

Magnetic Fields Produced by Currents

Physics II - Chapter 9

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Why calculate the field produced by a current?

Earlier work used a known magnetic field to predict forces. A wire, a coil, or a magnet design requires the opposite calculation: predict the vector field from a specified current path.

The physical roadmap

Biot–Savart adds the contributions of current elements. Straight wires and loops show how geometry enters. Ampère’s law then gives short calculations when symmetry fixes the field along a closed path.

The resulting field predicts mutual forces, coil performance, and sensor signals. Current reversal provides a useful direction and measurement check.

Revised Chapter 9: Sec. 9.1.

Learning route and prerequisites

- 1 Define a current element and its source-to-point vector, then construct wire and loop fields by integration.
- 2 Use closed-path circulation and symmetry to resolve extended conductors, solenoids, and toroids.
- 3 Apply the field at another current, then connect field specifications to heating and measurement uncertainty.
- 4 Extend the core to local current density and controlled corrections to coil models.

Prerequisites: vector components and cross products; definite integrals; conventional current; $d\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B}$; Ohm's law and $P = I^2 R_{\text{coil}}$. For a uniform wire, resistance grows with length and falls with area: $R_{\text{coil}} = \rho_e L_w / A_w$, with resistivity ρ_e , wire length L_w , and cross-sectional area A_w .

Revised Chapter 9: Sec. 9.2.

Learning goals

By the end of the core lecture, you should be able to:

- define the vectors in Biot–Savart and justify the direction of each field contribution;
- derive wire and coil fields, add components, and test finite-geometry approximations;
- evaluate signed circulation and decide whether symmetry permits an Ampère-law field calculation;
- calculate fields inside conductors, solenoids, and toroids, and the forces between currents;
- translate an air-core field target into an electrical load, and interpret a reversal measurement with uncertainty.

Advanced outcomes: infer a current profile, calculate finite-solenoid corrections, and quantify Helmholtz axial uniformity.

Revised Chapter 9: Sec. 9.1.

Steady sources, units, and force

Magnetostatic model: currents and charge distributions are steady. We calculate prescribed currents in vacuum, treating air as vacuum at introductory precision. A real steady circuit includes leads and a return path.

$$1 \text{ T} = 1 \text{ N A}^{-1} \text{ m}^{-1}, \quad \mu_0 \simeq 4\pi \times 10^{-7} \text{ T m/A}.$$

This numerical value of μ_0 is an approximation. A formula can be exact within an ideal model while the model itself approximates an apparatus.

$$d\mathbf{F} = I d\boldsymbol{\ell} \times \mathbf{B}_{\text{ext}}.$$

$d\boldsymbol{\ell}$ follows conventional current. The force uses the field of other sources, not the wire's singular filament self-field.

Revised Chapter 9: Sec. 9.3, Sec. 9.7.

Biot–Savart constructs the vector field

For an element at $\mathbf{x}'$ and an observation point $\mathbf{x}$,

$$\mathbf{r} = \mathbf{x} - \mathbf{x}', \quad r = |\mathbf{r}|, \quad d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{\ell} \times \mathbf{r}}{r^3}. \quad (9.1)$$

The cross product sets both direction and magnitude. Contributions vanish when the element and displacement are parallel.

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int_{\text{path}} \frac{d\boldsymbol{\ell} \times \mathbf{r}}{r^3}, \quad dB = \frac{\mu_0 I d\ell \sin \theta}{4\pi r^2}.$$

Integrate vectors at the same observation point. Use symmetry to cancel components before evaluating an integral.

Revised Chapter 9: Sec. 9.3.

Biot-Savart direction geometry

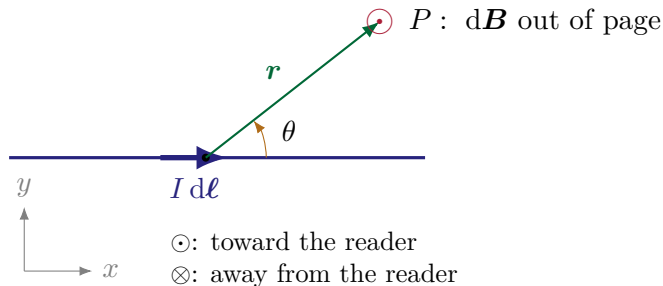


Figure 9.1: Biot-Savart geometry in the xy plane. A current element along $+x$ and a source-to-point vector with positive y component give $d\mathbf{B}$ along $+z$, out of the page. The field is shown at the observation point.

$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$: the positive y component of $\mathbf{r}$ fixes the out-of-page contribution. Its x component gives zero cross product.

Revised Chapter 9: Fig. 9.1.

Dimensions and two direction checks

Example 9.1

Show that Eq. (9.1) has the dimensions of magnetic field.

$$[dB] = \left(\frac{\text{T m}}{\text{A}} \right) \frac{\text{A m}}{\text{m}^2} = \text{T}.$$

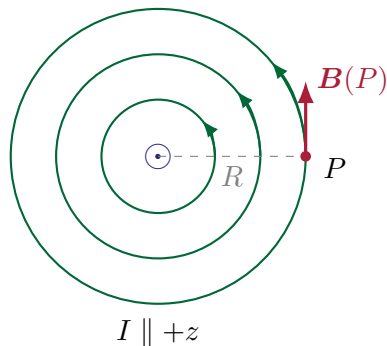
The unit vector is dimensionless. This checks the powers of length, although it does not prove the physical law.

Answered checks

At a point on the extension of the element, away from the source itself, $\sin \theta = 0$ and this element's contribution vanishes. Reversing $\mathbf{r}$ leaves r and the magnitude unchanged but reverses the cross product and hence the predicted direction.

Revised Chapter 9: Example 9.1, P2.

Circular field of a long straight current



cross section perpendicular
to the wire

Current emerges from the page.
Field arrows point counterclockwise.

Figure 9.2: Cross section through the circular field of a current along $+z$. Every field line lies in a plane perpendicular to the wire. The arrow at P is tangent to its circle. Line spacing here is illustrative; the magnitude is fixed by $B \propto 1/R$.

Rotations about the wire and translations along it leave the source unchanged. At a fixed radius the source field is azimuthal, with equal magnitude around the circle.

Revised Chapter 9: Fig. 9.2, Sec. 9.4.

Long-wire field by direct integration

Put the wire on z , with I along $+z$, and $P = (R, 0, 0)$.

$$d\ell = \hat{z}dz, \quad \mathbf{r} = R\hat{x} - z\hat{z}, \quad d\ell \times \mathbf{r} = R\hat{y}dz.$$

All contributions point along $+y$ at this observation point.

$$\begin{aligned} B_y &= \frac{\mu_0 I R}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{(R^2 + z^2)^{3/2}} \\ &= \frac{\mu_0 I}{4\pi R} \left[\frac{z}{\sqrt{R^2 + z^2}} \right]_{-\infty}^{\infty} = \frac{\mu_0 I}{2\pi R}. \end{aligned}$$

The endpoint bracket is $1 - (-1) = 2$. This ideal result applies outside the filament; it does not describe a real conductor's interior.

Revised Chapter 9: Sec. 9.4.

Finite segments and signed endpoint angles

For a segment $z_1 \leq z \leq z_2$, $z_2 > z_1$, and the same $P = (R, 0, 0)$,

$$B_y = \frac{\mu_0 I}{4\pi R} \left[\frac{z_2}{\sqrt{R^2 + z_2^2}} - \frac{z_1}{\sqrt{R^2 + z_1^2}} \right]. \quad (9.12)$$

Define $\theta_i = \arctan(z_i/R)$ from the perpendicular foot. Then

$$B_y = \frac{\mu_0 I}{4\pi R} (\sin \theta_2 - \sin \theta_1). \quad (9.13)$$

The bracket is positive because $z/\sqrt{R^2 + z^2}$ increases with z . A sum of two positive sines is valid only when the endpoints straddle the perpendicular foot.

Infinite endpoints give the long-wire result. A leg from 0 to $+\infty$ contributes half that field; its return path must still be specified.

Revised Chapter 9: Sec. 9.5.

