

Maxwell's Equations and Magnetism in Matter

Physics II - Chapter 12

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Why Maxwell completes the field picture

Earlier laws describe charge, steady currents and induction. A charging capacitor exposes the missing link: a changing electric flux must contribute to magnetic circulation. The completed theory connects circuits to waves and to magnetic matter.

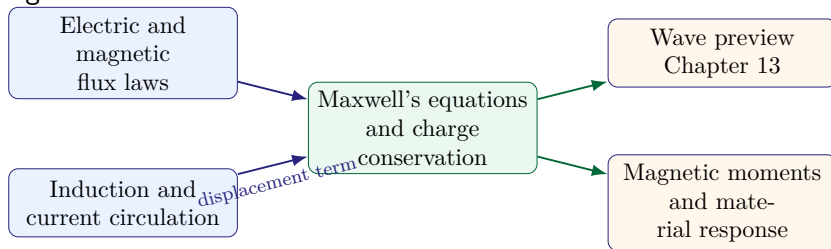


Figure 12.1: Roadmap: flux and circulation laws are unified before material response and the wave connection are developed. The detailed wave physics belongs to Chapter 13.

Chapter support: Revised Chapter 12; sec:ch12:motivation-roadmap.

Learning route and prerequisites



- 1 **Geometry and laws.** Distinguish normal flux from tangential circulation, then complete the four integral laws.
- 2 **Models and checks.** Use capacitor symmetry to calculate fields and use conservation to check surface independence.
- 3 **Magnetic matter.** Average dipole moments, define the internal response, and interpret domains and cycle loss.
- 4 **Applications and enrichment.** Infer fields from measurements, then connect local equations to interfaces, shape and energy.

Tools used in the core

Dot and cross products; the right-hand rule; area and line integrals; time derivatives; capacitor charge and voltage; Faraday's and steady Ampere's laws. Vector calculus is introduced in the advanced block.

Chapter support: Revised Chapter 12; sec:ch12:prior-knowledge.

Learning goals



After the core lecture, you should be able to:

- explain each Maxwell equation through its geometry, source terms and orientation;
- derive displacement current and capacitor magnetic fields, checking symmetry and time scales;
- use charge conservation to explain why two spanning surfaces give the same circulation;
- calculate dipole torque and energy, and distinguish B , H , M and linear susceptibility;
- interpret domains, remanence, coercivity and hysteresis energy without confusing their definitions;
- estimate a core flux swing and reconstruct field or loss from calibrated measurements.

Advanced enrichment adds local equations, dielectric current phase, interface conditions, demagnetization and electromagnetic energy balance.

Chapter support: Revised Chapter 12; [sec:ch12:motivation-roadmap](#).

Flux and circulation



Flux measures a field component *normal* to a surface. Circulation measures its component *tangent* to a closed path. They probe different properties of the same vector field.

$$\Phi_F = \int_S \mathbf{F} \cdot d\mathbf{A}, \quad \Gamma_F = \oint_C \mathbf{F} \cdot d\boldsymbol{\ell}.$$

For a uniform field on a flat area, $\Phi_F = FA \cos \theta$, with θ measured from the normal, not from the plane. For curved surfaces or nonuniform fields, integrate the local contributions.

Geometry before algebra

Identify S or C , orient it, identify the source, and only then ask whether symmetry permits taking a field component outside the integral.

Electric flux has units $\text{V m} = \text{N m}^2/\text{C}$; magnetic flux has units $\text{Wb} = \text{T m}^2$.
Electric circulation has units volts.

Chapter support: Revised Chapter 12; sec:ch12:flux-circulation.

Orientations and signs

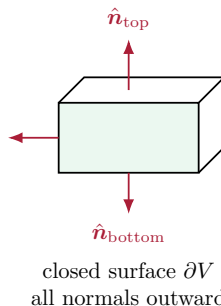
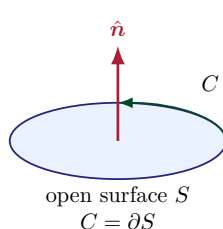


Figure 12.2: Orientation conventions. The open surface uses a normal tied to the positive loop direction. The closed surface uses outward normals. Arrows on a mathematical surface do not represent motion of the surface.

Choose counterclockwise traversal as viewed from $+z$ for an xy loop: its normal is $+\hat{z}$. Reversing both traversal and normal reverses both sides of a loop law, leaving the physical prediction unchanged.

Chapter support: Revised Chapter 12; sec:ch12:flux-circulation.

Example 12.1: uniform fields



Example 12.1: Flux and circulation for simple uniform fields

A uniform electric field $\mathbf{E} = E_0 \hat{\mathbf{z}}$ passes through a square surface of side a . The surface normal makes an angle θ with $\hat{\mathbf{z}}$. A separate circular loop of radius R lies in the xy -plane while a uniform magnetic field $\mathbf{B} = B_0 \hat{\mathbf{x}}$ is present. Find the electric flux through the square and the magnetic circulation around the circle.

Flux calculation. The surface area is a^2 , and its normal is at the stated angle to the electric field:

$$\Phi_E = \mathbf{E} \cdot \mathbf{A} = E_0 a^2 \cos \theta.$$

The flux is maximum for $\theta = 0$, zero for $\theta = 90^\circ$, and negative for a normal pointing against the field. Its units are electric field times area. The separate magnetic line integral is evaluated next.

Chapter support: Revised Chapter 12; ch12-flux-circulation.

Example 12.1: zero circulation



For the circular loop in Example 12.1, use angle ϕ in the xy plane and counterclockwise traversal:

$$d\ell = R(-\sin \phi \hat{x} + \cos \phi \hat{y}) d\phi.$$

Since $\mathbf{B} = B_0 \hat{x}$ is constant,

$$\oint_C \mathbf{B} \cdot d\ell = -B_0 R \int_0^{2\pi} \sin \phi d\phi = 0.$$

Independent check. For any closed path, $\oint d\ell = \mathbf{0}$, so a constant field has circulation $\mathbf{B} \cdot \mathbf{0} = 0$.

What the zero tells us

The field is nonzero on the loop. Zero circulation is an integral constraint, not proof that $\mathbf{B}$ vanishes. Nonzero open-surface flux also does not imply nonzero circulation.

Chapter support: Revised Chapter 12; ch12-flux-circulation.

Gauss's electric law: net sources



$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0}$$

Here $\epsilon_0 \simeq 8.854 \times 10^{-12} \text{ F/m}$ is vacuum permittivity. Outward normals fix the sign: positive net q_{enc} gives outward flux and negative net charge gives inward flux. Add enclosed charges algebraically. External charges change the local field but contribute zero *net* flux. A Gaussian surface is a mathematical choice.

Exact law, conditional shortcut

The law requires no symmetry. Finding a local field from it requires symmetry that fixes its direction and variation on the surface.

Answered check. Does zero flux imply zero field? No: a nonzero uniform field has canceling entering and leaving flux.

Chapter support: Revised Chapter 12; sec:ch12:gauss-electric.

Example 12.2: enclosed charge



Example 12.2: Electric flux through a closed surface

A closed surface encloses three small charged beads with charges $+2.0 \text{ nC}$, -5.0 nC and $+1.0 \text{ nC}$. A fourth bead of charge $+20 \text{ nC}$ is outside the surface. Find the net electric flux through the closed surface.

Source selection. Only enclosed charges enter Gauss's right-hand side:

$$q_{\text{enc}} = (2.0 - 5.0 + 1.0) \text{ nC} = -2.0 \times 10^{-9} \text{ C}.$$

$$\Phi_E = \frac{-2.0 \times 10^{-9}}{8.854 \times 10^{-12}} = -2.26 \times 10^2 \text{ N m}^2/\text{C}.$$

The net flux is inward; this does not require an inward field at every point. Moving the external bead or deforming the surface without crossing a charge changes no enclosed charge, so the net flux stays the same. The units are $\text{C}/(\text{F}/\text{m}) = \text{V m}$.

Chapter support: Revised Chapter 12; [ch12-gauss-electric-example](#).

Gauss's magnetic law: continuous flux

$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$$

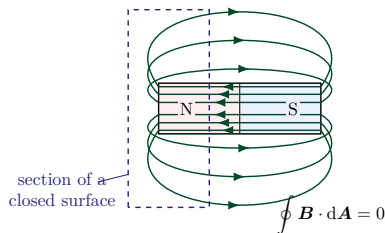


Figure 12.3: Continuous magnetic field loops, shown schematically. Outside the magnet the direction is N to S, and inside it is S to N. A closed Gaussian surface surrounding the N end also cuts the internal return flux. Its total flux is zero; the mathematical surface creates no new physical pole.

