

Capacitance, Capacitors, and Dielectrics

Physics II - Chapter 5

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Why store separated charge?

The same conductors can hold different separated charges at different voltages. We need a device relation that connects that state to geometry and material.

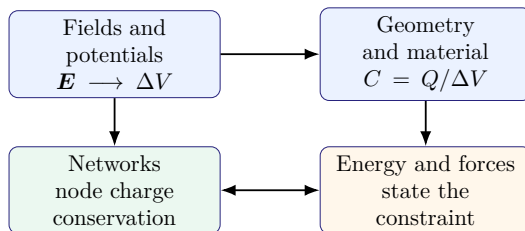


Figure 5.1: The chapter connects field calculations to circuit relations and then to energy, dielectric response and measurement. The advanced section develops the same connections with more general mathematics.

Roadmap. Find the field, obtain the voltage, define capacitance, and then use charge conservation and work to predict networks and dielectric changes.

Chapter support: Sections 5.1.

Learning route and prerequisites



The learning route

Field and potential determine a single capacitor. Node connections determine a network. Work determines stored energy. Polarization changes the charge-field relation. Measurements test the ideal model.

- **Prerequisites:** conductor equilibrium, Gauss's law, potential differences, integration, and algebraic charge conservation.
- **Advanced tools:** derivatives and a local stability test. New boundary and nonlinear-energy reasoning will be introduced before use.
- **Working convention:** Q is a plate-charge magnitude, $\Delta V = V_+ - V_- > 0$, and $\varepsilon_0 = 8.85 \times 10^{-12} \text{ F m}^{-1}$.

Unless a problem says otherwise, use linear capacitors, lossless static states, and initially neutral floating nodes. Charge sharing assumes a settled state with negligible leakage.

Chapter support: Sections 5.2.

Learning goals



By the end of the core lecture, you should be able to:

- derive capacitance from the field and potential for plane, cylindrical, and spherical geometries;
- identify circuit nodes, reduce series/parallel groups, and recover individual charges and voltages;
- calculate stored energy and distinguish it from voltage-source work;
- explain free and bound charge, and predict dielectric insertion under fixed charge or voltage;
- model partial filling, check local field limits, and interpret charge-sharing and calibration measurements;
- formulate and verify charge equations for floating nodes.

Advanced outcomes. Apply dielectric boundary conditions, integrate non-linear charging work, and distinguish actuator force from mechanical stability.

Chapter support: Sections 5.1.

Conductors, fields, and voltage signs



A conductor in electrostatic equilibrium has zero field in its material and one potential. A tangential surface field would move free charge.

$$V_b - V_a = - \int_a^b \mathbf{E} \cdot d\boldsymbol{\ell}.$$

For a plane gap, choose the lower plate at $z = 0$, $V_- = 0$, and the upper plate at $z = d$, $V_+ = \Delta V$:

$$V(z) = \frac{\Delta V}{d} z, \quad \mathbf{E} = -\frac{\Delta V}{d} \hat{\mathbf{z}}.$$

The field points downward, from higher to lower potential. Its magnitude is $E = \Delta V/d$.

A potential reference is a convention

Adding the same constant to every potential, including the reference, leaves all fields and terminal voltage differences unchanged.

Chapter support: Sections 5.2.

Capacitance measures a charge–voltage ratio



The two conductors carry $+Q$ and $-Q$. Their net charge is zero, while their separated charge and electric-field energy can be nonzero.

$$C = \frac{Q}{\Delta V}, \quad Q = C\Delta V.$$

For fixed geometry and a linear material, doubling Q doubles ΔV . Their ratio remains constant. A nonlinear material or moving electrode can change that relation.

$$1\text{ F} = 1\text{ C V}^{-1}, \quad 1\text{ }\mu\text{F} = 10^{-6}\text{ F}, \quad 1\text{ nF} = 10^{-9}\text{ F}, \quad 1\text{ pF} = 10^{-12}\text{ F}.$$

Physical meaning. Capacitance measures how much plate charge is required for a given terminal voltage. It depends on the entire conductor arrangement and the intervening material.

Chapter support: Sections 5.3.

Example 5.1: testing a linear capacitor



Example 5.1

A laboratory capacitor is charged to several potential differences. The measured charge magnitudes are $Q = 2.4 \text{ nC}$ at $\Delta V = 20 \text{ V}$ and $Q = 7.2 \text{ nC}$ at $\Delta V = 60 \text{ V}$. Estimate its capacitance and decide whether these two measurements are consistent with a linear capacitor.

Both measurements give a ratio of plate charge to terminal voltage:

$$C_1 = \frac{2.4 \times 10^{-9}}{20} \text{ F} = 120 \text{ pF}, \quad C_2 = \frac{7.2 \times 10^{-9}}{60} \text{ F} = 120 \text{ pF}.$$

An independent slope check gives

$$\frac{\Delta Q}{\Delta(\Delta V)} = \frac{(7.2 - 2.4) \text{ nC}}{(60 - 20) \text{ V}} = 120 \text{ pF}.$$

The data are consistent with $Q = C\Delta V$ through the origin. Two points cannot establish linearity over all voltages.

Chapter support: Sections 5.3; Example 5.1.

Parallel plates: geometry and direction

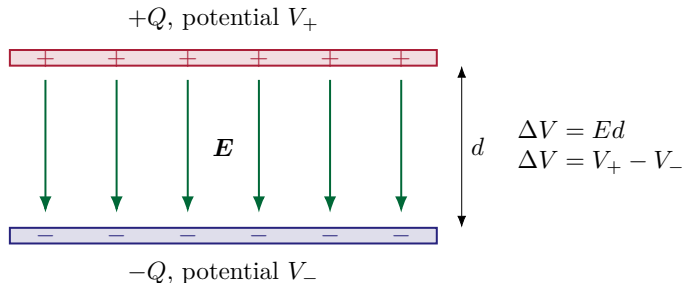


Figure 5.2: Parallel plates viewed in cross-section; each facing surface has area A . The field points from higher to lower potential. Edge fields are neglected in this model.

Model. Both plate dimensions must be much larger than d . Neglect fringing and treat air as vacuum when using ϵ_0 . The field in the gap is nearly uniform.

Chapter support: Sections 5.4.

Gauss's law and the factor of two



An isolated infinite sheet of surface charge density σ_f produces $\sigma_f/(2\epsilon_0)$ on each side. Opposite sheets reinforce each other in the gap and cancel outside:

$$E_{\text{gap}} = \frac{\sigma_f}{2\epsilon_0} + \frac{\sigma_f}{2\epsilon_0} = \frac{\sigma_f}{\epsilon_0}.$$

A second derivation uses a pillbox straddling a conducting surface. The field is zero on the metal side. The only flux is through its gap-facing area A_g :

$$\epsilon_0 E A_g = \sigma_f A_g.$$

Both arguments give the same field. The conductor-surface result already includes the correct total gap field. **Finite plates.** Q/A is an average density. Assuming it equals the local central density is part of the plane-field approximation.

Chapter support: Sections 5.4.

Parallel-plate capacitance from the field



Assign $+Q$ to the upper plate and $-Q$ to the lower one. Gauss's law and the potential integral give

$$\sigma_f = \frac{Q}{A}, \quad E = \frac{Q}{\epsilon_0 A},$$
$$\Delta V = \int_0^d E \, dz = \frac{Qd}{\epsilon_0 A}, \quad \boxed{C} = \boxed{\frac{\epsilon_0 A}{d}}.$$

- Larger area accommodates more charge at the same field and voltage.
- Larger separation requires more voltage for the same charge, so capacitance decreases.
- Units check: $(\text{F m}^{-1})(\text{m}^2)/\text{m} = \text{F}$.

The cancellation of Q explains why the linear capacitance is a geometric property. Edge and nearby-conductor effects belong to a more complete boundary problem.

Chapter support: Sections 5.4.

Example 5.2: a design estimate



Example 5.2

Two square conducting plates of side length 10.0 cm are separated by 1.00 mm in air. Estimate the capacitance, treating air as vacuum.