Endpoint geometry and coordinate orientation

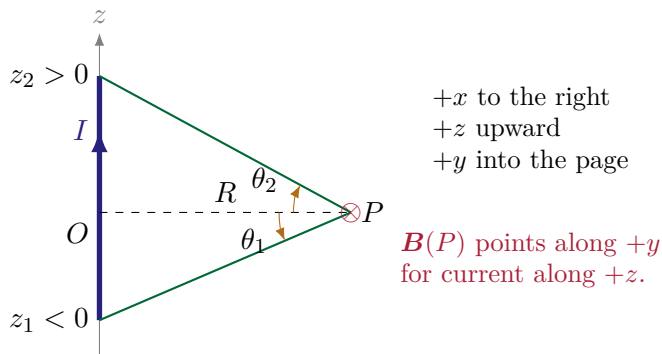


Figure 9.3: Finite-segment geometry in the xz plane. The signed angles are measured from the perpendicular at P , toward positive or negative z . The cross denotes a field into the page; the chosen axes are right-handed.

Here $\theta_1 < 0 < \theta_2$. The displayed cross denotes $+y$ into the page because the view is in the xz plane. It agrees with $\hat{z} \times \hat{x} = \hat{y}$.

Revised Chapter 9: Fig. 9.3.

Example 9.3: how long is long enough?

Example 9.3

A straight part of a circuit has total length 2ℓ , and a probe is on its perpendicular bisector at distance R . Determine its field relative to that of an infinite wire, and find a one-percent length criterion. The other parts of the circuit are to be evaluated separately.

Model: a steady filamentary segment, with the probe outside it. Set $z_1 = -\ell$ and $z_2 = \ell$:

$$B_{\text{seg}} = \frac{\mu_0 I}{2\pi R} \frac{\ell}{\sqrt{\ell^2 + R^2}}, \quad f = \frac{B_{\text{seg}}}{B_{\infty}} = \frac{\ell/R}{\sqrt{1 + (\ell/R)^2}}. \quad (9.14)$$

The straight-section deficit is $1 - f$, normalized to B_{∞} . The next slide solves the tolerance and checks the result.

Revised Chapter 9: Example 9.3.

Example 9.3: length criterion and error denominator

Write $q = \ell/R > 0$. The one-percent condition is

$$\frac{q}{\sqrt{1+q^2}} \geq 0.99 \quad \Rightarrow \quad q \geq \frac{0.99}{\sqrt{1-0.99^2}} = 7.02.$$

Thus $2\ell \geq 14.04R$ before rounding. If the error is instead divided by B_{seg} , then $1/f - 1 \leq 0.01$ and $q \geq 7.05$.

Numerical and limiting checks

For $2\ell = 0.40 \text{ m}$ and $R = 0.020 \text{ m}$, $q = 10$, $f = 0.9950$, and the deficit is 0.496%. For $q \gg 1$, $f = 1 - 1/(2q^2) + O(q^{-4})$, so doubling the half-length quarters the leading deficit.

These conditions constrain this segment contribution. A nearby return conductor can still matter.

Revised Chapter 9: Example 9.3.

Finite-wire approximation as a profile

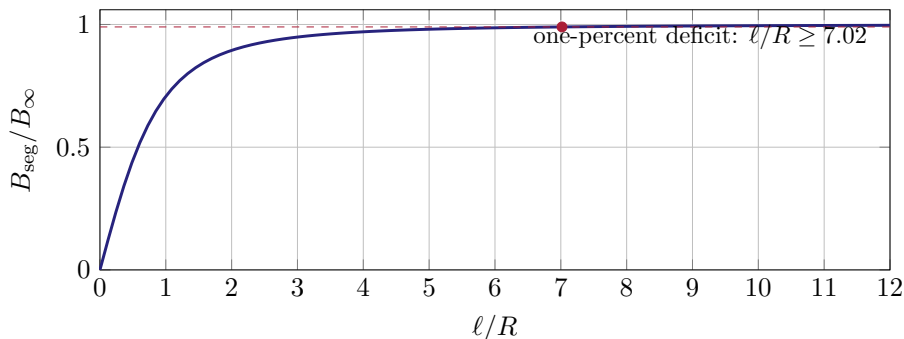


Figure 9.4: Finite-wire contribution on the perpendicular bisector, calculated from Eq. (9.14). Here 2ℓ is the full straight-section length. The horizontal dashed line is 0.99.

The curve approaches one monotonically. Extra length gives diminishing improvement because the leading deficit varies as $(R/\ell)^2$.

Revised Chapter 9: Fig. 9.4.

Practice P10: endpoints on the same side

Problem P10

A straight circuit segment lies on the z axis from $z_1 = 0.020$ m to $z_2 = 0.120$ m and carries 4.0 A along $+z$. Find its contribution at $(0.030, 0, 0)$ m. Why is a sum of two unsigned endpoint-angle sines incorrect here?

Model and direction: use the contribution of a steady filament segment in vacuum. The field at P is along $+y$ because each element has $d\ell \times \mathbf{r} = R dz \hat{\mathbf{y}}$.

$$\sin \theta_1 = \frac{0.020}{\sqrt{0.030^2 + 0.020^2}}, \quad \sin \theta_2 = \frac{0.120}{\sqrt{0.030^2 + 0.120^2}}.$$

Both angles are positive. Use their difference in Eq. (9.13). The complete answer follows on slide 18.

Revised Chapter 9: P10, Sec. 9.15.

P10 solution: signed subtraction

$$\begin{aligned} B_y &= \frac{(4\pi \times 10^{-7})(4.0)}{4\pi(0.030)} \left[\frac{0.120}{\sqrt{0.030^2 + 0.120^2}} - \frac{0.020}{\sqrt{0.030^2 + 0.020^2}} \right] \\ &= (1.3333 \times 10^{-5})(0.97014 - 0.55470) \text{ T} \\ &= 5.54 \times 10^{-6} \text{ T}. \end{aligned}$$

Answer: $B_{\text{seg}} = +5.54 \mu\text{T } \hat{y}$.

A sum of unsigned sines would include fictitious source length below the perpendicular foot. Independent integration of the positive kernel $R/(R^2 + z^2)^{3/2}$ gives the same positive result.

As the two endpoints approach each other, their difference tends to zero. The rest of the circuit remains unspecified, so this is a segment contribution.

Revised Chapter 9: P10.

Superposition requires vector components

Every source contributes at the same observation point:

$$\mathbf{B}_{\text{total}}(P) = \sum_j \mathbf{B}_j(P).$$

For long z-directed wires through (x_j, y_j) with signed currents I_j ,

$$\mathbf{B}_j(x, y) = \frac{\mu_0 I_j}{2\pi \rho_j^2} [-(y - y_j)\hat{\mathbf{x}} + (x - x_j)\hat{\mathbf{y}}], \quad \rho_j^2 = (x - x_j)^2 + (y - y_j)^2. \quad (9.17)$$

The bracket is the radial displacement rotated by 90° for positive current. Its length is ρ_j , recovering $B_j = \mu_0 |I_j| / (2\pi \rho_j)$.

Add magnitudes only when the fields have already been shown to be parallel in the same sense. Negative current reverses both components.

Revised Chapter 9: Sec. 9.5.

Two-wire geometry at an off-axis point

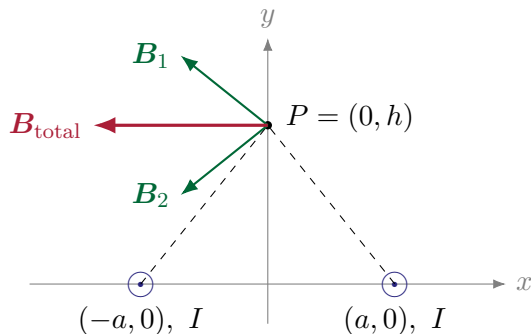


Figure 9.5: End view of two equal currents along $+z$. The field arrows at P are tangent to their respective source-centered circles. Dashed lines show source-to-observation geometry, not magnetic field lines.

At $P = (0, h)$ the vertical components are equal and opposite. Both horizontal components point left. The dashed lines are displacements, not field lines.

Revised Chapter 9: Fig. 9.5.

