

Electromagnetic Induction and Inductance

Physics II - Chapter 10

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How does changing magnetism drive a circuit?

A steady current creates a magnetic field. A changing linked flux creates an emf, allowing motion or another circuit to drive electrical response.

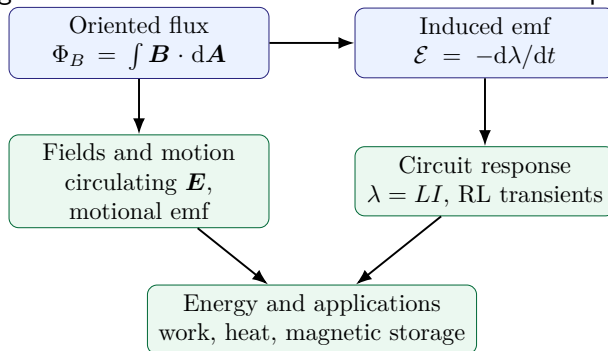


Figure 10.1: The chapter connects flux geometry to field and circuit descriptions, then tests both through energy conservation.

We will predict the response and check the energy supplied.

Learning route and prerequisites



A route from geometry to prediction

First choose an oriented surface and calculate its flux. Faraday's law then fixes the emf. The circuit model determines current, and an energy budget checks the result.

Tools we will refresh

Use the dot and cross products, surface and line integrals, differentiation, work and power, Ohm's law, and a first-order differential equation. Earlier magnetic-force laws supply the mechanical reaction.

The core develops flux, motion, induced fields, inductance, RL response, and applications. The advanced block extends moving-boundary reasoning, field geometry, coupled energy, and nonlinear linkage.

Learning goals



- Calculate signed flux and emf, and justify a current direction using the stated circuit assumptions.
- Derive motional emf and check mechanical input against heat and magnetic storage.
- Obtain an induced electric field from circulation and symmetry, including the exterior region.
- Calculate inductance, solve RL growth and switching, and verify the finite-time energy budget.
- Use flux overlap and pickup integration in coil calculations, and diagnose a numerical transient.

Advanced outcomes include integrating a nonuniform field, bounding mutual coupling, and distinguishing differential inductance from stored energy.

Prerequisite refresh: forces, power, and units



$$\mathbf{F}_B = q\mathbf{v}_q \times \mathbf{B},$$
$$P_{\text{mech}} = \mathbf{F} \cdot \mathbf{v},$$

$$\mathbf{F}_{\text{wire}} = I\boldsymbol{\ell} \times \mathbf{B},$$
$$P_R = I^2 R.$$

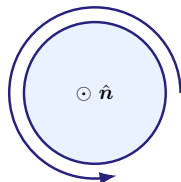
The vector $\boldsymbol{\ell}$ points along the chosen positive current. Mechanical power depends on the force component along the motion.

Unit checks

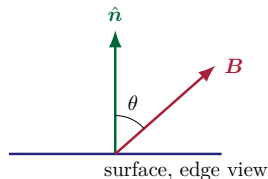
Flux: $1 \text{ Wb} = 1 \text{ T m}^2$. Emf: $1 \text{ V} = 1 \text{ Wb/s} = 1 \text{ J/C}$. Inductance: $1 \text{ H} = 1 \text{ Vs/A}$. Emf therefore has units of voltage.

A long air-core solenoid has $B \simeq \mu_0 n I$, with turn density $n = N/\ell_s$. This approximation neglects its end field.

Magnetic flux needs an orientation



View along $-\hat{n}$
positive traversal is counterclockwise



$$\Phi_B = BA \cos \theta$$

θ is measured from the normal

Figure 10.2: One orientation, shown in two views. The loop arrow fixes the normal; the angle in the dot product is between the field and that normal, not the plane.

$$\Phi_B = \int_S \mathbf{B} \cdot \hat{n} dA, \quad \Phi_B = BA \cos \theta \quad (\text{uniform, flat case}).$$

Reversing the reference normal reverses the signed flux. It does not alter the physical field.

Flux linkage and the changing quantity



For multiple turns, add their signed fluxes:

$$\lambda = \sum_{j=1}^N \Phi_{B,j} = N\Phi_B \quad \text{only when each turn links the same flux.}$$

For a uniform field and flat surface, differentiation separates the mechanisms:

$$\dot{\Phi}_B = A \cos \theta \dot{B} + B \cos \theta \dot{A} - BA \sin \theta \dot{\theta}.$$

Field strength, overlap area, and orientation can change together and their contributions can cancel. For a nonuniform field, integrate over the area first.

A useful surface check

Two consistently oriented spanning surfaces with the same boundary give the same flux because $\nabla \cdot \mathbf{B} = 0$. An open-surface flux can still be nonzero.

Example 10.1: nonuniform flux



Example 10.1. Problem. A circular loop of radius a lies in a field $B(r) = B_0(1 - r^2/a^2)$ perpendicular to the loop for $0 \leq r \leq a$. Find the flux through the loop.

Choose the normal along the stated field. Circular symmetry lets us sum annular strips of area $dA = 2\pi r dr$:

$$\begin{aligned}\Phi_B &= 2\pi B_0 \int_0^a \left(r - \frac{r^3}{a^2} \right) dr \\ &= 2\pi B_0 \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right]_0^a = \boxed{\frac{\pi B_0 a^2}{2}}.\end{aligned}$$

The area-average field is $B_0/2$, not its central value B_0 . The result has units of weber and tends to zero with the area. Choosing the opposite normal reverses its sign.

Faraday and Lenz: an oriented emf



On a fixed closed path, the induced electric field has circulation

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\dot{\Phi}_B, \quad \mathcal{E}_{\text{coil}} = -\dot{\lambda}.$$

An increasing positive flux gives negative emf in the selected traversal. A decreasing positive flux gives positive emf. The response opposes the change of linked flux.

Emf and current answer different questions

For a passive resistance-dominated loop with no other source, $I = \mathcal{E}/R$. With appreciable inductance or initial current, the circuit equation and its initial condition determine the current.

A constant nonzero flux gives zero emf. Zero instantaneous flux can have a nonzero slope and hence a nonzero emf.

Four Lenz-law direction cases

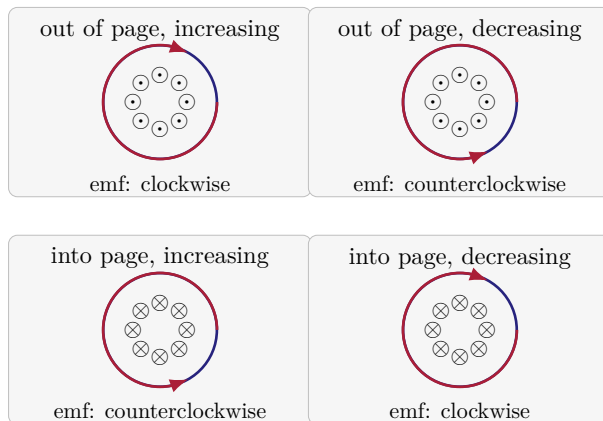


Figure 10.3: Four common Lenz-law cases for a circular loop. The arrows show the emf direction and the current in a resistance-dominated loop initially without current. The external field magnitude changes as stated; the induced response opposes that change.

Example 10.2: changing flux through a coil



Example 10.2. A rectangular coil with $N = 120$ turns has area $A = 3.0 \times 10^{-3} \text{ m}^2$. Its area normal is chosen to be in the positive z direction. A uniform magnetic field is also in the z direction and changes from 0.10 T to 0.46 T in 0.080 s . Find the average induced emf using the chosen positive orientation.