Convert the dimensions before forming the area:

$$A = (0.100 \text{ m})^2 = 1.00 \times 10^{-2} \text{ m}^2, \quad d = 1.00 \times 10^{-3} \text{ m}.$$

Then

$$C = (8.85 \times 10^{-12}) \frac{1.00 \times 10^{-2}}{1.00 \times 10^{-3}} \text{ F} = \boxed{88.5 \text{ pF}}.$$

Check. The gap-to-side ratio is 0.0100, supporting a central uniform-field estimate. It does not quantify the remaining fringe correction. **Interpretation.** A macroscopic plate area can still give a small capacitance. Thin gaps and large effective areas are needed for larger electrostatic capacitances.

Chapter support: Sections 5.4; Example 5.2.

Coaxial geometry and the Gaussian surface

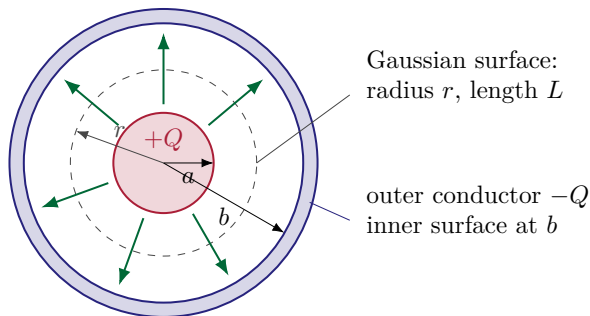


Figure 5.3: Cross-section of a long coaxial capacitor. The outer conductor is an annulus surrounding the inner conductor. The dashed circle is the cross-section of the cylindrical Gaussian surface. Radii refer to the facing conductor surfaces.

Assumptions. The axes are concentric, the conductors are equipotential, and $L \gg b$. Use a radial field in the gap and neglect end effects. Here b is the inner surface of the outer conductor.

Chapter support: Sections 5.5.

Cylindrical capacitance from a radial field



A coaxial Gaussian surface has curved area $2\pi rL$. With $a < r < b$,

$$E(r)(2\pi rL) = \frac{Q}{\epsilon_0}, \quad E(r) = \frac{Q}{2\pi\epsilon_0 Lr}.$$

The inner conductor is at higher potential. Integrating outward gives its positive terminal voltage:

$$\Delta V = \int_a^b E(r) dr = \frac{Q}{2\pi\epsilon_0 L} \ln \frac{b}{a}.$$

Thus

$$\boxed{C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}}, \quad \boxed{C' = \frac{C}{L} = \frac{2\pi\epsilon_0}{\ln(b/a)}}.$$

The logarithm is dimensionless and positive for $b > a$. C' has units F m^{-1} , rather than farads.

Chapter support: Sections 5.5.

Example 5.3: capacitance per length



Example 5.3

A coaxial cable has inner conductor radius $a = 0.50$ mm and inner radius of the outer conductor $b = 2.5$ mm. Estimate the capacitance per unit length in air.

The ratio is independent of the length unit when both radii use the same unit:

$$\frac{b}{a} = \frac{2.5 \text{ mm}}{0.50 \text{ mm}} = 5.0.$$

Consequently,

$$C' = \frac{2\pi(8.85 \times 10^{-12})}{\ln 5.0} \text{ F m}^{-1} = \boxed{34.6 \text{ pF m}^{-1}}.$$

Independent check. Substituting this C' into $\Delta V = (Q/L) \ln(b/a)/(2\pi\epsilon_0)$ recovers $Q/L = C'\Delta V$. A homogeneous dielectric would multiply C' by κ . For a finite cable, $C \simeq C'L$ also requires negligible end effects.

Chapter support: Sections 5.5; Example 5.3.

Concentric spheres: a different Gaussian area

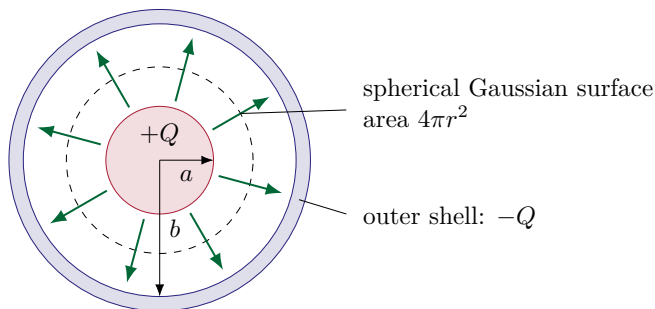


Figure 5.4: Central section of a spherical capacitor. The circular drawing is a section through spheres, whereas Fig. 5.3 is a section through cylinders. The distinction changes the Gaussian area and hence the radial dependence of the field.

A concentric sphere has radial symmetry, but its Gaussian area is $4\pi r^2$. That geometric change produces an inverse-square field rather than the coaxial inverse-radius field.

Chapter support: Sections 5.6.

Spherical capacitance and the isolated limit



For concentric conducting spheres with $a < r < b$,

$$E(r) = \frac{Q}{4\pi\epsilon_0 r^2}, \quad \Delta V = \frac{Q}{4\pi\epsilon_0} \int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right).$$

Therefore

$$C = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

Take $b \rightarrow \infty$ and write $a = R$ to obtain

$$C_{\text{isolated}} = 4\pi\epsilon_0 R.$$

This self-capacitance uses $V(\infty) = 0$ and assumes distant surroundings. A nearby grounded conductor changes the field and the capacitance.

Chapter support: Sections 5.6.

Example 5.4: small charge, large potential



Example 5.4

A conducting sphere of radius $R = 5.0 \text{ cm}$ carries charge $Q = 10 \text{ nC}$ and is far from other conductors. Estimate its capacitance and potential relative to infinity.

With $R = 0.050 \text{ m}$ and potential zero at infinity,

$$C = 4\pi(8.85 \times 10^{-12})(0.050) \text{ F} = \boxed{5.56 \text{ pF}},$$

$$V = \frac{10 \times 10^{-9} \text{ C}}{5.5606 \times 10^{-12} \text{ F}} = \boxed{1.80 \times 10^3 \text{ V}}.$$

Independent check. Directly evaluating $V = Q/(4\pi\epsilon_0 R)$ gives the same value without using the rounded capacitance. A small plate-charge magnitude can produce a high potential when capacitance is small. Voltage alone does not determine stored energy; that calculation also needs C or Q .

Chapter support: Sections 5.6; Example 5.4.

Curvature and geometric scaling



Thin curved gaps. Put $b = a + d$, with $d \ll a$:

$$C_{\text{cyl}} \simeq \frac{\varepsilon_0(2\pi aL)}{d} \quad \text{because } \ln(1 + d/a) \simeq d/a,$$
$$C_{\text{sph}} = \frac{\varepsilon_0(4\pi a^2)}{d} \left(1 + \frac{d}{a}\right).$$

Both leading terms recover the plane-gap formula using the inner facing area.

Answered scaling check

If every length doubles, area grows by four and gap grows by two, so C doubles. If only area doubles at fixed gap, C also doubles, for a different reason.

Environment check. A sphere of radius R inside a concentric grounded shell of radius $2R$ has $C = 8\pi\varepsilon_0 R$, twice its isolated value. The return conductor matters.

Chapter support: Sections 5.6.

Parallel connection follows the nodes

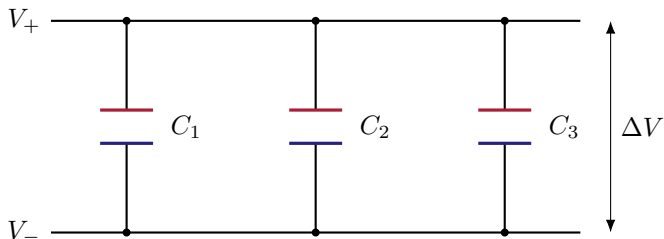


Figure 5.5: Parallel connection: every upper plate belongs to one node and every lower plate to the other. The node potentials, rather than the appearance of the drawing, determine the common voltage.

All connected upper terminals are one conductor at V_+ . All lower terminals are one conductor at V_- . Each component therefore sees the same voltage difference, regardless of how the diagram is drawn.

