FEM modelling of materials processing and properties

Theoretical bases and practical implementation in ABAQUS program.

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Literature

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Content

1. Theoretical basis of the linear theory of elasticity; the principle of virtual work; formulation of the boundary value problem; the basic principles for solving problems using the finite element method.

2. The theoretical foundations of the theory of small elastic plastic deformation; models of the mechanical properties of elastic-plastic materials; theorem on unloading.

3. The theoretical foundations of the theory of plastic flow of incompressible materials; the variation principle of Markov; the model materials under large deformations; features of using finite element method to solve problems in the theory of plastic flow

4. Basics of program ABAQUS; examples of solving problems of the theory of elasticity in the program ABAQUS; the problem of modelling the deformation of tool during metal forming processes

5. Modelling of residual stresses; the solution of elastic-plastic problems by using ABAQUS; examples solving of the cold metal forming problem.

Introduction

We concentrated on:

1. Theoretical basis of elasticity, elastic-plasticity, rigid and viscous plasticity;
2. Practical FEM procedures, that are integral part of CAD/CAE software;
3. Practical usage of FEM programs;
4. Developer of FEM procedures (?)

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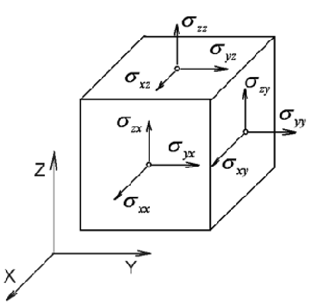
Lection 1.

Theoretical basis of the linear theory of elasticity; the principle of virtual work; formulation of the boundary value problem; the basic principles for solving problems using the finite element method.

Theoretical basis of the linear theory of elasticity

Stress tensor. Couchy stresses. Effective stress (intensity).



$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \sigma_0 \delta_{\mathrm{ij}} + \mathrm{s}_{\mathrm{ij}} = A_\sigma + D_\sigma$$

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\sigma_0 = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = \frac{1}{3}\sigma_{ij}\delta_{ij}$$

$$\overline{\sigma} = \sqrt{\frac{3}{2}\mathrm{s}_{\mathrm{ij}}\mathrm{s}_{\mathrm{ij}}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \left(\sigma_{yy} - \sigma_{zz}\right)^2 + \left(\sigma_{zz} - \sigma_{xx}\right)^2 + 6\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right)}$$

$$\overline{\tau} = T = \sqrt{\frac{1}{2}\mathrm{s}_{\mathrm{ij}}\mathrm{s}_{\mathrm{ij}}} = \frac{\overline{\sigma}}{\sqrt{3}}$$

Principals stress. Invariants of stress tensor.

$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad \sigma_{ij} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

Invariants:

$$\sigma_1 > \sigma_2 > \sigma_3$$

$$\sigma^I = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z = const$$

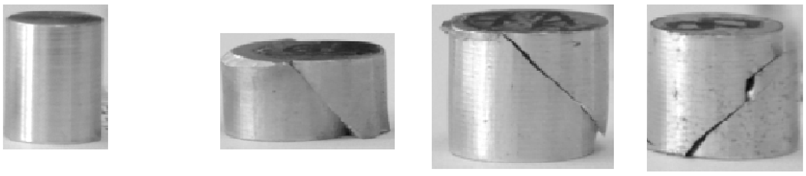
$$\sigma^{II} = \begin{vmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{vmatrix} + \begin{vmatrix} \sigma_{yy} & \sigma_{yz} \\ \sigma_{zy} & \sigma_{zz} \end{vmatrix} + \begin{vmatrix} \sigma_{zz} & \sigma_{zx} \\ \sigma_{xz} & \sigma_{xx} \end{vmatrix} = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 =$$

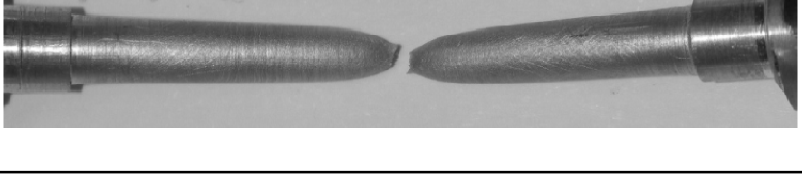
$$= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \sigma_{xy}^2 - \sigma_{xy}^2 - \sigma_{xy}^2 = const$$

$$\sigma^{III} = \left|\sigma_{ij}\right| = \sigma_1\sigma_2\sigma_3 = \sigma_x\sigma_y\sigma_z + 2\sigma_{xy}\sigma_{yz}\sigma_{zx} - \sigma_x\sigma_{yz}^2 - \sigma_y\sigma_{zx}^2 - \sigma_z\sigma_{xy}^2 = const$$

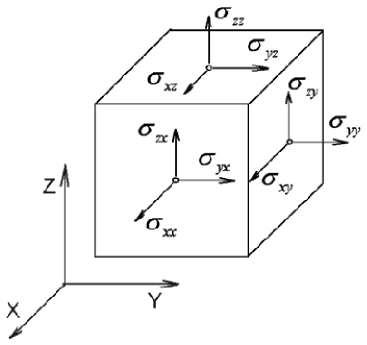
Maximal shear stresses

$$\sigma_{12} = \pm\frac{\sigma_1 - \sigma_2}{2} \qquad \sigma_{23} = \pm\frac{\sigma_2 - \sigma_3}{2} \qquad \sigma_{13} = \pm\frac{\sigma_1 - \sigma_3}{2}$$





Equilibrium equations



$$\left.\begin{array}{l}\dfrac{\partial\sigma_{xx}}{\partial x}+\dfrac{\partial\sigma_{xy}}{\partial y}+\dfrac{\partial\sigma_{xz}}{\partial z}+g_x=0\\ \dfrac{\partial\sigma_{yx}}{\partial x}+\dfrac{\partial\sigma_{yy}}{\partial y}+\dfrac{\partial\sigma_{yz}}{\partial z}+g_y=0\\ \dfrac{\partial\sigma_{zx}}{\partial x}+\dfrac{\partial\sigma_{zy}}{\partial y}+\dfrac{\partial\sigma_{zz}}{\partial z}+g_z=0\end{array}\right\}$$

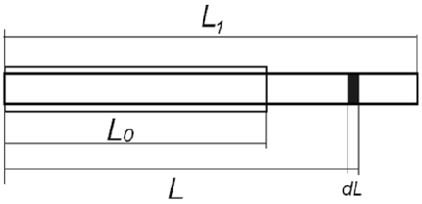
$$\int_V \frac{\partial \mathrm{P_i}}{\partial \mathrm{x_i}}\mathrm{dV}=\int_S \mathrm{P_i}\nu_\mathrm{i}\mathrm{dS}$$

$$\frac{\partial\sigma_{\mathrm{ij}}}{\partial \mathrm{x_j}}=\sigma_{\mathrm{ij,j}}=0$$

$$\int_V \frac{\partial\sigma_{\mathrm{ij}}}{\partial \mathrm{x_j}}\mathrm{dV}=\int_S \sigma_{\mathrm{ij}}\nu_\mathrm{i}\mathrm{dS}=0$$

Strains. Engineer approach. Logarithmical strains.

$$\varepsilon_L=\frac{L_1-L_0}{L_0}=\frac{\Delta L}{L_0}\qquad\qquad \varepsilon_L=\frac{L_1-L_0}{L_1}=\frac{\Delta L}{L_1}$$



$$e_L=\int_{L_0}^{L_1}\frac{dL}{L}=\ln\left(\frac{L_1}{L_0}\right)$$

$$e_L=\ln\left(\frac{L_1}{L_0}\right)=\ln\left(\frac{L_0+\Delta L}{L_0}\right)=\ln\left(1+\varepsilon_L\right)$$

$$e_L=\ln\left(1+\varepsilon_L\right)=\varepsilon_L-\frac{\varepsilon_L^2}{2}+\frac{\varepsilon_L^3}{3}-\frac{\varepsilon_L^4}{4}+\ldots\qquad\qquad e_L<\varepsilon_L$$

Cauchy equations (Augustin Louis Cauchy)

$$\vec{u} = (u_1, u_2, u_3)$$

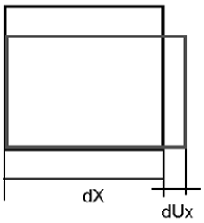
$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

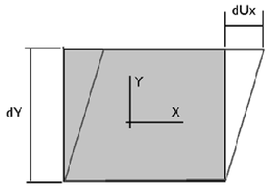
$$\dot{\varepsilon}_{ij} = \xi_{ij} = \frac{\partial \varepsilon_{ij}}{\partial t} = \frac{1}{2}\left(\dot{u}_{i,j} + \dot{u}_{j,i}\right)$$

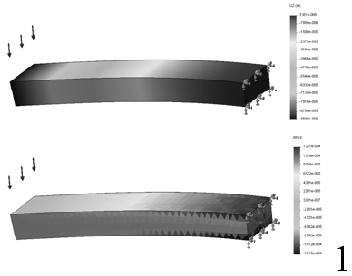
$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$







Strain state. Strain tensor.

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \qquad \varepsilon_{ij} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

Invariants of strain tensor.

$$\varepsilon^{I} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$\varepsilon^{II} = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{vmatrix} + \begin{vmatrix} \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_{zz} \end{vmatrix} + \begin{vmatrix} \varepsilon_{zz} & \varepsilon_{zx} \\ \varepsilon_{xz} & \varepsilon_{xx} \end{vmatrix} = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$$

$$\varepsilon^{III} = \left|\varepsilon_{ij}\right| = \varepsilon_1\varepsilon_2\varepsilon_3$$

Deviator and axiator

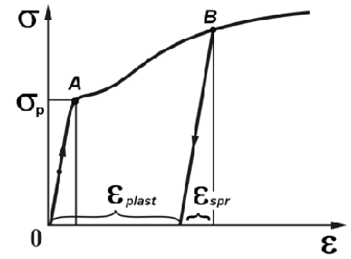
$$\varepsilon_{ij} = \varepsilon_0 \delta_{ij} + e_{ij} = A_\varepsilon + D_\varepsilon \qquad \varepsilon_0 = \frac{1}{3}\varepsilon_{ij}\delta_{ij} = \frac{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}{3}$$

Effective strain (strain intensity)

$$\bar{\varepsilon} = \sqrt{\frac{2}{3} e_{ij} e_{ij}}$$

$$\bar{\varepsilon} = \frac{\sqrt{2}}{3}\sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + \left(\varepsilon_y - \varepsilon_z\right)^2 + \left(\varepsilon_z - \varepsilon_x\right)^2 + 6\left(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2\right)}$$

Young's modulus. Poisson's ratio.



$$E = \frac{\bar{\sigma}}{\bar{\varepsilon}}$$

$$\varepsilon_2 = \varepsilon_3 = -\varepsilon_1 \nu = -\nu \frac{\sigma_1}{E}$$

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

$$\nu = \left|\frac{\varepsilon_2}{\varepsilon_1}\right| = \left|\frac{\varepsilon_3}{\varepsilon_1}\right|$$

Hooke's law.

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{yy} + \sigma_{xx}\right)\right]$$

$$\varepsilon_{xy} = \frac{1+\nu}{E}\sigma_{xy}$$

$$\varepsilon_{xz} = \frac{1+\nu}{E}\sigma_{xz}$$

$$\varepsilon_{zy} = \frac{1+\nu}{E}\sigma_{zy}$$

Hooke's law

$$\varepsilon_0 = \frac{\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}}{3} \qquad \sigma_0 = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \qquad \sigma_0 = \frac{E}{1-2\nu}\varepsilon_0 = 3k_V\varepsilon_0$$

$$\sigma_{ij} = \delta_{ij}3k_V\varepsilon_0 + \frac{E}{\left(1+\nu\right)}\left(\varepsilon_{ij} - \delta_{ij}\varepsilon_0\right)$$

$$\sigma_{xx} = 3k_V\varepsilon_0 + \frac{E}{\left(1+\nu\right)}\left(\varepsilon_{xx} - \varepsilon_0\right)$$

$$\sigma_{xy} = \frac{E}{\left(1+\nu\right)}\varepsilon_{xy}$$

$$k_V = \frac{E}{3\left(1-2\nu\right)}$$

$$D_\sigma = 2GD_\varepsilon$$
$$A_\sigma = 3k_V A_\varepsilon$$

$$\sigma_{ij} = \delta_{ij}3k_V\varepsilon_0 + 2G\left(\varepsilon_{ij} - \delta_{ij}\varepsilon_0\right) \qquad 2G = \frac{E}{\left(1+\nu\right)}$$

Boundary problem.

a) Cauchy equations :

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

b) Equilibrium equations:

$$\sigma_{ij,j} = 0$$

c) Stress-strain relationship:

$$\sigma_{ij} = \delta_{ij} 3k_V \varepsilon_0 + \frac{E}{(1+\nu)}\left(\varepsilon_{ij} - \delta_{ij}\varepsilon_0\right) \qquad (1)$$

Unit density of energy : $W\left(\varepsilon_{ij}\right) = \int_0^{\varepsilon_{ij}} \sigma_{ij}\left(\varepsilon_{ij}\right) d\varepsilon_{ij}$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}$$

The principle of virtual work

$$\int_{^tV} {}^t\sigma_{ij}\delta^t\varepsilon_{ij}\, d^tV = {}^tR \qquad {}^tR = \int_{^tV} {}^t\mathrm{f}_i^B\ \delta^t u_i\ d^tV + \int_{^tS} {}^t\mathrm{f}_i^S\ \delta^t u_i^S\ d^tS$$

${}^tV, {}^tS$ Current volume and surface at time t

$\delta^t u_i \quad \delta^t u_i^S$ Virtual displacements

$\delta^t\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial^t x_j} + \frac{\partial \delta u_j}{\partial^t x_i}\right)$ Virtual strains

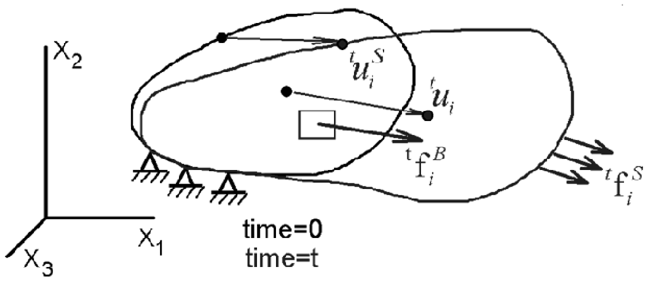
${}^t\mathrm{f}_i^B$ *and* ${}^t\mathrm{f}_i^S$ Body forces and surface forces

${}^t\sigma_{ij}$ Couchy stresses (forces per unit area at time t, actual physical stresses)

K.J.Bathe Finite Element Procedures for Solids and Structures (MIT Open Recourse)

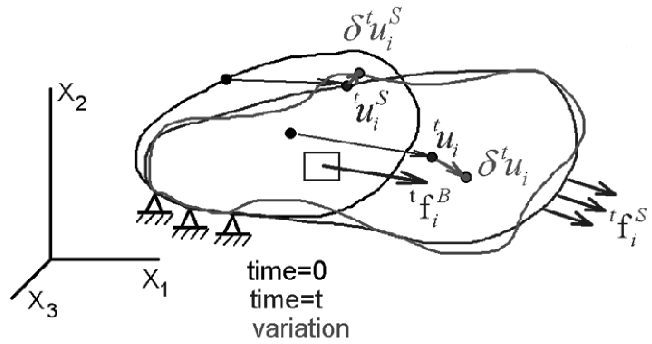
The principle of virtual work

$$\int_{{}^tV} {}^t\sigma_{ij}\,\delta^t\varepsilon_{ij}\,d^tV = {}^tR \qquad {}^tR = \int_{{}^tV} {}^t\mathrm{f}_i^B\,\delta u_i\,d^tV + \int_{{}^tS} {}^t\mathrm{f}_i^S\,\delta u_i\,d^tS$$



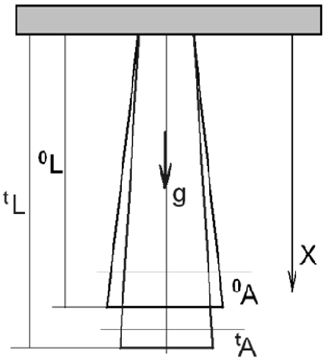
The principle of virtual work

$$\int_{{}^tV} {}^t\sigma_{ij}\,\delta^t\varepsilon_{ij}\,d^tV = {}^tR \qquad {}^tR = \int_{{}^tV} {}^t\mathrm{f}_i^B\,\delta u_i\,d^tV + \int_{{}^tS} {}^t\mathrm{f}_i^S\,\delta u_i\,d^tS$$



The principle of virtual work

$$\int_{{}^tV} {}^t\sigma_{ij}\delta{}^t\varepsilon_{ij}d{}^tV = \int_{{}^tL} {}^t\sigma\ \delta{}^t\varepsilon\ {}^tA\,dx \qquad {}^tR = \int_{{}^tV} {}^t\mathrm{f}_i^B\ \delta u_i\ d{}^tV = \int_{{}^tL} \rho g\delta u\ {}^tA\,d{}^tx$$



$$\int_{{}^tL} {}^t\sigma\ \delta{}^t\varepsilon\ {}^tA\,dx = \int_{{}^tL} \rho g\delta u\ {}^tA\,d{}^tx \qquad \delta{}^t\varepsilon = \frac{\partial\delta u}{\partial{}^tx}$$

$$\int_{{}^tL} \frac{\partial\delta u}{\partial{}^tx}\ {}^t\sigma\ {}^tA\,dx - \int_{{}^tL} \rho g\delta u\ {}^tA\,d{}^tx = 0$$

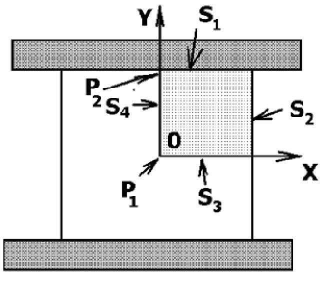
$$(uv)' = u'v + uv'$$

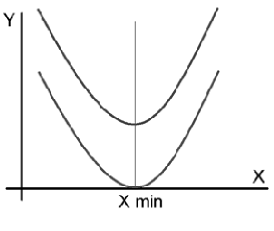
$$\int_{{}^tL}\left[\frac{\partial}{d{}^tx}\left({}^t\sigma\ {}^tA\right) + \rho g{}^tA\right]\delta u d{}^tx - \left[\left({}^t\sigma\ {}^tA\right)\delta u\right]\Big|_0^{{}^tL} = 0$$

$$\frac{\partial}{d{}^tx}\left({}^t\sigma\ {}^tA\right) + \rho g{}^tA = 0 \qquad \left[{}^t\sigma\ {}^tA\right]\Big|_{{}^tL} = 0$$

The principle of virtual work

$$\boxed{J = \int_V \frac{1}{2}E\bar{\varepsilon}^2 dV + \frac{3}{2}\int_V k_V\varepsilon_0^2 dV - \int_{S_1}\tau_{fric}u_\tau dS} - \boxed{\int_{S_1}\sigma_n u_n dS - \int_{S_2-S_4}\sigma_n u_n dS - \int_{S_2-S_4}\sigma_\tau u_\tau dS} \equiv 0$$





$$J = \int_V \frac{1}{2}E\bar{\varepsilon}^2 dV + \frac{3}{2}\int_V k_V\varepsilon_0^2 dV - \int_S \sigma_i u_i dS \xrightarrow[\text{min}]{}$$

Boundary problem in matrix form

$$\{\sigma\}=[D]\{\varepsilon\}$$

$$\{\varepsilon\}=D^{-1}\{\sigma\}$$

$$\{\sigma\}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{Bmatrix}=[D]\left(\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ 2\varepsilon_{xy}\end{Bmatrix}\right)$$

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}$$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_x(1-\nu)+\varepsilon_y\nu\right)$$

$$\varepsilon_x=\frac{\sigma_x}{E}-\frac{\nu\sigma_y}{E}-\frac{\nu\sigma_z}{E}$$

Material properties in matrix form

Plain strain

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}$$

Plain stress

$$[D]=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}$$

3d strain

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ & 1-\nu & \nu & 0 & 0 & 0\\ & & 1-\nu & 0 & 0 & 0\\ & & & (1-2\nu)/2 & 0 & 0\\ & & & & (1-2\nu)/2 & 0\\ sym & & & & & (1-2\nu)/2\end{bmatrix}$$

Strain – displacement relationships in matrix form

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u_x\\ u_y\end{Bmatrix}=[L]\{u\}$$

$$2\varepsilon_{xy}=\frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x}$$

$$\varepsilon_{xx}=\frac{\partial u_x}{\partial x}$$

$$[L]=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}$$

$$\varepsilon_{yy}=\frac{\partial u_y}{\partial y}$$

Strain – displacement relationships in discrete form.