Model and sign before arithmetic

All turns link the same uniform field, so $\lambda = NBA$. The $+z$ normal fixes counterclockwise traversal viewed from $+z$. Since $\Delta B > 0$, $\Delta\lambda > 0$ and the average emf is negative.

$$\overline{\mathcal{E}} = -\frac{\Delta\lambda}{\Delta t} = -NA\frac{B_f - B_i}{\Delta t}.$$

This determines an average. A waveform or a specified time dependence is needed for an instantaneous peak.

Example 10.2: linkage, emf, and checks



$$\begin{aligned}\Delta\lambda &= (120)(3.0 \times 10^{-3})(0.46 - 0.10) \\ &= 0.1296 \text{ Wb}, \\ \bar{\mathcal{E}} &= -\frac{0.1296}{0.080} \text{ Wb/s} = \boxed{-1.62 \text{ V}}.\end{aligned}$$

The emf is clockwise viewed from $+z$, opposite the selected positive traversal. Its magnitude is about 1.6 V at two significant figures.

Independent checks

One turn gives -13.5 mV and 120 turns give -1.62 V . The weber-per-second unit is volt. Reversing the reference normal reverses the numerical sign, while preserving the physical circulation.

Practice A1: an average does not fix a peak



Practice problem A1. A 75-turn coil of area $4.0 \times 10^{-3} \text{ m}^2$ has its positive normal parallel to a field decreasing from 0.20 to 0.05 T in 0.030 s. Find the signed average emf. Does this information determine the peak emf?

Attempt. Fix the sign of $\Delta\lambda$ before calculating.

$$\Delta\lambda = (75)(4.0 \times 10^{-3})(0.05 - 0.20) = -0.045 \text{ Wb},$$
$$\bar{\mathcal{E}} = -\frac{-0.045}{0.030} = \boxed{+1.5 \text{ V}}.$$

Solution and interpretation. The positive emf has the right-hand traversal associated with the field-aligned normal. A linear field fall gives constant +1.5 V. The same change concentrated in less time gives a larger peak, so the endpoints do not determine it. Units: $\text{Wb/s} = \text{V}$.

External drive, self flux, and current memory



Separate the externally imposed linkage from the current's own linkage. For a fixed linear loop with resistance R and constant inductance L ,

$$RI = -\frac{d}{dt}(\lambda_{\text{ext}} + LI) \implies L\dot{I} + RI = \mathcal{E}_{\text{ext}},$$

$$\mathcal{E}_{\text{ext}} = -\dot{\lambda}_{\text{ext}}.$$

An open pickup winding can show a signal while carrying almost no current. A closed RL loop retains its earlier current even when the external emf changes sign.

Counting the physics once

Including LI in total linkage and then adding a second self-induced emf counts self-induction twice. Either keep total flux in Faraday's equation or move the self term to the circuit side as above.

Fixed paths and moving conductors



A moving conductor has a magnetic driving term as well as an electric one:

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B[C(t)]}{dt}.$$

Here $\mathbf{u}$ is the local conductor velocity. The derivative follows the moving surface. A static field can therefore induce emf when the linked area changes.

One complete calculation

Evaluate the Lorentz-force integral on the actual conductors, or differentiate flux through the correctly moving boundary. For a sliding rod these give the same result and must not be added as separate effects.

Stationary rails have $\mathbf{u} = 0$. At sliding contacts, use each conductor segment's actual velocity; a fixed drawn contour is not automatically a material loop.

An isolated moving rod: voltage without current



A vertical rod moves right: $\mathbf{u} = v\hat{\mathbf{x}}$, $\mathbf{B} = -B\hat{\mathbf{z}}$. Bottom and top endpoints are a and b . Then

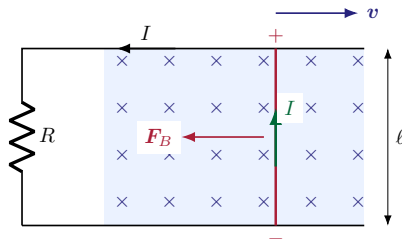
$$\mathbf{u} \times \mathbf{B} = vB\hat{\mathbf{y}}.$$

Positive charge is driven upward and electrons initially move downward. Surface charge builds an electric field until the open-circuit tangential force vanishes:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}, \quad V_b - V_a = -\int_a^b \mathbf{E} \cdot d\ell = Blv.$$

The top is positive. The motional contribution is Blv , but the total tangential Lorentz force is zero in equilibrium. Thus the open rod has no sustained longitudinal current.

A closed sliding-rod circuit



$$\mathbf{B} = -B\hat{z}, \quad \hat{n} = +\hat{z}$$

Figure 10.4: Closed sliding-rod circuit. Motion to the right increases the into-page external flux magnitude. The emf is counterclockwise, the rod current is upward, and the magnetic force is leftward. End signs indicate the rod's motional source polarity.

$$\Phi_{B,\text{ext}} = -Blx, \quad \mathcal{E}_{\text{ext}} = +Blv.$$

For negligible inductance, $I = Blv/R$ is counterclockwise.

Example 10.4: sliding rod on rails



Example 10.4. Neglect self-inductance and rail resistance except for the stated total R . A rod of length $\ell = 0.35 \text{ m}$ slides to the right at $v = 2.4 \text{ m/s}$ on conducting rails in a uniform magnetic field of magnitude $B = 0.80 \text{ T}$ into the page. The total circuit resistance is $R = 0.60 \Omega$. Find the induced emf magnitude, the induced current magnitude and the magnetic force on the rod.

Geometry and model

Use counterclockwise traversal and out-of-page normal as in Figure 10.4 on slide 18. The into-page flux is $-B\ell x$. Its magnitude increases, so the emf is positive and the current is upward in the moving rod.

$$\mathcal{E}_{\text{ext}} = B\ell v, \quad I = \frac{B\ell v}{R}, \quad \mathbf{F}_B = I\ell \times \mathbf{B}.$$

Example 10.4: force and energy transfer



$$|\mathcal{E}| = (0.80)(0.35)(2.4) = \boxed{0.672 \text{ V}},$$

$$I = 0.672/0.60 = \boxed{1.12 \text{ A}},$$

$$F_B = (1.12)(0.35)(0.80) = 0.3136 \text{ N} \simeq \boxed{0.314 \text{ N leftward}}.$$

At constant speed, the external force is rightward with the same magnitude. Keeping guard digits,

$$P_{\text{mech}} = 0.3136(2.4) = 0.75264 \text{ W} = I^2 R = (1.12)^2(0.60).$$

Both powers round to 0.753 W. The mechanical drive supplies resistor heat. This check independently confirms the drag direction and avoids discrepancies from rounding the force too early.

A loop crossing a finite field region

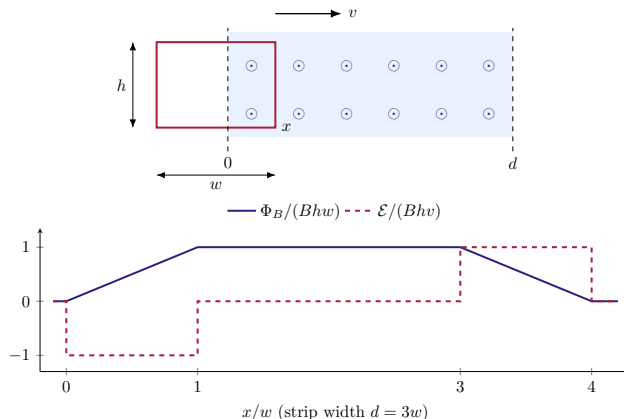


Figure 10.5: Geometry and calculated piecewise flux/emf for a loop crossing a sharp field boundary. The jumps are idealisations; fringing fields smooth the voltage transitions, and nonzero self-inductance smooths the current response.

Example 10.5: entry and exit waveform



Example 10.5. A square loop of side 0.10 m and resistance $0.20\ \Omega$ crosses a 0.30 m strip containing $B = 0.80\text{ T}$ out of the page. It moves right at 0.50 m/s . Neglect self-inductance. Determine the waveform and total heat generated in one passage.

