

Electric Potential and Potential Energy

Physics II - Chapter 4

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Why introduce an electric potential?

The field gives a local force direction. A scalar potential connects that field to work, voltage measurements and particle energy without solving the full trajectory.

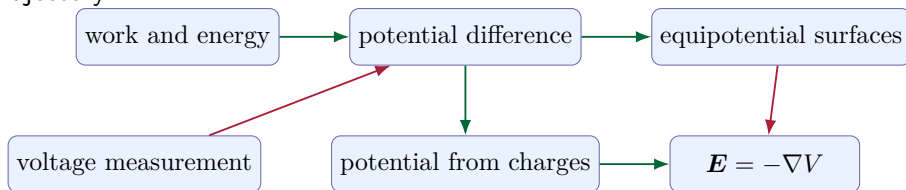


Figure 4.1: Roadmap for Chapter 4. The chapter begins with work and energy, defines potential difference, studies equipotentials, computes potentials from charge distributions, and recovers the field from the scalar potential.

Chapter support: Section 4.1; Figure 4.1.

Learning route and prerequisites



Route

Work $\rightarrow$ voltage $\rightarrow$ source potentials $\rightarrow$ spatial slopes $\rightarrow$ motion and conductors. Each stage turns one description of electrostatics into another.

Tools to refresh

Signed charges and $\mathbf{F} = q\mathbf{E}$; work and energy; dot products; integrals and derivatives; Coulomb's and Gauss's laws.

Model throughout the core

Electrostatics in vacuum, SI units, and static prescribed sources. A moving probe does not appreciably change its sources. A potential is external to the probe, excluding its self-field.

Chapter support: Conventions; Sections 4.2 and 4.8.

Learning goals



- 1 Calculate signed voltage and work from a field integral; explain why the endpoints suffice in electrostatics.
- 2 Compute potentials of discrete and continuous sources, with a stated reference and correct geometry.
- 3 Recover local field components and interpret equipotential spacing and potential slopes.
- 4 Use charge sign, pair energies and conservation to test accessible motion and assembly work.
- 5 Explain conductor potentials and infer a field from voltage readings with stated uncertainty.

The advanced block extends these tools to boundary data, constrained stability, induced sources and field energy.

Work, force and interaction energy



For a charge q displaced through a fixed electrostatic field,

$$dW_{\text{elec}} = q\mathbf{E} \cdot d\mathbf{l}, \quad \Delta U = -W_{\text{elec}}.$$

Positive electric work lowers interaction energy. The energy belongs to the charge–source configuration; assigning it to the particle is shorthand.

Include other forces once

The work–energy theorem gives

$$\Delta K = W_{\text{elec}} + W_{\text{other}}, \quad \Delta(K + U) = W_{\text{other}}.$$

For transport with equal initial and final kinetic energy and no dissipation, $W_{\text{app}} = \Delta U$. Otherwise it need not be.

Chapter support: Section 4.2; Eq. (4.5), work balance.

Example 4.1: account for all the work



Example 4.1 — complete statement

A fixed field does -0.72 mJ of work while a charge is carried from a to b . Its kinetic energy rises by 0.08 mJ . Determine the potential-energy change and the external work.

Conservative work determines ΔU ; total work determines ΔK :

$$\Delta U = -(-0.72) = +0.72 \text{ mJ},$$

$$W_{\text{app}} = \Delta K - W_{\text{elec}} = 0.08 + 0.72 = 0.80 \text{ mJ}.$$

Check: $W_{\text{app}} = \Delta K + \Delta U$. The supplied energy splits into 0.72 mJ of interaction energy and 0.08 mJ of kinetic energy.

Chapter support: Example 4.1; fixed source; no additional uncounted work.

Voltage is energy per unit charge



$$\Delta V = V_b - V_a = \frac{\Delta U}{q}, \quad \Delta U = q\Delta V, \quad 1 \text{ V} = 1 \text{ J/C}.$$

The voltage belongs to the source configuration and chosen endpoints. It is independent of the nonperturbing probe's charge and mass; its energy change is not.

Charge sign matters

In the same ΔV , positive and negative charges have opposite ΔU . The force points toward decreasing $U = qV$, while velocity also depends on the initial motion.

Voltage is a comparison between two points. Charge can flow and energy can be transferred; voltage is not a flowing substance.

Chapter support: Section 4.3; nonzero probe charge in the ratio definition.

From projected work to a line integral



$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\mathbf{l}.$$

On a short segment, $\Delta V \simeq -E \Delta l \cos \theta$. Only displacement along the field contributes. Adding such segments and taking their lengths to zero gives the integral.

Uniform field

The field can be taken outside the integral:

$$\Delta V = -\mathbf{E} \cdot (\mathbf{r}_b - \mathbf{r}_a) = -E_x \Delta x - E_y \Delta y - E_z \Delta z.$$

The field points toward decreasing V . Net displacement, rather than total path length, enters this result.

Chapter support: Section 4.3; line integral and uniform-vector relations.

Example 4.2: voltage between plates



Example 4.2 — complete statement

Two parallel plates are separated by $d = 4.0 \text{ mm}$. Between them the electric field is approximately uniform, $\mathbf{E} = 2.5 \times 10^4 \hat{\mathbf{i}} \text{ V/m}$. Point a is at the left plate and point b is at the right plate. Find $V_b - V_a$.

With $d\mathbf{l} = dx\hat{\mathbf{i}}$ and $d = 4.0 \times 10^{-3} \text{ m}$,

$$\Delta V = -\int_0^d E dx = -Ed = -(2.5 \times 10^4)(4.0 \times 10^{-3}) \text{ V} = -100 \text{ V}.$$

Check: $(\text{V/m})\text{m} = \text{V}$. The right plate has lower potential, consistent with the rightward field. Reversing the endpoints gives $+100 \text{ V}$.

Chapter support: Example 4.2; uniform interior approximation.

Practice P3: which displacement matters?

P3 — complete statement

A uniform field is $450\hat{j}$ V/m. Find ΔV for displacements $+0.20\hat{j}$ m, $-0.20\hat{j}$ m and $+0.20\hat{i}$ m. Can a curved path between the same endpoints change the answers?

Use the stated electrostatic vacuum model. First write the algebraic Δy for each displacement, then apply

$$\Delta V = -E_y \Delta y.$$

The x displacement is perpendicular to the field. For the final question, compare the endpoint change with the sum of projected segments along a curve.

Chapter support: P3; complete answer on slide 12.

P3: complete solution and path check



For $E_y = 450 \text{ V/m}$,

$$\Delta \mathbf{r} = +0.20 \hat{\mathbf{j}} \text{ m} : \quad \Delta V = -90 \text{ V},$$

$$\Delta \mathbf{r} = -0.20 \hat{\mathbf{j}} \text{ m} : \quad \Delta V = +90 \text{ V},$$

$$\Delta \mathbf{r} = +0.20 \hat{\mathbf{i}} \text{ m} : \quad \Delta V = 0.$$

For any curve between the same endpoints,

$$\Delta V = -450 \text{ V/m} \int_a^b dy = -450 \text{ V/m} (y_b - y_a).$$

Thus curvature cannot change any answer. Along x the motion stays on an equipotential; the zero voltage does not imply a zero field.

Chapter support: P3 and its selected detailed solution; statement on slide 11.

Path independence: a physical test



In electrostatics the same endpoints give the same work:

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0.$$

Otherwise a closed circuit through unchanged fixed sources could gain mechanical energy on every traversal. A conservative field defines a single-valued potential.

Example 4.4 — complete statement

Let $\mathbf{E} = (300\hat{\mathbf{i}} - 200\hat{\mathbf{j}})$ V/m, and let $a = (0, 0)$ and $b = (0.040, 0.030)$ m. Compare a straight path with a path that first travels horizontally and then vertically.

Chapter support: Example 4.4; compare the two calculations on slide 14.

