

Electric Flux and Gauss's Law

Physics II - Chapter 3

Hamzeh Khanpour

hamzeh.khanpour@agh.edu.pl

Faculty of Physics and Applied Computer Science
AGH University of Krakow

Lecturer information and contact details



Lecturer

Hamzeh Khanpour

High Energy Particle Physicist, AGH University of Krakow || CERN-TH

How to contact me

E-mail: hamzeh.khanpour@fis.agh.edu.pl; hamzeh.khanpour@agh.edu.pl

CERN e-mail: hamzeh.khanpour@cern.ch

Where to find me

Office & Room: D10 & 218

You are welcome to contact me or stop by for discussion at any time.

Website and course materials

<https://view.fis.agh.edu.pl/staff/hamzeh.khanpour/>

<https://home.agh.edu.pl/~hamzeh.khanpour/>

<https://hamzeh-khanpour.web.cern.ch/>

Why replace a field sum by a flux law?



Coulomb integration sums every source element. A flux law instead connects an entire closed boundary to its enclosed charge; symmetry can then determine the field.

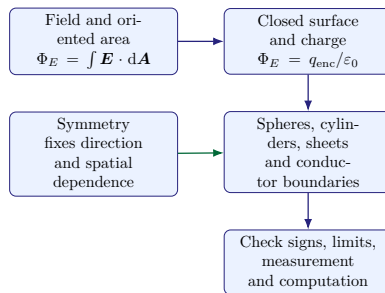


Figure 3.1: The chapter's reasoning sequence. The flux law supplies one constraint; symmetry or conductor equilibrium supplies the information needed to turn it into a field calculation.

Chapter support: Sec. 3.1

Learning route and prerequisites



- 1 **Describe geometry:** normals and dot products turn a vector field into signed surface contributions.
- 2 **Connect flux to sources:** close the surface and identify the algebraic enclosed charge.
- 3 **Calculate and test:** symmetry, superposition, or conductor equilibrium supplies the missing field information.

Prerequisites refreshed in this lecture

Vector components and $\mathbf{E} \cdot \hat{\mathbf{n}} = E \cos \theta$; areas and volumes; elementary integration; Coulomb's law and superposition.

Vacuum is assumed unless stated otherwise. We use signed components; $E = |\mathbf{E}|$ remains nonnegative.

Chapter support: Preface; conventions; Sec. 3.1

Learning goals



- Compute oriented flux through flat and curved surfaces and explain each sign.
- Distinguish total field, net flux, and algebraic enclosed charge.
- Derive spherical, cylindrical, and planar fields by establishing symmetry before evaluating the integral.
- Account for conductor surface and cavity charges; calculate a vacuum coaxial field.
- Check continuity, field jumps, dimensions, limiting cases, and numerical convergence.

Advanced enrichment, slides 61–70

Infer density from divergence, calculate open-surface flux with solid angle, integrate nonuniform density, and solve an insulating-cavity model by superposition.

Chapter support: Sec. 3.1 learning goals; Sec. 3.19

A surface needs an orientation



A scalar area tells us how large a patch is. Its normal tells us which field component pierces it:

$$\mathbf{A} = A\hat{n}, \quad d\mathbf{A} = \hat{n} dA.$$

The unit normal is dimensionless; A and dA have units m^2 . A curved surface needs a different local normal at each patch.

Orientation is part of the calculation

An open patch has two possible normals. Specify one. A closed surface uses the outward normal everywhere.

For a square in the xy plane, the normals are $\pm\hat{z}$. On a closed cube, the top face uses $+\hat{z}$ and the bottom uses $-\hat{z}$.

Chapter support: Sec. 3.2

Only the normal component pierces a patch

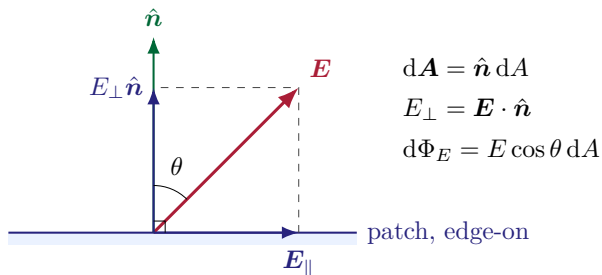


Figure 3.2: A common-tail decomposition of the field into components normal and tangent to a patch. The angle is measured from the normal, not from the surface. The area vector and the field component have different units and are not added to one another.

Chapter support: Sec. 3.2; Figure 3.2

Flux: signed field times projected area



For a flat patch in a uniform field,

$$\Phi_E = \mathbf{E} \cdot \mathbf{A} = EA \cos \theta = E_{\perp} A, \quad [\Phi_E] = \text{N m}^2/\text{C}.$$

The angle is measured from the chosen *normal*. The quantity $A \cos \theta$ is a signed projected area.

Orientation sets the sign

$\theta < 90^\circ$: positive flux.

$\theta = 90^\circ$: zero flux.

$\theta > 90^\circ$: negative flux.

What flux measures

The normal field integrated over area. It is neither field magnitude nor charge per time; charges need not cross the surface.

Chapter support: Sec. 3.3

Example 3.1: a tilted sensor plate



A rectangular plate of area $A = 0.120 \text{ m}^2$ is placed in a uniform electric field of magnitude $E = 850 \text{ N/C}$. The chosen area vector makes an angle 35° with the electric field. Find the electric flux through the plate.

Solution and orientation check

The flat plate and uniform field permit a single dot product:

$$\Phi_E = (850)(0.120) \cos 35^\circ = \boxed{83.6 \text{ N m}^2/\text{C}}.$$

The sign is positive because the chosen normal has a component along the field. The opposite normal gives $-83.6 \text{ N m}^2/\text{C}$; the field itself is unchanged. Also $|\Phi_E| < EA = 102 \text{ N m}^2/\text{C}$.

Chapter support: Example 3.1

Example 3.3: a nonuniform field on a rectangle



For varying fields or normals, add the contributions of small patches:

$$\Phi_E = \lim \sum_i (\mathbf{E}_i \cdot \hat{\mathbf{n}}_i) \Delta A_i = \int_S \mathbf{E} \cdot d\mathbf{A}.$$

A mathematical rectangular surface lies at $z = 0$, with $0 \leq x \leq a$ and $0 \leq y \leq b$. Its normal is $+\hat{\mathbf{z}}$. In its neighbourhood the field is

$$\mathbf{E} = \alpha z \hat{\mathbf{x}} + (E_0 + \alpha x) \hat{\mathbf{z}}, \quad E_0 = 200 \text{ N/C}, \quad \alpha = 600 \text{ N/(C m)}.$$

Find the flux for $a = 0.30 \text{ m}$ and $b = 0.20 \text{ m}$.

Chapter support: Sec. 3.3.1; Example 3.3

Example 3.3: integrate the normal component

Example 3.3 continued. On $z = 0$, the normal component is $E_z = E_0 + \alpha x$. The x component is tangent and does not contribute.

$$\begin{aligned}\Phi_E &= \int_0^b \int_0^a (E_0 + \alpha x) dx dy \\ &= b \left(E_0 a + \frac{\alpha a^2}{2} \right) \\ &= 0.20 \left[200(0.30) + \frac{600(0.30)^2}{2} \right] = \boxed{17.4 \text{ N m}^2/\text{C}}.\end{aligned}$$

Independent area-average check

E_z ranges from 200 to 380 N/C. Its linear variation has average 290 N/C, and $290(0.060) = 17.4 \text{ N m}^2/\text{C}$. A midpoint value is exact here because the variation is linear.