Use $+z$ normal and counterclockwise traversal. As in Figure 10.5, x is the leading-edge coordinate and the first boundary is reached at $t = 0$. Here $w = h = 0.10\text{ m}$ and $d = 0.30\text{ m}$.

$$\mathcal{E}(t) = \begin{cases} -0.040\text{ V}, & 0 < t < 0.20\text{ s}, \\ 0, & 0.20 < t < 0.60\text{ s}, \\ +0.040\text{ V}, & 0.60 < t < 0.80\text{ s}, \end{cases}$$

It is zero before entry and after exit. The pulse magnitude is Bhv and each pulse lasts w/v . The current is clockwise on entry and counterclockwise on exit.

Example 10.5: two pulses, positive heat



The current magnitude during each pulse is $|\mathcal{E}|/R = 0.040/0.20 = 0.20$ A. It vanishes in the interior interval in this zero-inductance model.

$$P_R = (0.20)^2(0.20) = 0.0080 \text{ W},$$

$$Q = P_R(2w/v) = (0.0080)(0.40) = \boxed{3.2 \times 10^{-3} \text{ J}}.$$

During both crossings, $F_B = IBh = 0.016$ N opposes motion. It acts over total distance $2w = 0.20$ m, so the drive supplies $F_B(2w) = 0.0032$ J.

Signed area and energy are different

The two signed voltage areas cancel because the initial and final flux are equal. Heat depends on $\int \mathcal{E}^2/R dt$ and stays positive. Fringing smooths voltage jumps and inductance smooths current.

Magnetic damping in the resistive approximation



For a freely sliding bar, motional current produces linear drag:

$$I = \frac{B\ell v}{R}, \quad F_B = -bv, \quad b = \frac{B^2\ell^2}{R} > 0.$$

Newton's equation separates into

$$m\dot{v} = -bv, \quad \frac{dv}{v} = -\frac{b}{m}dt, \quad v(t) = v_0 e^{-t/\tau_m}, \quad \tau_m = \frac{mR}{B^2\ell^2}.$$

When current can follow speed immediately

The circuit's relaxation time must be short: $L/R \ll \tau_m$. If this fails, current and motion are coupled dynamical variables rather than an instantaneous drag law.

The energy check is $\dot{K} = mv\dot{v} = -bv^2 = -I^2R$.

Example 10.6: damping a released bar



Example 10.6. A conducting bar of mass $m = 0.18 \text{ kg}$ slides without mechanical friction on horizontal rails separated by $\ell = 0.25 \text{ m}$. The circuit resistance is $R = 0.50 \Omega$, and a uniform magnetic field $B = 0.70 \text{ T}$ points perpendicular to the rails. The bar is given an initial speed $v_0 = 3.0 \text{ m/s}$ and then released. Find $v(t)$ in the resistance-dominated approximation $L/R \ll mR/(B^2\ell^2)$.

The drag equation on slide 24 gives

$$\tau_m = \frac{(0.18)(0.50)}{(0.70)^2(0.25)^2} = 2.9388 \text{ s},$$

$$v(t) = 3.0 \exp[-t/(2.94 \text{ s})] \text{ m/s}.$$

This returns v_0 at release and tends to zero without reversing. Substitution gives $m\dot{v} = -B^2\ell^2 v/R$. Thus the rod slows exponentially on the mechanical time scale, while the stated inequality ensures that current can follow it.

How does the energy reach the resistor?



For an individual charge at laboratory velocity $\mathbf{v}_q$,

$$q(\mathbf{v}_q \times \mathbf{B}) \cdot \mathbf{v}_q = 0.$$

Magnetic force does no net work on that charge. In a rod, laboratory motion includes both lattice motion and charge drift. Magnetic forces redirect charge motion and transmit a mechanical reaction to the lattice. The external agent supplies work, while electric fields and collisions transfer energy into the circuit and heat. At prescribed constant speed with constant L , the current equation gives

$$F_{\text{ext}} v = I\mathcal{E} = I^2 R + \frac{d}{dt} \left(\frac{1}{2} L I^2 \right).$$

Stored field energy can return to motion; determine the force from the actual current.

Induced electric fields exist without a wire



A fixed mathematical path in empty space can enclose changing flux:

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A}.$$

Electrostatic fields have zero closed-path circulation. A changing magnetic field can produce nonzero circulation, so its induced electric field cannot generally be represented globally by a single-valued electrostatic potential.

What the integral law does and does not determine

Faraday's law fixes circulation. To infer a field at each point, also use symmetry and boundary information. A conservative field from additional charges can be added without changing the circulation.

A wire supplies mobile charges that respond to the field. It does not create the need for the field law.

Solenoid symmetry: induced field inside and outside



Take an ideal long solenoid with $\mathbf{B} = B(t)\hat{\mathbf{z}}$ inside radius R , negligible outside. Exclude additional electrostatic fields and use a quasistatic approximation. Define $E_\phi > 0$ counterclockwise viewed from $+\mathbf{z}$.

$$2\pi r E_\phi = -\pi r^2 \dot{B} \quad (r < R),$$

$$2\pi r E_\phi = -\pi R^2 \dot{B} \quad (r > R).$$

$$E_\phi(r) = \begin{cases} -\frac{r}{2}\dot{B}, & r \leq R, \\ -\frac{R^2}{2r}\dot{B}, & r \geq R. \end{cases}$$

Both branches give $-R\dot{B}/2$ at the boundary. The outer path encloses changing flux even though its local magnetic field is negligible. Units are $\text{m T/s} = \text{V/m}$.

The induced-field geometry and radial dependence

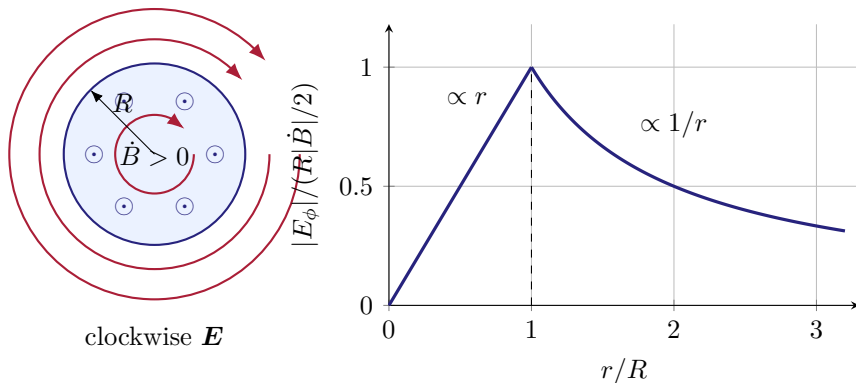


Figure 10.6: Induced field of an ideal long solenoid. Left: circulation for increasing out-of-page B . Right: the calculated magnitude, linear inside and inversely proportional to radius outside, continuous at $r = R$.

This ideal local approximation assumes negligible end effects and propagation delay. It is not an instantaneous far-field response of a finite source.

Example 10.9: two radii, equal field magnitudes



Example 10.9. A long solenoid produces a uniform magnetic field inside radius $R = 4.0\text{ cm}$ and negligible field outside. The field increases at rate $dB/dt = 0.50\text{ T/s}$. Estimate the induced electric-field magnitude at $r = 2.0\text{ cm}$ and at $r = 8.0\text{ cm}$.

Convert all lengths to metres and use the two branches on slide 28:

$$E(0.020) = \frac{0.020}{2}(0.50) = \boxed{5.0 \times 10^{-3}\text{ V/m}},$$

$$E(0.080) = \frac{(0.040)^2}{2(0.080)}(0.50) = \boxed{5.0 \times 10^{-3}\text{ V/m}}.$$

The equality follows from these selected radii, not a uniform induced field. Under the increasing $+z$ convention of slide 28, both signed components are negative and the field circulates clockwise. A boundary check gives $|E(R)| = 0.010\text{ V/m}$ from either expression, greater than both values.