Example 4.4: two paths, one voltage



Straight path: put $\mathbf{r}(t) = t(0.040\hat{\mathbf{i}} + 0.030\hat{\mathbf{j}})$ m, with $0 \leq t \leq 1$. Then

$$\Delta V = - \int_0^1 [300(0.040) - 200(0.030)] \text{ V } dt = -6.0 \text{ V}.$$

Horizontal then vertical:

$$\Delta V_x = -300(0.040) = -12 \text{ V}, \quad \Delta V_y = -(-200)(0.030) = +6.0 \text{ V}.$$

Their sum is again -6.0 V . The vertical leg raises the potential; taking the magnitude of the field times distance would lose this sign and the projection.

Independent check: $-\mathbf{E} \cdot \Delta \mathbf{r} = -6.0 \text{ V}$. Reversing the full route negates the answer.

Chapter support: Example 4.4; statement on slide 13.

Piecewise fields require piecewise integrals



An infinite insulating slab occupies $|x| < a$ with uniform positive density ρ . Gauss's law and reflection symmetry give

$$E_x = \begin{cases} \rho x / \epsilon_0, & |x| \leq a, \\ (\rho a / \epsilon_0) \operatorname{sgn}(x), & |x| > a. \end{cases}$$

Choose $V(0) = 0$. For $0 \leq x \leq a$, integrate the linear field. Beyond the face, retain $V(a)$ and integrate the constant field:

$$V(x) = \begin{cases} -\rho x^2 / (2\epsilon_0), & |x| \leq a, \\ -(\rho a / \epsilon_0)(|x| - a/2), & |x| \geq a. \end{cases}$$

Both V and V' match at the faces. An infinite slab has no finite infinity reference; the even potential has a maximum at its centre.

Chapter support: Section 4.3.3; infinite prescribed-density slab.

A slab: field shape and voltage area

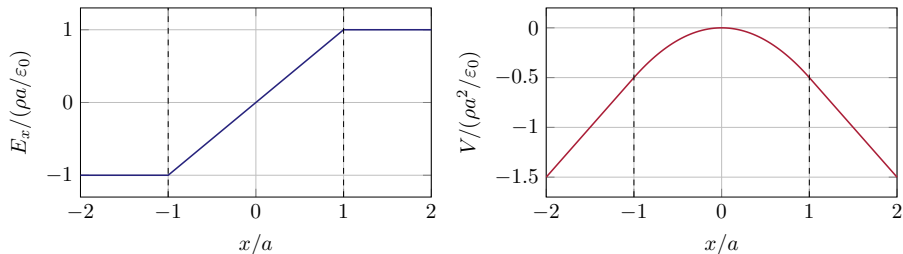


Figure 4.2: Calculated field and potential for an infinite uniformly charged slab, with $V(0) = 0$. Dashed lines locate the slab faces. The potential's slope is the negative field; a constant exterior field produces a linear exterior potential.

Voltage is the negative signed area under $E_x(x)$. A triangular interior area and a rectangular exterior area must be counted separately; an endpoint field value alone does not give the work.

Chapter support: Figure 4.2; slab model defined on slide 15.

Example 4.5: cross a slab face



Example 4.5 — complete statement

Take the ideal slab with $a = 5.0 \text{ mm}$ and $\rho = 2.0 \times 10^{-7} \text{ C/m}^3$. Find $V(2a) - V(0)$ and the electrical work on $+3.0 \text{ nC}$ moved from the centre to $x = 2a$.

For the infinite slab of slide 15, $\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m}$ and $E_s = \rho a / \epsilon_0 = 112.94 \text{ V/m}$.

$$\Delta V = -\frac{1}{2}E_s a - E_s a = -\frac{3}{2}E_s a = -0.847 \text{ V}.$$

$$W_{\text{elec}} = -q\Delta V = +2.54 \text{ nJ}.$$

Check: the exact piecewise potential at $x = 2a$ gives the same drop. Using E_s over the whole path gives -1.13 V , an overestimate of magnitude by one third.

Chapter support: Example 4.5; inherited slab geometry restored on slides 15–16.

A reference shift is not grounding



Relabel the potential

$V' = V + C$ leaves ΔV and $\mathbf{E} = -\nabla V$ unchanged. No charge needs to move. There is one common offset, not an independent zero at every point.

For an infinite line, a finite radial reference r_0 is essential:

$$V(r) - V(r_0) = -\frac{\lambda}{2\pi\epsilon_0} \ln(r/r_0).$$

The logarithm has a dimensionless argument. $V(\infty) = 0$ cannot be imposed on a divergent infinite-source model.

Chapter support: Section 4.4; finite references for infinite sources.

Ground a conductor

A connection to a reservoir permits charge exchange and imposes a physical boundary potential. Charge and nearby fields can change; grounded does not mean neutral.

Equipotentials constrain field direction

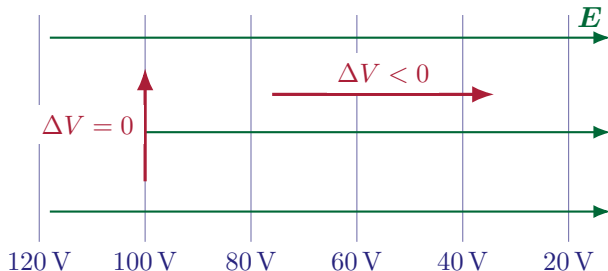


Figure 4.3: Equal voltage increments give equally spaced equipotentials in a uniform field. Red arrows indicate possible displacements, not trajectories. Only the component of displacement parallel to the field contributes to the voltage change. At a regular point, a tangential step has $dV = 0$, hence $\mathbf{E} \cdot d\mathbf{l} = 0$: the field is normal to the surface. A nonzero normal field is compatible with constant potential along it.

Chapter support: Section 4.5; Figure 4.3.

Contour spacing is a local field estimate



For neighbouring contours, measure the small separation Δn *normal* to them:

$$E \simeq \frac{|\Delta V|}{\Delta n}.$$

Closer contours imply a stronger field only when their voltage increments are equal. An oblique separation gives an underestimate. Finite endpoint voltage fixes an integral, not every local field value.

A qualified geometric rule

Different potential values cannot intersect. Branches of the same level set can meet where $\nabla V = 0$. For $V = A(x^2 - y^2)$ the zero lines $y = \pm x$ meet at the origin; there the field is zero and the level set has no unique normal.

Chapter support: Sections 4.5.1–4.5.2.

Practice: read a map before doing arithmetic



Example 4.6 — complete statement

Near a point P , locally parallel contours have values 18.0, 15.0 and 12.0 V, with consecutive normal separations of 6.0 mm. The unit normal $\hat{n}$ points from the 18.0 V contour toward the 12.0 V contour. Estimate the field and the electrical work on a charge $q = -3.0 \text{ nC}$ moving between these outer contours.

Attempt: distinguish the *local* spacing used for $\mathbf{E}$ from the *endpoint* difference used for work. Keep the original normal direction and the algebraic negative charge.

Chapter support: Example 4.6; complete solution on slide 22.

Example 4.6: field and work from the map



Treat each interval as locally uniform. The normal points toward lower potential, so

$$\mathbf{E} \simeq \frac{3.0 \text{ V}}{6.0 \times 10^{-3} \text{ m}} \hat{\mathbf{n}} = 500 \hat{\mathbf{n}} \text{ V/m.}$$

Across the outer contours, $\Delta V = 12.0 - 18.0 = -6.0 \text{ V}$:

$$\Delta U = (-3.0 \text{ nC})(-6.0 \text{ V}) = +18 \text{ nJ}, \quad W_{\text{elec}} = -18 \text{ nJ.}$$

Check: the negative charge is displaced along $\hat{\mathbf{n}}$ but its force is opposite that direction. The electrical work must be negative. Work is fixed by endpoints even if the local field estimate is approximate.

Chapter support: Example 4.6; complete statement on slide 21.