Chapter support: Example 3.3

Close the surface; choose outward normals

A sphere, cube, or capped cylinder is closed. A disk or an uncapped curved cylinder is open. For a closed surface, sum with outward normals:

$$\Phi_E = \oint_S \mathbf{E} \cdot d\mathbf{A}.$$

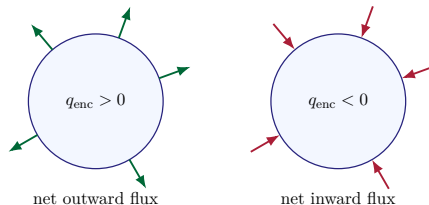


Figure 3.3: Radial examples illustrating the sign convention for closed-surface flux. The normal is outward; general fields may have both entering and leaving contributions. A positive enclosed charge produces positive net flux; a negative enclosed charge produces negative net flux.

Chapter support: Sec. 3.4; Figure 3.3

C1: zero net flux can hide a nonzero field



A uniform electric field threads a closed cube. Explain why the flux is zero even though the field is nonzero on every face. Which faces need not individually have zero flux?

Answer: opposite faces cancel

For any uniform field $\mathbf{E} = (E_x, E_y, E_z)$ and cube side L , each opposite pair contributes $-E_i L^2$ and $+E_i L^2$.

$$\Phi_E = (-E_x + E_x - E_y + E_y - E_z + E_z)L^2 = 0.$$

Faces with a nonzero field component along their normals have nonzero individual flux. For a field along $+x$, only the two x faces contribute: the left flux is negative and the right flux positive. A tangent face has zero flux despite a nonzero field.

Chapter support: Sec. 3.4; C1

Example 3.4: flux through a nonuniform-field cube



A cube of side $L = 0.40$ m occupies $0 \leq x \leq L$, $0 \leq y \leq L$, $0 \leq z \leq L$. The electric field is

$$\mathbf{E} = ax \hat{\mathbf{x}}, \quad a = 300 \text{ N}/(\text{C m}).$$

Find the net flux through the cube.

Solution face by face

At $x = 0$, $E_x = 0$. At $x = L$, $\hat{\mathbf{n}} = +\hat{\mathbf{x}}$ and $E_x = aL$. The other four faces have normals perpendicular to the field.

$$\Phi_E = 0 + (aL)L^2 = aL^3 = (300)(0.40)^3 = \boxed{19.2 \text{ N m}^2/\text{C}}.$$

The positive result is a surplus of outward flux. Its units are $\text{N}/(\text{C m}) \times \text{m}^3 = \text{N m}^2/\text{C}$; translating the cube along x would not change the difference between its opposite-face fluxes.

Chapter support: Example 3.4

Gauss's law: total field, enclosed charge



$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{enc}}}{\epsilon_0}$$

- S is closed and $d\mathbf{A}$ points outward.
- $\mathbf{E}$ is the **total field** from all charges, inside and outside.
- q_{enc} is the **algebraic sum** of all enclosed charge; it is signed once.

Here $\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)$. Both sides have flux units.

A constraint, before a shortcut

The law fixes an integral. It determines a local field only when additional information makes that integral evaluable. In matter it still uses total charge, including polarization charge, and ϵ_0 .

Chapter support: Sec. 3.5; conventions

Coulomb's field gives radius-independent flux



For an isolated point charge q , Coulomb's law gives

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}.$$

On a concentric sphere, $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and the signed radial component is constant:

$$\Phi_E = \frac{q}{4\pi\epsilon_0 r^2} \int_S dA = \frac{q}{4\pi\epsilon_0 r^2} (4\pi r^2) = \frac{q}{\epsilon_0}.$$

The inverse-square field and increasing spherical area cancel. The sign follows q ; a negative charge produces inward flux.

What this derivation establishes

It proves the result for a concentric sphere. Arbitrary closed surfaces need oriented projected areas, developed using solid angle in the advanced block. Superposition then adds charge contributions.

Chapter support: Sec. 3.5.1

An outside charge changes the local field

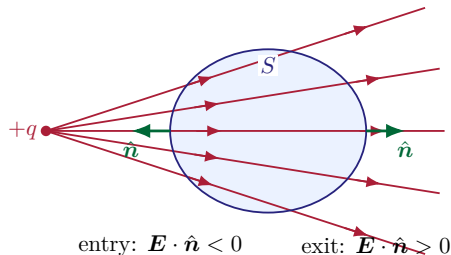


Figure 3.5: Straight radial field lines of an external positive point charge. The closed Gaussian surface does not bend the field. Each narrow bundle entering it also leaves it, so the two signed fluxes cancel even though the field strength changes along the bundle.

An external charge can change every local surface field. Its entering and leaving signed contributions cancel; only enclosed charge changes the net flux.

Chapter support: Sec. 3.6; Figure 3.5

Example 3.6: three surfaces, signed charge



Two point charges $q_1 = +6.0 \text{ nC}$ and $q_2 = -2.0 \text{ nC}$ are fixed in space. Find the net flux through three closed surfaces: S_A encloses only q_1 , S_B encloses only q_2 , and S_C encloses both charges.

Solution: add enclosed charges algebraically

S	q_{enc}	$\Phi_E \text{ (N m}^2/\text{C)}$
S_A	$+6.0 \text{ nC}$	$+6.78 \times 10^2$
S_B	-2.0 nC	-2.26×10^2
S_C	$+4.0 \text{ nC}$	$+4.52 \times 10^2$

Each value is $q_{\text{enc}}/\epsilon_0$, with $1 \text{ nC} = 10^{-9} \text{ C}$. The surface shapes do not matter. As a check, $\Phi_C = \Phi_A + \Phi_B$ before rounding. This does not make the local field uniform.

Chapter support: Example 3.6

Practice P3: calculate all six face fluxes



Student practice. Use outward normals on every face. Sources are in vacuum; use $\varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)$.

A cube occupies $0 < x, y, z < L$. In it, $\mathbf{E} = ax\hat{\mathbf{x}} + by\hat{\mathbf{y}} + E_0\hat{\mathbf{z}}$. Compute each face flux and infer its enclosed charge. Evaluate for $L = 0.20 \text{ m}$, $a = 150 \text{ N}/(\text{C m})$, $b = -50 \text{ N}/(\text{C m})$, and $E_0 = 80 \text{ N/C}$.

Route to the solution

Write the two x faces, then the two y faces, then the two z faces. Retain each signed contribution before summing. Use Gauss's law only after the surface integral is complete.

Chapter support: Sec. 3.17; P3

P3 solution: cancellation and enclosed charge



P3 continued. Every face has area L^2 . A constant tangential component contributes no flux.

face	$x = 0$	$x = L$	$y = 0$	$y = L$	$z = 0$	$z = L$
Φ_E	0	aL^3	0	bL^3	$-E_0L^2$	E_0L^2
value	0	1.20	0	-0.400	-3.20	3.20

All numerical fluxes are in $\text{N m}^2/\text{C}$. Hence

$$\Phi_E = (a + b)L^3 = 100(0.20)^3 = 0.800 \text{ N m}^2/\text{C},$$

$$q_{\text{enc}} = \epsilon_0 \Phi_E = 7.0832 \times 10^{-12} \text{ C} = \boxed{+7.08 \text{ pC}}.$$

The z terms cancel; they are not individually zero. As an independent check, subdividing into identical small boxes gives the same flux per volume, $a + b$, so $q_{\text{enc}} = \epsilon_0(a + b)L^3$.

Chapter support: P3; source solution Example 3.25

When does Gauss's law determine a field?



Gauss's law supplies one integral constraint. To infer a field, first establish the symmetry of the *physical sources and boundary conditions*.