Cutting an ordinary magnet produces two dipoles.

Chapter support: Revised Chapter 12; sec:ch12:gauss-magnetic.

Faraday's law and fixed paths



$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi_B}{dt}, \quad \Phi_B = \int_S \mathbf{B} \cdot d\mathbf{A}.$$

Here C and S are fixed, with matched orientations. The minus sign expresses Lenz's law. A changing flux gives nonconservative electric circulation even without a wire; a conducting loop detects it. Current additionally depends on the circuit impedance. For a moving or deforming material loop, charges also feel a magnetic force:

$$\mathcal{E} = \oint_{C(t)} (\mathbf{E} + \mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{A}.$$

Answered check. A loop moving in a static nonuniform field can have motional emf from $\mathbf{u} \times \mathbf{B}$. Its changing geometric flux does not imply electric circulation around a fixed path.

Chapter support: Revised Chapter 12; sec:ch12:faraday-law.

Circular induction: inside and outside



Assume axial symmetry and an approximately uniform $B_z(t)$ within radius R . Choose a counterclockwise circular path of radius r . Symmetry makes E_ϕ constant on it.

$$2\pi r E_\phi = -\dot{\Phi}_B, \quad \Phi_B = \begin{cases} B_z \pi r^2, & r \leq R, \\ B_z \pi R^2, & r \geq R. \end{cases}$$

$$E_\phi = -\frac{\dot{B}_z}{2} \begin{cases} r, & r \leq R, \\ R^2/r, & r \geq R. \end{cases}$$

Both branches give $-R\dot{B}_z/2$ at the edge; the axis value is zero. Outside the ideal field region, E_ϕ can be nonzero because the loop still encloses changing flux. The sharp field region is a long-solenoid approximation. A finite apparatus has fringing and return flux; magnetic field lines do not end at R .

Chapter support: Revised Chapter 12; eq:ch12-induced-e-piecewise.

Practice A2: decreasing magnetic flux



Student practice A2

† A uniform field B_z is confined approximately to a cylindrical region of radius $R = 4.0 \text{ cm}$ and decreases at 0.40 T/s . Assume axial symmetry. Find the signed counterclockwise electric circulation and E_ϕ at $r = 2.0 \text{ cm}$ and $r = 8.0 \text{ cm}$. Check continuity at R and explain the direction.

Model and sign convention

The specified signed field rate is $\dot{B}_z = -0.40 \text{ T/s}$. Choose positive circulation counterclockwise as viewed from $+z$, with normal $+\hat{z}$.

Use the area actually containing changing flux, not the entire loop area when $r > R$. Evaluate the circulation first, divide by $2\pi r$ to obtain E_ϕ , then check the common edge limit. The complete solution follows. The manuscript dagger marks a problem with a detailed source solution.

Chapter support: Revised Chapter 12; A2.

Practice A2: circulation and direction



Since $\dot{B}_z < 0$, Faraday's minus sign gives positive circulation:

$$\Gamma_E = -\pi \min(r^2, R^2) \dot{B}_z, \quad E_\phi = \frac{\Gamma_E}{2\pi r}.$$

r	Γ_E	E_ϕ
0.020 m	$5.03 \times 10^{-4} \text{ V}$	$+4.00 \times 10^{-3} \text{ V/m}$
0.080 m	$2.01 \times 10^{-3} \text{ V}$	$+4.00 \times 10^{-3} \text{ V/m}$

At $R = 0.040 \text{ m}$, both branches give $E_\phi = +8.00 \times 10^{-3} \text{ V/m}$. The circulation is counterclockwise; a test-loop current would oppose the negative change in signed $+z$ flux. Equal fields at the two requested radii are a coincidence of these data, not equal circulations. Check units: $r\dot{B}$ is $\text{m T/s} = \text{V/m}$. Substituting E_ϕ back into $2\pi r E_\phi$ recovers each flux derivative.

Chapter support: Revised Chapter 12; A2.

The charging-capacitor inconsistency

Steady Ampere's law, $\oint_C \mathbf{B} \cdot d\mathbf{\ell} = \mu_0 I_S$, fails during charge accumulation: changing the spanning surface cannot change one loop's circulation.

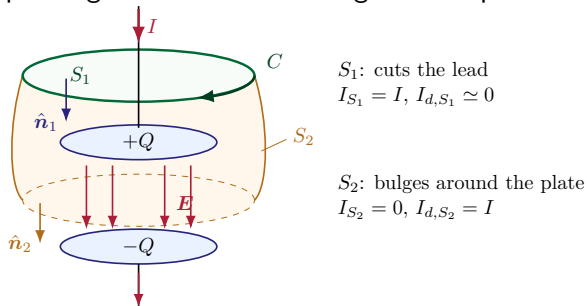


Figure 12.4: Two surfaces sharing the same oriented rim C . The flat disk S_1 is pierced by the lead. The bag-shaped S_2 passes around the positive plate and closes through the gap without crossing a conductor. In the ideal quasistatic model both yield $I_S + I_{d,S} = I$. The field arrows and surface normals point downward through the gap.

Chapter support: Revised Chapter 12; sec:ch12-ampere-before-maxwell.

Displacement current in a vacuum gap



$$I_d = \epsilon_0 \frac{d\Phi_E}{dt}$$

This term measures changing electric flux. It contributes to magnetic circulation but is not charge transport across empty space. Since $\epsilon_0 \Phi_E$ has units coulombs, I_d has units amperes. For an ideal vacuum parallel-plate capacitor, choose the gap normal along the increasing field and neglect fringing:

$$E = \frac{Q}{\epsilon_0 A}, \quad \Phi_E = EA = \frac{Q}{\epsilon_0}, \quad I_d = \frac{dQ}{dt} = I.$$

Answered check. A disconnected capacitor can retain a nonzero field with $I_d = 0$: its flux is constant. The displacement term depends on a derivative, not on the field magnitude alone.

Chapter support: Revised Chapter 12; sec:ch12-displacement-current.

Example 12.4: rate of field growth



Example 12.4: Displacement current in a charging capacitor

A parallel-plate capacitor with circular plates is being charged by a constant current $I = 25 \text{ mA}$. Edge effects are neglected. What is the displacement current between the plates? If the plate area is $A = 40 \text{ cm}^2$, what is dE/dt between the plates?

Model. The full-gap surface encloses the entire uniform field, so

$$I_d = I = 25 \text{ mA} = 2.5 \times 10^{-2} \text{ A}.$$

Convert $40 \text{ cm}^2 = 4.0 \times 10^{-3} \text{ m}^2$. Then

$$\dot{E} = \frac{I_d}{\epsilon_0 A} = \frac{2.5 \times 10^{-2}}{(8.854 \times 10^{-12})(4.0 \times 10^{-3})} = 7.1 \times 10^{11} \text{ V m}^{-1} \text{ s}^{-1}.$$

Multiplying this rate by $\epsilon_0 A$ recovers the charging current. The large rate reflects small capacitance per area in vacuum, not electrons crossing the gap. Its direction is the chosen increasing field direction.

Chapter support: Revised Chapter 12; [ch12-displacement-current-example](#).

Ampere-Maxwell law: one surface



$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \left(I_S + \varepsilon_0 \frac{d\Phi_{E,S}}{dt} \right)$$

Evaluate both terms on the *same* oriented surface. $\mu_0 \simeq 4\pi \times 10^{-7} \text{ H/m}$ is vacuum permeability. I_S includes all charge-transport currents, including bound currents in matter.

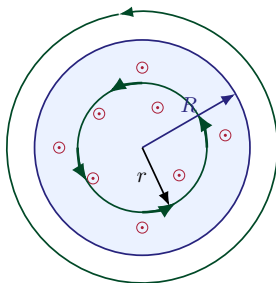
The two capacitor evaluations

Through the wire: $I_{S_1} = I$ and $I_{d,S_1} \simeq 0$. Through the vacuum gap: $I_{S_2} = 0$ and $I_{d,S_2} = I$. Each gives $\mu_0 I$; adding terms from different surfaces would incorrectly give $2\mu_0 I$.

Changing magnetic flux supplies electric circulation; changing electric flux supplies magnetic circulation. These coupled laws allow waves, but they also describe driven near fields such as a capacitor gap.