Practice P13: same and opposite currents

Problem P13

Two long wires at $x = -0.030$ m and $x = +0.030$ m each carry 5.0 A along $+z$. Find the field vector at $(x, y) = (0, 0.040)$ m, then reverse the right-hand current. Compare magnitudes and directions.

Model: infinitely long vacuum filaments. The common distance is

$$\rho = \sqrt{0.030^2 + 0.040^2} = 0.050 \text{ m.}$$

$$\mathbf{B}_1 = (-16\hat{x} + 12\hat{y}) \mu\text{T}, \quad \mathbf{B}_2 = (-16\hat{x} - 12\hat{y}) \mu\text{T}.$$

These follow from Eq. (9.17), with $\mu_0 I / (2\pi\rho^2) = 4.00 \times 10^{-4} \text{ T/m}$. Each magnitude is $20 \mu\text{T}$.

Revised Chapter 9: P13, Sec. 9.15.

P13 solution: a different component cancels

For two currents along $+z$,

$$\mathbf{B}_1 + \mathbf{B}_2 = -32 \mu\text{T} \hat{\mathbf{x}}.$$

Reversing the right-hand current changes the sign of its entire vector:

$$\mathbf{B}'_2 = (+16\hat{\mathbf{x}} + 12\hat{\mathbf{y}}) \mu\text{T}, \quad \mathbf{B}_1 + \mathbf{B}'_2 = +24 \mu\text{T} \hat{\mathbf{y}}.$$

Thus the first resultant is $32 \mu\text{T}$ leftward; the second is $24 \mu\text{T}$ upward. The geometry and individual magnitudes are unchanged.

Checks: equal currents give zero at the midpoint $h = 0$. For equal same-sense currents and $h \gg a$, $B \simeq \mu_0 I / (\pi h)$, the field of total current $2I$ at large distance.

Revised Chapter 9: P13.

Example 9.5: zeros of unequal wire fields

Example 9.5

Wires at $x = 0$ and $x = d$ carry currents I and $3I$ along $+z$. Find the field-zero point on the x axis. Then reverse the second current.

On the axis, with $x \neq 0, d$,

$$B_y(x) = \frac{\mu_0}{2\pi} \left(\frac{I}{x} + \frac{I_2}{x-d} \right).$$

Same sense: outside the pair both terms have the same sign. Between them, $1/x + 3/(x-d) = 0$ gives $x = d/4$.

Second current reversed: the terms reinforce between the wires. The equation $1/x - 3/(x-d) = 0$ gives $x = -d/2$.

At the second root the distances are $d/2$ and $3d/2$: the three-times stronger current is three times farther away. Direction and algebra agree in both cases.

Revised Chapter 9: Example 9.5.

Circular arcs and loop centers

At the center of an arc, every current element is perpendicular to the radius. Its field contribution is axial, in the same sense around that arc.

$$d\ell = R d\phi, \quad dB = \frac{\mu_0 I}{4\pi R} d\phi, \quad B_{\text{arc}} = \frac{\mu_0 I \phi}{4\pi R}.$$

The angular span ϕ is in radians. For one full loop, $\phi = 2\pi$:

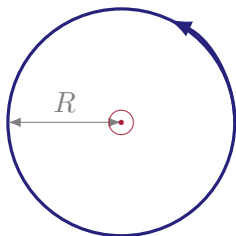
$$B_{\text{center}} = \frac{\mu_0 I}{2R}, \quad B_N = \frac{\mu_0 N I}{2R}.$$

Multiplication by N assumes coincident or tightly packed equal-radius turns. For appreciable winding thickness, sum the separate turns.

A radial lead contributes zero at the center because $d\ell \parallel \mathbf{r}$, provided the center is not on that lead.

Revised Chapter 9: Sec. 9.6.

Loop current and axial direction



counterclockwise current I
 $\mathbf{B}(0)$ and $\boldsymbol{\mu}$ point out of the page

Reversing the current reverses
both the field and the magnetic moment.

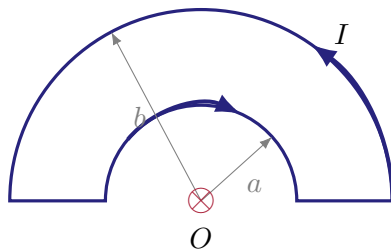
view from $+z$

Figure 9.6: The center field of a circular loop. The radius ends on the wire, and the dot explicitly shows the axial field direction perpendicular to the loop plane.

Curl the fingers with the current. The thumb gives the central field and the magnetic moment direction. Both reverse with the current.

Revised Chapter 9: Fig. 9.6.

A complete annular-sector circuit



radial joins: zero field at O
inner arc: into page
outer arc: out of page

Figure 9.7: A complete annular-sector circuit, not an isolated arc. The opposing arc contributions must be added with their signs. The joins follow radii and do not include the center.

Current continuity sets opposite senses on the inner and outer arcs. The next slide uses this exact closed-circuit geometry.

Revised Chapter 9: Fig. 9.7.

Example 9.8: two semicircles

Example 9.8

A thin wire forms two concentric semicircular arcs of radii $a = 0.040\text{ m}$ and $b = 0.080\text{ m}$ joined by radial sections, as in Figure 9.7. Current $I = 2.0\text{ A}$ runs counterclockwise along the outer arc. Find the field at the common center O .

The radial joins give zero at O . Outer current is counterclockwise ($+z$ field); inner current is clockwise ($-z$ field). With $\phi = \pi$,

$$B_z = \frac{\mu_0 I}{4} \left(\frac{1}{b} - \frac{1}{a} \right) = \frac{(4\pi \times 10^{-7})(2.0)}{4} (12.5 - 25) = -7.85\text{ }\mu\text{T}.$$

Answer: $7.9\text{ }\mu\text{T}$ into the page. The smaller arc dominates because its center contribution scales as $1/R$.

As $b \rightarrow a$ the fields cancel. Reversing current reverses the result. These checks use the whole closed path, not two independent same-sense arcs.

Revised Chapter 9: Example 9.8, Fig. 9.7.

Loop-axis geometry and component cancellation

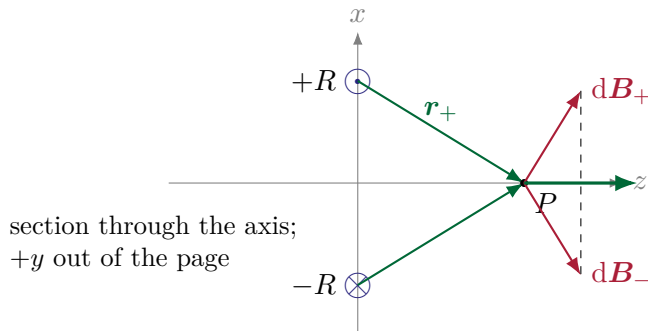


Figure 9.8: A meridional section through a circular loop. At the upper element ($x = R$), current is along $+y$; at the lower element it is along $-y$. Both contributions have positive axial components. Only two elements of the full loop are shown.

Opposite source elements have opposite transverse field components. Their axial components add even though the individual field vectors are tilted.

Revised Chapter 9: Fig. 9.8.

Derivation of the loop-axis field

On the axis, every element is at distance $r = \sqrt{R^2 + z^2}$ and has $d\ell \perp \mathbf{r}$. The axial projection introduces R/r :

$$dB_z = \frac{\mu_0 I d\ell}{4\pi r^2} \frac{R}{r}.$$

Since $\oint d\ell = 2\pi R$ and the denominator is constant,

$$B_z(z) = \frac{\mu_0 I R^2}{2(R^2 + z^2)^{3/2}}, \quad \frac{B_z(z)}{B_z(0)} = [1 + (z/R)^2]^{-3/2}. \quad (9.24)$$

At $z = 0$ this gives $\mu_0 I / (2R)$. At $|z| \gg R$ it becomes $\mu_0 I R^2 / (2|z|^3)$, a dipole falloff.

For the shown current, the field points along $+z$ on both sides of the loop. The even denominator does not reverse its direction at negative z .

Revised Chapter 9: Sec. 9.6.