- 1 Determine which components and coordinate dependences the symmetry permits.
- 2 Choose a closed surface with simple normals. Identify zero contributions and regions with a constant normal component.
- 3 Compute q_{enc} for that surface, solve the flux equation, and attach direction and domain.

Drawing a sphere does not create spherical symmetry

A charged ring, finite rod, or sphere beside an external charge can have a complicated field on an enclosing surface. The flux law remains exact even when its field shortcut fails.

Chapter support: Sec. 3.7; Secs. 3.7.1–3.7.2

Example 3.8: equal face fluxes are enough



A point charge q is fixed at the centre of a mathematical cube. Find the flux through one face without integrating the nonuniform field.

Solution: equal integrals, not a constant face field

Let L be the cube side. Gauss's law gives $\Phi_{\text{cube}} = q/\epsilon_0$. Rotations of the centred charge–cube configuration interchange the six faces. Their integrated fluxes are equal:

$$\Phi_{\text{face}} = \frac{q}{6\epsilon_0}.$$

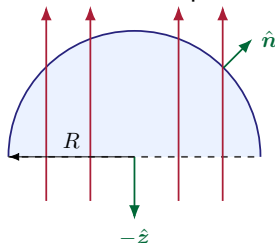
The field strength and its angle to the normal both vary across a face. Thus Φ_{face}/L^2 is an *average normal component*, not the field everywhere. Moving the charge off centre preserves total flux but destroys equal sharing. Multiplying the answer by six independently restores the closed-surface law.

Chapter support: Example 3.8

Close an open hemisphere mathematically



For an open curved surface, add a convenient closing surface and use the closed-surface constraint. Keep the added surface's outward normal.



curved surface: $+E_0\pi R^2$
added disk: $-E_0\pi R^2$

$$\mathbf{E} = E_0\hat{z}$$

Figure 3.4: An open upper hemisphere closed mathematically by a disk. The diagram is a meridional section. The disk's outward normal is downward, making its flux negative. The curved area is $2\pi R^2$, but its projected area is only πR^2 .

Chapter support: Sec. 3.5; Figure 3.4

Example 3.7: flux through the upper hemisphere



A hemisphere of radius R is placed in a uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The hemisphere is the curved upper half of a sphere, open at the equatorial plane. Its area vector is chosen outward from the missing full sphere. Find the flux through the curved hemispherical surface.

Solution: use the closure from slide 23

A uniform field has zero net closed-surface flux. The disk normal is $-\hat{\mathbf{z}}$, so

$$\Phi_{\text{disk}} = -E_0\pi R^2, \quad \Phi_{\text{curved}} + \Phi_{\text{disk}} = 0.$$

$$\boxed{\Phi_{\text{curved}} = +E_0\pi R^2}.$$

The projected area is πR^2 ; the curved scalar area $2\pi R^2$ cannot be multiplied by one normal component. Doubling R multiplies flux by four; reversing the field reverses its sign.

Chapter support: Example 3.7

Hemisphere check: integrate the changing normal



Example 3.7 checked independently. On the upper hemisphere, $0 \leq \theta \leq \pi/2$ is measured from $+z$. A ring has area $2\pi R^2 \sin \theta d\theta$, and its normal component is $E_0 \cos \theta$.

$$\begin{aligned}\Phi_{\text{curved}} &= 2\pi E_0 R^2 \int_0^{\pi/2} \cos \theta \sin \theta d\theta \\ &= 2\pi E_0 R^2 \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/2} = E_0 \pi R^2.\end{aligned}$$

The average normal component is $E_0/2$. The uniform field itself is not weaker at the equator; its projection becomes zero there.

General uniform-field check

$\Phi_E = \mathbf{E}_0 \cdot \int_S \hat{\mathbf{n}} dA$. The vector area of any closed boundary is zero: opposite oriented projections cancel. Its scalar area is still positive.

Chapter support: Sec. 3.5.2; Example 3.7

Spherical symmetry fixes the radial field



For an isolated, rotationally invariant source, the signed radial field depends only on r :

$$\mathbf{E}(r) = E_r(r)\hat{\mathbf{r}}.$$

On a concentric Gaussian sphere, both the field and outward normal are radial, and E_r is constant over the sphere:

$$\oint \mathbf{E} \cdot d\mathbf{A} = E_r(r) \int_S dA = 4\pi r^2 E_r(r),$$

$$\boxed{E_r(r)} = \boxed{\frac{q_{\text{enc}}(r)}{4\pi\epsilon_0 r^2}} \quad (r > 0).$$

The necessary extra information

The source and boundary conditions supply spherical symmetry. Gauss's law supplies the flux. Their combination supplies the local field. If $q_{\text{enc}} < 0$, $E_r < 0$ and the vector points inward.

Chapter support: Sec. 3.8

Source radius and Gaussian radius differ

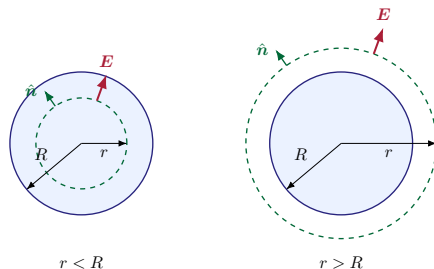


Figure 3.6: Cross-sections of concentric Gaussian spheres (dashed) inside and outside a uniformly charged solid sphere (blue), shown for $\rho > 0$. The source radius R is fixed; the Gaussian radius r selects the field point. Only the charge within the dashed surface enters $q_{\text{enc}}(r)$.

Chapter support: Sec. 3.8; Figure 3.6

Example 3.9: a uniformly charged thin shell



A thin spherical shell of radius R carries total charge Q uniformly over its surface. Find the electric field for $r > R$ and for $r < R$.

Solution: the enclosed charge changes at the shell

For $r < R$, $q_{\text{enc}} = 0$. For $r > R$, $q_{\text{enc}} = Q$. The spherical result from slide 26 gives

$$\mathbf{E}(r) = \begin{cases} \mathbf{0}, & r < R, \\ \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}, & r > R. \end{cases}$$

The interior field vanishes because *symmetry and* zero enclosed charge apply together. The exterior is the point-charge field. Negative Q reverses its direction. The ideal surface has distinct one-sided limits; no field value is assigned exactly on it.

Chapter support: Example 3.9

Example 3.10: count charge inside a solid sphere



A nonconducting sphere of radius R has uniform volume charge density ρ and total charge $Q = \rho(4\pi R^3/3)$. Find $\mathbf{E}(r)$ inside and outside the sphere.

Model and enclosed charge

The charge is spread through volume, not concentrated on a surface. Inside, only the ball of radius r contributes to q_{enc} :

$$q_{\text{enc}}(r) = \rho \frac{4\pi r^3}{3} = Q \frac{r^3}{R^3}, \quad r < R.$$

Outside, $q_{\text{enc}} = Q$. Insert each expression into $4\pi r^2 E_r = q_{\text{enc}}/\epsilon_0$. Do not use the full Q at an interior point.

Chapter support: Example 3.10

Example 3.10: the complete field and its limits



Example 3.10 continued. Dividing the enclosed-charge result by the Gaussian area gives

$$\mathbf{E}(r) = \begin{cases} \frac{\rho r}{3\epsilon_0} \hat{\mathbf{r}} = \frac{Qr}{4\pi\epsilon_0 R^3} \hat{\mathbf{r}}, & 0 < r < R, \\ \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}, & r > R. \end{cases}$$

- At the centre, $\mathbf{E} = \mathbf{0}$ by the direction-independent zero limit. The radial unit vector itself is undefined there.
- At R , both branches give $E_r = \rho R/(3\epsilon_0)$. There is no ideal surface sheet, so the field is continuous.
- Inside, charge grows as r^3 and area as r^2 : $E_r \propto r$. Outside, fixed charge gives $E_r \propto r^{-2}$.

