

Magnetic Fields and Magnetic Forces

Physics II - Chapter 8

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Why do moving charges bend?



Electric fields and circuit laws already describe charge and current. A sideways force on an electron beam or a current-carrying wire requires an additional field whose effect depends on motion.

The new physical structure

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$$

The field, velocity and charge sign together determine the force. The force bends a trajectory while leaving its speed unchanged when it acts alone.

The route is to predict particle tracks, use those tracks and Hall voltages to infer properties, then sum carrier forces to explain wire motion and coil rotation.

Chapter support: Sec. 8.1; motivation and roadmap

Learning route and prerequisites



Learning route

Force geometry establishes signs. Work and Newton's law turn the force into circular and helical motion. Force balance then supports selection and Hall inference. Summing carrier forces gives wire force and loop torque.

Tools to recall

Right-handed vectors and cross products; centripetal acceleration; electric potential and work; current density; torque and rotational energy. Derivatives and integrals connect these quantities.

The advanced sequence adds coupled differential equations, momentum inference and numerical error analysis after the core force models are established.

Chapter support: Preface; conventions; Secs. 8.1–8.14

Learning goals



- Determine signed force vectors and verify perpendicularity.
- Derive orbit radius, period and signed helix pitch under stated assumptions.
- Calculate finite-magnet deflections, mass-to-charge ratios and signed Hall quantities.
- Derive wire force and loop torque, then distinguish force, torque, stability and energy transfer.
- Use reversal and uncertainty propagation in a measurement model.

Advanced outcomes

Solve an analytic trajectory, infer transverse momentum from an arc, and distinguish numerical energy conservation from accurate orbital phase.

Chapter support: Chapter learning goals; advanced sections

Magnetic field as an operational quantity



A test charge probes the field at its instantaneous position:

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}, \quad F_B = |q|v_{\perp}B.$$

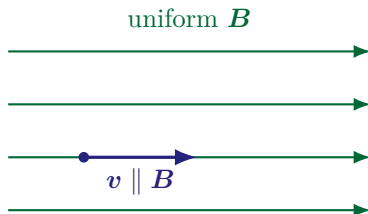
Here q is signed, $B = |\mathbf{B}| \geq 0$, and $v_{\perp}$ is the speed perpendicular to the field. The test charge must not appreciably disturb the sources.

$$1\text{ T} = 1 \frac{\text{N s}}{\text{C m}} = 1 \frac{\text{N}}{\text{A m}}.$$

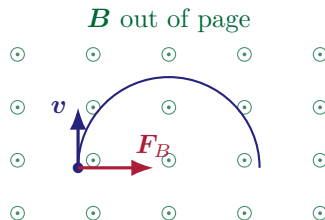
Answered checks. A stationary charge has no magnetic force. Parallel or antiparallel motion also gives zero force because the cross product vanishes. Zero force alone does not establish a zero field.

Chapter support: Secs. 8.1–8.2; conventions

Field direction and particle path



$$\mathbf{F}_B = 0$$



positive charge; curved path

Figure 8.1: A field specifies possible magnetic forces, not a unique trajectory. Left: parallel motion has zero magnetic force. Right: with $\mathbf{B}$ toward the reader and initial velocity upward, a positive charge curves to the right. The blue arc is the particle path, not a field line.

The field line gives the direction of $\mathbf{B}$. Initial velocity and charge sign are additional inputs needed to predict a path.

Chapter support: Sec. 8.1; Figure 8.1

Cross products and charge sign



$$|\mathbf{v} \times \mathbf{B}| = vB \sin \theta, \quad F_B = |q|vB \sin \theta.$$

The right-hand rule determines $\mathbf{v} \times \mathbf{B}$. Multiply by the signed charge afterward. A negative charge reverses the direction once.

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \quad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \quad \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}.$$

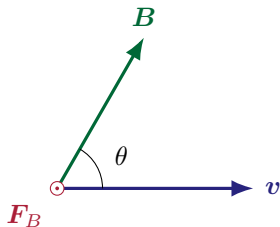
Reversing the order changes the sign. In particular, $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$.

Geometric check

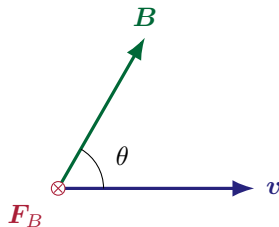
The magnetic force must be perpendicular to both velocity and field. Its magnitude is greatest for perpendicular motion and vanishes for parallel motion.

Chapter support: Sec. 8.2; vector product and right-hand rule

Right-hand geometry in the page



positive charge



negative charge

Figure 8.2: For velocity to the right and a field in the upper half of the page, $\mathbf{v} \times \mathbf{B}$ points out of the page. A positive charge has force out of the page (dot); a negative charge has force into it (cross). In both panels $+z$ points toward the reader.

A dot denotes an outward vector and a cross an inward vector. Both panels use $+z$ toward the reader.

Chapter support: Sec. 8.2; Figure 8.2

Example 8.1: proton and electron directions



Example 8.1 — complete statement.

A proton moves in the $+\hat{i}$ direction through a region where $\mathbf{B} = B\hat{k}$. Determine the direction of the magnetic force. Repeat for an electron with the same velocity.

For the proton $q = +e$:

$$\mathbf{F}_p = evB(\hat{i} \times \hat{k}) = -evB\hat{j}.$$

The electron has the same velocity and field but $q = -e$:

$$\mathbf{F}_e = +evB\hat{j}.$$

Both forces are perpendicular to $\hat{i}$ and $\hat{k}$. Their opposite directions follow from the charge sign, rather than a different field.

Chapter support: Example 8.1; Sec. 8.2

Component method for arbitrary directions



Write both vectors in one right-handed coordinate system. The determinant provides the cross product before the signed charge is applied:

$$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix}.$$

$$(\mathbf{v} \times \mathbf{B})_x = v_y B_z - v_z B_y,$$

$$(\mathbf{v} \times \mathbf{B})_y = v_z B_x - v_x B_z,$$

$$(\mathbf{v} \times \mathbf{B})_z = v_x B_y - v_y B_x.$$

After calculating $\mathbf{F}_B = q(\mathbf{v} \times \mathbf{B})$, test both $\mathbf{F}_B \cdot \mathbf{v} = 0$ and $\mathbf{F}_B \cdot \mathbf{B} = 0$ using unrounded components.

Chapter support: Sec. 8.2; component method

Example 8.3: a three-dimensional force



Example 8.3 — complete statement.

A particle of charge $q = +2e$ has velocity $\mathbf{v} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$ in units of 10^5 m/s. The magnetic field is $\mathbf{B} = 0.20\hat{\mathbf{j}} + 0.10\hat{\mathbf{k}}$ in tesla. Find the force vector.

In SI units, $\mathbf{v} = (3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})10^5$ m/s. Thus

$$\mathbf{v} \times \mathbf{B} = (4\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})10^4 \text{ N/C}.$$

With $q = 3.204 \times 10^{-19}$ C,

$$\mathbf{F}_B = (1.28\hat{\mathbf{i}} - 0.961\hat{\mathbf{j}} + 1.92\hat{\mathbf{k}})10^{-14} \text{ N}.$$

Independent checks. Unrounded dot products are proportional to $4(3) - 3(4) = 0$ and $-3(0.20) + 6(0.10) = 0$. The field has units N s/(C m), so $q\mathbf{v}\mathbf{B}$ has units N.

Chapter support: Example 8.3

Work, kinetic energy and the full Lorentz force



When both fields act,

$$\dot{\mathbf{p}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Here $\mathbf{p}$ is momentum and m is particle mass. The force law remains valid relativistically; the core uses $\mathbf{p} = m\mathbf{v}$ and neglects collisions and radiation. We use $c = 3.00 \times 10^8$ m/s.

$$\begin{aligned} \frac{dK}{dt} &= \mathbf{F} \cdot \mathbf{v} \\ &= q\mathbf{E} \cdot \mathbf{v} + q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}. \end{aligned}$$

The magnetic term vanishes at every instant, including in a nonuniform field. Magnetic deflection changes direction. Electric power can change speed. An induced electric field can do work when the magnetic field varies with time.