A single loop has limited axial uniformity

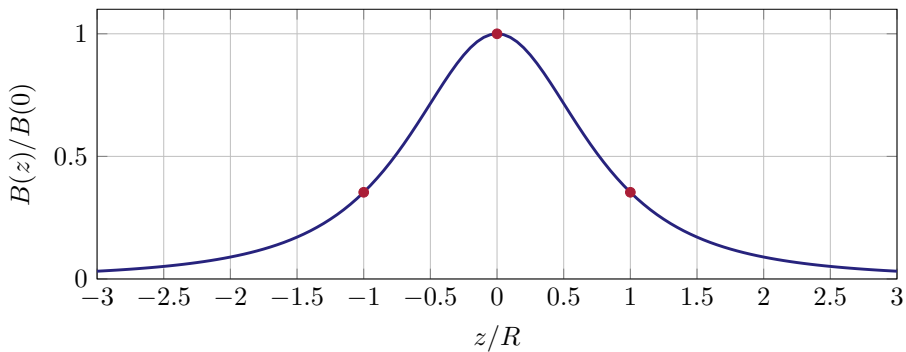


Figure 9.9: Calculated axial profile of a single circular loop. At $|z| = R$, the field is $2^{-3/2} = 0.354$ of its central value. This curve concerns the symmetry axis only.

Zero slope at the center gives local flatness, not uniformity over a radius-sized interval. The leading variation is $-3z^2/(2R^2)$.

Revised Chapter 9: Fig. 9.9.

Problem P16: coil field and half-field position

Problem P16

A circular coil has $N = 150$, $R = 0.040$ m, and $I = 0.20$ A. Find its center field and its field at $z = R$. Determine the axial displacement at which the field falls to half the center value.

Use N coincident thin turns in vacuum. The current sense was not specified; define $+\hat{n}$ by its right-hand rule.

$$B(0) = \frac{(4\pi \times 10^{-7})(150)(0.20)}{2(0.040)} = 4.71 \times 10^{-4} \text{ T.}$$

At $z = R$, multiply by $2^{-3/2} = 0.353553$:

$$B(R) = 1.67 \times 10^{-4} \text{ T.}$$

Both vectors point along $+\hat{n}$. The displacement condition is solved on slide 32.

Revised Chapter 9: P16, Sec. 9.15.

P16 solution: where the field falls by half

The turn count and current cancel from the ratio:

$$[1 + (z/R)^2]^{-3/2} = \frac{1}{2} \implies 1 + (z/R)^2 = 2^{2/3}.$$

Therefore

$$|z| = R\sqrt{2^{2/3} - 1} = (0.040)(0.766421) \text{ m} = 0.0307 \text{ m}.$$

Complete answer: $B(0) = 0.471 \text{ mT}$, $B(R) = 0.167 \text{ mT}$, and the half-field points are $z = \pm 30.7 \text{ mm}$, measured along the normal through the center.

Substitution into the normalized model returns exactly $1/2$. Both half-field vectors retain the same axial direction. Only their magnitudes are reduced.

Revised Chapter 9: P16.

Mutual force between long parallel currents

Put wire 1 at $x = 0$, wire 2 at $x = d$, both with current along $+z$. Wire 1 produces $\mathbf{B}_1 = +B_1\hat{\mathbf{y}}$ at wire 2:

$$B_1 = \frac{\mu_0 I_1}{2\pi d}, \quad \frac{\mathbf{F}_2}{L} = I_2 \hat{\mathbf{z}} \times B_1 \hat{\mathbf{y}} = -\frac{\mu_0 I_1 I_2}{2\pi d} \hat{\mathbf{x}}.$$

The force points toward wire 1. The field of wire 2 at wire 1 points along $-y$, giving the opposite force on wire 1.

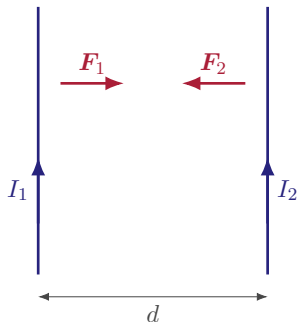
$$\frac{F}{L} = \frac{\mu_0 |I_1 I_2|}{2\pi d}.$$

Same-sense currents attract; opposite-sense currents repel. Use long conductors far from ends, and the external field at each conductor. The unit is $\text{A T} = \text{N/m}$.

Revised Chapter 9: Sec. 9.7.

Attraction and repulsion of parallel currents

same current direction: attraction



opposite current direction: repulsion

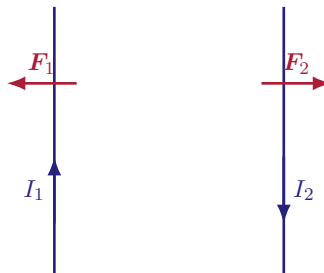


Figure 9.10: Forces between long parallel currents. The forces are equal in magnitude and opposite in direction, consistent with Newton's third law.

The forces are perpendicular to the wire directions. Reversing one current reverses the mutual-force sense without changing its magnitude at fixed current magnitudes.

Revised Chapter 9: Fig. 9.10.

Example 9.10: load between bus bars

Example 9.10

Two long parallel conductors are separated by $d = 8.0 \text{ cm}$ and carry currents $I_1 = I_2 = 250 \text{ A}$ in the same direction. Find the force per meter.

$$\frac{F}{L} = \frac{(4\pi \times 10^{-7})(250)^2}{2\pi(0.080)} = 0.156 \text{ N/m}.$$

The load is attractive. A 1.5 m supported section has $F = 0.234 \text{ N}$ before rounding beyond the model precision.

If both currents increase twentyfold, the load increases by $20^2 = 400$, to 62.5 N/m ; it does not increase twentyfold.

Check and scope: $F/L = I_2 B_1$ has unit N/m . Long straight conductors and fixed separation are assumed. A rapid fault also requires assessment of time dependence and current redistribution.

Revised Chapter 9: Example 9.10.

Ampère's law is a circulation law

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I_{\text{enc}}. \quad (9.39)$$

For a directed closed path C , add the tangential field component times each short path length. The result has unit T m .

Curl the right-hand fingers along C . The thumb sets the positive normal, so I_{enc} is the algebraic current through a spanning surface.

Local field and global circulation

The left side uses the total field, including external currents. External sources contribute zero net circulation around this path, even though they may contribute a nonzero local field.

Magnetic flux instead integrates a normal field through an area and has unit T m^2 . It is a different integral.

Revised Chapter 9: Sec. 9.8.

Orientation and enclosed-current signs

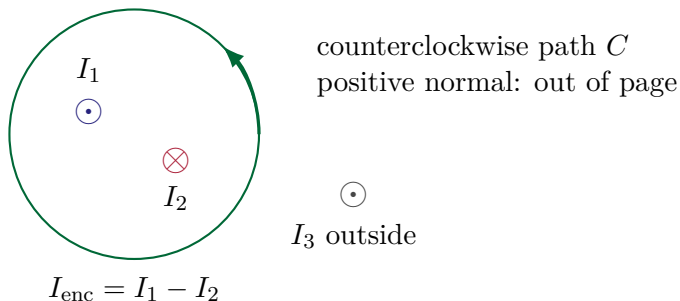


Figure 9.11: An oriented integration path and three source currents. A dot denotes outward current and a cross inward current. The external wire affects the local field but contributes zero net circulation around C .

Changing path orientation reverses both $d\ell$ and the positive normal. The field sources are unchanged, so the physical field does not reverse merely because the path is traversed differently.

Revised Chapter 9: Fig. 9.11.

Practice P3: circulation with an external current

Problem P3

A counterclockwise Amperian path encloses 9 A out of the page and 4 A into it. A 12 A current is outside. Find the circulation, state what changes when the path orientation reverses, and explain why the external current can affect $\mathbf{B}$ without appearing in I_{enc} .

Counterclockwise traversal gives an outward normal:

$$I_{\text{enc}} = 9 - 4 = 5 \text{ A}, \quad \oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0(5 \text{ A}) = 6.28 \times 10^{-6} \text{ T m}.$$

Clockwise traversal gives $I_{\text{enc}} = -5 \text{ A}$ and negative circulation of the same magnitude. It leaves the physical field unchanged.

The external 12 A source affects $\mathbf{B}$ along the path. Its positive and negative tangential contributions integrate to zero because it carries no current through the spanning surface. Units match $\mu_0 I$.