Voltage leads also enclose flux



For two paths P, Q from a to b , define $\mathcal{V}_P = - \int_P \mathbf{E} \cdot d\ell$. The comparison loop follows P and returns along Q :

$$\mathcal{V}_P - \mathcal{V}_Q = - \oint_C \mathbf{E} \cdot d\ell = \dot{\Phi}_B[C].$$

Example 10.10. Two routes for a pair of stationary voltage leads enclose a difference in effective area of 12 cm^2 perpendicular to a uniform field ramp of 30 T/s . Estimate the difference caused by induction.

$$|\Delta \mathcal{V}| = A |\dot{B}| = (12 \times 10^{-4})(30) = \boxed{0.036 \text{ V} = 36 \text{ mV}}.$$

The supplied data determine a magnitude. Its sign depends on routing orientation. Reducing the enclosed area reduces pickup; tightly paired or twisted leads cancel slowly varying flux. Greater meter resolution alone does not remove this signal.

Inductance packages self flux into a circuit quantity



In a fixed linear magnetic system, its own current produces proportional flux linkage:

$$\lambda_{\text{self}} = LI, \quad L = \frac{\lambda_{\text{self}}}{I}, \quad [L] = \text{Wb/A} = \text{H}.$$

Geometry and magnetic response determine L . An independent external field contributes separate linkage and must not enter this ratio.

Geometry includes the return path

A pair of conductors encloses field energy even without a wound coil. Moving the return wire changes the circuit inductance. The phrase “inductance of a wire” needs a specified return geometry.

At $I = 0$, use the linear coefficient or limiting derivative rather than $0/0$. Nonlinear or hysteretic cores require a richer model, introduced in the advanced block.

Long-solenoid inductance: field, flux, linkage

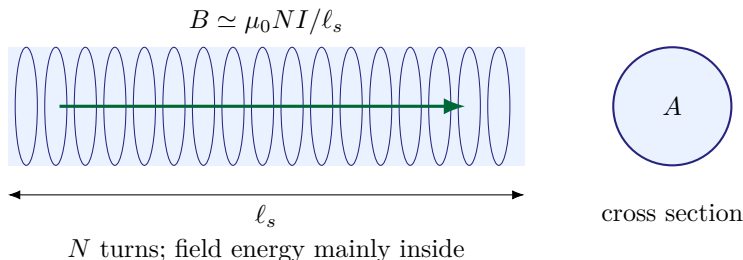


Figure 10.7: Long-solenoid model. The cross-sectional area is shown as an area, not a length. End and external fields are neglected in $L \simeq \mu_0 N^2 A / \ell_s$.

$$B \simeq \mu_0 \frac{N}{\ell_s} I, \quad \Phi_B = BA, \quad L = \frac{N\Phi_B}{I} \simeq \boxed{\frac{\mu_0 N^2 A}{\ell_s}}.$$

At fixed N , A , a longer solenoid has smaller L . At fixed turn density n , adding length adds turns and gives $L = \mu_0 n^2 A \ell_s$.

Example 10.11: solenoid inductance and linkage



Example 10.11. An air-core solenoid has $N = 800$ turns, length $\ell_s = 0.40$ m and cross-sectional area $A = 5.0 \times 10^{-4} \text{ m}^2$. Estimate its inductance. If it carries $I = 1.5$ A, estimate the flux through one turn.

Model and calculation route

Use the long, tightly wound air-core approximation, with end effects neglected. Calculate L from geometry, then find B , the one-turn flux, and total linkage at the specified current.

$$n = \frac{N}{\ell_s} = 2.0 \times 10^3 \text{ m}^{-1}, \quad L \simeq \mu_0 n^2 A \ell_s.$$

Numerical estimates use $\mu_0 \simeq 4\pi \times 10^{-7} \text{ H/m}$, adequate at the stated precision. The flux through one turn and the sum through all 800 turns are different quantities.

Example 10.11: numerical results and consistency



$$\begin{aligned} L &= (4\pi \times 10^{-7})(2000)^2(5.0 \times 10^{-4})(0.40) \\ &= 1.0053 \times 10^{-3} \text{ H} \simeq \boxed{1.0 \text{ mH}}, \end{aligned}$$

$$B = \mu_0 n I = 3.7699 \times 10^{-3} \text{ T},$$

$$\Phi_B = BA = \boxed{1.8850 \times 10^{-6} \text{ Wb}},$$

$$N\Phi_B = 1.5080 \times 10^{-3} \text{ Wb}.$$

Dividing the linkage by 1.5 A recovers 1.0053 mH. The inductance is independent of current in this linear model, while field and flux are proportional to it. At fixed length and area, doubling turns gives $4L$. Doubling length at fixed turn density gives $2L$. The constraints determine the scaling.

Self-induced emf and passive terminal voltage

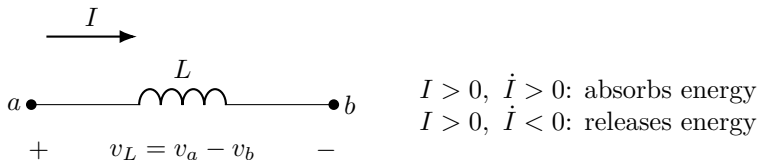


Figure 10.8: Reference polarity and current for an ideal inductor. The marked + is a voltage reference, not a claim that terminal a is always at higher potential.

$$\mathcal{E}_L = -\dot{\lambda} = -L\dot{I}, \quad v_L = v_a - v_b = +L\dot{I}.$$

The terminal drop balances the internal emf along the chosen current path. A real winding has $v_{ab} = R_w I + L\dot{I}$ in the lumped model. The absorbed power is $p_L = v_L I = LI\dot{I}$. Its sign depends on both factors: $p_L > 0$ when I^2 increases and $p_L < 0$ when field energy is released.

Example 10.13: a prescribed current ramp



Example 10.13. A 25 mH inductor is driven so that its current increases linearly from 0 to 4.0 A in 2.0 ms. What is the magnitude of the self-induced emf?

Take the increasing-current direction as positive. For an ideal constant- L inductor,

$$\begin{aligned}i &= \frac{4.0}{2.0 \times 10^{-3}} = 2.0 \times 10^3 \text{ A/s}, \\ \mathcal{E}_L &= -(25 \times 10^{-3})(2.0 \times 10^3) = -50 \text{ V}, \\ |\mathcal{E}_L| &= \boxed{50 \text{ V}}, \quad v_L = +50 \text{ V}.\end{aligned}$$

The source must balance this ideal-inductor drop. The integral $v_L \Delta t = 0.10 \text{ V s}$ equals $L\Delta I$, a separate volt-second check. Winding resistance would add a time-dependent $R_w I$ drop.

Finite voltage implies continuous current



Integrating $v_L = L\dot{I}$ gives

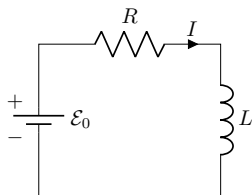
$$I(t_2) - I(t_1) = \frac{1}{L} \int_{t_1}^{t_2} v_L(t) dt.$$

Bounded voltage over a vanishing time interval gives vanishing current change. Current remains continuous across an ordinary switch, while voltage and current slope can jump. Across $100\ \mu\text{H}$, a $5.0\ \text{V}$ pulse lasting $4.0\ \mu\text{s}$ gives

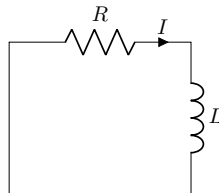
$$\Delta I = \frac{(5.0)(4.0 \times 10^{-6})}{100 \times 10^{-6}} = 0.20\ \text{A}.$$

Returning to the same current each cycle requires zero net inductor voltage-time area. A finite jump requires a voltage impulse; real interruption involves a discharge path, capacitance, or breakdown.