Chapter support: Secs. 8.2–8.3; Lorentz force and work

P9: Lorentz force with a negative charge



P9 — complete statement.

A particle with $q = -2.0 \text{ nC}$ has velocity $(3\hat{i} - 4\hat{j} + 2\hat{k}) \times 10^5 \text{ m/s}$ in $\mathbf{B} = (0.10\hat{i} + 0.20\hat{k}) \text{ T}$ and $\mathbf{E} = 50\hat{j} \text{ V/m}$. Calculate the magnetic force, total force and instantaneous electric power. Verify the magnetic power directly from the components.

Use the component method before multiplying by q :

$$\mathbf{v} \times \mathbf{B} = (-8\hat{i} - 4\hat{j} + 4\hat{k})10^4 \text{ N/C}.$$

Therefore

$$\mathbf{F}_B = (1.60\hat{i} + 0.80\hat{j} - 0.80\hat{k})10^{-4} \text{ N}.$$

The negative charge reverses every component. The electric contribution and power checks follow on slide 15.

Chapter support: P9; selected detailed solution P9

P9: total force and the power balance



The electric force is

$$\mathbf{F}_E = q\mathbf{E} = -1.00 \times 10^{-7} \hat{\mathbf{j}} \text{ N}.$$

Add it to the magnetic force from slide 14:

$$\mathbf{F} = (1.600\hat{\mathbf{i}} + 0.799\hat{\mathbf{j}} - 0.800\hat{\mathbf{k}})10^{-4} \text{ N}.$$

The extra intermediate digits expose the small electric contribution.

$$P_B = \mathbf{F}_B \cdot \mathbf{v} = 48 - 32 - 16 = 0 \text{ W},$$

$$P_E = (-2.0 \times 10^{-9})(50)(-4 \times 10^5) = 0.040 \text{ W}.$$

Kinetic energy is increasing. A much smaller electric force can supply all the power because power depends on the component along velocity.

Chapter support: P9; Sec. 8.3

Reversal and field reconstruction



One velocity leaves the field component along that velocity invisible. For fixed fields at one point, equal and opposite velocities separate the electric and magnetic contributions:

$$\frac{\mathbf{F}(\mathbf{v}) + \mathbf{F}(-\mathbf{v})}{2q} = \mathbf{E},$$
$$\mathbf{D}(\mathbf{v}) \equiv \frac{\mathbf{F}(\mathbf{v}) - \mathbf{F}(-\mathbf{v})}{2q} = \mathbf{v} \times \mathbf{B}.$$

The triple-product identity gives

$$\mathbf{B}_{\perp} = \frac{\mathbf{D} \times \mathbf{v}}{v^2}.$$

A second nonparallel probe direction supplies the missing component. Nearly parallel probes amplify errors in a weakly constrained direction. A force component along the probe velocity cannot be explained by a magnetic field.

Chapter support: Secs. 8.1–8.2; Eq. for inverse field

Example 8.4: reconstructing all field components



Example 8.4 — complete statement.

A known positive charge is probed at equal speed v along x and y . After subtracting the electric contribution by velocity reversal, the force data give

$$\frac{D(v\hat{i})}{v} = (-0.030\hat{j} + 0.020\hat{k}) \text{ T}, \quad \frac{D(v\hat{j})}{v} = (0.030\hat{i} - 0.010\hat{k}) \text{ T}.$$

Find $\mathbf{B}$ and identify a redundant measured component.

$$\hat{i} \times \mathbf{B} = B_y \hat{k} - B_z \hat{j} \text{ gives } B_y = 0.020 \text{ T and } B_z = 0.030 \text{ T}.$$

$$\hat{j} \times \mathbf{B} = B_z \hat{i} - B_x \hat{k} \text{ gives } B_x = 0.010 \text{ T and the same } B_z.$$

$$\mathbf{B} = (0.010\hat{i} + 0.020\hat{j} + 0.030\hat{k}) \text{ T}, \quad B = 0.0374 \text{ T}.$$

The redundant measurement is B_z . Its agreement tests the assumed common field and probe geometry. Each measured force is perpendicular to its probe velocity.

Chapter support: Example 8.4

Circular motion: force supplies the centripetal acceleration



For perpendicular motion in a prescribed uniform field, the magnetic force remains normal to velocity. Constant speed and Newton's law give

$$|q|vB = \frac{mv^2}{r}, \quad \boxed{r = \frac{mv}{|q|B}}.$$

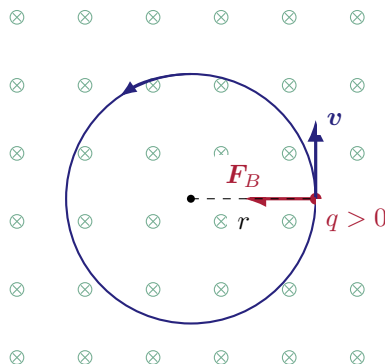
There is one inward force here. A separate additional “centripetal force” must not be added.

$$\omega_c = \frac{v}{r} = \frac{|q|B}{m}, \quad T_c = \frac{2\pi m}{|q|B}, \quad f_c = \frac{1}{T_c}.$$

In this nonrelativistic model the period is independent of speed. Charge sign determines the turning sense separately. Since $|q|B$ has units kg/s, the radius and period have units m and s.

Chapter support: Sec. 8.4; radius and period

Velocity is tangential and force is radial



B into page
counterclockwise orbit
 F_B radial inward

reverse q :
reverse the curvature

Figure 8.3: Uniform circular motion viewed perpendicular to the magnetic field. The velocity is tangential and the magnetic force points toward the centre at every point. Charge sign determines the sense of rotation.

For this view, $q > 0$ and the field is into the page. At the rightmost point the upward velocity gives an inward force to the left.

Chapter support: Sec. 8.4; Figure 8.3

Example 8.6: an electron orbit radius



Example 8.6 — complete statement.

An electron with speed $v = 3.0 \times 10^6$ m/s enters a region of uniform magnetic field $B = 0.020$ T perpendicular to its velocity. Estimate the radius of its path. Use $m_e = 9.11 \times 10^{-31}$ kg and $e = 1.602 \times 10^{-19}$ C.

The perpendicular uniform-field model gives

$$r = \frac{(9.11 \times 10^{-31})(3.0 \times 10^6)}{(1.602 \times 10^{-19})(0.020)} \text{ m} = 8.53 \times 10^{-4} \text{ m} \approx 0.85 \text{ mm}.$$

The speed is $0.010c$, supporting the nonrelativistic estimate. Radius is positive because it uses $|q|$; the electron curves oppositely to a positive charge in the same initial geometry.

Doubling B halves r at fixed speed. As B tends to zero, the large-radius limit approaches a straight path.

Chapter support: Example 8.6; Sec. 8.4

Example 8.7: what is held fixed?



Example 8.7 — complete statement.

Compare a proton with an alpha particle, taking $m_\alpha \simeq 4m_p$ and $q_\alpha = +2e$, for perpendicular motion in the same field.

At equal speed, $r = mv/(|q|B)$:

$$\frac{r_\alpha}{r_p} = \frac{4m_p/(2e)}{m_p/e} = 2.$$

At equal nonrelativistic kinetic energy, $v = \sqrt{2K/m}$:

$$r = \frac{\sqrt{2mK}}{|q|B}, \quad \frac{r_\alpha}{r_p} = \frac{\sqrt{4}}{2} = 1.$$

In both comparisons $T_\alpha/T_p = (4/2) = 2$. Both charges are positive, so equal initial directions give the same sense of rotation. Equal curvature therefore need not imply equal mass; charge state and the fixed quantity matter.