Chapter support: Revised Chapter 12; sec:ch12-ampere-maxwell.

Capacitor field direction



view from $+z$

$$\partial E_z / \partial t > 0$$

$\odot$: increasing field toward viewer

$B_\phi > 0$: counterclockwise

plate outline and concentric Amperian loops

Figure 12.5: End view of the ideal circular capacitor during increasing positive E_z . The dot symbols denote the direction of $\partial \mathbf{E} / \partial t$, and the green arrows show $\mathbf{B}$. Reversing the rate of change reverses the circulation.

Take $+z$ from the positively charging plate toward the negative plate. Positive $I = \dot{Q}$ increases E_z ; negative current reverses the signed B_ϕ , not its magnitude.

Chapter support: Revised Chapter 12; sec:ch12-capacitor-profile.

Deriving the capacitor magnetic profile



In the uniform vacuum-gap model, $J_{d,z} = I/(\pi R^2)$. For $r < R$, the enclosed current is Ir^2/R^2 ; for $r > R$, it is the full I .

$$2\pi r B_\phi = \mu_0 I_{d,\text{enc}},$$

$$B_\phi = \frac{\mu_0 I}{2\pi} \begin{cases} r/R^2, & 0 \leq r \leq R, \\ 1/r, & r \geq R. \end{cases}$$

Inside, enclosed area grows as r^2 while path length grows as r , so $B_\phi \propto r$. Outside, enclosed current stops growing and $B_\phi \propto 1/r$. At the axis, symmetry requires $B_\phi = 0$. Both branches meet at $B_R = \mu_0 I/(2\pi R)$, the maximum in this ideal model. Substitution into the circulation law independently checks both branches.

Chapter support: Revised Chapter 12; eq:ch12-capacitor-profile.

The radial profile and model limits

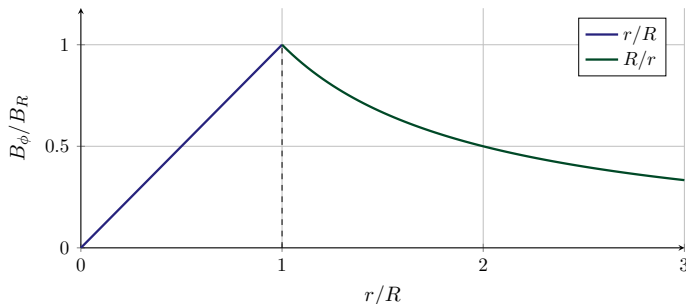


Figure 12.6: Calculated normalized profile for a positive current, with $B_R = \mu_0 I / (2\pi R)$. This is the uniform-gap, cylindrical, quasistatic model, not a full far-field solution for a finite capacitor and its return circuit.

The model requires $d \ll R$, uniform gap field, symmetric feed and return, and $\omega L_{\text{sys}}/c \ll 1$. A nearby return can invalidate $2\pi r B_\phi$ while the integral law remains true. Rapid drive needs spatial corrections; the finite circuit has a different distant field.

Chapter support: Revised Chapter 12; sec:ch12-capacitor-profile.

Practice A5: the enclosed-current fraction



Student practice A5

† A circular vacuum capacitor of radius $R = 5.0$ cm is charged with a positive displacement current of 0.40 A along $+z$. Under the ideal symmetric approximation, find B_ϕ at $r = 1.0$, 5.0 and 10.0 cm. Locate the maximum field and explain why applying the full enclosed current at the first radius is wrong.

Calculation route

At each radius, determine the enclosed displacement current before using $B_\phi = \mu_0 I_{d,\text{enc}} / (2\pi r)$. All requested components are positive for the specified $+z$ current.

The minimum information is the loop radius, the source radius and the uniform-current model. A total current alone does not determine the interior field if the radial current density is unspecified. The complete numerical solution follows.

Chapter support: Revised Chapter 12; A5.

Practice A5: profile and maximum



Use $I_{d,\text{enc}} = I \min(r^2/R^2, 1)$, with $I = 0.40 \text{ A}$ and $R = 0.050 \text{ m}$.

r	$I_{d,\text{enc}}$	B_ϕ
0.010 m	0.016 A	$+0.32 \mu\text{T}$
0.050 m	0.40 A	$+1.60 \mu\text{T}$
0.100 m	0.40 A	$+0.80 \mu\text{T}$

The maximum is at $r = R$: the interior branch rises and the exterior branch falls. At 1.0 cm, using the full 0.40 A would overestimate the field by $(R/r)^2 = 25$. **Checks.** Both formulas give $1.60 \mu\text{T}$ at R . Each table entry satisfies $2\pi r B_\phi = \mu_0 I_{d,\text{enc}}$. The field direction is counterclockwise as viewed from $+z$; the units of $\mu_0 I/r$ are tesla.

Chapter support: Revised Chapter 12; A5.

Voltage drive fixes current and phase



For a vacuum capacitor with prescribed voltage and constant capacitance,

$$C = \frac{\epsilon_0 \pi R^2}{d}, \quad Q = CV, \quad I_d = C\dot{V}.$$

If $V = V_0 \sin \omega t$, then $I_d = \omega C V_0 \cos \omega t$. Within the symmetric quasistatic model,

$$B_\phi(r, t) = \frac{\mu_0 r}{2\pi R^2} \omega C V_0 \cos \omega t \quad (r < R).$$

Current and B_ϕ lead V and $E_z = V/d$ by a quarter cycle. At voltage extrema the electric field is large but instantaneously stationary, so the displacement current and this magnetic field vanish.

A near field is not a plane wave

The capacitor fields do not generally obey $E_0 = cB_0$ or share the plane-wave phase relation. Geometry and drive determine their amplitudes and phases.

Chapter support: Revised Chapter 12; ch12-voltage-capacitor.

Problem A6: voltage-driven capacitor



Problem A6

† Let $V(t) = V_0 \sin \omega t$ across an ideal vacuum capacitor with $R = 3.0$ cm, $d = 0.50$ mm, $V_0 = 20$ V and $f = 100$ kHz. Find C , the current amplitude, and the magnetic-field amplitude at $r = R/2$. State their phases relative to V . For apparatus size $L_{\text{sys}} = 0.10$ m, estimate $\omega L_{\text{sys}}/c$.

Coordinates and model. Let positive voltage correspond to $+E_z$; take positive B_ϕ counterclockwise as viewed from $+z$. Use a symmetric feed, negligible fringing and a vacuum gap. **Reasoning route.** The plate area fixes C . Differentiating $Q = CV$ gives the current amplitude and phase. The enclosed-current formula gives the field at $R/2$. Finally compare propagation across L_{sys} with the drive period scale.

Chapter support: Revised Chapter 12; A6.

Problem A6: amplitudes and validity



Convert $R = 0.030$ m, $d = 5.0 \times 10^{-4}$ m, and $\omega = 2\pi \times 10^5$ s $^{-1}$:

$$C = \frac{(8.854 \times 10^{-12})\pi(0.030)^2}{5.0 \times 10^{-4}} = 5.01 \times 10^{-11} \text{ F},$$

$$I_{\text{pk}} = \omega CV_0 = 6.29 \times 10^{-4} \text{ A} = 0.629 \text{ mA},$$

$$B_{\text{pk}}(R/2) = \frac{\mu_0 I_{\text{pk}}}{4\pi R} = 2.10 \times 10^{-9} \text{ T} = 2.10 \text{ nT}.$$

Both current and signed field vary as $\cos \omega t$, leading V by $\pi/2$. They are maximum at a positive-going voltage zero crossing and vanish at voltage extrema. The scale $\omega L_{\text{sys}}/c = 2.09 \times 10^{-4} \ll 1$ supports quasistatic propagation, while $d/R = 0.0167$ supports the plate model. Neither check proves a symmetric return path. As a second check, $B_{\text{pk}} = \mu_0 \epsilon_0 r \omega V_0 / (2d)$ gives the same field.

Chapter support: Revised Chapter 12; A6.

Nonuniform field rates need integration

The area fraction r^2/R^2 holds only for uniform displacement-current density. For an axisymmetric vacuum field rate, an annulus contributes area $dA = 2\pi s ds$:

$$I_{d,\text{enc}}(r) = 2\pi\epsilon_0 \int_0^r \partial_t E_z(s) s ds,$$
$$B_\phi(r) = \frac{\mu_0\epsilon_0}{r} \int_0^r \partial_t E_z(s) s ds.$$

The second relation also requires magnetic symmetry and no charge current crossing the disk. A total current fixes the exterior circulation in this ideal geometry but does not specify the interior profile.