Revised Chapter 9: P3.

Example 9.12: zero circulation, nonzero field

Example 9.12

A circular integration path lies in a region of uniform magnetic field $\mathbf{B}_0 = B_0 \hat{x}$ and encloses no current. Does Ampere's law require $B_0 = 0$?

Parameterize the path with radius r by

$$\mathbf{r}_C = r(\cos \phi, \sin \phi, 0), \quad d\boldsymbol{\ell} = r(-\sin \phi, \cos \phi, 0)d\phi.$$

Then

$$\oint_C \mathbf{B}_0 \cdot d\boldsymbol{\ell} = -B_0 r \int_0^{2\pi} \sin \phi d\phi = 0.$$

Answer: B_0 may be nonzero. The tangential contributions cancel. Independently, for a constant field, $\mathbf{B}_0 \cdot \oint_C d\boldsymbol{\ell} = 0$ on any closed path.

Zero circulation is an integral constraint. A zero source field in a cable exterior needs symmetry and physical boundary conditions as well.

Revised Chapter 9: Example 9.12.

Symmetry makes circulation a field calculation

Example 9.11

Use Ampere's law to derive the field of a long straight wire carrying current I .

Cylindrical symmetry gives an azimuthal field of constant magnitude on a concentric circle of radius R . Thus

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = B(2\pi R) = \mu_0 I, \quad B = \frac{\mu_0 I}{2\pi R}.$$

This matches the independent Biot–Savart integral on slide 11.

The symmetry test

For a displaced wire inside a circular path, B is not constant or generally tangent to that path. The same circulation law holds, but replacing its left side by circumference times one field value is unjustified.

Identify field direction and constancy from the source symmetry before selecting a convenient path.

Revised Chapter 9: Example 9.11, Sec. 9.8.

Current density and the enclosed area

For an axisymmetric long conductor, let $J_z(r)$ be current per cross-sectional area, in A/m^2 .

$$I_{\text{enc}}(r) = 2\pi \int_0^r J_z(s) s \, ds, \quad B_\phi(r) = \frac{\mu_0 I_{\text{enc}}(r)}{2\pi r}.$$

Only material within the circular integration path contributes to its enclosed current. The annular area element is $2\pi s \, ds$.

For uniform density in a conductor of radius a ,

$$J_z = \frac{I}{\pi a^2}, \quad I_{\text{enc}}(r) = I \frac{r^2}{a^2} \quad (r < a).$$

Equal radial thicknesses do not enclose equal current: larger-radius annuli have greater area. The cylindrical symmetry still fixes the field direction.

Revised Chapter 9: Sec. 9.8.

Example 9.13: field inside a thick wire

Example 9.13

A long cylindrical wire of radius a carries total current I uniformly over its cross section. Find $B(r)$ both inside and outside the wire.

On each concentric path, $2\pi r B_\phi = \mu_0 I_{\text{enc}}$:

$$B_\phi(r) = \begin{cases} \frac{\mu_0 I}{2\pi a^2} r, & 0 \leq r < a, \\ \frac{\mu_0 I}{2\pi r}, & r \geq a. \end{cases}$$

Inside, $I_{\text{enc}} = I r^2 / a^2$; outside, $I_{\text{enc}} = I$. For current along $+z$, the field is along $+\hat{\phi}$ when nonzero.

At the axis $B = 0$; both branches give $\mu_0 I / (2\pi a)$ at the surface. The inner slope is positive and the outer slope negative, so the maximum occurs at $r = a$.

Revised Chapter 9: Example 9.13.

Continuity differs from differentiability

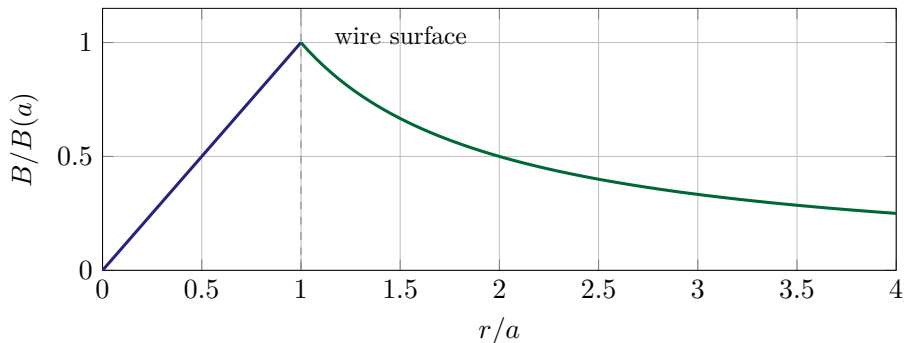


Figure 9.12: Field of a uniformly current-filled long cylinder. The two analytic branches meet at $r = a$, but their slopes differ. A filament model would wrongly extend the decreasing branch to the axis.

At $r = a$, the slopes are $+\mu_0 I/(2\pi a^2)$ and $-\mu_0 I/(2\pi a^2)$. A continuous field can have a discontinuous derivative when the ideal current density changes abruptly.

Revised Chapter 9: Fig. 9.12, Example 9.13.

Example 9.14: a thick coaxial return

Example 9.14

The inner conductor has radius a and uniform current $+I\hat{z}$. A concentric outer conductor occupies $b < r < c$ and carries uniform return current $-I\hat{z}$. The region $a < r < b$ is insulating. Find the field in all four radial regions, assuming infinite length, vacuum permeability, and no imposed background field.

Compute enclosed current before the field:

$$\frac{I_{\text{enc}}(r)}{I} = \begin{cases} r^2/a^2, & 0 \leq r < a, \\ 1, & a \leq r < b, \\ 1 - (r^2 - b^2)/(c^2 - b^2), & b \leq r < c, \\ 0, & r \geq c. \end{cases}$$

The return fraction is its enclosed annular area divided by total return area. Rotational and translational symmetry allow $B_\phi = \mu_0 I_{\text{enc}}/(2\pi r)$.

Revised Chapter 9: Example 9.14.

Example 9.14: field in every radial region

Dividing the enclosed current by the circular path length gives

$$B_{\phi}(r) = \frac{\mu_0 I}{2\pi} \begin{cases} r/a^2, & 0 \leq r < a, \\ 1/r, & a \leq r < b, \\ (c^2 - r^2)/[r(c^2 - b^2)], & b \leq r < c, \\ 0, & r \geq c. \end{cases} \quad (9.48)$$

Nonzero field is along $+\hat{\phi}$. At a the first two branches agree; at b the third equals $1/b$; at c it tends to zero.

Inside the return, the derivative of $c^2/r - r$ is $-c^2/r^2 - 1 < 0$. The field decreases monotonically.

As $c \rightarrow b$, a thin surface current replaces the finite return layer and the field jumps. Exterior cancellation here uses equal opposite currents and cylindrical symmetry.

Revised Chapter 9: Example 9.14.

Coaxial field confinement is geometric

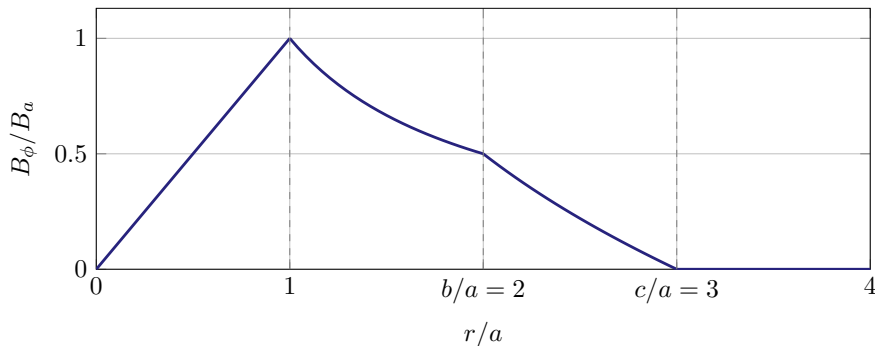
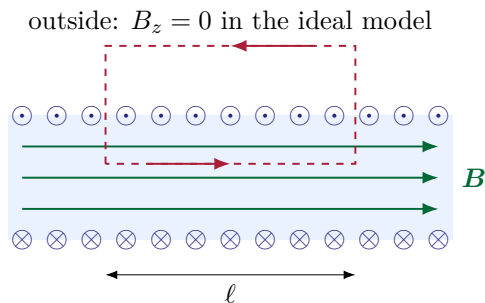


Figure 9.13: Calculated coaxial-cable field with $b = 2a$ and $c = 3a$, normalized by $B_a = \mu_0 I / (2\pi a)$. The four intervals are inner conductor, insulation, return conductor, and exterior. Currents are equal and opposite, with uniform densities in the two conductors.