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ 2\varepsilon_{xy}\end{Bmatrix}$$

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_{x}\\ \varepsilon_{y}\\ 2\varepsilon_{xy}\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u_x\\ u_y\end{Bmatrix}=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}\{U\}=[B]\{U\}$$

$$\{u\}=\begin{Bmatrix}u_x\\ u_y\end{Bmatrix}=\begin{Bmatrix}[N]\{U_x\}\\ [N]\{U_y\}\end{Bmatrix}=\begin{bmatrix}[N] & 0\\ 0 & [N]\end{bmatrix}\begin{Bmatrix}\{U_x\}\\ \{U_y\}\end{Bmatrix}=[\overline{N}]\{U\}$$

$$[B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}$$

$$\{\varepsilon\}=[B]\{U\}$$

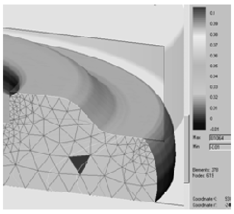
$$\varepsilon_{xx}=\frac{[N]}{\partial x}\{U_x\}$$

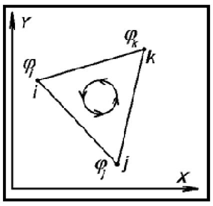
Example:

$$\varepsilon_x = \frac{\partial U_x}{\partial X}$$

$$\varepsilon_y = \frac{\partial U_y}{\partial Y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial U_x}{\partial Y} + \frac{\partial U_y}{\partial X}\right)$$





$$\boxed{N_i = \frac{1}{2A}\left(a_i + b_i X + c_i Y\right)}$$

$$\boxed{N_j = \frac{1}{2A}\left(a_j + b_j X + c_j Y\right)}$$

$$\boxed{N_k = \frac{1}{2A}\left(a_k + b_k X + c_k Y\right)}$$

$$U_x = U_{xi} N_i + U_{xj} N_j + U_{xk} N_k$$

$$U_y = U_{yi} N_i + U_{yj} N_j + U_{yk} N_k$$

$$\varepsilon_x = \frac{\partial U_x}{\partial X} = U_{xi}\frac{\partial N_i}{\partial X} + U_{xj}\frac{\partial N_j}{\partial X} + U_{xk}\frac{\partial N_k}{\partial X} = \frac{1}{2A}\left(U_{xi}b_i + U_{xj}b_j + U_{xk}b_k\right)$$

$$\varepsilon_y = \frac{\partial U_y}{\partial Y} = U_{yi}\frac{\partial N_i}{\partial Y} + U_{yj}\frac{\partial N_j}{\partial Y} + U_{yk}\frac{\partial N_k}{\partial Y} = \frac{1}{2A}\left(U_{yi}c_i + U_{yj}c_j + U_{yk}c_k\right)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial U_x}{\partial Y} + \frac{\partial U_y}{\partial X}\right) = \frac{1}{2}\left(U_{xi}\frac{\partial N_i}{\partial Y} + U_{xj}\frac{\partial N_j}{\partial Y} + U_{xk}\frac{\partial N_k}{\partial Y} + U_{yi}\frac{\partial N_i}{\partial X} + U_{yj}\frac{\partial N_j}{\partial X} + U_{yk}\frac{\partial N_k}{\partial X}\right) =$$

$$\frac{1}{4A}\left(U_{xi}c_i + U_{xj}c_j + U_{xk}c_k + U_{yi}b_i + U_{yj}b_j + U_{yk}b_k\right)$$

$$a_i = X_j Y_k - X_k Y_j$$
$$b_i = Y_j - Y_k$$
$$c_i = X_k - X_j$$

$$a_j = X_k Y_i - X_i Y_k$$
$$b_j = Y_k - Y_i$$
$$c_j = X_i - X_k$$

$$a_k = X_i Y_j - X_j Y_i$$
$$b_k = Y_i - Y_j$$
$$c_k = X_j - X_i$$

Principle of virtual work in matrix form

$$W = \int_V \frac{1}{2}\left(\{\varepsilon\}^T \{\sigma\}\right) dV$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$\{\varepsilon\} = [B]\{U\}$$

$$W_e = \int_{V_e} \frac{1}{2}\left(\{U\}^T [B]^T [D][B]\{U\}\right) dV$$

Principle of virtual work in matrix form

$$W_s = \{U\}^T\{P\} = \{P\}^T\{U\} \qquad \{P\} = \begin{Bmatrix}\{P_x\}\\ \{P_y\}\end{Bmatrix}$$

$$W_b = \int_{V_e}\left(u_x m_x + u_y m_y\right)dV = \int_{V_e}\{U\}^T\left[\overline{N}\right]^T\{M\}dV \qquad \{M\} = \begin{Bmatrix}m_x\\ m_y\end{Bmatrix}$$

$$\left[\overline{N}\right] = \begin{bmatrix} N_1 & N_2\ldots & N_p & 0 & 0\ldots & 0\\ 0 & 0\ldots & 0 & N_1 & N_2\ldots & N_p\end{bmatrix} = \begin{bmatrix}[N] & 0\\ 0 & [N]\end{bmatrix}$$

$$W_p = \int_{S_e}\left(u_x p_x + u_y p_y\right)dS = \int_{S_e}\{U\}^T\left[\overline{N}\right]^T\{p\}dS \qquad \{p\} = \begin{Bmatrix}p_x\\ p_y\end{Bmatrix}$$

$$W_e = \int_{V_e}\frac{1}{2}\{U\}^T[B]^T[D][B]\{U\}dV - \int_{V_e}\{U\}^T\left[\overline{N}\right]^T\{M\}dV - \int_{S_e}\{U\}^T\left[\overline{N}\right]^T\{p\}dS - \{U\}^T\{P\}$$

Stiffness matrix

$$\frac{\partial W_e}{\partial\{U\}} = \{U\}\int_{V_e}[B]^T[D][B]dV - \int_{V_e}\left[\overline{N}\right]^T\{M\}dV - \int_{S_e}\left[\overline{N}\right]^T\{p\}dS - \{P\} = 0$$

$$\frac{\partial W_e}{\partial\{U\}} = [K_e]\{U\} + \{F_e\} = 0$$

$$[K_e] = \int_{V_e}[B]^T[D][B]dV \qquad \{F_e\} = -\int_{V_e}\left[\overline{N}\right]^T\{M\}dV - \int_{S_e}\left[\overline{N}\right]^T\{p\}dS - \{P\}$$

$$[K]\{U\} + \{F\} = 0$$

$$[K] = \sum_{e=1}^{n_e}[K_e] \qquad \{F\} = \sum_{e=1}^{n_e}\{F_e\}$$

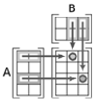
Stiffness matrix

$$[K]^e = \int_{V_e} [B]^T [D][B]\, dV$$

$$[B] = \begin{bmatrix} \frac{\partial[N]}{\partial x} & 0 \\ 0 & \frac{\partial[N]}{\partial y} \\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x} \end{bmatrix} \quad [B]^T = \begin{bmatrix} \frac{\partial[N]}{\partial x} & 0 & \frac{\partial[N]}{\partial y} \\ 0 & \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x} \end{bmatrix} \quad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

$$[K]^e = \int_{V_e} [B]^T [D][B] dV = \sum_{p=1}^{N_p} [B]^T [D][B] W_p \det|J|$$

Stiffness matrix



$$[K]^e = \int_{V_e} [B]^T [D][B] dV$$

$$[B]^T = \begin{bmatrix} \frac{\partial[N]^T}{\partial x} & 0 & \frac{\partial[N]^T}{\partial y} \\ 0 & \frac{\partial[N]^T}{\partial y} & \frac{\partial[N]^T}{\partial x} \end{bmatrix} \quad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \quad [B] = \begin{bmatrix} \frac{\partial[N]}{\partial x} & 0 \\ 0 & \frac{\partial[N]}{\partial y} \\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x} \end{bmatrix}$$

$$\int_{V_e} [B]^T [D][B] dV = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

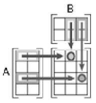
$$K_{11} = \int_{V_e} \left((1-\nu) \frac{\partial[N]^T}{\partial x} \frac{\partial[N]}{\partial x} + \frac{(1-2\nu)}{2} \frac{\partial[N]^T}{\partial y} \frac{\partial[N]}{\partial y} \right) dV$$

$$K_{12} = \int_{V_e} \left(\nu \frac{\partial[N]^T}{\partial x} \frac{\partial[N]}{\partial y} + \frac{(1-2\nu)}{2} \frac{\partial[N]^T}{\partial x} \frac{\partial[N]}{\partial y} \right) dV$$

$$K_{21} = \int_{V_e} \left(\nu \frac{\partial[N]^T}{\partial y} \frac{\partial[N]}{\partial x} + \frac{(1-2\nu)}{2} \frac{\partial[N]^T}{\partial y} \frac{\partial[N]}{\partial x} \right) dV$$

$$K_{22} = \int_{V_e} \left((1-\nu) \frac{\partial[N]^T}{\partial y} \frac{\partial[N]}{\partial y} + \frac{(1-2\nu)}{2} \frac{\partial[N]^T}{\partial x} \frac{\partial[N]}{\partial x} \right) dV$$

Stiffness matrix



$$[B]^T[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}\frac{\partial[N]^T}{\partial x} & 0 & \frac{\partial[N]^T}{\partial y}\\ 0 & \frac{\partial[N]^T}{\partial y} & \frac{\partial[N]^T}{\partial x}\end{bmatrix}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}=$$

$$=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}\frac{\partial[N]^T}{\partial x}(1-\nu) & \frac{\partial[N]^T}{\partial x}\nu & \frac{\partial[N]^T}{\partial y}(1-2\nu)/2\\ \frac{\partial[N]^T}{\partial y}\nu & \frac{\partial[N]^T}{\partial y}(1-\nu) & \frac{\partial[N]^T}{\partial x}(1-2\nu)/2\end{bmatrix}$$

$$[B]^T[D][B]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}\frac{\partial[N]^T}{\partial x}(1-\nu) & \frac{\partial[N]^T}{\partial x}\nu & \frac{\partial[N]^T}{\partial y}(1-2\nu)/2\\ \frac{\partial[N]^T}{\partial y}\nu & \frac{\partial[N]^T}{\partial y}(1-\nu) & \frac{\partial[N]^T}{\partial x}(1-2\nu)/2\end{bmatrix}\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}=$$

$$=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}$$

3d strain and stress state

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ 2\varepsilon_{xy}\\ 2\varepsilon_{yz}\\ 2\varepsilon_{zx}\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \\ & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u_x\\ u_y\\ u_z\end{Bmatrix}=[L]\{u\}=[B]\{U\}$$

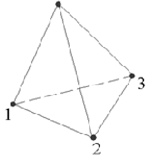
$$[L]=\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\end{bmatrix}$$

$$[B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0 & 0\\ 0 & \frac{\partial[N]}{\partial y} & 0\\ 0 & 0 & \frac{\partial[N]}{\partial z}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial z} & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial z} & 0 & \frac{\partial[N]}{\partial x}\end{bmatrix}$$

$$\{u\}=\begin{Bmatrix}u_x\\ u_y\\ u_z\end{Bmatrix}=\begin{bmatrix}[N] & 0 & 0\\ 0 & [N] & 0\\ 0 & 0 & [N]\end{bmatrix}\begin{Bmatrix}\{U_x\}\\ \{U_y\}\\ \{U_z\}\end{Bmatrix}=[\overline{N}]\{U\}$$

$$[N]=[N_1\ N_2\ N_3\ N_4]$$

$$N_1=L_1$$
$$N_2=L_2$$
$$N_3=L_3$$
$$N_4=L_4$$



$$\{\sigma\}=[D]\{\varepsilon\}$$

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ sym & & & & & \frac{(1-2\nu)}{2} \end{bmatrix}$$

$$\sigma_x=\frac{E}{(1+\nu)(1-2\nu)}\left(\varepsilon_x(1-\nu)+\varepsilon_y\nu+\varepsilon_z\nu\right)$$

$$\sigma_{xy}=\frac{E}{(1+\nu)(1-2\nu)}\left(2\varepsilon_{xy}\frac{1-2\nu}{2}\right)=2G\varepsilon_{xy}$$

Stiffness matrix for 3d elastic problem

$$[K_e]=\int_{V_e}[B]^T[D][B]\,dV$$

$$[B]^T=\begin{bmatrix} 0 & 0 & \frac{\partial[N]^T}{\partial z} & 0 & \frac{\partial[N]^T}{\partial y} & \frac{\partial[N]^T}{\partial x} \\ 0 & \frac{\partial[N]^T}{\partial y} & 0 & \frac{\partial[N]^T}{\partial x} & \frac{\partial[N]^T}{\partial z} & 0 \\ \frac{\partial[N]^T}{\partial x} & 0 & 0 & \frac{\partial[N]^T}{\partial y} & 0 & \frac{\partial[N]^T}{\partial z} \end{bmatrix} \quad [D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{(1-2\nu)}{2} & 0 & 0 \\ & & & & \frac{(1-2\nu)}{2} & 0 \\ sym & & & & & \frac{(1-2\nu)}{2} \end{bmatrix} \quad [B]=\begin{bmatrix} \frac{\partial[N]}{\partial x} & 0 & 0 \\ 0 & \frac{\partial[N]}{\partial y} & 0 \\ 0 & 0 & \frac{\partial[N]}{\partial z} \\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x} & 0 \\ 0 & \frac{\partial[N]}{\partial z} & \frac{\partial[N]}{\partial y} \\ \frac{\partial[N]}{\partial z} & 0 & \frac{\partial[N]}{\partial x} \end{bmatrix}$$

$$[B]^T[D]=\frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{c|c|c|c|c|c} (1-\nu)\frac{\partial[N]^T}{\partial x} & \nu\frac{\partial[N]^T}{\partial x} & \nu\frac{\partial[N]^T}{\partial x} & \frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial y} & 0 & \frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial z} \\ \hline \nu\frac{\partial[N]^T}{\partial y} & (1-\nu)\frac{\partial[N]^T}{\partial y} & \nu\frac{\partial[N]^T}{\partial y} & \frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial x} & \frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial z} & 0 \\ \hline \nu\frac{\partial[N]^T}{\partial z} & \nu\frac{\partial[N]^T}{\partial z} & (1-\nu)\frac{\partial[N]^T}{\partial z} & 0 & \frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial y} & \frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial x} \end{array}\right]$$

$$[B]^T[D][B]=\frac{E}{(1+\nu)(1-2\nu)}\left[\begin{array}{c|c|c}
\begin{array}{l}(1-\nu)\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial z}\end{array} &
\begin{array}{l}\nu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x}\end{array} &
\begin{array}{l}\nu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial z}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial x}\end{array} \\ \hline
\begin{array}{l}\nu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y}\end{array} &
\begin{array}{l}(1-\nu)\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial z}\end{array} &
\begin{array}{l}\nu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial z}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial y}\end{array} \\ \hline
\begin{array}{l}\nu\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial x}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial z}\end{array} &
\begin{array}{l}\nu\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial y}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial z}\end{array} &
\begin{array}{l}(1-\nu)\frac{\partial[N]^T}{\partial z}\frac{\partial[N]}{\partial z}+\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\\+\frac{1-2\nu}{2}\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}\end{array}
\end{array}\right]$$

Load vector

$$\{F_e\}=-\int_{V_e}\left[\overline{N}\right]^T\{M\}dV-\int_{S_e}\left[\overline{N}\right]^T\{p\}dS-\{P\}.$$

$$\left[\overline{N}\right]=\begin{bmatrix}[N] & 0 & 0\\ 0 & [N] & 0\\ 0 & 0 & [N]\end{bmatrix}$$

$$\{P\}=\begin{Bmatrix}\{P_x\}\\ \{P_y\}\\ \{P_z\}\end{Bmatrix}\qquad \{M\}=\begin{Bmatrix}m_x\\ m_y\\ m_z\end{Bmatrix}\qquad \{p\}=\begin{Bmatrix}p_x\\ p_y\\ p_z\end{Bmatrix}$$

Practical example: direct minimization of functional

$$J=\int_V\left(\int_0^{\bar\varepsilon}\bar\sigma(\bar\varepsilon)d\bar\varepsilon\right)d\mathrm{V}+\frac{3k_V}{2}\int_V\varepsilon_0{}^2d\mathrm{V}-\int_S\sigma_\tau U_\tau\mathrm{dS}\xrightarrow[\min(U_i)]{}$$

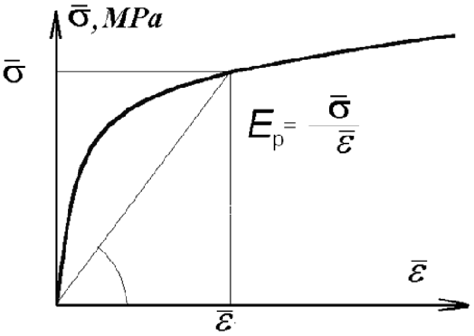
$$E_p=\frac{\bar\sigma(\bar\varepsilon)}{\bar\varepsilon}$$

$$k_V=\frac{E}{3(1-2\nu)}$$

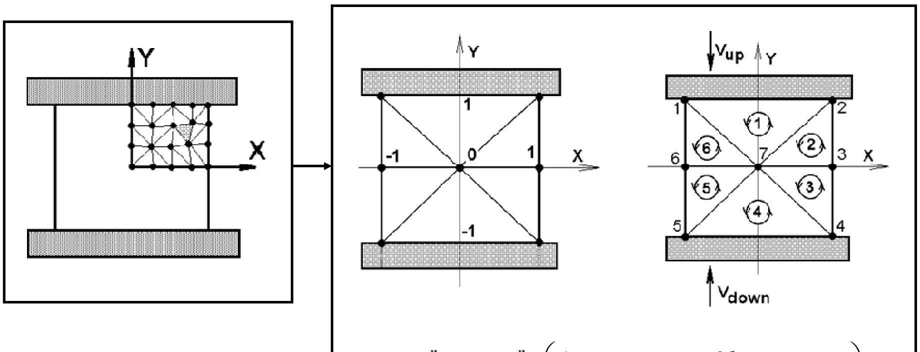
$$J=\frac{1}{2}\int_V E_p\bar\varepsilon^2d\mathrm{V}+\frac{3k_V}{2}\int_V\varepsilon_0{}^2d\mathrm{V}$$

$$\bar\varepsilon=\frac{\sqrt{2}}{3}\sqrt{(\varepsilon_x-\varepsilon_y)^2+(\varepsilon_x)^2+(\varepsilon_y)^2+6(\varepsilon_{xy}^2)}$$

$$\sigma_{ij}=\delta_{ij}3k_V\varepsilon_0+\frac{2\bar\sigma(\bar\varepsilon)}{3\bar\varepsilon}\varepsilon_{ij}$$

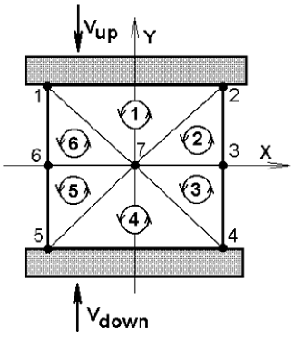


Example of discretisation



$$J=\sum_{i_e=1}^{n_e}J_e=\sum_{i_e=1}^{n_e}\left(\frac{1}{2}\int_V E_p\bar\varepsilon_e^2d\mathrm{V}+\frac{3k_V}{2}\int_V\varepsilon_{0e}{}^2d\mathrm{V}\right)$$

Data and structure:



n_e=6

n_h=7

$U_x(1)=0$; $U_y(1)=U_{up}$;	$Status_1 = 1$	$X_1=-1$ $Y_1=1$
$U_x(2)=0$; $U_y(2)=U_{up}$;	$Status_2 = 1$	$X_2=1$ $Y_2=1$
$U_x(3)=0$; $U_y(3)=0$;	$Status_3 = 0$	$X_3=1$ $Y_3=0$
$U_x(4)=0$; $U_y(4)=U_{down}$;	$Status_4 = 1$	$X_4=1$ $Y_4=-1$
$U_x(5)=0$; $U_y(5)=U_{down}$;	$Status_5 = 1$	$X_5=-1$ $Y_5=-1$
$U_x(6)=0$; $U_y(6)=0$;	$Status_6 = 0$	$X_6=-1$ $Y_6=0$
$U_x(7)=0$; $U_y(7)=0$;	$Status_7 = 0$	$X_7=0$ $Y_7=0$

nop(1,1)=1; nop(2,1)=7; nop(3,1)=2;
nop(1,2)=7; nop(2,2)=3; nop(3,2)=2;
nop(1,3)=3; nop(2,3)=7; nop(3,3)=4;
nop(1,4)=7; nop(2,4)=5; nop(3,4)=4;
nop(1,5)=7; nop(2,5)=6; nop(3,5)=5;
nop(1,6)=6; nop(2,6)=7; nop(3,6)=1.