Point-charge potential from its field



For a signed source Q in vacuum, $E_r = k_e Q/r^2$. Starting at a finite reference radius r_b ,

$$V(r) - V(r_b) = -k_e Q \int_{r_b}^r \frac{dr'}{r'^2} = k_e Q \left(\frac{1}{r} - \frac{1}{r_b} \right).$$

Taking $r_b \rightarrow \infty$ yields

$$V(r) = \frac{k_e Q}{r}, \quad V(\infty) = 0, \quad k_e = 8.99 \times 10^9 \text{ N m}^2/\text{C}^2.$$

r is a positive source-to-point distance. Its sign never comes from the observation coordinate. With this reference, the source sign sets the potential sign. **Check:** $-dV/dr = k_e Q/r^2$ recovers the radial field, including its sign.

Chapter support: Section 4.6; exclude the source point $r = 0$.

Point source: equal voltage steps

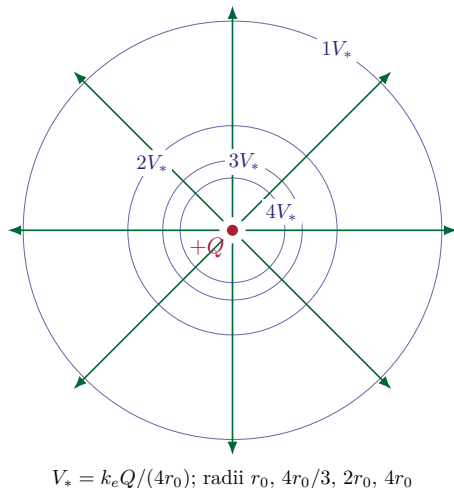


Figure 4.4: A plane through a positive point charge intersects its equipotential spheres in circles. The radii are calculated from $V = k_e Q / r$: equal potential increments crowd closer to the charge. Green arrows show radial field direction, not field magnitude.

Equal voltage increments correspond to smaller radial spacings near the source. The derivative determines field magnitude.

Chapter support: Figure 4.4; point-charge potential on slide 23.

Scalar superposition: square geometry

Signed scalar potentials add algebraically.

Example 4.7 — complete statement

Four charges lie at the corners of a square of side $a = 0.20\text{ m}$: $+2.0\text{ nC}$, -2.0 nC , $+3.0\text{ nC}$, and $+1.0\text{ nC}$. Find the potential at the centre.

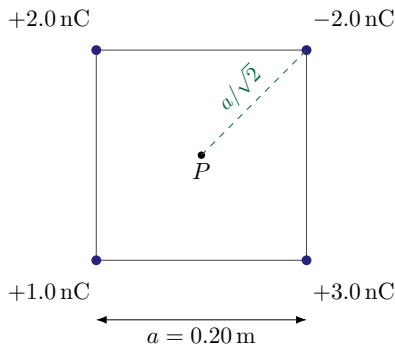


Figure 4.5: Geometry of Example 4.7. Each corner lies the same distance from P . Source signs enter an algebraic sum for V ; electric fields must also include their directions.

Chapter support: Example 4.7; Figure 4.5; solution on slide 26.

Example 4.7: add signs, not arrows



The centre-to-corner distance is

$$r = \sqrt{(a/2)^2 + (a/2)^2} = a/\sqrt{2} = 0.141421 \text{ m.}$$

The algebraic charge sum is $(2.0 - 2.0 + 3.0 + 1.0) \text{ nC} = 4.0 \text{ nC}$, hence

$$V(P) = \frac{(8.99 \times 10^9)(4.0 \times 10^{-9})}{0.20/\sqrt{2}} \text{ V} = 2.54 \times 10^2 \text{ V.}$$

Check: all distances are equal, so permuting the four charges leaves $V(P)$ unchanged. It can change the field because fields add as vectors. The potential units are $(\text{N m}^2/\text{C}^2)(\text{C})/\text{m} = \text{J/C}$.

Chapter support: Example 4.7; statement and original geometry on slide 25.

Example 4.9: locate zeros with distances



Example 4.9 — complete statement

Place $+Q$ at $x = 0$ and $-2Q$ at $x = d$, with $Q > 0$ and $d > 0$. Find all finite points on the x axis where $V = 0$, using zero potential at infinity.

Away from both sources,

$$V = k_e Q \left(\frac{1}{|x|} - \frac{2}{|x - d|} \right) = 0 \implies |x - d| = 2|x|.$$

For $x < 0$, $d - x = -2x$ gives $x = -d$. For $0 < x < d$, $d - x = 2x$ gives $x = d/3$. For $x > d$, the equation gives $x = -d$, outside that region. Thus the finite zeros are $-d$ and $d/3$. **Force check:** at $d/3$ both fields point right. At $-d$,

$$E_x = -k_e Q/d^2 + 2k_e Q/(2d)^2 = -k_e Q/(2d^2) \neq 0.$$

Zero potential is a reference-dependent value, not a force-free condition.

Chapter support: Example 4.9; original sources and infinity reference retained.

Dipole potential: exact to far field



Place $-q$ at $x = -d/2$ and $+q$ at $x = d/2$, with $q > 0$. Define $\mathbf{p} = qd\hat{\mathbf{i}}$ and measure θ from $\mathbf{p}$:

$$r_{\pm} = \sqrt{r^2 + d^2/4 \mp rd \cos \theta}, \quad V = k_e q \left(\frac{1}{r_+} - \frac{1}{r_-} \right).$$

For $d/r \ll 1$, $(1 + u)^{-1/2} \simeq 1 - u/2$ gives

$$\frac{1}{r_{\pm}} \simeq \frac{1}{r} \left(1 \pm \frac{d \cos \theta}{2r} \right), \quad V \simeq \frac{k_e p \cos \theta}{r^2}.$$

The neutral source cancels the monopole $1/r$ term. Its potential generally falls as $1/r^2$ and its field as $1/r^3$. On the perpendicular bisector the *exact* potential is zero by equal distances.

Chapter support: Section 4.7; source orientation differs from the separate Example 4.8.

Exact dipole contours

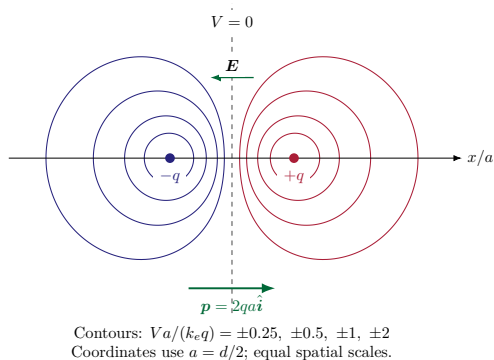


Figure 4.6: Calculated equipotential contours of the exact two-charge dipole. Positive contours surround $+q$ and negative contours surround $-q$; the perpendicular bisector has zero potential. At its regular points the field crosses from the positive side toward the negative side. The green field arrow shows direction only.

The zero bisector is crossed by the field; it is not field free. Near either charge use the exact two-source expression, not the distant dipole formula.

Chapter support: Figure 4.6; unchanged computed coordinate data.

Example 4.10: quantify far enough

Example 4.10 — complete statement

For $q = 2.0 \text{ nC}$, $d = 1.0 \text{ cm}$, evaluate the exact and dipole potentials at $x = r = 5.0 \text{ cm}$ on the positive axis. Estimate the relative approximation error.

On this axis, $r_+ = r - d/2$ and $r_- = r + d/2$:

$$V_{\text{exact}} = \frac{k_e q d}{r^2 - d^2/4} = 72.6 \text{ V}, \quad V_{\text{dip}} = \frac{k_e q d}{r^2} = 71.9 \text{ V}.$$

Combining the fractions provides an independent exact error formula:

$$\frac{V_{\text{exact}} - V_{\text{dip}}}{V_{\text{exact}}} = \frac{d^2}{4r^2} = 0.010 = 1.0\%.$$

The approximation underestimates the positive potential. This is model error, not measurement uncertainty; near a zero potential use absolute error rather than divide by zero.

Chapter support: Example 4.10; relative error denominator is the exact value.

Continuous sources: keep the point fixed



Replace the scalar sum by an integral:

$$V(\mathbf{r}) = k_e \int \frac{dq}{|\mathbf{r} - \mathbf{r}'|}.$$

The observation coordinate $\mathbf{r}$ stays fixed while the source coordinate $\mathbf{r}'$ runs over the object. Use $dq = \lambda dl'$, $\sigma dA'$ or $\rho d\tau'$ for a line, surface or volume.