Chapter support: Example 8.7; Sec. 8.4 radius scaling

Helical motion and signed pitch



Resolve velocity relative to a uniform field:

$$\mathbf{v} = \mathbf{v}_{\perp} + v_{\parallel} \hat{\mathbf{b}}, \quad \hat{\mathbf{b}} = \mathbf{B}/B.$$

The parallel component has no magnetic force and stays constant. The perpendicular motion is circular:

$$r_L = \frac{mv_{\perp}}{|q|B}, \quad T_c = \frac{2\pi m}{|q|B}, \quad h = v_{\parallel} T_c.$$

For pitch angle α measured from $+\mathbf{B}$,

$$v_{\perp} = v \sin \alpha, \quad v_{\parallel} = v \cos \alpha, \quad \frac{h}{r_L} = 2\pi \cot \alpha.$$

h is a signed axial advance, and r_L is the radius of the transverse projection. At $\alpha = 90^\circ$ the path is a circle. Exactly parallel motion is straight, with no actual gyration.

Chapter support: Sec. 8.4; helical motion and pitch angle

A helix and its transverse projection

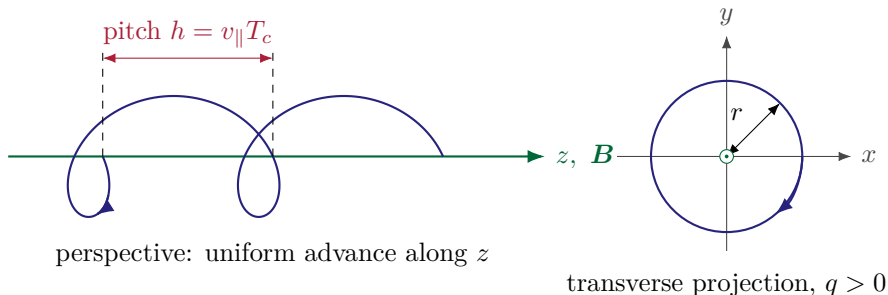


Figure 8.4: Helical motion for a positive charge with $v_{\parallel} > 0$. The perspective drawing is a projection of a circular helix with equal axial advances at equal phase. Viewed from $+z$ toward the origin, the transverse rotation is clockwise for $\mathbf{B} = B\hat{\mathbf{k}}$. Increasing $v_{\parallel}$ changes pitch without changing transverse radius. The parallel direction and charge sign control distinct aspects of the geometry.

Chapter support: Sec. 8.4; Figure 8.4

Practice P11: a proton helix



P11 — complete statement.

A proton enters $\mathbf{B} = 0.400\hat{\mathbf{k}}\text{ T}$ at speed $8.00 \times 10^5\text{ m/s}$ and angle 60.0° to $+z$, with its transverse velocity initially along $+x$. Find the transverse radius, period, signed pitch and initial force direction. Determine its position relative to entry after one quarter-period.

Model before calculation. Use $q = +e$, $m_p = 1.67 \times 10^{-27}\text{ kg}$, and $e = 1.602 \times 10^{-19}\text{ C}$. The speed is nonrelativistic, and the field is uniform.

$$v_{\perp} = v \sin 60^\circ = 6.93 \times 10^5\text{ m/s}, \quad v_{\parallel} = v \cos 60^\circ = 4.00 \times 10^5\text{ m/s}.$$

The positive pitch follows from $v_{\parallel} > 0$. Calculate the orbit quantities and locate the quarter-turn point before consulting slide 25.

Chapter support: P11; selected detailed solution P11

P11: orbit quantities and quarter-period position



Using the decomposed velocities from slide 24,

$$r_L = \frac{m_p v_{\perp}}{eB} = 1.81 \times 10^{-2} \text{ m},$$

$$T_c = \frac{2\pi m_p}{eB} = 1.64 \times 10^{-7} \text{ s},$$

$$h = v_{\parallel} T_c = 6.55 \times 10^{-2} \text{ m}.$$

Initially $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$, so the proton bends toward $-y$. Its transverse circle has centre $(0, -r_L)$ relative to entry. One quarter-turn carries it to $(r_L, -r_L)$ while it advances $h/4$ along z :

$$(x, y, z) = (0.0181, -0.0181, 0.0164) \text{ m}.$$

The check $h/r_L = 2\pi \cot 60^\circ = 3.63$ verifies the geometry independently. Total speed stays constant.

Chapter support: P11; Sec. 8.4

A finite magnet and a downstream screen

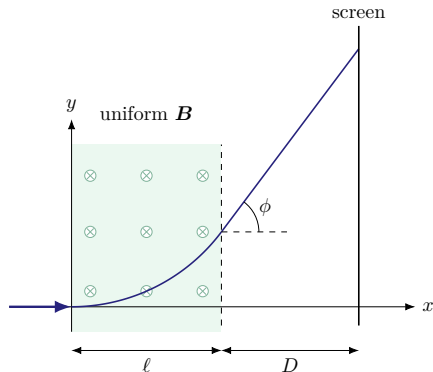


Figure 8.5: A circular arc inside a finite magnet followed by a straight tangent outside it. The drawing exaggerates the bending for clarity; the exact circular relation, rather than a small-angle sketch, determines the exit quantities.

Model: $q > 0$, entry along $+x$, $\mathbf{B} = -B\hat{\mathbf{k}}$ in $0 < x < \ell$. Fringe fields are neglected; the path outside is the exit tangent.

Chapter support: Sec. 8.4; Figure 8.5

Exact finite-magnet geometry and its approximation



The positive charge bends upward, with circle centre $(0, r)$:

$$x^2 + (y - r)^2 = r^2, \quad y(x) = r - \sqrt{r^2 - x^2}.$$

For $\ell < r$, the first forward exit satisfies

$$\sin \phi = \frac{\ell}{r}, \quad y_e = r - \sqrt{r^2 - \ell^2}, \quad \tan \phi = \frac{\ell}{\sqrt{r^2 - \ell^2}}.$$

At a screen a further horizontal distance D away,

$$y_s = y_e + D \tan \phi \simeq \frac{\ell^2}{2r} + \frac{D\ell}{r} \quad (\ell/r \ll 1).$$

The approximation follows by expanding the square root and tangent. If $\ell > r$, the orbit turns back before the far boundary. A straight flight time ℓ/v would then describe an impossible traversal.

Chapter support: Sec. 8.4; finite magnetic region

P14: finite-magnet deflection



P14 — complete statement.

A positive ion enters a finite field $\mathbf{B} = -B\hat{\mathbf{k}}$ along $+x$. Its radius is 0.250 m , and the field occupies $0 < x < 0.150\text{ m}$. A screen lies a further horizontal distance 0.200 m beyond the field. Find the exit angle, exit height and screen height. Quantify the error of the small-angle screen formula. Would the particle reach the far boundary if the magnet length were 0.300 m ?

For the first forward exit, $\ell/r = 0.600$ and

$$\sqrt{r^2 - \ell^2} = 0.200\text{ m}, \quad \phi = \sin^{-1}(0.600) = 36.9^\circ.$$

Thus $y_e = 0.250 - 0.200 = 0.0500\text{ m}$ and $\tan \phi = 0.750$. The screen height and validity checks follow on slide 29.

Chapter support: P14; selected detailed solution P14

P14: the downstream lever arm and a failed traversal



The exit tangent gives the exact screen height:

$$y_s = 0.0500 + (0.200)(0.750) = 0.200 \text{ m.}$$

The small-angle formula instead gives

$$y_{s,\text{approx}} = \frac{0.150^2}{2(0.250)} + \frac{(0.200)(0.150)}{0.250} = 0.165 \text{ m.}$$

With the exact value as the error denominator, its relative error is

$$\frac{0.165 - 0.200}{0.200} = -0.175 = -17.5\%.$$

For a 0.300 m magnet, $\ell > r = 0.250 \text{ m}$. The greatest forward x is r , so the particle never reaches that far boundary. Magnetic force still does zero work; the assumed traversal fails.

Chapter support: P14; Sec. 8.4

Crossed fields and the force-free velocity



For a collimated beam along $+x$, choose $\mathbf{E} = -E\hat{\mathbf{j}}$ and $\mathbf{B} = -B\hat{\mathbf{k}}$. A positive particle then has

$$F_y = q(Bv - E).$$

Zero deflection requires $v_0 = E/B$. Both forces reverse for a negative charge, preserving this selected velocity.

