

Electromagnetic Oscillations and Alternating Current

Physics II - Chapter 11

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Why do circuits oscillate?

A capacitor and an inductor store energy in different fields. Their reversible exchange produces oscillation. Resistance removes energy, while a source can maintain it.

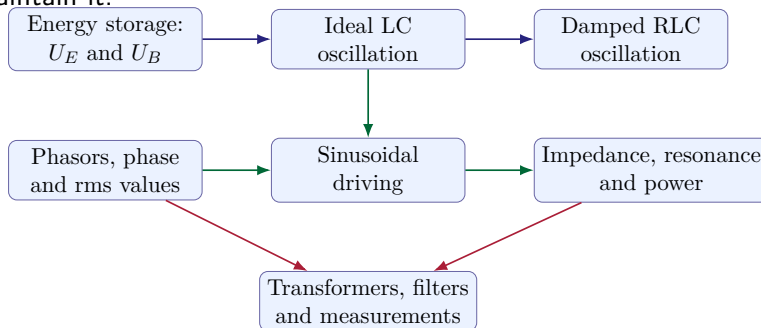


Figure 11.1: Conceptual roadmap for Chapter 11. The chapter begins with energy exchange in ideal LC circuits, adds damping, then develops the steady-state response of driven AC circuits using phase, phasors, impedance, power and resonance.

Chapter support: Chapter 11: [motivation-roadmap](#)

Learning route and prerequisites



- 1 **Free motion:** use element laws and Kirchhoff's loop rule to derive LC exchange and RLC damping.
- 2 **Driven motion:** use sinusoidal derivatives and phasor geometry to obtain current, phase and power.
- 3 **Applications:** use resonance and transformer relations, then identify what laboratory voltages actually measure.

Prerequisites: $q = C v_C$, $v_R = R i$, $v_L = L di/dt$; quadratic field energies; sine and cosine; differentiation and simple second-order differential equations. Complex numbers enter only in the advanced block.

Chapter support: Chapter 11: prior-knowledge

Learning goals



By the end of the core lecture, you should be able to:

- Derive the LC frequency and use initial conditions and energy conservation to predict charge and current.
- Classify free RLC damping and distinguish a transient from a driven steady response.
- Obtain rms values and derive the element phase relations.
- Add voltage phasors, calculate impedance and phase, and check average power by energy balance.
- Derive resonance bandwidth, interpret Q , and calculate transformer ratios.
- Choose a measurement model and infer circuit parameters from amplitude, width and signed phase.

Chapter support: Chapter 11: motivation-roadmap

Variables and sign conventions



The signed capacitor charge is $q(t)$ and the loop current is $i(t)$. Choose positive current *into* the plate whose charge is q , so $i = q'$.

$$v_C = q/C, \quad v_R = Ri, \quad v_L = Li'.$$

These are passive terminal voltage drops. The opposing self-induced emf is $-Li'$, consistent with Faraday's law.

Source as phase reference

$v_s = V_0 \cos \omega t$ and $i = I_0 \cos(\omega t - \phi)$. Positive ϕ means current lags. Peak amplitudes carry subscript 0; rms values have subscript rms. Use $\omega = 2\pi f$.

$R, L, C > 0$ are constant. Lumped elements require circuit dimensions small compared with the relevant wavelength.

Chapter support: Chapter 11: prior-knowledge

Reversible storage and irreversible loss



Charging the capacitor requires work against its existing voltage. Building current in the inductor requires work against its induced emf:

$$U_C = \int_0^q \frac{\bar{q}}{C} d\bar{q} = \frac{q^2}{2C}, \quad U_L = \int_0^i L \, di = \frac{1}{2} Li^2.$$

The two ideal elements store energy and can return it. A resistor instead converts electrical energy into internal energy at the non-negative rate

$$p_R = Ri^2.$$

A source contributes $v_s i$, which can have either sign. This energy accounting will check the loop equations.

Chapter support: Chapter 11: energy-storage

The four stages of LC exchange

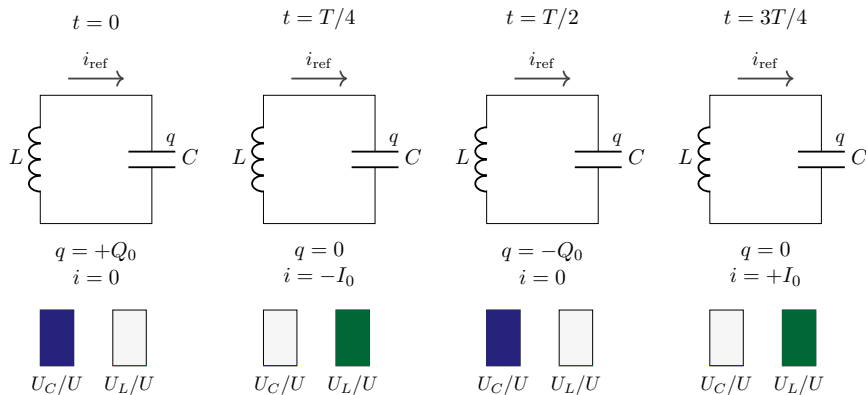


Figure 11.2: Four stages of an ideal closed LC loop starting with $q = +Q_0$ and $i = 0$. The same clockwise reference arrow is used in every panel; it enters the upper capacitor plate, so $i = dq/dt$. A negative listed current flows opposite to that arrow. The bars show energy fractions: charge polarity reverses after half a period, while electric energy returns to its maximum.

Chapter support: Chapter 11: energy-storage

Derivation of the ideal LC frequency



For the closed source-free loop, Kirchhoff's rule gives $v_L + v_C = 0$. With $i = q'$,

$$Lq'' + \frac{q}{C} = 0 \quad \Rightarrow \quad q'' + \omega_0^2 q = 0, \quad \omega_0 = \frac{1}{\sqrt{LC}}.$$

A trial cosine has $q'' = -\omega_0^2 q$ and therefore solves the equation. Thus

$$f_0 = \frac{1}{2\pi\sqrt{LC}}, \quad T_0 = 2\pi\sqrt{LC}.$$

The restoring term has a positive coefficient. Initially positive charge produces $i'(0) = -q(0)/(LC) < 0$, as the discharge direction requires.

The model neglects resistance, radiation, driving and distributed effects. Since $\text{HF} = \text{s}^2$, ω_0 has units s^{-1} .

Chapter support: Chapter 11: ideal-lc

Initial conditions determine the LC state



A second-order equation requires two initial values: $q_i = q(0)$ and $i_i = i(0)$.
Write

$$q(t) = A \cos \omega_0 t + B \sin \omega_0 t.$$

At zero, $A = q_i$. Differentiating gives $i(0) = \omega_0 B$, hence

$$q(t) = q_i \cos \omega_0 t + \frac{i_i}{\omega_0} \sin \omega_0 t.$$

The amplitude is $Q_0 = \sqrt{q_i^2 + (i_i/\omega_0)^2}$, and the peak current is $I_0 = \omega_0 Q_0$.
An initially charged capacitor with zero current has a cosine response. An initially uncharged capacitor with positive current begins with a positive sine response.

Chapter support: Chapter 11: state-variables

Energy conservation and quarter-cycle phase



For $q = Q_0 \cos(\omega_0 t + \delta)$, differentiation gives $i = -\omega_0 Q_0 \sin(\omega_0 t + \delta)$. Then

$$U_C = \frac{Q_0^2}{2C} \cos^2(\omega_0 t + \delta),$$
$$U_L = \frac{L\omega_0^2 Q_0^2}{2} \sin^2(\omega_0 t + \delta) = \frac{Q_0^2}{2C} \sin^2(\omega_0 t + \delta).$$

Since $L\omega_0^2 = 1/C$,

$$U_C + U_L = \frac{Q_0^2}{2C} = \text{constant}.$$

At maximum $|q|$, current is zero. At maximum $|i|$, charge is zero. The fields exchange energy while the state continues around the cycle.