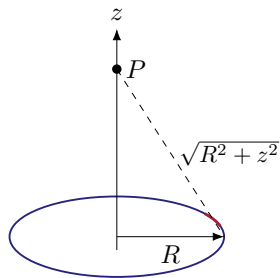
The modelling sequence

Specify the charge density; choose an element and limits; express its positive distance to the observation point; integrate; test units and a far-field limit. Scalar addition removes vector components, not geometry.

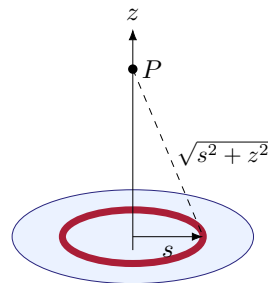
A uniform insulating distribution is prescribed charge data. A conducting object held at one voltage generally has a nonuniform surface charge and requires a boundary calculation.

Chapter support: Section 4.8; vacuum source integral.

Ring and disk: the integration geometry



(a) ring: $dq = \lambda R d\phi$



(b) disk ring: $dq = \sigma 2\pi s ds$

Figure 4.7: Perspective schematics for axial potential integrals. The disk is decomposed into annuli of radius s and radial width ds . The axial coordinate z remains fixed while s is integrated.

All elements of one ring have the same distance to the axial point. For the disk that distance changes with annular radius s while z remains fixed.

Chapter support: Figure 4.7; the two panels are unchanged.

A uniform ring: potential, field and limits



For a uniform ring of radius R and total charge Q , every element is at $\sqrt{R^2 + z^2}$:

$$V(z) = \frac{k_e}{\sqrt{R^2 + z^2}} \int dq = \frac{k_e Q}{\sqrt{R^2 + z^2}}, \quad V(\infty) = 0.$$

An axial derivative gives

$$E_z = -\frac{dV}{dz} = \frac{k_e Qz}{(R^2 + z^2)^{3/2}}.$$

Rotational symmetry cancels the transverse field. At the centre, $V = k_e Q/R$ but $E_z = 0$; far away, $V \simeq k_e Q/|z|$. **Scope check:** even a nonuniform ring has this axial potential for the same total Q , but its transverse field need not vanish. Data on a line determine only derivatives along that line.

Chapter support: Sections 4.8 and 4.8.2; lecture adaptation of Example 4.11.

Example 4.12: a finite rod on its axis



Example 4.12 — complete statement

A rod extends from $x = 0$ to $x = L$ and has linear charge density λ . Find the potential at $x = a > L$ on the same axis.

With the finite-source infinity reference,

$$\begin{aligned} V(a) &= k_e \lambda \int_0^L \frac{dx}{a-x} = k_e \lambda [-\ln(a-x)]_0^L \\ &= k_e \lambda \ln \left(\frac{a}{a-L} \right), \quad a > L. \end{aligned}$$

Checks: $k_e \lambda$ has units V, and the logarithm is dimensionless. For $a \gg L$, $-\ln(1 - L/a) \simeq L/a$, so $V \simeq k_e Q/a$ with $Q = \lambda L$. The line model diverges at $a \rightarrow L^+$; a real rod's radius limits its use there.

Chapter support: Example 4.12; no field-point distance is signed.

A rod on its perpendicular bisector



Let a uniform rod span $-L/2 \leq x' \leq L/2$ and observe it at $(0, b)$, $b > 0$. Its distance is $\sqrt{x'^2 + b^2}$:

$$\begin{aligned} V(b) &= k_e \lambda \int_{-L/2}^{L/2} \frac{dx'}{\sqrt{x'^2 + b^2}} \\ &= 2k_e \lambda \ln \left(\frac{L/2 + \sqrt{(L/2)^2 + b^2}}{b} \right). \end{aligned}$$

The factor two uses an even integrand. An antiderivative is $\operatorname{arsinh}(x'/b)$, equivalently the logarithmic expression evaluated at the limits. For $b \gg L$, $V \simeq k_e \lambda L/b$. For an infinite rod at fixed λ , the absolute potential diverges; differences between finite radii remain meaningful.

Chapter support: Section 4.8.3; vacuum, thin insulating line model.

Example 4.13: is a point model adequate?



Example 4.13 — complete statement

A thin insulating rod of length 0.40 m carries a uniformly distributed charge 8.0 nC. Calculate the potential 0.15 m from its centre on the perpendicular bisector. Is a point-charge model adequate there?

$\lambda = Q/L = 20 \text{ nC/m}$ and $\sqrt{(L/2)^2 + b^2} = 0.25 \text{ m}$. Therefore

$$V = 2(8.99 \times 10^9)(20 \times 10^{-9}) \ln 3 \text{ V} = 395 \text{ V}.$$

The point model gives $V_{\text{point}} = k_e Q/b = 479 \text{ V}$, so

$$\frac{V_{\text{point}} - V}{V} = 0.214 \simeq 21\%.$$

This is not a good small-error approximation here: b is not large relative to L . Most source elements are farther away than the centre, so concentrating all charge there must overestimate their potential.

Chapter support: Example 4.13; error relative to the extended-rod potential.

A charged disk assembled from annuli



For an insulating disk of radius R with prescribed uniform σ , an annulus of radius s contributes $dq = 2\pi\sigma s ds$. Hence

$$\begin{aligned} V(z) &= 2\pi k_e \sigma \int_0^R \frac{s ds}{\sqrt{s^2 + z^2}} \\ &= \frac{\sigma}{2\epsilon_0} \left[\sqrt{s^2 + z^2} \right]_0^R = \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2 + z^2} - |z| \right). \end{aligned}$$

The substitution $u = s^2 + z^2$ supplies the antiderivative. $\sqrt{z^2} = |z|$, so the potential is even on the two sides. For large $|z|$, avoid subtracting nearly equal terms:

$$V = \frac{\sigma R^2}{2\epsilon_0(\sqrt{R^2 + z^2} + |z|)}.$$

Chapter support: Section 4.8.4; Figure 4.7 on slide 32.

Disk field: distinguish two useful limits



For $z > 0$,

$$E_z = -\frac{dV}{dz} = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{R^2 + z^2}} \right).$$

The field reverses for $z < 0$. V is continuous but has a cusp at the ideal sheet; the one-sided field jump is σ/ϵ_0 .

Near and far

For $0 < z \ll R$, $E_z \rightarrow \sigma/(2\epsilon_0)$, the infinite-sheet field. For $|z| \gg R$, $V \simeq k_e Q/|z|$ with $Q = \pi R^2 \sigma$.

At fixed σ , taking $R \rightarrow \infty$ leaves a finite field but a divergent infinity-referenced potential. The approximation tolerance must be stated, not assumed from the word “near.”

Chapter support: Section 4.8.4; fixed surface density, not a fixed-voltage conductor.

Example 4.14: test an infinite-sheet estimate



Example 4.14 — complete statement

A uniformly charged disk has $R = 0.10\text{ m}$ and $\sigma = 4.0\text{ nC/m}^2$. Find V and E_z at $z = 0.020\text{ m}$ and compare the field with the infinite-sheet estimate.

Put $E_0 = \sigma/(2\epsilon_0) = 225.887\text{ V/m}$ and $D = \sqrt{R^2 + z^2} = 0.101980\text{ m}$:

$$V = E_0(D - z) = 18.5\text{ V}, \quad E_z = E_0(1 - z/D) = 182\text{ V/m}.$$

The deficit relative to the sheet is $(E_0 - E_z)/E_0 = z/D = 19.6\%$. For a deficit below 5%,

$$\frac{z/R}{\sqrt{1 + (z/R)^2}} < 0.05 \quad \Rightarrow \quad z/R < 0.0501.$$

At $z/R = 0.20$, the near-sheet approximation misses the specified tolerance. The far-field limit and the sign on the upper side independently check the result.