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + v_{\parallel} \hat{\mathbf{b}} \quad (\mathbf{E} \cdot \mathbf{B} = 0, \quad q \neq 0).$$

A parallel component is force-free as well. Calling E/B the total selected speed therefore presumes the specified beam direction. Apertures reject unwanted paths; the fields do not restore their speeds. Neutral particles are not speed-selected by this force balance.

Chapter support: Sec. 8.5; crossed fields and force-free family

Selector force directions

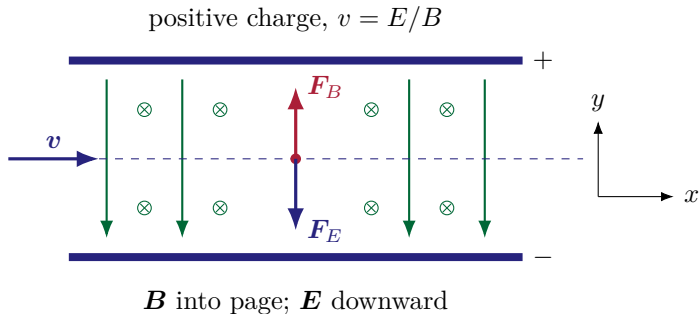


Figure 8.6: For a positive particle moving right, the magnetic force points up and the electric force down. Both arrows are attached to the particle. Reversing charge reverses both forces, so the same velocity remains undeflected.

For this positive beam, faster particles initially bend upward because $Bv > E$; slower ones bend downward. Reversing only B destroys cancellation for the same beam direction.

Chapter support: Sec. 8.5; Figure 8.6

Magnetic mass analysis and detector geometry

$$\frac{m}{|q|} = \frac{rB_a}{v} = \frac{rB_aB_s}{E}.$$

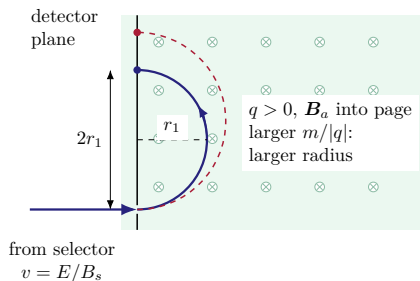


Figure 8.7: A schematic semicircular mass analyser. Two positive ions enter at the same point and speed. Their impact separation along the detector plane is $2(r_2 - r_1)$. The regions outside the analysing magnet, including the selector, are represented only schematically.

Chapter support: Sec. 8.5; Figure 8.7

Example 8.12: inferring an ion mass



Example 8.12 — complete statement.

Singly charged positive ions pass through a velocity selector with $E = 3.0 \times 10^4 \text{ V/m}$ and $B_s = 0.15 \text{ T}$, then enter an analyser field $B_a = 0.30 \text{ T}$. The observed circular radius is $r = 0.21 \text{ m}$. Estimate the ion mass.

First obtain speed from the selector, then use the analyser radius:

$$v = \frac{3.0 \times 10^4}{0.15} = 2.0 \times 10^5 \text{ m/s},$$
$$m = \frac{(1.602 \times 10^{-19})(0.30)(0.21)}{2.0 \times 10^5} = 5.0463 \times 10^{-26} \text{ kg} \approx 5.0 \times 10^{-26} \text{ kg}.$$

With $u = 1.6605 \times 10^{-27} \text{ kg}$ this is about $30u$. It is a mass estimate conditional on the singly charged state. Doubling the assumed charge doubles the inferred mass for the same speed and radius.

Chapter support: Example 8.12

Electric acceleration followed by magnetic bending



Define an accelerating potential magnitude $V_{\text{acc}} > 0$ so that $-q\Delta V = |q|V_{\text{acc}}$. From negligible initial kinetic energy,

$$\frac{1}{2}mv^2 = |q|V_{\text{acc}}, \quad mv = |q|Br.$$

Eliminate v to obtain

$$\boxed{\frac{|q|}{m} = \frac{2V_{\text{acc}}}{B^2 r^2}.$$

The electric region supplies energy; the magnetic region measures curvature. This model assumes perpendicular entry, a uniform analysing field, negligible losses and nonrelativistic motion.

At fixed accelerating voltage, $r \propto \sqrt{m/|q|}$. At fixed selected speed, $r \propto m/|q|$. A measured diameter must be halved before it is used as r .

Chapter support: Sec. 8.5; accelerating voltage and charge-to-mass ratio

Example 8.13: electron charge-to-mass inference



Example 8.13 — complete statement.

Electrons are accelerated through 250 V and bend in a field of 1.50 mT. A circular track has measured radius 3.56 cm. Estimate e/m_e and assess the nonrelativistic approximation.

Convert $B = 1.50 \times 10^{-3} \text{ T}$ and $r = 0.0356 \text{ m}$:

$$\frac{e}{m_e} = \frac{500}{(1.50 \times 10^{-3})^2 (0.0356)^2} = 1.75 \times 10^{11} \text{ C/kg.}$$

$$v = \sqrt{2(e/m_e)(250)} = 9.36 \times 10^6 \text{ m/s} = 0.0312c.$$

The speed supports the nonrelativistic model at this precision. The rounded reference constants give $e/m_e = 1.76 \times 10^{11} \text{ C/kg}$.

An approximately doubled radius would give an approximately four-times-smaller ratio. Check radius versus diameter, field calibration and entry geometry when that discrepancy occurs.

Chapter support: Example 8.13

Hall geometry and a signed voltage convention



A rectangular bar carries conventional current $I > 0$ along $+x$. Let $\mathbf{B} = B_z \hat{\mathbf{k}}$, width w along y , and thickness t along z . Edge a is at $y = -w/2$, edge b at $y = +w/2$.

$$V_H \equiv V_a - V_b = E_y w.$$

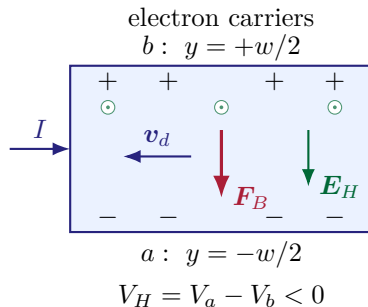
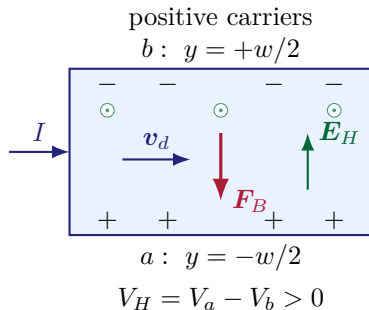
The positive meter terminal is connected to a . Since $\mathbf{E} = -\nabla V$, a positive E_y means potential decreases upward.

How equilibrium develops

Magnetic deflection creates transverse charge separation. Its electric field balances the transverse magnetic force when the open transverse circuit has no steady transverse current. Longitudinal electric driving and scattering still maintain the resistive current.

Chapter support: Sec. 8.6; signed Hall convention

Positive and negative Hall carriers



I along $+x$; $B_z > 0$ out of page; thickness t normal to page

Figure 8.8: Hall polarity for the explicitly defined voltage $V_a - V_b$. Charge accumulation is shown schematically. The red arrow is the initial magnetic force; the Hall electric force eventually opposes it. The positive and negative carrier cases have opposite Hall fields and opposite meter readings.

Chapter support: Sec. 8.6; Figure 8.8

Deriving the single-carrier Hall relation



For signed carrier charge q_c and density n ,

$$J_x = nq_c v_{dx} = \frac{I}{wt}.$$

Because $\mathbf{v}_d \times \mathbf{B} = -v_{dx} B_z \hat{\mathbf{j}}$, transverse balance gives

$$q_c(E_y - v_{dx} B_z) = 0 \implies E_y = v_{dx} B_z.$$

Thus

$$R_H = \frac{1}{nq_c}, \quad \boxed{V_H = \frac{IB_z}{nq_c t}}.$$

Width cancels at fixed current because it decreases drift speed and increases the voltage-integration distance in compensating proportions. The model assumes one effective carrier type, homogeneous current flow and open transverse contacts. Hall sign alone is insufficient without known wiring and field/current signs.