An inverse check within the same model

Differentiating rB_ϕ gives $\partial_t E_z = (\mu_0\epsilon_0 r)^{-1} \partial_r(rB_\phi)$. This is a local reconstruction; derivatives amplify measurement noise and need a regular limit at the axis.

Chapter support: Revised Chapter 12; sec:ch12-nonuniform-capacitor.

Example 12.7: integrating a radial profile



Example 12.7: The same total current with a different radial profile

In a quasistatic axisymmetric vacuum-field model, suppose that the instantaneous field rate is $\partial_t E_z = a(1 - s^2/R^2)$ for $0 \leq s \leq R$ and negligible beyond R . Derive the enclosed displacement current and magnetic field. Compare them with a uniform profile carrying the same total I_d .

Integrate for $r \leq R$. Weight the rate by annular area:

$$I_{d,\text{enc}} = 2\pi\epsilon_0 a \int_0^r \left(s - \frac{s^3}{R^2} \right) ds = 2\pi\epsilon_0 a \left(\frac{r^2}{2} - \frac{r^4}{4R^2} \right),$$

$$I_d = I_{d,\text{enc}}(R) = \frac{\pi\epsilon_0 a R^2}{2},$$

$$B_\phi = \frac{\mu_0 I_d}{2\pi R} (2x - x^3), \quad x = r/R.$$

For $r > R$, the enclosed current is the full I_d , so $B_\phi = \mu_0 I_d / (2\pi r)$. Both branches give $\mu_0 I_d / (2\pi R)$ at the edge.

Chapter support: Revised Chapter 12; ch12-profile-example.

Example 12.7: comparison and checks



For the same total I_d , the interior fractions are

$$\frac{I_{d,\text{enc}}}{I_d} = 2x^2 - x^4 \quad (\text{nonuniform}), \quad x^2 \quad (\text{uniform}).$$

At $x = 1/2$, they are $7/16$ and $1/4$, so the nonuniform magnetic field is larger by $7/4$. The concentrated central source matters even though the total current is unchanged. Differentiating the normalized field gives $2 - 3x^2$. Its maximum is at $x = \sqrt{2/3} \simeq 0.816$, inside the edge. At $x = 1$, its slope is -1 , matching the derivative of the exterior $1/x$ branch. **Independent check.** Differentiating $rB_\phi = \mu_0\epsilon_0 a(r^2/2 - r^4/4R^2)$ recovers the prescribed $a(1 - r^2/R^2)$ through the inverse formula. This instantaneous test profile is not claimed to be a flat equipotential-plate field or a complete time-dependent Maxwell solution without compatible sources and boundary data.

Chapter support: Revised Chapter 12; ch12-profile-example.

The four integral Maxwell equations



For fixed paths and surfaces, with all charge and charge-transport sources explicit:

$$\oint_{\partial V} \mathbf{E} \cdot d\mathbf{A} = q_{\text{enc}}/\epsilon_0$$
 electric source flux,

$$\oint_{\partial V} \mathbf{B} \cdot d\mathbf{A} = 0$$
 continuous magnetic flux,

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -d\Phi_B/dt$$
 electric circulation,

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0(I_S + \epsilon_0 d\Phi_E/dt)$$
 magnetic circulation.

The first two use closed surfaces; the last two use loops and spanning surfaces. Total sources include bound charge and current in matter. The Lorentz law $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ tells how fields act on charge. Source dynamics, material laws and initial or boundary data complete the physical model.

Chapter support: Revised Chapter 12; sec:ch12-maxwell-integral.

Charge conservation and surface independence



For a fixed volume, positive outward charge current reduces enclosed charge.
 $\mathbf{J}$ is charge-current density in A/m^2 :

$$\oint_{\partial V} \mathbf{J} \cdot d\mathbf{A} = -\dot{Q}_V.$$

Differentiate electric Gauss law for that volume and add the results:

$$\oint_{\partial V} (\mathbf{J} + \epsilon_0 \partial_t \mathbf{E}) \cdot d\mathbf{A} = -\dot{Q}_V + \dot{Q}_V = 0.$$

For two surfaces with the same oriented rim, S_1 and the oppositely oriented S_2 form a closed surface. Consequently,

$$\boxed{I_{S_1} + I_{d,S_1} = I_{S_2} + I_{d,S_2}}.$$

The separate terms change but their sum cannot. This proves compatibility with charge conservation; conservation alone does not uniquely fix every field equation. Experiment also establishes the theory.

Chapter support: Revised Chapter 12; sec:ch12-charge-conservation-maxwell.

The wave connection and its dimensions



In source-free vacuum the two time-dependent circulation laws remain coupled. They admit propagating field disturbances with

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \simeq 3.00 \times 10^8 \text{ m/s.}$$

Example 12.9: Checking the dimensions of the electromagnetic wave speed

Show that $1/\sqrt{\mu_0 \epsilon_0}$ has the dimension of speed.

Using $[\mu_0] = \text{N/A}^2$, $[\epsilon_0] = \text{C}^2/(\text{N m}^2)$ and $\text{A} = \text{C/s}$,

$$[\mu_0 \epsilon_0] = \frac{\text{C}^2}{\text{A}^2 \text{m}^2} = \frac{\text{s}^2}{\text{m}^2}, \quad [(\mu_0 \epsilon_0)^{-1/2}] = \text{m/s.}$$

This confirms dimensions, not the wave equation itself. The fields are simultaneous parts of one solution; they do not alternately switch one another on. The detailed wave development follows in Chapter 13.

Chapter support: Revised Chapter 12; sec:ch12-em-wave-preview.

Magnetic moment, torque and energy



For N equivalent planar turns, the area vector points along the current's right-hand-rule normal:

$$\mathbf{m} = N I \mathbf{A}, \quad \boldsymbol{\tau} = \mathbf{m} \times \mathbf{B}, \quad U = -\mathbf{m} \cdot \mathbf{B}.$$

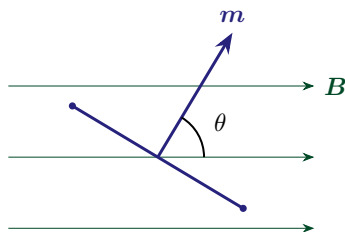
The loop area is its enclosed area, not the wire cross-section. Magnetic moment has units A m^2 ; mB has units joules or newton metres. For a spatially uniform field,

$$\mathbf{F}_{\text{net}} = I \left(\oint d\boldsymbol{\ell} \right) \times \mathbf{B} = \mathbf{0}.$$

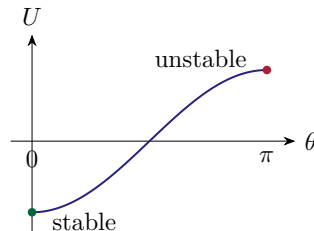
Equal opposite wire forces can still form a couple. Their moment gives $\tau = mB \sin \theta$. The signed torque reducing the angle is $\tau_\theta = -mB \sin \theta = -dU/d\theta$, which checks the energy formula. Atomic moments require quantum physics; the current-loop language is useful without treating spin as a rotating tiny sphere.

Chapter support: Revised Chapter 12; sec:ch12-magnetic-dipoles.

Stable and unstable dipole orientations



Loop seen edge-on; normal along m



$$U = -mB \cos \theta$$

Figure 12.7: The magnetic moment is normal to the loop plane. The orientation energy distinguishes stable alignment from unstable anti-alignment even though both have zero torque. The right-hand plot is schematic with a common positive scale mB .

Both $\theta = 0$ and π give zero torque, but alignment minimizes energy. Rotation can occur with zero net force. Translation of a fixed permanent moment instead requires a gradient: $\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$.

Chapter support: Revised Chapter 12; fig:ch12-dipole-energy.

Example 12.10: torque and alignment energy



Example 12.10: Torque on a small current loop

A rectangular loop has area $A = 3.0 \times 10^{-4} \text{ m}^2$ and carries current $I = 0.20 \text{ A}$. Its magnetic moment makes an angle 40° with a uniform field $B = 0.50 \text{ T}$. Find the torque magnitude and the energy difference between this orientation and perfect alignment.

Model and requested difference

This is one loop in an externally imposed uniform field, with the stated current fixed. The requested energy difference is $U(40^\circ) - U(0)$, not the work done by the magnetic torque during the reverse motion.

