rotation sense still indicates either finite-step error or an implementation mistake.

P38: a cycloid and reversible electric work

Define $v_D = E/B$ and $\Omega = qB/m > 0$. The equations are

$$\dot{v}_x = \Omega v_y, \quad \dot{v}_y = \Omega(v_D - v_x).$$

With $\mathbf{v}(0) = 0$, shift $u_x = v_x - v_D$. The pair (u_x, v_y) rotates with initial value $(-v_D, 0)$, giving

$$v_x = v_D(1 - \cos \Omega t), \quad v_y = v_D \sin \Omega t.$$

Integrating from the origin,

$$x = v_D \left(t - \frac{\sin \Omega t}{\Omega} \right), \quad y = \frac{v_D}{\Omega} (1 - \cos \Omega t).$$

The greatest speed is $2v_D$ at a half-period, and $y_{\max} = 2v_D/\Omega$. Direct evaluation of the kinetic energy gives

$$K = \frac{1}{2}m(v_x^2 + v_y^2) = mv_D^2(1 - \cos \Omega t) = qEy.$$

The final equality follows from $qEv_D/\Omega = mE^2/B^2$. It is also the electric work from the initial position; the magnetic force contributes zero work at every instant.

After one gyroperiod $T_c = 2\pi/\Omega$, the particle is again instantaneously at rest and $y = 0$, but $x = v_DT_c$. The average drift is therefore nonzero. During the first half-cycle the electric field does positive work, and during the second it takes that energy back. A cusp is an instant of zero velocity, not an interval of no displacement. This exact ideal trajectory is a useful numerical benchmark because it checks both force components and their energy roles.

P42: curvature precision and transverse momentum

Circle geometry gives

$$R = \frac{L^2}{8s} + \frac{s}{2} = \frac{0.500^2}{8(0.00300)} + 0.00150 = 10.4182 \text{ m}.$$

For a singly charged particle,

$$p_{\perp} [\text{GeV}/c] = 0.2998(1.20)(10.4182) = 3.748.$$

This is transverse momentum because the curvature is measured in the plane perpendicular to the field. The relation is valid relativistically without knowing the mass.

If $s = y_2 - (y_1 + y_3)/2$, independent equal position uncertainties give

$$\sigma_s^2 = \sigma_x^2 + \frac{1}{4}\sigma_x^2 + \frac{1}{4}\sigma_x^2, \quad \sigma_s = \sqrt{\frac{3}{2}}(40 \mu\text{m}) = 49.0 \mu\text{m}.$$

The derivative $\partial R/\partial s = -L^2/(8s^2) + 1/2$ differs negligibly here from the small-sagitta estimate $\sigma_R/R \simeq \sigma_s/s = 1.63\%$. Combining field and radius uncertainties gives

$$\frac{\sigma_{p_\perp}}{p_\perp} \simeq \sqrt{0.0020^2 + 0.0163^2} = 1.65\%,$$

so $p_\perp \simeq (3.75 \pm 0.06) \text{ GeV}/c$ within this measurement model. At $\alpha = 45^\circ$, $p = p_\perp / \sin \alpha = 5.30 \text{ GeV}/c$. If the pitch angle is uncertain, its contribution must also be propagated; it was treated as known here. Multiple scattering and detector alignment are further possible errors absent from the three-point ideal model.

P47: reflection and the loss cone

Initially,

$$K_{\perp 0} = K \sin^2 30^\circ = 1.00 \text{ keV}, \quad K_{\parallel 0} = K \cos^2 30^\circ = 3.00 \text{ keV}.$$

Conservation of the adiabatic magnetic moment implies $K_\perp(B) = K_{\perp 0} B/B_0$. Total kinetic energy remains 4.00 keV, so the parallel energy falls to zero at

$$B_m = \frac{B_0}{\sin^2 \alpha_0} = \frac{0.200}{0.250} = 0.800 \text{ T}.$$

Because $B_m < B_{\max} = 1.00 \text{ T}$, the ideal trajectory reflects before reaching the greatest field. At the turning point the speed is entirely transverse; it has not vanished.

The loss-cone boundary satisfies

$$\sin^2 \alpha_{\text{lc}} = \frac{B_0}{B_{\max}} = 0.200, \quad \alpha_{\text{lc}} = 26.6^\circ.$$

Particles directed toward that mirror with smaller acute pitch angles do not reflect in this field range. Reflection requires the field to vary little over a gyro-orbit and gyroperiod, typically $r_L/L_B \ll 1$, and requires negligible collisions and parallel electric work over the transit. The field must also have the transverse components required by $\nabla \cdot \mathbf{B} = 0$.

A single reflection does not establish long-term confinement. An open field line, collisions that scatter particles into the loss cone, drift across the device or time-dependent fields can still cause loss. The energy and pitch calculation identifies one necessary element of a confinement model, not a complete containment guarantee.

Pedagogical boundary

Chapter 9 develops the sources of magnetic fields, including the Biot–Savart and Ampère laws. Chapter 10 treats electromagnetic induction and magnetic energy. This chapter’s prescribed-field calculations do not replace those topics. The adiabatic mirror, transport tensor and relativistic rigidity are explicitly marked enrichment; mastering them is not a prerequisite for understanding the core force laws.

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