Chapter support: Example 4.14; $\epsilon_0 = 8.854 \times 10^{-12}\text{ F/m}$.

Recover a vector from local slopes



At fixed y, z , a small step in x gives $dV = -E_x dx$, so $E_x = -\partial V / \partial x$. Repeat for y and z :

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right).$$

Comparison of $dV = \nabla V \cdot d\mathbf{r}$ with the voltage integral proves the relation for every displacement.

Directional interpretation

Along a unit direction $\hat{\mathbf{s}}$, $dV/ds = \nabla V \cdot \hat{\mathbf{s}}$. The greatest decrease per distance is along $\mathbf{E}$. An equipotential tangent has zero directional slope.

A potential scan along one line gives one component, not automatically the whole field. The derivative has units $V/m = N/C$.

Chapter support: Section 4.9; local gradient derivation.

Potential slope and the spatial field

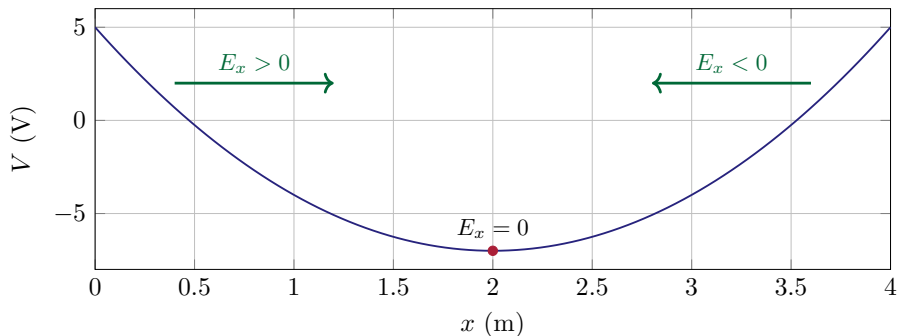


Figure 4.8: The potential in Example 4.15. The field component is the negative slope, and its direction is along the spatial x axis. A field vector is not a tangent vector drawn in the (x, V) plotting plane.

A rising $V(x)$ means $E_x < 0$ and a falling curve means $E_x > 0$. The arrows represent spatial directions, not tangent vectors in the (x, V) plotting plane.

Chapter support: Figure 4.8; Example 4.15 completed on slide 42.

Example 4.15: a stationary potential

Example 4.15 — complete statement

Let

$$V(x) = ax^2 + bx + c, \quad (1)$$

with $a = 3.0 \text{ V/m}^2$, $b = -12 \text{ V/m}$, and $c = 5.0 \text{ V}$.

Differentiate at fixed other coordinates:

$$E_x = -(2ax + b) = 12 \text{ V/m} - (6.0 \text{ V/m}^2)x.$$

Thus $E_x = 0$ at $x = 2.0 \text{ m}$, where $V = -7.0 \text{ V}$. Zero slope, rather than zero potential, is the force-free condition in this constrained coordinate.

$$U'' = qV'' = q(6.0 \text{ V/m}^2).$$

A positive charge has a restoring force near this minimum; a negative charge is unstable along x . A common constant added to V changes neither conclusion. Transverse stability requires the full spatial potential.

Chapter support: Example 4.15 and Section 4.9.3; no claim of a free three-dimensional trap.

Example 4.16: a two-dimensional voltage



Example 4.16 — complete statement

In a small region, let

$$V(x, y) = A(x^2 - y^2) + Bx + V_0, \quad (2)$$

where $A = 20 \text{ V/m}^2$, $B = -6.0 \text{ V/m}$ and $V_0 = 4.0 \text{ V}$. Find the field at $(0.10, 0.05) \text{ m}$ and estimate the potential change for displacement $\Delta \mathbf{r} = (1.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}) \text{ mm}$.

Find each partial derivative with the other coordinate held fixed. Then compare the local first-order estimate $-\mathbf{E} \cdot \Delta \mathbf{r}$ with the exact change of the polynomial.

Chapter support: Example 4.16; complete answer on slide 44.

Example 4.16: gradient and finite-step error



$$E_x = -2Ax - B, \quad E_y = 2Ay.$$

At $(0.10, 0.05) \text{ m}$,

$$\mathbf{E} = (2.0\hat{\mathbf{i}} + 2.0\hat{\mathbf{j}}) \text{ V/m}, \quad |\mathbf{E}| = 2.83 \text{ V/m}, \quad \theta = 45^\circ.$$

Convert the displacement to $(0.001\hat{\mathbf{i}} + 0.002\hat{\mathbf{j}}) \text{ m}$:

$$\Delta V_{\text{local}} = -\mathbf{E} \cdot \Delta \mathbf{r} = -0.0060 \text{ V}.$$

Direct subtraction of the polynomial supplies its exact quadratic correction,

$$A[(\Delta x)^2 - (\Delta y)^2] = -0.000060 \text{ V}, \quad \Delta V_{\text{exact}} = -6.06 \text{ mV}.$$

The -0.060 mV correction measures finite-step bias. V_0 cancels, and units in each correction are $(\text{V/m}^2)\text{m}^2 = \text{V}$.

Chapter support: Example 4.16; statement on slide 43.

Assembly energy counts every pair once



Bring charges from infinite separation one at a time, keeping already placed sources fixed. The first contributes no interaction energy; the next contributes its energy in the potential of earlier charges.

$$U = k_e \sum_{i < j} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_i q_i V_i.$$

V_i excludes the charge's self-potential. The second expression counts each pair twice before the factor $1/2$; the first counts $N(N-1)/2$ unordered pairs directly.

Interpret the signs

Like pairs contribute positive energy; unlike pairs contribute negative energy relative to separated charges. A negative total interaction energy alone proves neither static stability nor a bound trajectory: kinetic energy and escape routes also matter.

Chapter support: Section 4.10; self-energies excluded from this point-charge model.

Three charges: check the separations first

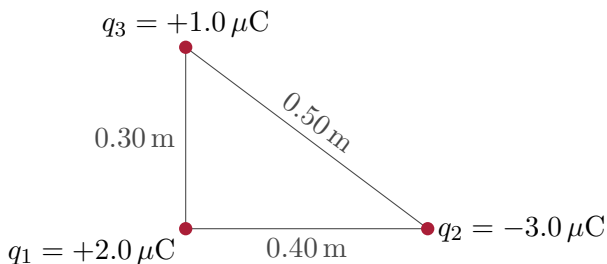


Figure 4.9: Three pair separations in the assembly calculation. The source positions are fixed while the interaction energy is evaluated.

Use three unordered pairs. The diagonal is $\sqrt{0.40^2 + 0.30^2} = 0.50 \text{ m}$, not either coordinate length. Quasistatic assembly work equals the interaction-energy increase.

Chapter support: Figure 4.9; complete Example 4.17 on slide 47.

Example 4.17: assembly can release energy



Example 4.17 — complete statement

Let $q_1 = +2.0 \mu\text{C}$ at the origin, $q_2 = -3.0 \mu\text{C}$ at $x = 0.40 \text{ m}$, and $q_3 = +1.0 \mu\text{C}$ at $y = 0.30 \text{ m}$. The separations are $r_{12} = 0.40 \text{ m}$, $r_{13} = 0.30 \text{ m}$, and $r_{23} = 0.50 \text{ m}$.

$$U = k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right).$$

Converting μC to 10^{-6}C , the three terms are -0.134850 , $+0.059933$ and -0.053940 J .

$$U = -0.128857 \text{ J} \simeq -0.129 \text{ J} = W_{\text{app,qs}}.$$

An external agent extracts net energy during quasistatic assembly. As a second check, assembling q_1 , then q_2 , then q_3 gives $W_2 = U_{12}$ and $W_3 = U_{13} + U_{23}$, the same sum. Released freely, the charges need not stop at these positions.

Chapter support: Example 4.17; geometry on slide 46; all three pair terms shown.

Energy conservation: the electron sign



With only electrostatic work, $K_i + qV_i = K_f + qV_f$, hence $\Delta K = -q\Delta V$.
The electronvolt is energy: $1\text{ eV} = 1.602 \times 10^{-19}\text{ J}$.

