

Chapter 9

Magnetic Fields Produced by Currents

First Edition

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Preface

This chapter introduces the magnetic fields produced by electric currents as part of the second volume of *Physics for Technical and Engineering Students*. Earlier chapters introduced charge, electric field, electric potential, capacitance, current, circuits, and the magnetic force on moving charges and current-carrying conductors. The present chapter completes the elementary magnetostatic circle: not only do magnetic fields act on currents, but currents themselves are sources of magnetic fields.

The physical motivation is broad and practical. A wire carrying current deflects a compass needle. A long coil becomes an electromagnet. Two parallel bus bars can attract or repel each other with forces large enough to matter in power engineering. Magnetic-field design is central to motors, magnetic sensors, induction devices, transformers, particle accelerators, magnetic resonance imaging, and detector magnets. The same equations also clarify why the magnetic field around a current is not radial like the electric field around a point charge, but circulates around the current.

The chapter develops two complementary tools. The Biot-Savart law is a constructive integral law: it builds the magnetic field from current elements. Ampere's law is a circulation law: it relates the line integral of the magnetic field around a closed path to the net steady current enclosed by that path. Ampere's law is exact in magnetostatics, but it is useful as a field-calculation shortcut only when the geometry has sufficient symmetry. This distinction is one of the central lessons of the chapter.

The exposition is written for Technical Physics, engineering, computer-science, applied-science, and space-science students. The core level is calculus-based undergraduate physics. Vectors, cross products, line integrals, symmetry arguments, and simple numerical modelling are introduced through physical calculations. A clearly separated enrichment develops vector calculus and field design for advanced undergraduates and beginning MSc students. Students who continue to electromagnetic induction, AC circuits, Maxwell equations, electromagnetic waves, materials, accelerator physics, or medical imaging will use the ideas introduced here repeatedly.

The chapter is designed for independent study as well as lectures and problem classes. The standard treatments in Halliday–Resnick–Walker, Serway–Jewett, and OpenStax provide complementary further reading [1, 2, 3]. Their overlapping topics are developed here with a common notation, explicit model limits, and engineering and computational applications.

The Guideline

When a current produces a magnetic field, begin by asking two questions before writing equations: what is the direction of the current element, and what is the direction from that element to the field point? The cross product in the Biot-Savart law and the orientation convention in Ampere's law are not details added after the calculation. They are the calculation.

Pedagogical boundary

This is a magnetostatic chapter. The currents are assumed steady, the charge distributions are not changing in time, and radiation is ignored. The displacement-current correction to Ampere's law and the full Maxwell-Ampere equation are postponed to the later chapter on Maxwell equations. This postponement is not a weakness: it allows us to understand the geometry and symmetry of current-generated magnetic fields before adding time-dependent fields.

Conventions and notation

Vectors are written in bold italic type. Cartesian unit vectors are written as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$. Cylindrical unit vectors are written as $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ when cylindrical symmetry is useful. The magnetic field is denoted by $\mathbf{B}$ and is measured in tesla, where $1 \text{ T} = 1 \text{ N A}^{-1} \text{ m}^{-1}$.

The current is denoted by I in this chapter. Some textbooks use i for current. The current element is $I d\boldsymbol{\ell}$, where $d\boldsymbol{\ell}$ points in the direction of conventional current. The displacement vector from the current element to the field point is denoted by $\mathbf{r}$, with magnitude r and unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$. The permeability constant is denoted by μ_0 . In introductory calculations we use

$$\mu_0 \simeq 4\pi \times 10^{-7} \text{ T m/A}.$$

This is an approximation, not an exact modern-SI definition. The vacuum permeability is experimentally determined; the approximation shown is far more accurate than the numerical precision used in this chapter [4]. Do not confuse an exact formula within an ideal source model with an exactly defined numerical constant.

Symbol	Meaning	SI unit or dimension
I	conventional electric current	A
$\mathbf{B}$	magnetic field	T
$d\boldsymbol{\ell}$	directed length element along a current path or integration loop	m
$I d\boldsymbol{\ell}$	current-length element in the Biot-Savart law	A m
$\hat{\mathbf{r}}$	unit vector from a current element to the field point	dimensionless
r	distance from current element to field point	m
μ_0	permeability constant, approximately $4\pi \times 10^{-7} \text{ T m/A}$	T m A^{-1}
I_{enc}	algebraic current enclosed by an oriented Amperian loop	A
n	number of turns per unit length of a solenoid	m^{-1}
N	total number of turns in a coil or toroid	dimensionless
R	radius of a wire loop, coil, solenoid, or toroid, depending on context	m
d	separation between two long parallel wires	m
$\boldsymbol{\mu}$	magnetic dipole moment of a current loop or coil	A m^2
$\mathbf{J}$	current density, used in advanced sections	A m^{-2}
$\mathbf{K}$	surface current density, used in advanced sections	A m^{-1}

Note

The symbol μ_0 is the permeability constant; the symbol $\boldsymbol{\mu}$ is a magnetic dipole moment. They are not the same object. The first is the vacuum permeability. The second describes a physical current loop or magnetized object.

Chapter 9

Magnetic Fields Produced by Currents

9.1 Currents as magnetic-field sources

Magnetostatics becomes conceptually complete only after we answer both halves of the interaction question. The previous magnetism chapter asked how a magnetic field acts on moving charge and on current. This chapter asks how currents create magnetic fields. The first observation is qualitative but profound: the field of a long straight current does not point away from the wire. It wraps around the wire in closed circles. This immediately tells us that magnetism is geometrically different from electrostatics.

We begin by building fields from small current elements. Straight wires and circular loops then provide two complementary examples of how geometry controls a field. Ampere's law reorganizes the same physics as a statement about circulation, giving short calculations for sufficiently symmetric sources. The distinction between a universally valid law and a useful symmetry shortcut is central to magnetic-field modelling.

A compass near a wire responds to the total field, including the ambient field. Turning on the current changes the compass orientation; reversing the current reverses the wire's contribution. This observation already teaches three ideas used throughout the chapter: currents are sources, field directions matter, and magnetic fields superpose. A compass traces the direction of the local field through the torque on its magnetic moment. It does not trace the trajectory of an electric charge.

Learning goals

After studying this chapter, the student should be able to:

- state the Biot-Savart law and identify each vector in it;
- use the right-hand rule to determine the magnetic-field direction around currents;
- derive and apply the magnetic field of a long straight wire;
- calculate fields at the center of circular arcs and loops;
- explain the force between two long parallel currents and determine whether it is attractive or repulsive;
- state Ampere's law in integral form and use the correct enclosed-current sign convention;
- decide when Ampere's law is a useful calculation method and when it is only a true but unhelpful integral statement;
- derive the standard ideal-field results for solenoids and toroids;
- describe a current loop as a magnetic dipole and connect this to torque in an external magnetic field;
- connect magnetic-field calculations to electromagnets, sensors, accelerator magnets, magnetic-resonance technology, and numerical field modelling.

Key message

A current distribution determines a vector field throughout space. Biot–Savart constructs that field from the source; Ampere’s law constrains its circulation. A force calculation then uses the field at the location of another current or moving charge.

9.2 Prerequisites and the conceptual shift from electrostatics

This chapter relies on several ideas from earlier work. From mechanics we use vector addition, cross products, line integrals, symmetry, and Newton’s third law. From the early Physics 2 chapters we use the concept of electric current, the Lorentz force on moving charges, the magnetic force on a current-carrying wire, and the idea that field laws often become simple only when the geometry has symmetry.

A useful comparison is with electrostatics. A point charge produces an electric field that points radially outward or inward, and Gauss’s law becomes powerful when the electric field has high symmetry. A long straight current produces a magnetic field tangent to circles around the wire, and Ampere’s law becomes powerful when the magnetic field is constant on parts of a closed loop and perpendicular to the remaining parts. The strategy is similar, but the geometry is different.

Idea	Electrostatic version	Magnetostatic version
Source	electric charge q	steady current I or current density $\mathbf{J}$
Basic field law	Coulomb law for point charges	Biot-Savart law for current elements
Symmetry integral	Gauss’s law, closed surfaces	Ampere’s law, closed loops
Typical field geometry	radial for a point charge	circular around a long straight current
Useful symmetry	spherical, cylindrical, planar	cylindrical, toroidal, long-solenoid symmetry
Common mistake	using Gauss’s law without symmetry	using Ampere’s law without symmetry
Later extension	time-varying fields and potentials	Maxwell-Ampere law with displacement current

Caution

Do not import electrostatic intuition too literally. Magnetic field lines of ordinary current distributions do not start on positive magnetic charge and end on negative magnetic charge. In classical electromagnetism they have no sources or sinks: a field line can close, extend indefinitely, or have more complicated nonclosing behavior. Magnetic north and south poles in bar magnets are not isolated magnetic charges in this introductory model.

Guided checks

Check 1. Why is it dangerous to guess that the field of a straight current points radially outward?

Answer. Because the magnetic field direction is fixed by a cross product and by experiment: it is tangent to circles around the wire, not radial.

Check 2. What earlier force law is needed when a wire is placed in an already-known magnetic field?

Answer. The magnetic force on a current-carrying wire, $d\mathbf{F} = I d\mathbf{\ell} \times \mathbf{B}$.

Check 3. Why is symmetry central in this chapter?

Answer. Without symmetry, the Biot-Savart integral or the Ampere line integral usually cannot be reduced to a simple algebraic expression.

9.3 The Biot-Savart law

The Biot-Savart law gives the magnetic field contribution from a small directed current element. For a steady current I flowing through a small element $d\mathbf{\ell}$, the magnetic field contribution at a point P is

$$d\mathbf{B}(P) = \frac{\mu_0}{4\pi} \frac{I d\mathbf{\ell} \times \hat{\mathbf{r}}}{r^2}, \quad (9.1)$$

where $\mathbf{r}$ points from the current element to the field point, $r = |\mathbf{r}|$, and $\hat{\mathbf{r}} = \mathbf{r}/r$. The total field is found by integrating along the current path:

$$\mathbf{B}(P) = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\mathbf{\ell} \times \hat{\mathbf{r}}}{r^2}. \quad (9.2)$$

These formulas describe prescribed steady currents in vacuum; air is treated as vacuum to the accuracy of our introductory examples. In matter, the same microscopic law uses all currents, including those associated with magnetization. Replacing the total current by the free current alone requires a material model, developed in Chapter 12.

The current element is an integration device, not an isolated physical source. In a steady circuit, charge does not accumulate at a wire end: the current continues through leads and a source. A partial path integral can be useful, but its result is only one contribution to the circuit field.

The vector product is essential. If $d\mathbf{\ell}$ points directly toward the field point, the contribution is zero because $d\mathbf{\ell} \times \hat{\mathbf{r}} = \mathbf{0}$. If the element is perpendicular to the direction toward the field point, the magnitude of the contribution is maximal.

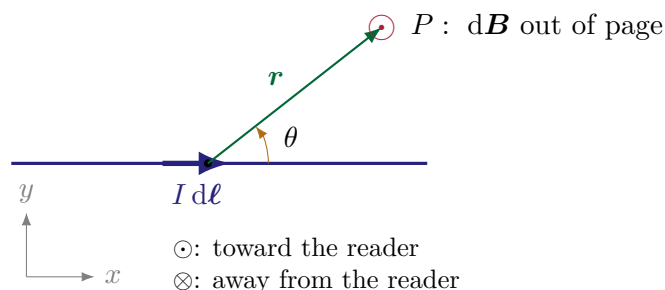


Figure 9.1: Biot-Savart geometry in the xy plane. A current element along $+x$ and a source-to-point vector with positive y component give $d\mathbf{B}$ along $+z$, out of the page. The field is shown at the observation point.

Figure 9.1 fixes the direction convention before any integration. The magnitude of the differential field is

$$dB = \frac{\mu_0}{4\pi} \frac{I d\ell \sin \theta}{r^2}, \quad (9.3)$$

where θ is the angle between $d\ell$ and $\hat{r}$. The direction follows from the right-hand rule for the cross product. Point the fingers of the right hand in the direction of the current element, curl them toward the field point direction, and the thumb gives the direction of $d\mathbf{B}$.

Method

Using the Biot-Savart law.

1. Draw the current path and choose coordinates.
2. Mark a source element $I d\ell$.
3. Draw the vector from the element to the field point.
4. Determine the direction of $d\mathbf{B}$ by the cross product.
5. Use symmetry to decide which components cancel and which add.
6. Integrate only the surviving component over the source.
7. Check units and limiting behavior.

Key message

The Biot-Savart law is local in source elements but nonlocal in calculation: every element of the current path contributes to the field at the chosen point. Good magnetic-field calculations are therefore often as much about symmetry as about integration.

Example 9.1: Dimensional check of the Biot-Savart law

Show that Eq. (9.1) has the dimensions of magnetic field.

Solution. The permeability constant has dimensions $[\mu_0] = \text{T m/A}$. The current element has dimensions $[Id\ell] = \text{A m}$. The factor $1/r^2$ has dimensions m^{-2} . The unit vector and sine factors are dimensionless. Therefore

$$[dB] = \frac{\text{T m}}{\text{A}} \frac{\text{A m}}{\text{m}^2} = \text{T}. \quad (9.4)$$

The equation is dimensionally consistent. This check does not prove the law, but it protects us from common errors such as forgetting one power of distance.

Guided checks

Check 1. What happens to $d\mathbf{B}$ if the field point lies exactly on the line of the current element?

Answer. The angle between $d\ell$ and $\hat{r}$ is 0 or π , so the cross product is zero.

Check 2. Can one replace the cross product by ordinary multiplication and add a direction later?

Answer. No. The cross product controls both the magnitude through $\sin\theta$ and the direction through the right-hand rule.

Check 3. Why is an isolated current element physically suspicious even though it is useful mathematically?

Answer. A steady current cannot simply start and stop in empty space; the element is a differential piece of a closed or externally connected current path.

9.4 Magnetic field of a long straight current

Consider a very long straight wire carrying current I along the $+z$ direction. At a point a perpendicular distance R from the wire, the magnetic field magnitude is

$$B(R) = \frac{\mu_0 I}{2\pi R}. \quad (9.5)$$

The direction is tangent to a circle centered on the wire. If the thumb of the right hand points in the direction of current, the curled fingers indicate the direction of $\mathbf{B}$.

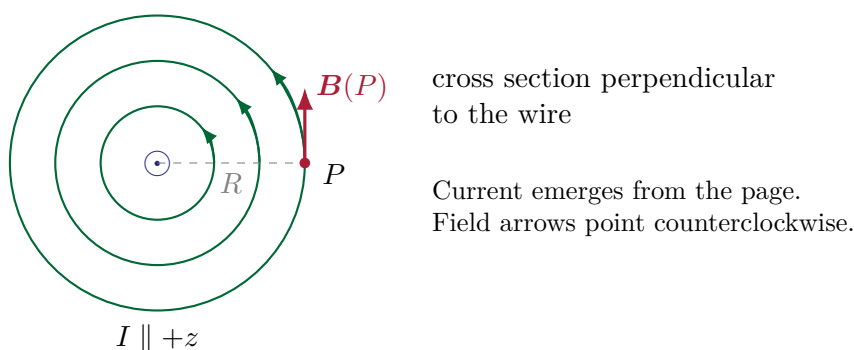


Figure 9.2: Cross section through the circular field of a current along $+z$. Every field line lies in a plane perpendicular to the wire. The arrow at P is tangent to its circle. Line spacing here is illustrative; the magnitude is fixed by $B \propto 1/R$.

Figure 9.2 separates the wire direction from the plane of the field lines. The result can be derived either from the Biot-Savart law or from Ampere's law. The Biot-Savart derivation is longer but more constructive. Let the wire lie on the z axis and the field point be at distance R from the axis. At $P = (R, 0, 0)$, $d\mathbf{\ell} = dz\hat{\mathbf{z}}$ and $\mathbf{r} = R\hat{\mathbf{x}} - z\hat{\mathbf{z}}$. Thus $d\mathbf{\ell} \times \mathbf{r} = Rdz\hat{\mathbf{y}}$. A source element at coordinate z contributes a field perpendicular to the plane containing the wire and the field point. Symmetry ensures that all contributions have the same azimuthal direction. The distance from the element to the field point is $r = \sqrt{R^2 + z^2}$ and $\sin \theta = R/r$. Thus

$$dB = \frac{\mu_0}{4\pi} \frac{I dz R}{(R^2 + z^2)^{3/2}}. \quad (9.6)$$

Integrating from $z = -\infty$ to $z = +\infty$ gives

$$B = \frac{\mu_0 I R}{4\pi} \int_{-\infty}^{+\infty} \frac{dz}{(R^2 + z^2)^{3/2}} \quad (9.7)$$

$$= \frac{\mu_0 I R}{4\pi} \left[\frac{z}{R^2 \sqrt{R^2 + z^2}} \right]_{-\infty}^{+\infty} = \frac{\mu_0 I}{2\pi R}. \quad (9.8)$$

Caution

The long-wire formula is not the field of every straight piece of wire. It assumes that the wire is long compared with the distance to the field point and that end effects are negligible. Near the end of a finite wire, the field is weaker than the infinite-wire result.

Example 9.2: Magnetic field near a current-carrying wire

A long straight wire carries $I = 12$ A. Find the magnetic field magnitude at $R = 4.0$ cm from the wire.

Solution. The long-wire approximation gives

$$B = \frac{\mu_0 I}{2\pi R} = \frac{(4\pi \times 10^{-7} \text{ T m/A})(12 \text{ A})}{2\pi(0.040 \text{ m})} = 6.0 \times 10^{-5} \text{ T.} \quad (9.9)$$

Thus $B = 60 \mu\text{T}$, large enough that ambient fields of tens of microtesla can matter in a measurement. For current along $+z$ and a point on the $+x$ axis, the field is along $+y$. The estimate assumes that distances to the ends and other circuit conductors are large compared with 0.040 m.

Guided checks

Check 1. Why does B decrease as $1/R$ for a long wire, while the electric field of a point charge decreases as $1/r^2$?

Answer. A long wire has cylindrical rather than spherical symmetry. The natural loop circumference grows as $2\pi R$, not as $4\pi r^2$.

Check 2. Does the divergence of the filament result as $R \rightarrow 0$ describe the interior of a real wire?

Answer. No. The current distribution across the wire must then be resolved. We will calculate that field after introducing Ampere's law.

Check 3. If the current direction is reversed, what changes?

Answer. The magnitude is unchanged, but the direction of the magnetic field reverses.

9.5 Finite wires and vector superposition

The infinite wire is a useful limit, but the integral that produced it can also answer a practical question: how long must a straight part of a circuit be before its endpoints can be ignored? Keeping the endpoints in the calculation provides both a usable formula and an estimate of the approximation error. It also prevents a common mistake in laboratory work: measuring the correct distance from a wire while overlooking a nearby bend or return conductor.

9.5.1 A formula with unambiguous endpoint signs

Let a straight segment lie on the z axis from $z = z_1$ to $z = z_2$, where $z_2 > z_1$, and let positive current flow along $+z$. Choose the observation point $P = (R, 0, 0)$, with $R > 0$. The origin is the foot of the perpendicular from P to the line containing the wire; it need not lie on the segment. A source point has position $z\hat{z}$, so

$$\mathbf{r} = R\hat{x} - z\hat{z}, \quad d\ell = dz\hat{z}, \quad d\ell \times \mathbf{r} = Rdz\hat{y}. \quad (9.10)$$

The direction is therefore $+y$ for every source element, irrespective of whether that element lies above or below the foot of the perpendicular. Only the magnitude remains to be integrated:

$$B_y(P) = \frac{\mu_0 I}{4\pi} \int_{z_1}^{z_2} \frac{R dz}{(R^2 + z^2)^{3/2}} \quad (9.11)$$

$$= \frac{\mu_0 I}{4\pi R} \left[\frac{z_2}{\sqrt{R^2 + z_2^2}} - \frac{z_1}{\sqrt{R^2 + z_1^2}} \right]. \quad (9.12)$$

The bracket is positive for $z_2 > z_1$ because $z/\sqrt{R^2 + z^2}$ increases with z . This provides a direction check without an angle convention.