RL topology: the current needs a closed path



(a) Source connected



(b) Source bypassed

Figure 10.9: Equivalent RL circuits before and after a switching operation. In (b) the battery is removed while a closed path through R remains. Simply opening the loop does not produce the ideal exponential discharge shown here. Current retains its direction at switching.

$$L\dot{I} + RI = V_s \quad (\text{source on}), \quad L\dot{I} + RI = 0 \quad (\text{source bypassed}).$$

Include the winding and discharge-path resistances in each switch state. An open loop is not the source-free closed RL circuit shown here.

Deriving RL relaxation for an arbitrary initial current



For constant source V_s and $R, L > 0$, define $I_\infty = V_s/R$ and the remaining gap $J = I_\infty - I$. Then

$$\dot{J} = -\frac{R}{L}J, \quad \int_{J(0)}^{J(t)} \frac{dJ'}{J'} = -\frac{R}{L} \int_0^t dt',$$

$$J(t) = J(0)e^{-t/\tau_L}, \quad \tau_L = \frac{L}{R},$$

$$I(t) = I_\infty + (I_0 - I_\infty)e^{-t/\tau_L}.$$

This returns I_0 at $t = 0$ and I_∞ at long times. Setting $I_0 = 0$ gives growth; setting $V_s = 0$ gives decay. Substitution verifies $L\dot{I} + RI = V_s$. For $R = 0$, solve the original equation instead: $I = I_0 + V_s t/L$. There is no finite resistive equilibrium.

The time constant and voltage sharing



For growth from zero,

$$I = \frac{V_s}{R}(1 - e^{-t/\tau_L}), \quad v_R = V_s(1 - e^{-t/\tau_L}), \quad v_L = V_s e^{-t/\tau_L}.$$

Initially all source voltage lies across the inductor and $\dot{i}(0) = V_s/L$. At long times the ideal inductor drop is zero and the resistor takes the source voltage.

What one time constant means

The remaining gap falls to $e^{-1} = 0.3679$ of its initial value. Growth reaches 63.2% of its final current; source-free decay retains 36.8%.

A larger R gives a shorter L/R but also a smaller V_s/R . Thus a shorter time constant alone does not determine a larger current. At $t \ll \tau_L$, $I \simeq V_s t/L$.

Example 10.14: current growth and inductor voltage



Example 10.14. A 12 V battery is connected in series with $R = 6.0\ \Omega$ and $L = 0.30\ \text{H}$. The switch is closed at $t = 0$, with zero initial current. Find the time constant, the final current, the current at $t = 0.10\ \text{s}$ and the inductor voltage at that time.

$$\begin{aligned}\tau_L &= 0.30/6.0 = \boxed{0.050\ \text{s}}, & I_\infty &= 12/6.0 = \boxed{2.0\ \text{A}}, \\ I(0.10) &= 2.0(1 - e^{-2}) = \boxed{1.73\ \text{A}}, & v_L(0.10) &= 12e^{-2} = \boxed{1.62\ \text{V}}.\end{aligned}$$

Since $0.10\ \text{s} = 2\tau_L$, the exponent is dimensionless. With guard digits, $v_R = 10.376\ \text{V}$ and $v_L = 1.624\ \text{V}$, whose sum is 12 V. Differentiating the current gives $L\dot{I} = 12e^{-2}$, independently verifying the same terminal voltage. Current begins at zero and remains below 2.0 A during growth.

Practice A12: a source reverses before its current



Practice problem A12. A series circuit with $R = 5.0\ \Omega$ and $L = 0.20\ \text{H}$ is driven from zero current by $+10\ \text{V}$ for $0.080\ \text{s}$, then by $-5.0\ \text{V}$. Find the current at switching, its derivative immediately after switching, and the elapsed time after reversal when current crosses zero.

Attempt. Use the first-stage solution to find the switching current. Then retain that current as the second-stage initial value. Take positive current in the direction driven by the first source and let t' measure time since reversal.

$$\tau_L = 0.040\ \text{s}, \quad I_s = 2.0(1 - e^{-2})\ \text{A},$$

$$I(t') = I_{\infty,2} + (I_s - I_{\infty,2})e^{-t'/\tau_L}, \quad I_{\infty,2} = -1.0\ \text{A}.$$

The current is continuous at reversal; its slope changes. The next slide solves all three requested quantities.

Practice A12: switching, slope, and zero crossing



$$I_s = 2.0(1 - e^{-2}) = \boxed{1.7293 \text{ A}},$$

$$i(0^+) = \frac{-5.0 - (5.0)(1.7293)}{0.20} = \boxed{-68.23 \text{ A/s}},$$

$$I(t') = -1.0 \text{ A} + (2.7293 \text{ A})e^{-t'/(0.040 \text{ s})}.$$

Set $I(t') = 0$ to obtain a dimensionless logarithm:

$$t'_0 = (0.040 \text{ s}) \ln\left(\frac{2.7293 \text{ A}}{1.0 \text{ A}}\right) = \boxed{0.04016 \text{ s}}.$$

Immediately after reversal, current is still positive while the slope is negative. Substituting $t' = 0$ recovers I_s , and substituting t'_0 gives zero. At long times the current tends to -1.0 A .

Magnetic energy: circuit and field descriptions



For a fixed ideal linear inductor, terminal power enters the field:

$$p_L = I v_L = L I \dot{I}, \quad U_B = \int_0^I L I' dI' = \boxed{\frac{1}{2} L I^2}.$$

In a long air-core solenoid, insert $L = \mu_0 n^2 A \ell_s$ and $B = \mu_0 n I$:

$$U_B = \frac{B^2}{2\mu_0} A \ell_s, \quad u_B = \boxed{\frac{B^2}{2\mu_0}}.$$

For a nonuniform vacuum field, integrate $U_B = \int B^2/(2\mu_0) dV$ over the field region, including leakage when relevant. Squaring an average field generally misses the energy. The density expression is for vacuum or air. Nonlinear and hysteretic magnetic media require their constitutive response and losses. This geometric model neglects carrier kinetic inductance.

RL traces: current, voltage, and energy

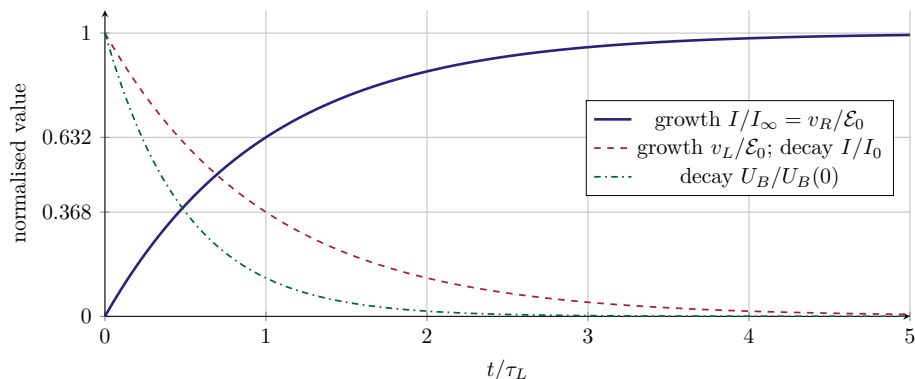


Figure 10.10: Exact normalised RL traces for the stated ideal circuits. The current scale is the final growth current or the initial decay current as indicated. The energy scale is the initial stored energy in decay.

Energy in decay has $U_B/U_0 = e^{-2t/\tau_L}$ because it is quadratic in current. Its half-value time is $(\tau_L/2) \ln 2$.