Calculation of functional

$$J=\sum_{i_e=1}^{n_e} J_e=\sum_{i_e=1}^{n_e}\left(\frac{1}{2}\int_V E_p\bar{\varepsilon}_e^2 dV+\frac{3k_V}{2}\int_V \varepsilon_{0e}{}^2 dV\right)\xrightarrow[\min(U_x,U_y)]{}$$

$$\bar{\varepsilon}=\frac{\sqrt{2}}{3}\sqrt{\left(\varepsilon_x-\varepsilon_y\right)^2+\left(\varepsilon_x\right)^2+\left(\varepsilon_y\right)^2+6\left(\varepsilon_{xy}^2\right)}$$

$$\boxed{\varepsilon_0=\frac{\varepsilon_x+\varepsilon_y}{3}}$$

$$\varepsilon_x=\frac{\partial U_x}{\partial X}=U_{xi}\frac{\partial N_i}{\partial X}+U_{xj}\frac{\partial N_j}{\partial X}+U_{xk}\frac{\partial N_k}{\partial X}=\frac{1}{2A}\left(U_{xi}b_i+U_{xj}b_j+U_{xk}b_k\right)$$

$$\varepsilon_y=\frac{\partial U_y}{\partial Y}=U_{yi}\frac{\partial N_i}{\partial Y}+U_{yj}\frac{\partial N_j}{\partial Y}+U_{yk}\frac{\partial N_k}{\partial Y}=\frac{1}{2A}\left(U_{yi}c_i+U_{yj}c_j+U_{yk}c_k\right)$$

$$\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial U_x}{\partial Y}+\frac{\partial U_y}{\partial X}\right)=\frac{1}{2}\left(U_{xi}\frac{\partial N_i}{\partial Y}+U_{xj}\frac{\partial N_j}{\partial Y}+U_{xk}\frac{\partial N_k}{\partial Y}+U_{yi}\frac{\partial N_i}{\partial X}+U_{yj}\frac{\partial N_j}{\partial X}+U_{yk}\frac{\partial N_k}{\partial X}\right)=$$

$$\frac{1}{4A}\left(U_{xi}c_i+U_{xj}c_j+U_{xk}c_k+U_{yi}b_i+U_{yj}b_j+U_{yk}b_k\right)$$

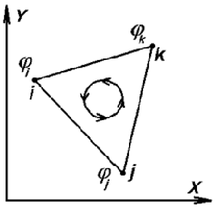
$$U_x=U_{xi}N_i+U_{xj}N_j+U_{xk}N_k$$

$$U_y=U_{yi}N_i+U_{yj}N_j+U_{yk}N_k$$

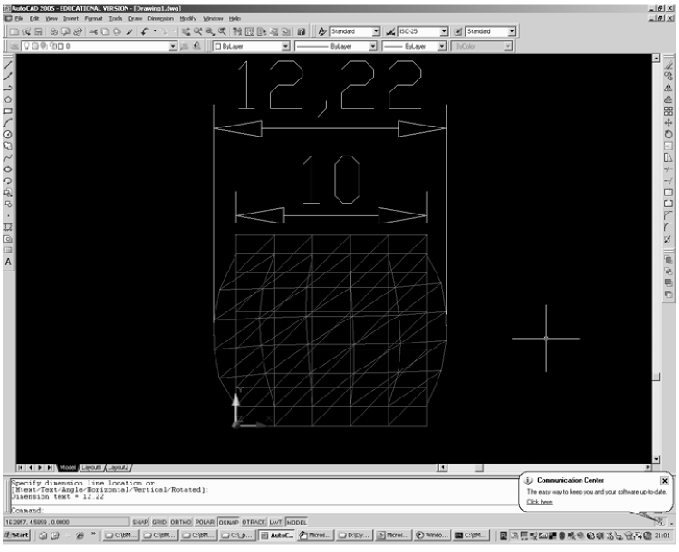
$$E_p = 1$$
$$3k_{Ve} = 10$$

$$J = \sum_{i_e=1}^{n_e} J_e = \sum_{i_e=1}^{n_e} \left(\frac{1}{2} E_p \bar{\varepsilon}_e^2 A_e + \frac{3k_{Ve}}{2} \varepsilon_{0e}^2 A_e \right)$$

$$2A_e = \begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix} =$$
$$= X_j Y_k + X_i Y_j + X_k Y_i - X_j Y_i - X_k Y_j - X_i Y_k$$



Implementation in AutoCAD environment (Visual Basic)



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Specify dimension line location or
[Mtext/Text/Angle/Horizontal/Vertical/Rotated]:
Dimension text = 12.22

Command:
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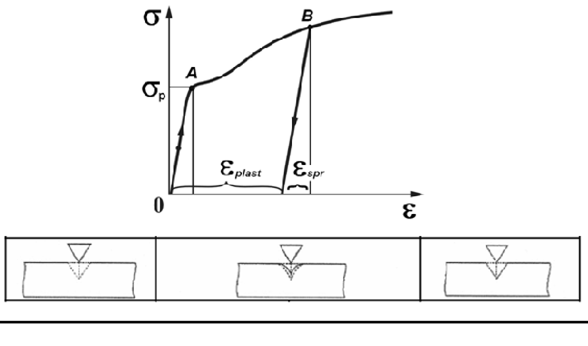
Lection 2

The theoretical foundations of the theory of small elastic plastic deformation; models of the mechanical properties of elastic-plastic materials; theorem on unloading.

The theoretical foundations of the theory of small elastic plastic deformation

Basics:

- The values of elastic and plastic deformations are near
- Volumetric strain is elastic
- The load is monotonic and simple
- Deformation values are small



Boundary problem.

a) Cauchy equations :

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

b) Equilibrium equations:

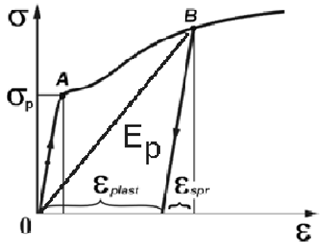
$$\sigma_{ij,j} = 0$$

$$E_p = \frac{\sigma}{\varepsilon}$$

c) Stress-strain relationship:

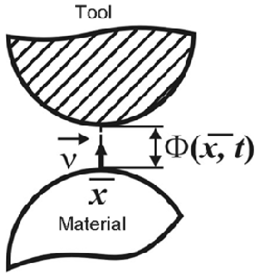
$$\sigma_{ij} = \delta_{ij} 3k_V \varepsilon_0 + \frac{E_p}{\left(1+\nu_p\right)}\left(\varepsilon_{ij} - \delta_{ij}\varepsilon_0\right)$$

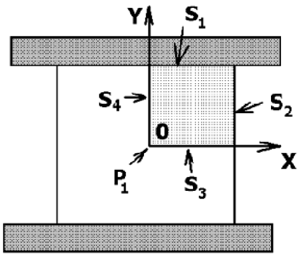
$$\nu_p = \frac{1}{2} - \frac{1-2\nu}{2E} E_p$$



Method of elastics solutions (A.A.Iliushin)

Non linearity contact conditions





For contact: $u_\nu(x,t) = \Phi(x,t)$

$\sigma_\nu(x,t) < 0$

For free surface: $u_\nu(x,t) < \Phi(x,t)$

$\sigma_\nu(x,t) = 0$

Non linearity of friction on S_1

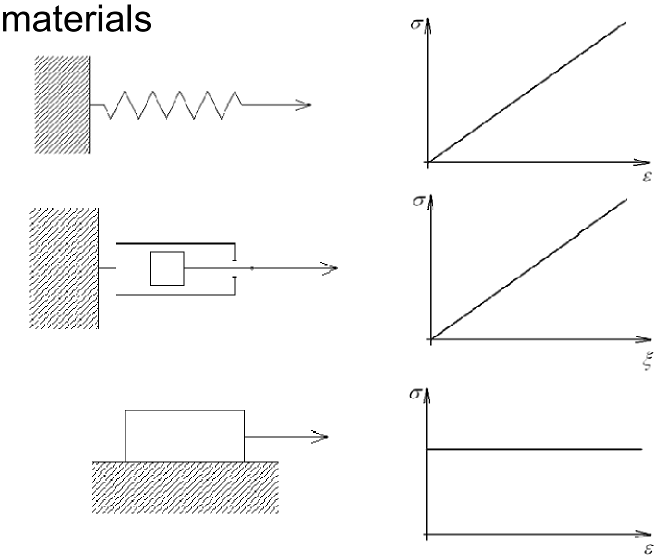
$$u_y = -u_{tool} \; ;$$

$$\tau_{fric} = \sigma_{xy} = \sigma_{fric}\frac{dU_x}{\left|dU_x\right|} = \sigma_{fric}\, sign\,\left(dU_x\right)$$

$$\sigma_{fric} = f_{tr}\frac{\sigma_p}{\sqrt{3}} \qquad \left|\vec{\sigma}_{fric}\right| = \mathrm{f}\left|\sigma_\nu\right|$$

$$\sigma_{fric} = f_{tr}\frac{\sigma_p}{\sqrt{3}}\left(1-\exp\left(-\frac{1.25\sigma_\nu}{\sigma_\tau}\right)\right)$$

Models of the mechanical properties of elastic-plastic materials

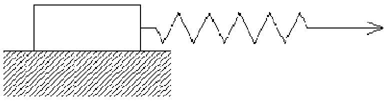


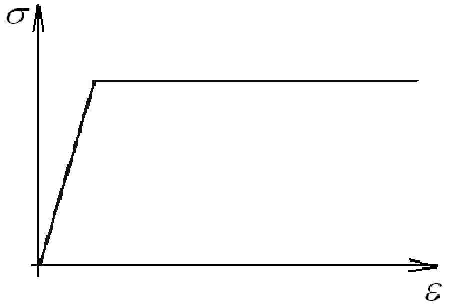
$$\overline{\sigma} = E\overline{\varepsilon}$$

$$\overline{\sigma} = \mu\dot{\varepsilon} = \mu\overline{\xi}$$

$$\overline{\sigma} = \sigma_p$$

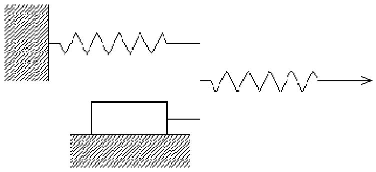
Elastic – plastic material

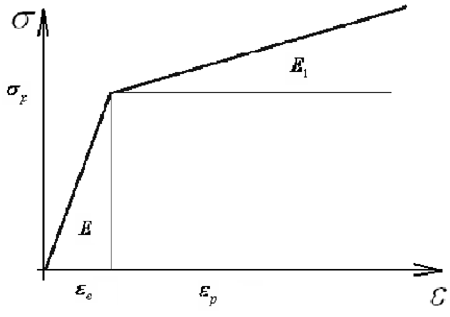




$$\begin{cases} \overline{\sigma} = E\overline{\varepsilon} & \forall\ \overline{\varepsilon} < \varepsilon_p \\ \overline{\sigma} = \sigma_p & \forall\ \overline{\varepsilon} \geq \varepsilon_p \end{cases}$$

Elastic- plastic material with hardening

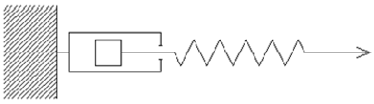




$$\begin{cases} \bar{\sigma} = E\bar{\varepsilon}_e \quad \forall\ \bar{\sigma} < \sigma_p \\ \bar{\sigma} = \sigma_p + E_1\varepsilon_p \quad \forall\ \bar{\sigma} \ge \sigma_p \end{cases}$$

Maxwell material

(James Clerk Maxwell proposed the model in 1867).

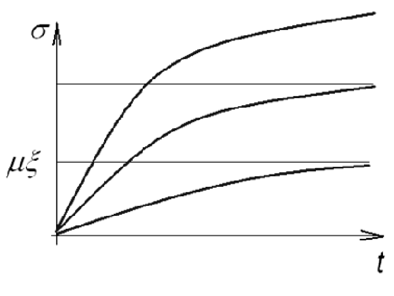


$$\varepsilon = \varepsilon_e + \varepsilon_p$$

$$\dot{\varepsilon} = \xi = \xi_e + \xi_p = \frac{d}{dt}\left(\frac{\sigma}{E}\right) + \frac{\sigma}{\mu} = \frac{1}{E}\frac{d\sigma}{dt} + \frac{\sigma}{\mu}$$

$$T = \frac{\mu}{E}$$

$$\frac{d\sigma}{dt} + \frac{\sigma}{T} = E\frac{d\varepsilon}{dt}$$



$$\sigma = \exp\left(-\frac{t}{T}\right)\left[E\int_0^t \frac{d\varepsilon}{dt}\exp\left(\frac{t}{T}\right)dt + \sigma_0\right]$$

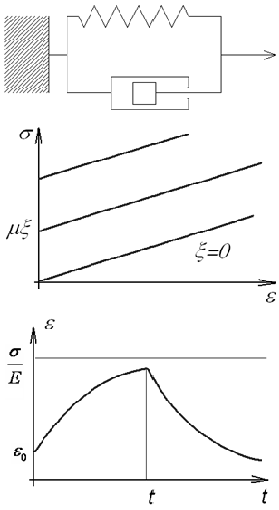
$$\sigma = \mu\xi\left(1 - \exp\left(-\frac{t}{T}\right)\right)$$

$$\sigma = \sigma_0 \exp\left(-\frac{t}{T}\right)$$

http://home.agh.edu.pl/~milenin/Dydaktyka/MN/L_8/LR_8_reo.htm

Kelvin–Voigt material

(named after the British physicist and engineer William Thomson, 1st Baron Kelvin and after German physicist Woldemar Voigt)



$$\sigma = \sigma_e + \sigma_p$$

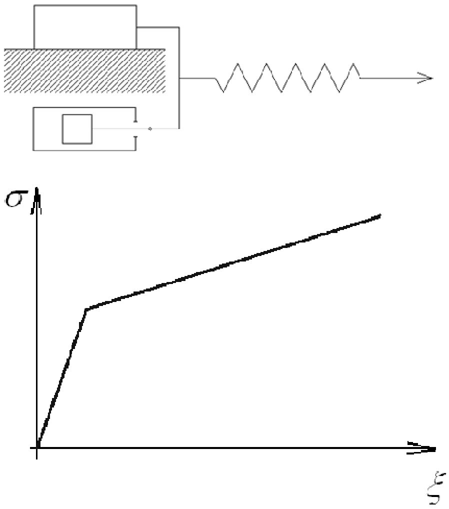
$$\sigma = E\varepsilon + \mu \frac{d\varepsilon}{dt}$$

$$T = \frac{\mu}{E}$$

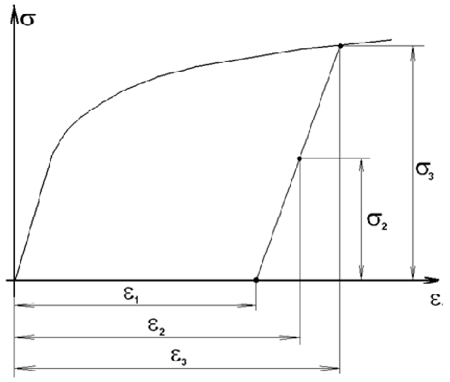
$$\varepsilon = \exp\left(-\frac{t}{T}\right)\left[\frac{1}{\mu}\int_0^t \sigma \exp\left(-\frac{t}{T}\right)dt + \varepsilon_0\right]$$

$$\varepsilon = \varepsilon_0 \exp\left(-\frac{t}{T}\right) + \frac{\sigma}{E}\left[1 - \exp\left(-\frac{t}{T}\right)\right]$$

Perzyna material



Theorem on unloading



Modeling of unloading

a) Solution of elastic -plastic problem for active loading σ_{ij} ε_{ij}

b) Simulation of elastic deformation of sample with inverse of load. ${\sigma^e}_{ij}$ ${\varepsilon_{ij}}^e$

c) Real stress and strain during unloading is equal sum of solution a) and b)

$$\varepsilon_{ij}^{odpr} = \varepsilon_{ij} + \varepsilon_{ij}^{e}$$

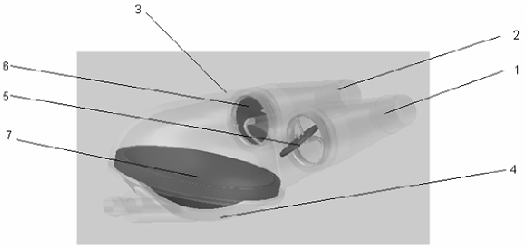
$$\sigma_{ij}^{odpr} = \sigma_{ij} + \sigma_{ij}^{e}$$

d) Final stage of deformation is determinate by analyzing of follow condition on contact surface $\sigma_n^{odpr} \le 0$

Theorem on unloading. Practical example.

Blood chamber of POLVAD EXT

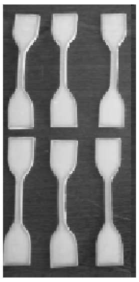
Blood chamber of POLVAD (*)

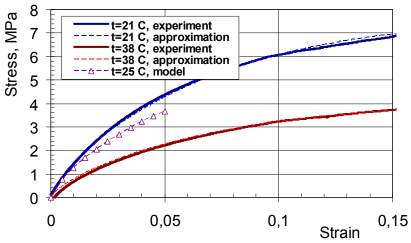


1 – inlet connector, 2 – outlet connector, 3 – blood chamber, 4 – pneumatic chamber,
5 – inlet valve, 6 – outlet valve, 7 – membrane in filling phase

---

Model of PU ChronoFlex C 55D*





$$\sigma = A\varepsilon^{n_1}\exp(n_2\varepsilon)\exp(-n_3 t)$$

- Testing machine MTS Criterion, standards: ISO 527-2 and ASTM D 638.
- 12 spacemen’s was used (6 for 21 C and 6 for 38 C),
- base of measurement was 20 mm;
- dimensions of samples h=4.09± 0.04mm, b=6.11 ±0.07 mm, L=35 mm;
- velocity of tension 10 mm/min.