Figure 9.3 defines signed endpoint angles by $\theta_i = \arctan(z_i/R)$, so $-\pi/2 < \theta_i < \pi/2$. Since $\sin \theta_i = z_i/\sqrt{R^2 + z_i^2}$,

$$B_y = \frac{\mu_0 I}{4\pi R} (\sin \theta_2 - \sin \theta_1). \quad (9.13)$$

When the perpendicular foot lies between the endpoints, $\theta_1 < 0 < \theta_2$. Only in that case may we replace the difference by the sum of two positive endpoint-angle sines. A drawing that puts both endpoints on the same side requires the signed difference.

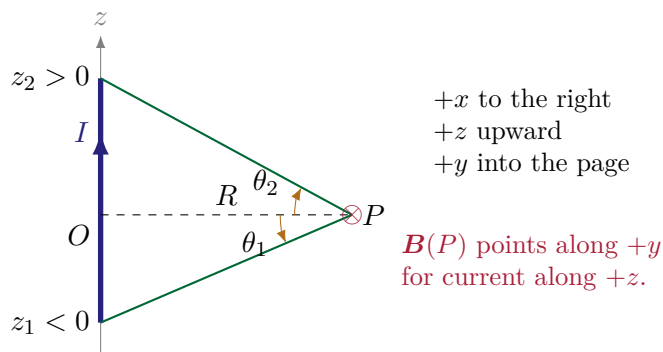


Figure 9.3: Finite-segment geometry in the xz plane. The signed angles are measured from the perpendicular at P , toward positive or negative z . The cross denotes a field into the page; the chosen axes are right-handed.

Two limiting cases are particularly useful. Letting $z_1 \rightarrow -\infty$ and $z_2 \rightarrow +\infty$ makes the bracket in Eq. (9.12) equal to 2 and recovers the infinite-wire result. Letting $z_1 = 0$ and $z_2 \rightarrow +\infty$ gives $B = \mu_0 I/(4\pi R)$, one half of the infinite-wire value. The latter is the contribution of a semi-infinite straight leg at a point beside its endpoint. It is not a claim that a steady current can terminate at that endpoint.

Caution

The field of a finite segment is one contribution to the field of a complete circuit. An isolated segment carrying steady current is not a self-contained magnetostatic source. Use Biot–Savart to add its contribution to those of the other circuit parts. Apply the magnetostatic Ampere law to the total field of the steady current distribution.

Example 9.3: How long is long enough?

A straight part of a circuit has total length 2ℓ , and a probe is on its perpendicular bisector at distance R . Determine its field relative to that of an infinite wire, and find a one-percent length criterion. The other parts of the circuit are to be evaluated separately.

Model and strategy. The source is filamentary, the current is steady, and the probe is outside the conductor. Put $z_1 = -\ell$ and $z_2 = \ell$ in the signed formula. The endpoint contributions are equal:

$$B_{\text{seg}} = \frac{\mu_0 I}{2\pi R} \frac{\ell}{\sqrt{\ell^2 + R^2}}, \quad f \equiv \frac{B_{\text{seg}}}{B_{\infty}} = \frac{\ell/R}{\sqrt{1 + (\ell/R)^2}}. \quad (9.14)$$

The fractional deficit relative to the infinite-wire value is $1 - f$. Requiring $1 - f \leq 0.01$ gives

$$\frac{\ell}{R} \geq \frac{0.99}{\sqrt{1 - 0.99^2}} = 7.02, \quad 2\ell \geq 14.0R. \quad (9.15)$$

If instead the error is normalized to the actual segment field, the condition is $(B_{\infty} - B_{\text{seg}})/B_{\text{seg}} \leq 0.01$, giving $\ell/R \geq 7.05$. The two conventions are close here but should not be mixed.

Interpretation and check. For a 0.40 m straight section and $R = 0.020$ m, $\ell/R = 10$ and $f = 0.9950$. The straight-section deficit is about 0.50%. This says nothing about the return lead: a long segment can have a small endpoint error while the circuit's return field remains important.

The normalized curve in Figure 9.4 shows that increasing length gives diminishing returns. Expanding for $\ell \gg R$ gives $f = 1 - \frac{1}{2}(R/\ell)^2 + \mathcal{O}((R/\ell)^4)$. Doubling the half-length reduces the leading endpoint deficit by a factor of four. This expansion is a useful design estimate, whereas Eq. (9.14) supplies the exact result within the segment model.

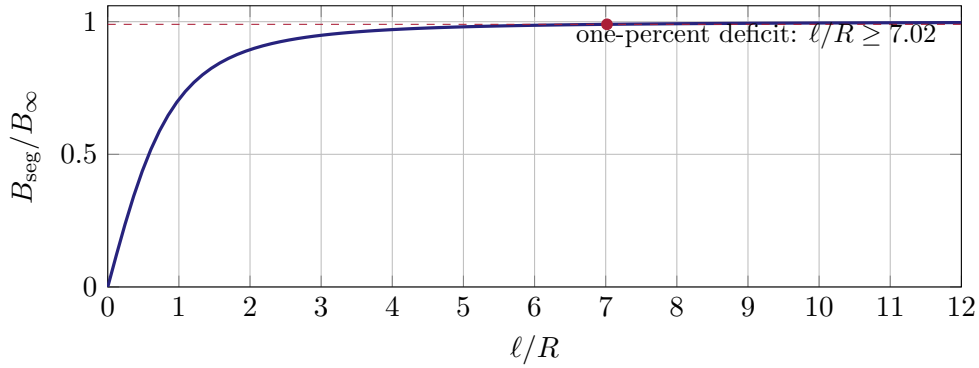


Figure 9.4: Finite-wire contribution on the perpendicular bisector, calculated from Eq. (9.14). Here 2ℓ is the full straight-section length. The horizontal dashed line is 0.99.

9.5.2 Adding fields requires components

For several conductors, superposition applies to the vector field:

$$\mathbf{B}_{\text{total}}(P) = \sum_j \mathbf{B}_j(P). \quad (9.16)$$

Adding magnitudes is justified only after establishing that the contributing fields are parallel with the same sense. Symmetry may cancel one component while reinforcing another. For long wires parallel to z , a convenient Cartesian expression makes the directions explicit. A wire through (x_j, y_j) carrying signed current I_j along $+z$ produces

$$\mathbf{B}_j(x, y) = \frac{\mu_0 I_j}{2\pi \rho_j^2} [-(y - y_j)\hat{\mathbf{x}} + (x - x_j)\hat{\mathbf{y}}], \quad \rho_j^2 = (x - x_j)^2 + (y - y_j)^2. \quad (9.17)$$

This is the circular field $\mu_0 I_j / (2\pi \rho_j)$ multiplied by its unit tangent. A negative I_j reverses both components. The expression is inapplicable on the ideal filament itself.

Example 9.4: Two wires away from the midpoint

Two long wires pass through $(x, y) = (-a, 0)$ and $(a, 0)$, each carrying current I out of the page. Determine the field at $P = (0, h)$ for $h > 0$, and evaluate it for $a = 0.030$ m, $h = 0.040$ m, and $I = 5.0$ A.

Geometry. Figure 9.5 shows why the vertical components cancel. The horizontal components both point to the left. Since each source is at distance $\sqrt{a^2 + h^2}$,

$$\mathbf{B}_1 = \frac{\mu_0 I}{2\pi(a^2 + h^2)}(-h\hat{x} + a\hat{y}), \quad (9.18)$$

$$\mathbf{B}_2 = \frac{\mu_0 I}{2\pi(a^2 + h^2)}(-h\hat{x} - a\hat{y}), \quad (9.19)$$

$$\mathbf{B}_{\text{total}} = -\frac{\mu_0 I h}{\pi(a^2 + h^2)}\hat{x} = -3.2 \times 10^{-5}\hat{x} \text{ T}. \quad (9.20)$$

Each field has magnitude $20 \mu\text{T}$, but their resultant is $32 \mu\text{T}$, not $40 \mu\text{T}$.

Checks. At $h = 0$ the midpoint field vanishes. For $h \gg a$, the two wires look like a single long wire carrying $2I$, giving $B \simeq \mu_0 I / (\pi h)$. If the right-hand current is reversed, the horizontal components cancel instead, and the result becomes $\mu_0 I a \hat{y} / [\pi(a^2 + h^2)]$.

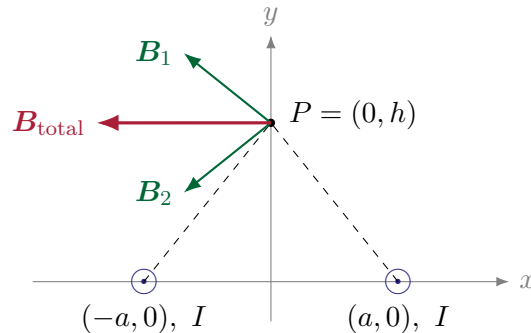


Figure 9.5: End view of two equal currents along $+z$. The field arrows at P are tangent to their respective source-centered circles. Dashed lines show source-to-observation geometry, not magnetic field lines.

Example 9.5: Where can two unequal wire fields cancel?

Wires at $x = 0$ and $x = d$ carry currents I and $3I$ along $+z$. Find the field-zero point on the x axis. Then reverse the second current.

Strategy. Identify directions in the three intervals before equating magnitudes. On the axis, Eq. (9.17) gives the signed component

$$B_y(x) = \frac{\mu_0}{2\pi} \left(\frac{I_1}{x} + \frac{I_2}{x-d} \right), \quad x \neq 0, d. \quad (9.21)$$

For $I_2 = 3I$, both contributions have the same sign outside the pair, so cancellation is possible only between the wires. Solving $I/x + 3I/(x-d) = 0$ gives $x = d/4$. For $I_2 = -3I$, the fields reinforce between the wires; solving $I/x - 3I/(x-d) = 0$ gives $x = -d/2$, on the outer side of the smaller current.

Interpretation. At the second zero, distances to the wires are $d/2$ and $3d/2$. The stronger current is three times farther away. A root must satisfy both the algebra and the direction analysis; equating unsigned fields without checking intervals can generate an incorrect location.

9.6 Circular arcs, loops, and coil-axis fields

A circular arc or loop is the second fundamental geometry. At the center of a circular arc of radius R , every current element is perpendicular to the radius vector toward the center. The field contributions are all perpendicular to the plane of the loop and have the same direction. For an arc, write $d\ell = R d\phi$. At its center, the magnitude from a small element is $dB = \mu_0 I d\phi / (4\pi R)$. Integrating over an angular span ϕ gives, with ϕ measured in radians,

$$B_{\text{arc}} = \frac{\mu_0 I \phi}{4\pi R}. \quad (9.22)$$

A complete circular loop has $\phi = 2\pi$, so at its center

$$B_{\text{center}} = \frac{\mu_0 I}{2R}. \quad (9.23)$$

For a coil of N tightly packed turns, multiply by N when each turn carries the same current and has essentially the same radius. If the winding has appreciable radial or axial thickness, sum the fields of the separate turns instead. Figure 9.7 illustrates why the directions, as well as the distances, must be specified for a composite circuit.

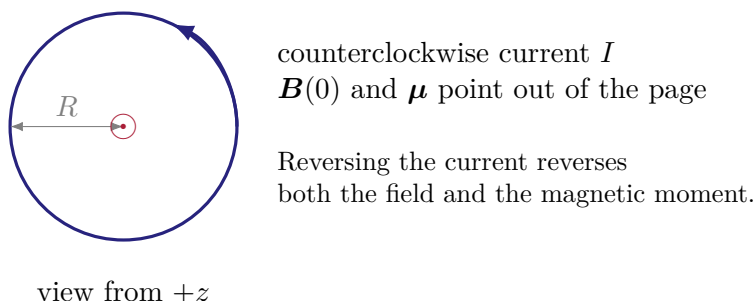


Figure 9.6: The center field of a circular loop. The radius ends on the wire, and the dot explicitly shows the axial field direction perpendicular to the loop plane.

Figure 9.6 shows the direction at the center. For points on the axis of a circular loop, the Biot-Savart law gives

$$B(z) = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}, \quad (9.24)$$

where z is measured from the center of the loop along its symmetry axis. The direction is along the magnetic dipole moment of the loop. This expression is important for coil design and for numerical benchmarks because it has a simple analytic form.

Key message

At the center of a circular loop, the Biot-Savart integral is simple because every element contributes in the same axial direction. Away from the axis, the same loop has a more complicated field and generally requires numerical integration or more advanced mathematical functions.

Example 9.6: Field at the center of a circular coil

A circular coil has $N = 80$ turns, radius $R = 6.0$ cm, and current $I = 0.75$ A. Estimate the magnetic field at its center.

Solution. Treat the turns as coincident circular loops:

$$B = N \frac{\mu_0 I}{2R} = 80 \frac{(4\pi \times 10^{-7} \text{ T m/A})(0.75 \text{ A})}{2(0.060 \text{ m})} = 6.3 \times 10^{-4} \text{ T}. \quad (9.25)$$

Thus the center field is approximately 0.63 mT. The direction is given by the curled-finger right-hand rule around the coil.

Example 9.7: Axial field of a loop and limiting checks

Use Eq. (9.24) to check the field at the center and far away from a circular loop.

Solution. At $z = 0$,

$$B(0) = \frac{\mu_0 I R^2}{2R^3} = \frac{\mu_0 I}{2R}, \quad (9.26)$$

which agrees with the center result. For $|z| \gg R$,

$$B(z) \simeq \frac{\mu_0 I R^2}{2|z|^3}. \quad (9.27)$$

The field still points along the same dipole axis on either side of the loop. The far field therefore decreases as $1/|z|^3$, characteristic of a dipole field, not as $1/z$ like a long wire. The different falloff reflects the fact that a closed loop has no infinite straight current extending through space.

Example 9.8: A closed circuit made of two semicircles

A thin wire forms two concentric semicircular arcs of radii $a = 0.040$ m and $b = 0.080$ m joined by radial sections, as in Figure 9.7. Current $I = 2.0$ A runs counterclockwise along the outer arc. Find the field at the common center O .

Model. All four parts belong to the same closed circuit. The radial sections do not pass through O , and their fields at O vanish because $d\ell \parallel \mathbf{r}$. Current continuity forces the inner arc to be clockwise, so its field points into the page. The outer arc contributes out of the page.

Calculation. Taking $+z$ out of the page,

$$B_z(O) = \frac{\mu_0 I}{4} \left(\frac{1}{b} - \frac{1}{a} \right) = -7.85 \times 10^{-6} \text{ T}. \quad (9.28)$$

Thus the resultant is $7.9 \mu\text{T}$ into the page. The smaller arc produces the larger magnitude because each element is closer to the center and the integrated center field scales as $1/R$.

Checks. As $b \rightarrow a$, the opposing arcs coincide and the net field tends to zero. Interchanging the current sense reverses B_z . Adding the two positive magnitudes would violate current continuity and the right-hand rule.

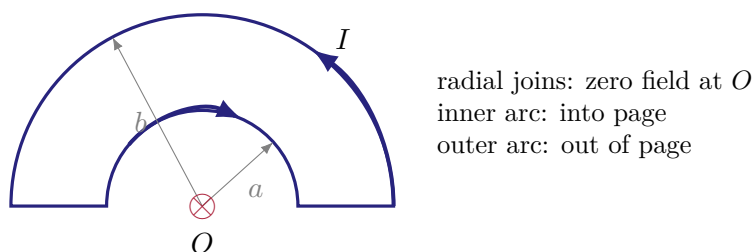


Figure 9.7: A complete annular-sector circuit, not an isolated arc. The opposing arc contributions must be added with their signs. The joins follow radii and do not include the center.

9.6.1 Why only the axial components survive

Let the loop lie in the xy plane, with counterclockwise current as viewed from $+z$. Its magnetic moment therefore points along $+z$. At $P = (0, 0, z)$, all elements are the same distance $r = \sqrt{R^2 + z^2}$ away, but their individual field vectors are not all parallel. Figure 9.8 shows a pair of diametrically opposite elements. Their transverse contributions cancel, and their axial contributions add.

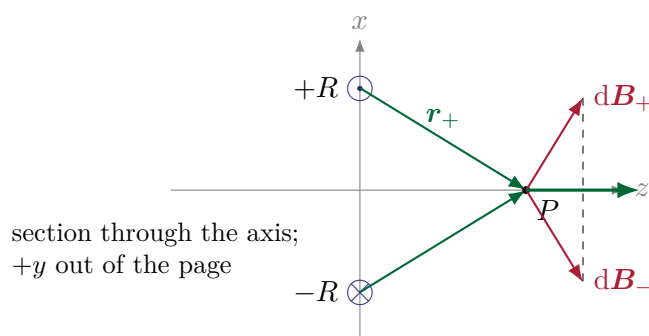


Figure 9.8: A meridional section through a circular loop. At the upper element ($x = R$), current is along $+y$; at the lower element it is along $-y$. Both contributions have positive axial components. Only two elements of the full loop are shown.

For every element, $d\ell$ is perpendicular to $\mathbf{r}$, so $dB = \mu_0 I d\ell / (4\pi r^2)$. The projection onto the axis

supplies an additional factor R/r :

$$dB_z = dB \frac{R}{r} = \frac{\mu_0 I R d\ell}{4\pi(R^2 + z^2)^{3/2}}. \quad (9.29)$$

The denominator is constant around the loop, and $\oint d\ell = 2\pi R$. Integrating yields Eq. (9.24). Omitting the projection factor would give the integral of the magnitudes, not the magnitude of the vector sum.

A Cartesian calculation gives the same result and provides an independent sign check. Parameterize the source by $\mathbf{r}' = (R \cos \phi, R \sin \phi, 0)$. Then

$$d\ell = R(-\sin \phi, \cos \phi, 0) d\phi, \quad (9.30)$$

$$\mathbf{r} = (-R \cos \phi, -R \sin \phi, z), \quad (9.31)$$

$$d\ell \times \mathbf{r} = (Rz \cos \phi, Rz \sin \phi, R^2) d\phi. \quad (9.32)$$

The sine and cosine terms integrate to zero over $0 \leq \phi \leq 2\pi$; the z component remains. This method is particularly useful when a symmetry argument is difficult to visualize.

The axial field has the same $+z$ direction on both sides of this loop, including at negative z . The symmetry-axis coordinate can be negative, whereas the magnitude in the denominator is positive. At the center, contributions are entirely axial; away from the center, a larger fraction cancels transversely. These geometric changes explain why the field decreases more quickly than a simple inverse-square estimate based only on distance would suggest.

The normalized profile is

$$\frac{B(z)}{B(0)} = [1 + (z/R)^2]^{-3/2}. \quad (9.33)$$

It is even in z , has zero slope at the origin, and approaches $(R/|z|)^3$ far away. Figure 9.9 makes clear that a single loop has a locally flat field but is not uniform over an axial length comparable with its radius.

