# My theorems [*]

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### Abstract

Publishing without (numerous) joint authors has numerous disadvantages, while the virtue of such behaviour (maybe the only one) is presented in the title. During my mathematical activity I have been interested mainly in the fixed point theory, and some details are contained in the second section. The first section consists of miscellaneous results. I apologize for not giving the citation data concerning the publications of other mathematicians, they can be found in the respective papers of mine.

Dear Reader, let me know if any of the results presented here have been obtained earlier by someone else. In such a case I will cry, but I will offer a rectification.

## 1 Miscellaneous results

A version of the Stokes Theorem (weak assumptions on continuity, and no problem of orientability).

**Theorem 1.1** ([28, Theorem 1]) *Let $A \subset \mathbb{R}^k$, $(k > 1)$ be a regular set [28, Definition 3] consisting of normal sets $A_1, \ldots, A_m$, and let $\Phi \colon A \to$*

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$\mathbb{R}^n$ *be a one to one mapping of class* $C^1$. *Assume that* $\overset{k-1}{\omega} = \sum b_I \bigwedge dy_I$ *is a continuous form on* $S = \Phi(A)$, *and of class* $C^1$ *on* $\bigcup \{\Phi(Int\,A_j)\colon j = 1,\ldots,m\}$. *Then the integrals in the following formula are convergent*

$$\int_S d\,\overset{k-1}{\omega} = \int_{\partial S} \overset{k-1}{\omega}\,,$$

*and the equality holds on condition that the orientations of* $\partial S = \Phi(Fr\,Int\,A)$ *and of* $S$ *coincide.*

As regards the compactness in function spaces, it seems more natural to consider mappings continuous on compacta instead of continuous mappings. Let us adopt the following notations: $Y^X$ is the family of all mappings on $X$ to $Y$; $\mathcal{C}_\mathcal{K} \subset Y^X$ is the family of all mappings continuous on compacta; $\mathcal{V}|\mathcal{K}$ is the uniformity of uniform convergence on compacta: $\mathcal{T}_\mathcal{K}$ is the compact open topology on $Y^X$.

**Theorem 1.2** ([23, Theorem 8]) *Let* $X$ *be a topological space, and* $(Y, \mathcal{V})$ *a uniform space. A family (closed in* $\mathcal{C}_\mathcal{K}$ *family)* $\mathcal{F} \subset \mathcal{C}_\mathcal{K}$ *is precompact (compact) in* $(Y^X, \mathcal{V}|\mathcal{K})$ *iff*
    *(i)* $\mathcal{F}$ *is equicontinuous on compacta*
    *(ii)* $\mathcal{F}(x)$ *is precompact (compact),* $\quad x \in X$.

**Theorem 1.3** ([23, Theorem 16]) *Let* $X$, $(Y, \mathcal{T})$ *be topological spaces, and* $Y$ *regular. A family* $\mathcal{F} \subset \mathcal{C}_\mathcal{K}$ *closed in* $\mathcal{C}_\mathcal{K}$ *is compact in* $(Y^X, \mathcal{T}_\mathcal{K})$ *iff*
    *(i)* $\mathcal{F}$ *is evenly continuous on compacta*
    *(ii)* $\mathcal{F}(x)$ *is compact,* $\quad x \in X$.

More "theoretical" versions of the above theorems are [23, Theorem 4, Theorem 13]. Some results concerning quasi-uniform spaces can be found in the second part of [15]; the first part of that paper is more advanced, and it is devoted to the mappings acting on the spaces of mappings.

The subsequent two results extend the celebrated theorems of Michael on continuous selections.

**Theorem 1.4** ([20, Theorem 7]) *Let* $X$ *be a paracompact space, and* $Y$ *a pseudo-metrizable topological subspace of a locally convex space. Assume* $F\colon X \to 2^Y$ *is an lsc mapping with closed, complete and convex values. Then* $F$ *admits a continuous multivalues selection (with closed, complete and convex values).*

**Theorem 1.5** ([20, Theorem 8]) *Let $X$ be a normal space, and let $Y$ be a pseudo-metrizable topological subspace of a locally convex space. Assume $F\colon X \to 2^Y$ is an lsc mapping with closed, complete and convex values, and $F(X)$ is precompact (i.e. totally bounded). Then $F$ admits a continuous multivalues selection (with closed, complete and convex values).*

In [20] theorems on retractions, and on continuous extensions are also presented (generalizations of Michael's, and Dugundji's results), and an extension of the Bartle-Graves theorem is proved.