Example 10.17: the same energy by two methods



Example 10.17. A long air-core solenoid has radius 2.0 cm, length 0.50 m and 1000 turns. It carries 2.0 A. Estimate its inductance and stored magnetic energy.

Convert the radius to 0.020 m. The long air-core model gives $A = \pi r_s^2 = 1.2566 \times 10^{-3} \text{ m}^2$ and $n = 2000 \text{ m}^{-1}$.

$$\begin{aligned} L &= \mu_0 n^2 A \ell_s = \boxed{3.1583 \text{ mH}}, \\ U_B &= \frac{1}{2} L (2.0)^2 = \boxed{6.3165 \text{ mJ}}, \\ B &= \mu_0 n I = 5.0265 \text{ mT}, \quad u_B = 10.0531 \text{ J/m}^3. \end{aligned}$$

Independently, $u_B(A\ell_s) = (10.0531)(6.2832 \times 10^{-4}) \text{ J} = 6.3165 \text{ mJ}$. At the input precision, report $L \simeq 3.2 \text{ mH}$ and $U_B \simeq 6.3 \text{ mJ}$. Both methods use the same declared ideal geometry.

Example 10.15: current decay and stored energy



Example 10.15. An inductor of $L = 0.80 \text{ H}$ initially carries $I_0 = 3.0 \text{ A}$ and is then connected across a resistor $R = 20 \Omega$. Find the current after 0.050 s and the initial magnetic energy.

The closed discharge loop has no source. Current starts at I_0 and follows the source-free branch of slide 40:

$$\tau_L = 0.80/20 = 0.040 \text{ s},$$

$$I(0.050) = 3.0e^{-0.050/0.040} = \boxed{0.860 \text{ A}},$$

$$U_B(0) = \frac{1}{2}(0.80)(3.0)^2 = \boxed{3.6 \text{ J}}.$$

Multiplying $L\dot{I} + RI = 0$ by I gives $\dot{U}_B = -RI^2$. Hence all initial field energy eventually becomes heat: $\int_0^\infty RI_0^2 e^{-2t/\tau_L} dt = LI_0^2/2$. The current retains its direction during this discharge.

Finite-time growth requires a full energy budget



Multiplying the growth equation by current gives $V_s I = RI^2 + \dot{U}_B$. With $I = I_\infty(1 - e^{-x})$, $x = t/\tau_L$, integrate the source and resistor powers:

$$W_s(t) = \frac{V_s^2}{R} [t - \tau_L(1 - e^{-x})],$$

$$Q_R(t) = \frac{V_s^2}{R} \left[t - 2\tau_L(1 - e^{-x}) + \frac{\tau_L}{2}(1 - e^{-2x}) \right],$$

$$U_B(t) = \frac{1}{2}LI_\infty^2(1 - e^{-x})^2.$$

The heat integrand contains $1 - 2e^{-x} + e^{-2x}$; retaining its cross term is essential. Direct subtraction gives $W_s - Q_R = U_B$. If the source remains connected, heat keeps accumulating while storage tends to a finite limit. Growth heat therefore needs a specified time interval.

Example 10.18: where the source energy goes



Example 10.18. Use the earlier 12 V, 6.0 Ω , 0.30 H circuit. How much energy has the source supplied after 0.050 s, and where is it?

Restored context. The source is connected at $t = 0$ with $I(0) = 0$ and remains connected. Here $\tau_L = 0.050$ s, $I_\infty = 2.0$ A, and $U_\infty = 0.60$ J.

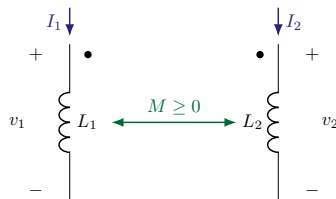
$$W_s(\tau_L) = \frac{2U_\infty}{e} = \boxed{0.4415 \text{ J}},$$

$$U_B(\tau_L) = U_\infty(1 - e^{-1})^2 = \boxed{0.2397 \text{ J}},$$

$$Q_R(\tau_L) = W_s - U_B = \boxed{0.2017 \text{ J}}.$$

Independently integrating Ri^2 gives the same heat. The field-energy fraction is 0.3996 of U_∞ . Both source and resistor powers start at zero because $I(0) = 0$, even though the inductor initially has 12 V across it. Displayed sums agree within rounding.

Mutual induction and winding references



Both reference currents enter dots.
Mutual terms in terminal voltages have positive signs.

Figure 10.11: Coupled-coil reference convention. Dots encode relative winding orientation; the central arrow denotes coupling, not a magnetic field line. With these references, $v_2 = L_2 \dot{i}_2 + M \dot{i}_1$.

$$\lambda_2 = M i_1 + L_2 i_2, \quad v_2 = M \dot{i}_1 + L_2 \dot{i}_2.$$

With both reference currents entering dots, $M \geq 0$ here. For an open secondary, $i_2 \simeq 0$: increasing positive i_1 makes its dot positive. The mutual Faraday emf along the current path is $-M \dot{i}_1$.

Flux overlap controls mutual inductance



A long primary solenoid has $B_1 \simeq \mu_0 N_1 I_1 / \ell_s$ in its central field area A_1 . A secondary surrounding that area has

$$\lambda_{21} = N_2 B_1 A_1, \quad M = \frac{\lambda_{21}}{I_1} \simeq \frac{\mu_0 N_1 N_2 A_1}{\ell_s}.$$

Enlarging a pickup beyond the ideal field region adds no linked flux. A smaller coaxial pickup inside that region links only its own area A_2 . Shared flux gives the ideal transformer emf ratio

$$|\mathcal{E}_2|/|\mathcal{E}_1| = N_2/N_1.$$

Mutual induction also enables contactless power transfer. Losses and load determine delivered power. Reciprocity $M_{12} = M_{21}$ applies to the fixed linear reciprocal system considered here.

Example 10.20: choosing the pickup flux area



Example 10.20. A long air-core primary has $N_1 = 1000$, $\ell_s = 0.50$ m and radius 2.0 cm. A coaxial $N_2 = 40$ turn pickup of radius 3.0 cm surrounds its central region. Find the approximate M and the open-circuit secondary voltage magnitude when $\dot{i}_1 = 20$ A/s.

The 3.0 cm pickup encloses the primary's entire 2.0 cm field radius. Use $A_1 = \pi(0.020)^2$ rather than the pickup area:

$$\begin{aligned} M &\simeq \frac{(4\pi \times 10^{-7})(1000)(40)\pi(0.020)^2}{0.50} \\ &= \boxed{1.2633 \times 10^{-4} \text{ H}}, \\ |v_2| &= M|\dot{i}_1| = \boxed{2.5266 \text{ mV}}. \end{aligned}$$

The pickup dot is positive with corresponding winding references. Using its area overestimates both results by $(3/2)^2 = 2.25$. Units H A/s = V check the voltage. The long-solenoid model omits leakage and end effects.

A rotating coil: flux and emf have different phases



For constant angular speed with the normal initially aligned with $\mathbf{B}$,

$$\lambda = NBA \cos \omega t, \quad \mathcal{E} = -\dot{\lambda} = NBA\omega \sin \omega t.$$

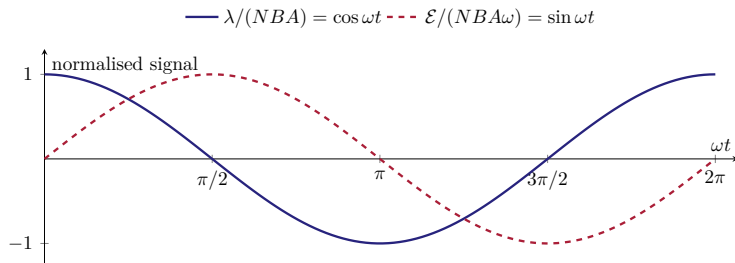


Figure 10.12: Flux linkage and emf for a coil rotating at constant angular speed. Curves are normalised separately; the emf amplitude grows with ω while the flux amplitude does not.