Example 4.3 — complete statement

An electron moves from point a to point b , where $V_b - V_a = +250\text{ V}$. Find ΔU and the kinetic-energy change if no non-electrostatic work is done.

For $q = -e$,

$$\Delta U = (-1.602 \times 10^{-19})(250)\text{ J} = -4.01 \times 10^{-17}\text{ J},$$

$$\Delta K = +4.01 \times 10^{-17}\text{ J} = 250\text{ eV}.$$

The electron gains kinetic energy at higher V because qV decreases. The two changes sum to zero; a positive charge would reverse their signs.

Chapter support: Example 4.3; Section 4.11; fixed sources and no non-electrostatic work.

Turning points are not equilibrium points

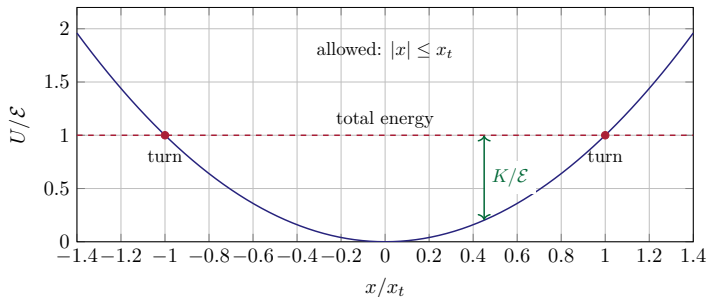


Figure 4.10: A constrained quadratic potential-energy landscape $U = \mathcal{E}(x/x_t)^2$. Turning points satisfy $K = 0$; equilibrium is at $x = 0$, where the force vanishes. The energy gap gives speed but not the sign of velocity.

In one-dimensional motion, $K(x) = \mathcal{E} - qV(x) \geq 0$. An intersection with the energy line gives $K = 0$; a horizontal tangent gives zero force. These conditions are different. Energy gives speed magnitude, while initial motion selects its sign.

Chapter support: Section 4.11.3; Figure 4.10 is a constrained local model.

Example 4.18: first test accessibility



Example 4.18 — complete statement

A particle has charge $q = -4.0 \text{ nC}$ and mass $m = 2.0 \text{ mg}$. It moves from a to b , where $V_b - V_a = -600 \text{ V}$. Its initial speed is 0.80 m/s . No non-electrostatic work is done.

$$\Delta U = (-4.0 \times 10^{-9})(-600) \text{ J} = +2.4 \mu\text{J},$$

$$K_i = \frac{1}{2}(2.0 \times 10^{-6})(0.80)^2 \text{ J} = 0.64 \mu\text{J}.$$

Then $K_f = K_i - \Delta U = -1.76 \mu\text{J}$ would be required. Since $K \geq 0$, the endpoint is inaccessible. In a monotonic one-dimensional path, turning occurs at

$$\Delta V_{\text{turn}} = K_i/q = -160 \text{ V}.$$

This specifies a voltage, not a position without $V(x)$. In general geometry the energy test excludes b but does not determine the trajectory.

Chapter support: Example 4.18 and Section 4.11.1; $2.0 \text{ mg} = 2.0 \times 10^{-6} \text{ kg}$.

Practice P27: where does the bead turn?



P27 — complete statement

A negative bead has $q = -4.0 \text{ nC}$, $m = 2.0 \text{ mg}$ and initial speed 0.80 m/s . It moves along a horizontal track where $V(x) = V_i - (2.0 \times 10^3 \text{ V/m})x$. Find its turning position. Assume the track is frictionless and exerts no tangential force.

The stated potential profile supplies the missing spatial information from Example 4.18. Take $x = 0$ at the bead's initial position and its initial velocity toward $+x$. **Attempt:** calculate K_i , write $q[V(x) - V_i]$, and set $K = 0$. Then verify the stopping distance with the force qE_x .

Chapter support: P27; electrostatic vacuum model; complete solution on slide 52.

P27: energy and force agree



With $m = 2.0 \times 10^{-6} \text{ kg}$, $K_i = 6.4 \times 10^{-7} \text{ J}$. The interaction energy rises by

$$q\Delta V = (-4.0 \times 10^{-9})(-2.0 \times 10^3)x = (8.0 \times 10^{-6} \text{ J/m})x.$$

At a turning point, $K_f = q\Delta V$, giving

$$x_{\text{turn}} = \frac{6.4 \times 10^{-7}}{8.0 \times 10^{-6}} \text{ m} = 0.080 \text{ m}.$$

Independent force check: $E_x = -V' = +2000 \text{ V/m}$, so $F_x = -8.0 \mu\text{N}$ and $a_x = -4.0 \text{ m/s}^2$. Then

$$x_{\text{stop}} = v_i^2 / (2|a_x|) = 0.080 \text{ m}.$$

The negative bead initially moves against its force and slows. The track's normal force does no work in the stated model.

Chapter support: P27; complete statement and added coordinate convention on slide 51.

A conductor at one potential

In electrostatic equilibrium, $\mathbf{E} = 0$ in the metal, so $\Delta V = -\int \mathbf{E} \cdot d\mathbf{l} = 0$ between its points. A tangential surface field would drive charge motion.

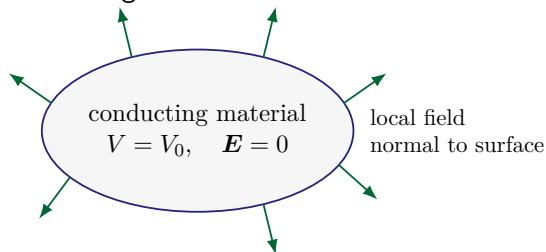


Figure 4.11: Surface field directions for a positively charged conductor in electrostatic equilibrium. The arrows use the normal to the illustrated ellipse; their lengths do not represent field magnitudes. No assumption of elliptical external equipotentials is made.

Immediately outside in vacuum, $\mathbf{E}_{\text{out}} = (\sigma/\epsilon_0)\hat{\mathbf{n}}$ with an outward normal and signed σ . Constant surface potential permits different normal field magnitudes.

Chapter support: Section 4.12; Figure 4.11; no arbitrary local curvature law assumed.

A conducting sphere is not a solid insulator

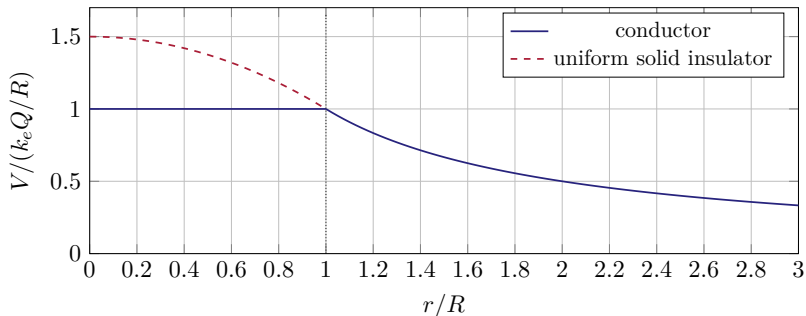


Figure 4.12: Calculated radial potentials for two spheres with the same radius and total charge. Both have $V(\infty) = 0$ and the same exterior potential. The conductor has constant interior potential; the uniform solid insulator has a quadratic interior potential.

For $r \leq R$, the conductor has $V = k_e Q/R$; the prescribed uniform solid insulator has $V = k_e Q(3 - r^2/R^2)/(2R)$. Both have $V = k_e Q/r$ outside. Only the conductor is field free throughout its interior.

Chapter support: Section 4.12.1; Figure 4.12; isolated spherical symmetry.

Example 4.22: an isolated conducting sphere



Example 4.22 — complete statement

An isolated conducting sphere has $R = 5.0 \text{ cm}$ and $Q = 2.0 \text{ nC}$. Find its potential, its surface field just outside, and the voltage change from the surface to $r = 20 \text{ cm}$.