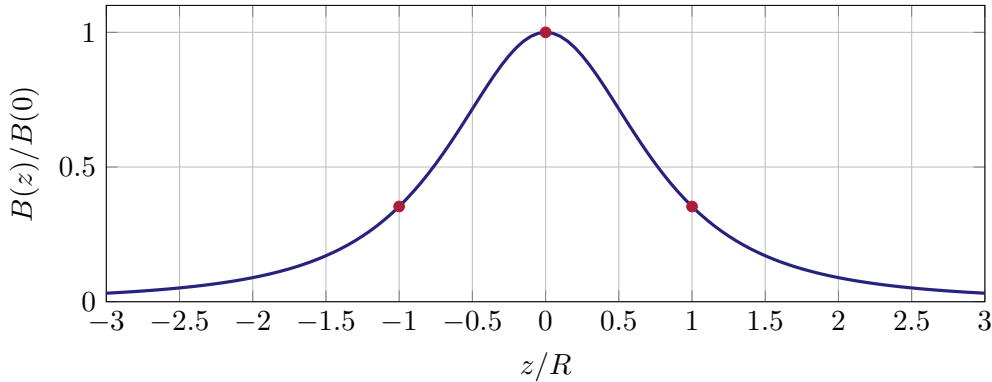


Figure 9.9: Calculated axial profile of a single circular loop. At $|z| = R$, the field is $2^{-3/2} = 0.354$ of its central value. This curve concerns the symmetry axis only.

Example 9.9: An axial tolerance for sensor placement

A sensor should experience a field within 1% of the central field of a thin circular coil of radius R . How far may it move along the axis?

Strategy. The coil current and turn number multiply both fields equally and cancel from their ratio. Solve $(1 + (z/R)^2)^{-3/2} \geq 0.99$:

$$\frac{|z|}{R} \leq \sqrt{0.99^{-2/3} - 1} = 0.0820. \quad (9.34)$$

For $R = 0.060$ m, the axial displacement must be at most 4.92 mm. The local expansion $B(z)/B(0) = 1 - \frac{3}{2}(z/R)^2 + \dots$ predicts $|z|/R \simeq \sqrt{0.01/1.5} = 0.0816$, close to the exact result.

Interpretation. A large field at one point does not guarantee a useful uniform region. The tolerance here describes axial displacement only. Off-axis uniformity requires additional analysis, developed in the advanced section.

9.7 Force between long parallel currents

Two long parallel wires carrying currents exert forces on each other. The calculation uses two steps. Wire 1 produces a magnetic field at the location of wire 2. Wire 2 then experiences a magnetic force because it carries current in that field.

If the separation is d , the field produced by wire 1 at wire 2 is

$$B_1 = \frac{\mu_0 I_1}{2\pi d}. \quad (9.35)$$

A length L of wire 2 experiences force magnitude

$$F_2 = I_2 L B_1, \quad (9.36)$$

so

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d}. \quad (9.37)$$

Figure 9.10 shows the two direction cases. Currents in the same direction attract. Currents in opposite directions repel.

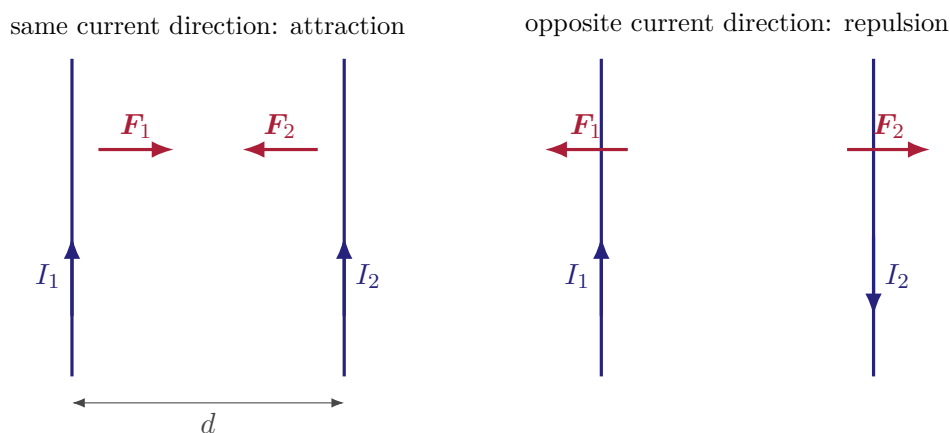


Figure 9.10: Forces between long parallel currents. The forces are equal in magnitude and opposite in direction, consistent with Newton's third law.

Caution

Do not say that wire 1 directly pulls wire 2 by “touching” it with a force at a distance. In the field description, wire 1 produces a magnetic field in space, and wire 2 experiences a force because its current is in that field. This distinction becomes essential in time-dependent electromagnetism.

Example 9.10: Force per unit length between bus bars

Two long parallel conductors are separated by $d = 8.0$ cm and carry currents $I_1 = I_2 = 250$ A in the same direction. Find the force per meter.

Solution.

$$\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi d} = \frac{(4\pi \times 10^{-7})(250)(250)}{2\pi(0.080)} \text{ N/m} = 1.56 \times 10^{-1} \text{ N/m}. \quad (9.38)$$

The force is attractive because the currents are in the same direction. For a supported length of 1.5 m the corresponding force is 0.234 N. If both current magnitudes increase by a factor of 20, the force increases by $20^2 = 400$, giving 62.5 N/m in the same geometric model. A transient use of this estimate assumes that field propagation delays and current redistribution are negligible on the time scale of interest; the steady result alone is not a complete fault analysis.

Guided checks

Check 1. If the current in one wire is reversed, what happens to the force?

Answer. Its magnitude is unchanged for the same current magnitudes, but attraction changes to repulsion.

Check 2. Why is the force per unit length proportional to both currents?

Answer. One current produces the magnetic field and the other current experiences the magnetic force; each step contributes one factor of current.

Check 3. Does Eq. (9.37) apply to short wires?

Answer. Only approximately when the wires are long compared with their separation and the section considered is far from the ends.

It is useful to check the sign explicitly. Put wire 1 at $x = 0$ and wire 2 at $x = d$, both carrying current along $+z$. Wire 1 produces $\mathbf{B}_1 = +B_1\hat{y}$ at wire 2, so $\mathbf{F}_2/L = I_2\hat{z} \times B_1\hat{y} = -I_2B_1\hat{x}$. Wire 2 is pulled toward wire 1. At wire 1 the field of wire 2 points along $-y$, giving a force along $+x$. The equality of magnitudes follows from the symmetric product I_1I_2 .

The force formula uses the *external* field at each wire: the field created by the other wire. The singular filament field at its own position cannot be inserted into $I d\boldsymbol{\ell} \times \mathbf{B}$ to obtain a self-force. Mechanical stresses within a finite wire or coil require an extended current distribution. This issue does not affect the mutual force between separated ideal wires.

9.8 Ampere’s law as a circulation law

Ampere’s law in magnetostatics states that

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}. \quad (9.39)$$

The integral is around a closed curve C , called an Amperian loop. The current I_{enc} is the algebraic net current passing through any surface bounded by that loop. The sign of I_{enc} is fixed by the orientation of the loop: curl the fingers of the right hand in the direction of integration; the thumb defines positive current through the loop.

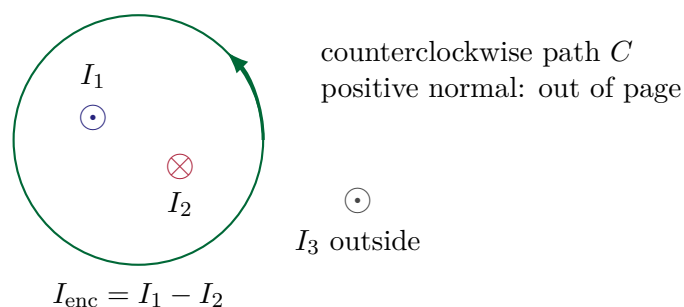


Figure 9.11: An oriented integration path and three source currents. A dot denotes outward current and a cross inward current. The external wire affects the local field but contributes zero net circulation around C .

Figure 9.11 illustrates the orientation and enclosed-current convention. Ampere's law is true for steady total currents, but it is not always a useful way to calculate $\mathbf{B}$. It becomes powerful when symmetry allows the integral to simplify. The standard pattern is:

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = B \oint_C d\ell = B(\text{simple length}). \quad (9.40)$$

This simplification requires $\mathbf{B}$ to be tangent to the path with constant magnitude, or zero/perpendicular on parts of the path. Without that, the integral relation alone does not determine the field.

Method

Ampere-law strategy.

1. Identify the symmetry of the current distribution.
2. Predict the direction of $\mathbf{B}$ from symmetry and the right-hand rule.
3. Choose an Amperian loop on which $\mathbf{B} \cdot d\boldsymbol{\ell}$ is simple.
4. Determine the algebraic enclosed current using the loop orientation.
5. Apply $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}$.
6. Check whether the result matches limiting cases and dimensions.

Caution

Ampere's law is not a universal shortcut. Drawing any random closed loop around a complicated current distribution gives a true equation, but usually not enough information to solve for B at each point on the loop.

Example 9.11: Long straight wire from Ampere's law

Use Ampere's law to derive the field of a long straight wire carrying current I .

Solution. Cylindrical symmetry tells us that $\mathbf{B}$ is tangent to circles centered on the wire and has constant magnitude on a circle of radius R . Choose such a circle as the Amperian loop. Then

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = B \oint d\ell = B(2\pi R). \quad (9.41)$$

The enclosed current is I , so

$$B(2\pi R) = \mu_0 I, \quad B = \frac{\mu_0 I}{2\pi R}. \quad (9.42)$$

This is much shorter than the Biot-Savart derivation because the symmetry is strong.

Guided checks

Check 1. What does the circle on the integral sign in Ampere's law mean?

Answer. It indicates a line integral around a closed path.

Check 2. Can I_{enc} be negative?

Answer. Yes. It is algebraic and depends on the orientation of the Amperian loop.

Check 3. Why is a circular Amperian loop natural for a long straight wire?

Answer. The current distribution has cylindrical symmetry, so B is tangent to such a loop and constant in magnitude on it.

9.8.1 What a circulation does and does not measure

To evaluate $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell}$, imagine dividing the path into short directed pieces. On each piece multiply the field component along the tangent by that piece's length, then add the results with their signs. In the limit of arbitrarily short pieces, this sum becomes the line integral. Its unit is T m, which agrees with $\mu_0 I$. It is not a flux: a flux uses the normal component through an area and has unit T m².

The magnetic field on the left-hand side is the *total* field, including contributions from currents outside the chosen surface. External currents are absent from I_{enc} because their total contribution to the closed-path circulation is zero, not because they produce no field on the path. Likewise, moving an enclosed wire within the boundary changes the local field around the path but leaves its circulation unchanged if the enclosed current is unchanged.

For a long wire at the center of a circular path, symmetry makes B constant and tangent, so the integral is $2\pi rB$. For a displaced wire inside the same circular path, the distances to the wire differ along the path and its field is generally not tangent to that path. The integral remains $\mu_0 I$, but replacing it by circumference times one local field value is incorrect. Even a centered square path gives the correct circulation, although the field magnitude and angle vary along each side.

Example 9.12: Zero circulation with a nonzero field

A circular integration path lies in a region of uniform magnetic field $\mathbf{B}_0 = B_0 \hat{x}$ and encloses no current. Does Ampere's law require $B_0 = 0$?

Calculation. Parameterize the path by $\mathbf{r}_C = r(\cos \phi, \sin \phi, 0)$, with $d\boldsymbol{\ell} = r(-\sin \phi, \cos \phi, 0)d\phi$. Then

$$\oint_C \mathbf{B}_0 \cdot d\boldsymbol{\ell} = -B_0 r \int_0^{2\pi} \sin \phi d\phi = 0. \quad (9.43)$$

Positive and negative contributions cancel even though the field has magnitude B_0 everywhere. Equivalently, for any closed path in a constant field, $\mathbf{B}_0 \cdot \oint d\boldsymbol{\ell} = 0$.

Lesson. A zero line integral is a constraint on the sum of tangential contributions. It is not a pointwise statement. To conclude that a coaxial cable or ideal toroid has zero external field, we need the appropriate symmetry and physical boundary conditions as well as zero enclosed current.

The surface spanning an Amperian loop is a mathematical choice. For a steady current distribution, charge conservation implies that the net current entering any volume equals the net current leaving it. Two spanning surfaces with the same edge therefore carry the same algebraic current: together they form a closed surface, whose net outward current is zero. A wire crossing a surface twice in opposite directions contributes zero net current through that surface. This is how the two legs of a toroidal turn cancel for a path surrounding the entire toroid.

Guided checks

Signed-current test. A counterclockwise loop encloses 7 A out of the page and 3 A into it. A further 10 A wire is outside the loop. The circulation is $\mu_0(7 - 3)$ A, while the local field contains contributions from all three wires. Traversing the loop clockwise changes the circulation and the signed enclosed current by a factor of -1 .

9.8.2 Distributed current in a cylindrical conductor**Example 9.13: Field inside and outside a thick wire with uniform current density**

A long cylindrical wire of radius a carries total current I uniformly over its cross section. Find $B(r)$ both inside and outside the wire.

Solution. Cylindrical symmetry implies that $\mathbf{B}$ is azimuthal and has constant magnitude on a circular path of radius r . Ampere's law gives

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = B(2\pi r) = \mu_0 I_{\text{enc}}. \quad (9.44)$$

For $r \geq a$, the enclosed current is I , so

$$B(r) = \frac{\mu_0 I}{2\pi r}, \quad r \geq a. \quad (9.45)$$

For $r < a$, uniform current density gives $I_{\text{enc}} = I(r^2/a^2)$, hence

$$B(r) = \frac{\mu_0 I}{2\pi a^2} r, \quad r < a. \quad (9.46)$$

The field grows linearly from zero at the central axis and joins continuously to the external $1/r$ law at $r = a$. Its derivative is not continuous there: the slope changes from $+\mu_0 I/(2\pi a^2)$ inside to $-\mu_0 I/(2\pi a^2)$ just outside. The surface is therefore the location of the maximum field. Figure 9.12 distinguishes continuity from differentiability.

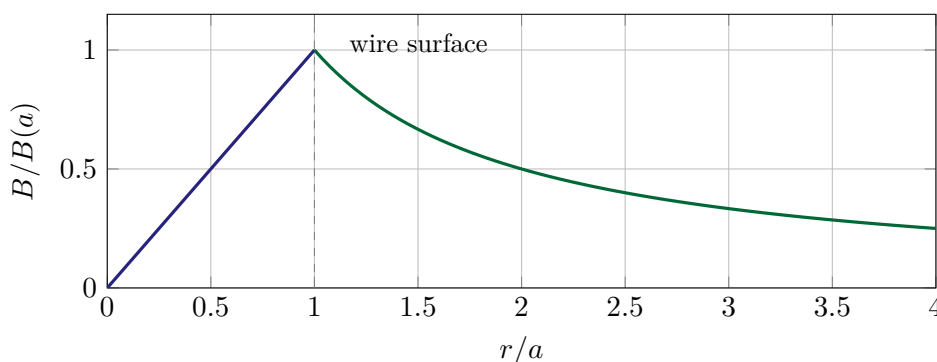


Figure 9.12: Field of a uniformly current-filled long cylinder. The two analytic branches meet at $r = a$, but their slopes differ. A filament model would wrongly extend the decreasing branch to the axis.

The internal result depends on the current distribution. A hollow conductor with no current in its central cavity has zero source field in that cavity under the same cylindrical symmetry. A nonuniform current density generally gives a different internal power law. The external field still depends only on total current for an axisymmetric infinitely long source. Thus a single external measurement cannot determine how the current is distributed inside the conductor.

9.8.3 Coaxial return currents and field confinement

A practical circuit needs a return path. A coaxial geometry surrounds the outgoing current with an axisymmetric return current. It combines exact cancellation outside the ideal cable with a nonzero magnetic field in the space between conductors. This is a geometrical cancellation, not evidence that a conducting tube automatically blocks every static magnetic field.

Example 9.14: A coaxial cable with a thick return conductor

The inner conductor has radius a and uniform current $+I\hat{z}$. A concentric outer conductor occupies $b < r < c$ and carries uniform return current $-I\hat{z}$. The region $a < r < b$ is insulating. Find the field in all four radial regions, assuming infinite length, vacuum permeability, and no imposed background field.

Strategy. The current distribution is invariant under rotations about z and translations along z . The source field is azimuthal and constant on a concentric circle. Compute the enclosed current first, then use $B_\phi = \mu_0 I_{\text{enc}} / (2\pi r)$.

Enclosed current. The inner density is $I/(\pi a^2)$ and the outer density is $-I/[\pi(c^2 - b^2)]$. Thus

$$I_{\text{enc}}(r) = \begin{cases} Ir^2/a^2, & 0 \leq r < a, \\ I, & a \leq r < b, \\ I - I(r^2 - b^2)/(c^2 - b^2), & b \leq r < c, \\ 0, & r \geq c. \end{cases} \quad (9.47)$$

The factor in the third region is an *area fraction*, not a radial thickness fraction. Dividing by $2\pi r$ gives

$$B_\phi(r) = \frac{\mu_0 I}{2\pi} \begin{cases} r/a^2, & 0 \leq r < a, \\ 1/r, & a \leq r < b, \\ (c^2 - r^2)/[r(c^2 - b^2)], & b \leq r < c, \\ 0, & r \geq c. \end{cases} \quad (9.48)$$

The direction for nonzero field is $+\hat{\phi}$.

Boundary checks. The field is zero at the axis, continuous at a and b , and tends continuously to zero at c . At each boundary the adjacent expressions agree. In the outer material the increasing enclosed return current causes a faster decrease than $1/r$.

Thin-shell limit. The original thin outer conductor model is recovered as $c \rightarrow b$. The field then drops discontinuously from $\mu_0 I / (2\pi b)$ just inside the sheet to zero outside. This allowed jump belongs to an ideal surface current. The finite-thickness model in Figure 9.13 resolves it over a nonzero radial interval.

If the outer return current has magnitude $I_{\text{ret}} \neq I$, the axisymmetric source field outside is $\mu_0(I - I_{\text{ret}})\hat{\phi}/(2\pi r)$. In a finite apparatus, any missing current returns elsewhere, and that path must be included in the total field. For two separated parallel wires carrying $+I$ and $-I$, the net enclosed current of a surrounding loop is also zero, but the field outside is generally nonzero. Far from the pair it falls approximately as separation divided by distance squared. This comparison isolates the role of cylindrical symmetry in coaxial cancellation.