Maybe the following criterion for uniformly convex spaces can be useful:

**Theorem 1.6** ([26, Theorem 13]) *A normed space $(X, \|\cdot\|)$ is uniformly convex iff for each $\beta > 0$ there exists a $\delta > 0$ such that for each $u, x \in X$ any of the following conditions is satisfied:*

$(i)$   $\|x - u\|$, $\|x + u\| \le 1 + \delta$, $2\|x\| = \beta$ imply $\|u\| \le 1 - \delta$,
$(ii)$   $\|x - u\|$, $\|x + u\| \le 1 + \delta$, $2\|x\| = \beta$ imply $\|u\| \le 1$,
$(iii)$   $\|x - u\|$, $\|x + u\| \le 1$, $2\|x\| = \beta$ imply $\|u\| \le 1 - \delta$,
$(iv)$   $\|u\|$, $\|x\| \le 1 + \delta$, $\|x - u\| = \beta$ imply $\|(x + u)/2\| \le 1 - \delta$,
$(v)$   $\|u\|$, $\|x\| \le 1 + \delta$, $\|x - u\| = \beta$ imply $\|(x + u)/2\| \le 1$,
$(vi)$   $\|u\|$, $\|x\| \le 1$, $\|x - u\| = \beta$ imply $\|(x + u)/2\| \le 1 - \delta$.

In [27] an "order" reasoning is applied to mappings satisfying the triangle inequality. Consequently, extensions of the Ekeland, and the Brøndsted variational principles, and of the Takahashi fixed point theorem are obtained.

**Definition 1.7** ([27, Definition 1]) *Let $X$ be nonempty, and let $\varphi\colon X \times X \to \mathbb{R}$ be a mapping satisfying the triangle inequality. Then $(X, \varphi)$ is a structure if $\varphi(\cdot, x_0)$ has a finite lower bound for an $x_0 \in X$.*

It is worth noting that if $\varphi(\cdot, x_0)$ is bounded below for an $x_0 \in X$, then $\varphi(\cdot, x)$ is bounded below for each $x \in X$, as

$$\varphi(\cdot, x_0) \le \varphi(\cdot, x) + \varphi(x, x_0).$$

Therefore, each local structure is a global structure, and it is sufficient to consider the structures in [27]. The idea of completeness for structures was presented in [27, Definition 3].

The results of [30] are not so much abstract as those of [27]. First, let us present a somewhat more precise version of [30, Theorem 2] which is an extension of the well-known Brézis-Browder theorem. The proof is trivial (see [30]).

**Theorem 1.8** *Let $(X, \varphi)$ be a structure. Then for $\gamma = \inf\{\varphi(z, x_0) : z \in X\}$, each sequence $(\varphi(x_n, x_0))_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \varphi(x_n, x_0) = \gamma$, and each $x \in X$, we have $0 \le \varphi(x, x_0) - \gamma \le \liminf_{n \to \infty} \varphi(x, x_n)$.*

If for $\varphi \colon X \times X \to \mathbb{R}$ the triangle inequality is satisfied, then relation $\preccurlyeq$ is defined as follows:

$$y \preccurlyeq x \text{ iff } \varphi(y, x) \le 0, \quad x, y \in X \tag{1}$$

In addition, for $A \ne \emptyset$ ($A \subset X$) we adopt

$$\gamma = \inf\{\varphi(z, x_0) : z \in A\}, \quad m(A, \varphi) = \{z \in A : \varphi(z, x_0) = \gamma\}.$$

**Theorem 1.9** ([30, Theorem 9]) *Assume $(X, \varphi)$ is a complete structure. Then for $\preccurlyeq$ as in (1), and each maximal chain $A \subset X$ containing a predecessor of $x_0$, the set $m(A, \varphi)$ is nonempty, and for each $x \in m(A, \varphi)$, $y \in A$ with $\varphi(y, x) \le 0$ (i.e. $y \preccurlyeq x$) the following conditions hold: $\varphi(y, x_0) = \varphi(x, x_0) = \gamma$ (i.e. $y \in m(A, \varphi)$), and $\varphi(y, x) = 0$.*

The previous theorem, and the Altman one seem to be independent.