Chapter support: Example 3.10

Read the profile: continuity versus a jump

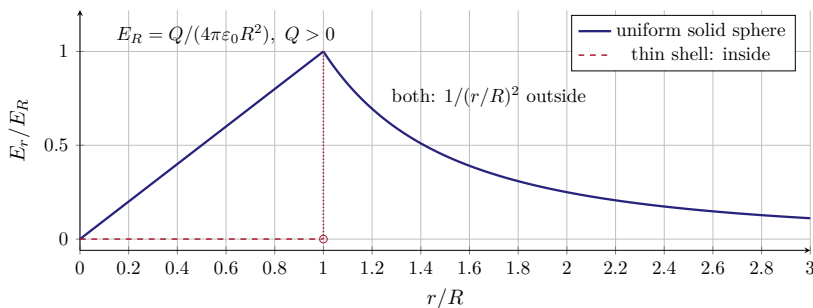


Figure 3.7: Calculated field profiles for equal positive Q and radius R . A uniform volume charge gives a continuous field at R ; an ideal surface shell gives a jump. The dotted vertical segment indicates a discontinuity, not field values assigned at the singular sheet.

Chapter support: Sec. 3.8.1; Figure 3.7

Practice P7: density, fields, and a sketch



Student practice. Use vacuum permittivity $\epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)$, with no applied field. A requested field includes its direction.

A uniformly charged insulating sphere has $R = 0.10 \text{ m}$ and $Q = 12.0 \text{ nC}$. Find its density and fields at $r = 0.05 \text{ m}$, 0.10 m , and 0.20 m . Sketch the signed radial field.

Keep the model visible

First infer the uniform density from total charge and volume. Then decide which field branch applies at each observation radius. The sketch must show a positive linear rise, the surface maximum, and an exterior inverse-square decrease.

Chapter support: Sec. 3.17; P7

P7 solution: numerical values and directions



P7 continued. Convert $Q = 12.0 \text{ nC} = 12.0 \times 10^{-9} \text{ C}$ and use $R = 0.10 \text{ m}$.

$$\rho = \frac{3Q}{4\pi R^3} = \boxed{2.86 \times 10^{-6} \text{ C/m}^3}.$$

With $k = 1/(4\pi\epsilon_0)$, the inside field is kQr/R^3 and the outside field is kQ/r^2 :

$r \text{ (m)}$	$E_r \text{ (N/C)}$	direction
0.05	5.39×10^3	outward
0.10	1.08×10^4	outward
0.20	2.70×10^3	outward

Both branches agree at R . The field at $R/2$ is half the surface field; at $2R$ it is one quarter. These ratios independently check the numerical values. The requested sketch follows on slide 34.

Chapter support: P7; source solution Example 3.27

P7 solution: the requested signed-field sketch



P7 sketch. The blue curve gives E_r/E_R versus r/R , with $R = 0.10\text{ m}$ and $E_R = 1.08 \times 10^4\text{ N/C}$. The red shell curve is retained for comparison.

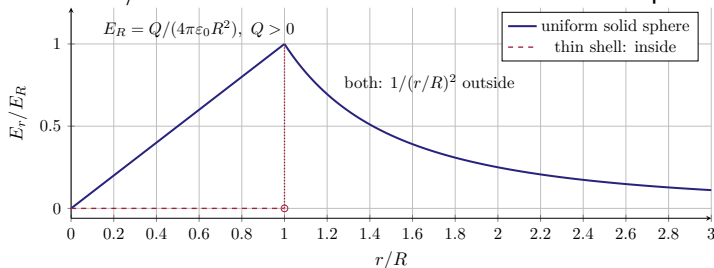


Figure 3.7: Calculated field profiles for equal positive Q and radius R . A uniform volume charge gives a continuous field at R ; an ideal surface shell gives a jump. The dotted vertical segment indicates a discontinuity, not field values assigned at the singular sheet.

Chapter support: P7; Figure 3.7

Cylindrical symmetry and zero cap flux



Axial translations remove z dependence; rotations remove azimuthal dependence. Reflections exclude axial and azimuthal field components. The remaining component is radial and depends only on r .

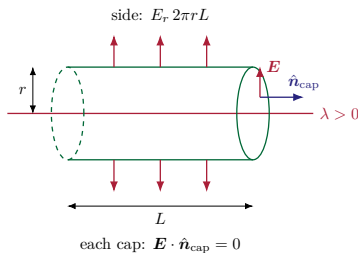


Figure 3.8: A closed coaxial Gaussian cylinder. The charged line continues beyond both caps. Radial fields are tangent to the caps; the cap normal shown is axial. The nonzero flux comes from the curved side.

Chapter support: Sec. 3.9; Figure 3.8

Example 3.12: an ideal infinite line



An ideal infinitely long line carries uniform linear charge density λ . Find the electric field at distance r from the line.

Solution: the arbitrary Gaussian length cancels

The coaxial cylinder has side area $2\pi rL$, zero cap flux, and charge $q_{\text{enc}} = \lambda L$.

$$2\pi rL E_r = \frac{\lambda L}{\epsilon_0} \implies \boxed{\mathbf{E}(r) = \frac{\lambda}{2\pi\epsilon_0 r} \hat{\mathbf{r}}}$$

Positive λ gives outward field; negative λ gives inward field. Substitution back into the side flux gives $\lambda L/\epsilon_0$, independent of radius. The physical line extends beyond both caps: L is a mathematical choice, not its total length.

Chapter support: Example 3.12

Example 3.13: a uniformly charged cylinder



A very long nonconducting cylinder of radius R has uniform volume charge density ρ . Find the electric field inside and outside the cylinder.

The closed coaxial surface has flux $2\pi rLE_r$. Count charged volume separately in each radial region:

$$q_{\text{enc}} = \begin{cases} \rho\pi r^2L, & r < R, \\ \rho\pi R^2L, & r > R. \end{cases}$$

Therefore

$$\mathbf{E}(r) = \begin{cases} \frac{\rho r}{2\epsilon_0} \hat{\mathbf{r}}, & 0 < r < R, \\ \frac{\rho R^2}{2\epsilon_0 r} \hat{\mathbf{r}}, & r > R. \end{cases}$$

At the axis $\mathbf{E} = \mathbf{0}$. Both branches give $E_r(R) = \rho R/(2\epsilon_0)$: the field is continuous without a surface sheet. Its sign follows ρ .

Chapter support: Example 3.13

Why spheres and cylinders have different factors

Uniform sphere

Enclosed charge $\propto r^3$.

Flux-carrying area $\propto r^2$.

$E_r = \rho r / (3\epsilon_0)$ inside.

$E_r \propto r^{-2}$ outside.

Uniform cylinder

Enclosed line charge $\propto r^2$.

Side area per length $\propto r$.

$E_r = \rho r / (2\epsilon_0)$ inside.

$E_r \propto r^{-1}$ outside.

The factors $1/3$ and $1/2$ follow from geometry. Units alone cannot choose between them.

Finite sources require a separate approximation test

Use the infinite model only in a suitable central region; near ends, axial field and cap flux can appear. For a uniform rod of half-length ℓ and the same λ , $E_{\text{finite}}/E_{\infty} = 1/\sqrt{1 + (r/\ell)^2}$ on its perpendicular bisector; optional Sec. 3.19.13 quantifies the error.

Chapter support: Sec. 3.9.1; Sec. 3.19.13 reading

Planar symmetry: both pillbox caps contribute



In-plane translations and rotations leave an infinite uniform sheet unchanged. They exclude tangential field. Reflection across the sheet gives equal magnitudes with opposite directions for the source alone.