9.9 Solenoids

A solenoid is a long coil of wire with many turns. If the turns are tightly wound, the length is much larger than the radius, and the observation point is not near the ends, the magnetic field inside is approximately uniform:

$$B = \mu_0 n I, \quad (9.49)$$

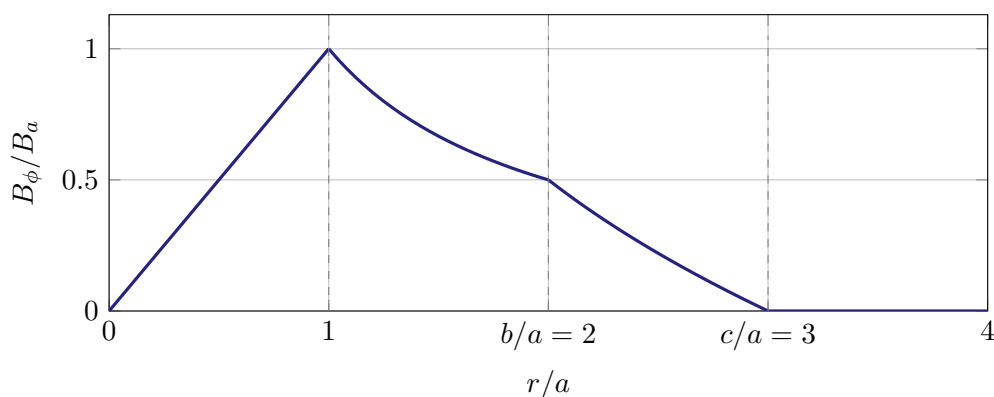


Figure 9.13: Calculated coaxial-cable field with $b = 2a$ and $c = 3a$, normalized by $B_a = \mu_0 I / (2\pi a)$. The four intervals are inner conductor, insulation, return conductor, and exterior. Currents are equal and opposite, with uniform densities in the two conductors.

where $n = N/L$ is the number of turns per unit length. For an infinitely long cylindrical sheet carrying purely azimuthal surface current, the source field outside is exactly zero and the interior field is exactly uniform. A finite, closely wound air-core solenoid approximates this model near its center. A single helical wire also transports current along its axis as it advances from turn to turn; that component and its return lead are neglected in the purely azimuthal model.

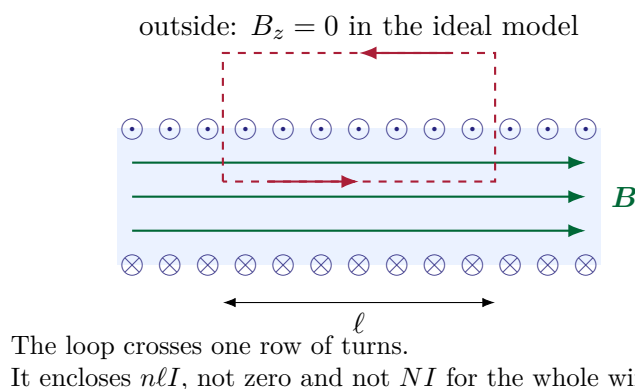


Figure 9.14: Longitudinal section of a long solenoid. The upper row carries current out of the page, the lower row into it, producing an interior field to the right. One long side of the Amperian rectangle is inside and the other is outside.

The derivation uses the rectangular Amperian loop in Figure 9.14 with one long side inside the solenoid and one outside. For an ideal long solenoid, the outside contribution is negligible, and the short sides are perpendicular to the field. If the inside length of the loop is ℓ , the line integral is approximately $B\ell$. The loop crosses only the upper winding sheet. The number of turns pierced by the loop is $n\ell$, each carrying current I , so $I_{\text{enc}} = n\ell I$. Ampere's law gives

$$B\ell = \mu_0 n\ell I, \quad B = \mu_0 nI. \quad (9.50)$$

Caution

The ideal-solenoid formula is an approximation. Near the ends of a real solenoid the field spreads outward and is not exactly uniform. The approximation improves when the solenoid is long compared with its radius and the turns are closely spaced.

Example 9.15: Design current for a solenoid field

A laboratory solenoid has $N = 1200$ turns over length $L = 0.60$ m. What current is required to produce an interior field of approximately $B = 15$ mT?

Solution. The turn density is

$$n = \frac{N}{L} = \frac{1200}{0.60 \text{ m}} = 2.0 \times 10^3 \text{ m}^{-1}. \quad (9.51)$$

Using $B = \mu_0 n I$,

$$I = \frac{B}{\mu_0 n} = \frac{15 \times 10^{-3} \text{ T}}{(4\pi \times 10^{-7} \text{ T m/A})(2.0 \times 10^3 \text{ m}^{-1})} = 6.0 \text{ A}. \quad (9.52)$$

This estimate ignores heating, winding resistance, end effects, and the field contribution of leads. Those are engineering design issues, not contradictions of the magnetostatic model.

Guided checks

Check 1. What parameter controls the ideal-solenoid field more directly: total turn number or turn density?

Answer. The ideal formula uses turn density $n = N/L$, not total turns alone.

Check 2. Why do the short sides of the rectangular Amperian loop not contribute in the ideal derivation?

Answer. They are chosen perpendicular to the interior field direction, so $\mathbf{B} \cdot d\mathbf{\ell} = 0$ there.

Check 3. Why is the outside contribution neglected?

Answer. For a very long tightly wound solenoid, the external field is small compared with the interior field away from the ends.

The law applied to this rectangle actually determines the difference $B_{\text{in}} - B_{\text{out}} = \mu_0 n I$. Setting the exterior source field to zero also uses the ideal geometry and the absence of an imposed background field. Adding a uniform axial background changes both long-side contributions by the same amount and leaves their difference unchanged. This is another reason not to infer a field from the enclosed current without the full physical setting.

The absence of R from the ideal formula can seem surprising. At fixed turn density and current, making each loop larger weakens its individual central contribution, but more distant turns then contribute appreciably over a longer axial range. In the infinite continuous-winding limit these effects compensate. For a finite coil they do not compensate exactly, so its radius matters. The advanced section derives the finite-solenoid field rather than applying the infinite result near an end.

For a long finite solenoid, the axial field near an open end approaches one half of the central ideal value. At the center, turns on both sides contribute; at the end, approximately one half of that long stack remains. The field then spreads into the exterior and returns around the winding. No field line terminates at the end face. A photograph or sketch of a finite solenoid therefore should not be interpreted as the zero-exterior-field infinite model.

9.10 Toroids

A toroid is a coil wound around a doughnut-shaped support. Here the support and interior have vacuum permeability; a magnetic core requires additional material physics. For an ideal toroid with N turns carrying current I , symmetry suggests that the magnetic field inside the winding region is tangent to circular paths around the central axis. An Amperian circle of radius r inside the toroid

gives

$$B(r)(2\pi r) = \mu_0 NI, \quad (9.53)$$

so

$$B(r) = \frac{\mu_0 NI}{2\pi r}. \quad (9.54)$$

Figure 9.15 identifies which crossings contribute in each region. The field is not exactly uniform across the cross section because it depends on $1/r$. It is stronger on the inner side of the toroid than on the outer side. For the ideal continuous poloidal winding, the field is zero in the central hole and outside the winding region. This conclusion includes the symmetry and the absence of external sources. A real toroidal winding has discrete turns, finite pitch, and leads, which can produce a leakage field.

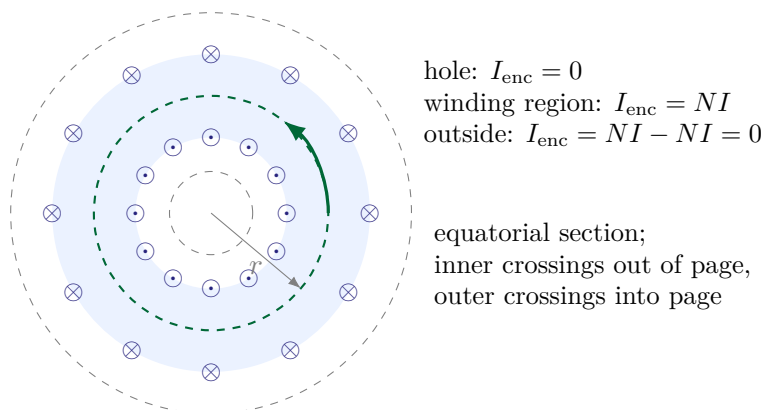


Figure 9.15: Three Amperian paths for an ideal toroidal winding. Current crosses the equatorial plane outward at the inner edge and inward at the outer edge. A path between the edges links each turn once; an exterior path encloses both crossings.

Key message

A long solenoid is designed to make an approximately uniform field in a straight interior region. A toroid is designed to guide the magnetic field around a closed path and reduce external leakage. Both results are Ampere-law successes, but their geometries and design goals differ.

Example 9.16: Toroid field at inner and outer radii

A toroid has $N = 500$ turns and current $I = 1.8$ A. The winding occupies radii from $r_{\text{in}} = 4.0$ cm to $r_{\text{out}} = 6.0$ cm. Estimate the magnetic field just inside the winding region at both radial boundaries.

Solution.

$$B(r_{\text{in}}) = \frac{\mu_0 NI}{2\pi r_{\text{in}}} = \frac{(4\pi \times 10^{-7})(500)(1.8)}{2\pi(0.040)} = 4.5 \times 10^{-3} \text{ T}, \quad (9.55)$$

$$B(r_{\text{out}}) = \frac{\mu_0 NI}{2\pi r_{\text{out}}} = \frac{(4\pi \times 10^{-7})(500)(1.8)}{2\pi(0.060)} = 3.0 \times 10^{-3} \text{ T}. \quad (9.56)$$

The field is larger on the inner side because the same circulation is distributed over a shorter circular path.

To quantify when a single mean-radius estimate is adequate, write the radial limits as $r_m - t/2$ and $r_m + t/2$. Relative to $B(r_m)$, the inner and outer fields are

$$\frac{B_{\text{in}}}{B(r_m)} = \frac{1}{1 - t/(2r_m)}, \quad \frac{B_{\text{out}}}{B(r_m)} = \frac{1}{1 + t/(2r_m)}. \quad (9.57)$$

For $t \ll r_m$, the departures are approximately $\pm t/(2r_m)$. The full variation across the radial thickness is approximately t/r_m of the mean-radius field. The numerical example has $t/r_m = 0.4$, so treating that toroid as radially uniform would be poor modelling.

A thin toroid also connects directly to the long-solenoid result. Along a circle of mean circumference $2\pi r_m$, the local turn density is $n = N/(2\pi r_m)$. Substituting this into $\mu_0 n I$ gives the toroidal field at the mean radius. Bending a long solenoid into a large ring therefore recovers the same local physics while changing the global field geometry.

9.11 Current loops as magnetic dipoles

A current loop behaves as a magnetic dipole. For a flat loop of area A carrying current I , the magnetic dipole moment is

$$\boldsymbol{\mu} = IA \hat{\mathbf{n}}, \quad (9.58)$$

where $\hat{\mathbf{n}}$ is determined by the right-hand rule. For a coil of N identical turns,

$$\boldsymbol{\mu} = NIA \hat{\mathbf{n}}. \quad (9.59)$$

In an external magnetic field, a current loop experiences torque

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}_{\text{ext}}. \quad (9.60)$$

This torque law was introduced earlier. The new point here is that the same current loop also produces its own magnetic field, with a far-field pattern analogous in shape to the field of a small bar magnet.

The dipole concept links microscopic and macroscopic magnetism. At the level of this course, we treat an electron orbit or spin only qualitatively. A current loop is the classical model that prepares the language of magnetic moments, magnetization, and magnetic materials.

Note

A current loop has a north-like side and a south-like side, but this does not mean the loop contains isolated magnetic charges. The familiar loop-field pattern passes through the loop and returns outside it. The dipole language describes the field pattern and torque response.

9.11.1 Reading the field around the entire loop

The axial formula describes a line of observation points, not the whole field. Figure 9.16 shows a meridional section calculated from the circular-loop Biot–Savart integral. Inside the loop the field points along the magnetic moment. In the exterior part of the equatorial plane, it points oppositely. The field curves continuously between these regions; there is no surface of magnetic charge at the loop.

A compass or a small freely rotating test coil tends to align with the local field, so it can reveal this changing direction. Its own field should be weak enough not to disturb the source appreciably. A magnetic force on a moving charge, in contrast, is perpendicular to both its velocity and the local field; the particle does not generally follow a field line.

Far from a localized loop, its size and detailed shape become difficult to resolve. The leading field is specified by $\boldsymbol{\mu}$ and falls as distance cubed in the denominator. Near the conductor, the local wire geometry matters and the ideal filament field grows without bound as the wire is approached. These are different limits: a dipole approximation must not be extended into the source region. The advanced section derives the dipole field and quantifies its axial accuracy.

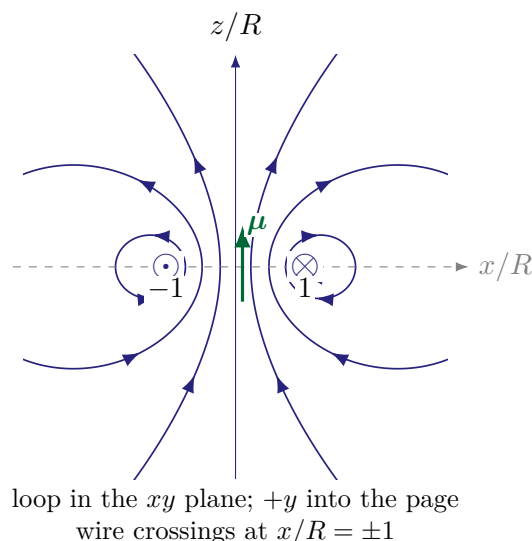


Figure 9.16: Calculated field-line traces in a plane through the loop axis. Arrows follow $\mathbf{B}$. The dots and crosses indicate the prescribed current, not magnetic charges. Some traces continue beyond the frame. Seeds are chosen for readability, so their spacing is not a calibrated measure of field strength.

Example 9.17: Magnetic moment of a rectangular coil

A rectangular coil has $N = 40$ turns, side lengths 4.0 cm and 7.0 cm, and current 0.25 A. Find the magnetic dipole moment magnitude.

Solution. The area of each turn is

$$A = (0.040 \text{ m})(0.070 \text{ m}) = 2.8 \times 10^{-3} \text{ m}^2. \quad (9.61)$$

The magnetic moment magnitude is

$$\mu = NIA = (40)(0.25 \text{ A})(2.8 \times 10^{-3} \text{ m}^2) = 2.8 \times 10^{-2} \text{ A m}^2. \quad (9.62)$$

The direction is perpendicular to the coil plane by the right-hand rule.

Guided checks

Check 1. How is the direction of μ chosen for a loop?

Answer. Curl the fingers of the right hand in the direction of current; the thumb gives μ .

Check 2. Why does multiplying by N make sense for a tightly wound coil?

Answer. Each turn contributes nearly the same magnetic moment, so the moments add.

Check 3. Is a magnetic dipole the same as a pair of isolated magnetic charges?

Answer. No. It is a field pattern and torque response; magnetic field lines have no endpoints; they need not all form closed curves.

9.12 Engineering models: field, heat, and mechanical load

A field specification is usually accompanied by geometrical, electrical, or mechanical constraints. The analytical laws remain useful because they reveal how these constraints compete. Increasing the current strengthens a field linearly, while resistive heating and many magnetic forces grow quadratically. Increasing turn number may reduce the required current, but also changes wire length and winding size. The correct comparison must say what is held fixed.

Example 9.18: A calibration coil with a power budget

An air-core circular coil has $N = 200$ turns of mean radius $R = 0.050$ m. The copper-wire cross-sectional area is $A_w = 0.50 \text{ mm}^2$, and for this estimate take the resistivity as $\rho_e = 1.7 \times 10^{-8} \Omega \text{ m}$. Estimate the current, voltage, and resistive power needed for a center field of 2.0 mT.

Assumptions. The winding dimensions are small enough compared with R to model all turns at their mean radius; lead length is neglected. The given resistivity is treated as constant, so the result is an initial electrical estimate, not a prediction of operating temperature.

Field requirement. From the loop-center result,

$$NI = \frac{2RB}{\mu_0} = 159 \text{ A}, \quad I = 0.796 \text{ A}. \quad (9.63)$$

The product NI is commonly called the ampere-turn requirement; turn number is dimensionless.

Electrical calculation. The wire length and resistance are

$$L_w \simeq 2\pi RN = 62.8 \text{ m}, \quad (9.64)$$

$$R_{\text{coil}} = \rho_e \frac{L_w}{A_w} = \frac{(1.7 \times 10^{-8})(62.8)}{0.50 \times 10^{-6}} \Omega = 2.14 \Omega. \quad (9.65)$$

Consequently $V = IR_{\text{coil}} = 1.70 \text{ V}$ and $P = I^2 R_{\text{coil}} = 1.35 \text{ W}$. The current density in the wire is approximately 1.59 A/mm^2 .

Interpretation. These numbers do not establish a permissible operating temperature: cooling, insulation, mounting, and resistance changes with temperature have not been specified. At fixed resistance, a target field twice as large requires twice the current and four times the power. At fixed applied voltage, heating-induced resistance changes instead reduce current and field. A regulated current supply maintains the field more directly, provided it has sufficient voltage range.

Suppose the same mean coil radius and total copper volume V_w are retained while N is changed. Since $L_w \simeq 2\pi RN$, the available wire area becomes $A_w = V_w/L_w$. Hence

$$R_{\text{coil}} = \frac{\rho_e L_w^2}{V_w} = \frac{\rho_e (2\pi R)^2 N^2}{V_w}. \quad (9.66)$$

At fixed allowed resistive power P , the current scales as $I \propto 1/N$, so the product NI and the ideal center field remain unchanged. More turns primarily trade current against voltage in this simplified fixed-volume comparison. With fixed wire gauge instead, copper volume is not fixed and a different scaling follows. This distinction is more useful than the unqualified statement that more turns always give a better magnet.

9.12.1 A loop in a nonuniform source field

The field of a long straight conductor provides a simple bridge between field design and mechanical response. Unlike a uniform field, it can exert a net force on a closed loop. Figure 9.17 defines a rectangular loop of height h with its near side a distance a from a long straight source wire and width w . The source current I_s points upward, and the loop's near side carries upward current I_ℓ .

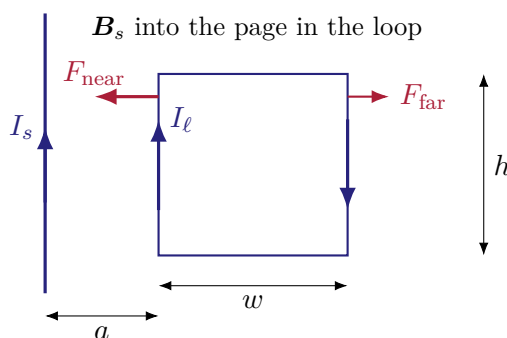


Figure 9.17: A rectangular loop near a long straight wire. The near and far vertical sides experience unequal horizontal forces. The top and bottom contributions cancel because they traverse the same range of distances with opposite current directions.

Example 9.19: Net force on a rectangular current loop

Find the net force on the loop of Figure 9.17, and evaluate it for $I_s = 20$ A, $I_\ell = 3.0$ A, $h = 0.10$ m, $a = 0.020$ m, and $w = 0.030$ m.

Strategy. Use the source wire's field at each side, not a single average field for the whole loop. The near-side current is parallel to the source current, so its force is attractive. The far-side current is antiparallel, giving a weaker repulsion.

$$F_{\text{near}} = \frac{\mu_0 I_s I_\ell h}{2\pi a}, \quad F_{\text{far}} = \frac{\mu_0 I_s I_\ell h}{2\pi(a+w)}, \quad (9.67)$$

$$F_{\text{net}} = \frac{\mu_0 I_s I_\ell h}{2\pi} \left(\frac{1}{a} - \frac{1}{a+w} \right) = 3.6 \times 10^{-5} \text{ N}. \quad (9.68)$$

The net force points toward the source wire. On the horizontal sides, the local force is vertical. At each distance x , equal current elements on the top and bottom experience opposite vertical forces, so their integrated sum vanishes.

Limit check. If $a \gg w$, the bracket is approximately w/a^2 and the force decreases as a^{-2} . If $w \rightarrow 0$ at fixed h and current, the loop area and net force both vanish. A uniform external field would give exactly zero net force on a closed loop, even though it could produce torque.