Exterior cancellation applies to this symmetric, balanced source with no imposed background field.

Revised Chapter 9: Fig. 9.13.

The ideal solenoid and its integration path



The loop crosses one row of turns.

It encloses $n\ell I$, not zero and not NI for the whole winding.

Figure 9.14: Longitudinal section of a long solenoid. The upper row carries current out of the page, the lower row into it, producing an interior field to the right. One long side of the Amperian rectangle is inside and the other is outside.

Revised Chapter 9: Fig. 9.14, Sec. 9.9.

Solenoid field and physical boundary conditions

For a continuous, purely azimuthal infinite winding, let $n = N/L$ and let the long sides of the rectangle have length ℓ .

$$(B_{\text{in}} - B_{\text{out}})\ell = \mu_0(n\ell I), \quad B_{\text{in}} - B_{\text{out}} = \mu_0 nI.$$

The transverse sides give zero dot product. With no imposed background field, the ideal exterior source field is zero and $B_{\text{in}} = \mu_0 nI$.

A finite tightly wound air-core coil approximates this result near its center when $L \gg R$. At a long coil's open end the axial field approaches half the ideal interior value.

Model limits

A helical wire also transports current axially, and its leads contribute fields. Radius, end position, turn pitch, and return geometry matter when the infinite winding approximation is insufficient.

Revised Chapter 9: Sec. 9.9.

Example 9.15: required solenoid current

Example 9.15

A laboratory solenoid has $N = 1200$ turns over length $L = 0.60$ m. What current is required to produce an interior field of approximately $B = 15$ mT?

$$n = \frac{1200}{0.60 \text{ m}} = 2000 \text{ m}^{-1}, \quad I = \frac{B}{\mu_0 n} = \frac{15 \times 10^{-3}}{(4\pi \times 10^{-7})(2000)} \text{ A} = 5.97 \text{ A}.$$

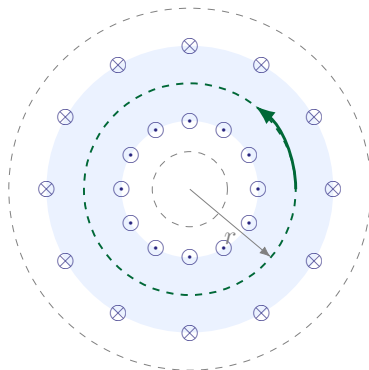
The introductory estimate is 6.0 A. Reinsert the unrounded current into $\mu_0 n I$ to recover 15 mT.

This assumes a long, closely wound air-core solenoid and a point well inside. The stated N and L give turn density, but the radius was not supplied, so they do not quantify the end correction.

Heating, voltage requirement, finite geometry, and leads need further information. Increasing N at fixed L differs from increasing L at fixed N .

Revised Chapter 9: Example 9.15.

Toroidal winding: three enclosed-current cases



hole: $I_{\text{enc}} = 0$

winding region: $I_{\text{enc}} = NI$

outside: $I_{\text{enc}} = NI - NI = 0$

equatorial section;
inner crossings out of page,
outer crossings into page

Figure 9.15: Three Amperian paths for an ideal toroidal winding. Current crosses the equatorial plane outward at the inner edge and inward at the outer edge. A path between the edges links each turn once; an exterior path encloses both crossings.

Revised Chapter 9: Fig. 9.15, Sec. 9.10.

Toroid law and problem P25

For an ideal continuous poloidal winding with vacuum permeability,

$$2\pi r B_\phi(r) = \mu_0 N I, \quad B_\phi(r) = \frac{\mu_0 N I}{2\pi r} \quad \text{inside the winding region.}$$

The field decreases radially. The hole and exterior have zero source field in this ideal geometry without external sources.

Problem P25

An ideal toroid has $N = 600$, $I = 1.2 \text{ A}$, and radial limits 0.060 m and 0.090 m . Find the inner, mean-radius, and outer fields, evaluated within the winding region. Quantify the largest fractional departure from the mean-radius value.

Use $r_m = (0.060 + 0.090)/2 = 0.075 \text{ m}$, and compare local fields with $B(r_m)$, not with each other.

Revised Chapter 9: Sec. 9.10, P25.

P25 solution: mean-radius error

Here $NI = 720 \text{ A}$ and $\mu_0 NI / (2\pi) = 1.44 \times 10^{-4} \text{ T m}$:

$$B(r_{\text{in}}) = 2.40 \text{ mT}, \quad B(r_m) = 1.92 \text{ mT}, \quad B(r_{\text{out}}) = 1.60 \text{ mT}.$$

With $B(r_m)$ as denominator,

$$\frac{B_{\text{in}} - B_m}{B_m} = +25.0\%, \quad \frac{B_{\text{out}} - B_m}{B_m} = -16.7\%.$$

The largest fractional departure is 25.0%, at the inner boundary.

Checks: each value gives the same $2\pi rB = \mu_0 NI$. The radial thickness is 0.030 m , so $t/r_m = 0.40$; a uniform mean-radius model is poor here. All nonzero vectors follow the winding's right-hand-rule azimuthal sense.

Revised Chapter 9: P25.

A current loop is a magnetic dipole

For a flat coil with area A per turn,

$$\boldsymbol{\mu} = NIA\hat{\mathbf{n}}, \quad [\mu] = \text{A m}^2, \quad \boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}_{\text{ext}}.$$

The right-hand rule selects $\hat{\mathbf{n}}$. The moment describes the loop's leading distant field and its torque response in an external field.

For a circular loop, $\mu = I\pi R^2$ and its axial far field is

$$B_{\text{axial}} \simeq \frac{\mu_0 \mu}{2\pi |z|^3}, \quad |z| \gg R.$$

On either side of the axis the field points along the moment. The exterior equatorial return field points oppositely. The dipole limit cannot describe points near the wire.

In a uniform external field a closed loop has zero net force, although it may have torque. Magnetic field lines have no sources or sinks; they need not all close.

Revised Chapter 9: Sec. 9.11, Sec. 9.12.

Example 9.18: a field target and power budget

Example 9.18

An air-core circular coil has $N = 200$ turns of mean radius $R = 0.050$ m. The copper-wire cross-sectional area is $A_w = 0.50 \text{ mm}^2$, and for this estimate take the resistivity as $\rho_e = 1.7 \times 10^{-8} \Omega \text{ m}$. Estimate the current, voltage, and resistive power needed for a center field of 2.0 mT .

Model: turns at a common mean radius, negligible lead length, and the stated constant resistivity. This predicts an initial electrical load, not an operating temperature.

$$NI = \frac{2RB}{\mu_0} = 159.15 \text{ A}, \quad I = \frac{159.15}{200} = 0.796 \text{ A}.$$

Use the actual copper cross section for resistance:

$$L_w \simeq 2\pi RN, \quad R_{\text{coil}} = \rho_e L_w / A_w, \quad V = IR_{\text{coil}}, \quad P = I^2 R_{\text{coil}}.$$

Revised Chapter 9: Example 9.18, Sec. 9.12.

Example 9.18: current, voltage, and heating

Convert $A_w = 0.50 \text{ mm}^2 = 0.50 \times 10^{-6} \text{ m}^2$.

$$L_w = 2\pi(0.050)(200) = 62.8 \text{ m},$$

$$R_{\text{coil}} = \frac{(1.7 \times 10^{-8})(62.8319)}{0.50 \times 10^{-6}} = 2.14 \, \Omega,$$

$$V = (0.795775)(2.13628) = 1.70 \text{ V},$$

$$P = (0.795775)^2(2.13628) = 1.35 \text{ W}.$$

Answer: $I = 0.796 \text{ A}$, $V = 1.70 \text{ V}$, and $P = 1.35 \text{ W}$. The wire current density is 1.59 A/mm^2 .