Chapter support: Sections 5.7.

Common voltage makes capacitances add



In a parallel connection, the terminal voltage is common:

$$Q_i = C_i \Delta V, \quad Q_{\text{terminal}} = \sum_i Q_i = \left(\sum_i C_i \right) \Delta V.$$

The equivalent two-terminal relation is

$$C_{\text{eq}} = \sum_i C_i.$$

What the equivalent charge means

Q_{terminal} is the charge supplied to one joined terminal conductor. It is neither the zero net charge of the complete device nor the charge on every individual component.

The largest capacitance carries the largest charge at a common voltage. For positive components, C_{eq} exceeds each individual value.

Chapter support: Sections 5.7.

Example 5.5: distributing charge



Example 5.5

Three capacitors, $C_1 = 2.0 \mu\text{F}$, $C_2 = 5.0 \mu\text{F}$ and $C_3 = 8.0 \mu\text{F}$, are connected in parallel across $\Delta V = 12 \text{ V}$. Find the equivalent capacitance and the charge on each capacitor.

All components share the stated 12 V:

$$C_{\text{eq}} = (2.0 + 5.0 + 8.0) \mu\text{F} = \boxed{15.0 \mu\text{F}}.$$

$$Q_1 = (2.0)(12) \mu\text{C} = 24 \mu\text{C},$$

$$Q_2 = (5.0)(12) \mu\text{C} = 60 \mu\text{C},$$

$$Q_3 = (8.0)(12) \mu\text{C} = 96 \mu\text{C}.$$

Charge check. $24 + 60 + 96 = 180 \mu\text{C} = C_{\text{eq}}(12 \text{ V})$. Each negative terminal has the opposite charge. The common-voltage condition produces unequal charge magnitudes. The largest capacitor holds the largest separated charge.

Chapter support: Sections 5.7; Example 5.5.

Series connection and a neutral island

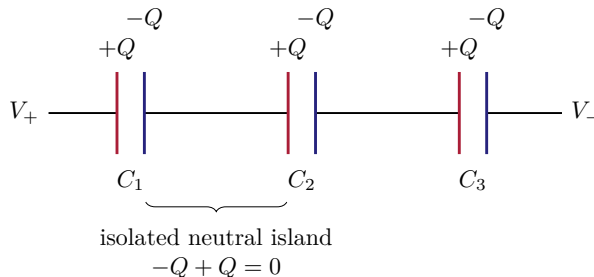


Figure 5.6: Series connection of three capacitors. Each internal island consists of a wire and the two plates it joins. Initially neutral isolated islands enforce equal charge magnitudes. The plate gaps are the only breaks in conduction paths. The isolated internal conductor includes its connecting wire and both attached plates. Its conserved net charge is initially zero. No conduction charge crosses a dielectric gap.

Chapter support: Sections 5.8.

Equal charge and reciprocal addition



For an internal island joining the negative plate of capacitor 1 to the positive plate of capacitor 2,

$$-Q_1 + Q_2 = 0 \implies Q_1 = Q_2 = Q.$$

The terminal voltage is the sum of the component drops:

$$\Delta V = \sum_i \frac{Q}{C_i} = Q \sum_i \frac{1}{C_i}.$$

Hence

$$\boxed{\frac{1}{C_{\text{eq}}} = \sum_i \frac{1}{C_i}}, \quad C_{\text{eq}} = \frac{C_1 C_2}{C_1 + C_2} \text{ for two capacitors.}$$

A smaller capacitance takes a larger voltage because $\Delta V_i = Q/C_i$. With positive components, C_{eq} is below the smallest component. Nonzero trapped island charge requires a signed node equation instead.

Chapter support: Sections 5.8.

Example 5.6: voltage division in series



Example 5.6

A $3.0\ \mu\text{F}$ capacitor and a $6.0\ \mu\text{F}$ capacitor are connected in series across $18\ \text{V}$. Find the equivalent capacitance, the charge on each capacitor and the potential difference across each capacitor.

The internal island is initially neutral, so both charge magnitudes are equal:

$$C_{\text{eq}} = \frac{(3.0)(6.0)}{3.0 + 6.0} \mu\text{F} = 2.0 \mu\text{F}, \quad Q = (2.0)(18) \mu\text{C} = 36 \mu\text{C}.$$

The component voltage differences are

$$\Delta V_1 = \frac{36}{3.0} \text{V} = 12 \text{V}, \quad \Delta V_2 = \frac{36}{6.0} \text{V} = 6 \text{V}.$$

Checks. $12 + 6 = 18\ \text{V}$, and $2.0\ \mu\text{F} < 3.0\ \mu\text{F}$. The smaller capacitance receives twice the voltage, which matters when checking component ratings.

Chapter support: Sections 5.8; Example 5.6.

Mixed network: identify both terminal nodes

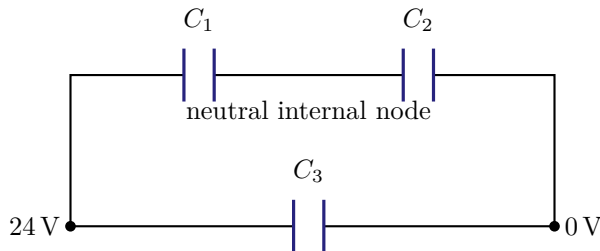


Figure 5.7: The mixed network used in Example 5.7: C_1 and C_2 form one series branch, in parallel with C_3 . Wires meeting at the marked nodes are electrically connected.

The upper branch has a neutral internal node, so its two capacitors can be reduced in series. That branch and the lower capacitor share both terminal nodes and are therefore in parallel. **Workflow.** Reduce forward, solve the terminal relation, then expand backward using the same charge or voltage constraint.

Chapter support: Sections 5.9.

Example 5.7: reduce the mixed network



Example 5.7

Capacitors $C_1 = 4.0 \mu\text{F}$ and $C_2 = 12.0 \mu\text{F}$ are connected in series. That series pair is connected in parallel with $C_3 = 6.0 \mu\text{F}$. The whole network is connected across 24 V . Find the equivalent capacitance and the charge on each capacitor.

Reduce the upper series branch first:

$$C_{12} = \frac{(4.0)(12.0)}{4.0 + 12.0} \mu\text{F} = 3.0 \mu\text{F}.$$

The parallel branches then give

$$C_{\text{eq}} = C_{12} + C_3 = \boxed{9.0 \mu\text{F}}, \quad Q_{\text{terminal}} = C_{\text{eq}}(24 \text{ V}) = 216 \mu\text{C}.$$

The terminal charge is distributed between branches. It must not be assigned to every original component. Recover those charges on the next slide.

Chapter support: Sections 5.9; Example 5.7.

Example 5.7: expand backward and check



Example 5.7, continued. Both branches have 24 V across them:

$$Q_3 = (6.0 \mu\text{F})(24 \text{ V}) = \boxed{144 \mu\text{C}},$$

$$Q_{12} = (3.0 \mu\text{F})(24 \text{ V}) = 72 \mu\text{C}.$$

The upper neutral island enforces

$$\boxed{Q_1 = Q_2 = 72 \mu\text{C}}, \quad \Delta V_1 = \frac{72}{4.0} \text{ V} = 18 \text{ V}, \quad \Delta V_2 = \frac{72}{12.0} \text{ V} = 6 \text{ V}.$$

Independent checks. The upper drops sum to 24 V. The source-terminal charge is $Q_1 + Q_3 = 216 \mu\text{C}$. The upper island has $-Q_1 + Q_2 = 0$. The two charges in the series branch are a repeated magnitude on different capacitors, rather than two separate contributions to the terminal charge.

Chapter support: Sections 5.9; Example 5.7.

A floating node can carry trapped charge



Let C_1 connect a fixed potential V_0 to island X , and C_2 connect X to ground. Define q_X as the conserved net charge of that island.

$$q_X = C_1(V_X - V_0) + C_2 V_X, \quad \boxed{V_X = \frac{C_1 V_0 + q_X}{C_1 + C_2}}.$$

Each term is signed charge on an island-facing plate. Their sum must equal the specified island charge.