9.12.2 Choosing the model for an instrument

A straight conductor gives an accessible field with a strong position dependence, useful for illustrating geometry but sensitive to probe placement. A single circular coil provides a well-defined center field. A long solenoid offers a larger accessible region of nearly uniform axial field. A toroid confines its main field to a closed interior region, making it less suitable when a sensor must be placed in an open bore. A Helmholtz pair deliberately reduces the central field curvature; its uniformity is derived in the enrichment section.

Detector tracking illustrates why knowing the field is as important as generating it. In a nearly uniform axial field, the transverse momentum obeys $p_\perp = |q|B\rho$, where ρ is the radius of the projected track perpendicular to the field. A fractional field calibration bias produces the same fractional momentum bias when the measured curvature is fixed. This connection uses the force law from Chapter 8; it does not require a full detector design here. Spatial field maps and finite-coil corrections matter when the uniform-field approximation is inadequate.

Ferromagnetic cores, saturation, hysteresis, and shielding by magnetic materials belong to Chapter 12. Inductance and magnetic energy belong to Chapter 10. The present source calculations determine the vacuum or air-core field on which those later models build.

9.13 Laboratory connection: measuring a wire field

A Hall probe provides a quantitative test of the long-wire prediction. Its calibrated output is a component of $\mathbf{B}$ along a sensitive axis $\hat{\mathbf{n}}_s$, not automatically the magnitude of the field. With the probe axis tangent to the expected circular field, an ideal measurement would give $\mu_0 I / (2\pi R)$. Rotating the probe by angle α away from that direction multiplies the signal by $\cos \alpha$.

A useful arrangement uses a long straight measured section, a stable DC current, and a translation stage that controls the distance from the *wire axis*. Record the actual current at every setting. Keep the probe orientation fixed, and document the return path. A return conductor at distance D typically produces a field of order $\mu_0 I / (2\pi D)$, so making it ten times farther away than the measured wire only suppresses its contribution to about ten percent in a comparable orientation. Return geometry should be calculated or measured rather than assumed irrelevant.

9.13.1 Current reversal separates signal from offset

A simple measurement model is

$$b(I, R) = g \hat{\mathbf{n}}_s \cdot \mathbf{B}_{\text{circuit}}(I, R) + b_0, \quad (9.69)$$

where g is the calibration gain and b_0 combines a stable background component and an additive sensor offset. For equal and opposite current settings in a linear source,

$$b_{\text{odd}} = \frac{b(+I, R) - b(-I, R)}{2}, \quad b_{\text{even}} = \frac{b(+I, R) + b(-I, R)}{2}. \quad (9.70)$$

The odd part retains the current-dependent field; the even part estimates b_0 . Reversal does not remove gain errors, orientation errors, or the return-wire field: the latter reverses with the desired signal. It also fails to isolate a constant offset if the background drifts appreciably between readings.

If the positive and negative settings have different current magnitudes, fit the readings to $b = kI + b_0$ using the measured signed currents. Taking an uncorrected half-difference would then bias the field estimate. Current-induced heating, hysteretic nearby material, or sensor nonlinearity may also invalidate the simple linear model; these effects can be diagnosed by repeating the current sweep in both directions.

Example 9.20: Reversal data and a field-law test

The following *synthetic teaching data* represent a probe near a wire carrying ± 5.00 A. They are not experimental measurements. All field entries are calibrated components in μT .

R (m)	$b(+I)$	$b(-I)$	b_{odd}	b_{even}
0.020	68.4	-31.8	50.1	18.3
0.030	51.0	-15.8	33.4	17.6
0.040	43.2	-7.0	25.1	18.1
0.060	34.5	1.1	16.7	17.8

Prediction. The long-wire coefficient is $k = \mu_0 I / (2\pi) = 1.00 \times 10^{-6} \text{ T m}$. Thus $b_{\text{odd}} = k/R$ predicts 50.0, 33.3, 25.0, and $16.7 \mu\text{T}$. The data agree with these values at the displayed precision, while the even part remains close to $18 \mu\text{T}$.

Fit. Set $x_i = 1/R_i$ and $y_i = b_{\text{odd},i}$. For equal independent field uncertainties and negligible uncertainty in x_i , minimizing $\sum_i (y_i - kx_i)^2$ gives

$$\hat{k} = \frac{\sum_i x_i y_i}{\sum_i x_i^2} = 1.00228 \times 10^{-6} \text{ T m}. \quad (9.71)$$

The corresponding introductory estimate is $\hat{\mu}_0 = 2\pi\hat{k}/I = 1.2595 \times 10^{-6} \text{ T m/A}$. A through-origin fit tests a model in which all non-wire odd contributions are negligible. An intercept should not be forced to zero merely to improve apparent agreement.

Interpretation. Agreement with $1/R$ checks a geometry model at laboratory precision. It is not a competitive determination of the fundamental constant. Finite length, probe position, calibration gain, return currents, and field averaging over the probe's sensitive area can dominate the uncertainty.

The comparison is displayed in Figure 9.18. Plotting against $1/R$ turns the ideal model into a straight line, making deviations easier to interpret. A systematic bend at small R could indicate an axis-distance offset, a finite probe size, or a failure of the thin-wire approximation; it should not immediately be attributed to a failure of Biot–Savart.

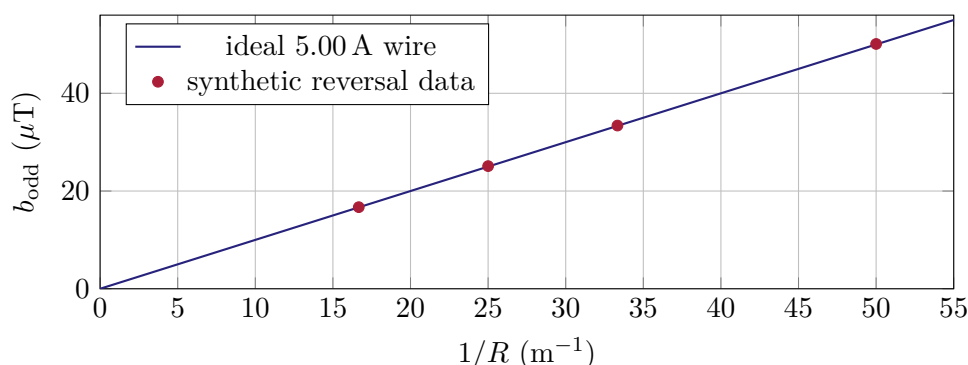


Figure 9.18: Synthetic current-reversal data compared with the ideal long-wire law. The plotted data are generated teaching values and do not establish the accuracy of a real instrument.

9.13.2 Uncertainty and model discrepancy

If independent standard uncertainties in I and R are σ_I and σ_R , first-order propagation for $B = k_0 I / R$, with $k_0 = \mu_0 / (2\pi)$, gives

$$\left(\frac{\sigma_B}{B}\right)^2 = \left(\frac{\sigma_I}{I}\right)^2 + \left(\frac{\sigma_R}{R}\right)^2. \quad (9.72)$$

This follows by differentiating the model: $\partial B/\partial I = B/I$ and $\partial B/\partial R = -B/R$, then adding independent variances. Correlated uncertainties require covariance terms. A worst-case bound instead adds absolute fractional error bounds linearly. These are different questions and should not use the same symbol without explanation.

For $I = (5.00 \pm 0.02)$ A and $R = (20.0 \pm 0.3)$ mm, interpreted as independent standard uncertainties, the relative model-input uncertainty is $\sqrt{0.004^2 + 0.015^2} = 1.55\%$. The distance dominates. If each field reading has independent standard uncertainty $0.4 \mu\text{T}$, the odd part has uncertainty $\sqrt{0.4^2 + 0.4^2}/2 = 0.283 \mu\text{T}$, or 0.57% at $50 \mu\text{T}$. Under independent input and readout errors, their combination is about 1.65% , before calibration and geometry systematics.

A shared error in the position of the wire axis is not four independent distance errors. It shifts all R_i in a correlated way. One may fit $b_{\text{odd}} = k/(R_{\text{nominal}} + \delta)$ if the data span enough distances to constrain both k and the offset δ . Parameter correlation and a finite-length correction should then be reported. The advanced computational module shows how a known circuit geometry can replace an oversimplified $1/R$ model.

Method

A useful laboratory report states the circuit and probe geometry, records signed currents and paired readings, calculates the source-field prediction including the dominant geometric corrections, and plots residuals. Report which uncertainty terms are random, which are shared calibration errors, and which arise from approximations in the field model.

9.14 Chapter summary and formula summary

Summary

- A steady current produces a magnetic field whose direction is determined by a cross product and a right-hand rule.
- The Biot-Savart law builds $\mathbf{B}$ from current elements and is broadly applicable but often requires integration.
- A long straight current produces $B = \mu_0 I/(2\pi R)$ with circular field lines around the wire.
- A circular loop produces $B = \mu_0 I/(2R)$ at its center and behaves as a magnetic dipole.
- Parallel currents in the same direction attract; opposite currents repel; the force per length is $F/L = \mu_0 I_1 I_2/(2\pi d)$ for long wires.
- Ampere's law relates magnetic circulation to enclosed steady current: $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}$.
- Ampere's law is a powerful calculation method only when symmetry makes the line integral simple.
- Ideal long solenoids have approximately uniform interior field $B = \mu_0 n I$.
- Ideal continuous toroidal windings have $B(r) = \mu_0 N I/(2\pi r)$ in the winding region and zero exterior source field; real windings can leak.
- Real devices require correction for finite geometry, materials, heating, and measurement uncertainty.

Formula summary

These formulas assume prescribed steady currents and vacuum permeability. Wire and solenoid results require the geometries stated beside them; arc formulas apply at the center and loop-axis formulas only on the axis. ϕ is in radians, current magnitudes are positive unless explicitly signed, and all force formulas use the field of other sources.

$$\begin{aligned} d\mathbf{B} &= \frac{\mu_0}{4\pi} \frac{I d\boldsymbol{\ell} \times \hat{\mathbf{r}}}{r^2}, \\ B_{\text{long wire}} &= \frac{\mu_0 I}{2\pi R}, \\ B_{\text{arc center}} &= \frac{\mu_0 I \phi}{4\pi R}, \\ B_{\text{loop center}} &= \frac{\mu_0 I}{2R}, \\ B_{\text{loop axis}}(z) &= \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}, \\ \frac{F}{L} &= \frac{\mu_0 I_1 I_2}{2\pi d}, \\ \oint \mathbf{B} \cdot d\boldsymbol{\ell} &= \mu_0 I_{\text{enc}}, \\ B_{\text{solenoid}} &\simeq \mu_0 n I, \\ B_{\text{toroid}}(r) &\simeq \frac{\mu_0 N I}{2\pi r}, \\ \boldsymbol{\mu} &= N I A \hat{\mathbf{n}}, \quad \boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}_{\text{ext}}. \end{aligned}$$

9.15 Problems for practice and independent study

Unless specified otherwise, use $\mu_0 \simeq 4\pi \times 10^{-7} \text{ T m/A}$, steady prescribed currents, and vacuum permeability. Ideal straight wires are infinitely long. Include a direction, an assumption, and a check with each quantitative answer. Problems P1–P8 emphasize concepts, P9–P25 analytical methods, and P26–P40 applications, measurement, and synthesis. Detailed solutions to a representative selection follow the problem set.

- P1.** A vertical wire carries current upward. Use east, north, and up as a right-handed coordinate system to determine the field at a point east of the wire. Explain what happens when the current is reversed.
- P2.** In a Biot–Savart diagram, the source-to-point vector is accidentally reversed. Which of the magnitude, direction, and distance dependence is affected? Explain using the cross product.
- P3.** A counterclockwise Amperian path encloses 9 A out of the page and 4 A into it. A 12 A current is outside. Find the circulation, state what changes when the path orientation reverses, and explain why the external current can affect $\mathbf{B}$ without appearing in I_{enc} .
- P4.** A student claims that $I_{\text{enc}} = 0$ implies $\mathbf{B} = 0$ along the path. Give an explicit counterexample and identify the missing assumption in the claim.
- P5.** Why can radial leads give zero field at an arc center while tangential leads do not? State why neither observation permits omission of the rest of an unspecified circuit.
- P6.** A square and a circle have the same area and carry the same current with parallel normals. Compare their dipole moments and far fields. Must their center fields agree? Derive the ratio

of the square's center field to the circle's using the finite-wire formula.

- P7.** A wire carrying current along $+z$ produces circular field lines. Does an electron initially moving tangent to one of those lines follow the line? Discuss its instantaneous magnetic force.
- P8.** Distinguish an exact infinite current-sheet solenoid model from a finite helical winding. Name two source features that can produce an exterior field omitted in the ideal model.
- P9.** A wire carries 8.0 A. Find its field at 2.0 cm and 10 cm. A probe is actually 2.0 cm from the surface of a wire of radius 2.0 mm. Quantify the bias from treating that surface distance as the distance to the axis.
- P10.** A straight circuit segment lies on the z axis from $z_1 = 0.020$ m to $z_2 = 0.120$ m and carries 4.0 A along $+z$. Find its contribution at $(0.030, 0, 0)$ m. Why is a sum of two unsigned endpoint-angle sines incorrect here?
- P11.** Derive the contribution from a semi-infinite wire at a point beside its endpoint. Explain which parts of a physical circuit must still be specified before the complete field is known.
- P12.** On the perpendicular bisector of a segment of total length $20R$, find the fractional deficit relative to the infinite-wire value. Find the total length needed for a deficit below 0.1%.
- P13.** Two long wires at $x = -0.030$ m and $x = +0.030$ m each carry 5.0 A along $+z$. Find the field vector at $(x, y) = (0, 0.040)$ m, then reverse the right-hand current. Compare magnitudes and directions.
- P14.** Wires at $x = 0$ and $x = d$ carry I and $4I$ along $+z$. Find every finite zero of the field on the x axis. Repeat when the second current is $-4I$.
- P15.** A closed annular-sector circuit has inner radius a , outer radius $b > a$, and angular span ϕ . The outer arc current is counterclockwise viewed from $+z$. Derive the center field. Evaluate for $a = 0.030$ m, $b = 0.090$ m, $\phi = 120^\circ$, and $I = 1.5$ A.
- P16.** A circular coil has $N = 150$, $R = 0.040$ m, and $I = 0.20$ A. Find its center field and its field at $z = R$. Determine the axial displacement at which the field falls to half the center value.
- P17.** For fixed current, double a circular loop's radius. Find the factors by which its center field, moment, and axial far field at a fixed distance much larger than both radii change. Explain why the answers differ.
- P18.** Two long wires separated by 5.0 cm carry 10 A and 15 A in opposite directions. Find force per unit length and the force on a 0.80 m section far from the ends. If both currents triple, how does the force change?
- P19.** A long cylindrical wire of radius $a = 2.0$ mm carries 6.0 A uniformly. Find the field at $r = 0$, $a/2$, a , and $2.5a$. Sketch the complete profile and state where its derivative changes.
- P20.** An infinitely long cylindrical shell occupies $a < r < b$ and carries uniform axial current density with total current I . Find the field in the cavity, within the material, and outside. Check both boundaries.
- P21.** For the thick coaxial cable of Eq. (9.48), show that the field is continuous at every boundary and decreases monotonically through the return conductor. Why would equal radial thicknesses not contain equal amounts of return current?
- P22.** A coaxial cable has $a = 1.0$ mm, $b = 3.0$ mm, $c = 4.0$ mm, and $I = 8.0$ A. Find B_ϕ at $r = 0.50$, 2.0, 3.5, and 5.0 mm. If the return current is only 7.5 A in the same coaxial region, what is

its exterior source-field contribution at 5.0 mm?

- P23.** An air-core solenoid needs 25 mT with current limited to 5.0 A. Find the ideal required turn density in turns per metre and per centimetre. State why total turn count cannot be determined without a length.
- P24.** A long solenoid is stretched at fixed total turns and current, increasing its length by 20%. Predict the ideal field change. What changes if additional turns are supplied so that turn density stays fixed?
- P25.** An ideal toroid has $N = 600$, $I = 1.2$ A, and radial limits 0.060 m and 0.090 m. Find the inner, mean-radius, and outer fields, evaluated within the winding region. Quantify the largest fractional departure from the mean-radius value.
- P26.** A rectangular coil has 50 turns, sides 3.0 cm and 6.0 cm, and current 0.40 A. Find its moment and maximum torque in an external 0.30 T field. Does that uniform field exert a net force on the whole loop?
- P27.** A long source wire and nearby rectangular loop have the geometry of Figure 9.17. Reverse the loop current and derive the new force direction. Show how the force scales if all geometrical lengths are doubled while currents remain fixed.
- P28.** Repeat the calibration-coil calculation for $N = 100$, $R = 0.060$ m, wire area 0.40 mm^2 , given resistivity $1.7 \times 10^{-8} \Omega \text{ m}$, and a target field of 1.0 mT. Find I , wire resistance, voltage, and power under the thin-winding approximation.
- P29.** With a fixed coil radius and fixed wire gauge, doubling N doubles the resistance. Compare the center field change at fixed current, fixed voltage, and fixed resistive power. Which comparison also changes the copper volume?
- P30.** A single-axis probe is misaligned by 15° from the local field. Calculate the fractional signal loss. For small angles in radians, derive the leading loss and explain why it is quadratic.
- P31.** Readings with equal and opposite currents are $b_+ = 72.0 \pm 0.4 \mu\text{T}$ and $b_- = -28.0 \pm 0.4 \mu\text{T}$, where the stated values are independent standard uncertainties. Find odd and even parts and their uncertainties. List two errors that current reversal does not remove.
- P32.** A probe at $R = 0.040$ m measures a current-dependent field component of $50 \mu\text{T}$. Infer the current assuming an isolated long-wire model. How does a 1% gain bias propagate into the inferred current?
- P33.** A current and distance have independent standard uncertainties of 0.5% and 2.0%. Find the standard uncertainty in the predicted wire field. Compare with a worst-case bound when those percentages are instead error limits.
- P34.** In the synthetic data table of Example 9.20, compute $R_i b_{\text{odd},i}$ for each row. Compare the spread with the ideal coefficient. Explain why four consistent values do not by themselves prove that return-current and gain errors are negligible.
- P35.** For two opposite long currents at $x = \pm a$, calculate the far field on the perpendicular bisector. Show its leading dependence on a/y^2 and explain why zero enclosed net current does not make it vanish at every exterior point.
- P36.** A tangent compass experiment places a wire field perpendicular to a horizontal background B_h . Derive $\tan \theta = B_{\text{wire}}/B_h$. For $I = 3.0$ A, $R = 0.030$ m, and given $B_h = 25 \mu\text{T}$, find the deflection. What does the compass follow if the two fields are not perpendicular?

- P37.** The thin-wire formula predicts a field proportional to $1/R$. Suppose distances are incorrectly recorded from the wire surface rather than the axis. Describe the curvature this produces in a plot against reciprocal recorded distance, and formulate a corrected fitting function with one distance-offset parameter.
- P38.** A current loop with radius $R = 0.050$ m is used to estimate a field at $z = 0.25$ m. Compare its exact axial field with its leading dipole approximation by expanding the core loop-axis formula. Determine the relative error normalized to the exact field.
- P39.** A field sensor inside a solenoid is rotated through 180° while the current remains fixed. A probe's electronic offset does not reverse under this rotation but its measured field component does. Derive how paired readings isolate the field component. Contrast this operation with reversing the coil current.
- P40.** Write a short model-selection argument for each task: field inside a round current-filled wire, field at the center of an arc-and-lead circuit, field at an arbitrary point near a square loop, and force on a second wire. State the appropriate law and the symmetry or geometry that makes it useful.