Chapter support: Chapter 11: ideal-lc

Example 11.1: LC frequency



Example 11.1. A capacitor of capacitance $C = 0.50 \mu\text{F}$ is charged to $V_0 = 12 \text{ V}$ and then connected to an ideal inductor of inductance $L = 20 \text{ mH}$. Find the natural frequency and the maximum current.

Model and frequency. The ideal released loop conserves field energy. In SI units, $L = 0.020 \text{ H}$ and $C = 5.0 \times 10^{-7} \text{ F}$.

$$\omega_0 = \frac{1}{\sqrt{(0.020)(5.0 \times 10^{-7})}} = 1.00 \times 10^4 \text{ rad s}^{-1},$$

$$f_0 = \omega_0 / (2\pi) = 1.59 \times 10^3 \text{ Hz}.$$

The peak-current calculation and independent energy check follow on slide 13.

Chapter support: Chapter 11: lc-frequency-example; Example 11.1

Example 11.1: current and energy check



The initial charge is

$$Q_0 = CV_0 = (0.50 \times 10^{-6})(12) = 6.00 \mu\text{C}.$$

Differentiating the charge solution yields the non-negative current amplitude

$$I_0 = \omega_0 Q_0 = (1.00 \times 10^4)(6.00 \times 10^{-6}) = 0.0600 \text{ A}.$$

Independent check by energy:

$$\frac{1}{2}CV_0^2 = 3.60 \times 10^{-5} \text{ J} = \frac{1}{2}LI_0^2.$$

Thus $I_0 = 60.0 \text{ mA}$. With the chosen $i = q'$ convention the first current is negative, even though its amplitude is positive.

Chapter support: Chapter 11: lc-frequency-example; Example 11.1

Practice P2: magnetic energy at release



P2. An ideal loop has $L = 25 \text{ mH}$ and $C = 4.0 \mu\text{F}$. Initially $q = 0$ and $i = +20 \text{ mA}$. Find $q(t)$, its amplitude, and the time of its first positive maximum. Check the initial energy.

Attempt. Use the general initial-value form from slide 10. Decide which coefficient vanishes before calculating. All elements are ideal and linear; use $i = q'$ and time in seconds.

The complete solution is on slide 15.

Chapter support: Chapter 11: [sec:ch11-practice-problems; P2](#)

P2: solution and energy check



The angular frequency is $\omega_0 = [(0.025)(4.0 \times 10^{-6})]^{-1/2} = 3162.28 \text{ rad s}^{-1}$.
Because $q_i = 0$,

$$q(t) = (6.325 \mu\text{C}) \sin[(3162.28 \text{ s}^{-1})t], \quad Q_0 = \frac{0.020}{3162.28} = 6.325 \mu\text{C}.$$

Its derivative is $i(t) = (20.0 \text{ mA}) \cos \omega_0 t$, which verifies both initial values.
The first positive charge maximum is at

$$t = \frac{\pi}{2\omega_0} = 0.497 \text{ ms}.$$

Initially $U = Li_i^2/2 = 5.00 \mu\text{J}$. At the charge maximum, $Q_0^2/(2C) = 5.00 \mu\text{J}$.
Magnetic energy has become electric energy.

Chapter support: Chapter 11: sec:ch11-selected-solutions; P2

The energy ellipse in phase space

$$\frac{q^2}{Q_0^2} + \frac{i^2}{I_0^2} = 1, \quad I_0 = \omega_0 Q_0.$$

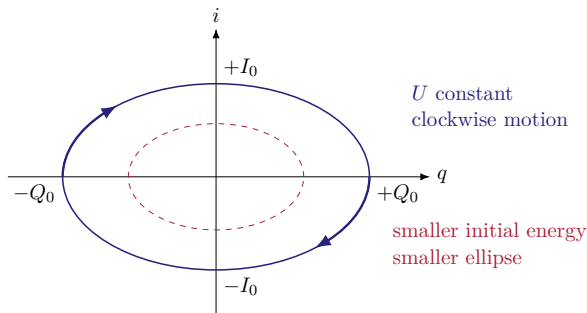


Figure 11.3: Constant-energy curves $q^2/Q_0^2 + i^2/I_0^2 = 1$, with $I_0 = \omega_0 Q_0$. For $i = q'$, the motion is clockwise: at $q = +Q_0$ one has $i = 0$ but $i' = -q/(LC) < 0$. The dashed curve represents another initial energy, not damping within a lossless trajectory.

Chapter support: Chapter 11: energy-ellipse

Mechanical analogy



The two equations have the same restoring structure:

$$Lq'' + q/C = 0, \quad mx'' + kx = 0.$$

Circuit	Mechanics	Role
q	x	State coordinate
$i = q'$	$v = x'$	Rate of change
L	m	Inertia coefficient
$1/C$	k	Restoring coefficient
$Li^2/2$	$mv^2/2$	Kinetic-like energy
$q^2/(2C)$	$kx^2/2$	Potential-like energy
R	b	Dissipation

The analogy concerns equations and energy. Charge is a circuit variable, and current is its rate of change.

Chapter support: Chapter 11: mechanical-analogy

Resistance gives a damped free oscillator



For a source-free series RLC loop,

$$Lq'' + Rq' + q/C = 0, \quad q'' + 2\alpha q' + \omega_0^2 q = 0, \quad \alpha = \frac{R}{2L}.$$

Multiplying by $i = q'$ converts the loop equation to an energy balance:

$$\frac{d}{dt} \left(\frac{Li^2}{2} + \frac{q^2}{2C} \right) = -Ri^2 \leq 0.$$

Stored energy decreases monotonically. It has zero instantaneous loss rate whenever current is zero, even if the circuit will ring later.

For underdamping the charge envelope is exactly $e^{-\alpha t}$. Instantaneous total energy is generally not a pure exponential. Its cycle average follows approximately $e^{-2\alpha t}$ for weak damping.

Chapter support: Chapter 11: damped-rlc

The three free damping regimes



Try $q = e^{st}$. The characteristic equation and roots are

$$s^2 + 2\alpha s + \omega_0^2 = 0, \quad s_{\pm} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}.$$

- $\alpha < \omega_0$: complex conjugate roots give underdamped motion with $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.
- $\alpha = \omega_0$: a repeated root gives $(A + Bt)e^{-\omega_0 t}$, the critically damped response.
- $\alpha > \omega_0$: two negative real roots give an overdamped sum of exponentials.

The threshold is $R_c = 2\sqrt{L/C}$. These labels describe *free transients*. A continuously driven sinusoid remains at its source frequency.

Chapter support: Chapter 11: critical-damping

Free response for identical initial conditions

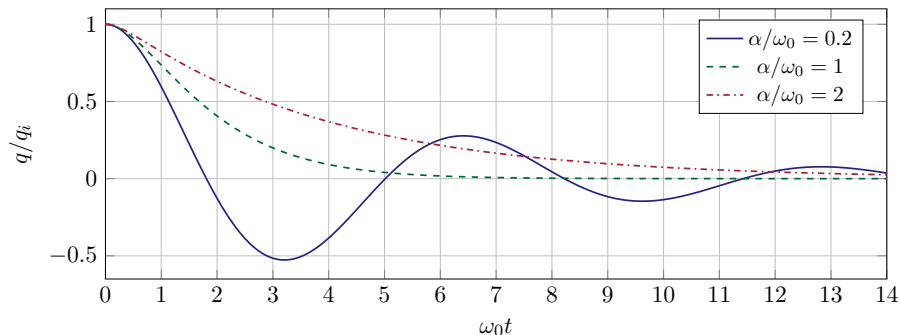


Figure 11.4: Exact normalized free responses for $q(0) = q_i > 0$, $i(0) = 0$. Underdamping permits repeated reversals; critical damping gives the fastest non-oscillatory return for this standard release; strong overdamping leaves a slower tail. All curves have zero initial slope.