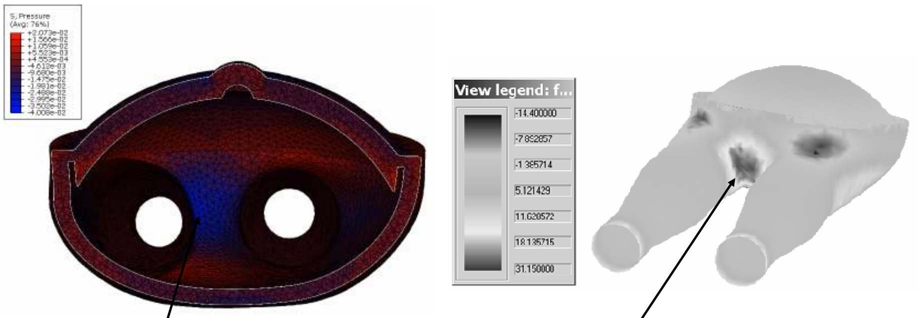
A=	104,5357
n_1=	0,750084
n_2=	-3,36967
n_3=	0,037283

* Research was perform in Institute of Artificial Hertz, Zabrze, Poland

POLVAD

ABAQUS

Authors code

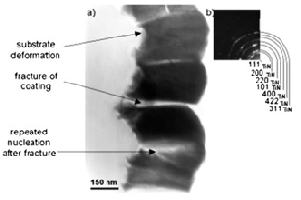


Distribution of pressure (mean stress)

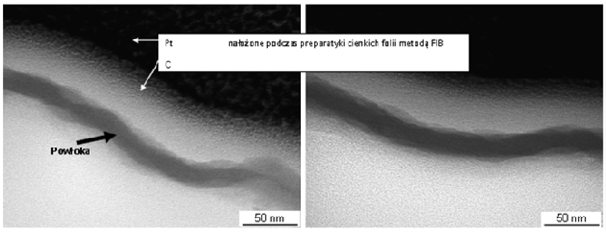
ABAQUS: 35 kPa,
Authors code: 31 kPa,

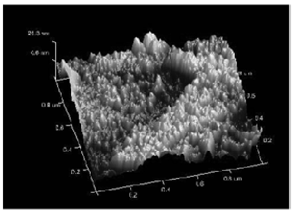
ABAQUS: 14 kPa,
Authors code: 13 kPa,

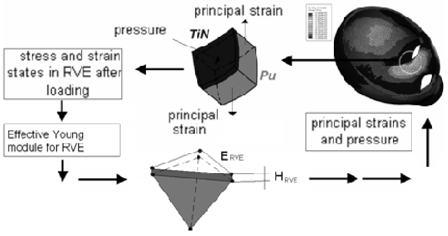
Finite element solution:Microscale



7 a microstructure (TEM) of 350 nm TiN coating deposited on polyurethane and b electron diffraction computed using program described in Ref. 19







stress and strain states in RVE after loading

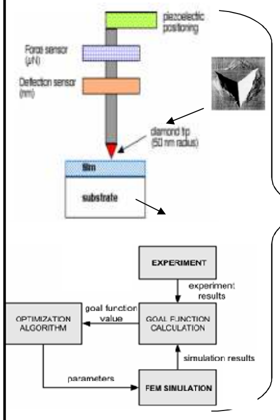
Effective Young module for RVE

principal strains and pressure

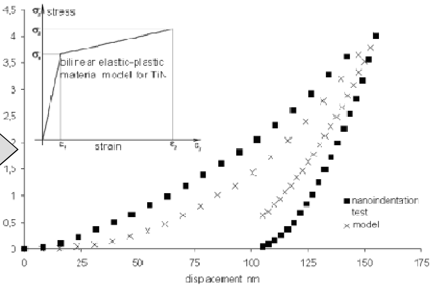
•M. Kopernik, A. Milenin, R. Major and J. M. Lackner Identification of material model of TiN using numerical simulation of nanoindentation test // Materials Science and Technology (2011) VOL 27 NO 3, p.604-616, DOI 10.1179/026708310X12756557336193

Finite element solution: Microscale

Micro Representative Volume model of wall of VAD



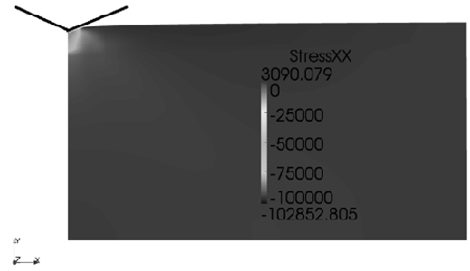
$\varepsilon_1 = 0.009, \sigma_1 = 2{,}614$ GPa
$\varepsilon_2 = 0.166, \sigma_2 = 9{,}107$ GPa



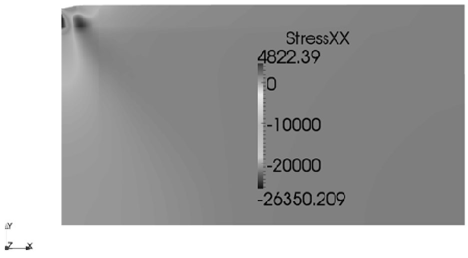
Kopernik, M., Milenin, A., Major, R., Lackner, J.M., 2011, Identification of material model of TiN using numerical simulation of nanoindentation test, Materials Science and Technology, 26 (in press).

Numerical simulation of nanoindentation test

Loading

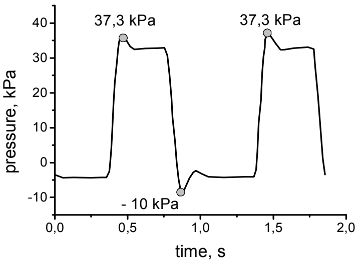


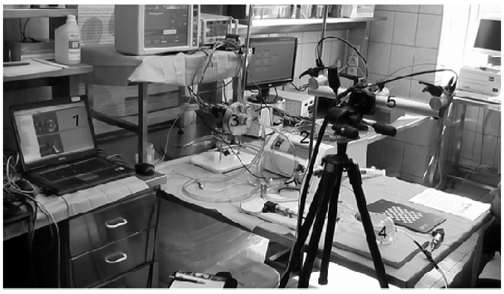
Unloading

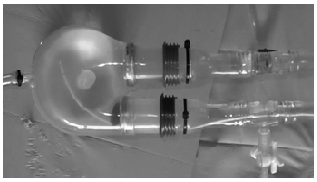


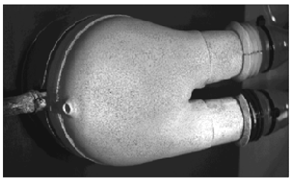
Experimental VALIDATION

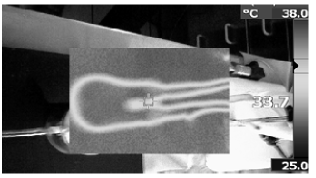
DIGITAL IMAGE CORRELATION







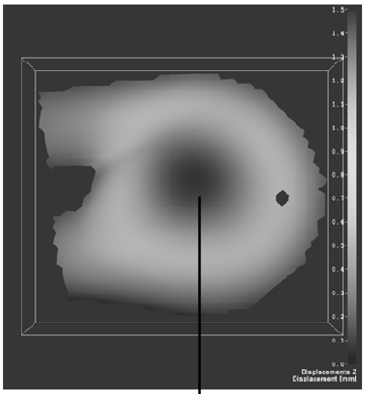




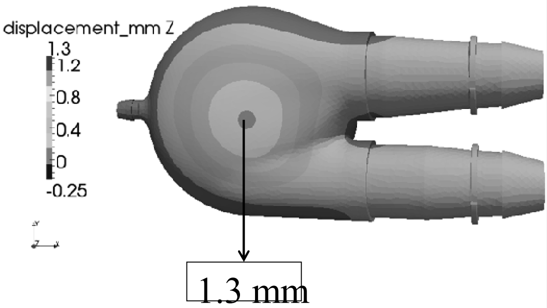
Experimental VALIDATION

DIC versus FEM

The *Z*-directional displacement on the external surface of the POLVAD_EXT at a pressure of 37.3 kPa.



1.4 mm



1.3 mm

Lection 3

The theoretical foundations of the theory of plastic flow of incompressible materials; the variation principle of Markov; the model materials under large deformations; features of using finite element method to solve problems in the theory of plastic flow

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The theoretical foundations of the theory of plastic flow of incompressible materials

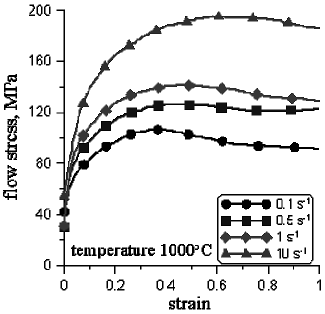
$$\xi_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

$$\sigma_{ij,j} = 0$$

$$\sigma_{ij} = \delta_{ij}\sigma_0 + \frac{2\overline{\sigma}}{3\overline{\xi}}\xi_{ij} \qquad s_{ij} = \frac{2\overline{\sigma}}{3\overline{\xi}}\xi_{ij} = 2\mu\xi_{ij}$$

$$\xi_0 = 0$$

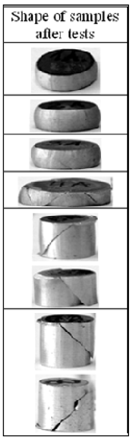
$$\xi_0 = \frac{1}{3}div(\vec{v}) = 0$$



$$\overline{\sigma} = \overline{\sigma}\left(\overline{\varepsilon}, \overline{\xi}, t\right)$$

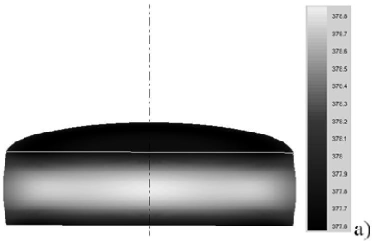
$$\mu = \frac{\overline{\sigma}}{3\overline{\xi}}$$

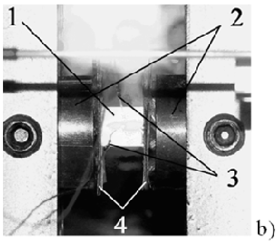
Mechanical properties of the workable metal

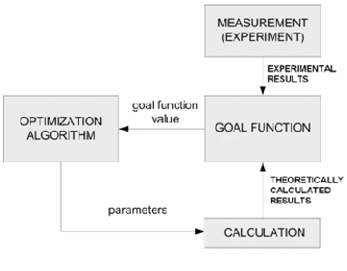


$$\overline{\sigma} = \overline{\sigma}\left(\overline{\varepsilon}, \overline{\xi}, t\right)$$

Problems:






MEASUREMENT (EXPERIMENT)

EXPERIMENTAL RESULTS

GOAL FUNCTION

goal function value

OPTIMIZATION ALGORITHM

parameters

CALCULATION

THEORETICALLY CALCULATED RESULTS

$$\delta = \sum_{m}^{m_{test}} \sum_{n=1}^{n_{pnt}} \left(P_{mn}^{calc} - P_{mn}^{exp}\right)^2$$

$$\overline{\sigma} = \overline{\sigma}\left(\overline{\varepsilon}, \overline{\xi}, t\right)$$

$$\overline{\sigma} = A\overline{\varepsilon}^{n} \exp\left(-B\overline{\varepsilon}\right)\overline{\xi}^{m} \exp\left(-Ct\right)$$

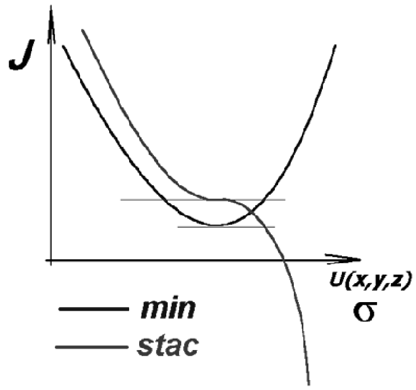
The variation principle of Markov

Elastic-plastic formulation

$$J=\int_V\int_0^{\bar\varepsilon}\bar\sigma(\bar\varepsilon)d\bar\varepsilon dV+\frac{3}{2}\int_V k_V\varepsilon_0^2dV-\int_{S_\sigma}\sigma_i\,u_i\,dS$$

Flow formulation

$$J=\int_V\left(\int_0^{\bar\xi}\bar\sigma(\bar\xi)d\bar\xi\right)dV+\int_V\sigma_0\xi_0dV-\int_S\sigma_iv_idS$$



Penalty method

$$J=\int_V\left(\int_0^{\bar\xi}\bar\sigma(\bar\xi)d\bar\xi\right)dV+K_{pen}\int_V\xi_0^2dV-\int_S\sigma_iv_idS$$

K

Features of using finite element method to solve problems in the theory of plastic flow

$$\{\sigma\} = \sigma_0[I]^T + \{s\} = \sigma_0[I]^T + [D]\{\xi\}$$

$$[I] = [1 \quad 1 \quad 0] \qquad [I]^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

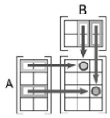
$$\{\xi\} = \begin{Bmatrix} \xi_{xx} \\ \xi_{yy} \\ 2\xi_{xy} \end{Bmatrix} \qquad \xi_0 = \frac{1}{3}[I]\{\xi\} = \frac{1}{3}[1 \quad 1 \quad 0]\begin{Bmatrix} \xi_{xx} \\ \xi_{yy} \\ 2\xi_{xy} \end{Bmatrix} \qquad \sigma_0 = \frac{1}{3}[I]\{\sigma\} = \frac{1}{3}[1 \quad 1 \quad 0]\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ 2\sigma_{xy} \end{Bmatrix}$$

$$W = \int_V \frac{1}{2}\{\xi\}^T\{s\}dV + \int_V \sigma_0[I]\{\xi\}dV \qquad \mathrm{J}(\mathrm{v_i}, \sigma_0) = \frac{1}{2}\int_V \mu\bar{\xi}^2 \mathrm{d}V + \int_V \xi_0\sigma_0 \mathrm{d}V - \int_S \sigma_i \nu_\mathrm{i}\, \mathrm{d}S$$

$$W_p = \int_S \{v\}^T\{p\}dS$$

$$J = \int_V \frac{1}{2}\{\xi\}^T\{s\}dV + \int_V \sigma_0[I]\{\xi\}dV - \int_S \{v\}^T\{p\}dS \qquad \{p\} = \begin{Bmatrix} p_x \\ p_y \end{Bmatrix}$$

$$J = \int_V \frac{1}{2}\{\xi\}^T\{s\}dV + \int_V \sigma_0[I]\{\xi\}dV - \int_S \{v\}^T\{p\}dS$$



$$\{s\} = [D]\{\xi\} = \begin{Bmatrix} 2\mu\xi_{xx} \\ 2\mu\xi_{yy} \\ \mu 2\xi_{xy} \end{Bmatrix} \qquad \{\xi\} = \begin{Bmatrix} \xi_{xx} \\ \xi_{yy} \\ 2\xi_{xy} \end{Bmatrix}$$

$$\sigma_x = \sigma_0 + 2\mu\xi_x$$
$$\sigma_y = \sigma_0 + 2\mu\xi_y$$
$$\sigma_{xy} = \mu 2\xi_{xy}$$

$$[D] = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \qquad \{s\} = \begin{Bmatrix} 2\mu\xi_{xx} \\ 2\mu\xi_{yy} \\ \mu 2\xi_{xy} \end{Bmatrix}$$

$$J = \int_V \frac{1}{2}\{\xi\}^T[D]\{\xi\}dV + \int_V \sigma_0[I]\{\xi\}dV - \int_S \{v\}^T\{p\}dS$$

$$\{\xi\}=\begin{Bmatrix}\xi_x\\ \xi_y\\ 2\xi_{xy}\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}v_x\\ v_y\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}[N]\{\mathrm{v}_\mathrm{x}\}\\ [N]\{\mathrm{v}_\mathrm{y}\}\end{Bmatrix}=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}\begin{Bmatrix}\{\mathrm{v}_\mathrm{x}\}\\ \{\mathrm{v}_\mathrm{y}\}\end{Bmatrix}=[B]\{\mathrm{v}\}$$

$$v_x=[N]\{\mathrm{v}_\mathrm{x}\}$$
$$v_y=[N]\{\mathrm{v}_\mathrm{y}\}$$

$$\{\xi\}=[B]\{\mathrm{v}\}$$

$$[B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}$$

$$\{v\}=\begin{Bmatrix}v_x\\ v_y\end{Bmatrix}=\begin{bmatrix}[N] & 0\\ 0 & [N]\end{bmatrix}\begin{Bmatrix}\{\mathrm{v}_x\}\\ \{\mathrm{v}_y\}\end{Bmatrix}=[\overline{N}]\{\mathrm{v}\}$$

$$[\overline{N}]=\begin{bmatrix}N_1 & N_2\ldots & N_p & 0 & 0\ldots & 0\\ 0 & 0\ldots & 0 & N_1 & N_2\ldots & N_p\end{bmatrix}=\begin{bmatrix}[N] & 0\\ 0 & [N]\end{bmatrix}$$

Approximation of mean stress: $\sigma_0=[N]\{\sigma_0\}$ $\sigma_0=[H]\{\sigma_0\}$

$$\xi_0=[I]\{\xi\}=[1\quad 1\quad 0]\begin{Bmatrix}\xi_{xx}\\ \xi_{yy}\\ 2\xi_{xy}\end{Bmatrix}=[I][B]\{\mathrm{v}\}=[E]\{\mathrm{v}\}$$

$$[E]=[I][B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & \frac{\partial[N]}{\partial y}\end{bmatrix}$$

$$\{s\}=[D][B]\{\mathrm{v}\}$$

$$J=\int_{\mathrm{V}}\frac{1}{2}\{\mathrm{v}\}^T[B]^T[D][B]\{\mathrm{v}\}dV+\int_{\mathrm{V}}\sigma_0[E]\{\mathrm{v}\}dV-\int_S\{\mathrm{v}\}^T\left[\overline{N}\right]^T\{p\}dS=0$$

$$J=\int_{\mathrm{V}}\frac{1}{2}\{\mathrm{v}\}^T[B]^T[D][B]\{\mathrm{v}\}dV+\int_{\mathrm{V}}[H]\{\sigma_0\}[E]\{\mathrm{v}\}dV-\int_{\mathrm{S}}\{\mathrm{v}\}^T\left[\overline{N}\right]^T\{p\}dS=0$$

$$\frac{\partial J}{\partial\{\mathrm{v}\}}=\left(\int_{\mathrm{V}}[B]^T[D][B]dV\right)\{\mathrm{v}\}+\left(\int_{\mathrm{V}}[E]^T[H]dV\right)\{\sigma_0\}-\int_{\mathrm{S}}\left[\overline{N}\right]^T\{p\}dS=0$$

$$\frac{\partial J}{\partial\{\sigma_0\}}=\left(\int_{\mathrm{V}}[H]^T[E]dV\right)\{\mathrm{v}\}=0$$

$$[K]\{\mathrm{v},\sigma_0\}+\{F\}=0$$

Stiffness matrix [K]:

$$[B]^T[D][B]=\begin{bmatrix}\frac{\partial[N]^T}{\partial x} & 0 & \frac{\partial[N]^T}{\partial y}\\ 0 & \frac{\partial[N]^T}{\partial y} & \frac{\partial[N]^T}{\partial x}\end{bmatrix}\begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}$$

$$[B]^T[D]=\begin{bmatrix}2\mu\frac{\partial[N]^T}{\partial x} & 0 & \mu\frac{\partial[N]^T}{\partial y}\\ 0 & 2\mu\frac{\partial[N]^T}{\partial y} & \mu\frac{\partial[N]^T}{\partial x}\end{bmatrix}$$

$$[B]^T[D][B]=\begin{bmatrix}2\mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} & \mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y}\\ \mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y} & 2\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}+\mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}\end{bmatrix}$$

Stiffness matrix [K]:

$$\frac{\partial J}{\partial\{\mathrm{v}\}}=\left(\int_{\mathrm{V}}[B]^T[D][B]dV\right)\{\mathrm{v}\}+\left(\int_{\mathrm{V}}[E]^T[H]dV\right)\{\sigma_0\}-\int_{\mathrm{S}}[\overline{N}]^T\{p\}dS=0$$

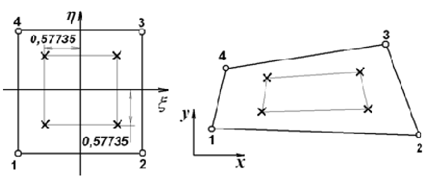
$$\frac{\partial J}{\partial\{\sigma_0\}}=\left(\int_{\mathrm{V}}[H]^T[E]dV\right)\{\mathrm{v}\}=0$$