The neutral case is one initial condition

When $q_X = 0$, the island plate charges have equal magnitudes and opposite signs. With trapped charge, equal magnitudes would generally violate conservation.

The connectivity defines the equation. Initial charge defines its right-hand side. Changing initial charge does not change the component capacitances.

Chapter support: Sections 5.9.1.

Example 5.8: a precharged internal island



Example 5.8

Take $C_1 = 2.0 \mu\text{F}$, $C_2 = 4.0 \mu\text{F}$, $V_0 = 12 \text{ V}$ and $q_X = +6.0 \mu\text{C}$. Find the island potential and the plate charges attached to it.

Use the signed node relation, with microfarads and microcoulombs:

$$2.0(V_X - 12) + 4.0V_X = 6.0 \implies \boxed{V_X = 5.0 \text{ V}}.$$

The two island-facing charges are

$$q_{X,1} = 2.0(5.0 - 12) \mu\text{C} = \boxed{-14 \mu\text{C}}, \quad q_{X,2} = 4.0(5.0) \mu\text{C} = \boxed{+20 \mu\text{C}}.$$

Checks. Their algebraic sum is $+6.0 \mu\text{C}$. The voltage drops are 7.0 V and 5.0 V , which sum to 12 V . Setting $q_X = 0$ instead gives $V_X = 4.0 \text{ V}$ and island charges -16 and $+16 \mu\text{C}$.

Chapter support: Sections 5.9.1; Example 5.8.

Charging work is an integral

At intermediate charge q , the voltage is $v(q) = q/C$. Reversible charging at fixed geometry requires $dW = v(q) dq$:

$$U = \int_0^Q \frac{q}{C} dq = \frac{Q^2}{2C}.$$

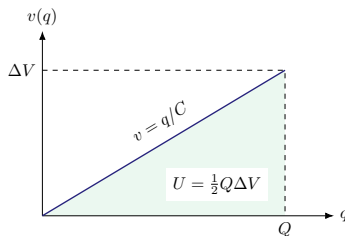


Figure 5.8: Charging work for a linear capacitor is the triangular area $\int_0^Q v dq$. The rectangle $Q\Delta V$ would count every charge increment at the final voltage. Axes are normalized schematic coordinates; the relation itself is exact for constant C .

Chapter support: Sections 5.10.

Stored energy and source work



For a constant linear capacitance, substitute $Q = C\Delta V$ into charging work:

$$U = \frac{Q^2}{2C} = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}Q\Delta V.$$

Choose the form that exposes the controlled quantity. Doubling charge at fixed C , or doubling voltage at fixed C , multiplies energy by four.

A fixed battery supplies a different amount of work

If a source at V_s charges an initially uncharged capacitor, it supplies $W_s = V_s Q = CV_s^2$. The final stored energy is $CV_s^2/2$. The remaining half transfers to other degrees of freedom during the transient.

Resistance-limited charging transfers that difference to heat. A reversible variable-voltage source can instead supply just the stored-energy increase. The final state alone does not specify the charging path.

Chapter support: Sections 5.10.

Electric-field energy density



For plane plates, use $C = \varepsilon_0 A/d$ and $\Delta V = Ed$:

$$U = \frac{1}{2} \frac{\varepsilon_0 A}{d} (Ed)^2 = \frac{1}{2} \varepsilon_0 E^2 (Ad).$$

Dividing by the gap volume Ad gives

$$\boxed{u_E = \frac{1}{2} \varepsilon_0 E^2} \quad \text{in vacuum,} \quad U = \int u_E d\tau.$$

For a reversible linear isotropic dielectric at fixed material state,

$$u_E = \frac{1}{2} \varepsilon E^2.$$

This macroscopic energy includes reversible polarization response. The vacuum contribution alone is $\varepsilon_0 E^2/2$. Nonuniform fields require the volume integral. Units are J m^{-3} .

Chapter support: Sections 5.11.

Example 5.10: field and energy density



Example 5.10

An air-filled parallel-plate capacitor has plate separation $d = 0.80 \text{ mm}$ and is connected to a 400 V source. Estimate the electric field and the energy density between the plates.

Neglect fringing and use the total terminal voltage across the plane gap:

$$E = \frac{400 \text{ V}}{0.80 \times 10^{-3} \text{ m}} = \boxed{5.0 \times 10^5 \text{ V m}^{-1}}.$$

Then

$$u_E = \frac{1}{2}(8.85 \times 10^{-12})(5.0 \times 10^5)^2 \text{ J m}^{-3} = \boxed{1.11 \text{ J m}^{-3}}.$$

Check. For any area A in the ideal model, $u_E Ad$ equals $\frac{1}{2}(\epsilon_0 A/d)(400 \text{ V})^2$. The local and terminal energy descriptions agree. This is energy per field volume, rather than total energy.

Chapter support: Sections 5.11; Example 5.10.

Changing the gap follows two energy paths

At fixed plate area, $C \propto 1/d$. The constraint selects the energy path.

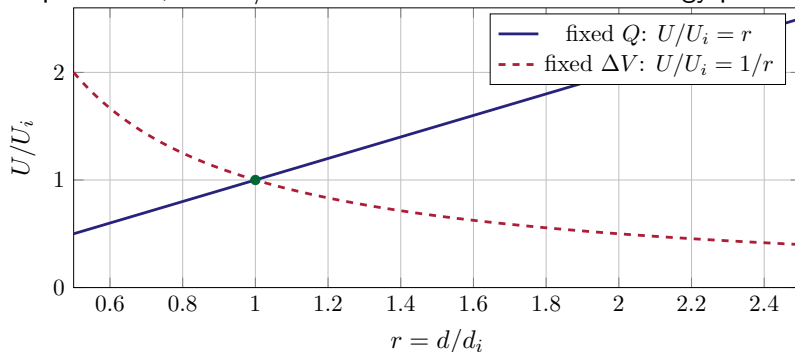


Figure 5.9: Stored energy as the gap changes from a common initial state. The source is part of the energy accounting for the dashed curve. The graph does not imply that force follows the negative slope of capacitor energy along both paths. A connected source belongs in the energy balance.

Chapter support: Sections 5.11.

Example 5.11: disconnect before motion



Example 5.11

A capacitor of initial capacitance 100 pF is at 100 V . Its gap is doubled slowly. Compare (a) disconnecting it before the motion and (b) keeping an ideal 100 V source attached. Neglect fringing and all dissipation.

Initial state and part (a). The initial quantities are

$$Q_i = (100 \text{ pF})(100 \text{ V}) = 10 \text{ nC}, \quad U_i = \frac{1}{2} C_i V_i^2 = 0.500 \mu \text{ J}.$$

Doubling the gap gives $C_f = 50 \text{ pF}$. Isolation fixes Q :

$$Q_f = 10 \text{ nC}, \quad V_f = Q_i / C_f = 200 \text{ V}, \quad U_f = Q_i^2 / (2 C_f) = 1.00 \mu \text{ J}.$$

The external agent supplies $W_{\text{ext}} = \Delta U = +0.500 \mu \text{ J}$. The unchanged plane field occupies twice the gap volume. Part (b) keeps the source attached.

Chapter support: Sections 5.11; Example 5.11.

Example 5.11: keep the source connected



Example 5.11, part (b). Keep $V = 100 \text{ V}$ while C falls from 100 to 50 pF :

$$Q_f = 5.0 \text{ nC}, \quad U_f = 0.250 \mu \text{ J}.$$

Take source work as positive when delivered to the capacitor. Then

$$\begin{aligned} W_s &= V(Q_f - Q_i) = (100 \text{ V})(-5.0 \text{ nC}) = -0.500 \mu \text{ J}, \\ \Delta U &= U_f - U_i = -0.250 \mu \text{ J}, \\ W_{\text{ext}} &= \Delta U - W_s = +0.250 \mu \text{ J}. \end{aligned}$$

Conservation check. $W_s + W_{\text{ext}} = \Delta U$. The source receives $0.500 \mu \text{ J}$: 0.250 comes from field energy and 0.250 from mechanical work. The plates attract in both experiments, so separating them requires positive external work under either constraint.