First find $m = IA$. Use the cross product for the torque magnitude and use the scalar product to compare the two energies. A positive energy difference means that the tilted orientation stores more potential energy than alignment.

Chapter support: Revised Chapter 12; [ch12-torque-loop](#).

Example 12.10: solution and stability check



The loop moment is $m = (0.20)(3.0 \times 10^{-4}) = 6.0 \times 10^{-5} \text{ A m}^2$.

$$\tau = mB \sin 40^\circ = (6.0 \times 10^{-5})(0.50) \sin 40^\circ = 1.9 \times 10^{-5} \text{ N m},$$

$$\begin{aligned}\Delta U &= U(40^\circ) - U(0) = mB(1 - \cos 40^\circ) \\ &= (6.0 \times 10^{-5})(0.50)(1 - \cos 40^\circ) = 7.0 \times 10^{-6} \text{ J}.\end{aligned}$$

The torque tends to reduce θ ; its vector direction is $\mathbf{m} \times \mathbf{B}$. Without a specified orientation plane, the statement fixes a magnitude rather than a unique Cartesian torque component. **Independent check.** Integrating the quasistatic external torque from 0 to 40° gives $\int mB \sin \theta d\theta = mB(1 - \cos 40^\circ)$. Both torque and energy difference vanish at alignment, as required.

Chapter support: Revised Chapter 12; [ch12-torque-loop](#).

Magnetization and internal material fields



Magnetization is the vector dipole moment per volume, averaged over many atoms:

$$\mathbf{M} = \frac{\Delta \mathbf{m}}{\Delta V}, \quad \mathbf{m}_{\text{sample}} = V \mathbf{M} \quad \text{for uniform } \mathbf{M}.$$

Large individual moments may cancel in that average. Define the auxiliary field inside the specimen by

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \quad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}).$$

$\mathbf{H}$ and $\mathbf{M}$ have units A/m; $\mathbf{B}$ has units tesla. Internal $\mathbf{H}$ is not generally the field present before inserting the sample. For a linear, isotropic, reversible response at fixed temperature,

$$\mathbf{M} = \chi_m \mathbf{H}, \quad \mu_r = 1 + \chi_m, \quad \mathbf{B} = \mu_0 \mu_r \mathbf{H}.$$

A scalar constant susceptibility describes a specified regime; it is not a general ferromagnetic history law.

Chapter support: Revised Chapter 12; sec:ch12-material-definition.

Three principal magnetic responses

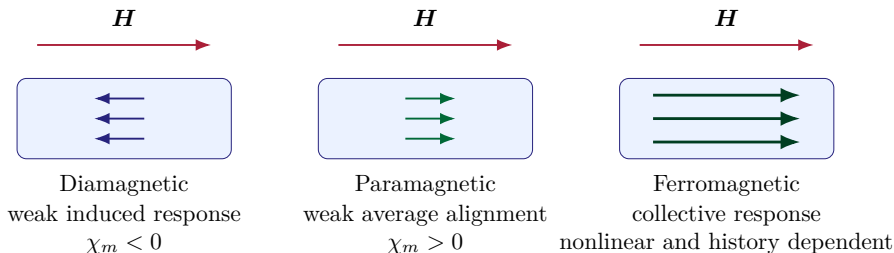


Figure 12.8: Qualitative directions of net magnetization. The arrows inside each sample represent the average $\mathbf{M}$, not a literal alignment of every atomic moment. The simple dia- and paramagnetic pictures assume weak, isotropic response; ferromagnetic response cannot generally be represented by a single constant χ_m . The sign refers to $\mathbf{M}$ relative to internal $\mathbf{H}$. The classes describe different microscopic responses. Opposing ordered sublattices define antiferromagnets; unequal opposing moments give ferrimagnets a net response. Many core ferrites are ferrimagnetic; their resistivity separately affects eddy loss.

Chapter support: Revised Chapter 12; fig:ch12-material-response.

Diamagnetism, thermal disorder and Curie law



Diamagnetism. The field induces an opposing electronic response, giving small negative χ_m for ordinary weak diamagnets. It can persist in a constant applied field; it is not a transient eddy current. **Paramagnetism.** Permanent microscopic moments partially align with the field, while thermal disorder opposes ordering. In an appropriate weak-field temperature range,

$$\chi_m = \frac{C}{T}, \quad x = \frac{mB}{k_B T}.$$

C is the Curie constant, T is absolute temperature and k_B is Boltzmann's constant. The parallel–antiparallel energy separation is $2mB$; $x \ll 1$ signals weak alignment of independent moments. Curie's law is a model, not a universal law for every paramagnet. Interactions, conduction electrons and saturation can change the temperature dependence; extrapolating C/T to zero temperature is unjustified.

Chapter support: Revised Chapter 12; sec:ch12-paramagnetism.

Example 12.14: the atomic energy scale



Example 12.14: An atomic energy-scale check

Take $m = 9.27 \times 10^{-24} \text{ A m}^2$, approximately one Bohr magneton, $B = 1.0 \text{ T}$ and $T = 300 \text{ K}$. Estimate x and the parallel–antiparallel energy difference.

With the manuscript value $k_B \simeq 1.381 \times 10^{-23} \text{ J/K}$,

$$k_B T = (1.381 \times 10^{-23})(300) = 4.14 \times 10^{-21} \text{ J},$$

$$2mB = 2(9.27 \times 10^{-24})(1.0) = 1.85 \times 10^{-23} \text{ J},$$

$$x = \frac{9.27 \times 10^{-24}}{4.143 \times 10^{-21}} = 2.24 \times 10^{-3}.$$

The separation is only $2x = 0.00448$ of the thermal scale. Independent moments are therefore weakly polarized, despite their nonzero individual moments. Strong ferromagnetic response requires collective interactions. The units $\text{A m}^2 \text{ T} = \text{J}$ check the comparison. Cooling at fixed B increases x , but the independent-moment model must be reassessed before extrapolation.

Chapter support: Revised Chapter 12; ch12-thermal-example.

Collective order and domain populations

Below the Curie temperature, exchange favors local collective order. Domain moments can cancel macroscopically; a field redistributes their volumes or directions.

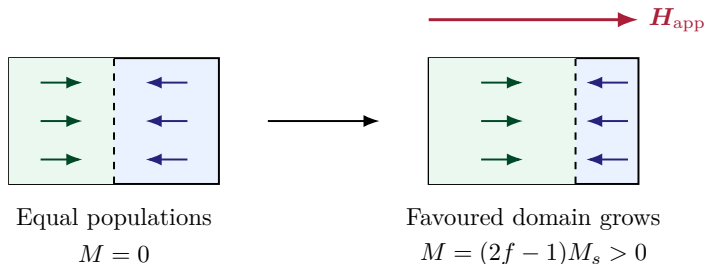


Figure 12.10: Two-domain volume model, with moments along $\pm x$. A field can move a wall so that the favourable volume fraction f grows. The drawing illustrates domain-wall motion; real domain patterns also involve rotation, anisotropy and pinning.

Chapter support: Revised Chapter 12; sec:ch12-ferromagnetism.

Example 12.15: a volume-weighted average



Example 12.15: Magnetization from partial domain alignment

A small ferromagnetic sample consists, in a simplified model, of two domain populations. A fraction 0.70 of the volume has magnetization magnitude $M_s = 8.0 \times 10^5$ A/m along $+x$, and the remaining fraction has the same magnitude along $-x$. Estimate the net magnetization.

Average signed vector components over the two volume fractions:

$$M_x = fM_s + (1 - f)(-M_s) = (2f - 1)M_s.$$

For the stated populations,

$$M_x = (2 \times 0.70 - 1)(8.0 \times 10^5) = +3.2 \times 10^5 \text{ A/m}.$$

The net moment points along $+x$. Changing the favorable volume fraction changes the average without changing each domain's M_s . For $f = 1/2$ the average vanishes; for $f = 1$ it reaches M_s . These limits independently check the averaging formula.

Chapter support: Revised Chapter 12; ch12-extra-domain-average.

Hysteresis: axes, intercepts and history

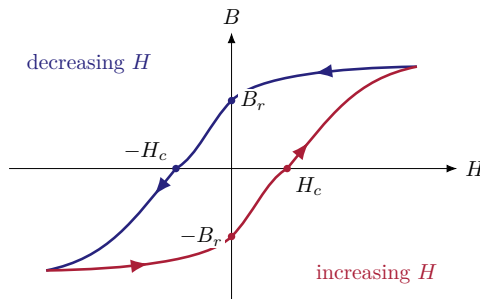


Figure 12.9: Schematic closed B - H hysteresis loop with its physical traversal. Remanent flux densities lie on the vertical axis; coercive fields defined by $B = 0$ lie on the horizontal axis. The curve is illustrative, not measured data or a quantitative constitutive model.