Chapter support: Chapter 11: critical-damping

Example 11.3: damping regime and frequency



Example 11.3. A series RLC circuit has $L = 80 \text{ mH}$, $C = 2.0 \mu\text{F}$ and $R = 40 \Omega$. Determine whether the free response is underdamped, and find ω_0 and ω_d .

Solution. Convert $L = 0.080 \text{ H}$ and $C = 2.0 \times 10^{-6} \text{ F}$:

$$\omega_0 = (LC)^{-1/2} = 2500 \text{ rad s}^{-1},$$

$$R_c = 2\sqrt{L/C} = 400 \Omega > 40 \Omega,$$

$$\alpha = R/(2L) = 250 \text{ s}^{-1},$$

$$\omega_d = \sqrt{2500^2 - 250^2} = 2487.47 \text{ rad s}^{-1}.$$

The response is underdamped. Substitution gives $\omega_d^2 + \alpha^2 = \omega_0^2$; the frequency approaches ω_0 as $R \rightarrow 0$.

Chapter support: Chapter 11: damping-example; Example 11.3

A correct damped initial-value solution



For $\alpha < \omega_0$, write $q = e^{-\alpha t}(A \cos \omega_d t + B \sin \omega_d t)$. At zero,

$$A = q_i, \quad i_i = -\alpha A + \omega_d B.$$

Thus both initial values require

$$q(t) = e^{-\alpha t} \left[q_i \cos \omega_d t + \frac{i_i + \alpha q_i}{\omega_d} \sin \omega_d t \right].$$

When $i_i = 0$, differentiation gives

$$i(t) = -\frac{q_i \omega_0^2}{\omega_d} e^{-\alpha t} \sin \omega_d t.$$

Simply multiplying the undamped cosine by $e^{-\alpha t}$ would impose $i(0) = -\alpha q_i$. It would not describe release from rest.

Chapter support: Chapter 11: critical-damping

Example 11.4: the first current extremum



Example 11.4. Use $L = 80 \text{ mH}$, $C = 2.0 \mu\text{F}$ and $R = 40 \Omega$ from Example 11.3. The capacitor initially has 10 V and the current is zero. Find the first extremum of current and compare its magnitude with the lossless prediction.

Restored context. Example 11.3 established an underdamped loop with $\omega_0 = 2500 \text{ rad s}^{-1}$, $\alpha = 250 \text{ s}^{-1}$ and $\omega_d = 2487.47 \text{ rad s}^{-1}$. Here $q_i = CV_i = 20.0 \mu\text{C}$.

Use the zero-initial-current formula on slide 22 and find $di/dt = 0$. The signed extremum, lossless comparison and loop-voltage check follow on slide 24.

Chapter support: Chapter 11: damped-initial-example; Example 11.4

Example 11.4: solution and loop check



Differentiating the current on slide 22 gives the extremum condition

$$\omega_d \cos \omega_d t - \alpha \sin \omega_d t = 0.$$

The first extremum at positive time occurs at

$$t_* = \frac{\tan^{-1}(\omega_d/\alpha)}{\omega_d} = 0.591 \text{ ms}.$$

Substitution gives $i(t_*) = -43.1 \text{ mA}$, compared with the lossless amplitude $q_i\omega_0 = 50.0 \text{ mA}$.

At the extremum $v_L = Li' = 0$, so the free loop requires

$$v_C = -Ri = +1.73 \text{ V}.$$

The capacitor is still charged when $|i|$ is greatest. As $R \rightarrow 0$, $t_* \rightarrow \pi/(2\omega_0)$ and the lossless peak is recovered.

Chapter support: Chapter 11: damped-initial-example; Example 11.4

Rms is a mean-square quantity



The average of a symmetric sinusoid is zero, but resistor heating depends on its square. For any periodic quantity,

$$x_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T x^2(t) dt}.$$

For $x = X_0 \cos \omega t$, $\cos^2 \theta = (1 + \cos 2\theta)/2$ yields

$$x_{\text{rms}} = X_0/\sqrt{2}, \quad P_R = I_{\text{rms}}^2 R.$$

If $v = V_{\text{dc}} + v_{\text{ac}}$ with $\langle v_{\text{ac}} \rangle = 0$, then

$$V_{\text{rms}}^2 = V_{\text{dc}}^2 + V_{\text{ac,rms}}^2.$$

The $1/\sqrt{2}$ conversion applies to a sinusoid. Rms heating equivalence holds for any periodic waveform in a constant ohmic resistance.

Chapter support: Chapter 11: sinusoidal-rms

Example 11.6: a pulse train



Example 11.6. A $100\ \Omega$ resistor receives a periodic voltage that is 12 V for one quarter of each cycle and zero otherwise. Find its mean voltage, rms voltage and average power. Compare it with a zero-mean sinusoid producing the same power.

Model. Over a period T , $v = 12\text{ V}$ for $T/4$, and $v = 0$ for the remainder. Weight each level by its time fraction:

$$\langle v \rangle = \frac{1}{4}(12) = 3.00\text{ V}, \quad V_{\text{rms}} = \sqrt{\frac{1}{4}(12)^2} = 6.00\text{ V}.$$

The pulse train contains a DC component. Its heating and equal-power sinusoidal comparison are completed on slide 27.

Chapter support: Chapter 11: rms-waveform-example; Example 11.6

Example 11.6: heating and peak comparison



For the constant $100\ \Omega$ resistor,

$$P = \frac{V_{\text{rms}}^2}{R} = \frac{6.00^2}{100} = 0.360\ \text{W}.$$

A zero-mean sinusoid delivering the same power must also have $V_{\text{rms}} = 6.00\ \text{V}$, hence peak voltage

$$V_{0,\sin} = 6.00\sqrt{2} = 8.49\ \text{V}.$$

The original pulse train peaks at $12\ \text{V}$. Equal heating therefore does not imply equal peak stress.

As a separate measurement check, its AC-only rms is

$$V_{\text{ac,rms}} = \sqrt{6.00^2 - 3.00^2} = 5.20\ \text{V}.$$

The mean, total rms and AC-only rms answer different questions about the same signal.

Chapter support: Chapter 11: rms-waveform-example; Example 11.6

Resistor response to a sinusoid



Choose the current as a temporary phase reference: $i = I_0 \cos \omega t$. Ohm's law then gives

$$v_R = RI_0 \cos \omega t, \quad V_{R0} = I_0 R.$$

Voltage and current are in phase. The instantaneous power is

$$p_R = RI_0^2 \cos^2 \omega t \geq 0, \quad \langle p_R \rangle = \frac{1}{2} RI_0^2 = RI_{\text{rms}}^2.$$

At fixed resistance this relation holds at each instant. Rms Ohm's law follows because both sinusoidal amplitudes receive the same conversion factor.

The current reference used to derive element voltages can later be rotated so that the source voltage becomes the phase reference.

Chapter support: Chapter 11: [simple-elements-ac](#)

Inductor voltage leads current



Starting with $i = I_0 \cos \omega t$, the passive inductor law gives

$$\begin{aligned} v_L &= L \frac{di}{dt} = -\omega L I_0 \sin \omega t \\ &= \omega L I_0 \cos(\omega t + \pi/2). \end{aligned}$$

Thus

$$X_L = \omega L, \quad V_{L0} = I_0 X_L.$$

Voltage leads current by 90° , or current lags voltage by 90° . Faster variation requires more voltage for the same current amplitude.

An ideal inductor has zero average power: $v_L i = d(Li^2/2)/dt$ integrates to zero over a steady cycle. A perfectly lossless driven inductor may also retain a constant current offset fixed by its initial state.

Chapter support: Chapter 11: [simple-elements-ac](#)

Capacitor current leads voltage



Use $v_C = V_{C0} \cos \omega t$. Since $q = Cv_C$,

$$\begin{aligned} i &= C \frac{dv_C}{dt} = -\omega CV_{C0} \sin \omega t \\ &= \omega CV_{C0} \cos(\omega t + \pi/2). \end{aligned}$$

Thus

$$X_C = \frac{1}{\omega C}, \quad V_{C0} = I_0 X_C.$$

Current leads voltage by 90° . Faster variation of capacitor voltage produces more current for the same voltage amplitude.