Chapter support: Sec. 8.6; signed Hall derivation

P17: signed Hall inference



P17 — complete statement.

A single-carrier Hall bar has $w = 4.00 \text{ mm}$, $t = 0.200 \text{ mm}$ and $I = 50.0 \text{ mA}$ along $+x$. At $B_z = +0.500 \text{ T}$, $V_a - V_b = -3.00 \text{ mV}$. Assuming $|q_c| = e$, find the carrier sign, R_H , density and signed drift velocity. Calculate E_y and verify the transverse force balance on an individual carrier.

Here $I > 0$, $B_z > 0$ and $V_H < 0$. The single-carrier model therefore gives $q_c = -e$.

$$R_H = \frac{V_H t}{IB_z} = \frac{(-3.00 \times 10^{-3})(2.00 \times 10^{-4})}{(0.0500)(0.500)} = -2.40 \times 10^{-5} \text{ m}^3/\text{C}.$$

Density, drift velocity and the individual carrier-force check continue on slide 40.

Chapter support: P17; selected detailed solution P17

P17: density, drift and transverse force balance



$$n = \frac{1}{e|R_H|} = 2.60 \times 10^{23} \text{ m}^{-3},$$

$$J_x = \frac{0.0500}{(0.00400)(0.000200)} = 6.25 \times 10^4 \text{ A/m}^2,$$

$$v_{dx} = \frac{J_x}{n(-e)} = -1.50 \text{ m/s},$$

$$E_y = \frac{V_H}{w} = -0.750 \text{ V/m} = v_{dx} B_z.$$

$$F_{E,y} = (-e)(-0.750) = +1.20 \times 10^{-19} \text{ N},$$

$$F_{B,y} = -q_c v_{dx} B_z = -1.20 \times 10^{-19} \text{ N}.$$

The forces on an individual electron cancel. Drift is opposite to current; $V_a < V_b$ agrees with the downward Hall field.

Chapter support: P17; Sec. 8.6

Carrier forces add to a force on the wire



An active straight segment has length L , cross section A_c , and nA_cL mobile carriers. Summing their forces gives

$$\mathbf{F} = nA_cLq_c\mathbf{v}_d \times \mathbf{B}.$$

Use $\mathbf{J} = nq_c\mathbf{v}_d$ and $I = JA_c$ to obtain

$$\boxed{\mathbf{F} = I\mathbf{L} \times \mathbf{B}}, \quad F = ILB \sin \theta.$$

$\mathbf{L}$ follows conventional current. Carrier sign is already included in that direction, so an additional electron minus sign would be incorrect.

Carrier–lattice interactions and electric forces transmit the force to the neutral conductor. Specify the external field and the object of interest: an active segment and a complete circuit are different systems.

Chapter support: Sec. 8.7; carrier-to-wire derivation

The active length inside the field

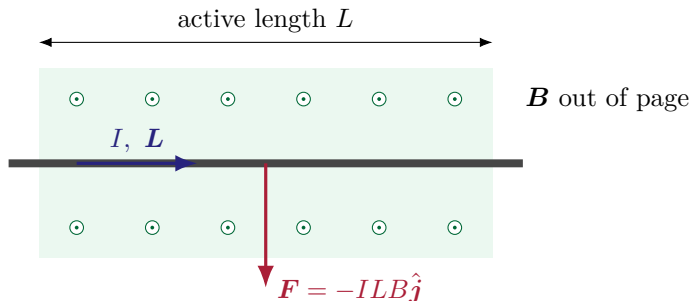


Figure 8.9: A straight segment carrying conventional current rightward in a field out of the page feels a downward force. Only the length inside the stated field region contributes to the displayed ILB estimate.

The drawing concerns one selected segment. Return leads contribute only where their local external field is significant and must be included when the complete circuit is analysed.

Chapter support: Sec. 8.7; Figure 8.9

Differential wire force and the endpoint theorem



For a curved wire, use local directed elements:

$$d\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B}, \quad \mathbf{F} = I \int_A^C d\boldsymbol{\ell} \times \mathbf{B}.$$

If $\mathbf{B}$ is uniform, it factors out:

$$\mathbf{F} = I(\mathbf{r}_C - \mathbf{r}_A) \times \mathbf{B}.$$

The net force then depends on endpoint displacement, rather than arc length. For a closed loop it is zero, although torque may remain nonzero. For a nonuniform field, keep the field inside the integral. A linear profile $B_z = B_0(1 + x/L)$ along a straight $+x$ segment gives $F_y = -\frac{3}{2}IB_0L$; using the exit field $2B_0$ would overestimate the magnitude by one third.

Chapter support: Sec. 8.7; finite wires and force transfer

Example 8.16: force on a semicircular arc



Example 8.16 — complete statement.

A wire consists of a semicircular arc of radius a carrying current from the left end to the right end along the upper arc. It lies in the xy plane. A uniform field $\mathbf{B} = B\hat{\mathbf{k}}$ is applied. Find the net force on the arc.

Uniformity permits the endpoint reduction:

$$\int_A^C d\ell = 2a\hat{\mathbf{i}}, \quad \boxed{\mathbf{F} = -2laB\hat{\mathbf{j}}.}$$

An independent parameterization runs from $\varphi = \pi$ to 0:

$$d\ell = a(-\sin\varphi\hat{\mathbf{i}} + \cos\varphi\hat{\mathbf{j}})d\varphi.$$

Then $F_x = laB \int_{\pi}^0 \cos\varphi d\varphi = 0$ and $F_y = laB \int_{\pi}^0 \sin\varphi d\varphi = -2laB$. A straight segment between the same endpoints has the same net force in this uniform field.

Chapter support: Example 8.16

Magnetic moment and a rotational couple



Equal and opposite forces can cancel in translation while producing a rotational couple. Define the loop's normal by curling the right hand with conventional current:

$$\boldsymbol{\mu} = NIA\hat{n}, \quad [\mu] = \text{A m}^2.$$

The moment is perpendicular to the plane of the loop. For N identical turns it is proportional to their total oriented area.

The angle that matters

Let θ be the angle between $\boldsymbol{\mu}$ and $\mathbf{B}$. A field in the loop's plane corresponds to $\theta = 90^\circ$, rather than zero. An angle stated relative to the plane is complementary.

The next drawing shows the current-normal convention and a tilted-loop view before the torque is derived.

Chapter support: Sec. 8.8; magnetic moment

Loop forces and the tilted lever arm

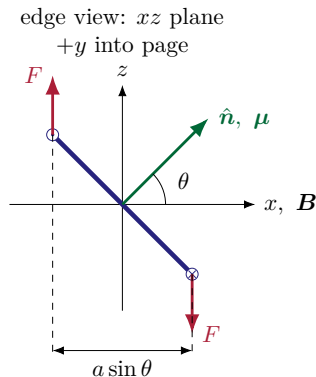
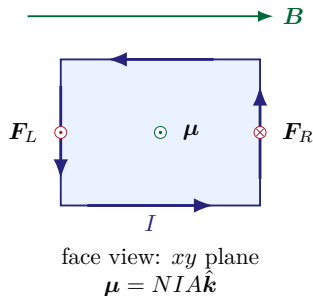


Figure 8.10: Left: a loop in the xy plane with counterclockwise current has moment out of the page. A rightward field gives forces out of the page on the left side and into it on the right. Right: a tilted loop viewed along $+y$ has two active sides perpendicular to the view; their force lines are separated by $a \sin \theta$. The two panels show different loop orientations.