At $H = 0$ read B_r ; at $B = 0$ read coercivity. Since $B = \mu_0(H + M)$, $B = 0$ permits nonzero M . Saturated M still allows B to increase with H ; the schematic's flattened ends are illustrative.

Chapter support: Revised Chapter 12; sec:ch12-hysteresis-reading.

Why the loop area is energy density



For a thin uniform toroid, let core volume be $V_c = A\ell$ and the winding have N turns. Neglect winding loss and leakage. With passive terminal voltage,

$$H\ell = NI, \quad v = NAB, \quad \frac{v\ell}{V_c} = H\dot{B}.$$

Integrating magnetic power over a closed cycle gives

$$\boxed{w_{\text{loss}} = \oint H dB}, \quad \boxed{P = fV_c w_{\text{loss}}}.$$

The physical traversal of a passive cycle gives positive work. Units are $(\text{A/m})\text{T} = \text{J/m}^3$. For an M - H loop, $\oint H dB = \mu_0 \oint H dM$, so an extra μ_0 is required. A quasistatic area describes hysteretic work. A dynamic measured area may also contain eddy and other rate-dependent losses; adding those again can double count. Linear scaling with f assumes an unchanged cycle.

Chapter support: Revised Chapter 12; eq:ch12-hysteresis-energy.

Problem A12: quasistatic and dynamic loss



Problem A12

† A quasistatic B – H loop has area 80 J/m^3 per cycle in a core of volume $2.5 \times 10^{-4} \text{ m}^3$. Estimate the hysteresis power at 60 Hz if this area remains representative. A dynamic loop measured at that frequency instead has area 110 J/m^3 . What loss does its area imply, and why must the two results not simply be added?

Comparison model. The quasistatic estimate assumes the stated area remains representative at the operating frequency. The dynamic area is measured at that frequency and describes its corresponding magnetic cycle loss. The area must be multiplied by the same core volume and cycles per second in each calculation. Comparing the two powers is meaningful only under comparable flux swing, waveform, temperature and calibration. The solution follows.

Chapter support: Revised Chapter 12; A12.

Problem A12: loss and double counting



The quasistatic energy per core cycle and its power estimate are

$$W_{\text{qs}} = 80(2.5 \times 10^{-4}) = 0.020 \text{ J}, \quad P_{\text{hyst}} \simeq 60(0.020) = 1.20 \text{ W}.$$

The dynamic loop gives

$$W_{\text{dyn}} = 110(2.5 \times 10^{-4}) = 0.0275 \text{ J}, \quad P_{\text{dyn}} = 60(0.0275) = 1.65 \text{ W}.$$

Do not add 1.20 and 1.65 W: the latter is the cycle loss already represented by the dynamic measurement. Under comparable conditions their 0.45 W difference estimates extra dynamic loss, but does not uniquely identify its mechanism. Winding copper loss generally lies outside this core-loop area and needs separate accounting. The dimensional check is $(\text{J}/\text{m}^3)(\text{m}^3)(\text{s}^{-1}) = \text{W}$.

Chapter support: Revised Chapter 12; A12.

Core flux swing is set by volt-seconds



All turns are assumed to link the same nearly uniform core flux. The applied magnetic winding voltage balances the induced emf:

$$v = N A \dot{B}, \quad B(t) - B(t_0) = \frac{1}{NA} \int_{t_0}^t v(t') dt'.$$

For $B = B_{\text{pk}} \sin \omega t$, the voltage is $NA\omega B_{\text{pk}} \cos \omega t$, giving

$$V_{\text{rms}} = \frac{2\pi}{\sqrt{2}} f N A B_{\text{pk}} \simeq 4.44 f N A B_{\text{pk}}.$$

The coefficient is specific to a sinusoid. At fixed rms voltage, reducing frequency increases the flux swing. Permeability determines the required magnetizing current, not the flux fixed by this voltage integral. Nonzero net volt-seconds per cycle add a flux offset. A practical saturation criterion concerns a material curve and operating conditions; it is not an absolute Maxwell-law ceiling on B .

Chapter support: Revised Chapter 12; sec:ch12-engineering-connections.

Problem E1: operating limit and flux drift



Problem E1

† A transformer winding has $N = 400$ turns and a nearly uniform core area $A = 2.0 \times 10^{-4} \text{ m}^2$. Its sinusoidal voltage is $12 V_{\text{rms}}$. Neglect winding drop. Find B_{pk} at 50 Hz and the minimum frequency consistent with a chosen 1.0 T limit. If a switching error produces a net 0.020 V s per cycle, calculate the flux-density drift per cycle before nonlinear effects alter the waveform.

Assumptions

Use the given sinusoidal rms voltage across the magnetic part of the winding. All turns link the effective area; leakage, winding drops and waveform distortion are neglected. The chosen 1.0 T is a design limit for this problem.

Find the alternating peak field from the sinusoidal relation. For the switching error, integrate the signed voltage over one full cycle rather than inserting it into a sinusoidal amplitude formula.

Chapter support: Revised Chapter 12; E1.

Problem E1: frequency and drift solution



With $NA = 400(2.0 \times 10^{-4}) = 0.080 \text{ m}^2$,

$$B_{\text{pk}} = \frac{12}{4.44(50)(0.080)} = 0.676 \text{ T},$$

$$f_{\text{min}} = \frac{12}{4.44(0.080)(1.0)} = 33.8 \text{ Hz},$$

$$\Delta B_{\text{cycle}} = \frac{0.020 \text{ V s}}{0.080 \text{ m}^2} = +0.25 \text{ T}.$$

The frequency inequality is $f \geq f_{\text{min}}$ within the model. Substituting 33.8 Hz into the voltage relation returns approximately 1.0 T, an independent threshold check. The positive drift adds an offset to the alternating swing each cycle before nonlinear effects alter the waveform. It can cause saturation despite an acceptable first-cycle swing. The exact sinusoidal coefficient is $\sqrt{2}\pi$; 4.44 is its rounded value. Units of volt-seconds per area are tesla.

Chapter support: Revised Chapter 12; E1.

Core loss and magnetic shielding



Thin insulated laminations interrupt conducting eddy-current paths; a narrow quasistatic loop reduces irreversible magnetic work. These address distinct losses. For a sheet of thickness a in a nearly uniform sinusoidal parallel field, Faraday's law gives $E_y = -x\dot{B}_z$. Averaging Ohmic heating over thickness and a cycle yields

$$\langle p_e \rangle = \frac{\sigma_c a^2 \omega^2 B_{pk}^2}{24}.$$

At fixed B_{pk} and frequency, halving thickness quarters this ideal loss. The model assumes uniform penetration and negligible field perturbation; it fails when skin effects become significant. A high-permeability shield diverts flux around a region. Seams, open ends and a narrow saturated section can dominate performance; permeability alone supplies no shielding factor. Conductive screening instead depends on time-varying induced currents. Soft cores favor easy low-loss reversal; permanent magnets need resistance to demagnetization.

Chapter support: Revised Chapter 12; sec:ch12-engineering-connections.

Inductive and capacitive pickup



Magnetic coupling

$$\mathcal{E} = -A \cos \theta \dot{B}.$$

A stationary wiring loop links changing magnetic flux. Reduce its enclosed area by keeping outgoing and return conductors close. Twisting helps cancel slowly varying external flux over successive twists. The induced current or noise voltage also depends on the receiving impedance; emf alone does not specify either.

Chapter support: Revised Chapter 12; [ch12-coupling](#).

Electric coupling

$$I_s = C_s \dot{V}.$$

Stray capacitance between insulated conductors permits a changing-field contribution without conduction through the gap. The receiver voltage depends on its impedance and on loading. Increase separation or use a suitably connected screen to reduce C_s . Reducing loop area alone does not eliminate this coupling.