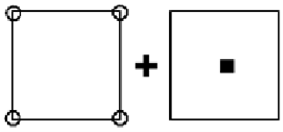
$$[K]=\int_V\left[\begin{array}{c|c|c} \begin{gathered}2\mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\\+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\end{gathered} & \mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x} & \frac{\partial[N]^T}{\partial x}[H] \\ \hline \mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y} & \begin{gathered}2\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}+\\+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\end{gathered} & \frac{\partial[N]^T}{\partial y}[H] \\ \hline [H]^T\frac{\partial[N]}{\partial x} & [H]^T\frac{\partial[N]}{\partial y} & 0 \end{array}\right]dV.$$

Load vector:

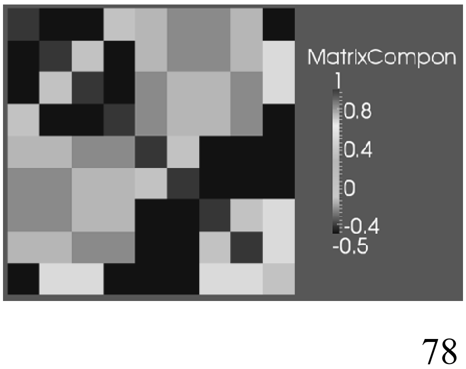
$$\{F\}=\int_S\begin{Bmatrix}[N]^T p_x\\ [N]^T p_y\\ 0\end{Bmatrix}dS$$

Graphic interpretation of local stiffness matrix [K]:



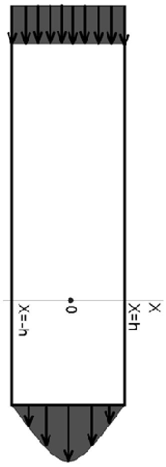


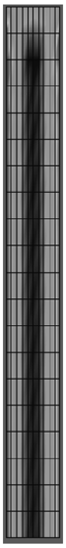
$$[K]=\int_V\left[\begin{array}{c|c|c} \begin{gathered}2\mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\\+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\end{gathered} & \mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x} & \frac{\partial[N]^T}{\partial x}[H] \\ \hline \mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y} & \begin{gathered}2\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}+\\+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}\end{gathered} & \frac{\partial[N]^T}{\partial y}[H] \\ \hline [H]^T\frac{\partial[N]}{\partial x} & [H]^T\frac{\partial[N]}{\partial y} & 0 \end{array}\right]dV.$$

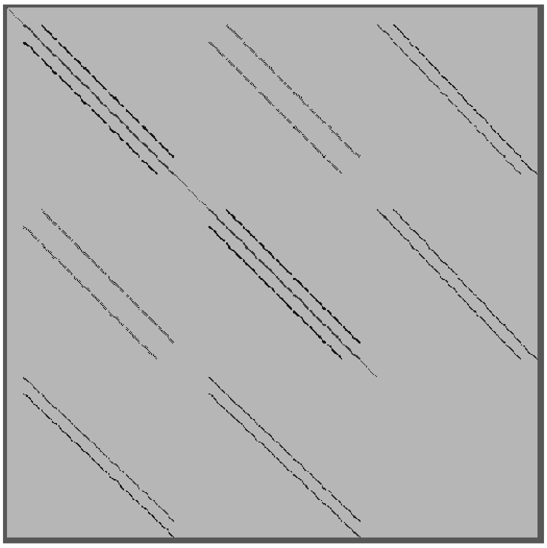


Graphic interpretation of global stiffness matrix [K]:

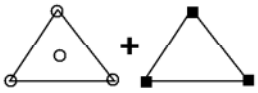
$$v_z = \frac{3}{2}\frac{B}{h^3}\left(h^2 - x^2\right)$$



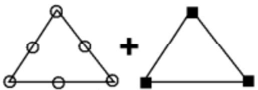




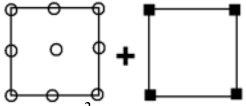
Stabile interpolations *N* (○) and *H* (■)



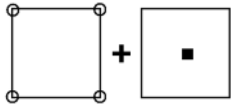
O(h) T3B1 / 3C



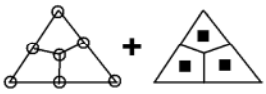
O(h^2) T6 / 3C



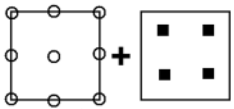
O(h^2) Q9 / 4C



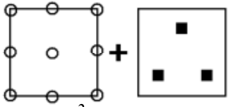
O(h2) T6B1 / 3D



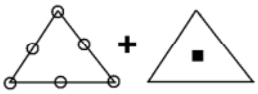
O(h) Q4 / 1D



O(h^2) Q9 / 4D



O(h^2) Q9 / 3D



O(h) T6 / 1D

Example of FORTRAN code

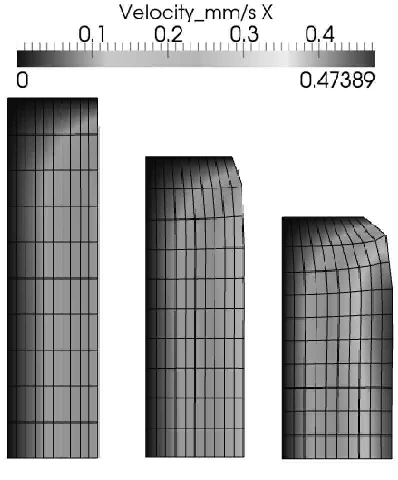
```
          DO P=1, ELSlv%N_p
                    DO N=1,NBN
                              Row1 = N;
                              Row2 = NBN + N;
                              Row3 = 2*NBN + N;
                              DO I=1,NBN
                                        C1 = I;
                                        C2 = NBN + I;
                                        C3 = 2*NBN + I;
feSM(Row1,C1)=feSM(Row1,C1) + m*(2*Ndx(N)*Ndx(i)+Ndy(N)*Ndy(i))*DetJ;
feSM(Row1,C2)=feSM(Row1,C2) + m*Ndx(i)*Ndy(N)*DetJ;

if (i<=NBNp) feSM(Row1,C3)=feSM(Row1,C3) + Ndx(N)*detJ*Hk(i,p);

feSM(Row2,C1)=feSM(Row2,C1) + m*Ndx(N)*Ndy(i)*DetJ;
feSM(Row2,C2)=feSM(Row2,C2) + m*(2*Ndy(N)*Ndy(i)+Ndx(N)*Ndx(i))*DetJ

if (i<=NBNp) feSM(Row2,C3) = feSM(Row2,C3) + Ndy(N)*detJ*Hk(i,p);

                    if (N<=NBNp) then
feSM(Row3,C1) = feSM(Row3,C1) + Ndx(i)*detJ*Hk(n,p);
feSM(Row3,C2) = feSM(Row3,C2) + Ndy(i)*detJ*Hk(n,p);
                     end if
                              END DO
                    END DO
          END DO
```



Features of using finite element method to solve problems in the theory of plastic flow. Practical Implementation.

Development and Validation of a Mathematical Model of Warm Drawing Process of Magnesium Alloys in Heated Dies

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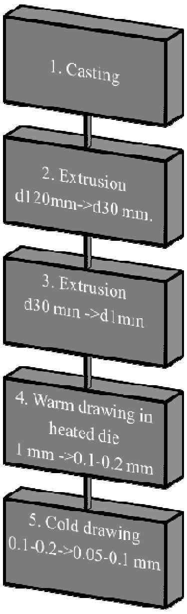
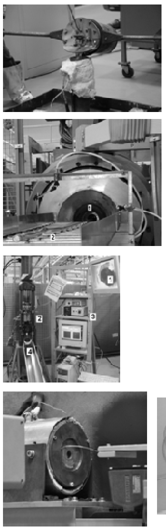
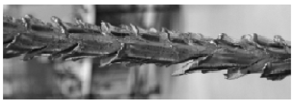
*Faculty of Metal Engineering and Industrial Computer Science, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059, Krakow, Poland, milenin@agh.edu.pl

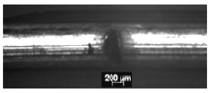
[1] A. Milenin, D.J. Byrska, O. Gryfin The multi-scale physical and numerical modeling of fracture phenomena in the MgCa0.8 alloy// Computers and Structures 89 (2011) 1038–1049 (Proc. 6th MIT Conferense, 06.2011)
[2] A. MILENIN, P. KUSTRA Mathematical model of warm drawing process of magnesium alloys in heated dies// STEEL RESEARCH INTERNATIONAL vol. 81 no. 9 spec. ed. s. 1251–1254 2010
[3] Milenin, A., Seitz, J.-M., Bach, Fr.-W., Bormann, D., Kustra, P., 2010a. Production of thin wires of magnesium alloys for surgical applications. Proc. Conf. Wire Expo 2010 Milwaukee, USA, pp. 61-70.

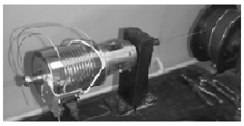
Introduction. Technology.


1. Casting
2. Extrusion d120mm->d30 mm.
3. Extrusion d30 mm ->d1mm
4. Warm drawing in heated die 1 mm ->0.1-0.2 mm
5. Cold drawing 0.1-0.2->0.05-0.1 mm









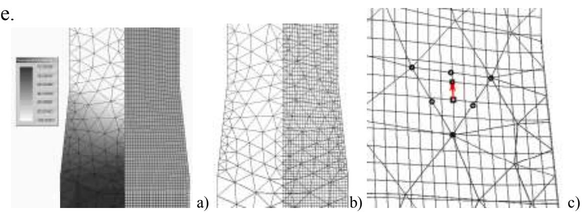
A. Milenin, P. Kustra, J.-M. Seitz, Fr.-W. Bach, D. Bormann Development and Validation of a Mathematical Model of Warm Drawing Process of Magnesium Alloys in Heated Dies // Proc. Conf. InterWire 2011, Atlanta, Georgia, USA, May 2011, Wire Ass. Int. Inc

FEM Model of Wire Drawing. Model of Metal Deformation.

The FE code Drawing2d developed by Milenin (2005, 2008) is used in the present work. The FE model solves a boundary problem considering such phenomena as metal deformation, heat transfer in a die and in a wire, metal heating due to deformation and friction. Solution of the boundary problem is obtained by using variation principle of rigid-plastic theory:

$$J = \int_V \int_0^{\bar{\xi}} \bar{\sigma}(\bar{\varepsilon}, \bar{\xi}, t) d\bar{\xi} dV + \int_V \sigma_0 \xi_0 dV - \int_S \sigma_\tau v_\tau dS ,$$

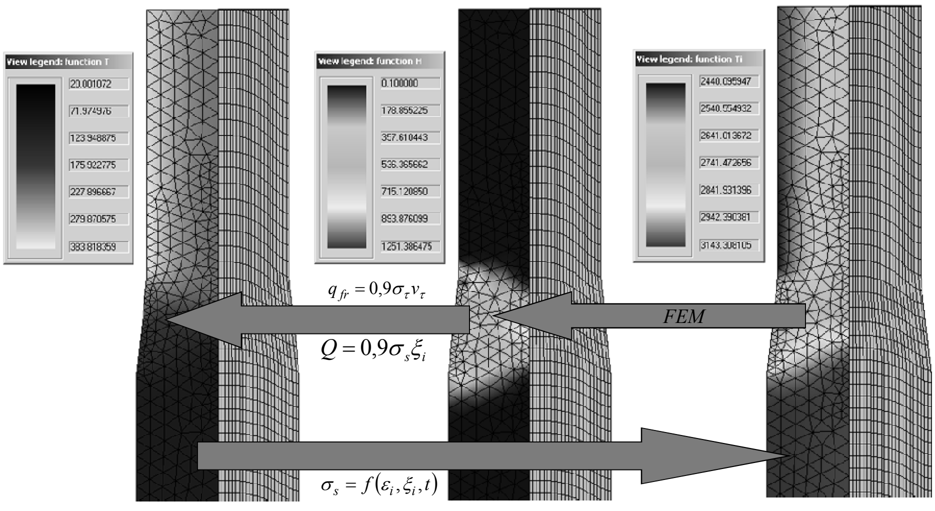
where: $\bar{\xi}$ – strain rate, $\bar{\sigma}$ – yield stress, $\bar{\varepsilon}$ – effective strain, t – temperature, V – volume, σ_0 – mean stress, ξ_0 – volumetric strain rate; S – contact area between alloy and die, σ_τ – friction stress, v_τ – alloy slip velocity along area of die.



The scheme to the determination flow lines point location; a – velocity field in drawing direction and flow lines mesh; b – the flow line mesh placed on FEM mesh; c – the scheme to the determination the next point of current flow lines.

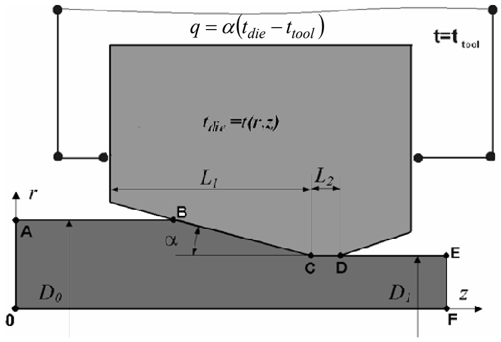
Milenin A., Kustra P.: The multiscale FEM simulation of wire fracture phenomena during drawing of Mg alloy, Steel Research International, ISSN 1611-3683. - 79(2008) spec. ed. s. 717–722.
Milenin A.: Program komputerowy Drawing2d – narzędzie do analizy procesów technologicznych ciągnienia wielostopniowego, Hutnik, No 2, 2005. - s. 100-104 (In Polish).

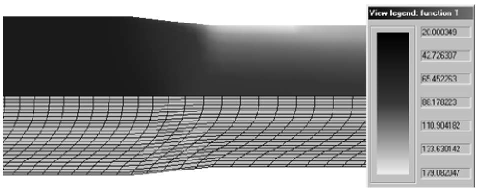
The algorithm of FEM model of metal deformation



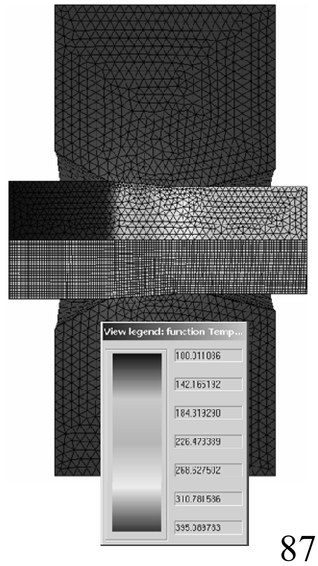
FEM Solution of Thermal Problem in Die

Heating





FEM grids



FEM Couple Solution of Thermal Problem in Metal and Die

THE FEM SOLUTION OF THE THERMAL PROBLEM IN METAL

Thermal problem is solved by applying the following method. The passage of the section through the zone of deformation is simulated. For this section at each time step the non-stationary temperature problem is examined:

$$\lambda\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right) + Q_d = c\rho\frac{dt}{d\tau}$$

where: $Q_d = 0.9\sigma_s\xi_i$ – deformation power, c – specific heat; ρ – alloy density, τ – time, λ – thermal conductivity coefficient (the following values are used for MgCa0.8 alloy: c = 624 J /kgK, ρ = 1738 kg /m^3, λ = 126 J /mK). Heat exchange between the alloy and the die is defined as:

$$q_{conv} = \alpha(t - t_{die})$$

where: t_{die} – die temperature, α – heat exchange coefficient.

The generation of heat from friction is calculated according to the formula:

$$q_{fr} = 0.9\sigma_\tau v_\tau \; .$$

FEM SOLUTION OF THERMAL PROBLEM IN THE DIE

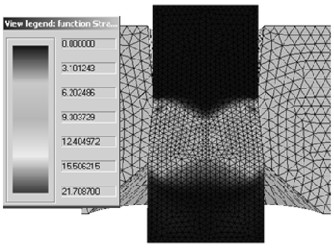
The model of temperature distribution in the die is based on the solution of Fourier equation in the cylindrical coordinate system:

$$\lambda\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial y^2}\right) + Q_h = 0$$

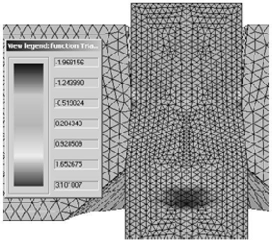
where: Q_h – power of the heating element, r, y cylindrical coordinates.
The heat Q_h is generated in the finite elements, which correspond to the position of heating device. For the areas, which are in contact with the metal, the temperature of the alloy is obtained from the solution of the thermal problem for the metal.

Preliminary Simulations for Determination of Materials Tests Conditions

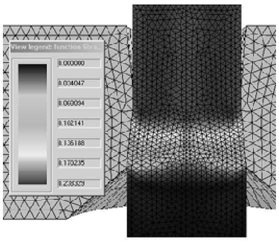
Strain rate, 1/s



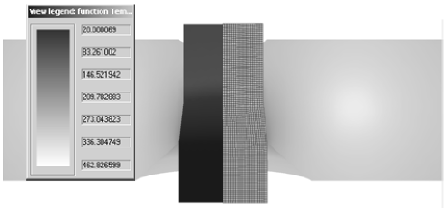
Triaxility factor

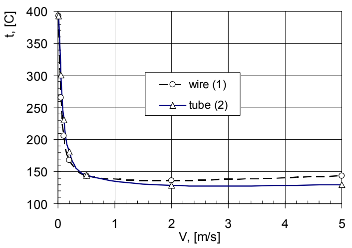


Strain intensity



Temperature, v=0.05 m/s





Yield Stress and Ductility Models

Yield Stress Model. For obtaining the model of flow stress the load-displacement curves from upsetting tests were used. Model of yield stress was proposed as a modified Henzel-Spittel equation:

$$\boxed{\sigma_s = Ae^{-m_1 t}\varepsilon_i^{m_2}\xi_i^{m_3}\left(\frac{t-20}{280}\right)^{m_6} e^{\frac{m_4}{\varepsilon_i}}\left(1+\varepsilon_i\right)^{m_5 t}e^{m_7\varepsilon_i}\xi_i^{m_8 t}t^{m_9}}$$

where: A, m_1 - m_9 – empirical coefficients.

Ductility Model. The key parameter, which presents fracture is called **ductility function**. This parameter is defined by the following formula:

$$\psi = \frac{\varepsilon_i}{\varepsilon_p(k,t,\xi_i)} < 1$$

where: k – triaxility factor, $k = \sigma_0 / \sigma_s$.

Critical deformation function $\varepsilon_p(k,t,\xi_i)$ is obtained on the basis of experimental results for the upsetting and the tension tests.

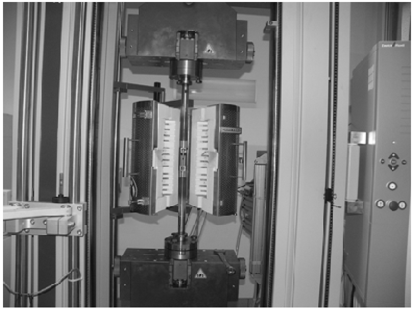
$$\psi = \int_0^{\tau}\frac{\xi_i}{\varepsilon_p(k,t,\xi_i)}d\tau \approx \sum_{m=1}^{m=m_\tau}\frac{\xi_i^{(m)}}{\varepsilon_p(k,t,\xi_i)}\Delta\tau^{(m)}$$

where: τ – time of deformation, $\Delta\tau^{(m)}$ – time increment, $\xi_i^{(m)}$ – the values of the strain rate in the current time, m – a index number of time step during numerical integration along the flow line.

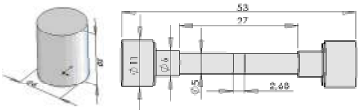
The following function of critical deformation is proposed:

$$\boxed{\varepsilon_p = d_1\exp(-d_2 k)\exp(d_3 t)\xi_i^{d_4}}$$

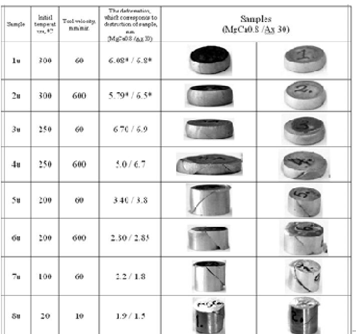
Materials tests and data processing



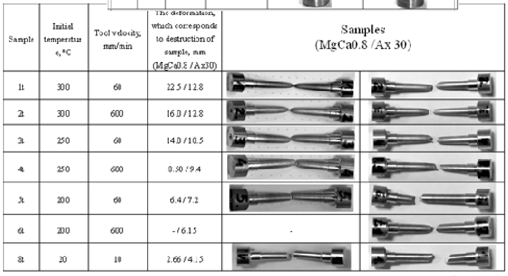
Upsetting and tensile tests were performed on the Zwick Z250 machine at the AGH University of Science and Technology.