9.16 Selected detailed solutions

These solutions show how a direction choice and a model check accompany the calculation. They do not replace the diagrams and assumptions requested in the problem set.

P3: signed circulation and an external current

Counterclockwise traversal corresponds to a surface normal out of the page. The signed enclosed current is $9 - 4 = 5$ A, so

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0(5 \text{ A}) = 6.28 \times 10^{-6} \text{ T m}.$$

The external 12 A wire contributes to the field at each point on C , but its tangential contributions sum to zero around this path. Clockwise traversal reverses every $d\boldsymbol{\ell}$ and the associated positive normal, so both circulation and signed enclosed current reverse. The source field itself does not change when we merely change a mathematical integration convention.

P6: equal moments do not imply equal near fields

Let the square have side $2a$. Its area is $4a^2$, so an equal-area circle has radius $R = 2a/\sqrt{\pi}$. Both moments have magnitude $4Ia^2$ and the same direction. Their leading dipole far fields agree because the leading term depends only on the moment. Higher multipole terms need not agree.

At the center of the square, each side is at distance a and subtends endpoint angles $\pm 45^\circ$. Thus

$$B_{\text{square}} = 4 \frac{\mu_0 I}{4\pi a} (2 \sin 45^\circ) = \frac{\sqrt{2} \mu_0 I}{\pi a}.$$

The circle gives $B_{\text{circle}} = \mu_0 I / (2R) = \sqrt{\pi} \mu_0 I / (4a)$. Their ratio is

$$\frac{B_{\text{square}}}{B_{\text{circle}}} = \frac{4\sqrt{2}}{\pi^{3/2}} = 1.016.$$

The difference is small in this particular comparison but is not zero. The dipole moment is a compact description of the distant field, not a full description of the source geometry.

P10: both endpoints on the same side

Here z_1 and z_2 are both positive, so their signed-angle sines must be subtracted. Equation (9.12) gives

$$B_y = \frac{(4\pi \times 10^{-7})(4.0)}{4\pi(0.030)} \left[\frac{0.120}{\sqrt{0.030^2 + 0.120^2}} - \frac{0.020}{\sqrt{0.030^2 + 0.020^2}} \right] \\ = 5.54 \times 10^{-6} \text{ T.}$$

The direction is $+y$, because $\hat{z} \times \hat{x} = \hat{y}$. The bracket is $0.9701 - 0.5547 = 0.4154$. Adding these positive sines would incorrectly include source length on the opposite side of the perpendicular foot. The result above is the segment contribution only; the return path remains unspecified.

P13: two different cancellations of components

The distance from either wire to the point is 0.050 m. Each field magnitude is $\mu_0(5.0)/(2\pi \times 0.050) = 20 \mu\text{T}$. For currents both along $+z$, their components are

$$\mathbf{B}_1 = (-16\hat{x} + 12\hat{y}) \mu\text{T}, \quad \mathbf{B}_2 = (-16\hat{x} - 12\hat{y}) \mu\text{T}.$$

The resultant is $-32\hat{x} \mu\text{T}$. Reversing the right-hand current changes the second vector's sign, so the new resultant is $+24\hat{y} \mu\text{T}$. Symmetry cancels a different component in each case. Equal source-field magnitudes do not imply a zero resultant unless their directions are opposite.

P16: a coil field and a specified reduction

The center value is

$$B(0) = \frac{\mu_0 NI}{2R} = \frac{(4\pi \times 10^{-7})(150)(0.20)}{2(0.040)} = 4.71 \times 10^{-4} \text{ T.}$$

At $z = R$, the normalized factor is $2^{-3/2}$, so $B(R) = 1.67 \times 10^{-4} \text{ T}$. To obtain half the center value, solve

$$(1 + (z/R)^2)^{-3/2} = \frac{1}{2}, \quad |z| = R\sqrt{2^{2/3} - 1} = 0.0307 \text{ m.}$$

Both positive and negative positions give the same magnitude and the same axial direction for a fixed current sense. This is a check on the even dependence of the loop field on z .

P19: why the field peaks at the surface

Inside the conductor, the enclosed current is Ir^2/a^2 ; outside, it is I . The surface field is

$$B(a) = \frac{\mu_0 I}{2\pi a} = \frac{(2 \times 10^{-7})(6.0)}{0.0020} = 6.0 \times 10^{-4} \text{ T.}$$

At the requested radii, the fields are respectively 0, 3.0×10^{-4} , 6.0×10^{-4} , and $2.4 \times 10^{-4} \text{ T}$. The internal slope is positive and constant; the external derivative is negative. Their values at the surface are $+\mu_0 I/(2\pi a^2)$ and $-\mu_0 I/(2\pi a^2)$. The field is continuous there, but the slope changes abruptly in the ideal uniform-density model.

P22: enclosed current in a finite-thickness return

The four observation radii lie inside the inner conductor, in the insulation, within the return conductor, and outside the cable. In order,

$$B_\phi(0.50 \text{ mm}) = (2 \times 10^{-7})(8.0) \frac{0.00050}{(0.0010)^2} = 0.800 \text{ mT}, \\ B_\phi(2.0 \text{ mm}) = \frac{(2 \times 10^{-7})(8.0)}{0.0020} = 0.800 \text{ mT}.$$

At $r = 3.5 \text{ mm}$, the enclosed fraction of return current is $(3.5^2 - 3.0^2)/(4.0^2 - 3.0^2) = 3.25/7$. Therefore $I_{\text{enc}} = 8(1 - 3.25/7) = 4.286 \text{ A}$ and

$$B_\phi = \frac{(2 \times 10^{-7})(4.286)}{0.0035} = 0.245 \text{ mT}.$$

With equal returns, the exterior field at 5.0 mm is zero. With a 7.5 A coaxial return, the residual is 0.50 A, giving $20\ \mu\text{T}$ along $+\hat{\phi}$ from this coaxial portion. A physical circuit must carry the remaining return current elsewhere, and its field must also be added.

P25: the limits of a mean-radius toroid model

Here $NI = 720\ \text{A}$, so $B(r) = (1.44 \times 10^{-4}\ \text{T m})/r$. The inner, mean-radius, and outer values are 2.40, 1.92, and 1.60 mT. Relative to the mean-radius value, the deviations are +25.0% and -16.7%. The mean radius is 0.075 m, and the radial thickness is not negligible compared with it. Reporting only 1.92 mT would hide the nonuniformity relevant to a local field measurement.

P28: a magnetic specification translated into a circuit load

The current needed at the coil center is

$$I = \frac{2RB}{\mu_0 N} = \frac{2(0.060)(0.0010)}{(4\pi \times 10^{-7})(100)} = 0.955\ \text{A}.$$

The wire length is $2\pi RN = 37.7\ \text{m}$ and its resistance is

$$R_{\text{coil}} = \frac{(1.7 \times 10^{-8})(37.7)}{0.40 \times 10^{-6}} = 1.60\ \Omega.$$

Thus $V = 1.53\ \text{V}$ and $P = 1.46\ \text{W}$. The radius determines the field geometry; the wire area determines the resistance. Neither quantity should be substituted for the other. Heating and finite winding thickness can change these estimates, so they are a starting point for design rather than an operating-temperature assessment.

P31: what reversal can remove

The odd and even components are $(72 - (-28))/2 = 50.0\ \mu\text{T}$ and $(72 + (-28))/2 = 22.0\ \mu\text{T}$. Each has standard uncertainty

$$\sigma = \frac{1}{2}\sqrt{0.4^2 + 0.4^2} = 0.283\ \mu\text{T}.$$

A stable additive electronic offset and a stable ambient field component enter the even part. A gain bias multiplies the odd part, and a return-conductor field reverses with the source current, so neither is removed. The independent-reading assumption is needed for the quoted uncertainty; shared drift may correlate the readings.

P34: a diagnostic transformation of the synthetic data

Multiplying each source-field estimate by its axis distance gives, in $\mu\text{T m}$,

$$1.002, \quad 1.002, \quad 1.004, \quad 1.002.$$

These are close to the ideal value $1.000\ \mu\text{T m}$ for 5.00 A. The spread is small, but a common gain factor would multiply every row by the same amount. A return geometry producing a similar distance dependence could also bias the coefficient. Consistency across distances is a useful model test, not a proof that every systematic error is absent.

P38: checking a far-field approximation

The source-to-point ratio is $|z|/R = 5$. The dipole approximation is $B_{\text{dip}} = \mu_0 IR^2/(2|z|^3)$ and the exact value is $B_{\text{exact}} = \mu_0 IR^2/[2(R^2 + z^2)^{3/2}]$. Their ratio is

$$\frac{B_{\text{dip}}}{B_{\text{exact}}} = (1 + 1/25)^{3/2} = 1.06060.$$

The relative overestimate is 6.06%. Expanding in R^2/z^2 gives $1 + 3R^2/(2z^2) + \dots$, so the leading estimate of the error is 6.00%. The comparison requires no current value because it is a ratio of fields from the same source.

9.17 Advanced Topics for Technical and Engineering Students

The core chapter used symmetry and one-dimensional integrals to obtain fields of wires and coils. This enrichment develops the same ideas in three connected directions: local field equations, controlled corrections to ideal coil models, and numerical evaluation of complete current paths. The additional mathematics is introduced where it is used. Students may read the finite-solenoid and numerical sections independently of the vector-potential derivation.

Pedagogical boundary

The source remains a prescribed steady current distribution in vacuum or nonmagnetic surroundings. Time-dependent induction and electromagnetic energy are developed in Chapter 10, and magnetization and the Maxwell displacement-current term in Chapter 12. No result below should be interpreted as a complete model of a ferromagnetic or rapidly pulsed magnet.

9.18 Current density, local circulation, and boundary conditions

9.18.1 From a filament to a volume source

For a bundle of thin current tubes, an area element $dA_{\perp}$ perpendicular to the local current carries $dI = JdA_{\perp}$. A short length of tube occupies volume $dV = dA_{\perp}d\ell$, so $dI d\ell = JdV$. Substituting into Biot–Savart and adding all tubes gives

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3x'. \quad (9.73)$$

The integration variable $\mathbf{x}'$ labels the source, and $\mathbf{x}$ labels the observation point. The integrand is a vector. The source-to-point separation is $\mathbf{x} - \mathbf{x}'$, never the reverse. The units are $(\text{T m/A})(\text{A/m}^2)(\text{m})(\text{m}^3)/\text{m}^3 = \text{T}$.

For a thin sheet, the corresponding source is a surface-current density $\mathbf{K}$ in amperes per metre. It is current per unit width transverse to the flow, not current per unit area. Replacing Jd^3x' by $\mathbf{K}dS'$ gives

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_S \frac{\mathbf{K}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} dS'. \quad (9.74)$$

The sheet model describes a tightly wound solenoid with $K_{\phi} = nI$. These descriptions form a hierarchy: line current when the wire radius is unresolved, surface current when a thin layer is unresolved, and volume current when the conductor cross section matters.

9.18.2 Charge conservation and the differential law

Charge conservation states that a net outward current from a volume reduces the charge inside it. Locally,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_q}{\partial t}. \quad (9.75)$$

Here ρ_q is charge density, distinct from the electrical resistivity ρ_e used earlier. In magnetostatics $\partial \rho_q / \partial t = 0$, so $\nabla \cdot \mathbf{J} = 0$. This is the differential statement that current has no endpoints.

Stokes's theorem converts a line integral around an oriented boundary into an integral of the curl over its spanning surface:

$$\oint_{\partial S} \mathbf{B} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} dS. \quad (9.76)$$

The current through that same surface is $\int_S \mathbf{J} \cdot \hat{\mathbf{n}} dS$. Since Ampere's law holds for every sufficiently small surface,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0. \quad (9.77)$$

The second equation expresses the absence of magnetic charge. Curl and divergence describe different local properties: the first measures circulation per unit area; the second measures outward flux per unit volume. Neither is simply the visual bending of a plotted curve.

For example, outside a long straight wire the field lines are circles, yet the local curl is zero. In cylindrical coordinates, for an azimuthal field independent of ϕ and z ,

$$(\nabla \times \mathbf{B})_z = \frac{1}{r} \frac{d}{dr}(rB_\phi). \quad (9.78)$$

For $B_\phi = C/r$, the product $rB_\phi = C$ is constant and its derivative vanishes. A closed loop around the wire nevertheless has nonzero circulation. A surface spanning that loop must encounter the excluded current-carrying region; it cannot be wholly inside the current-free domain while avoiding the wire. There is no conflict with Stokes's theorem.

Example 9.21: Recovering current density from a field profile

An axisymmetric long conductor has an internal field $B_\phi(r) = \beta r^2$ for $0 \leq r < a$, where β has units T/m². Determine the current density and total current, and match the exterior field.

Derivation. Equation (9.78) gives

$$J_z(r) = \frac{1}{\mu_0 r} \frac{d}{dr}(\beta r^3) = \frac{3\beta r}{\mu_0}. \quad (9.79)$$

The total current is

$$I = \int_0^a J_z(r) 2\pi r dr = \frac{2\pi\beta a^3}{\mu_0}. \quad (9.80)$$

Outside, $B_\phi = \mu_0 I / (2\pi r) = \beta a^3 / r$. At $r = a$ this equals βa^2 , agreeing with the internal branch. The nonuniform density is proportional to r , whereas a uniform density would produce $B_\phi \propto r$.

Experimental meaning. Reconstructing J requires differentiating a measured field profile. Differentiation amplifies measurement noise, so a fitted physical model or controlled smoothing is needed. The derivative should not be interpreted at a mathematical surface carrying a sheet current without accounting for the possible field jump.

More generally, $J_z(r) = J_0(r/a)^p$ for $p \geq 0$ produces

$$I = \frac{2\pi J_0 a^2}{p+2}, \quad B_\phi(r) = \frac{\mu_0 J_0 r^{p+1}}{(p+2)a^p} = \frac{\mu_0 I}{2\pi a} \left(\frac{r}{a}\right)^{p+1}, \quad r < a. \quad (9.81)$$

The exterior field is the same for all these profiles at fixed total I . This explicitly illustrates what the external measurement cannot determine.

9.18.3 A current sheet and its field discontinuity

Take an infinite sheet at $z = 0$ carrying $\mathbf{K} = K\hat{\mathbf{x}}$. The sheet has no preferred x or y position. Contributions from parallel current filaments cancel normal components and reinforce a field parallel to y . The right-hand rule gives $\mathbf{B} = -B\hat{\mathbf{y}}$ above the sheet and $\mathbf{B} = +B\hat{\mathbf{y}}$ below it. Figure 9.19 shows the directions in a cross section perpendicular to the current.

Choose a rectangular loop with two long sides of length ℓ along these fields. Its circulation is $2B\ell$, while the current passing through the rectangle is $K\ell$. Hence

$$\mathbf{B}(z) = \begin{cases} -\mu_0 K \hat{\mathbf{y}}/2, & z > 0, \\ +\mu_0 K \hat{\mathbf{y}}/2, & z < 0. \end{cases} \quad (9.82)$$

A field evaluated exactly on the ideal sheet has two one-sided limits; the model does not assign a unique continuous value there. The independence of $|z|$ reflects the infinite extent. A finite sheet eventually has a decreasing distant field.

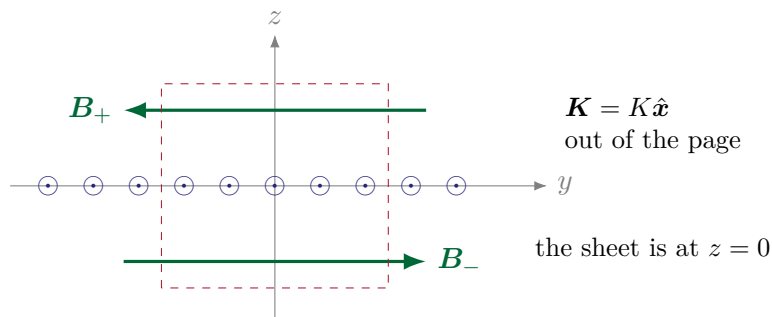


Figure 9.19: A yz -plane cross section of an infinite current sheet located at $z = 0$. The currents point along $+x$ out of the page. The two field limits have equal magnitude and opposite directions.

This argument extends to an arbitrary locally flat current sheet. Let $\hat{n}$ point from side 1 to side 2. Shrinking the height of an Amperian rectangle across the sheet gives the tangential boundary condition

$$\hat{n} \times (\mathbf{B}_2 - \mathbf{B}_1) = \mu_0 \mathbf{K}. \quad (9.83)$$

A thin closed pillbox and $\nabla \cdot \mathbf{B} = 0$ give continuity of the normal component, $\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$. For the sheet above, $\hat{n} = \hat{z}$ and $\mathbf{B}_2 - \mathbf{B}_1 = -\mu_0 K \hat{y}$, so the cross product gives $+\mu_0 K \hat{x}$, as required.

Example 9.22: Resolving a sheet into a current-carrying slab

Replace the ideal sheet by a slab $-t/2 < z < t/2$ carrying uniform $\mathbf{J} = J\hat{x}$. Find its field and recover the sheet limit at fixed $K = Jt$.

Local calculation. Symmetry gives $\mathbf{B} = B_y(z)\hat{y}$ and $B_y(0) = 0$. The x component of curl is $-dB_y/dz = \mu_0 J$, so inside

$$B_y(z) = -\mu_0 Jz. \quad (9.84)$$

Outside, $J = 0$ and B_y is constant on each side. Matching continuously gives $B_y = -\mu_0 Jt/2$ above and $+\mu_0 Jt/2$ below. As $t \rightarrow 0$ at fixed $Jt = K$, the smooth internal transition narrows to the discontinuity of Eq. (9.82).

Interpretation. A discontinuity can be a controlled idealization of a thin region, not a failure of the field law. The same distinction separates the finite-thickness coaxial return conductor from its thin-shell limit.

9.19 Vector potential and the magnetic dipole field

9.19.1 A potential that automatically has zero divergence

Because the divergence of a curl vanishes, writing

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (9.85)$$

automatically satisfies Gauss's law for magnetism. The vector potential $\mathbf{A}$ has units Tm. It is not unique: replacing it by $\mathbf{A} + \nabla\chi$ leaves $\mathbf{B}$ unchanged because the curl of a gradient is zero. This freedom is called gauge freedom. A field calculation must therefore distinguish the physically observable $\mathbf{B}$ from a particular representation of $\mathbf{A}$.

For localized steady currents in vacuum, a useful representation is

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'. \quad (9.86)$$

To verify it, take the curl with respect to the field coordinate $\mathbf{x}$. The source current is independent of that derivative, and

$$\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -\frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (9.87)$$

Therefore $\nabla \times [\mathbf{J}/|\mathbf{x} - \mathbf{x}'|] = \mathbf{J} \times (\mathbf{x} - \mathbf{x}')/|\mathbf{x} - \mathbf{x}'|^3$, reproducing Eq. (9.73). For divergence-free localized currents whose boundary contribution vanishes, this representation also satisfies the Coulomb-gauge condition $\nabla \cdot \mathbf{A} = 0$.

Combining $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ with $\mathbf{B} = \nabla \times \mathbf{A}$ and the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ gives

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \quad (9.88)$$

in this gauge. Each Cartesian component obeys a Poisson equation, linking magnetostatic source calculation to the electrostatic methods introduced earlier. Boundary conditions remain essential: a homogeneous solution can represent an additional external field.