Chapter support: Sections 5.11.1; Example 5.11.

A dielectric polarizes in the plate field

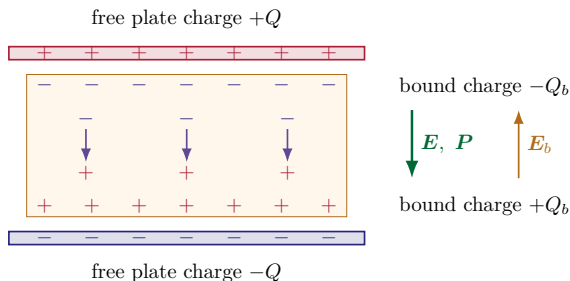


Figure 5.10: Polarization for a positively charged upper plate. Dipole moments point from negative to positive charge, downward along $\mathbf{E}$. The negative upper bound charge and positive lower bound charge produce an opposing field $\mathbf{E}_b$. Small clearances separate the schematic labels; the full-fill formula neglects air gaps. Microscopic charges displace or align. Bound charge changes the field. Full filling by a linear homogeneous dielectric gives $C = \kappa C_0$; a source may separately supply free plate charge.

Chapter support: Sections 5.12.

Free charge fixes displacement flux



The polarization $\mathbf{P}$ is dipole moment per unit volume. Its outward-normal surface charge is $\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}$.

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \oint \mathbf{D} \cdot d\mathbf{A} = Q_{f, \text{enc}}.$$

For a linear isotropic dielectric,

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \varepsilon \mathbf{E}, \quad \kappa = 1 + \chi_e.$$

Plane symmetry gives

$$D = \frac{Q}{A}, \quad E = \frac{Q}{\kappa \varepsilon_0 A}, \quad Q_b = \left(1 - \frac{1}{\kappa}\right) Q.$$

$\mathbf{E}$ is the total macroscopic field. Ordinary Gauss's law still includes free plus bound charge. Q_b denotes the magnitude on either dielectric face.

Chapter support: Sections 5.12.

Example 5.12: count free and bound charge



Example 5.12

An ideal fully filled capacitor has $A = 20 \text{ cm}^2$, $Q = 30 \text{ nC}$ and $\kappa = 3$. Determine D , E and the bound surface charge magnitude.

Convert $A = 20 \text{ cm}^2 = 2.0 \times 10^{-3} \text{ m}^2$. Gauss's law fixes displacement:

$$D = \frac{30 \times 10^{-9}}{2.0 \times 10^{-3}} \text{ C m}^{-2} = 1.5 \times 10^{-5} \text{ C m}^{-2},$$

$$E = \frac{D}{3\epsilon_0} = 5.65 \times 10^5 \text{ V m}^{-1},$$

$$P = \frac{2}{3}D = 1.0 \times 10^{-5} \text{ C m}^{-2},$$

$$Q_b = PA = \boxed{20 \text{ nC}}.$$

The negative bound face adjoins the positive conductor. **Check.** $E = (Q - Q_b)/(\epsilon_0 A)$, using $Q - Q_b = 10 \text{ nC}$, gives the same field. Free plate charge stays unchanged.

Chapter support: Sections 5.12; Example 5.12.

Dielectric insertion: which quantity is fixed?



For full insertion in a fixed plane gap, $C_f = \kappa C_0$ under either protocol. The electrical constraint determines the other changes.

Quantity	Isolated: fixed Q	Connected: fixed ΔV
Q_f/Q_i	1	κ
$\Delta V_f/\Delta V_i$	$1/\kappa$	1
E_f/E_i	$1/\kappa$	1
U_f/U_i	$1/\kappa$	κ

With fixed charge, polarization reduces the field and voltage. With fixed voltage, the source supplies additional free charge, compensating the bound-charge field. **Energy accounting.** An increase in stored energy with the source attached is compatible with an inward dielectric force. The source supplies both stored energy and mechanical work.

Chapter support: Sections 5.13.

Example 5.13: isolated dielectric insertion



Example 5.13

An air-filled capacitor has capacitance $C_0 = 200 \text{ pF}$ and is initially at $\Delta V_i = 100 \text{ V}$. A dielectric with $\kappa = 4.0$ fully fills the capacitor. Compare the final charge, voltage and stored energy for (a) an isolated capacitor and (b) a capacitor kept connected to a 100 V source. Assume quasistatic, reversible insertion with the plate geometry fixed.

The initial state and final capacitance are

$$Q_i = C_0 V_i = 20.0 \text{ nC}, \quad U_i = \frac{1}{2} C_0 V_i^2 = 1.00 \mu\text{J}, \quad C_f = 800 \text{ pF}.$$

Part (a), isolated. $Q_f = Q_i = 20.0 \text{ nC}$, so

$$V_f = 25.0 \text{ V}, \quad U_f = 0.250 \mu\text{J}.$$

Reversible mechanical output is $U_i - U_f = 0.750 \mu\text{J}$. Both voltage and energy fall by $1/\kappa$.

Chapter support: Sections 5.13; Example 5.13.

Example 5.13: connected insertion and work



Example 5.13, part (b). With the source attached, $V_f = 100 \text{ V}$ and $C_f = 800 \text{ pF}$:

$$Q_f = 80.0 \text{ nC},$$

$$U_f = 4.00 \mu \text{ J}.$$

The source supplies

$$W_s = V(Q_f - Q_i) = (100 \text{ V})(60.0 \text{ nC}) = 6.00 \mu \text{ J}.$$

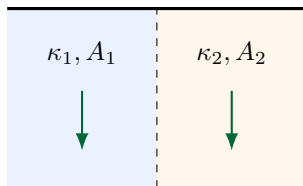
The stored-energy increase is $\Delta U = 3.00 \mu \text{ J}$. The field's mechanical output is therefore

$$W_{\text{mech}} = W_s - \Delta U = 3.00 \mu \text{ J}.$$

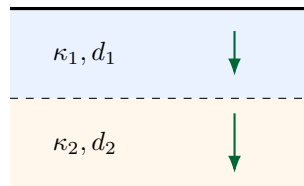
Check. Both protocols pull the slab inward. Fixed charge has no electrical work input. Fixed voltage receives source work. These endpoint balances assume reversible quasistatic motion, as the statement specifies.

Chapter support: Sections 5.13; Example 5.13.

Partial filling: lateral regions or layers?



(a) common ΔV
lateral regions: parallel



(b) common $D_n = Q/A$
stacked layers: series

Figure 5.11: Partial dielectric filling in the plane-field approximation. The internal dashed lines mark dielectric interfaces, not conductors. Panel (b) illustrates differing field magnitudes; the lengths of those arrows are qualitative.

Lateral regions span the same terminal voltage. Stacked regions have common normal D when the interface has no free charge. The dielectric interface carries polarization charge rather than an added metal plate.

Chapter support: Sections 5.14.

Example 5.14: side-by-side dielectrics



Example 5.14

An ideal parallel-plate capacitor of area A and separation d is half filled by a dielectric κ_1 and half by a dielectric κ_2 , with the materials side by side across the area. Find the capacitance.

Each half spans the full gap and has the same terminal voltage:

$$C_1 = \frac{\kappa_1 \epsilon_0 A}{2d}, \quad C_2 = \frac{\kappa_2 \epsilon_0 A}{2d}.$$

The regions contribute in parallel, giving

$$C = \frac{\epsilon_0 A}{2d} (\kappa_1 + \kappa_2).$$

Check. Equal constants recover $\kappa \epsilon_0 A/d$; both constants equal to one recover the empty capacitor. The plane-field model uses an area-weighted arithmetic average and neglects interface edge effects.

Chapter support: Sections 5.14; Example 5.14.

Example 5.15: stacked dielectric layers



Example 5.15

An ideal parallel-plate capacitor of area A is filled by two dielectric slabs stacked along the field direction. Slab 1 has thickness d_1 and dielectric constant κ_1 ; slab 2 has thickness d_2 and dielectric constant κ_2 . Find the capacitance.