Checks: $VI = I^2 R_{\text{coil}}$, and the unrounded current returns 2.0 mT in the field model. At fixed resistance a doubled field needs twice the current and four times the power. Cooling and temperature-dependent resistance require another model.

Revised Chapter 9: Example 9.18.

A loop in a nonuniform source field

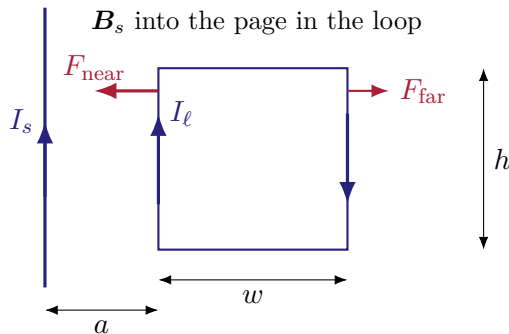


Figure 9.17: A rectangular loop near a long straight wire. The near and far vertical sides experience unequal horizontal forces. The top and bottom contributions cancel because they traverse the same range of distances with opposite current directions.

Define $+x$ to the right and $+y$ upward; $+z$ is out of the page. Near-side current is parallel to I_s , so that side is attracted more strongly than the far side is repelled.

Revised Chapter 9: Fig. 9.17.

Example 9.19: net force on a rectangular loop

Example 9.19

Find the net force on the loop of Figure 9.17, and evaluate it for $I_s = 20$ A, $I_\ell = 3.0$ A, $h = 0.10$ m, $a = 0.020$ m, and $w = 0.030$ m.

Use the source-wire field at each vertical side. Taking $+x$ away from the wire,

$$F_{\text{near}} = \frac{\mu_0 I_s I_\ell h}{2\pi a} = 60.0 \mu\text{N}, \quad F_{\text{far}} = \frac{\mu_0 I_s I_\ell h}{2\pi(a+w)} = 24.0 \mu\text{N}.$$

$$\mathbf{F}_{\text{net}} = -\frac{\mu_0 I_s I_\ell h}{2\pi} \left(\frac{1}{a} - \frac{1}{a+w} \right) \hat{\mathbf{x}} = -36.0 \mu\text{N} \hat{\mathbf{x}}.$$

Top and bottom forces cancel point by point in x , leaving 3.6×10^{-5} N toward the source wire. As $w \rightarrow 0$, the net force vanishes; for $a \gg w$, it scales as w/a^2 .

Revised Chapter 9: Example 9.19, Fig. 9.17.

Current reversal and uncertainty: practice P31

For a linear source and stable offset, $b_{\pm} = \pm b_s + b_0$:

$$b_{\text{odd}} = \frac{b_+ - b_-}{2}, \quad b_{\text{even}} = \frac{b_+ + b_-}{2}.$$

If reading errors are independent, propagation of either combination gives

$$\sigma_{\text{odd}}^2 = \sigma_{\text{even}}^2 = \frac{\sigma_+^2 + \sigma_-^2}{4}.$$

Problem P31

Readings with equal and opposite currents are $b_+ = 72.0 \pm 0.4 \mu\text{T}$ and $b_- = -28.0 \pm 0.4 \mu\text{T}$, where the stated values are independent standard uncertainties. Find odd and even parts and their uncertainties. List two errors that current reversal does not remove.

A single-axis probe measures a calibrated field component along its sensitive axis. Misalignment multiplies the signal by $\cos \alpha$.

Revised Chapter 9: Sec. 9.13, P31.

P31 solution: retained signal and remaining errors

$$b_{\text{odd}} = \frac{72.0 - (-28.0)}{2} = 50.0 \mu\text{T}, \quad b_{\text{even}} = \frac{72.0 + (-28.0)}{2} = 22.0 \mu\text{T}.$$

Each standard uncertainty is

$$\sigma = \frac{1}{2} \sqrt{0.4^2 + 0.4^2} = 0.283 \mu\text{T}.$$

Thus $b_{\text{odd}} = (50.0 \pm 0.3) \mu\text{T}$ and $b_{\text{even}} = (22.0 \pm 0.3) \mu\text{T}$ when uncertainties are rounded.

Two errors that remain: a gain bias still multiplies the signal; the return-conductor field reverses with the desired source field. Orientation errors and drift also require separate treatment.

Independent check: $b_{\text{even}} + b_{\text{odd}} = 72.0$ and $b_{\text{even}} - b_{\text{odd}} = -28.0$ in μT . Correlated reading errors would require covariance terms in the uncertainty formula.

Revised Chapter 9: P31.

Core synthesis: choosing and checking a model

- **Build a field:** Biot–Savart needs the directed complete source path and vector superposition.
- **Use symmetry:** Ampère's law gives signed circulation. A field shortcut also needs justified direction, constancy, and boundary conditions.
- **Predict a force:** evaluate the external field at the other current and apply the cross product.
- **Check an apparatus model:** compare finite geometry, return paths, resistance, and calibration assumptions with the requested accuracy.

Reading: revised Chapter 9, core sections and practice set; further textbook pointers are listed in the companion reference audit.

Next connection: these source fields prepare the later treatment of induction and magnetic energy. The following ten slides first refine current and coil models within magnetostatics.

Revised Chapter 9: Sec. 9.14, Sec. 9.12.

Advanced: local circulation and current density

Steady charge conservation gives $\nabla \cdot \mathbf{J} = 0$. Stokes's theorem converts the circulation law to a local statement:

$$\oint_{\partial S} \mathbf{B} \cdot d\boldsymbol{\ell} = \int_S (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} dS = \mu_0 \int_S \mathbf{J} \cdot \hat{\mathbf{n}} dS.$$

Thus $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$: curl measures local circulation per area. Magnetic flux has no sources, $\nabla \cdot \mathbf{B} = 0$: divergence measures outward flux per volume.

For an axisymmetric azimuthal field,

$$J_z(r) = \frac{1}{\mu_0 r} \frac{d}{dr} (r B_\phi(r)).$$

For $B_\phi = C/r$, the local curl vanishes. A linking path has nonzero circulation because its spanning surface crosses the current region.

Revised Chapter 9: Sec. 9.18, Sec. 9.18.2.

Example 9.21: reconstructing a current profile

Example 9.21

An axisymmetric long conductor has an internal field $B_\phi(r) = \beta r^2$ for $0 \leq r < a$, where β has units T/m^2 . Determine the current density and total current, and match the exterior field.

Use the local relation taught on slide 61:

$$J_z = \frac{1}{\mu_0 r} \frac{d(\beta r^3)}{dr} = \frac{3\beta r}{\mu_0}, \quad I = 2\pi \int_0^a J_z r dr = \frac{2\pi\beta a^3}{\mu_0}.$$

Outside,

$$B_\phi(r) = \frac{\mu_0 I}{2\pi r} = \frac{\beta a^3}{r} \quad (r \geq a).$$

At a , both fields equal βa^2 . Independently, integrating J_z to radius r gives $I_{\text{enc}} = 2\pi\beta r^3/\mu_0$ and reconstructs $B_\phi = \beta r^2$.

The units of J_z are A/m^2 . Differentiating measured data amplifies noise, so inference needs a fitted profile or controlled smoothing.

Revised Chapter 9: Example 9.21.

Finite solenoid as a stack of loops

Let R be winding radius, L its length, and $n = N/L$. A source slice at z' contains $n dz'$ turns. Superpose their on-axis fields:

$$B_z(z) = \frac{\mu_0 n I R^2}{2} \int_{-L/2}^{L/2} \frac{dz'}{[R^2 + (z - z')^2]^{3/2}}.$$

Set $u = z - z'$ and keep the reversed integration limits. Since $\int R^2(R^2 + u^2)^{-3/2} du = u/\sqrt{R^2 + u^2}$,

$$B_z(z) = \frac{\mu_0 n I}{2} \left[\frac{z + L/2}{\sqrt{R^2 + (z + L/2)^2}} - \frac{z - L/2}{\sqrt{R^2 + (z - L/2)^2}} \right]. \quad (9.97)$$

At the center, $B_z(0) = \mu_0 n I (L/2) / \sqrt{R^2 + (L/2)^2}$. This is exact on the axis of the continuous azimuthal current-sheet model; discrete turns and leads require corrections.