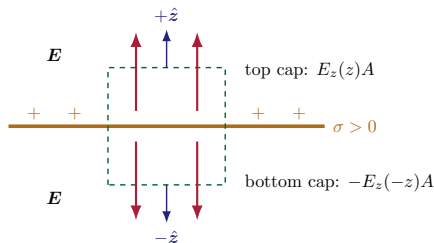


Figure 3.9: Side view of a pillbox straddling an isolated uniform sheet. The outward normals on the two caps point in opposite directions. For positive charge, both cap fluxes are positive. The dashed boundary is a mathematical surface, not an enclosure made of metal.

Chapter support: Sec. 3.10; Figure 3.9

Example 3.14: the field of one infinite sheet



An ideal infinite sheet carries uniform surface charge density σ . Find the electric field on either side.

Solution: two caps, one enclosed sheet

For $\sigma > 0$, both cap fields point along their outward normals. The side flux is zero:

$$2EA = \frac{\sigma A}{\epsilon_0} \implies E = \frac{\sigma}{2\epsilon_0}.$$

For either charge sign, on a sheet at $z = 0$,

$$\boxed{\mathbf{E}(z) = \frac{\sigma}{2\epsilon_0} \operatorname{sgn}(z) \hat{\mathbf{z}}}, \quad z \neq 0.$$

The magnitude is $|\sigma|/(2\epsilon_0)$, away from positive charge and toward negative charge. The absence of distance dependence belongs to the infinite-source model. Substituting both signed cap fields restores $\sigma A/\epsilon_0$.

Chapter support: Example 3.14

Example 3.15: two oppositely charged sheets



Two large parallel sheets carry surface charge densities $+\sigma$ and $-\sigma$, with $\sigma > 0$. Neglect edge effects. Find the field between the plates and outside them.

Solution by superposition

Place the positive sheet at $z = 0$ and the negative one at $z = d$. Each has field magnitude $\sigma/(2\epsilon_0)$.

region	$+\sigma$ sheet	$-\sigma$ sheet	E_z
$z < 0$	$-$	$+$	0
$0 < z < d$	$+$	$+$	σ/ϵ_0
$z > d$	$+$	$-$	0

Thus the gap field points from the positive plate to the negative plate; the ideal exterior field is zero. Opposite components cancel outside and add inside. Finite edges produce fringing, excluded by the statement.

Chapter support: Example 3.15

Example 3.16: unequal prescribed sheets



Superposition must add components, not magnitudes. With $+z$ fixed, each isolated sheet contributes $+\sigma_i/(2\epsilon_0)$ above it and $-\sigma_i/(2\epsilon_0)$ below it. Two prescribed, infinite insulating sheets lie at $z = 0$ and $z = d$. Their fixed surface charge densities are $\sigma_1 = +6.0 \text{ nC/m}^2$ and $\sigma_2 = -2.0 \text{ nC/m}^2$. Determine the field in all three regions, with no externally applied field.

The density is prescribed

These are insulating sheets with fixed charges. They do not redistribute like conductor surfaces. Resolve all three regions before doing arithmetic.

Chapter support: Example 3.16

Example 3.16: solve with signed components



Example 3.16 continued. The signed sum in each region is

$$E_z = \begin{cases} -(\sigma_1 + \sigma_2)/(2\epsilon_0), & z < 0, \\ (\sigma_1 - \sigma_2)/(2\epsilon_0), & 0 < z < d, \\ (\sigma_1 + \sigma_2)/(2\epsilon_0), & z > d. \end{cases}$$

With $(\sigma_1, \sigma_2) = (+6.0, -2.0) \times 10^{-9} \text{ C/m}^2$,

$$E_z = (-226, +452, +226) \text{ N/C}$$

in the three regions, respectively. Negative means $-z$ direction; positive means $+z$.

Boundary and charge check

Before rounding, the jumps are $+\sigma_1/\epsilon_0 = +678 \text{ N/C}$ at 0 and $\sigma_2/\epsilon_0 = -226 \text{ N/C}$ at d . The nonzero exterior field reflects the net density $+4.0 \text{ nC/m}^2$.

Chapter support: Example 3.16

Example 3.17: derive the normal-field jump



An ideal surface charge density σ lies in the plane $z = 0$. Use a pillbox to find the discontinuity in the normal electric field across the sheet.

Solution: one fixed normal, two signed components

Use one comparison normal $+\hat{z}$ from minus to plus. The pillbox cap normals are $+\hat{z}$ and $-\hat{z}$:

$$\Phi_E = (E_z^+ - E_z^-)A, \quad q_{\text{enc}} = \sigma A.$$

For bounded local fields the side flux vanishes with height. Thus

$$E_z^+ - E_z^- = \sigma/\epsilon_0.$$

The jump fixes a *difference*, leaving an added uniform field undetermined. Substituting the isolated-sheet fields $E_z^\pm = \pm\sigma/(2\epsilon_0)$ checks the result.

Chapter support: Example 3.17; Sec. 3.10

From a sheet to a uniform volume slab



Let fixed uniform density $\rho > 0$ occupy $-a < x < a$, with infinite extent in y, z and no applied field. Reflection selects $E_x(-x) = -E_x(x)$ and $E_x(0) = 0$. For caps at $\pm x$ with $0 < x < a$,

$$2AE_x(x) = \frac{\rho(2xA)}{\epsilon_0} \implies E_x(x) = \frac{\rho x}{\epsilon_0}.$$

Outside, the entire charged thickness $2a$ is enclosed:

$$E_x(x) = \begin{cases} -\rho a/\epsilon_0, & x < -a, \\ \rho x/\epsilon_0, & -a \leq x \leq a, \\ +\rho a/\epsilon_0, & x > a. \end{cases}$$

The field points away from the positive slab and is continuous at both boundaries. Its slope changes where the volume density changes; there is no surface sheet to create a jump.

Chapter support: Sec. 3.10.2

A finite density step changes slope, not field

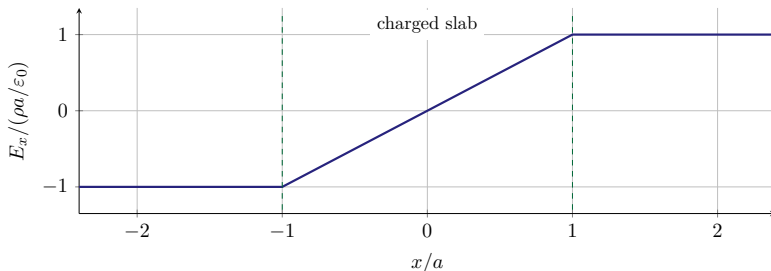


Figure 3.10: Signed field of a uniformly charged infinite slab, with $\rho > 0$ and no applied background field. The dashed lines mark the material boundaries. Negative E_x on the left means the field points toward decreasing x , away from the positive slab.

For the thin-slab limit, hold $\sigma = 2a\rho$ fixed. The exterior fields stay $\pm\sigma/(2\epsilon_0)$ as the continuous transition contracts into a sheet jump. Holding ρ fixed instead makes the exterior field vanish.

Chapter support: Sec. 3.10.2; Figure 3.10

Electrostatic equilibrium in conducting material



Free charges in an ideal conductor rearrange until its macroscopic bulk field vanishes at electrostatic equilibrium:

$$\mathbf{E} = \mathbf{0} \quad \text{in the conducting material.}$$

- 1 A Gaussian surface wholly in metal has zero flux and no enclosed excess charge. Excess charge resides at interfaces.
- 2 A tangential surface field would drive charge along the boundary. The equilibrium exterior field is normal to a smooth surface.
- 3 Surface charge supports a nonzero exterior field.