Chapter support: Sec. 8.8; Figure 8.10

Deriving torque from opposite sides



Take active sides of length b along $\pm y$, separated by $a\hat{\mathbf{u}}$ in $\mathbf{B} = B\hat{\mathbf{i}}$, where

$$\hat{\mathbf{u}} = \sin \theta \hat{\mathbf{i}} - \cos \theta \hat{\mathbf{k}}, \quad \hat{\mathbf{n}} = \hat{\mathbf{u}} \times \hat{\mathbf{j}} = \cos \theta \hat{\mathbf{i}} + \sin \theta \hat{\mathbf{k}}.$$

The side at $+(a/2)\hat{\mathbf{u}}$ carries current along $+y$ and feels $-lbB\hat{\mathbf{k}}$. The opposite side feels $+lbB\hat{\mathbf{k}}$. Their torques add:

$$\boldsymbol{\tau} = 2 \left[\frac{a}{2} \hat{\mathbf{u}} \times (-lbB\hat{\mathbf{k}}) \right] = labB \sin \theta \hat{\mathbf{j}}.$$

The remaining pair has zero net torque about the centre. With $A = ab$ and N turns,

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}, \quad \tau = NIAB \sin \theta.$$

The two forces and their half-separation arms explain the factor, avoiding an erroneous extra two.

Chapter support: Sec. 8.8; four-side torque derivation

Example 8.18: rectangular-coil torque



Example 8.18 — complete statement.

A rectangular coil has $N = 80$ turns, area $A = 2.5 \times 10^{-3} \text{ m}^2$ and current $I = 0.60 \text{ A}$. It is placed in a uniform magnetic field $B = 0.40 \text{ T}$. The angle between μ and B is 60° . Find the torque magnitude.

The moment and torque magnitudes are

$$\mu = (80)(0.60)(2.5 \times 10^{-3}) = 0.120 \text{ A m}^2,$$

$$\tau = (0.120)(0.40) \sin 60^\circ = 4.16 \times 10^{-2} \text{ N m} \approx 4.2 \times 10^{-2} \text{ N m}.$$

The upper bound is $\mu B = 0.0480 \text{ N m}$ at 90° . The stated angle is between moment and field. At alignment or opposition the torque is zero, but those two equilibria have different stability.

Chapter support: Example 8.18

Dipole energy and stability



Hold μ and the prescribed external field fixed. Choose θ to increase away from alignment. The generalized restoring torque is

$$\tau_{\theta} = -\mu B \sin \theta = -\frac{dU}{d\theta}.$$

Integrate and choose $U = 0$ at 90° :

$$U = -\mu B \cos \theta = -\boldsymbol{\mu} \cdot \mathbf{B}.$$

At 0° , U is a minimum and alignment is stable. At 180° , U is a maximum and opposition is unstable. Both have zero torque, so zero torque does not prove stability.

An undamped loop released away from alignment can overshoot and oscillate. Dissipation is required for settling. For a powered coil this is an effective mechanical potential at fixed moment, not the complete energy of all sources and fields.

Chapter support: Sec. 8.9; dipole energy

Energy and signed angular torque

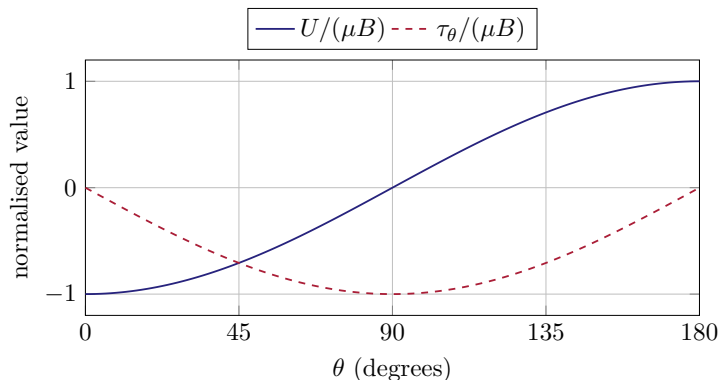


Figure 8.11: Calculated energy and signed angular torque for a fixed magnetic moment. Alignment at 0° is stable; opposition at 180° is unstable. The energy rises as the dipole is turned away from alignment, while the torque opposes that change.

Chapter support: Sec. 8.9; Figure 8.11

Example 8.19: quasistatic work to reverse a coil



Example 8.19 — complete statement.

A coil with fixed $\mu = 0.050 \text{ A m}^2$ is slowly turned from alignment to opposition in a field $B = 0.80 \text{ T}$. Find the external work in a quasistatic, lossless rotation.

Slow lossless rotation makes the external agent balance the magnetic torque. Its mechanical work equals the increase of the effective potential:

$$\begin{aligned} W_{\text{ext}} &= U(\pi) - U(0) = +\mu B - (-\mu B) \\ &= 2(0.050)(0.80) = 0.080 \text{ J}. \end{aligned}$$

Equivalently, $W_{\text{ext}} = \int_0^\pi \mu B \sin \theta \, d\theta = 2\mu B$. The field torque performs -0.080 J during the imposed rotation. The units $\text{A m}^2 \times \text{T} = \text{J}$ provide a second check.

The calculation holds the moment fixed. If the current-maintaining circuit belongs to the system, its energy exchange must also be counted.

Chapter support: Example 8.19

Example 8.20: motor torque and a thermal constraint



Current switching sustains motor torque. Peak single-coil torque is $NIAB$. Resistive heating is $P = VI = I^2R$, since $V = IR$; the electrical source supplies mechanical output and losses.

Example 8.20 — complete statement.

A coil has $N = 100$, $A = 8.0 \times 10^{-4} \text{ m}^2$ and operates in $B = 0.25 \text{ T}$. Its resistance is $R = 4.0 \Omega$. What current gives maximum torque 0.020 N m , and is it consistent with an assumed continuous Joule-heating allowance of 3.0 W ?

$$I = \frac{0.020}{100(8.0 \times 10^{-4})(0.25)} = 1.0 \text{ A}.$$

Heating is $I^2R = 4.0 \text{ W}$, exceeding the 3.0 W allowance. The current limit $I_{\max} = \sqrt{3.0/4.0} = 0.866 \text{ A}$ gives $\tau_{\max} = 0.0173 \text{ N m}$. The target requires a design change under these assumptions.

Chapter support: Sec. 8.10; Example 8.20

Loudspeaker force in a radial field



A voice coil has tangential current elements in an approximately radial gap field. Their local cross products all point axially. The closed-loop zero-force theorem does not apply because the vector field changes direction around the coil.

$$L_{\text{active}} = N2\pi a, \quad F \simeq BL_{\text{active}}I \equiv k_F I.$$

The force factor k_F has units N/A. Active length is the sum of circumferences inside the useful gap, rather than the axial height of the coil.

$$M\ddot{x} + b\dot{x} + kx = k_F I(t).$$

The mechanical mass, damping, suspension and acoustic load determine motion. A force amplitude alone does not determine sound pressure. When the coil leaves the uniform part of the gap, the force factor can depend on position.

Chapter support: Sec. 8.10; voice-coil model

Moving-coil instruments and field geometry



A torsion spring gives restoring torque $-\kappa\phi$, with ϕ in radians. In a radial-field design the magnetic torque stays approximately $NIAB$:

$$\kappa\phi = NIAB, \quad \frac{\phi}{I} = \frac{NAB}{\kappa}.$$

This geometry produces an approximately linear scale over its designed range. For a uniform fixed-direction field with the moment initially perpendicular to it, deflection toward alignment changes the torque:

$$\kappa\phi = NIAB \cos \phi.$$

The response is approximately linear only at small ϕ . Static sensitivity states the eventual reading. Inertia and damping determine the transient response and settling time. Mechanical stability, heating and current dynamics remain separate parts of an instrument model.

Chapter support: Sec. 8.10; radial-field meter; advanced instrument reading

Reversal measurements and uncertainty



A force sensor or balance measures a signed response with an offset:

$$F_{\text{meas}} = F_0 + sI, \quad |s| = LB.$$

For stable conditions, the half-difference removes a current-even offset:

$$\frac{F_{\text{meas}}(I) - F_{\text{meas}}(-I)}{2} = sI.$$

A balance supporting the magnet measures the reaction, whereas a wire-mounted sensor measures wire force. A displayed mass difference Δm between opposite currents corresponds to $2F/g$:

$$B = \frac{g|\Delta m|}{2|I|L}.$$

For independent small standard uncertainties and sufficiently well-known g ,

$$\frac{\sigma_B}{B} = \sqrt{(\sigma_{\Delta m}/\Delta m)^2 + (\sigma_I/I)^2 + (\sigma_L/L)^2}.$$

Chapter support: Sec. 8.11; current balance and propagation

Practice P24: balance reversal and field uncertainty



P24 — complete statement.