Revised Chapter 9: Sec. 9.20.

Example 9.24: center and end corrections

Example 9.24

A solenoid has $R = 0.030$ m, $L = 0.120$ m, $N = 240$, and $I = 0.50$ A. Calculate its center field and its axial end field, and compare the center field with the infinite-solenoid estimate.

$$n = 240/0.120 = 2000 \text{ m}^{-1} \text{ and } B_{\infty} = \mu_0 n I = 1.25664 \text{ mT}.$$

$$f_c = \frac{0.060}{\sqrt{0.030^2 + 0.060^2}} = 0.89443, \quad B(0) = 1.124 \text{ mT}.$$

$$B(L/2) = \frac{B_{\infty}}{2} \frac{0.120}{\sqrt{0.030^2 + 0.120^2}} = 0.610 \text{ mT}.$$

The center deficit is $1 - f_c = 10.6\%$ of B_{∞} . The ideal estimate overstates the actual center by $1/f_c - 1 = 11.8\%$.

The $L/R = 4$ coil is not well approximated by the infinite model for precise calibration. Independent loop-stack integration confirms both values; for $L/R \rightarrow \infty$, the center and end ratios tend to 1 and $1/2$.

Revised Chapter 9: Example 9.24.

Finite-solenoid profiles and the long limit

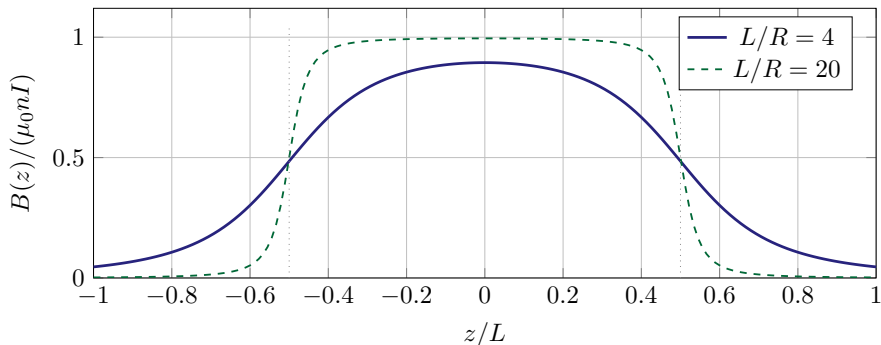
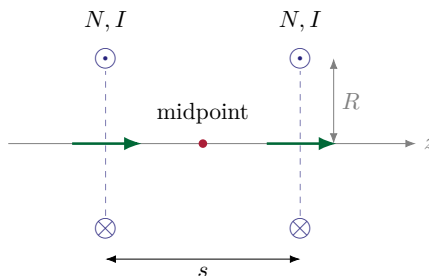


Figure 9.20: Continuous-winding finite-solenoid profiles from Eq. (9.97). The dotted lines locate the ends. Normalization uses the infinite-solenoid value for the same nI .

The endpoints are at $z/L = \pm 1/2$. A longer winding has a broader central plateau, but its end field still approaches half the interior ideal value. A center correction is not a whole-bore uniformity guarantee.

Revised Chapter 9: Fig. 9.20.

Helmholtz geometry: equal same-sense coils



section through the common axis; $+y$ out of page
dots and crosses: current at the two crossings of each coil

Figure 9.21: Two equal coaxial coils viewed in an axial section. Each dashed line spans a circular coil; it is not a straight wire. The current crossings give magnetic moments and central fields along $+z$ (green arrows). Helmholtz spacing is $s = R$.

Each coil has N turns and current I . Both moments point along $+z$. Reflection about the midpoint cancels the first axial derivative for every spacing; removing the second derivative requires a particular s/R .

Revised Chapter 9: Fig. 9.21, Sec. 9.21.

Why Helmholtz spacing equals the radius

Superpose the two loop-axis fields:

$$B_s(z) = \frac{\mu_0 N I R^2}{2} \left\{ [R^2 + (z - s/2)^2]^{-3/2} + [R^2 + (z + s/2)^2]^{-3/2} \right\}.$$

For $f(u) = (R^2 + u^2)^{-3/2}$,

$$f''(u) = \frac{3(4u^2 - R^2)}{(R^2 + u^2)^{7/2}}, \quad B_s''(0) = \frac{3\mu_0 N I R^2 (s^2 - R^2)}{(R^2 + s^2/4)^{7/2}}.$$

The field is even, so $B_s'(0) = 0$. Setting $s = R$ also removes the quadratic term. Smaller spacing gives a central axial maximum; larger spacing gives a minimum.

$$B_0 = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 N I}{R}, \quad \frac{B_R(z)}{B_0} = 1 - \frac{144}{125} (z/R)^4 + O((z/R)^6).$$

The first surviving variation is quartic, rather than the single loop's quadratic variation.

Revised Chapter 9: Sec. 9.21.

Coil-pair strength and profile shape

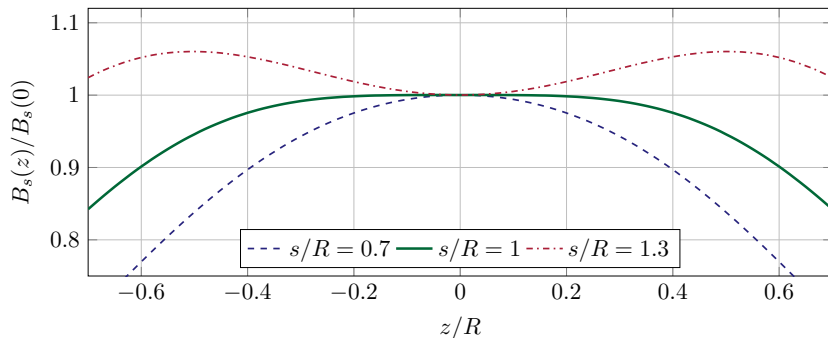


Figure 9.22: Calculated axial profiles of equal coil pairs. Only the Helmholtz spacing eliminates the central quadratic variation. All curves use Eq. (9.103) and their respective midpoint normalization.

Each curve is divided by its own midpoint field. The green curve suppresses central curvature. At fixed NI , bringing coils closer can increase the midpoint field while degrading uniformity.

Revised Chapter 9: Fig. 9.22.

Advanced challenge: a specified uniform interval

Example 9.25

A Helmholtz pair has $R = 0.10\text{ m}$, $N = 100$ turns per coil, and $I = 0.50\text{ A}$. Determine its center field and estimate the axial interval over which the field differs from its center value by less than 1%.

The coils are thin, coincident-turn sources with spacing $s = R$. Start with the quartic estimate from slide 67:

$$B_0 = \left(\frac{4}{5}\right)^{3/2} \frac{(4\pi \times 10^{-7})(100)(0.50)}{0.10} = 0.4496\text{ mT}.$$

$$\frac{144}{125}(z/R)^4 < 0.01 \quad \Rightarrow \quad |z|/R \lesssim 0.305.$$

This gives a roughly 30.5 mm axial half-length. The complete exact-model check and final interval follow on slide 70.

Revised Chapter 9: Example 9.25.

Helmholtz challenge: exact interval and checks

Set $u = z/R$. The exact normalized on-axis field is

$$F(u) = \frac{1.25^{3/2}}{2} \left\{ [1 + (u - 0.5)^2]^{-3/2} + [1 + (u + 0.5)^2]^{-3/2} \right\}.$$

Solving $F(u) = 0.99$ gives $u = \pm 0.31375$. With $R = 0.10$ m, the connected interval with strictly less than one-percent departure is bounded by $z \simeq \pm 31.4$ mm.

$ z /R$	Exact deficit	Quartic estimate
0.2	0.176%	0.184%
0.3	0.844%	0.933%

The center field is 0.4496 mT along $+z$. The approximation improves closer to the midpoint. This axial interval does not establish a three-dimensional uniform volume: off-axis components, winding thickness, unequal currents, and misalignment require further analysis.

Revised Chapter 9: Example 9.25, Sec. 9.21.