(a) (b)
Shape and dimension of samples for upsetting test (a) and tensile test (b).

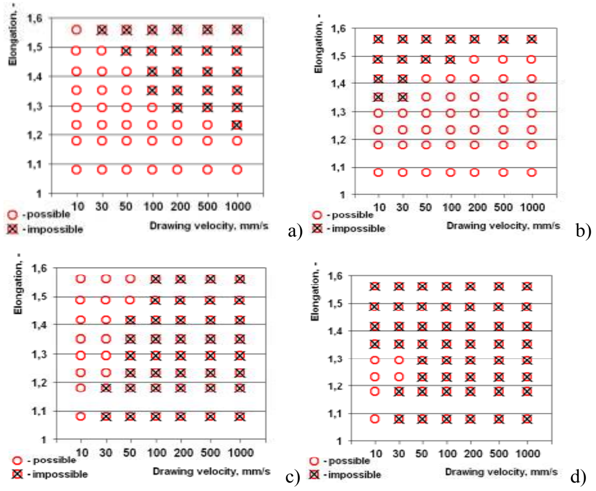


Sample	Initial temperature, °C	Tool velocity, mm/min	The deformation, which corresponds to destruction of sample, mm (MgCa0.8 /Ax 30)	Samples (MgCa0.8 /Ax 30)	
1u	300	60	6.08* / 6.8*		
2u	300	600	5.79* / 6.5*		
3u	250	60	6.70 / 6.9		
4u	250	600	5.0 / 6.7		
5u	200	60	3.40 / 3.8		
6u	200	600	2.50 / 2.85		
7u	100	60	2.2 / 1.8		
8u	20	10	1.9 / 1.5		



Sample	Initial temperature, °C	Tool velocity, mm/min	The deformation, which corresponds to destruction of sample, mm (MgCa0.8 /Ax30)	Samples (MgCa0.8 /Ax 30)	
1t	300	60	22.5 / 12.8		
2t	300	600	16.0 / 12.8		
3t	250	60	14.0 / 10.5		
4t	250	600	8.50 / 9.4		
5t	200	60	6.4 / 7.2		
6t	200	600	- / 6.15	-	
8t	20	10	2.66 / 4.15		

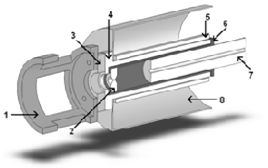
Determination of wire drawing schedule

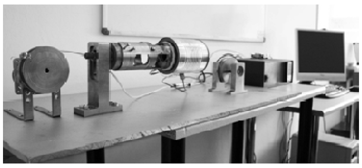


Maps of possible elongations per pass: (a) – because the fracture criteria (13); (b) – because the relationship σ_y/σ_s criteria (wire breaking); (c) – because temperature conditions; (d) – summary map

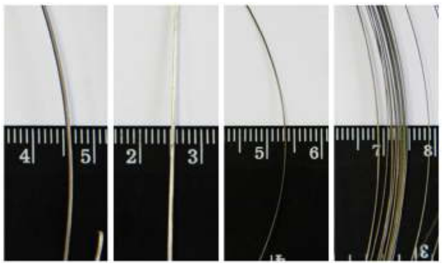
Production of thin wire

Equipment for drawing (construction of AGH, Krakow)





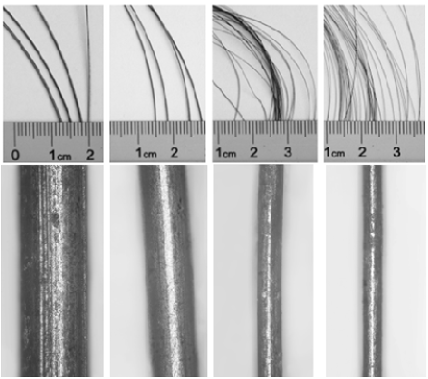
Wire after drawing



(a) (b) (c) (d)

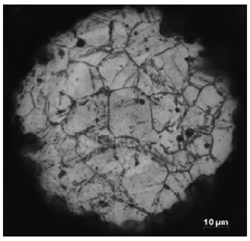
Diameters if wire from Ax30 alloy after drawing: (a) 0,761 – pass 3 , (b) 0,694 – pass 4, (c) 0,306 – pass 13, (d) 0,233 – pass 16.

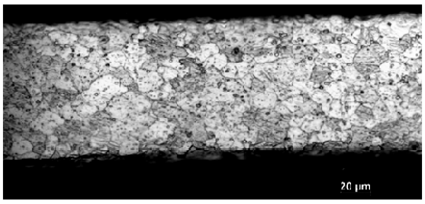
Production of thin wire



(a) (b) (c) (d)

Vire after drawing process of MgCa0.8 in heated die: (a) o0.634 mm, (b) o0.402 mm, (c) 0.162 mm, (d) o0.100 mm

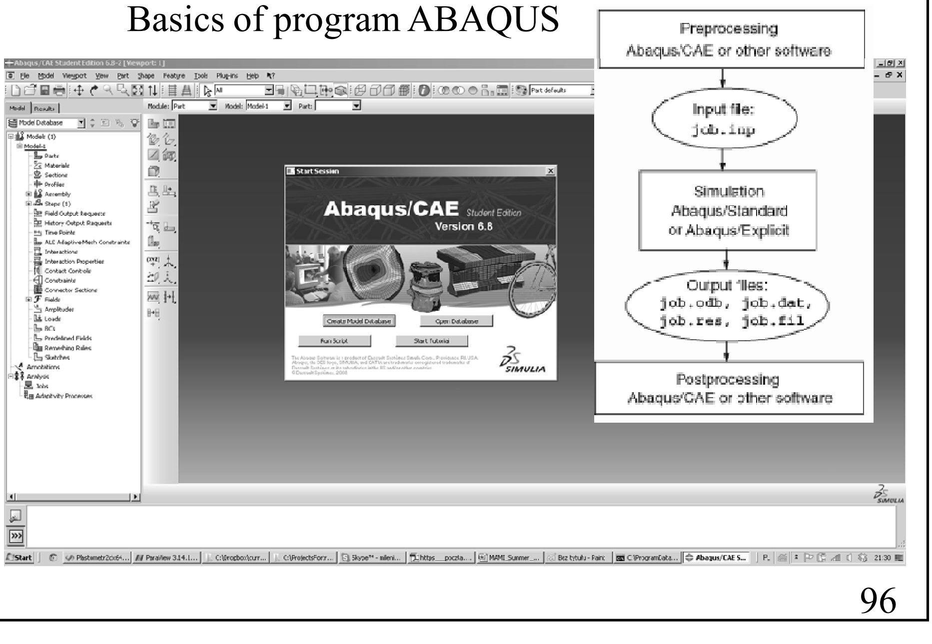




Lection 4

Basics of program ABAQUS; examples of solving problems of the theory of elasticity in the program ABAQUS; the problem of modelling the deformation of tool during metal forming processes.

Basics of program ABAQUS



Preprocessing
Abaqus/CAE or other software

Input file:
job.inp

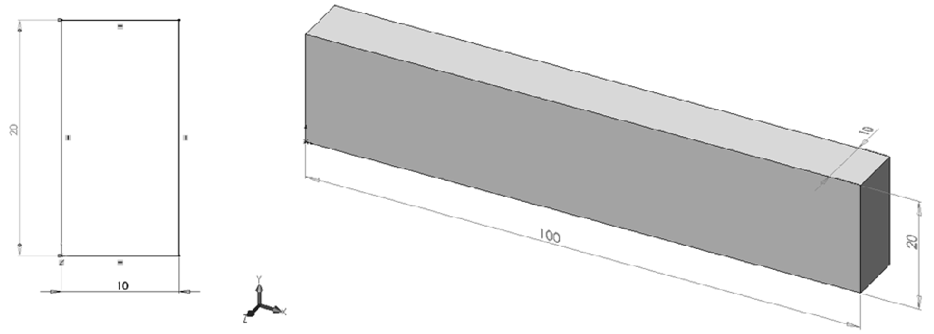
Simulation
Abaqus/Standard
or Abaqus/Explicit

Output files:
job.odb, job.dat,
job.res, job.fil

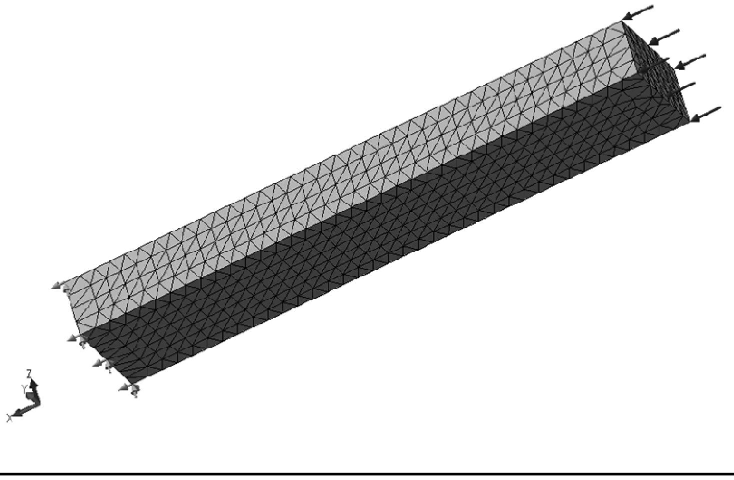
Postprocessing
Abaqus/CAE or other software

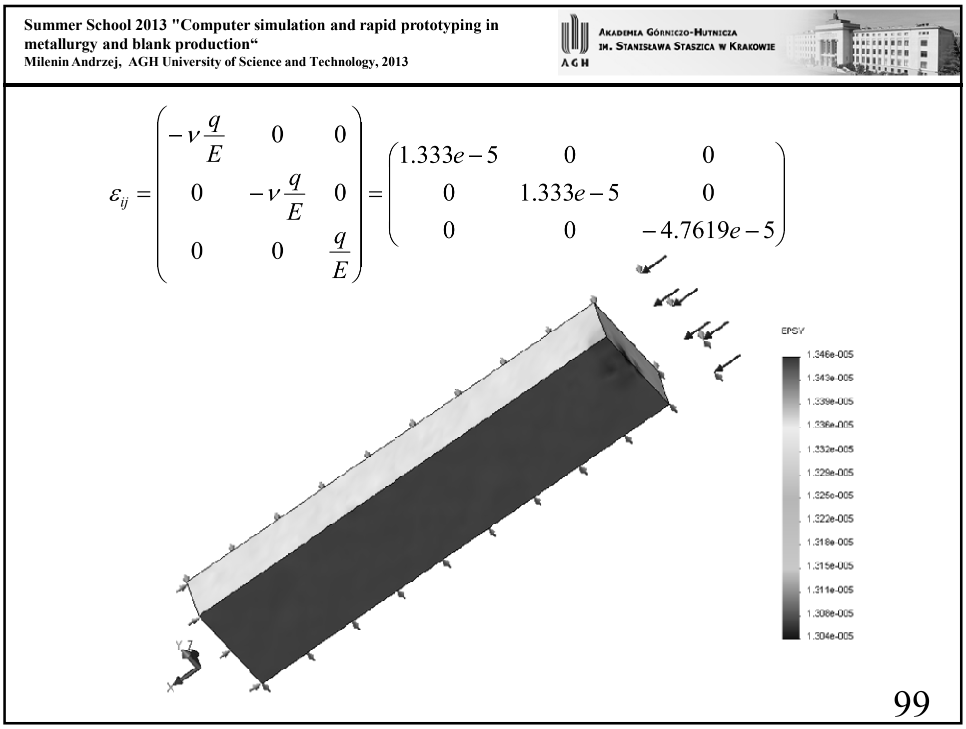
Examples of solving problems of the theory of elasticity in the program ABAQUS

Upsetting

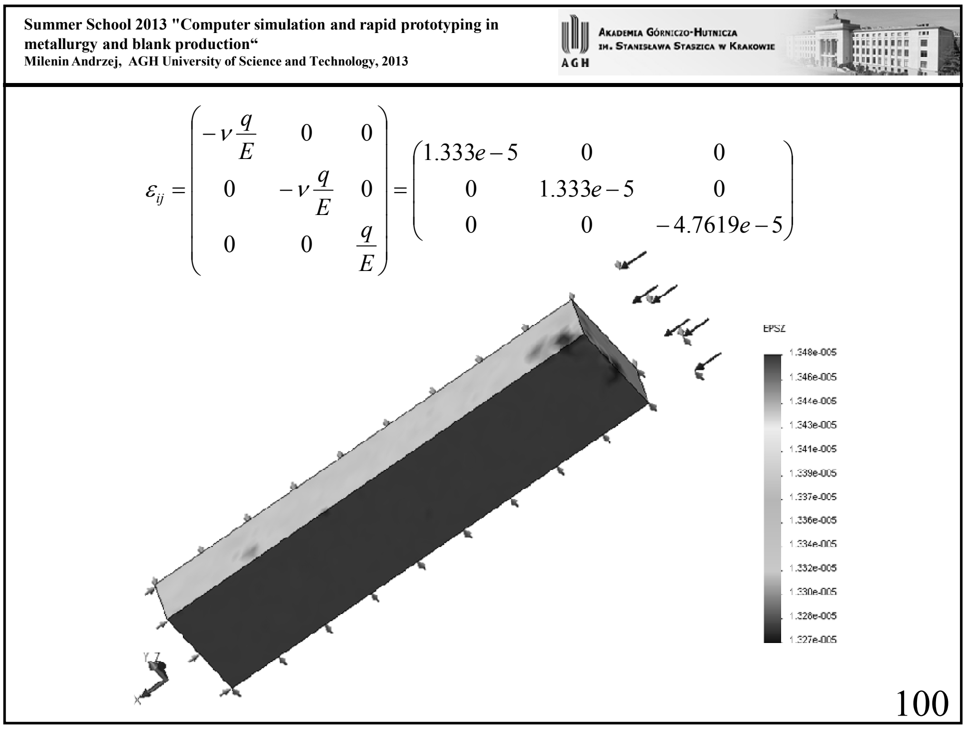


$$q = -10000000Pa = -10MPa$$



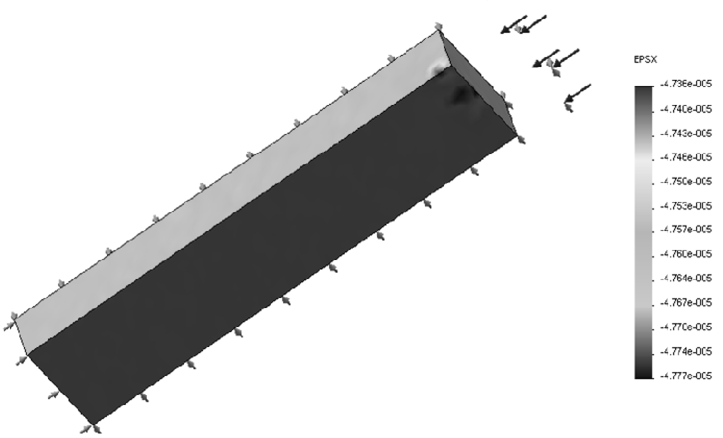


$$\varepsilon_{ij}=\begin{pmatrix}-\nu\dfrac{q}{E} & 0 & 0\\ 0 & -\nu\dfrac{q}{E} & 0\\ 0 & 0 & \dfrac{q}{E}\end{pmatrix}=\begin{pmatrix}1.333e-5 & 0 & 0\\ 0 & 1.333e-5 & 0\\ 0 & 0 & -4.7619e-5\end{pmatrix}$$

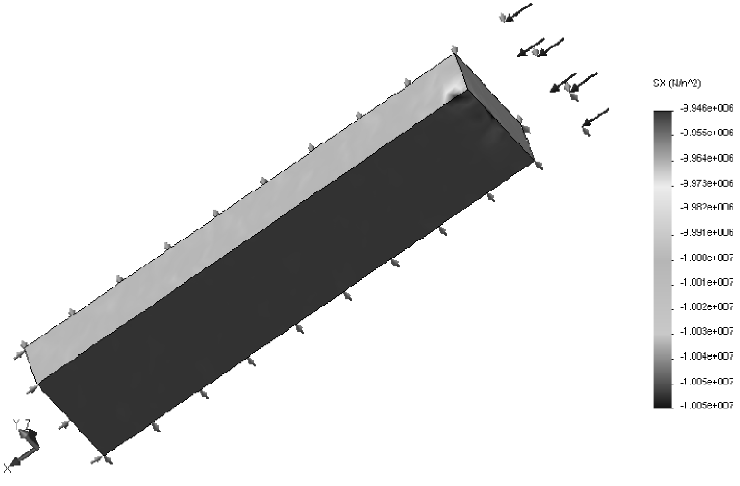


$$\varepsilon_{ij}=\begin{pmatrix}-\nu\dfrac{q}{E} & 0 & 0\\ 0 & -\nu\dfrac{q}{E} & 0\\ 0 & 0 & \dfrac{q}{E}\end{pmatrix}=\begin{pmatrix}1.333e-5 & 0 & 0\\ 0 & 1.333e-5 & 0\\ 0 & 0 & -4.7619e-5\end{pmatrix}$$

$$\varepsilon_{ij} = \begin{pmatrix} -\nu\dfrac{q}{E} & 0 & 0 \\ 0 & -\nu\dfrac{q}{E} & 0 \\ 0 & 0 & \dfrac{q}{E} \end{pmatrix} = \begin{pmatrix} 1.333e-5 & 0 & 0 \\ 0 & 1.333e-5 & 0 \\ 0 & 0 & -4.7619e-5 \end{pmatrix}$$

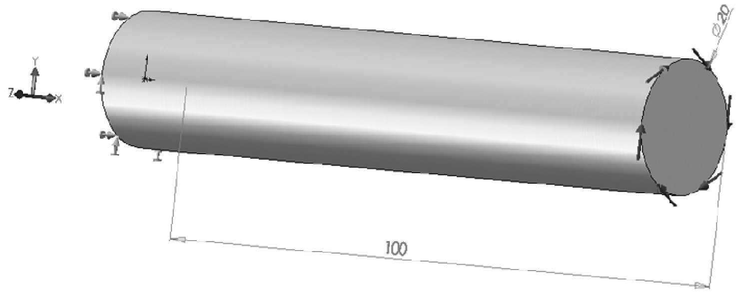


$$\sigma_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$



Twisting

$$M = 1000Nm$$



$$\tau = \frac{M_x}{I_0}R$$

$$\varepsilon = \frac{M_x R}{2GI_0}$$

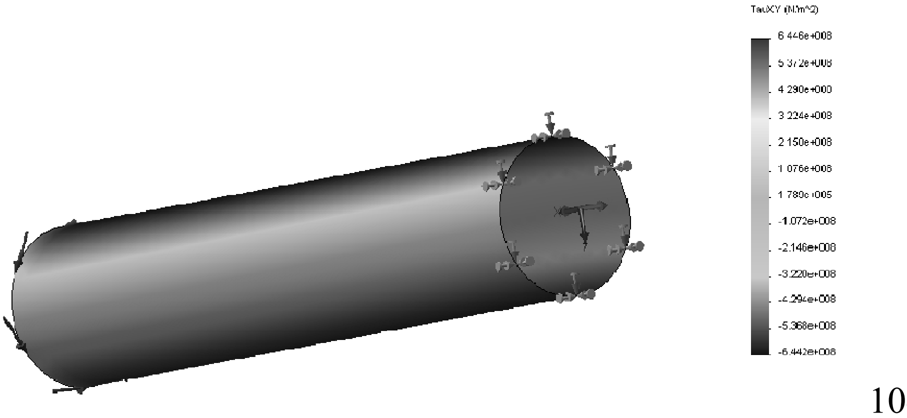
$$I_0 = \frac{\pi R^4}{2} = \frac{\pi d^4}{32} = \frac{\pi 0.020^4}{32} = 0{,}00000001570795 = 15{,}70795\text{e-9m}^4$$

$$G = 79000MPa = 79\text{ e9 } Pa$$

$$\sigma_{ij} = \begin{pmatrix} 0 & -\frac{M_x}{I_0}z & \frac{M_x}{I_0}y \\ -\frac{M_x}{I_0}z & 0 & 0 \\ \frac{M_x}{I_0}y & 0 & 0 \end{pmatrix}$$

$$\tau = \frac{M_x}{I_0}R$$

$$\tau = \frac{M_x}{I_0}R = \frac{1000Nm}{15{,}70795\text{e-9m}^4}0{,}010m = 0{,}636\text{e9Pa} = 636\text{MPa}$$