A magnet rests on a balance while an independently supported 0.0800 m wire crosses its gap perpendicularly. On reversing the current from $+3.00\text{ A}$ to -3.00 A , the balance reading changes by 1.20 g . Find B . If the standard uncertainties in the reading difference, current and length are 0.04 g , 0.02 A and 0.0010 m , find the combined relative uncertainty in B .

The mass difference corresponds to opposite reaction forces:

$$B = \frac{9.81(1.20 \times 10^{-3})}{2(3.00)(0.0800)} = 0.0245\text{ T}.$$

$$\frac{\sigma_B}{B} = \sqrt{(0.04/1.20)^2 + (0.02/3.00)^2 + (0.0010/0.0800)^2} = 0.0362.$$

Thus $B = (0.0245 \pm 0.0009)\text{ T}$ at one standard uncertainty. The stated 0.04 g already refers to the difference. The sign of a balance change depends on mounting; the supplied change determines the field magnitude.

Chapter support: P24; selected detailed solution P24

Calibration slope and an offset

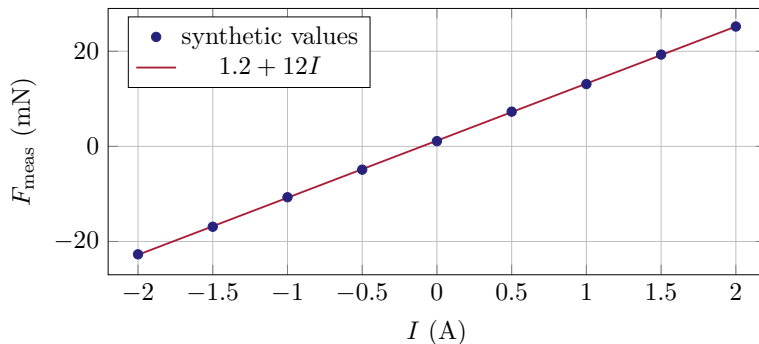


Figure 8.12: Synthetic calibration data and their fitted linear model. The slope measures LB ; the intercept measures an offset. Deviations are much smaller than the plotted full force range, so a residual plot would be needed to inspect them clearly.

The synthetic fit gives $s = 12.0 \text{ mN/A}$ and $F_0 = 1.20 \text{ mN}$. At $L = 0.150 \text{ m}$, $B = |s|/L = 0.0800 \text{ T}$.

Chapter support: Sec. 8.11; Figure 8.12; synthetic Table 8.1

Trajectory simulation and a physical time scale



The nonrelativistic model integrates $\dot{\mathbf{v}} = (q/m)(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and $\dot{\mathbf{r}} = \mathbf{v}$. Choose diagnostics before trusting an orbit plot: speed, radius, parallel velocity, period and initial curvature.

Example 8.23 — complete statement.

An electron moves in a uniform magnetic field $B = 5.0 \text{ mT}$. Estimate the cyclotron period and suggest a time step that resolves the motion with about 100 steps per revolution.

$$\omega_c = \frac{eB}{m_e} = 8.79 \times 10^8 \text{ s}^{-1}, \quad T_c = \frac{2\pi}{\omega_c} = 7.15 \times 10^{-9} \text{ s}.$$

About 100 steps gives $\Delta t = T_c/100 \approx 7 \times 10^{-11} \text{ s}$. Resolving a turn is a starting choice, not proof of numerical accuracy. The advanced block checks energy and accumulated phase separately.

Chapter support: Sec. 8.12; Example 8.23

Answered checks on magnetic models



- ① **Does a nonuniform magnetic field necessarily change speed?** No. The magnetic power remains zero. The acceleration direction and magnitude can still change as the local field changes.
- ② **Can zero net loop force coexist with torque?** Yes. Separated opposite forces form a couple. Uniformity over the entire closed path is required for the zero-force theorem.
- ③ **Does zero torque imply stable alignment?** No. Opposition also has zero torque but is an energy maximum. Damping is needed for settling after a release.
- ④ **What if Hall current and field both reverse?** The product IB_z is unchanged, so the ideal Hall voltage is unchanged. Reversing either alone changes its sign; swapping leads changes only the displayed sign.

Chapter support: Guided checks; P3, P4, P5, P6 concepts (lecture checks)

Core synthesis and the next physical question



- The particle law acts on signed charge and instantaneous velocity. Radius measures perpendicular momentum; pitch carries parallel motion.
- A Hall voltage and a balance reading require a defined measurement geometry, lead convention and system boundary before they support inference.
- Carrier forces give wire force. Opposite wire forces can form a loop torque, and fixed-moment energy distinguishes stable from unstable orientation.

Bridge

Here the field is prescribed or measured. Chapter 9 asks how currents produce magnetic fields through the Biot–Savart and Ampère laws. Chapter 10 develops induction and magnetic energy. The advanced pages now deepen the present trajectory and inference models.

Reading: revised Chapter 8 and its bibliography.

Chapter support: Sec. 8.14 synthesis; final chapter boundary; bibliography

Advanced: the uniform-field trajectory equations



For $\mathbf{B} = B\hat{\mathbf{k}}$, $\mathbf{E} = 0$, define the signed frequency $\Omega = qB/m$. The nonrelativistic equations are

$$\dot{v}_x = \Omega v_y, \quad \dot{v}_y = -\Omega v_x, \quad \dot{v}_z = 0.$$

Differentiating gives $\ddot{v}_x = -\Omega^2 v_x$. Initial conditions determine

$$\begin{aligned} v_x &= v_{x0} \cos \Omega t + v_{y0} \sin \Omega t, \\ v_y &= v_{y0} \cos \Omega t - v_{x0} \sin \Omega t. \end{aligned}$$

This is a transverse rotation. Squaring and adding verifies constant $v_x^2 + v_y^2$. Integrate each velocity to obtain position, retaining the initial position. The sign of Ω controls the sense of rotation, while $\omega_c = |\Omega|$ is positive.

Chapter support: Sec. 8.13.1; coordinate solution

Example 8.24: an analytic circular trajectory



Example 8.24 — complete statement.

A positive particle starts at the origin with $\mathbf{v}(0) = v_0 \hat{\mathbf{i}}$ in a uniform field $\mathbf{B} = B \hat{\mathbf{k}}$. Find $x(t)$ and $y(t)$.

With $\Omega = qB/m > 0$,

$$v_x = v_0 \cos \Omega t, \quad v_y = -v_0 \sin \Omega t.$$

Integrating from zero position gives

$$x = \frac{v_0}{\Omega} \sin \Omega t, \quad y = \frac{v_0}{\Omega} (\cos \Omega t - 1).$$

Let $r = v_0/\Omega$. Then $x^2 + (y+r)^2 = r^2$: the centre is $(0, -r)$. Differentiation recovers the initial velocity and the two force equations. At fixed time, $\Omega \rightarrow 0$ gives $x \rightarrow v_0 t$ and $y \rightarrow 0$, the straight-line limit.

Chapter support: Example 8.24; Sec. 8.13.1

Crossed fields as drift plus gyromotion



In constant perpendicular fields, define

$$\mathbf{v}_D = \frac{\mathbf{E} \times \mathbf{B}}{B^2}.$$

The identity $(\mathbf{E} \times \mathbf{B}) \times \mathbf{B} = -B^2 \mathbf{E}$ gives $\mathbf{v}_D \times \mathbf{B} = -\mathbf{E}$. Substitute $\mathbf{v} = \mathbf{v}_D + \mathbf{u}$:

$$m\dot{\mathbf{u}} = q[\mathbf{E} + (\mathbf{v}_D + \mathbf{u}) \times \mathbf{B}] = q\mathbf{u} \times \mathbf{B}.$$

The remaining motion is a uniform magnetic rotation. Its mean transverse velocity over one gyroperiod is zero, so the mean guiding-centre motion is $\mathbf{v}_D$, independent of mass and charge sign.