Maximum flux has zero emf; its steepest change gives maximum emf.

Example 10.21: a generator design estimate



Example 10.21. A small generator coil has $A = 2.5 \times 10^{-3} \text{ m}^2$ and rotates at 50 Hz in a field of 0.30 T. How many turns are needed for an emf amplitude of approximately 20 V?

With $\omega = 2\pi f = 314.159 \text{ rad/s}$, use $\mathcal{E}_0 = NBA\omega$:

$$N = \frac{20}{(0.30)(2.5 \times 10^{-3})(314.159)} = 84.883.$$

Choose approximately 85 turns. Their ideal amplitude is 20.0277 V, within about 0.138% of the target. Real design also needs winding space, resistance, load, and mechanical-loss estimates.

An emf is not a power source by itself

With a purely resistive load and negligible inductance, $I = \mathcal{E}/R_{\text{tot}}$ and $\tau_{\text{ext}}\omega = I\mathcal{E}$. Mechanical drive supplies the electrical output. At open circuit the emf can remain large while delivered power is nearly zero.

Eddy-current braking and restricted current paths

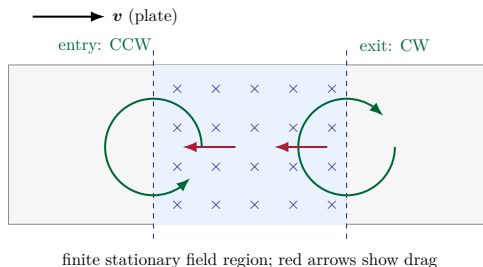


Figure 10.13: Qualitative eddy-current loops near the edges of a finite into-page field region as the plate moves right. The two circulations differ, but the force on the portions inside the field is leftward in both cases. These loops illustrate direction, not a solved current-density map.

A solid conductor supports distributed currents near field boundaries. Slots or insulated laminations restrict large loops and reduce unwanted heating. Uniform translation wholly inside a static uniform field need not produce these currents.

Pickup calibration: integrate the signed voltage



For negligible loading, define the sensor polarity to match positive winding emf. Faraday's law gives

$$\Delta\lambda = - \int \mathcal{E}(t) dt, \quad \Delta B = - \frac{\int \mathcal{E} dt}{NA} \quad (\text{uniform aligned field}).$$

Example 10.23. An illustrative, noise-free voltage trace has a triangular positive pulse of height 0.80 V and full duration 20 ms . A 200-turn coil has area $2.0 \times 10^{-4} \text{ m}^2$. Determine the field change for a uniform aligned field and estimate the bias from a residual 2.0 mV offset over a 0.10 s integration window.

The pulse is synthetic, not measured data. Subtract a stationary baseline before integration. A residual offset contributes throughout the full integration window, including time outside the pulse.

Example 10.23: pulse area and offset bias



The triangle area is voltage times time:

$$\mathcal{A}_V = \frac{1}{2}(0.80)(0.020) = 0.0080 \text{ V s},$$

$$\Delta\lambda = \boxed{-0.0080 \text{ Wb}},$$

$$\Delta B = -\frac{0.0080}{(200)(2.0 \times 10^{-4})} = \boxed{-0.20 \text{ T}}.$$

A positive residual 2.0 mV offset over 0.10 s adds $2.0 \times 10^{-4} \text{ V s}$, so

$$\delta B = -\frac{2.0 \times 10^{-4}}{0.040} = \boxed{-0.0050 \text{ T}}, \quad \frac{|\delta B|}{|\Delta B|} = \boxed{2.5\%}.$$

The inferred decrease becomes too large. Faster motion changes pulse height and width, but preserves its signed area for the same endpoints. A pickup measures change; an absolute initial field needs separate information.

A numerical RL model needs accuracy as well as stability



Euler gives $I_{k+1} = I_k + \Delta t(V - RI_k)/L$. With $e_k = I_k - I_\infty$ and $h = \Delta t/\tau_L$,

$$e_{k+1} = (1 - h)e_k.$$

Stability requires $0 < h < 2$, but $1 < h < 2$ creates spurious oscillation. Accuracy needs smaller steps. For $R = 10\ \Omega$, $L = 0.50\ \text{H}$, $V = 5.0\ \text{V}$, $I(0) = 0$, compare at fixed $t = \tau_L = 0.050\ \text{s}$:

h	Steps	I_{Euler} (A)	Relative error
0.20	5	0.336160	6.36%
0.10	10	0.325661	3.04%
0.05	20	0.320757	1.49%
0.025	40	0.318384	0.735%

The exact current is 0.316060 A. Error is $(I_{\text{Euler}} - I_{\text{exact}})/I_{\text{exact}}$. Halving the step approximately halves the error.

Core synthesis and bridge to the next chapters



A complete induction calculation

Choose a traversal and normal. Compute the linked flux and its change. Use the actual conductor motion where needed. Solve the circuit with its initial condition, then verify an energy budget.

Field and motion

$\mathcal{E} = -\dot{\lambda}$; moving conductors require $\mathbf{u} \times \mathbf{B}$. Symmetry supplies local fields from circulation.

Chapter 11 develops AC response and resonance. Chapters 12 and 13 extend the field description through Maxwell's equations and electromagnetic waves. **Further reading.** The chapter bibliography points to Halliday–Resnick–Walker, Serway–Jewett, and OpenStax Volume 2. The companion notes identify optional chapter exercises and the topics selected for the ten advanced slides.

Circuit and energy

$v_L = L\dot{I}$, $\tau_L = L/R$, $U_B = LI^2/2$. The current's history and discharge path matter.

Advanced: transporting flux with a material boundary



Stokes' theorem on a fixed surface yields the local law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

For a moving material surface, changes also arise from ribbons swept by its boundary. With paired contour and normal orientation,

$$\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{A} = \int_{S(t)} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A} - \oint_{C(t)} (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}.$$

Combining these relations gives the moving-conductor emf of slide 16. The partial derivative describes field change at a fixed point; the total flux derivative follows the boundary. A right-moving rod in $+z$ field has downward $\mathbf{u} \times \mathbf{B}$. For counterclockwise traversal its motional integral is negative, while the swept flux derivative is positive, checking the transport sign.

Example 10.26: a rotating rod has no single speed



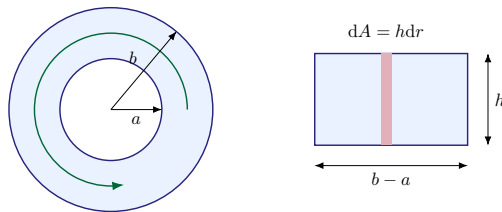
Example 10.26. A conducting rod of length a rotates counterclockwise in the xy plane about one end O , with constant angular speed $\omega > 0$ in $\mathbf{B} = B\hat{z}$. Find the open-circuit outer-end potential relative to the pivot.

At radius r , $\mathbf{u} = \omega r \hat{\phi}$, so $\mathbf{u} \times \mathbf{B} = B\omega r \hat{r}$. Open-circuit equilibrium gives $E_r = -B\omega r$:

$$V(a) - V(0) = - \int_0^a E_r dr = \int_0^a B\omega r dr = \boxed{\frac{1}{2} B\omega a^2}.$$

The outer end is positive. With the source's illustrative $a = 0.20$ m, $B = 0.50$ T, $\omega = 30$ s⁻¹, the result is 0.30 V. Using the tip speed for every element doubles the answer. A closed circuit with outward rod current has opposing torque $|\tau_B| = \int_0^a rIB dr = IBa^2/2$ and $|\tau_B|\omega = I\mathcal{E}$, an independent power check.