No free interface charge gives $D_n = Q/A$:

$$E_i = \frac{Q}{\kappa_i \varepsilon_0 A}, \quad \Delta V = \frac{Q}{\varepsilon_0 A} \left(\frac{d_1}{\kappa_1} + \frac{d_2}{\kappa_2} \right).$$

Therefore

$$C = \frac{\varepsilon_0 A}{d_1/\kappa_1 + d_2/\kappa_2}.$$

Check. Equal constants recover $C = \kappa \varepsilon_0 A / (d_1 + d_2)$.

Chapter support: Sections 5.14; Example 5.15.

Example 5.16: a small air gap matters



An air gap g and dielectric thickness t are stacked layers:

$$C = \frac{\epsilon_0 A}{g + t/\kappa}, \quad E_{\text{air}} = \frac{\Delta V}{g + t/\kappa}, \quad E_{\text{diel}} = E_{\text{air}}/\kappa.$$

Example 5.16

A capacitor has area 10 cm^2 and separation 1.0 mm . A sheet of thickness 0.90 mm and $\kappa = 9$ leaves a total air gap of 0.10 mm . Find C and the two fields at $\Delta V = 100 \text{ V}$.

Here $A = 10^{-3} \text{ m}^2$ and $g + t/\kappa = 0.20 \text{ mm}$. Thus

$$C = 44.3 \text{ pF}, \quad E_{\text{air}} = 5.0 \times 10^5 \text{ V m}^{-1}, \quad E_{\text{diel}} = 5.56 \times 10^4 \text{ V m}^{-1}.$$

Check. The air and dielectric drops are each 50 V , summing to 100 V . A completely filled gap would give 79.7 pF . A small void can strongly change capacitance and concentrate the local field.

Chapter support: Sections 5.14.1; Example 5.16.

Breakdown uses the local field



Breakdown occurs when an insulating material becomes conducting. A permissible field is a specified model input, with real dependence on material state, geometry, defects, and time.

$$\Delta V \lesssim E_{\text{allow}} d \quad \text{for a uniform plane gap,} \quad u_{\text{max}} \simeq \frac{1}{2} \kappa \epsilon_0 E_{\text{allow}}^2.$$

- In a layered system, use each region's field and its own allowable value. An average voltage-to-gap ratio can miss the maximum.
- Large κ increases capacitance, but usable energy density also depends quadratically on the field limit.
- Equal capacitances can have different voltage ratings. Stored energy depends on both capacitance and voltage.

Leakage, electrode resistance, lead inductance, and dielectric absorption require additional models. The electrostatic estimate does not determine transient performance.

Chapter support: Sections 5.15.

Practice: energy density at an allowable field

Problem 35

Compare energy densities $u = \frac{1}{2}\epsilon E^2$ for two dielectrics: $\kappa = 2$, $E_{\max} = 200 \text{ MV m}^{-1}$ and $\kappa = 20$, $E_{\max} = 30 \text{ MV m}^{-1}$.

Attempt before the next slide. For each material, use its own specified permissible field in $u = \frac{1}{2}\kappa\epsilon_0 E^2$. Convert megavolts per metre to SI units before squaring. The exercise compares ideal field-volume energy densities. Treat the field limits as supplied illustrative inputs and the dielectrics as reversible, linear, and isotropic.

Questions guiding the comparison

Does a tenfold increase in relative permittivity overcome the decrease in permitted field? Compare the product $\kappa E_{\max}^2$, then interpret the numerical ratio. The complete calculation and answer follow on slide 50.

Chapter support: Sections 5.22; Problem 35.

Problem 35: material comparison



Problem 35, solution. The two specified field limits give

$$u_1 = \frac{1}{2}(2)(8.85 \times 10^{-12})(200 \times 10^6)^2 = \boxed{3.54 \times 10^5 \text{ J m}^{-3}},$$

$$u_2 = \frac{1}{2}(20)(8.85 \times 10^{-12})(30 \times 10^6)^2 = \boxed{7.97 \times 10^4 \text{ J m}^{-3}}.$$

Using unrounded values,

$$\frac{u_1}{u_2} = \frac{2(200)^2}{20(30)^2} = 4.44.$$

Material 1 stores about 4.44 times the ideal energy density at its stated field limit, despite its smaller κ . **Check and interpretation.** The permittivity and squared-field factors have the required energy-density units. The common $\varepsilon_0/2$ cancels in the independent ratio check. Electrode and packaging volume would lower complete-device energy density.

Chapter support: Sections 5.23; Problem 35.

Charge sharing conserves conductor charge



After joining corresponding terminals, conserve signed charge on each newly connected conductor. For two linear capacitors with initial signed voltages V_1, V_2 ,

$$Q_{\text{joined}} = C_1 V_1 + C_2 V_2 = (C_1 + C_2) V_f,$$

$$V_f = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2}.$$

The final common voltage is a capacitance-weighted average. For opposite initial polarities, one signed initial voltage is negative.

Static endpoint and transient process

Negligible leakage preserves conductor charge. During settling, heat and possibly radiation reduce field energy. An exactly lossless inductive model can oscillate instead.

Chapter support: Sections 5.16.

Example 5.18: infer an unknown capacitance



Example 5.18

A known capacitor $C_1 = 1.00 \mu\text{F}$ is charged to $V_i = 12.0 \text{ V}$ and disconnected from the source. It is then connected in parallel with an initially uncharged capacitor C_2 . The final common voltage is $V_f = 4.00 \text{ V}$. Find C_2 , assuming negligible charge leakage and negligible meter loading after the redistribution has settled.

Conserve charge on the joined terminal:

$$C_1 V_i = (C_1 + C_2) V_f, \quad C_2 = C_1 \left(\frac{V_i}{V_f} - 1 \right).$$

Substitute the stated voltages:

$$C_2 = (1.00 \mu\text{F})(12.0/4.00 - 1) = 2.00 \mu\text{F}.$$

Check. Final charges 4.00 and $8.00 \mu\text{C}$ sum to the initial $12.0 \mu\text{C}$. Energy accounting follows on slide 53.

Chapter support: Sections 5.16; Example 5.18.

Example 5.18: where the energy goes



Example 5.18, energy check. Evaluate the two endpoint states:

$$U_i = \frac{1}{2}(1.00 \mu\text{F})(12.0 \text{ V})^2 = 72.0 \mu\text{J},$$

$$U_f = \frac{1}{2}(3.00 \mu\text{F})(4.00 \text{ V})^2 = 24.0 \mu\text{J}.$$

The released field energy is $48.0 \mu\text{J}$, transferred during settling. For arbitrary signed initial voltages, expansion of the energy difference gives

$$U_i - U_f = \frac{1}{2} \frac{C_1 C_2}{C_1 + C_2} (V_1 - V_2)^2 \geq 0.$$

Putting $V_1 = 12.0 \text{ V}$ and $V_2 = 0$ into this second expression also gives $48.0 \mu\text{J}$. The difference vanishes if both initial voltages agree. Electrostatics determines the released amount, rather than its detailed time dependence.

Chapter support: Sections 5.16.1; Example 5.18.

Practice: fit a capacitance measurement



A simple measurement model separates active capacitance from an approximately constant stray term:

$$C_{\text{meas}}(d) = \frac{\varepsilon_0 A}{d} + C_s.$$

A plot against $1/d$ has slope $\varepsilon_0 A$ and intercept C_s , if this model applies.

Problem 27

A plate fixture produces $C_{\text{meas}} = 99 \text{ pF}$ at $d = 1.0 \text{ mm}$ and 54 pF at $d = 2.0 \text{ mm}$. Fit $C_{\text{meas}} = \varepsilon_0 A/d + C_s$ to these two points. Predict the capacitance at $d = 0.50 \text{ mm}$ and explain why a third measurement is valuable.

Attempt. Write the two equations with d in millimetres and C in picofarads. Solve for the slope and intercept, then convert the slope to SI units to infer A . The complete answer follows on slide 55.

Chapter support: Sections 5.16.2; Problem 27.