9.19.2 The leading far field of a compact loop

For a filamentary closed loop, Eq. (9.86) becomes $\mathbf{A} = \mu_0 I \oint d\ell' / (4\pi |\mathbf{x} - \mathbf{x}'|)$. At a distance $r = |\mathbf{x}|$ much larger than the source size,

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{x}'}{r^2} + \mathcal{O}\left(\frac{|\mathbf{x}'|^2}{r^3}\right). \quad (9.89)$$

The leading term in the loop integral vanishes because $\oint d\ell' = 0$. Define

$$\boldsymbol{\mu} = \frac{I}{2} \oint \mathbf{x}' \times d\ell'. \quad (9.90)$$

For a planar loop this equals $IA\hat{\mathbf{n}}$. To identify the next term, note that $\oint d(x'_i x'_j) = 0$, so $\oint x'_i dx'_j = -\oint x'_j dx'_i$. This antisymmetry reduces the first-order source integral to the magnetic moment and gives

$$\mathbf{A}_{\text{dip}}(\mathbf{x}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{\mu} \times \hat{\mathbf{r}}}{r^2}. \quad (9.91)$$

Taking its curl, at points away from the source, yields

$$\mathbf{B}_{\text{dip}}(\mathbf{x}) = \frac{\mu_0}{4\pi r^3} [3(\boldsymbol{\mu} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \boldsymbol{\mu}]. \quad (9.92)$$

This is the leading nonzero term for a compact source with nonzero magnetic moment. If the total moment vanishes, higher multipoles determine the distant field. The formula is not valid through the wire or at the source center.

For $\boldsymbol{\mu} = \mu \hat{\mathbf{z}}$, spherical components are

$$B_r = \frac{\mu_0 \mu}{4\pi r^3} 2 \cos \theta, \quad B_\theta = \frac{\mu_0 \mu}{4\pi r^3} \sin \theta. \quad (9.93)$$

Along either half of the axis, the field points in the $+z$ direction with magnitude $\mu_0 \mu / (2\pi r^3)$. In the equatorial plane, it points along $-z$ with half that magnitude at the same r . The reversal between axial and equatorial directions describes the returning dipole field; it is not evidence of magnetic charges.

Example 9.23: How far away is the dipole approximation accurate?

For a circular loop, compare the exact axial magnitude with the dipole approximation and find the distance for a relative error below 1%, normalized to the exact field.

Comparison. With $\mu = I\pi R^2$,

$$\frac{B_{\text{dip}}}{B_{\text{exact}}} = \left(1 + \frac{R^2}{z^2}\right)^{3/2}, \quad z \neq 0. \quad (9.94)$$

The dipole expression overestimates the exact axial field. Requiring this ratio to be at most 1.01 gives

$$\frac{|z|}{R} \geq \left(1.01^{2/3} - 1\right)^{-1/2} = 12.26. \quad (9.95)$$

At $|z| = 5R$, the overestimate is $(1.04)^{3/2} - 1 = 6.06\%$. Thus a distance that looks visually large compared with the loop may still be insufficient for a stringent accuracy requirement.

Check. The binomial expansion gives a leading relative error $3R^2/(2z^2)$, predicting $|z|/R \simeq \sqrt{150} = 12.25$ for one percent. The agreement confirms the asymptotic reasoning.

9.19.3 Reflection symmetry needs a vector convention

Magnetic field is an axial vector: it transforms like a cross product, not like an ordinary displacement. Consequently a blanket statement that the field component normal to a mirror plane vanishes is unsafe. For a circular loop in the xy plane, the field in that plane has a nonzero normal component B_z , while in-plane components cancel. The cross product in Biot–Savart supplies the correct parity.

For an axisymmetric pair of identical same-sense coils at $z = \pm s/2$, the field components obey $B_z(r, -z) = B_z(r, z)$ and $B_r(r, -z) = -B_r(r, z)$. Hence $B_r(r, 0) = 0$, not $B_z(r, 0) = 0$. Reversing one coil changes these parities. Such symmetry checks are powerful tests of a numerical field map when the current directions are stated explicitly.

9.20 Finite solenoids: deriving and testing the ideal limit

A finite solenoid can be constructed as a stack of circular loops. This approach makes the end correction a consequence of the same Biot–Savart law used in the core chapter. Let the cylindrical winding have radius R , length L , and uniform turn density $n = N/L$. Measure the observation coordinate z from its center. A thin slice at source coordinate z' contains ndz' turns, each contributing the loop-axis field.

9.20.1 Integrating the loop stack

For current sense that produces field along $+z$, the total axial component is

$$B(z) = \frac{\mu_0 n I R^2}{2} \int_{-L/2}^{L/2} \frac{dz'}{[R^2 + (z - z')^2]^{3/2}}. \quad (9.96)$$

Put $u = z - z'$, so $dz' = -du$. The antiderivative of $R^2/(R^2 + u^2)^{3/2}$ is $u/\sqrt{R^2 + u^2}$. Keeping the reversed limits gives

$$B(z) = \frac{\mu_0 n I}{2} \left[\frac{z + L/2}{\sqrt{R^2 + (z + L/2)^2}} - \frac{z - L/2}{\sqrt{R^2 + (z - L/2)^2}} \right]. \quad (9.97)$$

This is exact for the continuous cylindrical current-sheet model, and an approximation to a closely spaced collection of turns. It is an on-axis expression, not a formula for every point in the bore.

At the center,

$$B(0) = \mu_0 n I \frac{L/2}{\sqrt{R^2 + (L/2)^2}}. \quad (9.98)$$

The dimensionless ratio L/R controls the center correction. At fixed n and I , taking $L/R \rightarrow \infty$ gives $\mu_0 n I$. At fixed total turns N instead, taking $L \rightarrow 0$ while the radius stays fixed gives $\mu_0 N I / (2R)$: the stack collapses into a thin circular coil. Specifying the fixed quantity is essential when taking a limit.

At an end, $z = L/2$, Eq. (9.97) becomes

$$B(L/2) = \frac{\mu_0 n I}{2} \frac{L}{\sqrt{R^2 + L^2}} \rightarrow \frac{\mu_0 n I}{2} \quad (L \gg R). \quad (9.99)$$

Far beyond the complete coil, $|z| \gg L, R$, the field tends to the dipole result with total magnetic moment $\mu = N I \pi R^2$. Thus one formula connects a loop, a long solenoid, and a distant dipole.

Example 9.24: A short solenoid used for calibration

A solenoid has $R = 0.030$ m, $L = 0.120$ m, $N = 240$, and $I = 0.50$ A. Calculate its center field and its axial end field, and compare the center field with the infinite-solenoid estimate.

Calculation. The turn density is $n = 2000 \text{ m}^{-1}$, giving $\mu_0 n I = 1.257 \text{ mT}$. The center correction is

$$f_c = \frac{0.060}{\sqrt{0.030^2 + 0.060^2}} = 0.89443, \quad (9.100)$$

so $B(0) = 1.124 \text{ mT}$. At the end,

$$B(L/2) = \frac{1.257 \text{ mT}}{2} \frac{0.120}{\sqrt{0.030^2 + 0.120^2}} = 0.610 \text{ mT}. \quad (9.101)$$

The center is 10.6% below the infinite-solenoid value; equivalently, the infinite-solenoid estimate is 11.8% above the finite center field. These percentages have different denominators.

Interpretation. A coil only four radii long has an appreciable center correction and a much larger position dependence near its ends. Using its nominal turn density is insufficient for a precise sensor calibration. Finite-length modelling should precede a claim of field uniformity.

The calculated profiles in Figure 9.20 compare two aspect ratios. The horizontal coordinate is normalized to the full coil length, so the ends occur at $z/L = \pm 1/2$. The long coil has a wider central plateau in these units, but neither profile is perfectly uniform.

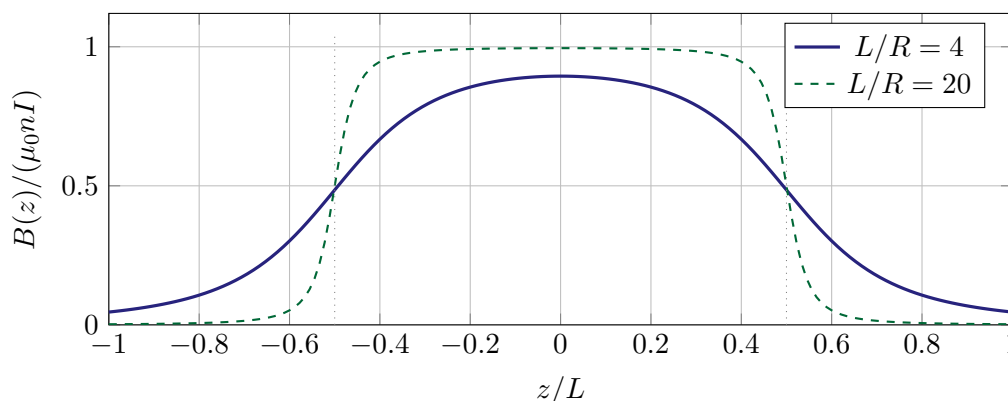


Figure 9.20: Continuous-winding finite-solenoid profiles from Eq. (9.97). The dotted lines locate the ends. Normalization uses the infinite-solenoid value for the same nI .

9.20.2 Discrete turns and geometric corrections

For N discrete circular turns at positions z_j , the corresponding on-axis sum is

$$B_N(z) = \frac{\mu_0 I R^2}{2} \sum_{j=1}^N \frac{1}{[R^2 + (z - z_j)^2]^{3/2}}. \quad (9.102)$$

Placing the turns at slice midpoints $z_j = -L/2 + (j-1/2)L/N$ makes this a midpoint approximation to the continuous stack. It is especially accurate when the pitch L/N is small compared with the length scale over which a single-turn contribution varies. Small pitch relative to R is a useful first check on the axis. Very near a wire, local geometry sets a smaller scale and the continuous-winding approximation can fail despite a large total number of turns.

If turns have different radii or currents, use their individual R_j and I_j . Leads and the axial advance of a helical winding require a full path model. This separation of corrections is valuable: a finite-length error, a discrete-turn error, and a lead-field error are distinct and need not decrease at the same rate when the coil design changes.

9.21 Helmholtz coils: uniformity as a design objective

A calibration region often needs a known field that changes little when a sensor is moved. Two equal coils can be positioned so that the field variations of one partly compensate those of the other. Let each coil have N turns, radius R , and current I , with centers at $z = \pm s/2$. The currents have the same sense when viewed from a common direction along the axis. Both moments then point along $+z$.

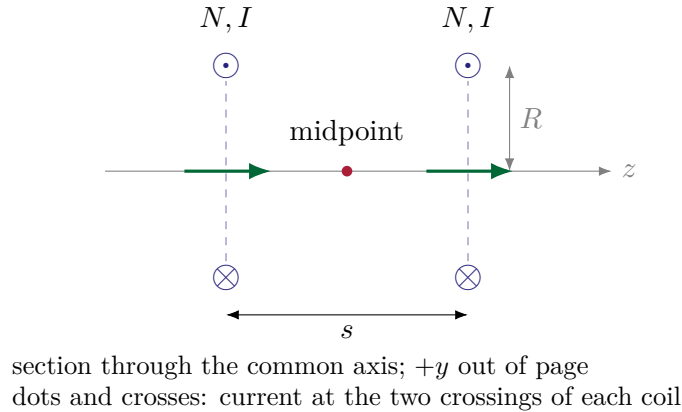


Figure 9.21: Two equal coaxial coils viewed in an axial section. Each dashed line spans a circular coil; it is not a straight wire. The current crossings give magnetic moments and central fields along $+z$ (green arrows). Helmholtz spacing is $s = R$.

Figure 9.21 defines the geometry. Superposition gives the exact on-axis field in the coincident-turn model:

$$B_s(z) = \frac{\mu_0 N I R^2}{2} \left\{ [R^2 + (z - s/2)^2]^{-3/2} + [R^2 + (z + s/2)^2]^{-3/2} \right\}. \quad (9.103)$$

Here N is the number of turns *in each coil*, not the pair total. The field is even in z , so $B'_s(0) = 0$ for every spacing. This first-derivative cancellation follows from reflection symmetry and does not by itself select the Helmholtz spacing.

9.21.1 Why the separation equals the radius

Let $f(u) = (R^2 + u^2)^{-3/2}$. Differentiation gives

$$f''(u) = \frac{3(4u^2 - R^2)}{(R^2 + u^2)^{7/2}}. \quad (9.104)$$

Using $u = s/2$ for the two equal contributions at the midpoint,

$$B_s''(0) = \frac{3\mu_0 N I R^2 (s^2 - R^2)}{(R^2 + s^2/4)^{7/2}}. \quad (9.105)$$

The quadratic variation therefore vanishes when $s = R$. At smaller separation, the center is an axial maximum; at larger separation, it is an axial minimum. The Helmholtz choice lies between these two regimes.

At $s = R$, the central magnitude and normalized Taylor expansion are

$$B_0 = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 N I}{R}, \quad (9.106)$$

$$\frac{B_R(z)}{B_0} = 1 - \frac{144}{125} \left(\frac{z}{R}\right)^4 + \mathcal{O}\left((z/R)^6\right). \quad (9.107)$$

Odd terms vanish by symmetry, and the quadratic term vanishes by the spacing choice. The first remaining variation is quartic. For a single coil the leading variation is quadratic, $-3z^2/(2R^2)$, so the improvement near the midpoint is substantial.

The exact normalized curves in Figure 9.22 compare spacings at fixed radius. Each curve is divided by its own midpoint field; the plot compares shape, not absolute field strength. Moving coils closer together increases the midpoint field for fixed NI but worsens its curvature. An optimal geometry depends on whether strength or uniformity is the objective.

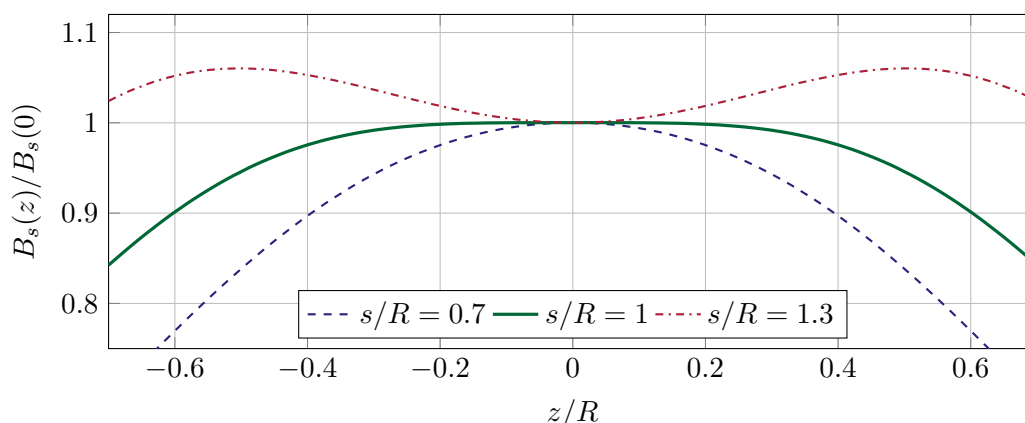


Figure 9.22: Calculated axial profiles of equal coil pairs. Only the Helmholtz spacing eliminates the central quadratic variation. All curves use Eq. (9.103) and their respective midpoint normalization.

Example 9.25: A specified uniform interval

A Helmholtz pair has $R = 0.10$ m, $N = 100$ turns per coil, and $I = 0.50$ A. Determine its center field and estimate the axial interval over which the field differs from its center value by less than 1%.

Magnitude. Equation (9.106) gives $B_0 = 0.4496$ mT. The quartic estimate requires $(144/125)(z/R)^4 \leq 0.01$, or $|z|/R \lesssim 0.305$. Solving the exact normalized expression gives $|z|/R \leq 0.31375$ for the connected interval around the origin. The exact axial half-length is therefore 31.4 mm.

Checks. At $z = 0.2R$ the exact fractional reduction is 0.176%; at $0.3R$ it is 0.844%. The quartic approximation gives 0.184% and 0.933%, respectively. Its accuracy improves closer to the center.

Scope. This is an axial specification. It does not establish a uniform spherical or cylindrical volume, nor does it account for finite winding thickness, misalignment, unequal currents, or magnetic surroundings.

9.21.2 From the axis to a small three-dimensional region

An axial curve alone cannot describe a measurement volume. In a current-free, axisymmetric region with no azimuthal field, let $b(z) = B_z(0, z)$. Near the axis, write

$$B_z(r, z) = b(z) + a_2(z)r^2 + \cdots, \quad B_r(r, z) = c_1(z)r + c_3(z)r^3 + \cdots. \quad (9.108)$$

The radial field must vanish at the axis; regularity makes B_z even and B_r odd in a signed radial coordinate. The vacuum equations give

$$\frac{1}{r} \frac{\partial(rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0, \quad \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = 0. \quad (9.109)$$

Matching the lowest powers yields $c_1 = -b'/2$, $a_2 = -b''/4$, and $c_3 = b'''/16$. Continuing once more gives

$$B_z(r, z) = b(z) - \frac{r^2}{4}b''(z) + \frac{r^4}{64}b^{(4)}(z) + \cdots, \quad (9.110)$$

$$B_r(r, z) = -\frac{r}{2}b'(z) + \frac{r^3}{16}b'''(z) + \cdots. \quad (9.111)$$

For a Helmholtz pair, both $b'(0)$ and $b''(0)$ vanish. The leading three-dimensional field departure is of fourth order in distance from the midpoint. Using $c = 144/125$, the local expansion becomes

$$\frac{B_z}{B_0} = 1 - \frac{c}{R^4} \left(z^4 - 3r^2z^2 + \frac{3}{8}r^4 \right) + \cdots, \quad (9.112)$$

$$\frac{B_r}{B_0} = \frac{c}{R^4} \left(2rz^3 - \frac{3}{2}r^3z \right) + \cdots. \quad (9.113)$$

A specified vector-field error, $|\mathbf{B} - B_0\hat{\mathbf{z}}|/B_0$, includes direction as well as magnitude. Checking only $||\mathbf{B}| - B_0|/B_0$ can miss a tilt. Both measures are useful, but the engineering specification must say which is required.

9.21.3 Opposite currents produce a field gradient

If the coil at $+s/2$ has moment along $+z$ and the coil at $-s/2$ has moment along $-z$, the on-axis field is the *difference* of the two terms in Eq. (9.103). It is odd in z , vanishes at the midpoint, and has slope

$$G = \left. \frac{dB_z}{dz} \right|_0 = \frac{3\mu_0 N I R^2 s}{2(R^2 + s^2/4)^{5/2}}. \quad (9.114)$$

Near the midpoint, Eq. (9.111) gives $B_z \simeq Gz$ and $B_r \simeq -Gr/2$. A purely axial field $Gz\hat{\mathbf{z}}$ would have nonzero divergence and cannot be the complete vacuum field. The accompanying radial component is required by Gauss's law for magnetism. This is a useful example of local field equations revealing information that an axis-only plot hides.

9.22 Numerical Biot–Savart modelling and reproducibility

Numerical integration is most valuable when a source lacks the symmetry needed for a short analytic calculation. A rectangular loop, a coil lead, and a displaced coil pair can all be treated by the same source integral. The main challenge is then to distinguish errors in the assumed geometry from errors in evaluating the integral.