The current represents changing plate charge; charge does not cross the ideal dielectric. Average power is zero because $v_C i = d(q^2/2C)/dt$ is a periodic energy derivative.

Chapter support: Chapter 11: simple-elements-ac

Element phasors in the phase plane

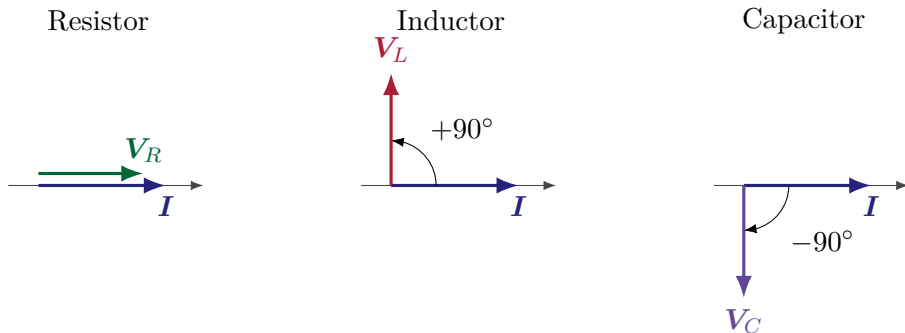


Figure 11.5: Element voltage phasors relative to current. Positive phase is counterclockwise. Inductor voltage leads; capacitor voltage lags. Current and voltage use independent drawing scales, since they have different units. The phasor arrows describe phase relationships, not spatial directions in a wire.

Frequency check: doubling ω doubles X_L and halves X_C . The phase shifts remain $\pm 90^\circ$ for ideal elements.

Chapter support: Chapter 11: simple-elements-ac

Driven response and the decaying transient



A source changes the loop equation to

$$Lq'' + Rq' + q/C = V_0 \cos \omega t.$$

Linearity allows $q = q_{\text{tr}} + q_{\text{ss}}$. The transient satisfies the free equation and enforces the initial charge and current. For $R > 0$, it decays.

The steady response oscillates at the *source* frequency ω . Phasors describe that response after settling, rather than the switching transient.

Time scale

An underdamped charge envelope has e-folding time $\tau_A = 1/\alpha = 2L/R$. High- Q circuits can need many periods to settle. In strong overdamping, the slow root instead gives a long RC tail.

Chapter support: Chapter 11: phasors-impedance

Series impedance from voltage addition



All series elements carry the same current. With current as the horizontal phasor reference, V_R is horizontal, V_L is upward and V_C is downward. Therefore

$$V_0^2 = (I_0 R)^2 + [I_0(X_L - X_C)]^2.$$

Divide by I_0^2 to obtain

$$Z = \sqrt{R^2 + (X_L - X_C)^2}, \quad I_0 = V_0/Z.$$

Geometry also gives

$$\tan \phi = \frac{X_L - X_C}{R}, \quad -\pi/2 < \phi < \pi/2 \quad (R > 0).$$

With the source as reference, $i = I_0 \cos(\omega t - \phi)$. Net inductive reactance gives $\phi > 0$ and lagging current; net capacitive reactance gives leading current.

Chapter support: Chapter 11: phasors-impedance

The series voltage triangle

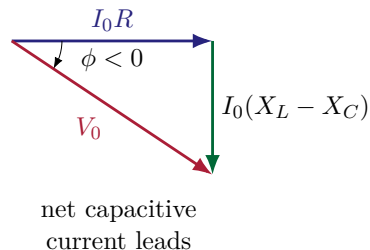
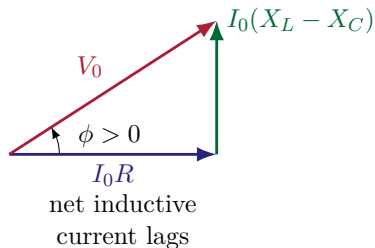


Figure 11.6: Head-to-tail addition of resistor and net reactive voltages. The two panels use the current direction as the horizontal phase reference. The vertical component can be positive or negative; its sign fixes the interpretation of the phase angle.

The vector sum enforces Kirchhoff's law at every instant. Adding peak magnitudes as scalars would discard the phase differences.

Chapter support: Chapter 11: phasor-by-hand

Example 11.9: model and reactances



Example 11.9. A series circuit has $R = 30\ \Omega$, $L = 0.20\ \text{H}$ and $C = 50\ \mu\text{F}$. It is driven by $v_s(t) = 120 \cos(400t)\ \text{V}$. Find the current amplitude and phase.

Model. Seek the sinusoidal steady state at the stated angular frequency $400\ \text{rad s}^{-1}$. The $120\ \text{V}$ is a peak amplitude.

$$X_L = (400)(0.20) = 80\ \Omega, \quad X_C = \frac{1}{(400)(50 \times 10^{-6})} = 50\ \Omega.$$

The signed net reactance is $+30\ \Omega$, so the current must lag the source. The impedance, time function and check follow on slide 36.

Chapter support: Chapter 11: series-rlc-example; Example 11.9

Example 11.9: amplitude, phase and check



The voltage triangle gives

$$Z = \sqrt{30^2 + 30^2} = 42.426 \, \Omega, \quad I_0 = 120/Z = 2.83 \, \text{A}.$$

Because $\tan \phi = 30/30 = 1$, $\phi = 45^\circ = \pi/4$. The time function is

$$i(t) = 2.83 \cos(400t - \pi/4) \, \text{A}.$$

Check the source phasor. The resistor voltage is 84.85 V peak. The net reactive voltage is also 84.85 V peak. Their perpendicular sum has magnitude $\sqrt{84.85^2 + 84.85^2} = 120 \, \text{V}$.

The current lags by half a right angle, consistent with the positive net inductive reactance.

Chapter support: Chapter 11: series-rlc-example; Example 11.9

Example 11.10: a capacitive response



Example 11.10. A series circuit has $R = 50\ \Omega$, $L = 40\ \text{mH}$ and $C = 10\ \mu\text{F}$. It is driven at $\omega = 1000\ \text{rad s}^{-1}$ with rms voltage $24\ \text{V}$. Find I_{rms} and state whether the current leads or lags.

Solution. Use $L = 0.040\ \text{H}$ and $C = 10 \times 10^{-6}\ \text{F}$:

$$X_L = 40\ \Omega, \quad X_C = 100\ \Omega, \quad X_L - X_C = -60\ \Omega.$$

$$Z = \sqrt{50^2 + (-60)^2} = 78.10\ \Omega, \quad I_{\text{rms}} = 24/Z = 0.307\ \text{A}.$$

The angle is $\phi = \tan^{-1}(-60/50) = -50.2^\circ$. Current leads the source by 50.2° .

Check: $(I_{\text{rms}}R)^2 + [I_{\text{rms}}(X_L - X_C)]^2 = 24^2\ \text{V}^2$, using unrounded values.

Chapter support: Chapter 11: capacitive-rlc-example; Example 11.10

Instantaneous power and average real power



Multiplying the driven loop equation by $i = q'$ gives

$$v_s i = \frac{dU}{dt} + Ri^2, \quad U = \frac{q^2}{2C} + \frac{Li^2}{2}.$$

For $v_s = V_0 \cos \omega t$ and $i = I_0 \cos(\omega t - \phi)$, the product-to-sum identity yields

$$p(t) = \frac{V_0 I_0}{2} [\cos \phi + \cos(2\omega t - \phi)].$$

The second term averages to zero, and periodic stored energy has zero net change. Hence

$$P = V_{\text{rms}} I_{\text{rms}} \cos \phi = I_{\text{rms}}^2 R.$$

The power factor is $\cos \phi = R/Z$. The apparent power VI uses rms magnitudes. Negative instantaneous source power means energy return; resistor heating remains non-negative.