Problem 27: slope, offset, and a prediction



Problem 27, solution. Write $C_{\text{meas}} = m/d + C_s$, with d in millimetres:

$$99 = m + C_s, \quad 54 = m/2 + C_s \quad \implies \quad m = 90 \text{ pF mm}, \quad \boxed{C_s = 9 \text{ pF}}.$$

Because $1 \text{ pF mm} = 10^{-15} \text{ F m}$,

$$\boxed{A = \frac{90 \times 10^{-15}}{8.85 \times 10^{-12}} \text{ m}^2 = 1.017 \times 10^{-2} \text{ m}^2 \simeq 102 \text{ cm}^2}.$$

At $d = 0.50 \text{ mm}$, predict $C_{\text{meas}} = 90/0.50 + 9 = \boxed{189 \text{ pF}}$. **Checks.** Substitution recovers both supplied measurements. Two points fit two parameters exactly, so they cannot test the model. A third reading can expose changing stray coupling, fringing, or a separation error.

Chapter support: Sections 5.23; Problem 27.

General networks become linear systems



For a capacitor between nodes i and j , its charge on the i plate is

$$q_i^{(ij)} = C_{ij}(V_i - V_j).$$

Sum signed contributions to enforce the specified floating-node charge:

$$q_i = \sum_j C_{ij}(V_i - V_j).$$

Move fixed-voltage terms to the right. Each diagonal sums incident capacitances; an off-diagonal is the negative connecting capacitance. **Checks.** Substitute into each node equation, recover terminal charge, and compare summed branch energies with terminal energy. Residuals test algebra; separate topology and model checks test physical assumptions.

Chapter support: Sections 5.17.

Example 5.24: a two-node bridge

The floating nodes X and Y are initially neutral.

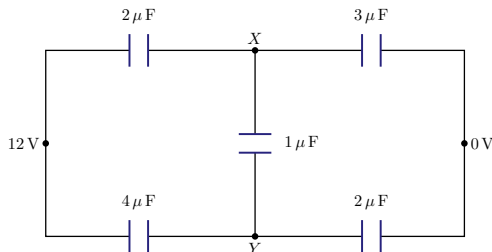


Figure 5.12: A five-capacitor bridge. The central $1\mu\text{F}$ capacitor connects two floating nodes and makes their charge equations coupled. All branch values are shown; the outside terminals are fixed.

Example 5.24

For Fig. 5.12, use charge conservation to determine V_X , V_Y and the terminal capacitance.

Chapter support: Sections 5.17; Example 5.24.

Example 5.24: solve and verify the bridge



Example 5.24, solution. In microfarads and volts, neutral-node conservation gives

$$2(V_X - 12) + 3V_X + (V_X - V_Y) = 0,$$

$$4(V_Y - 12) + 2V_Y + (V_Y - V_X) = 0.$$

Thus $6V_X - V_Y = 24$ and $-V_X + 7V_Y = 48$, giving

$$V_X = \frac{216}{41} \text{ V} = 5.268 \text{ V},$$

$$V_Y = \frac{312}{41} \text{ V} = 7.610 \text{ V}.$$

$$Q_A = 2(12 - V_X) + 4(12 - V_Y) = \frac{1272}{41} \mu\text{C}.$$

$$C_{\text{eq}} = \frac{Q_A}{12 \text{ V}} = \frac{106}{41} \mu\text{F} = 2.585 \mu\text{F}.$$

Checks. Node residuals are zero. Summed branch energy is $186.15 \mu\text{J} = \frac{1}{2} C_{\text{eq}} (12 \text{ V})^2$. Since $V_Y > V_X$, the central branch's Y plate is positive.

Chapter support: Sections 5.17; Example 5.24.

Capacitors as reserves and sensors



A charge reserve. Delivering charge δQ while permitting a voltage drop δV needs

$$C \geq \frac{\delta Q}{\delta V}.$$

An energy reserve. Delivering W while falling from V_i to V_f needs

$$C \geq \frac{2W}{V_i^2 - V_f^2}.$$

Energy remaining at V_f cannot be counted as delivered energy. **A displacement sensor.** For $C(d) = \varepsilon A/d$,

$$C(d_0 + \delta d) \simeq C_0 \left(1 - \frac{\delta d}{d_0}\right), \quad \left. \frac{dC}{dd} \right|_{d_0} = -\frac{C_0}{d_0}.$$

The next relative term is $(\delta d/d_0)^2$. Stray capacitance and the measurement constraint can dominate the observed signal. These electrostatic relations specify states and ideal bounds, rather than a complete transient response.

Chapter support: Sections 5.18.

Core synthesis and the route to circuits



Core method. State the geometry, material, electrical constraint, and initial node charge. Derive or select a suitable capacitance model. Apply signed conservation and include source work when capacitance changes. Finish with units and limiting cases.

$$Q = C\Delta V, \quad U = \frac{1}{2}Q\Delta V, \quad u_E = \frac{1}{2}\varepsilon E^2 \quad (\text{linear, reversible}).$$

The next chapter introduces current, resistance, and power. For signed terminal charge, take current positive into the terminal defining the voltage:

$$I = \frac{dQ}{dt} = C \frac{dv}{dt} \quad (C \text{ fixed}), \quad I = C\dot{v} + v\dot{C} \quad (C \text{ time dependent}).$$

Reading. Use the revised Chapter 5 for additional exercises, measurement uncertainty, and advanced extensions. Its bibliography points to Halliday–Resnick–Walker, Serway–Jewett, and OpenStax. The next ten slides deepen the established electrostatic models.

Chapter support: Sections 5.20, 5.24.

Advanced: dielectric boundary conditions



Advanced Topics for Technical and Engineering Students A nonuniform polarization contributes bound volume charge:

$$\rho_b = -\nabla \cdot \mathbf{P}, \quad \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}}_{\text{outward}}.$$

At an interface, choose $\hat{\mathbf{n}}$ from medium 1 into medium 2. A thin pillbox and a narrow electrostatic loop give

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_{f,\text{interface}}, \quad \mathbf{E}_{2t} = \mathbf{E}_{1t}.$$

The pillbox uses Gauss's law for free charge. The loop uses $\oint \mathbf{E} \cdot d\ell = 0$. With no free interface charge, normal D is continuous, while normal E can change. The symbols t and n denote tangential and normal components. Bound interface charge remains present even when the free-charge jump is zero.

Chapter support: Sections 5.21.1.

Example 5.28: bound charge at an interface



Example 5.28

Two plane layers carry the same downward displacement D and have $\kappa_1 = 2$ above and $\kappa_2 = 6$ below, with no free interfacial charge. Find their fields and the net bound charge density at the contact surface.

Take the interface normal downward, as specified. Displacement continuity and the material relations give

$$E_1 = \frac{D}{2\varepsilon_0}, \quad E_2 = \frac{D}{6\varepsilon_0}, \quad E_1 = 3E_2, \quad P_1 = \frac{D}{2}, \quad P_2 = \frac{5D}{6}.$$

Opposed outward normals give interface contributions $+P_1$ and $-P_2$:

$$\sigma_{b,\text{net}} = P_1 - P_2 = -\frac{D}{3}.$$

Check. Ordinary Gauss's law gives $\varepsilon_0(E_{2n} - E_{1n}) = -D/3$, the same bound charge. D stays continuous because free interface charge is zero.

Chapter support: Sections 5.21.1; Example 5.28.

Advanced: nonlinear charge–voltage laws



A reversible nonlinear capacitor has $Q = Q(v)$ and $Q(0) = 0$. At a specified bias,

$$C_{\text{sec}}(v) = \frac{Q(v)}{v}, \quad C_{\text{diff}}(v) = \frac{dQ}{dv}.$$

The secant relates a state to the origin. The differential relates a small perturbation: $\delta Q \simeq C_{\text{diff}}(v_0)\delta v$. Charging work must use the entire characteristic:

$$U(v) = \int_0^v v' \frac{dQ(v')}{dv'} dv'.$$

Integration by parts defines coenergy and an independent identity:

$$U^*(v) = \int_0^v Q(v') dv', \quad U + U^* = vQ(v).$$

For constant linear C , both integrals equal $Cv^2/2$. A reversible nonlinear law generally gives different areas. Hysteresis or leakage requires an expanded model.