Remote other objects permit spherical symmetry:

$$V_s = k_e Q/R = 359.6 \text{ V} \simeq 360 \text{ V},$$

$$E_s = k_e Q/R^2 = 7.19 \times 10^3 \text{ V/m} \quad \text{outward},$$

$$V(0.20) - V(R) = k_e Q(1/0.20 - 1/0.050) \text{ m}^{-1} \simeq -270 \text{ V}.$$

Check: $E_s = V_s/R$ with the infinity reference, and integrating the exterior field returns the same voltage. The metal has nonzero potential with zero internal field; this geometric E_s/V_s relation is not universal.

Chapter support: Example 4.22; $R = 0.050 \text{ m}$, $Q = 2.0 \times 10^{-9} \text{ C}$.

What a voltage-mapping experiment measures

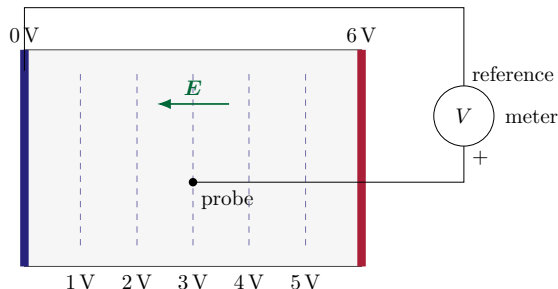


Figure 4.13: A low-voltage sheet-mapping arrangement. The central contours are drawn for an ideal uniform sheet with full-width parallel electrodes and insulating side edges. The meter reads 3 V at the shown probe position. The drawing specifies measurement connections, not a complete power-supply circuit.

The meter reports its positive-input potential minus its reference-input potential; reversing the leads reverses the sign. This steady-current sheet is an analogue: uniform conductivity gives $\nabla \cdot \mathbf{J} = 0$ and $\mathbf{J} = -\sigma_c \nabla V$, hence $\nabla^2 V = 0$.

Chapter support: Section 4.13; Figure 4.13; same boundary data, two-dimensional analogue.

A field estimate has spacing and noise errors



For smooth potential at accurately known positions $x \pm h$,

$$E_x^{(h)} = -\frac{V(x+h) - V(x-h)}{2h} = -V'(x) - \frac{h^2}{6}V^{(3)}(x) + \mathcal{O}(h^4).$$

Taylor expansions at $x+h$ and $x-h$ cancel the even powers in the numerator. The leading spacing error is second order; the sample separation is $2h$. For independent readings of equal standard uncertainty u_V ,

$$u(E_x) = \frac{\sqrt{u_V^2 + u_V^2}}{2h} = \frac{u_V}{\sqrt{2}h}.$$

Smaller spacing reduces smooth-function bias but amplifies reading noise. A common additive offset cancels; a scale error does not. Position error and correlated readings need additional terms.

Chapter support: Section 4.13.3; do not apply a smooth stencil across a surface-field jump.

Example 4.24: a field with stated uncertainty

Example 4.24 — complete statement

An illustrative pair of readings is $V(x+h) = 4.80 \text{ V}$ and $V(x-h) = 5.20 \text{ V}$ at $h = 2.0 \text{ mm}$. Each has independent standard uncertainty 0.02 V . Estimate the field component and its reading uncertainty.

The sample separation is $2h = 0.0040 \text{ m}$:

$$E_x = -\frac{4.80 - 5.20}{0.0040} \text{ V/m} = +100 \text{ V/m},$$

$$u(E_x) = \frac{0.02}{\sqrt{2}(0.0020)} \text{ V/m} = 7.07 \text{ V/m}.$$

Report $E_x = (100 \pm 7) \text{ V/m}$, with one standard reading uncertainty only. The positive field agrees with decreasing potential toward $+x$. Halving h doubles this noise contribution; position error and finite-spacing bias are not included.

Chapter support: Visible Example 4.24; internal label ch4-adv-ex-1 occurs in the [core manuscript](#).

Core synthesis: connect four descriptions



Energy and endpoints

$$\Delta U = q\Delta V, \quad \Delta K = -q\Delta V.$$

The second relation assumes only electrical work. Test $K \geq 0$ before computing a speed. **Prescribed sources**

$$V = k_e \sum_i q_i / r_i \quad \text{or} \quad k_e \int dq / R.$$

State geometry and reference.

Work and local field

$$\Delta V = - \int \mathbf{E} \cdot d\mathbf{l}, \quad \mathbf{E} = -\nabla V.$$

Field follows slope, not potential value. **Assembly**

$$U = k_e \sum_{i < j} q_i q_j / r_{ij}.$$

Count pairs once; omit point self-energies. One conductor has one equilibrium potential.

Chapter support: Sections 4.14–4.15; this sheet summarizes, rather than replaces, the derivations.

From potential to electrode boundary data



Knowing a voltage difference constrains work and an average field; local field structure also depends on charge density and the whole geometry. The advanced block now asks how to calculate V when electrode values are prescribed.

Bridge to Chapter 5

For the isolated sphere, $Q/V_s = 4\pi\epsilon_0 R$ depends on geometry. The next chapter develops capacitance, conductors in pairs, stored energy and dielectric response using this voltage language.

Further reading in this chapter: Sections 4.11–4.13 extend the energy and conductor applications; P1–P40 supply optional practice. The manuscript cites Halliday–Resnick–Walker Vol. 2, Ch. 24; Serway–Jewett, Ch. 25; and OpenStax Vol. 2, Ch. 7.

Chapter support: Section 4.12.4 and manuscript bibliography; textbook pointers were not separately consulted.

Advanced: charge density and boundary potential



Gauss's local law and the gradient relation give

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \quad \nabla^2 V = -\rho/\epsilon_0.$$

The Laplacian is the sum of second spatial derivatives, with units V/m^2 . In a charge-free region it is zero; the field itself need not be zero. **Boundary data:** specified potential is a Dirichlet condition. Specified normal derivative is a Neumann condition. A floating conductor instead has constant unknown potential and a total-charge constraint.

Why the construction is decisive

Two solutions with the same density and boundary potentials differ by a harmonic w with zero boundary value. Integration by parts gives $\int |\nabla w|^2 d\tau = 0$, so $w = 0$: the Dirichlet solution is unique.

Chapter support: Section 4.19; use this source-and-boundary test in the next challenge.

Advanced challenge: a charged electrode gap

Example 4.25 — complete statement

An ideal laterally infinite vacuum slab occupies $0 < x < d$, has prescribed uniform volume charge density ρ_0 , and is bounded by electrodes at $V(0) = 0$ and $V(d) = V_b$. Find $V(x)$ and $E_x(x)$.

Solution: integrate $V'' = -\rho_0/\epsilon_0$ twice:

$$V = -\frac{\rho_0 x^2}{2\epsilon_0} + Ax + B, \quad B = 0, \quad A = \frac{V_b}{d} + \frac{\rho_0 d}{2\epsilon_0}.$$

$$V(x) = \frac{V_b x}{d} + \frac{\rho_0 x(d-x)}{2\epsilon_0}, \quad E_x = -\frac{V_b}{d} + \frac{\rho_0}{\epsilon_0}(x - d/2).$$

Both endpoint potentials are satisfied; $V'' = -\rho_0/\epsilon_0$ and $E'_x = \rho_0/\epsilon_0$. Also $\int_0^d E_x dx = -V_b$. Bias fixes the average field, while density shapes the local field.

Chapter support: Example 4.25; complete statement, solution and independent residual/boundary checks.

A zero field can hide a saddle

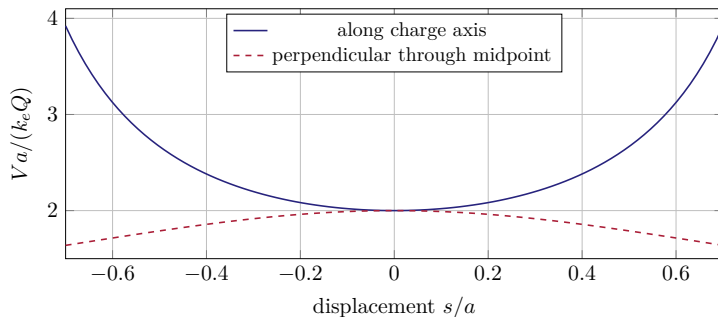


Figure 4.14: Exact potential cuts for two equal positive charges at $x = \pm a$. The origin is a minimum along x and a maximum along a transverse line: a saddle of the full potential.