Search coils infer changes in flux



With matched winding traversal, normal and lead polarity, a stationary search coil in a uniform field gives

$$v_s = -N_s A_s \cos \theta \dot{B}.$$

For a normal sinusoidal field $B = B_0 \sin(2\pi ft)$,

$$v_{s,0} = 2\pi f N_s A_s B_0, \quad B_0 = \frac{v_{s,0}}{2\pi f N_s A_s}.$$

This uses amplitudes consistently, not an instantaneous signed voltage. High instrument impedance and negligible loading are assumed. Integrating voltage recovers a flux change, not an unknown constant field. For independent small standard uncertainties and an exact counted turn number,

$$\left(\frac{u_B}{B_0}\right)^2 = \left(\frac{u_v}{v_{s,0}}\right)^2 + \left(\frac{u_f}{f}\right)^2 + \left(\frac{u_A}{A_s}\right)^2.$$

This follows by differentiating the logarithm of the amplitude relation and adding independent variances.

Chapter support: Revised Chapter 12; eq:ch12-search-coil.

Example 12.19: calibrated field inference



Example 12.19: Inferring a field with a search coil

A coil has $N_s = 200$ turns, $A_s = (1.50 \pm 0.03) \times 10^{-4} \text{ m}^2$ and its normal is aligned with a sinusoidal field at $f = (60.0 \pm 0.1) \text{ Hz}$. The measured voltage amplitude is $(0.226 \pm 0.004) \text{ V}$. Treat the quoted uncertainties as independent standard uncertainties and neglect loading. Find B_0 .

Interpretation of the data. The area, frequency and voltage values are supplied hypothetical measurement inputs with standard uncertainties. The integer turn count is treated as exact. Neglect loading as specified and use a normal, uniform field. Compute the amplitude first. Combine fractional uncertainties in quadrature, then round the central value to the precision justified by its uncertainty. The solution also checks the bias of an unrecognized tilt.

Chapter support: Revised Chapter 12; ch12-search-measurement.

Example 12.19: uncertainty and alignment



The inferred field amplitude is

$$B_0 = \frac{0.226}{2\pi(60.0)(200)(1.50 \times 10^{-4})} = 1.998 \times 10^{-2} \text{ T.}$$

$$\frac{u_B}{B_0} = \sqrt{\left(\frac{0.004}{0.226}\right)^2 + \left(\frac{0.1}{60.0}\right)^2 + \left(\frac{0.03}{1.50}\right)^2} = 0.0268.$$

Thus $u_B = 0.535 \text{ mT}$. The rounded result is $B_0 = (20.0 \pm 0.5) \text{ mT}$. An unknown 5° tilt lowers the inferred amplitude by $1 - \cos 5^\circ = 0.38\%$. This is geometric bias, not repeatable random noise. At exact alignment the first derivative of $\cos \theta$ vanishes; away from it, angle uncertainty contributes $|\tan \theta| u_\theta$ in radians. Substituting the unrounded B_0 into $2\pi f N_s A_s B_0$ returns 0.226 V . The result presumes the stated loading and spatial-uniformity model.

Chapter support: Revised Chapter 12; [ch12-search-measurement](#).

Reconstructing a toroidal B-H loop

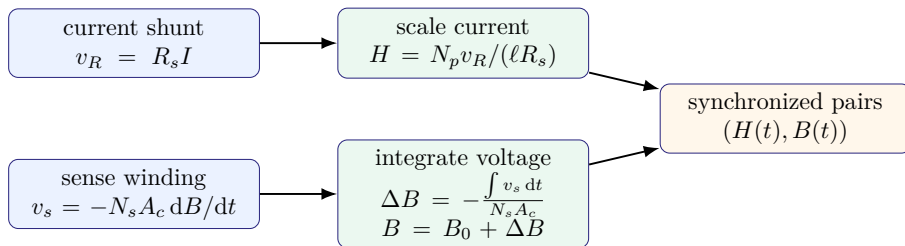


Figure 12.11: Measurement logic for a toroidal loop. A primary current determines the approximate H , while integrated secondary emf determines changes in B . The two signals require calibrated gain, known polarity and a common time base.

Use a thin, nearly uniform toroid; retain the time order of synchronized H and B samples. An unknown constant B_0 changes remanence intercepts but not the differential dB in a closed-loop area. A voltage offset causes integrator drift; a relative channel delay can create a false area even for a reversible sample. Calibrate gain, polarity and timing before interpreting loss. Cycle closure is a diagnostic only for an actually periodic physical state.

Chapter support: Revised Chapter 12; sec:ch12-loop-measurement.

Problem E6: a time-ordered loss cycle



Problem E6

† One cycle has the ordered (H, B) pairs: $(100, 0.4)$, $(-100, 0.4)$, $(-100, -0.4)$, $(100, -0.4)$, $(100, 0.4)$, in A/m and tesla. Use the trapezoidal line integral to estimate the dissipated energy density. Calculate the average power in $2.0 \times 10^{-5} \text{ m}^3$ at 50 Hz. Explain why sorting these points by H destroys the required information.

Numerical rule. Close the path explicitly and preserve its recorded order:

$$w \simeq \sum_j \frac{H_j + H_{j+1}}{2} (B_{j+1} - B_j).$$

This rule integrates $H dB$ along the path rather than along a sorted graph. The supplied rectangle is synthetic geometry for a calculation check, not measured material data. The complete edge-by-edge calculation follows.

Chapter support: Revised Chapter 12; E6.

Problem E6: cycle integral and power



The horizontal edges have $\Delta B = 0$, so they contribute no work. The two vertical edges give

$$w_2 = (-100)(-0.4 - 0.4) = +80 \text{ J/m}^3,$$

$$w_4 = (+100)(0.4 - (-0.4)) = +80 \text{ J/m}^3,$$

$$w = 160 \text{ J/m}^3,$$

$$P = 50(160)(2.0 \times 10^{-5}) = 0.160 \text{ W}.$$

Independent geometric check. The rectangle area $4H_c B_r$ equals 160 J/m^3 . Sorting by H mixes the increasing and decreasing branches and discards the traversal that distinguishes them. Reversing the complete path reverses the signed work; the given passive traversal is positive. Taking an absolute value cannot replace a sensor-polarity check.

Chapter support: Revised Chapter 12; E6.

Core synthesis and the bridge to Chapter 13



Closed surfaces measure flux; fixed loops measure circulation. The displacement term makes magnetic circulation consistent with charge accumulation and surface independence. Magnetic matter adds moment averaging, internal fields and domain history; calibrated measurements test that response.

A complete field calculation

State geometry and orientation, select the source bookkeeping, justify symmetry, declare the material law, and check units, limits and conservation. A correct formula outside its model can give a wrong prediction.

Further study. Read the manuscript's pulse-integration convergence, integrator-drift and incremental-inductance sections. **Bridge to Chapter 13.** Coupled field dynamics provide the foundation for wave propagation, polarization and energy transport. The advanced block now develops local equations and material reasoning.

Chapter support: Revised Chapter 12; [sec:ch12-summary](#).

Advanced: local field equations



New mathematics. Divergence is outward flux per unit volume; curl is oriented circulation per unit area. The divergence and Stokes theorems state

$$\oint_{\partial V} \mathbf{F} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{F} dV, \quad \oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A}.$$

If the integral laws hold for arbitrary small volumes and fixed surfaces in a smooth region, they give

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\epsilon_0, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\partial_t \mathbf{B}, & \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial_t \mathbf{E}. \end{aligned}$$

Here ρ and $\mathbf{J}$ are total charge and charge-current densities. Divergence of a curl is zero, so the final equation gives $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$: positive outward current divergence lowers charge density.

Chapter support: Revised Chapter 12; sec:ch12-local-maxwell.

Advanced: free and bound source accounting



Polarization $\mathbf{P}$ is electric dipole moment per volume, in C/m^2 . Together with magnetization it represents bound sources:

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad \mathbf{J}_b = \partial_t \mathbf{P} + \nabla \times \mathbf{M}.$$

Define $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$. Substituting the free-plus-bound sources and collecting terms gives

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \partial_t \mathbf{D}.$$

Magnetic Gauss law and Faraday's law retain their $\mathbf{B}$ and $\mathbf{E}$ forms. Material displacement current $\partial_t \mathbf{D}$ already includes polarization current; adding it again double counts. Constitutive laws supply additional physics. A simple homogeneous, isotropic, instantaneous model uses $\mathbf{D} = \varepsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$. Nonlinearity, dispersion and hysteresis require more information.

Chapter support: Revised Chapter 12; sec:ch12-macroscopic-maxwell.