Chapter support: Chapter 11: ac-power

Power exchange during a cycle

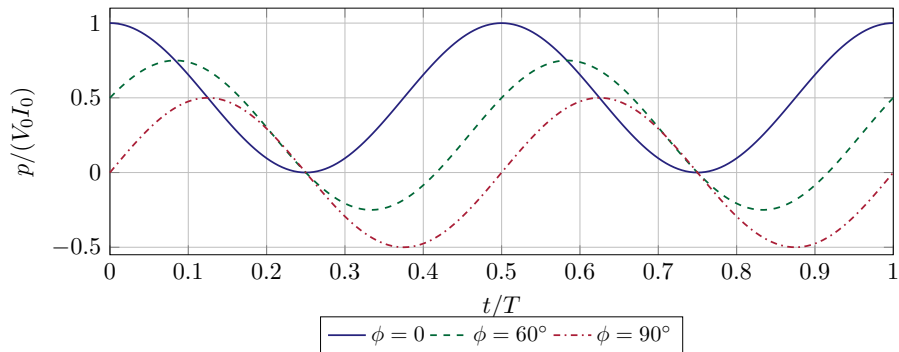


Figure 11.7: Calculated instantaneous power for equal peak-product normalization. The mean levels are $1/2$, $1/4$ and zero, respectively. The reactive cases contain intervals of energy return; the resistor case is non-negative throughout.

Chapter support: Chapter 11: ac-power

Example 11.12: average power



Example 11.12. Use the circuit in Example 11.9. Find the rms current and average power.

Restored context: $R = 30\ \Omega$, $L = 0.20\ \text{H}$ and $C = 50\ \mu\text{F}$ (slides 35–36). The source is $v_s = 120 \cos(400t)\ \text{V}$. The unrounded results are $I_0 = 2\sqrt{2}\ \text{A}$ and $\phi = \pi/4$.

$$V_{\text{rms}} = 120/\sqrt{2} = 84.85\ \text{V}, \quad I_{\text{rms}} = I_0/\sqrt{2} = 2.00\ \text{A}.$$

$$P = (84.85)(2.00) \cos 45^\circ = 120\ \text{W}.$$

Independent resistor check: $I_{\text{rms}}^2 R = (2.00)^2(30) = 120\ \text{W}$. The ideal reactive elements return their energy over each full cycle.

Chapter support: Chapter 11: average-power-example; Example 11.12

Driven current resonance



For a fixed source amplitude and constant ideal series elements,

$$I_0(\omega) = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}.$$

The denominator is smallest when the two reactances cancel:

$$\omega L = 1/(\omega C) \implies \omega = \omega_0 = 1/\sqrt{LC}.$$

At this current resonance, $Z = R$, $\phi = 0$ and $I_0 = V_0/R$.

The capacitor and inductor remain active energy stores. Their equal voltage phasors cancel in the loop sum. The free frequency ω_d is generally below ω_0 , while the driven current peak stays at ω_0 for the stated model.

Chapter support: Chapter 11: resonance

Resistance controls peak height and width

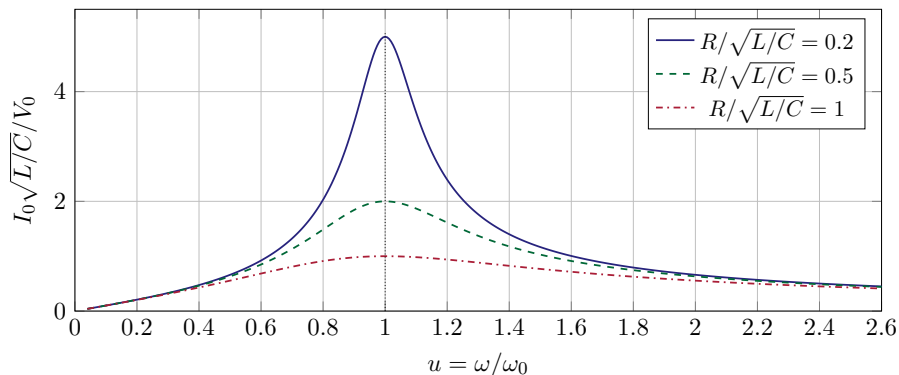


Figure 11.8: Calculated current responses for the same L , C and source amplitude. The normalization is common to all three curves, so their different peak heights are meaningful. Decreasing resistance increases the peak and narrows the bandwidth; each current maximum remains at ω_0 .

Chapter support: Chapter 11: resonance

Exact half-power bandwidth



The resistor-power ratio is

$$\frac{P(\omega)}{P_{\max}} = \frac{R^2}{R^2 + (\omega L - 1/\omega C)^2}.$$

Half power requires $|\omega L - 1/(\omega C)| = R$, so the positive roots are

$$\omega_{1,2} = \frac{\sqrt{R^2 + 4L/C} \mp R}{2L}.$$

Subtracting and multiplying the roots gives the exact relations

$$\Delta\omega = R/L, \quad \omega_1\omega_2 = \omega_0^2.$$

The current at half power is $I_{\max}/\sqrt{2}$. The peak frequency is the geometric mean of the half-power frequencies. Approximate symmetry $\omega_{1,2} \simeq \omega_0 \mp R/(2L)$ requires a narrow peak.

Chapter support: Chapter 11: resonance

Quality factor and voltage magnification



For the constant-element series model,

$$Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}.$$

At current resonance, $I_{\text{rms}} = V_{s,\text{rms}}/R$, hence

$$V_{L,\text{rms}} = V_{C,\text{rms}} = QV_{s,\text{rms}}.$$

Reactive voltages can greatly exceed the source voltage. Their opposite phases enforce cancellation.

The free response is underdamped when $Q > 1/2$, since $\alpha = \omega_0/(2Q)$. A high- Q approximation needs $Q \gg 1$.

At driven resonance, $U = LI_0^2/2$ and the loss per cycle is $RI_0^2 T_0/2$, so $2\pi U/E_{\text{loss/cycle}} = Q$.

Chapter support: Chapter 11: resonance

Practice P9: a broad current resonance



P9. For $L = 10 \text{ mH}$, $C = 1.0 \mu\text{F}$ and $R = 200 \Omega$, find the exact half-power frequencies, bandwidth and Q . Compare the arithmetic and geometric means of the half-power frequencies. Classify the free response.

Attempt. Use the exact half-power roots, not the narrow-peak approximation. Assume ideal linear lumped elements and a fixed source amplitude. The convention is $i = q'$.

The complete calculation and free-response classification follow on slide 46.

Chapter support: Chapter 11: [sec:ch11-practice-problems; P9](#)

P9: exact roots and free-response classification



Here $\omega_0 = 10^4 \text{ rad s}^{-1}$ and $Q = \omega_0 L/R = 0.500$. The exact roots give

$$\begin{aligned}\omega_1 &= 4142.14 \text{ rad s}^{-1}, & f_1 &= 659.24 \text{ Hz}, \\ \omega_2 &= 24142.14 \text{ rad s}^{-1}, & f_2 &= 3842.34 \text{ Hz}.\end{aligned}$$

Therefore $\Delta\omega = 20000 \text{ rad s}^{-1}$ and $\Delta f = 3183.10 \text{ Hz}$.

$$\frac{\omega_1 + \omega_2}{2} = 14142.14 \text{ rad s}^{-1}, \quad \sqrt{\omega_1 \omega_2} = 10000 \text{ rad s}^{-1}.$$

The means differ substantially. Here $R_c = 2\sqrt{L/C} = 200 \Omega = R$, so the free response is critically damped. The driven current still peaks at ω_0 .

Check: the two roots give net reactances $-R$ and $+R$, respectively, and $\Delta\omega = R/L$.

Chapter support: Chapter 11: [sec:ch11-selected-solutions](#); P9

Example 11.14: resonance current



Example 11.14. A series RLC circuit has $L = 25 \text{ mH}$, $C = 0.40 \mu\text{F}$ and $R = 12 \Omega$. It is driven by a source with rms voltage 10 V . Find the resonance frequency and the rms current at resonance.