Identify the physical region and state

A charged cavity can have a field while its surrounding metal has none. A resistive wire carrying steady current is outside this macroscopic equilibrium model.

Chapter support: Sec. 3.11

One nonzero cap at a conductor boundary

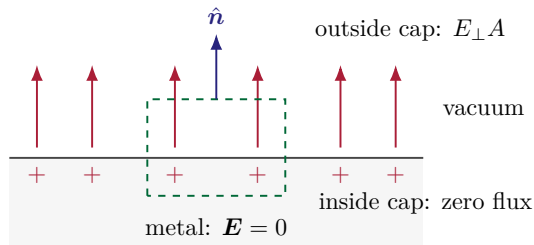


Figure 3.11: Local conductor boundary, shown for $\sigma > 0$. One cap of the thin Gaussian pillbox lies in the metal, where the macroscopic field vanishes. The other lies in vacuum. In general $E_{\perp} = \sigma/\epsilon_0$ is a signed component; the field magnitude is $|\sigma|/\epsilon_0$.

The metal cap contributes zero; the vacuum cap contributes $E_{\perp} A$. With $q_{\text{enc}} = \sigma A$, Gauss's law gives $E_{\perp} = \sigma/\epsilon_0$. This is one nonzero cap, whereas an isolated sheet has two.

Chapter support: Sec. 3.11; Figure 3.11

P12: a negative conductor surface density



P12. Assume a smooth vacuum interface and electrostatic equilibrium; the normal points outward from the metal.

A conductor has local surface density -4.50 nC/m^2 . Find the signed outward normal component and the magnitude of the field just outside. What is the tangential component?

Solution: retain the sign of surface density

$$E_{\perp} = \frac{-4.50 \times 10^{-9}}{8.854 \times 10^{-12}} = \boxed{-508 \text{ N/C}}.$$

The magnitude is 508 N/C ; the vector points inward toward the metal. The tangential component is zero. Substituting the signed field into $\epsilon_0 E_{\perp} = \sigma$ restores the negative surface density. A negative component does not mean a negative magnitude.

Chapter support: P12; source solution Example 3.28

A charged cavity: account for each surface

A Gaussian surface entirely in the metal encloses the cavity charge and inner wall. Equilibrium supplies zero field on that surface, so $q + Q_{\text{inner}} = 0$.

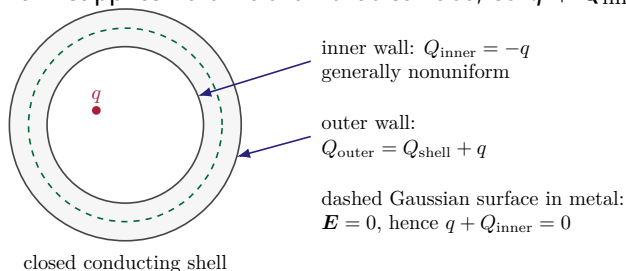


Figure 3.12: Charge accounting for an isolated shell, with no contact between the internal charge and the metal. The Gaussian surface lies entirely in the conducting material. Its zero flux determines the total charge on the inner wall, even for an off-centre charge. No guessed field lines are needed.

Chapter support: Sec. 3.12; Figure 3.12

Practice P13: induced totals without symmetry



Student practice P13. Assume equilibrium and no wall contact.

An isolated conductor has net charge -3.0 nC and a closed cavity containing a point charge $+5.0 \text{ nC}$. Find the inner and outer surface charges. Which conclusions survive displacement of the point charge away from the cavity centre?

Solution. Zero flux in metal fixes the inner-wall charge; conservation of the conductor's charge fixes the outer-wall charge:

$$Q_{\text{inner}} = -q = \boxed{-5.0 \text{ nC}},$$

$$Q_{\text{outer}} = Q_{\text{conductor}} - Q_{\text{inner}} = -3.0 - (-5.0) = \boxed{+2.0 \text{ nC}}.$$

Both totals survive displacement of the charge. Its inner-wall density generally becomes nonuniform and is not fixed by Gauss's law. Check: $-5.0 + 2.0 = -3.0 \text{ nC}$.

Chapter support: P13; source solution Example 3.28

Example 3.20: a central cavity charge



A conducting spherical shell has inner radius a , outer radius b , and net charge Q_{shell} . A point charge q is placed at the centre of the spherical cavity. Assume no external sources and an isolated conductor. Determine the net charges on the inner and outer conducting surfaces and find the field in each radial region.

Separate the three physical regions

The cavity $0 < r < a$ contains the point charge. The metal $a < r < b$ has zero field. The exterior $r > b$ sees the entire system. Central placement and no external sources supply spherical symmetry.

Use an in-metal Gaussian sphere to determine Q_{inner} first. Then use conservation of Q_{shell} and finally evaluate the field for each region.

Chapter support: Example 3.20

Example 3.20: fields in all radial regions



Example 3.20 continued. Zero flux in the metal gives $q + Q_{\text{inner}} = 0$; conservation gives $Q_{\text{outer}} = Q_{\text{shell}} + q$.

$$Q_{\text{inner}} = -q, \quad Q_{\text{outer}} = Q_{\text{shell}} + q.$$

Writing $k = 1/(4\pi\epsilon_0)$, the complete field is

$$\mathbf{E}(r) = \begin{cases} kq\hat{r}/r^2, & 0 < r < a, \\ \mathbf{0}, & a < r < b, \\ k(Q_{\text{shell}} + q)\hat{r}/r^2, & r > b. \end{cases}$$

For either sign of charge, each radial coefficient is signed. The point $r = 0$ is singular; surface values use one-sided limits. The jumps check the surface charges: at a , the radial jump is $-kq/a^2 = Q_{\text{inner}}/(4\pi a^2\epsilon_0)$; at b , it is $k(Q_{\text{shell}} + q)/b^2$. The surface charges sum to Q_{shell} .

Chapter support: Example 3.20

Shielding and grounding add boundary information

Empty, fully enclosed cavity

Zero cavity charge gives zero net wall charge, but that alone does not prove zero field. One connected conductor fixes a single boundary potential. Electrostatic uniqueness then selects the constant-potential, zero-field solution throughout the empty cavity.

Isolated conductor

Its net charge Q is conserved. A cavity charge q gives $Q_{\text{inner}} = -q$ and $Q_{\text{outer}} = Q + q$.

Grounded enclosure

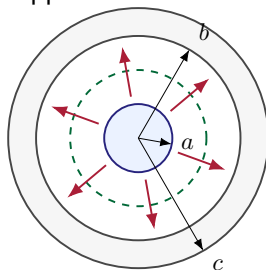
It can exchange charge. With no exterior sources and zero potential at infinity, $Q_{\text{outer}} = 0$; the conductor's net charge is $-q$.

The inner-wall total stays $-q$ in either case. An open pail or apertured cage is a different boundary problem. The potential proof is optional reading in Sec. 3.19.12.

Chapter support: Sec. 3.12.1; Sec. 3.19.12 reading

Coaxial geometry separates metal and vacuum

Assume an infinitely long, coaxial system in vacuum, electrostatic equilibrium, and no external sources. Cylindrical symmetry applies in the gap and exterior; equilibrium applies in both metals.



inner conductor: $+\lambda$ per length
vacuum gap: $a < r < b$

outer conductor, inner face: $-\lambda$
outer face: $\lambda + \lambda_{\text{out}}$

outer conductor net: λ_{out}
 $\mathbf{E} = 0$ in both metals

Figure 3.13: Cross-section of an ideal infinitely long coaxial system. Arrows show the gap field for $\lambda > 0$. The exterior field vanishes when $\lambda_{\text{out}} = -\lambda$ and no external field is applied. The dashed circle represents a coaxial cylindrical Gaussian surface.