Nonuniform field geometry: the toroid



plan view; azimuthal B

radial cross section

Figure 10.14: Toroid geometry. Flux through a winding turn crosses the radial rectangle. The field varies as $1/r$, so using a single mean radius is an additional approximation.

In the ideal continuous-winding air-core model, Ampere's law gives

$$B(r) 2\pi r = \mu_0 N I, \quad B(r) = \frac{\mu_0 N I}{2\pi r} \quad (a < r < b).$$

A turn links the radial rectangle, $dA = h dr$, rather than the central hole.

Geometry and energy integrals before a design estimate



Integrating across the toroid's radial rectangle gives

$$\Phi_B = \int_a^b \frac{\mu_0 N I h}{2\pi r} dr, \quad L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}.$$

For a coaxial return path, $B = \mu_0 I / (2\pi r)$ between the conductors. Excluding internal conductor fields and end effects,

$$U'_B = \int_a^b \frac{B^2}{2\mu_0} 2\pi r dr = \frac{\mu_0 I^2}{4\pi} \ln \frac{b}{a}, \quad L'_{\text{ext}} = \frac{2U'_B}{I^2}.$$

Example 10.27. Estimate the inductance of an air-core toroid with $N = 500$, $a = 3.0$ cm, $b = 5.0$ cm and $h = 1.0$ cm. Also find the external inductance of 2.0 m of coaxial cable with $b/a = 3$.

The prime means per unit length. For a thin cross section, $\ln(b/a) \simeq (b - a)/r_m$, where $r_m = (a + b)/2$.

Example 10.27: exact model and approximate geometry



For the toroid, convert a, b, h to 0.030, 0.050, 0.010 m:

$$\begin{aligned} L &= (2.0 \times 10^{-7})(500)^2(0.010) \ln(5/3) \\ &= \boxed{2.5541 \times 10^{-4} \text{ H}}. \end{aligned}$$

The mean-radius estimate uses $r_m = 0.040$ m and $(b - a)/r_m = 0.50$, giving $L_m = 2.5000 \times 10^{-4}$ H. Its relative shortfall is

$$\frac{L - L_m}{L} = \boxed{2.12\%}.$$

$$L'_{\text{ext}} = (2.0 \times 10^{-7}) \ln 3 = 0.21972 \mu\text{H/m}, \quad L_{\text{ext}}(2.0 \text{ m}) = \boxed{0.43944 \mu\text{H}}.$$

Both logarithms are dimensionless. The cable has inductance without turns; the toroid gains N^2 . The external cable value omits internal magnetic energy, capacitance, and propagation.

Coupled energy limits mutual inductance



For fixed linear reciprocal windings using dot-entering references,

$$\lambda_1 = L_1 i_1 + M i_2, \quad \lambda_2 = M i_1 + L_2 i_2.$$

Summing $i_1 \dot{\lambda}_1 + i_2 \dot{\lambda}_2$ and integrating gives

$$U_B = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2.$$

Complete the square to test all allowed current combinations:

$$U_B = \frac{L_1}{2} \left(i_1 + \frac{M}{L_1} i_2 \right)^2 + \frac{1}{2} \left(L_2 - \frac{M^2}{L_1} \right) i_2^2.$$

For $L_1 > 0$, passivity requires $M^2 \leq L_1 L_2$ and $k = |M|/\sqrt{L_1 L_2} \leq 1$. The mutual term can be negative while total energy stays nonnegative. Efficiency also depends on losses and load.

Advanced challenge 10.28: measure coupling in series



Example 10.28. Two linear coils have $L_1 = 4.0 \text{ mH}$ and $L_2 = 9.0 \text{ mH}$. A series-aiding connection is measured as 23 mH . Find M , the coupling magnitude, and the predicted series-opposing inductance.

Attempt. Insert $I_1 = I_2 = I$ for aiding and $I_1 = -I_2 = I$ for opposing into the energy on slide 66.

$$L_+ = L_1 + L_2 + 2M, \quad L_- = L_1 + L_2 - 2M.$$

$$M = \frac{23 - 4 - 9}{2} = \boxed{5.0 \text{ mH}},$$

$$k = \frac{5.0}{\sqrt{4.0 \times 9.0}} = \boxed{0.833}, \quad L_- = 4.0 + 9.0 - 10.0 = \boxed{3.0 \text{ mH}}.$$

Checks. The bound $|M| \leq 6.0 \text{ mH}$ is satisfied. Also $M = (23 - 3)/4 = 5.0 \text{ mH}$ and $L_+ + L_- = 2(L_1 + L_2)$, consistent with the fixed linear model.

Nonlinear linkage: slope, ratio, and energy differ



For a fixed reversible single-valued relation $\lambda(I)$,

$$L_{\text{sec}} = \frac{\lambda}{I}, \quad L_{\text{diff}} = \frac{d\lambda}{dI}, \quad v_L = L_{\text{diff}} \dot{I}.$$

The local slope controls a small current change. The ratio characterises total linkage relative to current. At zero current use the appropriate limit. Power integration gives stored energy and the related coenergy:

$$U_B(I) = \int_0^I I' \frac{d\lambda}{dI'} dI', \quad W'_B(I) = \int_0^I \lambda(I') dI'.$$

Integration by parts gives $U_B + W'_B = I\lambda$. Only in the linear case are the two equal to $LI^2/2$. Neither $I\lambda/2$ nor $L_{\text{diff}} I^2/2$ is a general stored-energy formula.

Example 10.29: energy in a nonlinear model



Example 10.29. Consider the reversible model $\lambda(I) = L_0 I + \alpha I^3$ with $L_0 > 0$ and $\alpha > 0$. Find both inductances and stored energy. This model illustrates nonlinearity; it is not a saturation model.

$$L_{\text{sec}} = L_0 + \alpha I^2,$$

$$L_{\text{diff}} = L_0 + 3\alpha I^2,$$

$$U_B = \int_0^I I' (L_0 + 3\alpha I'^2) dI' = \frac{1}{2} L_0 I^2 + \frac{3}{4} \alpha I^4.$$

For the source's illustrative $L_0 = 0.10 \text{ H}$, $\alpha = 0.020 \text{ H/A}^2$, $I = 2.0 \text{ A}$:

$$L_{\text{sec}} = 0.18 \text{ H}, \quad L_{\text{diff}} = 0.34 \text{ H}, \quad \boxed{U_B = 0.44 \text{ J}}.$$

Differentiating U_B gives $I L_{\text{diff}}$, checking the power law. The shortcut $I\lambda/2 = 0.36 \text{ J}$ underestimates storage. The positive cubic term raises the slope and does not model saturation.

Changing geometry: magnetic force and source work



In a lossless locally linear system with coordinate x , $\lambda = L(x)I$. Differentiation and energy conservation give

$$v_L = L\dot{I} + I\frac{dL}{dx}\dot{x}, \quad Id\lambda = dU_B + F_x dx.$$

Insert $U_B = LI^2/2$: $Id\lambda = LI dI + I^2 dL$ and $dU_B = LI dI + I^2 dL/2$. Therefore

$$\boxed{F_x = \frac{1}{2}I^2 \frac{dL}{dx}} = \left(\frac{\partial W'_B}{\partial x} \right)_I.$$

For the source's scale estimate, constant $I = 2.0$ A, $dL/dx = 0.80$ H/m, and $\dot{x} = 0.10$ m/s give $F_x = 1.6$ N toward increasing L , $v_L = 0.16$ V, and source power $Iv_L = 0.32$ W. Half the input, 0.16 W, increases field energy and half becomes mechanical power. At fixed current the source supplies both; an energy derivative that ignores the source gives the wrong force sign.