Solution. With $L = 0.025 \text{ H}$ and $C = 0.40 \times 10^{-6} \text{ F}$,

$$\omega_0 = (LC)^{-1/2} = 1.00 \times 10^4 \text{ rad s}^{-1}, \quad f_0 = 1.59 \times 10^3 \text{ Hz}.$$

At resonance the reactive voltages cancel, so

$$I_{\text{rms}} = 10/12 = 0.833 \text{ A}.$$

Check: $X_L = \omega_0 L = 250 \Omega = X_C$, giving $Z = 12 \Omega$ and unity power factor. Both VI and $I^2 R$ give $P = 8.33 \text{ W}$.

Chapter support: Chapter 11: resonance-example; Example 11.14

Example 11.15: large reactive voltages



Example 11.15. For the resonant circuit of Example 11.14, find the rms voltage across the inductor and capacitor at resonance.

Restored context: $L = 25 \text{ mH}$, $C = 0.40 \mu\text{F}$, $R = 12 \Omega$, $V_{s,\text{rms}} = 10 \text{ V}$; $\omega_0 = 10^4 \text{ rad s}^{-1}$ (slide 47).

$$X_L = X_C = 250 \Omega, \quad I_{\text{rms}} = 10/12 \text{ A}.$$

$$V_{L,\text{rms}} = V_{C,\text{rms}} = (10/12)(250) = 208 \text{ V}.$$

The voltage phasors are equal and opposite; their sum is zero. The source supplies the 10 V resistor drop.

Check: $Q = 250/12 = 20.83$ and $QV_s = 208 \text{ V}$. This ideal prediction requires component limits to be checked at the stated operating frequency.

Chapter support: Chapter 11: large-reactive-voltage-example; Example 11.15

Transformer physics and ideal ratios



Changing core flux links two electrically isolated windings. If each turn links the same flux,

$$v_p = N_p \frac{d\Phi}{dt}, \quad v_s = N_s \frac{d\Phi}{dt}, \quad \frac{V_s}{V_p} = \frac{N_s}{N_p}.$$

Corresponding terminal polarities fix the signed voltages. The displayed ratio uses consistent peak or rms magnitudes.

Neglect winding resistance, leakage flux, core losses and magnetizing current. The load-current magnitudes then obey

$$\frac{I_s}{I_p} = \frac{N_p}{N_s}.$$

For a resistive load $V_p I_p = V_s I_s$ is real-power balance. For a reactive load the ideal phase angle is preserved, so average power is $VI \cos \phi$ on each side.

Chapter support: Chapter 11: transformers

The coupled and isolated transformer circuits

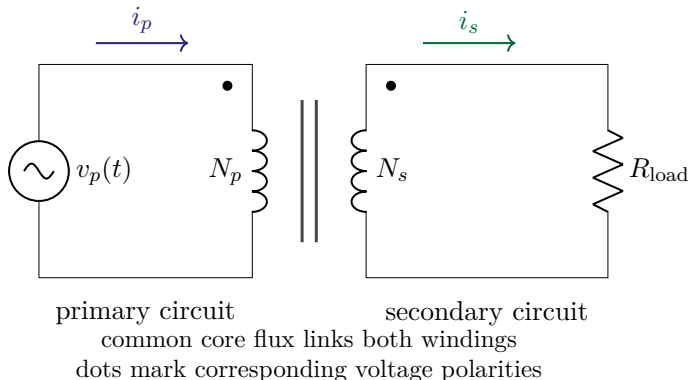


Figure 11.9: Ideal transformer with a resistive load. The two windings are magnetically coupled and electrically isolated; the parallel lines denote the core. Current enters the dotted primary terminal and leaves the dotted secondary terminal toward the load. The winding symbols are schematic and do not encode a numerical turns ratio.

Chapter support: Chapter 11: transformers

Example 11.18: step-down transformation



Example 11.18. An ideal transformer is used to step down from 230 V rms to 12 V rms. The primary has $N_p = 1150$ turns. Find the required number of secondary turns. If the secondary supplies 2.0 A rms, find the primary current.

Solution. The voltage ratio determines the turns:

$$N_s = N_p V_s / V_p = 1150(12/230) = 60.$$

The inverse current ratio gives

$$I_p = I_s (N_s / N_p) = 2.0(60/1150) = 0.104 \text{ A rms.}$$

For the resistive-load context used in the chapter, the power check is $V_s I_s = 24.0 \text{ W} = V_p I_p$ before rounding.

Reducing voltage increases the load-side current magnitude at conserved ideal power. A finite secondary voltage does not require a finite load current in the open-secondary limit.

Chapter support: Chapter 11: transformer-example; Example 11.18

Reflected load and the voltage–frequency constraint



The load resistance R_L seen through an ideal transformer becomes

$$R_{\text{in}} = \frac{V_p}{I_p} = \left(\frac{N_p}{N_s} \right)^2 R_L.$$

The physical resistor stays the same, but the source sees a changed voltage-to-current ratio.

For sinusoidal flux $\Phi = A_c B_{\text{max}} \sin(2\pi ft)$, with core area A_c and peak flux density B_{max} , Faraday's law gives

$$V_{p,\text{rms}} = \frac{2\pi}{\sqrt{2}} f N_p A_c B_{\text{max}} \simeq 4.44 f N_p A_c B_{\text{max}}.$$

At fixed voltage, reducing frequency increases peak flux density. A turns ratio alone cannot establish the acceptable frequency or power rating.

A steady nonzero primary voltage causes ideal flux to ramp. In a real core this can lead to saturation and excessive current.

Chapter support: Chapter 11: case-reflected-resistance

P14: receiving voltage and line loss



For a unity-power-factor load and a purely resistive line, $I = P_{\text{load}}/V_{\text{recv}}$. The line dissipates $I^2 R_{\text{line}}$, and the sending voltage must include the line drop.

P14. A single-phase unity-power-factor load receives 10 kW at 1000 V rms through an outgoing-and-return line resistance of 4.0Ω . Find line current, line loss and sending-end voltage. Repeat for a receiving-end voltage of 5000 V at the same delivered power.

Model. All voltages and currents are rms. The line resistance includes both conductors. The sending source is adjusted to maintain the specified receiving voltage. The two cases are solved on slide 54.

Chapter support: Chapter 11: [sec:ch11-practice-problems](#); P14

P14: two transmission-voltage calculations



At a 1000 V receiving voltage,

$$I_1 = 10000/1000 = 10.0 \text{ A}, \quad P_{\text{line},1} = 10.0^2(4.0) = 400 \text{ W},$$

$$V_{\text{send},1} = 1000 + (10.0)(4.0) = 1040 \text{ V}.$$

At a 5000 V receiving voltage,

$$I_2 = 2.00 \text{ A}, \quad P_{\text{line},2} = 2.00^2(4.0) = 16.0 \text{ W},$$

$$V_{\text{send},2} = 5000 + (2.00)(4.0) = 5008 \text{ V}.$$

A fivefold receiving voltage reduces loss by 25. Independent checks are $1040(10) = 10000 + 400$ and $5008(2) = 10000 + 16$, in watts.

Chapter support: Chapter 11: [sec:ch11-selected-solutions](#); P14

A current-monitor measurement

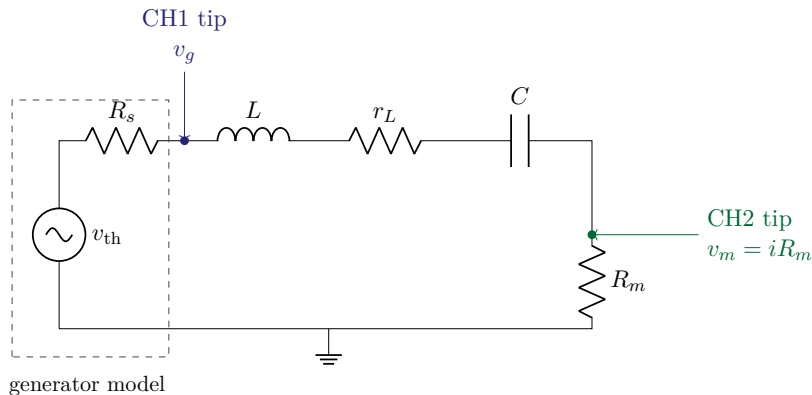


Figure 11.10: A series RLC measurement with a low-side current monitor. Both oscilloscope channel returns connect to the common bottom node. CH1 measures the loaded generator terminal voltage, not the ideal internal emf v_{th} . The monitor is part of the circuit and adds damping.