Chapter support: Sec. 3.13.1; Figure 3.13

Engineering E1: an unbalanced coaxial assembly



Engineering E1. Use $\varepsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/(\text{N m}^2)$, electrostatic equilibrium, and the infinite coaxial model from slide 55, without external sources. A long coaxial assembly has inner conductor radius $a = 1.0 \text{ mm}$ and outer conductor inner/outer radii $b = 4.0 \text{ mm}$, $c = 5.0 \text{ mm}$. The inner conductor carries $+2.0 \text{ nC/m}$ and the outer conductor has net -1.0 nC/m . Find the charge per length on each outer-conductor face, all radial field regions, and the field at $r = 2.0 \text{ mm}$ and 10 mm . The gap is vacuum.

Solution route

Start with a Gaussian cylinder in the outer metal to obtain its inner-face charge. Use the outer conductor's *net* charge to obtain its outer-face charge. Only then calculate the gap and exterior fields.

Chapter support: Sec. 3.17; E1

E1 solution: face charges and regional fields



E1 continued. In the outer metal, zero flux requires $\lambda_b = -2.0 \text{ nC/m}$. Its net charge is -1.0 nC/m , so $\lambda_c = +1.0 \text{ nC/m}$.

$$E(r) = \begin{cases} 0, & r < a, \\ \frac{2.0 \times 10^{-9}}{2\pi\epsilon_0 r} \hat{r}, & a < r < b, \\ 0, & b < r < c, \\ \frac{1.0 \times 10^{-9}}{2\pi\epsilon_0 r} \hat{r}, & r > c. \end{cases}$$

$$E_r(2.0 \text{ mm}) = 1.80 \times 10^4 \text{ N/C},$$

$$E_r(10 \text{ mm}) = 1.80 \times 10^3 \text{ N/C}.$$

Both fields point outward. Check: $-2.0 + 1.0 = -1.0 \text{ nC/m}$ on the outer conductor. The whole assembly has $+1.0 \text{ nC/m}$, explaining its nonzero exterior field. Both metal regions still have zero field.

Chapter support: E1; source solution Example 3.31

Numerical flux: sum signed midpoint patches



For a point charge $q \neq 0$ away from the boundary of a cube $[-a, a]^3$, evaluate Coulomb's field at every face-patch midpoint:

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_q}{|\mathbf{r} - \mathbf{r}_q|^3}.$$

With N^2 patches per face, $\Delta A = (2a/N)^2$,

$$\Phi_N = \sum_{\text{six faces}} \sum_{i,j} (\mathbf{E}_{ij} \cdot \hat{\mathbf{n}}_{\text{face}}) \Delta A, \quad f_N = \frac{\epsilon_0 \Phi_N}{q}.$$

Use outward normals $\pm\hat{\mathbf{x}}$, $\pm\hat{\mathbf{y}}$, $\pm\hat{\mathbf{z}}$. Do not replace signed dot products by absolute values.

Physical and numerical tests

The exact f is one for an inside charge and zero for an outside charge. Moving an interior charge changes the surface field but not its exact net flux. Uniform rescaling of all lengths leaves f_N unchanged.

Chapter support: Sec. 3.14; midpoint flux sum

Convergence tests distinguish error from charge



Synthetic midpoint results for $[-1, 1]^3$. Positions are in units of the cube half-width. Every face has N^2 samples.

N	centre (0, 0, 0)	off centre (0.6, 0.2, -0.3)	near face (0.95, 0, 0)	outside (1.5, 0, 0)
8	1.003829	1.003797	0.743690	-0.001234
16	1.000957	1.000956	0.895752	-0.000310
32	1.000239	1.000239	0.988639	-0.000077
64	1.000060	1.000060	0.999938	-0.000019
128	1.000015	1.000015	1.000006	-0.000005
exact	1	1	1	0

For smooth face integrands, doubling N reduces midpoint error by about four. Near the face, the $N = 8$ patch width $0.25a$ exceeds the charge-face distance $0.05a$: refinement must first resolve the narrow peak. Use $|f_N - 1|$ inside and $|f_N|$ outside. A relative error divided by the exact exterior flux is undefined. A correct total flux alone does not certify the local field.

Chapter support: Sec. 3.14.2; Table 3.2

Core synthesis and the bridge to potential



Before accepting a result

Choose normals and count enclosed charge. Justify symmetry before taking a field component outside the integral. Check signs, units, boundary continuity or jumps, limits, and numerical resolution.

Zero net flux determines zero *algebraic charge*. A zero field needs symmetry, equilibrium, or boundary information.

Bridge to Chapter 4: electric potential

Electrostatics also has zero closed-loop circulation, allowing a scalar potential. Chapter 4 develops work, potential difference, and equipotentials; our radial fields become inputs to potential integrals.

Reading: Secs. 3.15–3.18 and the bridge; optional charge measurement and Faraday-pail experiments in Sec. 3.13. The next ten slides deepen selected Chapter 3 models.

Chapter support: Secs. 3.15–3.18; Sec. 3.20 bridge

Advanced: outgoing flux per shrinking volume



Advanced Topics for Technical and Engineering Students. Divergence is the outgoing flux per unit volume as a smooth-field box shrinks. For opposite faces normal to x ,

$$\Delta\Phi_x = [E_x(x + \Delta x/2) - E_x(x - \Delta x/2)] \Delta y \Delta z \simeq \frac{\partial E_x}{\partial x} \Delta V.$$

Adding all three pairs and dividing by volume gives

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \boxed{\rho/\epsilon_0}.$$

The divergence theorem sums these local balances; internal faces cancel. The divergence has units $\text{N}/(\text{C m})$. Zero divergence need not mean zero field. At a point charge or ideal sheet, ordinary smooth derivatives fail: retain the enclosed singular charge or its surface jump.

Chapter support: Sec. 3.19.1

Advanced Example 3.35: infer a source density



Suppose in a region the electric field is modelled by

$$\mathbf{E} = (\alpha x \hat{\mathbf{x}} + \beta y \hat{\mathbf{y}} - 2\alpha z \hat{\mathbf{z}}),$$

where $\alpha = 100 \text{ N}/(\text{C m})$ and $\beta = 40 \text{ N}/(\text{C m})$. Find the charge density required by Gauss's law.

Solution and independent box check

$$\nabla \cdot \mathbf{E} = \alpha + \beta - 2\alpha = \beta - \alpha = -60 \text{ N}/(\text{C m}),$$

$$\rho = \varepsilon_0(\beta - \alpha) = -5.31 \times 10^{-10} \text{ C}/\text{m}^3.$$

For a rectangular box of volume V , opposite-face subtraction gives exact flux $(\alpha + \beta - 2\alpha)V$, so $q = \rho V$ independently checks the derivative result. The negative sign means net convergence. The diagonal linear field has zero curl, but Gauss's law still cannot fix an added uniform field without boundary data.

Chapter support: Example 3.35; Sec. 3.19.3

Advanced: solid angle measures oriented projection

For a point charge, oriented projected area divided by squared source distance defines solid angle:

$$d\Omega = \frac{\hat{\mathbf{s}} \cdot \hat{\mathbf{n}}}{s^2} dA, \quad d\Phi_E = \frac{q}{4\pi\epsilon_0} d\Omega.$$

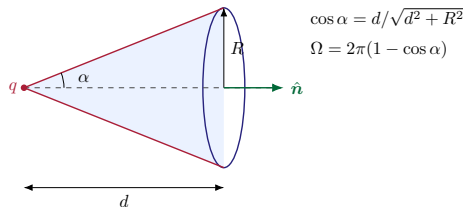


Figure 3.14: A point charge views an axial disk through a cone of half-angle α . The chosen disk normal points away from the charge, giving positive oriented solid angle. The shaded meridional section is a geometric construction, not a conducting object.