Two like charges have zero field at the midpoint, yet the curvatures have opposite signs along and across the charge axis. For a positive probe the axial force restores, while transverse forces drive it away. A one-dimensional scan cannot establish a free trap.

Chapter support: Section 4.20.1; Figure 4.14.

Ring stability: examine all directions



Expand the positive uniform ring's axial potential near its centre:

$$V(0,0,z) \simeq \frac{k_e Q}{R} - \frac{k_e Q}{2R^3} z^2, \quad |z| \ll R.$$

The centre is charge free. Rotation symmetry and $\nabla^2 V = 0$ fix the full quadratic term:

$$V \simeq \frac{k_e Q}{R} + \frac{k_e Q}{4R^3} (x^2 + y^2) - \frac{k_e Q}{2R^3} z^2.$$

A negative probe minimizes $U = qV$ along z but maximizes it transversely. A support must supply the transverse constraint. The mean-value property of a harmonic potential excludes a strict interior maximum or minimum unless it is constant. Thus static external electrostatic forces alone cannot make an isolated stable equilibrium of a fixed point charge in a charge-free region; constraints and other forces change that conclusion.

Chapter support: Section 4.20.2; relevant Earnshaw theorem and small-displacement regime.

Example 4.26: a constrained ring oscillation

Example 4.26 — complete statement

A bead with negative charge q and mass m is constrained to the axis of a fixed positive uniform ring. For $|z| \ll R$, find its angular frequency of small oscillations. Ignore gravity and friction.

From slide 64, $E_z \simeq k_e Q z / R^3$. Since $q = -|q|$,

$$m\ddot{z} = -\frac{k_e Q |q|}{R^3} z, \quad \boxed{\omega = \sqrt{\frac{k_e Q |q|}{m R^3}}}.$$

For the source example's $Q = 5.0 \text{ nC}$, $|q| = 1.0 \text{ nC}$, $R = 0.050 \text{ m}$ and $m = 1.0 \text{ mg}$,

$$\omega = 19.0 \text{ s}^{-1}, \quad T = 2\pi/\omega = 0.331 \text{ s}.$$

Checks: $qV''(0) > 0$ and $k_e Q |q| / (m R^3)$ has units s^{-2} . Without the axial support, the transverse instability remains; this is not a free electrostatic trap.

Chapter support: Example 4.26; nonperturbing probe; $|z| \ll R$.

Grounded plane: an image is a construction

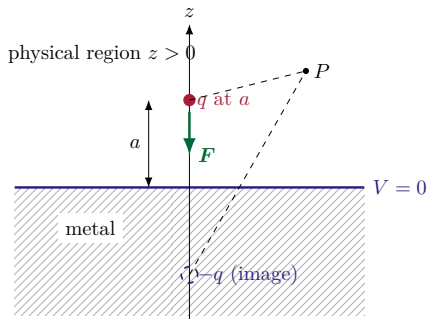


Figure 4.15: Image construction for a point charge above an infinite grounded plane. The image reproduces the upper-half-space field; it is not a real source in the metal. The force on the real charge points toward the plane for either sign of q .

In $z > 0$, the real charge at a and image $-q$ at $-a$ give the same distances on the plane, hence $V = 0$. Their potential has the correct real singularity and remote limit, and solves Laplace's equation elsewhere. Uniqueness establishes it in the physical upper region.

Chapter support: Section 4.22; Figure 4.15; the grounded metal itself has $V = 0$.

Induced sources: force and energy differ



The image represents the induced charges' field. At the real charge its distance is $2a$, so the self-field-excluded force is

$$\mathbf{F} = -\frac{k_e q^2}{4a^2} \hat{\mathbf{k}}.$$

On the plane, radial coordinate s gives $\sigma(s) = -qa/[2\pi(s^2 + a^2)^{3/2}]$, whose integral is $Q_{\text{ind}} = -q$. The reservoir supplies this charge. When a changes, the induced distribution changes too. Integrate the *actual* force from infinite height:

$$U(a) = -\int_{\infty}^a F_z(a') da' = -\frac{k_e q^2}{4a} = \frac{1}{2} q V_{\text{ind}}, \quad V_{\text{ind}} = -\frac{k_e q}{2a}.$$

The naive real-image pair energy is twice too negative. Differentiating the physical U returns $-dU/da = -k_e q^2/(4a^2)$; the attraction holds for either sign of q .

Chapter support: Sections 4.22.2–4.22.3; grounded infinite plane, responding source.

Example 4.28: lift against induced attraction



Example 4.28 — complete statement

A charge $q = 1.0 \text{ nC}$ is moved quasistatically from $a_1 = 1.0 \text{ cm}$ to $a_2 = 3.0 \text{ cm}$ above the ideal plane. Calculate the required external work and the initial force magnitude.

Using the physical energy from slide 67,

$$W_{\text{app}} = \Delta U = \frac{k_e q^2}{4} \left(\frac{1}{a_1} - \frac{1}{a_2} \right) = 1.50 \times 10^{-7} \text{ J}.$$

$$|F(a_1)| = \frac{k_e q^2}{4a_1^2} = 2.25 \times 10^{-5} \text{ N} \quad \text{toward the plane.}$$

Checks: integrating $k_e q^2 / (4a^2)$ from a_1 to a_2 gives the same positive work. It is below the constant-initial-force estimate $|F(a_1)|(a_2 - a_1) = 4.50 \times 10^{-7} \text{ J}$, because the attraction decreases with height.

Chapter support: Example 4.28; quasistatic electrical work; gravity and support energy excluded.

Field energy: retain the boundary term



Scale a smooth source from 0 to ρ . At intermediate fraction η the potential is ηV , so integrating assembly work gives

$$U = \frac{1}{2} \int \rho V \, d\tau.$$

Use $\rho = \epsilon_0 \nabla \cdot \mathbf{E}$ and $\nabla \cdot (V \mathbf{E}) = -E^2 + V \nabla \cdot \mathbf{E}$:

$$U = \frac{\epsilon_0}{2} \int_{\Omega} E^2 d\tau + \frac{\epsilon_0}{2} \oint_{\partial\Omega} V \mathbf{E} \cdot d\mathbf{A}.$$

For an isolated localized source over all space with $V(\infty) = 0$, the remote term vanishes and $u_E = \epsilon_0 E^2/2$. Finite regions and maintained electrodes require boundary and supply-energy accounting. Positive total field energy includes self-energies. The core's pair energy excludes them; its negative interaction terms therefore cause no contradiction. Point-source self-energies diverge.

Chapter support: Section 4.23; fixed geometry, linear vacuum electrostatics.

Example 4.29: charge a finite sphere



Example 4.29 — complete statement

Build charge Q on an isolated conducting sphere of fixed radius R .

$$U = \int_0^Q \frac{k_e Q'}{R} dQ' = \frac{k_e Q^2}{2R}.$$

Independent field calculation: $E = 0$ inside the metal and $E_r = k_e Q/r^2$ outside, hence

$$U = \frac{\epsilon_0}{2} \int_R^\infty \left(\frac{k_e Q}{r^2} \right)^2 4\pi r^2 dr = \frac{k_e Q^2}{2} \int_R^\infty \frac{dr}{r^2} = \frac{k_e Q^2}{2R}.$$

For $Q = 2.0 \text{ nC}$ and $R = 5.0 \text{ cm}$, $U = 3.60 \times 10^{-7} \text{ J} = QV_s/2$. Charging samples potentials from zero to the final value; QV_s would double count the work. At fixed Q , $U \propto 1/R$, exposing the point-source self-energy limit.

Chapter support: Example 4.29; all-space isolated-source energy and finite radius.