Chapter support: Chapter 11: lab-computational

Two measured inputs imply two response models

Let R_s be source resistance, r_L winding resistance and R_m the monitor. Write $X = \omega L - 1/(\omega C)$.

$$I_{\text{rms}} = \frac{V_{\text{th,rms}}}{\sqrt{(R_s + r_L + R_m)^2 + X^2}}.$$

This model holds the internal source amplitude fixed. The measured external-load ratio instead obeys

$$\frac{V_{m,\text{rms}}}{V_{g,\text{rms}}} = \frac{R_m}{\sqrt{(r_L + R_m)^2 + X^2}}.$$

The second expression excludes R_s because CH1 measures voltage *after* its drop. The monitor itself adds damping.

A bandwidth extracted from a response must use the resistance belonging to that response. Identifying the input voltage is part of identifying the physical model.

Chapter support: Chapter 11: lab-computational

E1: what the oscilloscope measures



E1. A generator has open-circuit voltage 2.0 V rms and $R_s = 50 \Omega$. It drives a series resonator with $R_m = 20 \Omega$ and coil resistance 10Ω . At resonance find current, terminal voltage and monitor voltage. Which resistance follows from the ratio V_m/V_g ?

Solution. With reactive cancellation at resonance,

$$I = \frac{2.0}{50 + 20 + 10} = 25.0 \text{ mA rms.}$$

$$V_g = I(20 + 10) = 0.750 \text{ V}, \quad V_m = I(20) = 0.500 \text{ V.}$$

Thus $V_m/V_g = 2/3$ and $R_m/(V_m/V_g) = 30 \Omega$, the external loss resistance. Using $V_m/V_{th} = 1/4$ would instead infer 80Ω .

Check: the internal 1.25 V drop plus the 0.750 V terminal voltage equals 2.0 V . All values use rms conventions.

Chapter support: Chapter 11: [sec:ch11-selected-solutions](#); E1

Inference from resonance and signed timing



A known effective capacitance and measured series current peak give

$$L = [(2\pi f_0)^2 C]^{-1}, \quad R = 2\pi L \Delta f.$$

The resistance belongs to the chosen amplitude model. Stray capacitance and source termination can bias the inference.

A noninductive monitor has $v_m = R_m i$. If its peak occurs $\Delta t > 0$ after the reference-voltage peak, current lags:

$$\phi = 360^\circ \frac{\Delta t}{T}.$$

For independent small standard uncertainties,

$$u_\phi = |\phi| \sqrt{(u_{\Delta t}/\Delta t)^2 + (u_T/T)^2}.$$

The formula follows by differentiating the time-ratio relation and adding independent variance contributions.

Chapter support: Chapter 11: case-inductance-from-resonance

E4: signed phase with uncertainty



E4. A current-monitor trace peaks (0.100 ± 0.002) ms after the terminal-voltage trace. The period is (0.800 ± 0.004) ms. Treat uncertainties as independent standard uncertainties. Find ϕ , its uncertainty, and the sign of net reactance.

Solution. A later current peak gives positive ϕ , so

$$\phi = 360^\circ(0.100/0.800) = +45.0^\circ,$$

$$u_\phi = 45.0^\circ \sqrt{(0.002/0.100)^2 + (0.004/0.800)^2} = 0.928^\circ.$$

Report $\phi = (45.0 \pm 0.9)^\circ$ with standard uncertainty. The net reactance is positive: current lags and the load is net inductive.

Check: the delay is one eighth of a period, which is 45° . Equal time units cancel in the ratio.

Chapter support: Chapter 11: [sec:ch11-selected-solutions](#); E4

Core synthesis and the bridge to fields



- **Storage and loss:** LC fields exchange conserved energy. Resistance gives $dU/dt = -Ri^2$ in a free loop.
- **Driving and phase:** impedance determines the steady amplitude and angle. Rms power is $VI \cos \phi = I^2 R$ for the ideal series model.
- **Resonance and measurement:** specify the source, topology and observable. Bandwidth and peak height reveal loss only after defining the measured input.

The advanced block develops complex algebra, distinct resonance peaks, filter loading and ringdown inference.

Bridge: Chapter 12 develops Maxwell's equations and magnetic matter; Chapter 13 develops electromagnetic waves. Their fields extend the lumped circuit description into space.

Further practice: the chapter's unselected problems and bibliography provide reading beyond this 70-page lecture.

Chapter support: Chapter 11: [sec:ch11-bridge](#)

Advanced: complex steady-state amplitudes



Represent a real sinusoid as $v(t) = \Re[\tilde{V}e^{i\omega t}]$. A complex amplitude records magnitude and phase; differentiation multiplies it by $i\omega$.

Insert $q = \Re[\tilde{q}e^{i\omega t}]$ into the driven charge equation:

$$(-\omega^2 L + i\omega R + 1/C)\tilde{q} = \tilde{V}.$$

Since $\tilde{I} = i\omega\tilde{q}$,

$$\tilde{V} = \underbrace{[R + i(\omega L - 1/\omega C)]}_{\mathcal{Z}} \tilde{I}.$$

Thus $\mathcal{Z}_R = R$, $\mathcal{Z}_L = i\omega L$ and $\mathcal{Z}_C = -i/(\omega C)$. The magnitude is $Z = |\mathcal{Z}|$ and the phase is $\arg \mathcal{Z} = \phi$.

These are peak phasors unless explicitly labelled rms. Taking the real part recovers the measurable waveform.

Chapter support: Chapter 11: driven-ode-derivation

Example 11.28: complex impedance



Example 11.28. A series circuit has $R = 100\ \Omega$, $L = 50\text{ mH}$ and $C = 2.0\ \mu\text{F}$ at angular frequency $\omega = 2000\text{ rad s}^{-1}$. Find the complex impedance.

Solution. The two positive reactance magnitudes are

$$X_L = (2000)(0.050) = 100\ \Omega, \quad X_C = [(2000)(2.0 \times 10^{-6})]^{-1} = 250\ \Omega.$$

$$\mathcal{Z} = (100 - i150)\ \Omega, \quad |\mathcal{Z}| = \sqrt{100^2 + 150^2} = 180.3\ \Omega,$$

$$\arg \mathcal{Z} = \tan^{-1}(-150/100) = -56.3^\circ.$$

Current leads the source in this capacitive response. Check: $|\mathcal{Z}| \cos \phi = 100\ \Omega$ and $|\mathcal{Z}| \sin \phi = -150\ \Omega$, using unrounded values.

Chapter support: Chapter 11: complex-impedance-example; Example 11.28

Resonance depends on the observable



Let $u = \omega/\omega_0$ and $D = (1 - u^2)^2 + u^2/Q^2$. The series model gives

$$V_{C0}/V_0 = D^{-1/2}, \quad V_{L0}/V_0 = u^2 D^{-1/2}.$$

For the capacitor peak, differentiate D :

$$D' = 2u[2(u^2 - 1) + Q^{-2}] = 0.$$

A nonzero peak requires $Q > 1/\sqrt{2}$. For the inductor, differentiating u^4/D gives $4D - uD' = 0$, or $4(1 - u^2) + 2u^2/Q^2 = 0$. The results are

$$\omega_C = \omega_0 \sqrt{1 - 1/(2Q^2)}, \quad \omega_L = \frac{\omega_0}{\sqrt{1 - 1/(2Q^2)}}.$$

The free ringdown frequency is $\omega_d = \omega_0 \sqrt{1 - 1/(4Q^2)}$. Current peaks at ω_0 . For $1/2 < Q \leq 1/\sqrt{2}$, free ringing exists while no capacitor-voltage peak exists.