The lab speed can vary because the electric field does work. This nonrelativistic decomposition assumes constant fields and speeds well below c ; a parallel electric component adds separate parallel acceleration.

Chapter support: Sec. 8.13.2; drift reduction

P38: a particle released from rest in crossed fields



P38 — complete statement.

A positive ion is released from rest in $\mathbf{E} = E\hat{\mathbf{j}}$ and $\mathbf{B} = B\hat{\mathbf{k}}$, with $E, B > 0$. Derive $x(t), y(t), v_x(t), v_y(t)$. Determine the greatest speed and transverse excursion. Show that $K = qEy$ and find the displacement after one gyroperiod. Explain why a nonzero average drift is compatible with periodically vanishing speed.

Put $v_D = E/B$, $\Omega = qB/m > 0$. Since $u_x(0) = -v_D$ and $u_y(0) = 0$, the rotation from slide 61 gives

$$v_x = v_D(1 - \cos \Omega t), \quad v_y = v_D \sin \Omega t.$$

Taking the release point as the origin and integrating,

$$x = v_D \left(t - \frac{\sin \Omega t}{\Omega} \right), \quad y = \frac{v_D}{\Omega} (1 - \cos \Omega t).$$

The remaining extrema, energy and displacement requests are answered on slide 65.

Chapter support: P38; selected detailed solution P38

P38: energy, extrema and nonzero average drift



From the velocity solution,

$$v^2 = 2v_D^2(1 - \cos \Omega t), \quad v_{\max} = 2v_D, \quad y_{\max} = \frac{2v_D}{\Omega}.$$

Both maxima occur at a half-period. The kinetic energy is

$$K = \frac{1}{2}mv^2 = mv_D^2(1 - \cos \Omega t) = qEy,$$

because $qEv_D/\Omega = m(E/B)^2$. This is exactly the electric work from release. The magnetic term has zero power at every instant.

After $T_c = 2\pi/\Omega$,

$$\Delta \mathbf{r} = (v_D T_c) \hat{\mathbf{i}}, \quad y = 0, \quad \mathbf{v} = 0.$$

The particle takes energy from the electric field in the first half-cycle and returns it in the second. A cusp is an instant of rest, so periodically zero speed is compatible with a nonzero net displacement and average drift.

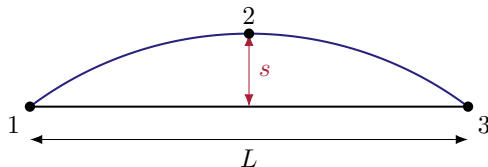
Chapter support: P38; Sec. 8.13.2

Momentum from curvature and sagitta

For perpendicular momentum, $|\dot{\mathbf{p}}| = p_{\perp} v_{\perp} / R = |q| v_{\perp} B$, hence

$$p_{\perp} = |q| B R, \quad p_{\perp} [\text{GeV}/c] \simeq 0.2998 |z| B [\text{T}] R [\text{m}].$$

The bending law is relativistically valid. A chord of length L and sagitta s obeys $(R - s)^2 + (L/2)^2 = R^2$, giving $R = L^2/(8s) + s/2$.



circular arc
 $R = L^2/(8s) + s/2$

Figure 8.15: Three position samples define a sagitta for a symmetric arc. The curve is drawn from a circle, with its bending enlarged for readability. Small errors in the midpoint relative to the two endpoints become important when the sagitta is small.

Chapter support: Secs. 8.13.3 and 8.13.5; Figure 8.15

Advanced challenge P42: track momentum and precision



P42 — complete statement.

A track in a 1.20 T field has transverse chord $L = 0.500\text{ m}$ and sagitta $s = 3.00\text{ mm}$. Find the exact circle radius and transverse momentum for charge magnitude e . Each of three equally precise position measurements has independent uncertainty $\sigma_x = 40\text{ }\mu\text{m}$; the sagitta is the middle coordinate minus the mean of the endpoint coordinates. Estimate momentum uncertainty when field uncertainty is 0.20% and chord uncertainty is negligible. If the pitch angle is 45° , what is the total momentum?

Before substituting numbers, propagate the sagitta estimator:

$$\sigma_s^2 = \sigma_x^2 + \frac{1}{4}\sigma_x^2 + \frac{1}{4}\sigma_x^2 = \frac{3}{2}\sigma_x^2.$$

For small sagitta and independent errors,

$$\left(\frac{\sigma_{p_\perp}}{p_\perp}\right)^2 \simeq \left(\frac{\sigma_B}{B}\right)^2 + \left(\frac{\sigma_s}{s}\right)^2.$$

Chapter support: P42; selected detailed solution P42

P42: radius, momentum and the error budget



$$R = \frac{0.500^2}{8(0.00300)} + \frac{0.00300}{2} = 10.4182 \text{ m},$$

$$p_{\perp} = 0.2998(1.20)(10.4182) = 3.748 \text{ GeV}/c,$$

$$\sigma_s = \sqrt{\frac{3}{2}}(40 \mu\text{m}) = 49.0 \mu\text{m}.$$

The exact derivative is $\partial R/\partial s = -L^2/(8s^2) + 1/2$. It gives $\sigma_R/R \simeq 1.63\%$ here. Combining field and sagitta uncertainty,

$$\frac{\sigma_{p_{\perp}}}{p_{\perp}} \simeq \sqrt{0.0020^2 + 0.0163252^2} = 1.64\%, \quad p_{\perp} \simeq (3.75 \pm 0.06) \text{ GeV}/c.$$

For known $\alpha = 45^\circ$, total momentum is $p = p_{\perp}/\sin \alpha = 5.30 \text{ GeV}/c$. The geometric correction is only 0.0144%, much smaller than the measurement error. Angle uncertainty, alignment and scattering are additional effects outside this stated model.

Chapter support: P42; Sec. 8.13.5

Numerical magnetic rotation: norm versus phase



In pure uniform B , forward Euler adds a perpendicular increment. With $\delta = \omega_c \Delta t$ and transverse velocity,

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \frac{q\Delta t}{m} \mathbf{v}_n \times \mathbf{B}, \quad v_{n+1}^2 = (1 + \delta^2) v_n^2.$$

The magnetic midpoint step instead satisfies

$$\frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{\Delta t} = \frac{q}{m} \left(\frac{\mathbf{v}_{n+1} + \mathbf{v}_n}{2} \right) \times \mathbf{B}.$$

Dotting with $\mathbf{v}_{n+1} + \mathbf{v}_n$ proves norm conservation. The Boris magnetic rotation implements this step. Its positive angle per step is $2 \tan^{-1}(\delta/2)$, smaller than δ .

$$\frac{K_N^{\text{Euler}}}{K_0} = (1 + \delta^2)^N, \quad \Delta\phi = N[\delta - 2 \tan^{-1}(\delta/2)].$$

Correct norm alone does not ensure arrival at the correct orbital phase.

Chapter support: Sec. 8.13.7; Euler growth, midpoint and Boris phase

Worked extension: ten turns of numerical motion



Original teaching extension — complete statement.

A nonrelativistic particle moves perpendicular to uniform $\mathbf{B}$ with $\mathbf{E} = 0$. Over ten gyroperiods, compare forward Euler and the magnetic midpoint (Boris) velocity updates at $n = 100$ and $n = 200$ steps per turn. Calculate their kinetic-energy ratios and the Boris accumulated phase lags. Explain the effect of halving the step.

Use $N = 10n$ and $\delta = 2\pi/n$ in the formulas from slide 69:

n	Euler K_N/K_0	Boris K_N/K_0	Boris lag (rad)
100	51.42	1.000000	0.02066
200	7.192	1.000000	0.005167

Midpoint preserves magnetic energy. The expansion $\Delta\phi \simeq 20\pi^3/(3n^2)$ explains the fourfold reduction in phase lag. Euler still gains energy artificially. Check both invariants and matched-time positions.

Chapter support: Original teaching extension; Sec. 8.13.7 and Table 8.2