Advanced: current phase and relaxation



For constant real σ and ε in a homogeneous linear medium, choose $E(t) = E_0 \cos \omega t$:

$$J_c = \sigma E_0 \cos \omega t, \quad J_D = -\varepsilon \omega E_0 \sin \omega t.$$

The terms are in quadrature. Their total amplitude is $E_0 \sqrt{\sigma^2 + (\varepsilon \omega)^2}$. For positive coefficients the phase advance is $\delta = \tan^{-1}(\varepsilon \omega / \sigma)$. Free-charge continuity and Gauss law give

$$\partial_t \rho_f = -\nabla \cdot (\sigma \mathbf{E}) = -\frac{\sigma}{\varepsilon} \rho_f, \quad \rho_f(t) = \rho_f(0) e^{-t/\tau_q}, \quad \tau_q = \frac{\varepsilon}{\sigma}.$$

The amplitude ratio is $\omega \tau_q$. Bulk excess charge relaxes by flowing elsewhere, often to boundaries; total conserved charge does not disappear. Spatially varying coefficients or dielectric dispersion invalidate this simple decay model.

Chapter support: Revised Chapter 12; eq:ch12-relaxation.

Advanced challenge Ch2: a lossy dielectric



Advanced attempt/solve challenge Ch2

† A weakly conducting linear dielectric has $\epsilon_r = 4.0$, $\sigma = 2.0 \times 10^{-4} \text{ S/m}$ and is driven at $\omega = 1.0 \times 10^7 \text{ rad/s}$ with field amplitude $E_0 = 100 \text{ V/m}$. Find the conduction and material displacement current-density amplitudes, their ratio, the total current-density amplitude and its phase relative to E . Find the bulk charge-relaxation time and compare it with $1/\omega$.

Model and convention. Set $E(t) = E_0 \cos \omega t$. Use the homogeneous, constant real ϵ and σ model, with no additional dispersive dielectric loss. All requested current densities are amplitudes except when their time dependence is stated explicitly. Compute $\epsilon = 4\epsilon_0$, find the two component amplitudes, and combine them by their phase rather than by arithmetic addition. The same coefficients determine the independent bulk relaxation-time check. The complete solution follows.

Chapter support: Revised Chapter 12; Ch2.

Advanced challenge Ch2: full solution



With $\varepsilon = 4(8.854 \times 10^{-12}) = 3.5416 \times 10^{-11} \text{ F/m}$,

$$J_{c,0} = \sigma E_0 = 0.0200 \text{ A/m}^2,$$

$$J_{D,0} = \varepsilon \omega E_0 = 0.0354 \text{ A/m}^2,$$

$$J_{D,0}/J_{c,0} = 1.77, \quad J_0 = \sqrt{0.0200^2 + 0.035416^2} = 0.0407 \text{ A/m}^2,$$

$$\delta = \tan^{-1}(1.7708) = 60.5^\circ.$$

Thus $J_{\text{total}} = J_0 \cos(\omega t + \delta)$ leads E . Adding amplitudes would incorrectly give 0.0554 A/m^2 . The relaxation time is $\tau_q = \varepsilon/\sigma = 1.77 \times 10^{-7} \text{ s}$, while $1/\omega = 1.00 \times 10^{-7} \text{ s}$. Their ratio 1.77 checks the current-amplitude ratio independently. Both terms must be retained; the Ohmic term dissipates energy while this ideal real-permittivity term is reversible.

Chapter support: Revised Chapter 12; Ch2.

Advanced: fields across an interface



Let $\hat{n}$ point from medium 1 to medium 2. A pillbox shrinking across the interface isolates normal flux; its two faces have opposite normals:

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f, \quad \hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0.$$

A shrinking rectangular loop has opposite traversal on its two long sides. For bounded field time derivatives, its enclosed flux vanishes, while a free current sheet can supply finite current:

$$\hat{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0}, \quad \hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f.$$

Here σ_f is free surface charge in C/m^2 and $\mathbf{K}_f$ is free sheet current in A/m . With no free current sheet, B_n and H_t are continuous, not generally both components of $\mathbf{B}$. These are ordinary-interface conditions without singular magnetic sources.

Chapter support: Revised Chapter 12; sec:ch12-boundaries.

Advanced Example 12.24: field refraction



Example 12.24: How a magnetic field changes direction at an interface

Two linear isotropic media with $\mu_1 = \mu_0$ and $\mu_2 = 4\mu_0$ meet at a plane with no free surface current. Immediately in medium 1, $B_{1n} = 0.20 \text{ mT}$ and $B_{1t} = 0.10 \text{ mT}$. Find the field immediately in medium 2.

Continuity of B_n and $H_t = B_t/\mu$ gives

$$B_{2n} = 0.20 \text{ mT}, \quad B_{2t} = \frac{\mu_2}{\mu_1} B_{1t} = 0.40 \text{ mT},$$

$$B_2 = \sqrt{0.20^2 + 0.40^2} = 0.447 \text{ mT}.$$

For angles measured from the normal,

$$\tan \theta_2 = \frac{\mu_2}{\mu_1} \tan \theta_1, \quad \theta_1 = 26.6^\circ, \quad \theta_2 = 63.4^\circ.$$

The field is more tangential in medium 2. Check $B_{2t}/(4\mu_0) = B_{1t}/\mu_0$ and unchanged normal flux. This local constraint is not optical Snell's law or a global field solution.

Chapter support: Revised Chapter 12; ch12-boundary-angle.

Advanced: shape and apparent susceptibility



For a uniformly magnetized ellipsoid along a principal axis,

$$H_{\text{int}} = H_{\text{appl}} - N_d M, \quad M = \chi_m H_{\text{int}}.$$

N_d is dimensionless; the SI principal factors sum to one. A sphere has $N_d = 1/3$. Rearranging gives

$$\chi_a \equiv \frac{M}{H_{\text{appl}}} = \frac{\chi_m}{1 + N_d \chi_m}, \quad \chi_m = \frac{\chi_a}{1 - N_d \chi_a}.$$

The specimen's own field opposes the applied field. Nonellipsoids generally need a nonuniform-field treatment.

Advanced problem Ch5

† A spherical, reversible, linear sample has measured $M/H_{\text{appl}} = 1.5$. Infer its intrinsic susceptibility using $N_d = 1/3$. Find its magnetization and internal field at $H_{\text{appl}} = 200 \text{ A/m}$. Explain why interpreting the apparent ratio as intrinsic susceptibility would be wrong.

Chapter support: Revised Chapter 12; sec:ch12-demag.

Advanced problem Ch5: intrinsic response



The measured apparent ratio is $\chi_a = 1.5$ and the sphere has $N_d = 1/3$. Hence

$$\chi_m = \frac{1.5}{1 - (1/3)(1.5)} = 3.0.$$

At the stated applied field,

$$M = \chi_a H_{\text{appl}} = 1.5(200) = 300 \text{ A/m},$$

$$H_{\text{int}} = 200 - \frac{300}{3} = 100 \text{ A/m}.$$

Independent check. $M/H_{\text{int}} = 300/100 = 3.0$ reproduces the inferred intrinsic susceptibility. Dividing by the applied field instead underestimates it because the specimen changes its own internal field. The inversion becomes sensitive to uncertainty as $N_d \chi_a \rightarrow 1$. A ratio above $1/N_d$ cannot represent this positive constant reversible-susceptibility model; it signals inconsistent data or a failed assumption. In the limit $N_d \rightarrow 0$, apparent and intrinsic susceptibilities agree.

Chapter support: Revised Chapter 12; Ch5.

Advanced: electromagnetic energy balance



The source-explicit curl equations also conserve energy. Dot Ampere-Maxwell with $\mathbf{E}/\mu_0$, dot Faraday with $\mathbf{B}/\mu_0$, and use

$$\nabla \cdot (\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{B}).$$

Collecting the field derivatives gives

$$\boxed{\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{J} \cdot \mathbf{E}}, \quad u = \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}, \quad \mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}.$$

$\mathbf{S}$ is energy flux in W/m^2 . Positive $\mathbf{J} \cdot \mathbf{E}$ is power delivered to matter. Net inward energy flow supplies stored field energy and work on charge. For a charging vacuum capacitor, $\mathbf{E} \parallel \hat{\mathbf{z}}$ and $\mathbf{B} \parallel \hat{\phi}$ make $\mathbf{S}$ radial and inward when $E\dot{E} > 0$. No charge crosses the gap; energy can still enter it. The full chapter checks $P_{\text{in}} = VI$ in the quasistatic model.

Chapter support: Revised Chapter 12; sec:ch12-poynting.