Chapter support: Sections 5.21.5.

Example 5.30: two capacitances at one bias



Example 5.30

A reversible model has $Q = av + bv^3$, where $a = 20 \text{ pF}$ and $b = 0.010 \text{ pF V}^{-2}$. Find both capacitances and the stored energy at $v = 10 \text{ V}$.

Differentiate the given characteristic before substitution:

$$C_{\text{sec}} = a + bv^2, \quad C_{\text{diff}} = a + 3bv^2.$$

At $v = 10 \text{ V}$, $bv^2 = 1.0 \text{ pF}$. Consequently,

$$C_{\text{sec}} = 21 \text{ pF}, \quad C_{\text{diff}} = 23 \text{ pF}, \quad Q = 210 \text{ pC}.$$

Check. The tangent slope exceeds the secant slope because the cubic term is positive. The cubic coefficient has units F V^{-2} , ensuring that bv^3 is charge. This is an assumed reversible model at fixed geometry. Its differential capacitance is positive in the specified range. The stored-energy integral is evaluated on slide 65.

Chapter support: Sections 5.21.5; Example 5.30.

Example 5.30: stored energy and coenergy



Example 5.30, continued. Use the derivative of the charge law in charging work:

$$U = \int_0^v v'(a + 3bv'^2) dv' = \frac{1}{2}av^2 + \frac{3}{4}bv^4.$$

At 10 V,

$$U = 1.000 \text{ nJ} + 0.075 \text{ nJ} = 1.075 \text{ nJ}.$$

The coenergy and independent identity give

$$U^* = \frac{1}{2}av^2 + \frac{1}{4}bv^4 = 1.025 \text{ nJ}, \quad U + U^* = 2.100 \text{ nJ} = vQ.$$

Using $\frac{1}{2}C_{\text{sec}}v^2$ would give 1.050 nJ, while $\frac{1}{2}C_{\text{diff}}v^2$ would give 1.150 nJ. Neither replaces the work integral. The coefficient units also ensure that each bv^4 term has energy units.

Chapter support: Sections 5.21.5; Example 5.30.

Advanced: force and the controlled variable



Let x change $C(x)$, with force F_x positive toward increasing x . For fixed free charge,

$$F_x = - \left(\frac{\partial}{\partial x} \frac{Q^2}{2C} \right)_Q = \frac{Q^2}{2C^2} \frac{dC}{dx}.$$

At fixed voltage, source work is $v dQ = v^2 dC$. Subtract $dU = v^2 dC/2$ to obtain mechanical work:

$$F_x dx = \frac{1}{2} v^2 dC, \quad \boxed{F_x = \frac{1}{2} v^2 \frac{dC}{dx}}.$$

The state formula agrees, but a path has constant v or constant Q . Equivalently, the fixed-voltage mechanical potential is $U - vQ = -Cv^2/2$. For plane plates and $x = d$, $C'(d) = -\epsilon A/d^2$. Therefore $F_d = -\epsilon A v^2/(2d^2) < 0$: the plates attract. Differentiating positive capacitor energy alone at fixed voltage gives the wrong sign.

Chapter support: Sections 5.21.8.

Example 5.31: instantaneous actuator force



Example 5.31

A plate of area $A = 1.0 \text{ mm}^2$ has an air gap $d = 2.0 \mu\text{m}$ and applied voltage 20 V . Estimate the instantaneous attractive force in the ideal plane model.

Convert $A = 10^{-6} \text{ m}^2$ and $d = 2.0 \times 10^{-6} \text{ m}$. The gap field is $E = v/d = 10^7 \text{ V m}^{-1}$.

$$p = \frac{1}{2} \epsilon_0 E^2 = 442.5 \text{ Pa}, \quad |F_d| = pA = 4.43 \times 10^{-4} \text{ N} = 0.443 \text{ mN}.$$

The signed force is negative in the separation coordinate, so it closes the gap.

Independent check. $C = \epsilon_0 A/d = 4.425 \text{ pF}$ and $U = Cv^2/2 = 0.885 \text{ nJ}$. Then $U/d = 4.425 \times 10^{-4} \text{ N}$, agreeing before rounding. An instantaneous force estimate does not establish a stable position or a permissible voltage. The ideal calculation omits surface effects and small-gap breakdown physics.

Chapter support: Sections 5.21.8; Example 5.31.

Advanced: force balance and stability

Let x close gap d_0 against spring force kx . With $F_e = \varepsilon_0 AV^2 / [2(d_0 - x)^2]$, equilibrium requires $kx = F_e$ and stability requires $k > F'_e$.

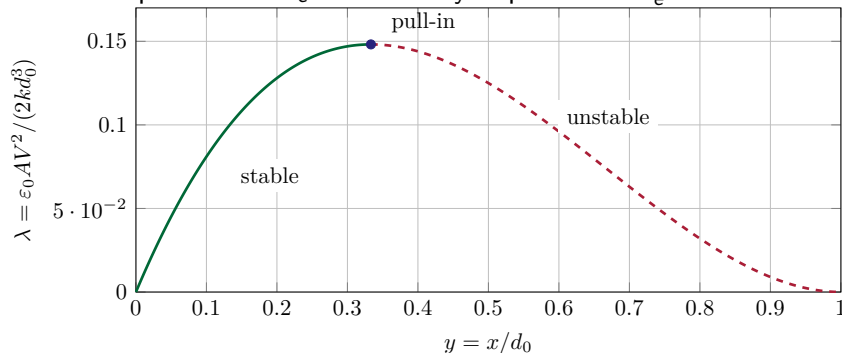


Figure 5.14: Equilibria of the ideal voltage-driven plane actuator. The solid branch is stable, the dashed branch unstable. Above $\lambda = 4/27$, there is no noncontact equilibrium in the model. Surface forces, finite plate rigidity and stops are omitted.

Chapter support: Sections 5.21.

Advanced challenge: pull-in of a plate

Problem 48

A voltage-driven plate of area 0.50 mm^2 is supported by a spring of stiffness 2.0 N m^{-1} at initial gap $10 \mu\text{m}$. Derive and calculate its ideal pull-in displacement and voltage. Explain why an equilibrium below pull-in still requires a stability test.

Derivation. Combine equilibrium and marginal stability:

$$kx = \frac{\epsilon_0 AV^2}{2(d_0 - x)^2}, \quad k = \frac{\epsilon_0 AV^2}{(d_0 - x)^3}.$$

Divide the first equation by the second. The common voltage and material factors cancel:

$$x = \frac{d_0 - x}{2} \quad \Rightarrow \quad \boxed{x_{\text{PI}} = \frac{d_0}{3}}.$$

The voltage and stability check follow on slide 70.

Chapter support: Sections 5.21, 5.22; Problem 48.

Problem 48: pull-in and stability check



Problem 48, numerical solution. Convert $A = 0.50 \times 10^{-6} \text{ m}^2$ and $d_0 = 10^{-5} \text{ m}$. Then

$$x_{\text{PI}} = 3.33 \mu\text{m}, \quad V_{\text{PI}} = \sqrt{\frac{8kd_0^3}{27\varepsilon_0 A}}.$$

$$V_{\text{PI}} = \sqrt{\frac{8(2.0)(10^{-5})^3}{27(8.85 \times 10^{-12})(0.50 \times 10^{-6})}} \text{ V} = 11.6 \text{ V}.$$

Check. $kd_0^3/(\varepsilon_0 A)$ has units V^2 . The curve $\lambda = y(1-y)^2$ peaks at $y = 1/3$, $\lambda = 4/27$. At equilibrium,

$$\frac{dF_e}{dx} = \frac{2kx}{d_0 - x}, \quad k > \frac{dF_e}{dx} \iff x < \frac{d_0}{3}.$$

For $0 < V < V_{\text{PI}}$, the smaller-displacement equilibrium is stable and the larger one is unstable. At the ideal threshold stability is marginal.

Chapter support: Sections 5.23; Problem 48.