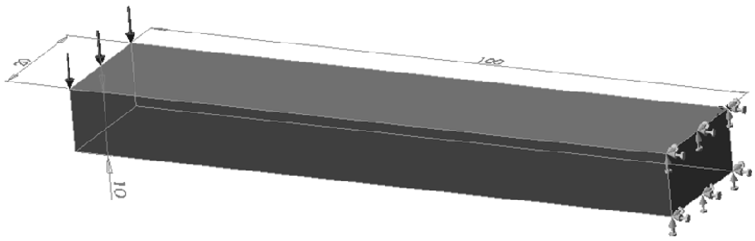
Bending

$$P = 100N$$

$$\sigma_x^{\max} = \frac{M_y}{I_y} z^{\max}$$

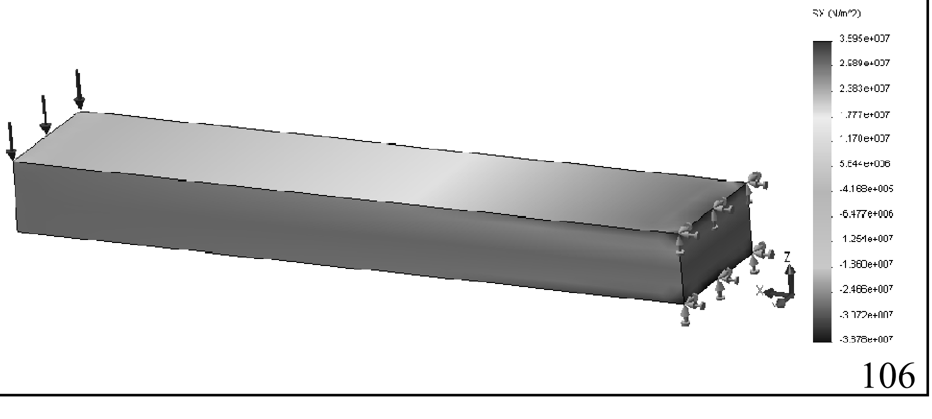
$$M = Pl = 100N \cdot 0.1m = 10Nm$$

$$I_y = \frac{bh^3}{12} = \frac{0.02m \cdot (0.01m)^3}{12} = 1.667e - 9m^4$$



$$\sigma_{ij} = \begin{pmatrix} \frac{M_y}{I_y} z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_x^{\max} = \frac{M_y}{I_y} z^{\max} = \frac{10Nm}{1.667e - 9m^4} \frac{0.01m}{2} = 30.0MPa$$



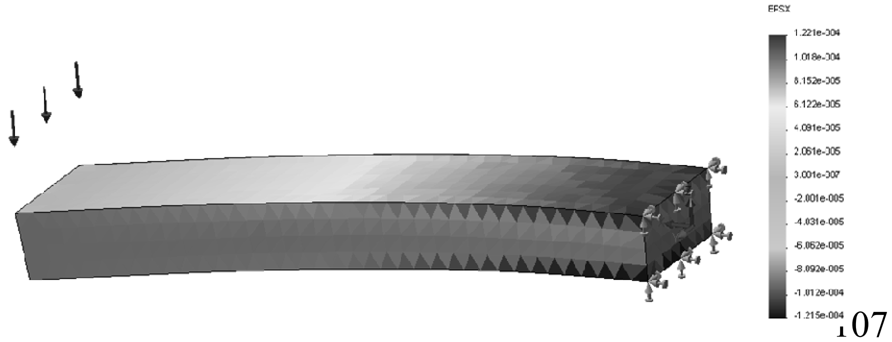
$$\varepsilon_{ij} = \begin{pmatrix} \frac{M_y}{EI_y} z & 0 & 0 \\ 0 & -\nu \frac{M_y}{EI_y} z & 0 \\ 0 & 0 & -\nu \frac{M_y}{EI_y} z \end{pmatrix}$$

$$I_y = \frac{bh^3}{12} = \frac{0.02m \cdot (0.01m)^3}{12} = 1.667e - 9m^4$$

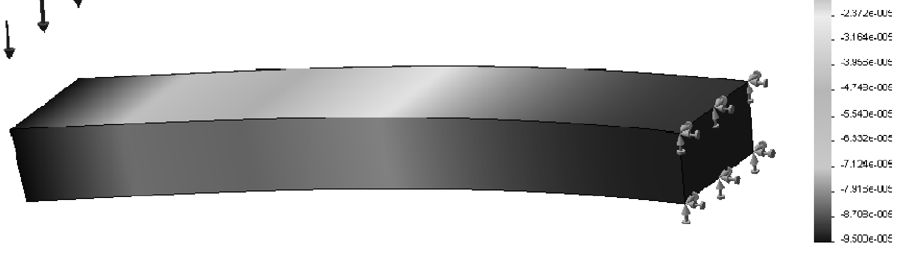
$$E = 210e9Pa$$

$$\nu = 0.28$$

$$\frac{M_y}{EI_y}\frac{h}{2} = \frac{10Nm}{210e9\frac{N}{m^2}1.667e - 9m^4}\frac{0.01m}{2} = 1.43e - 4$$

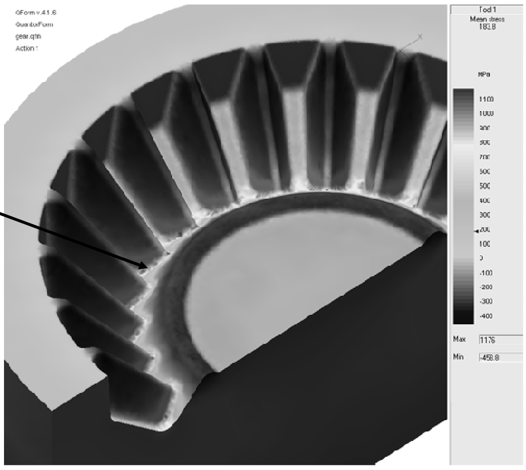


$$f = \frac{M_y}{2EI_y}a^2 = \frac{10Nm}{2 \cdot 210e9\frac{N}{m^2}1.667e - 9m^4}(0.1m)^2 = 1.428e - 4m$$



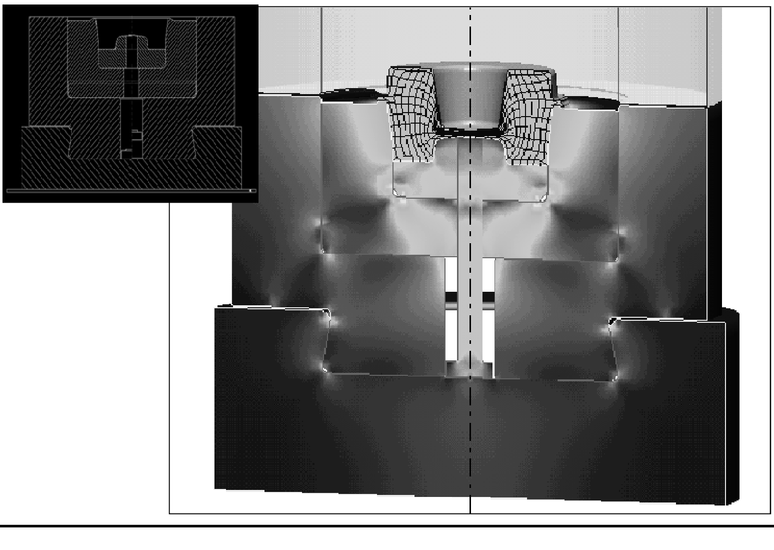
The problem of modelling the deformation of tool during metal forming processes

- **Maximum Stress Shown Where Dies Were Cracking**



Die stress in solid die block (Qform)

Effective stress distribution in assembled die

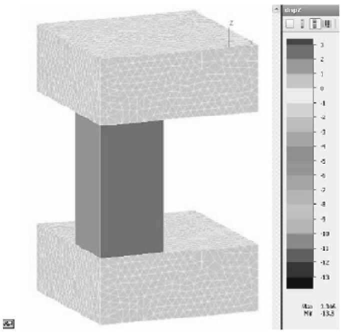


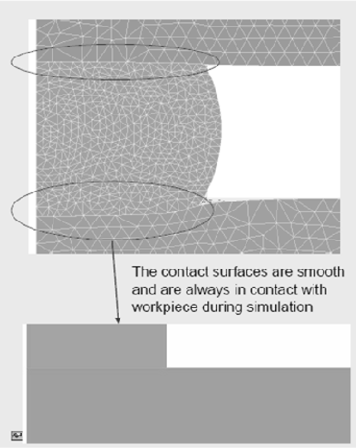
Quantor Form QFORM2D/3D

Coupled mechanical simulation of the tools and the forged part

Sample of upsetting in “soft” dies

Workpiece – steel
Upper tool – $E^* = 2150$ MPa ($E_{\text{real}}/100$)
Lower tool – $E^* = 430$ MPa ($E_{\text{real}}/500$)




The contact surfaces are smooth and are always in contact with workpiece during simulation

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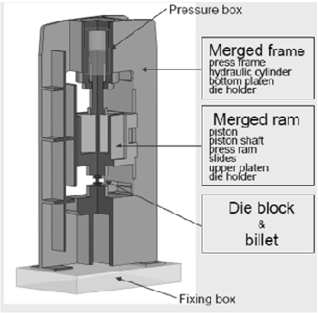
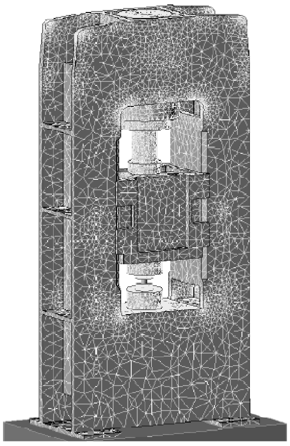
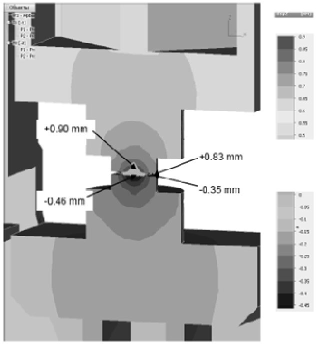
Finite element mesh for coupled deformation and thermal problem

Design scheme of hydraulic press 10 MN



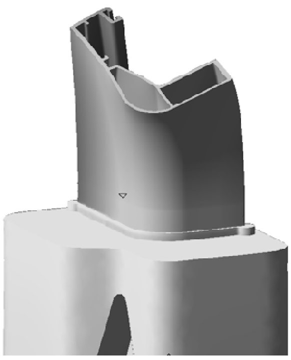


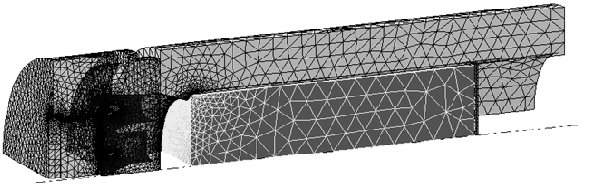
Vertical displacements in the die block



Total deformation in gutter area is about 1.2 mm, additional deformation of die cavity is ~0.1 mm

Influence of tool deformation on metal flow (Qform-Extrusion)

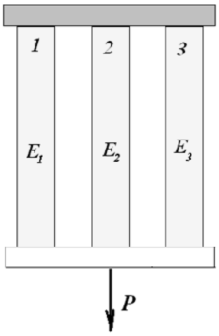


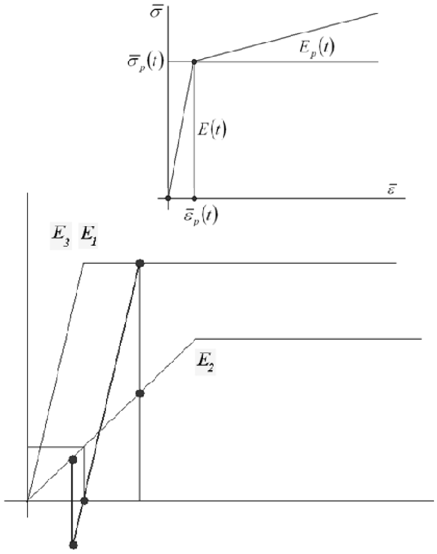


Lection 5

Modelling of residual stresses; the solution of elastic-plastic problems by using ABAQUS; examples solving of the cold metal forming problem.

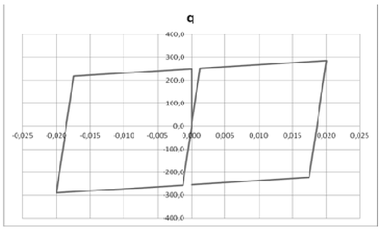
Modelling of residual stresses

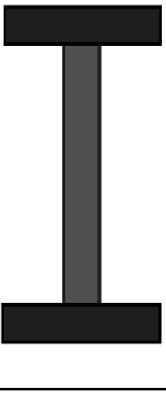


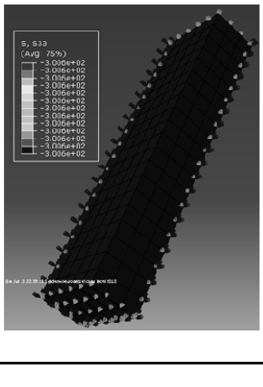


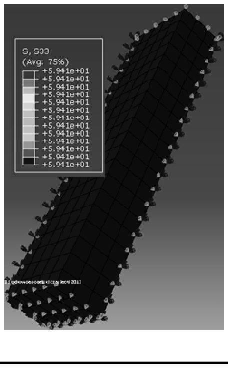
Modelling of residual stresses

t = 100 C => 1000 C => 100 C









Modelling of residual stresses

MATHEMATICAL MODEL OF THE RESIDUAL STRESSES IN HOT-ROLLED SHEETS FOR LASER CUTTING

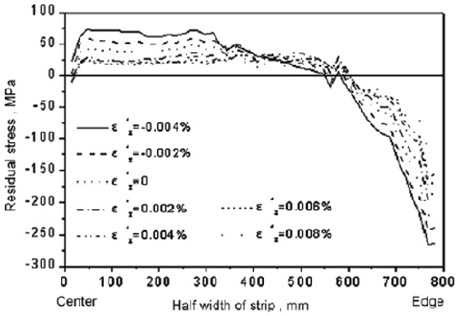
1.Introduction
2.Thermal model
3.Model of residual stresses
    3.1. Material model
    3.2. Fast model of residual stresses
4. Example of simulation
5. Conclusions

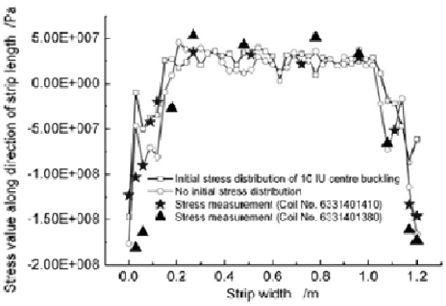
The problem of residual stresses becomes of practical importance when the laser cutting of sheets is applied. High values of residual stresses lead to deformation (bending and twisting) of sheets during laser cutting.

Influence factors:
- Local thermal expansion of the metal during uneven changes of temperature;
- Dilatation due to phase transformations;
- Unloading processes;
- Stress relaxation.

Assumptions:
- Residual stresses occur mainly in the process of cooling of sheets at lower temperatures, when elastic-plastic material properties increase;
- Only longitudinal stresses have a significant influence in the rolled strips.

Existing FEM solutions. Problem of calculation of residual stresses is generally solved by using the FEM and commercial programs, for example ABAQUS. This led to the following difficulties:
- Increase of the calculation time;
- It is not possible to use the model in the control system of the rolling mill;
- Accuracy of results is limited.





Cai, Z., 1998. Research on buckling behavior of hot-rolled strip after cooling, PhD Dissertation, Northeast University, 6

X. Wang, F. Li , Q. Yang, A. He FEM analysis for residual stress prediction in hot rolled steel strip during the run-out table cooling Applied Mathematical Modelling 37 (2013) 586–609

Thermal model

$$c_{eff}(t)\rho(t)\frac{dt}{d\tau}=\mathrm{div}\left[k(t)\,\mathrm{grad}(t)\right]+q_{def}$$

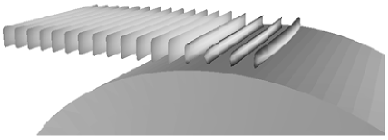
$$q_{def}=0.9\bar{\sigma}\bar{\xi}$$

Boundary condition for rolling and cooling

$$Q_{conv}=\alpha(t-t_{\infty})$$

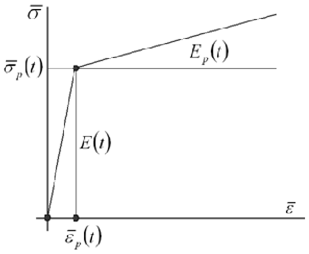
Basic equations and FEM model

$$[H]\{t\}+[C]\frac{\partial}{\partial\tau}\{t\}+\{P\}=0$$



Model of residual stresses

Material model



$$\overline{\sigma}(t,\overline{\varepsilon}) = E(t)\overline{\varepsilon} \qquad \text{for } \overline{\varepsilon} \le \overline{\varepsilon}_p(t)$$

$$\overline{\sigma}(t,\overline{\varepsilon}) = \overline{\sigma}_p(t) + E_p(t)\left(\overline{\varepsilon} - \frac{\overline{\sigma}_p(t)}{E(t)}\right) \qquad \text{for } \overline{\varepsilon} > \overline{\varepsilon}_p(t)$$

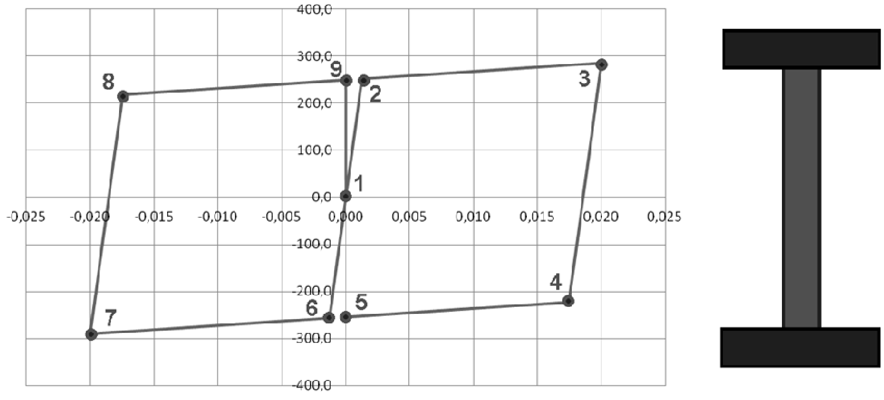
$$\overline{\varepsilon}_p(t) = \frac{\overline{\sigma}_p(t)}{E(t)}$$

where $\overline{\varepsilon}$ - effective strain, $\overline{\varepsilon}_p(t)$ - boundary of plasticity, $E_p(t)$ modulus of plasticity, $\overline{\sigma}_p(t)$ - yield stress, $E(t)$ - Young modulus.

$$\Delta\overline{\sigma}(t,\overline{\varepsilon}) = E(t)\Delta\overline{\varepsilon} \qquad \text{for } \overline{\varepsilon} \le \overline{\varepsilon}_p(t)$$

$$\Delta\overline{\sigma}(t,\overline{\varepsilon}) = E_p(t)\Delta\overline{\varepsilon} \qquad \text{for } \overline{\varepsilon} > \overline{\varepsilon}_p(t)$$

To validate the model, a numerical test for a rod with fixed ends was performed.
- Rod temperature changed from 1200 C to 200 C and then back up to 1200 C.
- It was assumed in this test that the mechanical characteristics of the material are independent of temperature.