Chapter support: Chapter 11: resonance-types

Example 11.29: four characteristic frequencies

Example 11.29. Let $L = 10 \text{ mH}$, $C = 1.0 \mu\text{F}$ and $R = 50 \Omega$. Distinguish the undamped frequency, free ringdown frequency and the two reactive-voltage peak frequencies.

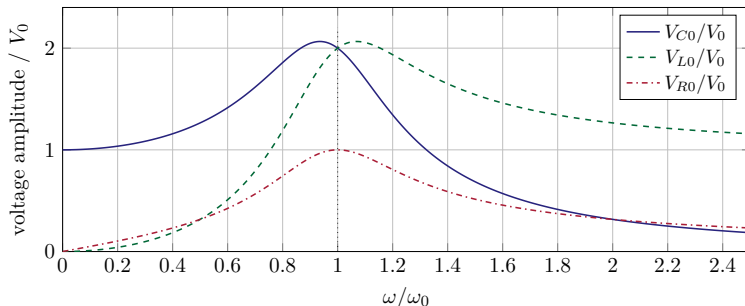


Figure 11.11: Different driven observables for the same series RLC circuit with $Q = 2$. The resistor voltage, and therefore current, peaks at $\omega = \omega_0$; the capacitor and inductor voltages peak below and above it. All curves are calculated from the same impedance model.

Chapter support: Chapter 11: four-frequencies; Example 11.29

Example 11.29: frequencies and checks

With $L = 0.010 \text{ H}$, $C = 1.0 \times 10^{-6} \text{ F}$ and $R = 50 \Omega$,

$$\omega_0 = 10000 \text{ rad s}^{-1}, \quad Q = \omega_0 L / R = 2.00, \quad \alpha = 2500 \text{ s}^{-1}.$$

The distinct frequencies are

$$\omega_d = \sqrt{10000^2 - 2500^2} = 9682 \text{ rad s}^{-1},$$

$$\omega_C = 10000 \sqrt{1 - 1/8} = 9354 \text{ rad s}^{-1},$$

$$\omega_L = 10000 / \sqrt{1 - 1/8} = 10690 \text{ rad s}^{-1}.$$

The undamped scale and driven current peak are both 10000 rad s^{-1} . Check: $\omega_C \omega_L = \omega_0^2$, and substitution into the corresponding amplitude derivatives gives zero.

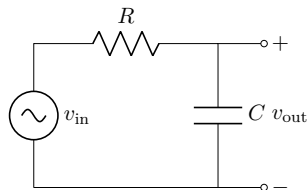
A capacitor-voltage scan and a current-monitor scan therefore locate different peaks in this same damped circuit.

Chapter support: Chapter 11: four-frequencies; Example 11.29

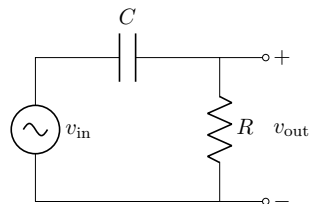
RC transfer functions and output topology

A transfer function is $H = \tilde{V}_{\text{out}}/\tilde{V}_{\text{in}}$. The unloaded complex divider gives

$$H_C = \frac{1}{1 + i\omega RC}, \quad H_R = \frac{i\omega RC}{1 + i\omega RC}.$$



low-pass: output across C



high-pass: output across R

Figure 11.13: Two RC filters using the same components but different placement relative to the common return. The output is defined between the indicated terminals. A finite load across those terminals changes the voltage divider.

For capacitor output, $|H_C| = [1 + (\omega RC)^2]^{-1/2}$ and $\arg H_C = -\tan^{-1}(\omega RC)$. Low frequencies pass; high frequencies attenuate.

Chapter support: Chapter 11: filters

Advanced challenge A2: a loaded RC divider



A finite R_{load} sits in parallel with C . Its impedance obeys

$$\mathcal{Z}_b^{-1} = R_{\text{load}}^{-1} + i\omega C.$$

Using $H = \mathcal{Z}_b / (R + \mathcal{Z}_b)$ gives

$$H = \frac{G_0}{1 + i\omega\tau}, \quad G_0 = \frac{R_{\text{load}}}{R + R_{\text{load}}}, \quad \tau = C(R \parallel R_{\text{load}}).$$

The relative cutoff satisfies $|H| = G_0/\sqrt{2}$, so $\omega_c\tau = 1$.

A2. An RC low-pass filter with $R = 10\text{ k}\Omega$, $C = 100\text{ nF}$ is loaded by $R_{\text{load}} = 20\text{ k}\Omega$. The source resistance is negligible. Find the low-frequency gain, cutoff relative to that gain, and gain and phase at 100 Hz .

Attempt: calculate all four quantities before reading slide 68.

Chapter support: Chapter 11: filter-design; A2

A2: gain, cutoff and phase



At low frequency the capacitor is open, so

$$G_0 = \frac{20}{10 + 20} = \frac{2}{3}, \quad R \parallel R_{\text{load}} = 6.667 \text{ k}\Omega.$$

$$\tau = (6.667 \times 10^3)(100 \times 10^{-9}) = 0.6667 \text{ ms}, \quad f_c = \frac{1}{2\pi\tau} = 238.7 \text{ Hz}.$$

At 100 Hz,

$$|H| = \frac{2/3}{\sqrt{1 + (100/238.7)^2}} = 0.6149,$$

$$\arg H = -\tan^{-1}(100/238.7) = -22.73^\circ.$$

Check: as $\omega \rightarrow 0$, $H \rightarrow 2/3$; as $R_{\text{load}} \rightarrow \infty$, $H \rightarrow 1/(1 + i\omega RC)$. The unloaded cutoff 159.2 Hz is not the loaded cutoff.

Chapter support: Chapter 11: [sec:ch11-selected-solutions](#); A2

Ringdown: exact logarithmic decrement



Same-polarity peaks are separated by $T_d = 2\pi/\omega_d$. For the underdamped linear model,

$$\delta = \frac{1}{N} \ln \frac{|A_n|}{|A_{n+N}|} = \alpha T_d, \quad \frac{A_{n+N}}{A_n} = e^{-\alpha N T_d}.$$

From $\omega_0^2 = \omega_d^2 + \alpha^2$ and $Q = \omega_0/(2\alpha)$,

$$Q = \sqrt{(\pi/\delta)^2 + 1/4}, \quad \Delta f = \frac{2\alpha}{2\pi}.$$

Example 11.36. A low-voltage resonator has same-polarity capacitor-voltage peaks decreasing from 4.0 V to 1.0 V over 12 damped periods. The measured period is 0.200 ms. Infer Q , α , ω_0 and the series-model bandwidth.

Its inference and a measured-amplitude check conclude on slide 70.

Chapter support: Chapter 11: case-ringdown; Example 11.36

Example 11.36: loss inferred from ringdown



The amplitude ratio is 4 over 12 damped periods, and $T_d = 2.00 \times 10^{-4}$ s. Therefore

$$\delta = \ln 4/12 = 0.1155245,$$

$$\alpha = \delta/T_d = 577.6 \text{ s}^{-1},$$

$$\omega_d = 2\pi/T_d = 31415.93 \text{ rad s}^{-1},$$

$$\omega_0 = \sqrt{\omega_d^2 + \alpha^2} = 31421.24 \text{ rad s}^{-1},$$

$$Q = \omega_0/(2\alpha) = 27.20, \quad \Delta f = \alpha/\pi = 183.9 \text{ Hz}.$$

Independent check: $e^{-\alpha(12T_d)} = 0.2500$, reproducing the measured peak ratio. The exact Q relation agrees with the inferred frequency and decay rate.

A driven scan should agree only for the same terminated circuit. Removing the generator can change the damping resistance. Fit several peaks away from the noise floor to test a single exponential envelope.

Chapter support: Chapter 11: case-ringdown-example; Example 11.36