Chapter support: Sec. 3.19.4; Figure 3.14

Advanced Example 3.37: flux through an axial disk



A point charge q lies on the axis of a circular disk of radius R , a perpendicular distance $d > 0$ from its plane. Choose the disk normal to point away from the charge. Find the flux.

Annular integration. At disk radius s , project the Coulomb field onto the normal and use annular area:

$$E_n(s) = \frac{q}{4\pi\epsilon_0} \frac{d}{(d^2 + s^2)^{3/2}}, \quad dA = 2\pi s \, ds.$$

The full flux integral is

$$\Phi_E = \frac{q}{2\epsilon_0} \int_0^R \frac{d s}{(d^2 + s^2)^{3/2}} \, ds.$$

The normal projection varies across the flat surface. A centre field times total area is generally insufficient.

Chapter support: Example 3.37

Example 3.37: limits and an independent check



Example 3.37 continued. Since the antiderivative is $-d/\sqrt{d^2 + s^2}$,

$$\Phi_E = \frac{q}{2\epsilon_0} \left(1 - \frac{d}{\sqrt{d^2 + R^2}} \right), \quad \Omega = 2\pi \left(1 - \frac{d}{\sqrt{d^2 + R^2}} \right).$$

- Independent geometric check: $\Omega = \int_0^{2\pi} \int_0^\alpha \sin \theta \, d\theta \, d\phi = 2\pi(1 - \cos \alpha)$, with $\cos \alpha = d/\sqrt{d^2 + R^2}$.
- For $d \gg R$, $1 - d/\sqrt{d^2 + R^2} \simeq R^2/(2d^2)$, so $\Phi_E \simeq [q/(4\pi\epsilon_0 d^2)]\pi R^2$.
- For $R \gg d$, $\Omega \rightarrow 2\pi$ and $\Phi_E \rightarrow q/(2\epsilon_0)$. An open disk does not enclose half the charge.

Reversing the normal reverses flux. The $d \rightarrow 0^+$ limit is one-sided; a point source exactly on the disk is excluded.

Chapter support: Example 3.37; Sec. 3.19.4

Advanced challenge 3.38: a nonuniform sphere



Advanced attempt/solve challenge. Radial variation preserves spherical symmetry but changes the enclosed-charge integral. A layer at r' has volume $4\pi r'^2 dr'$:

$$q_{\text{enc}}(r) = 4\pi \int_0^r \rho(r') r'^2 dr', \quad E_r(r) = \frac{q_{\text{enc}}(r)}{4\pi\epsilon_0 r^2}.$$

A sphere of radius R has charge density

$$\rho(r) = \rho_0 \left(1 - \frac{r}{R}\right), \quad 0 \leq r \leq R.$$

Find the total charge and the electric field inside.

The prime marks the integration coordinate, not the observation radius. Assume the prescribed source alone, with no central point charge or surface sheet. Integrate the actual density; multiplying the boundary density by the entire interior volume would be incorrect.

Chapter support: Sec. 3.19.6; Example 3.38

Example 3.38: integrate density, then infer field

Example 3.38 continued. The layer integral gives

$$q_{\text{enc}}(r) = 4\pi\rho_0 \int_0^r \left(1 - \frac{r'}{R}\right) r'^2 dr' = 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4R}\right).$$

$$Q = q_{\text{enc}}(R) = \frac{\pi\rho_0 R^3}{3},$$

$$\mathbf{E}(r) = \frac{\rho_0}{\epsilon_0} \left(\frac{r}{3} - \frac{r^2}{4R}\right) \hat{\mathbf{r}} \quad (0 < r < R).$$

Independent checks

$\mathbf{E} \rightarrow \mathbf{0}$ at the centre. At R , $E_r = \rho_0 R / (12\epsilon_0) = Q / (4\pi\epsilon_0 R^2)$ matches the exterior field. Differentiation gives

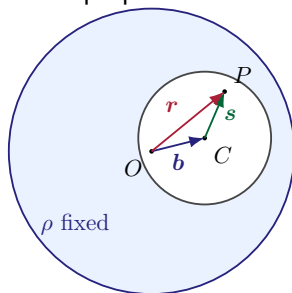
$$\frac{\epsilon_0}{r^2} \frac{d}{dr} (r^2 E_r) = \rho_0 (1 - r/R),$$

This restores the density; the field direction follows the sign of ρ_0 .

Chapter support: Example 3.38; Sec. 3.19.6

Advanced: decompose an offset insulating cavity

The offset cavity destroys the final system's spherical symmetry. Decompose its fixed charge distribution into two symmetric spheres, calculate each interior field, and then superpose.



full sphere: $+\rho$

removed sphere: $-\rho$

$$\mathbf{r} = \mathbf{b} + \mathbf{s}$$

$$\mathbf{E}_{\text{cavity}} = \rho \mathbf{b} / (3\epsilon_0)$$

Figure 3.16: Cross-section of a cavity removed from a uniformly charged insulator. Vector addition relates positions measured from the two centres. The negative-density sphere is used only in the superposition calculation; the physical cavity contains no charge.

Chapter support: Sec. 3.19.11; Figure 3.16

Advanced Example 3.44: complete cavity statement



A nonconducting sphere has fixed uniform volume density ρ . A spherical cavity of radius a is removed, entirely inside the original sphere. Let $\mathbf{b}$ point from the original centre O to the cavity centre C . Find the field at any point P inside the cavity. Assume the remaining charge does not rearrange.

Model before algebra

Represent the removed material by a mathematical sphere of density $-\rho$ centred at C , added to the full sphere of density $+\rho$ centred at O . Define $\mathbf{r} = \overrightarrow{OP}$, $\mathbf{s} = \overrightarrow{CP}$, and $\mathbf{b} = \overrightarrow{OC}$. The unchanged geometry on slide 68 gives $\mathbf{r} = \mathbf{b} + \mathbf{s}$.

Every cavity point lies inside both calculation spheres. Their fields therefore use the *interior* uniform-sphere result from slide 30. There are no mobile charges to enforce conducting-cavity shielding.

Chapter support: Example 3.44; Figure 3.16 on slide 68

Example 3.44: a uniform field without shielding

Example 3.44 continued. The two interior fields are

$$\mathbf{E}_{\text{full}} = \frac{\rho}{3\epsilon_0} \mathbf{r}, \quad \mathbf{E}_{\text{removed}} = -\frac{\rho}{3\epsilon_0} \mathbf{s}.$$

Using $\mathbf{r} - \mathbf{s} = \mathbf{b}$,

$$\mathbf{E}_{\text{cavity}} = \frac{\rho}{3\epsilon_0} \mathbf{b}.$$

It is uniform throughout the cavity and, for $\rho > 0$, points from O toward C . For the source's illustrative $\rho = 1.0 \mu\text{C}/\text{m}^3$ and $|\mathbf{b}| = 2.0 \text{ cm}$,

$$E = \frac{(1.0 \times 10^{-6})(2.0 \times 10^{-2})}{3(8.854 \times 10^{-12})} \simeq \boxed{7.5 \times 10^2 \text{ N/C}}.$$

Check: $\mathbf{b} = 0$ gives zero field; $\rho\mathbf{b}/\epsilon_0$ has field units. A cavity can have nonzero uniform field yet zero closed flux. Fixed insulating charge permits this result; a closed empty conducting cavity has different boundary conditions.

Chapter support: Example 3.44; Sec. 3.19.11