9.22.1 A midpoint approximation to a wire path

Represent the current path by ordered vertices $\mathbf{x}_0, \dots, \mathbf{x}_M$. To describe a closed loop, include the first vertex again at the end. Segment j has directed length $\Delta\ell_j = \mathbf{x}_{j+1} - \mathbf{x}_j$ and midpoint $\mathbf{m}_j = (\mathbf{x}_{j+1} + \mathbf{x}_j)/2$. For an observation point $\mathbf{x}$, put $\mathbf{r}_j = \mathbf{x} - \mathbf{m}_j$. The midpoint approximation is

$$\mathbf{B}_{\text{mid}}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \sum_{j=0}^{M-1} \frac{\Delta\ell_j \times \mathbf{r}_j}{|\mathbf{r}_j|^3}. \quad (9.115)$$

This formula uses the directed chord, not an undirected segment length. If the complete vertex order is reversed, the field reverses. A point on the ideal wire is excluded because the filament model is singular.

There are two approximations when a smooth curve is replaced by chords. First, the physical path has changed into a polygon. Second, each straight-segment integral is replaced by a midpoint value. Both errors generally decrease as the segmentation is refined at observation points a finite distance from the wire. Improving only the quadrature leaves the polygon geometry error.

9.22.2 Integrating each straight segment exactly

Let a segment run from $\mathbf{a}$ to $\mathbf{b}$ with length $d = |\mathbf{b} - \mathbf{a}|$ and unit tangent $\hat{\mathbf{u}} = (\mathbf{b} - \mathbf{a})/d$. Define the projection $t = (\mathbf{x} - \mathbf{a}) \cdot \hat{\mathbf{u}}$ and perpendicular displacement $\boldsymbol{\rho} = \mathbf{x} - \mathbf{a} - t\hat{\mathbf{u}}$. The finite-wire integral gives

$$\mathbf{B}_{\text{seg}} = \frac{\mu_0 I}{4\pi} \frac{\hat{\mathbf{u}} \times \boldsymbol{\rho}}{\rho^2} \left[\frac{t}{\sqrt{\rho^2 + t^2}} - \frac{t-d}{\sqrt{\rho^2 + (t-d)^2}} \right], \quad \rho > 0. \quad (9.116)$$

The derivation is Eq. (9.12) in a coordinate system aligned with that segment. Summing this expression is exact for a polygonal filament in exact arithmetic. At points collinear with a segment but beyond its endpoints, that segment contributes zero. On the segment itself the filament model cannot be used.

The program below implements both methods using only Python's standard library. All coordinates are in metres, current in amperes, and output in tesla. The clearance parameter excludes observation points too close to the ideal wire; it does not regularize the source or model a conductor of that radius. For extreme aspect ratios or points almost collinear with distant segments, the difference in square brackets may lose floating-point precision; a production implementation would use stabilized formulas or adaptive quadrature.

```

"""Chapter 9: pure-Python Biot-Savart benchmarks; SI units throughout."""
from math import pi, sin, cos, sqrt
MU0 = 4*pi*1e-7

def sub(a, b):
    return tuple(x-y for x, y in zip(a, b))

def dot(a, b):
    return sum(x*y for x, y in zip(a, b))

def cross(a, b):
    return (a[1]*b[2]-a[2]*b[1],
            a[2]*b[0]-a[0]*b[2],
            a[0]*b[1]-a[1]*b[0])

def field(path, point, current, method="exact", clearance=1e-6):
    """Open paths give partial circuit contributions. Close loops explicitly.
    Reject points within clearance of any segment; do not omit segments.
    """
    if method not in ("exact", "midpoint"):
        raise ValueError("Unknown integration method")
    total = [0.0, 0.0, 0.0]
    for a, b in zip(path[:-1], path[1:]):
        dl = sub(b, a)
        length = sqrt(dot(dl, dl))
        if length == 0:
            raise ValueError("Zero-length segment")
        u = tuple(v/length for v in dl)
        ap = sub(point, a)
        t = dot(ap, u)
        tc = min(length, max(0.0, t))
        near = tuple(a[k]+tc*u[k] for k in range(3))
        sep = sub(point, near)
        if dot(sep, sep) <= clearance**2:
            raise ValueError("Point too close to filament model")
        if method == "midpoint":
            mid = tuple((a[k]+b[k])/2 for k in range(3))
            r = sub(point, mid)
            vec = cross(dl, r)
            factor = dot(r, r)**(-1.5)
        else:
            rho = tuple(ap[k]-t*u[k] for k in range(3))
            rho2 = dot(rho, rho)
            if rho2 == 0.0: # collinear extension, outside segment
                continue
            q = t-length
            factor = (t/sqrt(rho2+t*t)
                      - q/sqrt(rho2+q*q))/rho2
            vec = cross(u, rho)
        for k in range(3):
            total[k] += factor*vec[k]
    return tuple(MU0*current*v/(4*pi) for v in total)

def circle(radius, segments, z=0.0):
    pts = [(radius*cos(2*pi*j/segments),
            radius*sin(2*pi*j/segments), z)
            for j in range(segments)]
    return pts+[pts[0]]

if __name__ == "__main__":
    radius, current = 0.10, 2.0
    reference = MU0*current/(2*radius)
    print("M      midpoint/circle      exact-polygon/circle")
    for m in (8, 16, 32, 64, 128):

```

```

path = circle(radius, m)
bmid = field(path, (0, 0, 0), current, "midpoint")[2]
bpoly = field(path, (0, 0, 0), current, "exact")[2]
print(f"{m:3d} {bmid/reference:16.9f} {bpoly/reference:20.9f}")
a = 0.10
square = [(-a,-a,0), (a,-a,0), (a,a,0), (-a,a,0), (-a,-a,0)]
bsq = field(square, (0,0,0), current)[2]
analytic = sqrt(2)*MU0*current/(pi*a)
print("Square Bz (T):", bsq, "analytic:", analytic)

```

9.22.3 A convergence table with an analytic explanation

At the center of a circle of radius R , $B_{\text{circle}} = \mu_0 I / (2R)$. An inscribed regular M -gon has apothem $R \cos \alpha$ and side length $2R \sin \alpha$, where $\alpha = \pi/M$. Every segment contributes in the same axial direction. The midpoint method therefore gives

$$\frac{B_{\text{mid}}}{B_{\text{circle}}} = \frac{\sin \alpha}{\alpha \cos^2 \alpha} = 1 + \frac{5}{6} \alpha^2 + \mathcal{O}(\alpha^4). \quad (9.117)$$

Exact integration along the polygon gives instead

$$\frac{B_{\text{poly}}}{B_{\text{circle}}} = \frac{\tan \alpha}{\alpha} = 1 + \frac{1}{3} \alpha^2 + \mathcal{O}(\alpha^4). \quad (9.118)$$

Both overestimate the circular-loop result because the chords pass closer to the center; the midpoint approximation adds a further error. The results below were calculated by the included program and checked against these two analytic ratios.

M	$B_{\text{mid}}/B_{\text{circle}}$	$B_{\text{poly}}/B_{\text{circle}}$	midpoint error (%)
8	1.141692329	1.054786175	14.1692
16	1.032899238	1.013052368	3.28992
32	1.008079374	1.003225197	0.80794
64	1.002010931	1.000803965	0.20109
128	1.000502179	1.000200846	0.05022

Doubling M reduces the leading errors by approximately four, as expected for an M^{-2} convergence law. More digits printed by a computer do not imply more physical accuracy: at $M = 8$, a perfectly repeatable midpoint result is still about 14% away from the circle model.

Example 9.26: A square-loop benchmark with no curve approximation

A square loop has side $2a$ and counterclockwise current I . Find the center field and compare it with the program's result for $a = 0.10$ m and $I = 2.0$ A.

Analytic result. Each side is at perpendicular distance a and has endpoint angles $\pm 45^\circ$. Its field is $\mu_0 I \sqrt{2} / (4\pi a)$ out of the plane. All four contributions add:

$$B_{\text{square}} = \frac{\sqrt{2} \mu_0 I}{\pi a} = 1.13137 \times 10^{-5} \text{ T}. \quad (9.119)$$

The exact-segment program gives the same value to floating-point precision using just four sides. Subdividing those straight sides should not change that result appreciably. In contrast, the midpoint method must subdivide them to approximate the integral accurately.

Purpose. This test separates quadrature error from geometry error. For the circle, refining chords changes both; for the square, the physical geometry is already exact with four vertices and closure.

9.22.4 Validation beyond one scalar value

A center-field test alone can hide a direction error or exploit a special symmetry accidentally. Use a small set of tests with distinct purposes: reverse the path and check the full field vector changes sign; evaluate the circular loop at several positive and negative axial positions; translate the complete geometry and observation point together and check the field is unchanged; and compare the square-loop center result with its analytic value. Off-axis reflection checks test components that vanish at the center.

A further test links the source calculation to Ampere's law. Take a closed square source loop in the xy plane with its right-hand side at $x = a$ carrying current along $+y$. An integration circle centered at $(a, 0, 0)$ in the xz plane, with radius $q < a$, links this side once. Parameterize it as

$$\mathbf{x}_C(\theta) = (a + q \cos \theta, 0, -q \sin \theta), \quad 0 \leq \theta \leq 2\pi. \quad (9.120)$$

The orientation gives normal $+y$. Numerically integrate the *total field of all four sides* along this path. The result should approach $\mu_0 I$. Refining the integration path tests a second numerical operation independent of source discretization. Using just an isolated finite straight segment would not be a valid test of the complete magnetostatic circulation law.

9.22.5 A computational mini-project: useful field volume

For the Helmholtz pair, construct two closed polygonal loops at $z = \pm R/2$ and add their fields, multiplying by the turn count of each coil. Evaluate a meridional grid $(x, 0, z)$ with $0 \leq x \leq 0.5R$ and $|z| \leq 0.5R$. A nonzero signed current on each loop fixes its moment direction.

At every grid point compute both

$$e_{\text{mag}} = \frac{||\mathbf{B}| - B_0|}{B_0}, \quad e_{\text{vec}} = \frac{|\mathbf{B} - B_0 \hat{\mathbf{z}}|}{B_0}. \quad (9.121)$$

Identify the connected region containing the origin in which the chosen error is below 0.01. Repeat with twice as many segments and a finer observation grid. A boundary that moves appreciably under refinement is not yet a reliable design result. Verify the axial intercept against $|z|/R = 0.31375$ from the exact formula, and compare the small-distance map with Eq. (9.111).

Finally, change the coil separation by 5%, or make one current 1% larger, and determine which change most strongly alters the chosen region. Unequal currents break the even axial symmetry and introduce a central gradient. A useful report contains the field specification, geometry, convergence evidence, and a physical explanation of the change. A colored field image alone is not a quantitative result.

Caution

Never repair a near-wire singularity by silently skipping a current segment. This changes the source and can create a plausible but incorrect field. Exclude invalid observation points explicitly, increase resolution where the filament model is valid, or replace the filament by a justified finite cross-sectional current distribution.

9.23 Advanced problems and computational projects

The problems below build directly on the enrichment. State which results are exact within a specified ideal model and which are approximations. Use A11 and A12 as longer computational assignments if desired.

- A1.** A cylindrical conductor has $J_z(r) = J_0(1 - r^2/a^2)$ for $r < a$ and zero current density outside. Express J_0 in terms of total current I , derive $B_\phi(r)$ everywhere, and find the location and magnitude of the maximum field.

- A2.** Two infinite sheets at $z = \pm d/2$ carry opposite surface currents: the upper sheet has $+K\hat{x}$ and the lower sheet $-K\hat{x}$. Find the vector field in all three regions. Verify the boundary jump at each sheet.
- A3.** Derive the finite-solenoid center correction for fixed N , R , and L . Find the aspect ratio L/R needed for the center field to be within 1% below the value $\mu_0(N/L)I$. Explain why this condition does not ensure one-percent uniformity over the whole bore.
- A4.** For the finite solenoid in Example 9.24, determine the field at $z = L/4$. Compare it with the center field and the infinite-solenoid value. Which denominator should be used for a uniformity specification around the center?
- A5.** Differentiate Eq. (9.103) twice and verify Eq. (9.105). At fixed R , expand the normalized central curvature to first order in the spacing error $\epsilon = (s - R)/R$.
- A6.** For an opposite-current coil pair, show that the cubic axial term vanishes at $s = \sqrt{3}R$. Hint: differentiate $f(u) = (R^2 + u^2)^{-3/2}$ three times. Compare the goal of a linear gradient with the goal of a uniform nonzero field.
- A7.** Starting from the source-free divergence and curl equations, derive the first two nontrivial terms in Eq. (9.111). For a Helmholtz pair, find the leading field variation in its midpoint plane $z = 0$ and compare it with the axial quartic variation at the same distance from the origin.
- A8.** A thin insulating disk of radius R carries a prescribed uniform surface charge density σ and rotates rigidly with angular velocity $\omega\hat{z}$. Treat the charge distribution as continuous and nonrelativistic. Derive its center magnetic field and magnetic moment by resolving it into circular rings. Express both in terms of total charge Q . Explain why this source is steady despite motion of the charge.
- A9.** Use the dipole field to compare axial and equatorial magnitudes at equal distance. Derive the field-line equation in a meridional plane from $dr/(rd\theta) = B_r/B_\theta$, and state why it must not be extended through a finite current source.
- A10.** Derive the exact center-field ratio $\tan(\pi/M)/(\pi/M)$ for an inscribed regular polygon. Compare its large- M error with that of midpoint integration. Explain why a parametrized circular-arc quadrature can give an exact center value even with few equally weighted samples, without being accurate at general off-axis points.
- A11.** Implement the closed-square circulation test described in the numerical section with $a = 0.10$ m, $q = 0.025$ m, and $I = 2.0$ A. Use exact straight-segment source fields. Refine only the integration circle, and tabulate its circulation divided by $\mu_0 I$. Reverse the path orientation as a direction test.
- A12.** Complete the Helmholtz useful-volume project using both magnitude and vector error criteria. Report the axial intercept, a transverse intercept, and the effect of a 5% spacing error. Distinguish observation-grid resolution, source-polygon error, and physical winding-model error.

9.24 Selected advanced solutions

A1: a current profile whose field maximum is inside

Integrating the current density over the cross section gives

$$I = 2\pi J_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r \, dr = \frac{\pi J_0 a^2}{2}, \quad J_0 = \frac{2I}{\pi a^2}.$$

For $r < a$, $I_{\text{enc}} = \pi J_0 [r^2 - r^4/(2a^2)]$. Ampere's law then gives

$$B_\phi(r) = \frac{\mu_0 I}{\pi a^2} \left(r - \frac{r^3}{2a^2}\right), \quad r < a; \quad B_\phi(r) = \frac{\mu_0 I}{2\pi r}, \quad r \geq a.$$

The two branches agree at a . Differentiating the internal branch gives a maximum at $r = a\sqrt{2/3} = 0.8165a$. Relative to $B(a) = \mu_0 I/(2\pi a)$,

$$\frac{B_{\text{max}}}{B(a)} = \frac{4}{3} \sqrt{\frac{2}{3}} = 1.0887.$$

The maximum need not lie at the surface when the density is nonuniform. Here J_z tends to zero at the surface, and the two field derivatives also match there. That behavior differs from the sharp uniform-current boundary of the core example.

A2: confinement between two current sheets

For the upper sheet carrying $+K\hat{x}$, the field is $-\mu_0 K\hat{y}/2$ above it and $+\mu_0 K\hat{y}/2$ below it. The lower sheet has reversed current, so its signs reverse. Above both sheets the fields cancel; between them they add along $+y$; below both they cancel:

$$\mathbf{B} = \begin{cases} 0, & z > d/2, \\ \mu_0 K\hat{y}, & -d/2 < z < d/2, \\ 0, & z < -d/2. \end{cases}$$

At the upper sheet, with upward normal, the jump is $-\mu_0 K\hat{y}$; crossing with $\hat{z}$ gives $+\mu_0 K\hat{x}$. At the lower sheet the jump is $+\mu_0 K\hat{y}$, giving $-\mu_0 K\hat{x}$. Both agree with the prescribed sheet currents. Perfect cancellation here depends on infinite planar symmetry.

A3: a center-field criterion

Set $\alpha = L/(2R)$. The center-to-ideal ratio is $f = \alpha/\sqrt{1+\alpha^2}$. Requiring $f \geq 0.99$ yields $\alpha \geq 7.018$, or $L/R \geq 14.04$. This is a center-field condition, not a uniformity condition. Even when this bound holds, the field near an end remains approximately half the infinite-solenoid field. The useful region must be defined by a position-dependent comparison such as $|B(z) - B(0)|/B(0)$.

A5: sensitivity to coil spacing

Divide Eq. (9.105) by $B_s(0) = \mu_0 N I R^2 / (R^2 + s^2/4)^{3/2}$. In terms of $q = s/R$,

$$\frac{R^2 B_s''(0)}{B_s(0)} = \frac{3(q^2 - 1)}{(1 + q^2/4)^2}.$$

For $q = 1 + \epsilon$, this is $(96/25)\epsilon + \mathcal{O}(\epsilon^2)$. The Taylor series for the normalized field consequently acquires a quadratic term $(48/25)\epsilon(z/R)^2$. A positive spacing error makes the midpoint an axial minimum; a negative one makes it a maximum. The curvature cancellation is geometrical and is sensitive to the separation-to-radius ratio, whereas changing both equal currents simply rescales the field.

A6: a nearly linear gradient

For $f(u) = (R^2 + u^2)^{-3/2}$,

$$f'''(u) = \frac{15u(3R^2 - 4u^2)}{(R^2 + u^2)^{9/2}}.$$

The opposite-current pair is odd about the origin, so its first potentially nonlinear term is cubic. Its cubic coefficient is proportional to $f'''(s/2)$ and therefore vanishes for $s = \sqrt{3}R$ (excluding the zero-separation cancellation of the entire source). The leading nonlinearity is then of fifth order. This spacing optimizes local gradient linearity, while the same-sense spacing $s = R$ cancels quadratic curvature in a nonzero central field. Different field objectives select different arrangements.

A8: converting rotating charge into steady current

A ring of radius r and width dr contains charge $dq = \sigma 2\pi r dr$. Its rotation period is $2\pi/\omega$, so it carries current

$$dI = \frac{\omega dq}{2\pi} = \sigma \omega r dr.$$

The ring's center contribution is $dB = \mu_0 dI/(2r) = \mu_0 \sigma \omega dr/2$. Integrating from 0 to R gives

$$B(0) = \frac{\mu_0 \sigma \omega R}{2} = \frac{\mu_0 Q \omega}{2\pi R}.$$

For $Q\omega > 0$, the field points along $+z$. The vanishingly small rings do not produce a divergent integral because their currents also vanish in proportion to radius.

Each ring has moment $d\mu = \pi r^2 dI = \pi \sigma \omega r^3 dr$, hence

$$\mu = \frac{\pi \sigma \omega R^4}{4} = \frac{Q \omega R^2}{4}.$$

Outer rings are weighted much more strongly in the moment than in the center field. The continuous axisymmetric charge and current densities are independent of time even though individual charges move. The model assumes a prescribed uniform distribution, rigid rotation, and speeds $\omega R \ll c$; it is not a model of an electron's intrinsic spin.

A9: dipole field lines away from the source

Using $B_r/B_\theta = 2 \cot \theta$ gives $dr/r = 2 \cot \theta d\theta$, hence $r = C \sin^2 \theta$ for each non-axial field line. The axial magnitude is twice the equatorial magnitude at equal distance. These curves describe the ideal dipole field outside its source region. A finite current loop must replace the singular central idealization; the far-field approximation cannot determine the field inside the loop from this line equation alone.

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