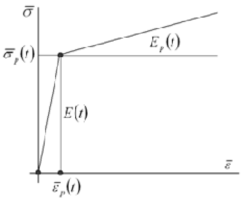


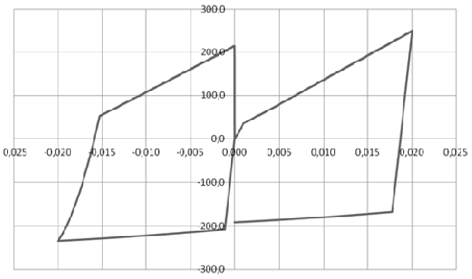
Simulation result of loading and unloading of a fixed rod with material properties that do not depend on temperature

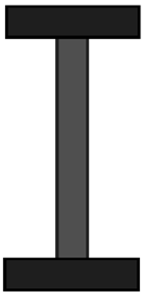
$$E(t) = E_{20}\exp[(t-20)a_1]\left(1 + a_2\left(\frac{t-20}{1000}\right)^3\right)$$

$$\overline{\sigma}_p(t) = \overline{\sigma}_{p20}\exp[(t-20)b_1]\left(1 + b_2\left(\frac{t-20}{1000}\right)^3\right)$$

$$E_p(t) = E_{p20}\exp[(t-20)c_1]\left(1 + c_2\left(\frac{t-20}{1000}\right)^3\right)$$







The simulation results of loading and unloading of a fixed rod with the material properties dependent on the temperature

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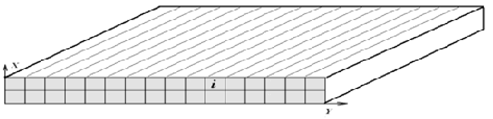
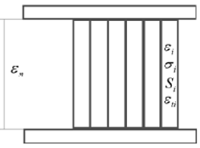
Beam model of residual stresses in hot-rolled sheets

The model of residual stresses is based on the assumption that all components of the stress tensor except tension along the length of the sheet are zero. In this case, the sheet can be presented in the form of the investigated earlier rod, fixed at the ends. In addition to the thermal deformation of each rod, all the rods are exposed to the average strain of the sheet ε_m that is a result of the changing length of the sheet in the cooling process.





If in the rod *i* an increment of temperature Δt and the corresponding increment of thermal deformation $\Delta\varepsilon_t$ appear, the total increment of the deformation of the rod will be equal to:

$$\Delta\overline{\varepsilon}_i = \Delta\varepsilon_m - \Delta\varepsilon_t$$

and the current strain is

$$\overline{\varepsilon}_i = \sum_{\tau=1}^{n}\Delta\overline{\varepsilon}_{i,\tau}$$

Where *n* - number of increments of temperature (and thermal load).

The increment of the thermal deformation is determined taking into account phase transformations. This relation can be represented in a general form:

$$\Delta\varepsilon_t = f(t, \Delta t)$$

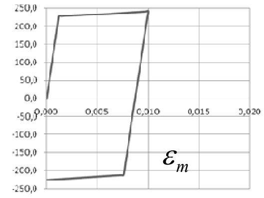
Increment of stress -

$$\Delta\overline{\sigma}_i = \overline{\sigma}(t, \overline{\varepsilon}_i, \Delta\varepsilon_i)$$

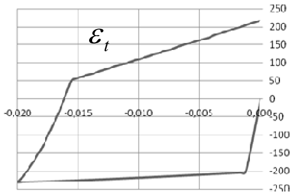
Determination of $\Delta\varepsilon_m$ is carried out on the basis of the conditions of equilibrium of the system of rods:

$$\sum_{i=1}^{k}\overline{\sigma}_i\left(t,\varepsilon_i,\Delta\varepsilon_t,\Delta\varepsilon_m\right)S_i=0$$

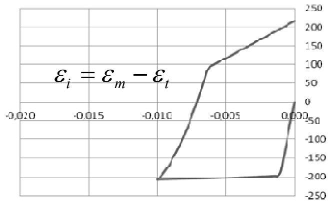
Where i – number of the rod, k – the number of rods, S_i – cross section area of the rod i, $\overline{\sigma}_i\left(t,\varepsilon_i,\Delta\varepsilon_t,\Delta\varepsilon_m\right)$ - rod model.



The scheme of loading and unloading of the rod under average strain of sheet



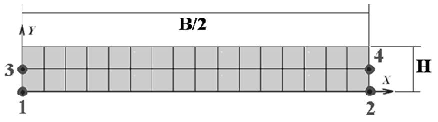
The scheme of loading and unloading of the rod under thermal deformation

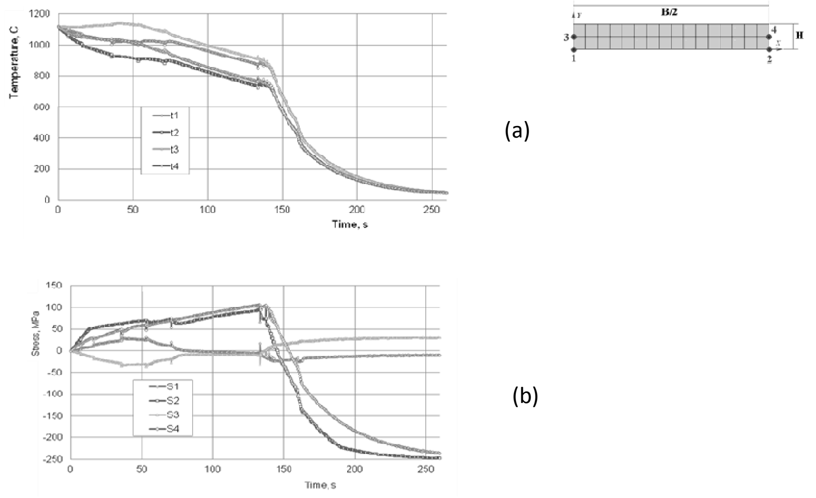


The scheme of loading and unloading of the rod under thermal deformation of the sheet and the average strain

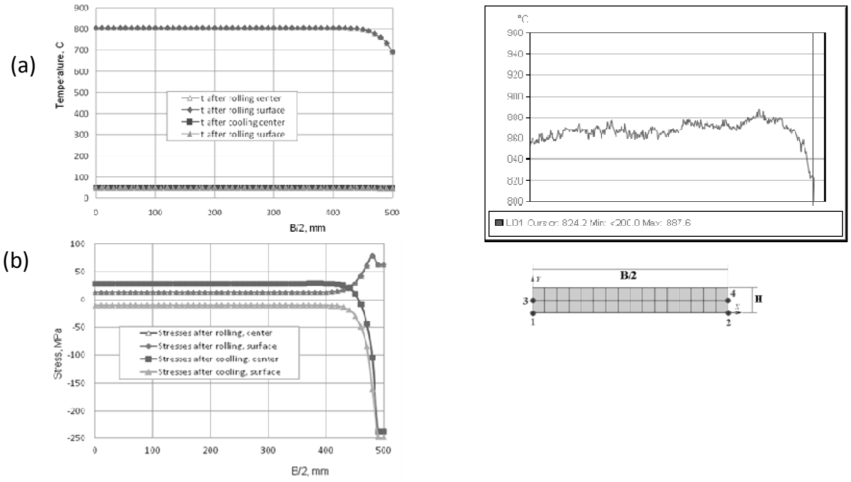
The process of the rolling of 1.48 mm thick sheet in a continuous 11 stands mill was selected as an example. The mill contains two groups of stands. The first five of the stands are a roughing train and the remaining stands form a finishing train. The time of transport of the strip between the trans was about 60 seconds.

Stand	Initial thickness, mm	Final thickness, mm	Rolling velocity, m/s
1	168.0	142.0	0.70
2	142.0	90.0	1.07
3	90.0	50.0	1.10
4	50.0	33.0	1.30
5	33.0	19.0	1.88
6	19.0	9.12	1.42
7	9.12	5.46	2.37
8	5.46	3.37	3.83
9	3.37	2.31	5.60
10	2.31	1.76	7.37
11	1.76	1.48	8.73



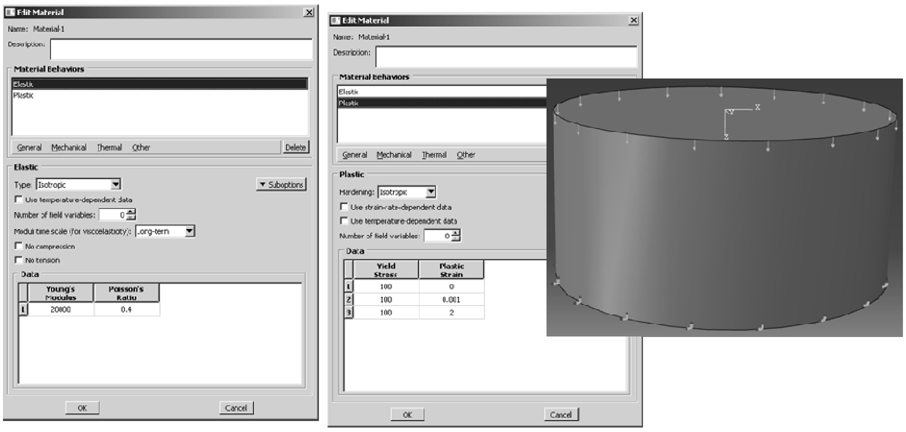


The simulation results for temperature (a) and thermal stress (b).



Temperature distribution (a) and longitudinal stresses (b) along the width of the strip.

The solution of elastic-plastic problems by using ABAQUS;



Edit Material

Name: Material-1

Description:

Material Behaviors: Elastic, Plastic

Elastic

Type: Isotropic

☐ Use temperature-dependent data

Number of field variables: 0

Moduli time scale (for viscoelasticity): Long-term

☐ No compression

☐ No tension

Data

	Young's Modulus	Poisson's Ratio
1	20000	0.4

Edit Material

Name: Material-1

Description:

Material Behaviors: Elastic, Plastic

Plastic

Hardening: Isotropic

☐ Use strain-rate-dependent data

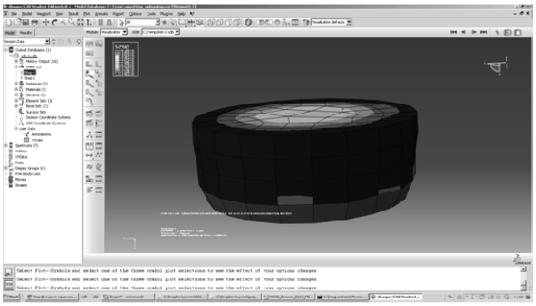
☐ Use temperature-dependent data

Number of field variables: 0

Data

	Yield Stress	Plastic Strain
1	100	0
2	100	0.001
3	100	2

Loading



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Select Plot->Symbols and select one of the three symbol plot selections to see the effect of your options changes.
Select Plot->Symbols and select one of the three symbol plot selections to see the effect of your options changes.
Select Plot->Symbols and select one of the three symbol plot selections to see the effect of your options changes.
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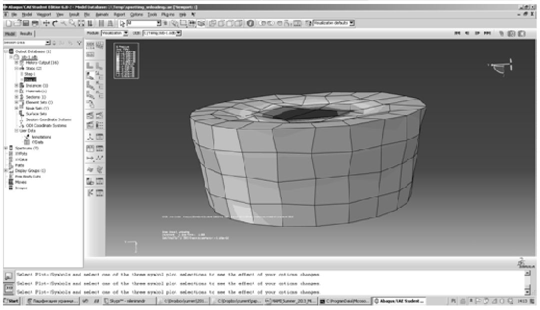
Unloading



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Select Plot->Symbols and select one of the three symbol plot selections to see the effect of your options changes.
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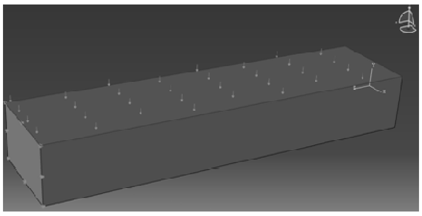

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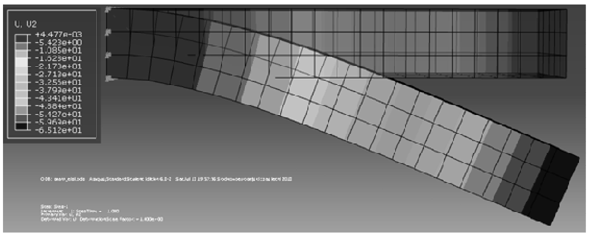
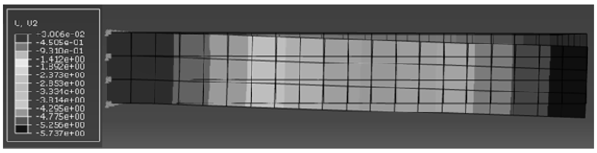
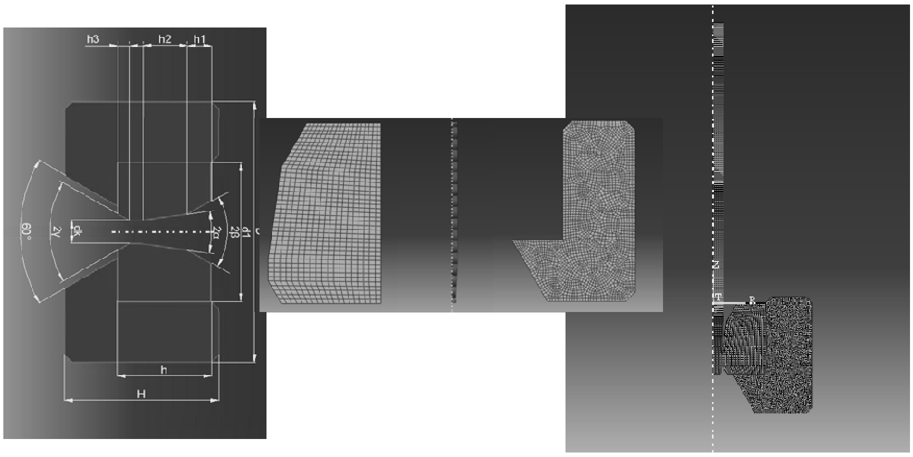
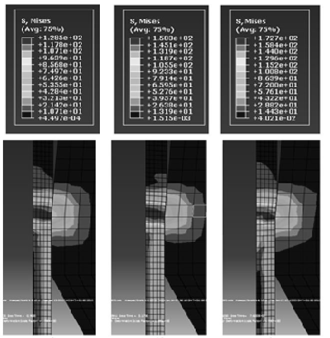


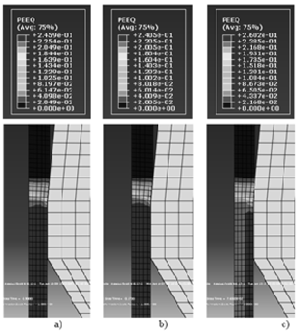
Unloading

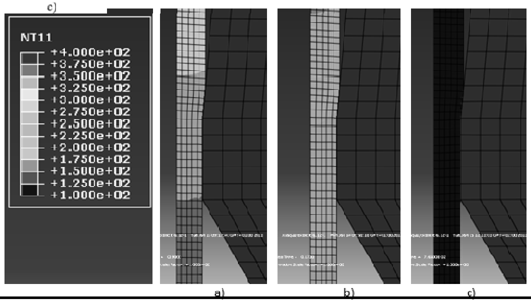


Examples solving of the wire drawing problem

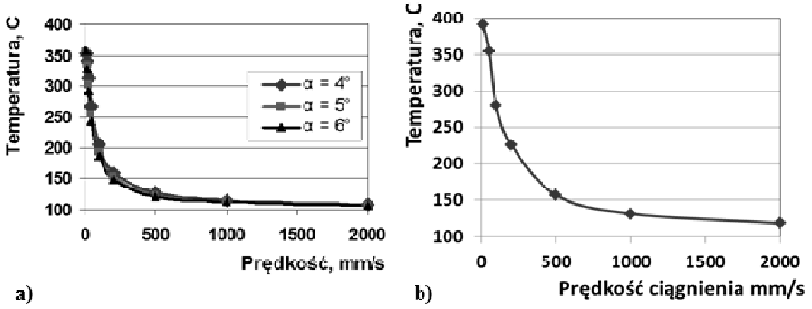






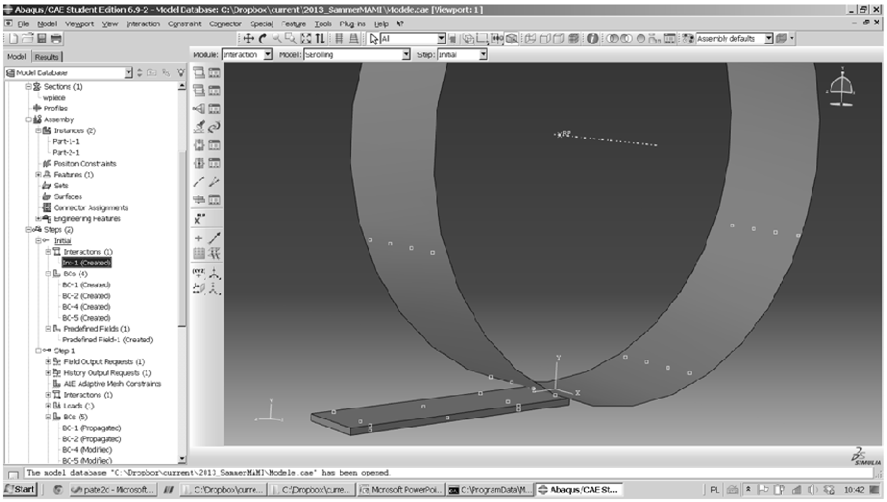


Examples solving of the wire drawing problem



a) b)

Examples solving of the plate rolling problem



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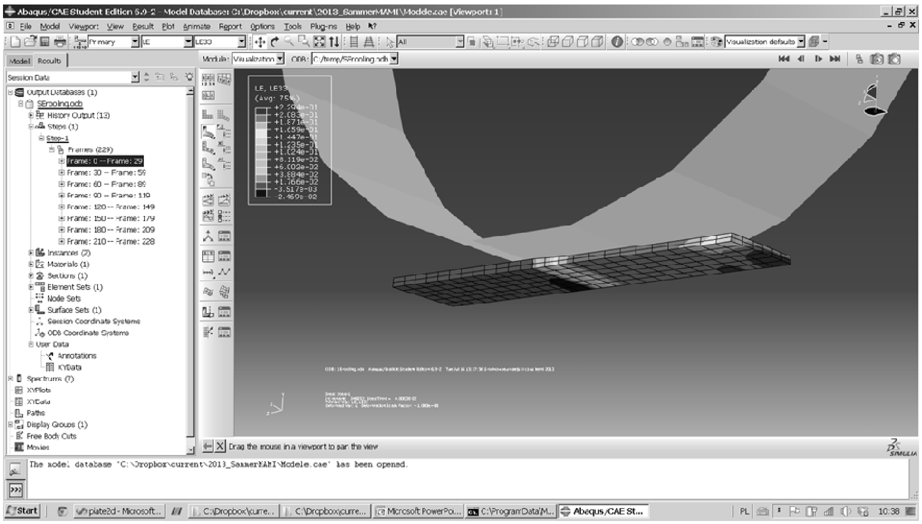
Examples solving of the plate rolling problem



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The model database 'C:\Dropbox\current\2013_SammerMAMI\Modele.cae' has been opened.
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Conclusions

1. Theories of elasticity, elastic-plasticity, rigid and viscous plasticity are absolutely necessary for understanding of FEM algorithms and interpretation of FEM results;
2. FEM techniques is depend on kinds of boundary problem.
3. Critical interpretation of results of FEM commercial programs, usage of analytical solutions for validation of FEM programs;
4. Illustration of FEM procedures.

Thank You for attention