**Definition 1.10** ([30, Definition 14]) *Let $q \colon X \times X \to [0, \infty)$ be a mapping. Then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, q)$ if for each $\epsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that each $m, n \in \mathbb{N}$, $n_0 < m < n$ yield $q(x_n, x_m) < \epsilon$.*

Now, let us present an extension of the notion of $\omega$-distance due to Kada-Suzuki-Takahashi.

**Definition 1.11** ([30, Definition 15]) *Let $(X, d)$ be a metric space. A mapping $q \colon X \times X \to [0, \infty)$ is a d-istance in $X$ if $q$ satisfies the triangle inequality, and the following conditions hold:*

*$q(\cdot, x)$ is lower semicontinuous, $x \in X$,*

*each Cauchy sequence in $(X, q)$ is a Cauchy sequence in $(X, d)$.*

It is worth noting that the further results of [30] were extended in [44] to the case of uniform spaces.

If a mapping $q \colon X \times X \to \mathbb{R}$ satisfies the triangle inequality, and $\psi \colon X \to \mathbb{R}$ is a mapping, then $\varphi \colon X \times X \to \mathbb{R}$ defined by

$$\varphi(y, x) = \psi(y) + q(y, x) - \psi(x), \quad x, y \in X$$

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also satisfies the triangle inequality, and (1) has the following form:

$$y \preccurlyeq x \text{ iff } \psi(y) + q(y, x) - \psi(x) \leq 0, \quad x, y \in X. \qquad (2)$$

In [30, Lemma 18] conditions are presented under which $(X, \varphi)$ is a complete structure.

The subsequent theorem is a far extension of a result of Kada-Suzuki-Takahashi.

**Theorem 1.12** ([30, Theorem 21]) *Let $(X, d)$ be a complete metric space, $q$ a d-istance in $X$, and $\psi \colon X \to \mathbb{R}$ a lower semicontinuous mapping bounded below. Let $\preccurlyeq$ be defined by (2), and assume that the following holds:*

*for each $x \in X \setminus B$ there exists a $y \in X \setminus \{x\}$ such that $y \preccurlyeq x$.*

*Then for any $x_0 \in X \setminus B$, each maximal chain $A \subset X$ containing $x_0$ has a unique smallest element $x$, in addition satisfying:*

*(i)* $\psi(x) = \inf\{\psi(z) \colon z \in X\}$ *(i.e. $x \in B$),*
*(ii)* $\psi(x) + q(x, x_0) - \psi(x_0) = \inf\{\psi(z) + q(z, x_0) - \psi(x_0) \colon z \in A\} \leq 0$,
*(iii)* $0 < \psi(y) + q(y, x) - \psi(x), \quad y \in X \setminus \{x\}$.

The subsequent theorem is another interesting result on variations.

**Theorem 1.13** ([30, Theorem 22]) *Let $(X, d)$ be a complete metric space, $q$ a d-istance in $X$, and $\psi \colon X \to \mathbb{R}$ a lower semicontinuous mapping bounded below. Let $\preccurlyeq$ be defined by (2), and assume that the following condition holds:*

*for each $x \in X$ there exists a $y$ such that $y \preccurlyeq x$.*

*Then for any $x_0 \in X$, each maximal chain $A \subset X$ containing $x_0$ has a unique smallest element $x$, in addition satisfying:*

*(i)* $\psi(x) = \inf\{\psi(z) \colon z \in A\}$,
*(ii)* $\psi(x) + q(x, x_0) - \psi(x_0) = \inf\{\psi(z) + q(z, x_0) - \psi(x_0) \colon z \in A\} \leq 0$,
*(iii)* $0 < \psi(y) + q(y, x) - \psi(x), \quad y \in X \setminus \{x\}$,
*(iv)* $q(x, x) = 0$.

Now, let us present an extension of the Ekeland variational principle.

**Theorem 1.14** ([30, Theorem 25]) *Let $(X, d)$ be a complete metric space, $q$ a d-istance in $X$, and $\psi \colon X \to \mathbb{R}$ a lower semicontinuous mapping bounded below. Then the following are satisfied:*

*(i) for each $x_0 \in X$ there exists an $x \in X$ such that $\psi(x) \leq \psi(x_0)$, and $\psi(x) - q(y, x) < \psi(y)$, for each $y \in X \setminus \{x\}$,*

*(ii) for any $\epsilon > 0$, and each $x_0 \in X$ with $q(x_0, x_0) = 0$, and $\psi(x_0) \leq \epsilon + \inf\{\psi(z) : z \in X\}$ there exists an $x \in X$ such that $\psi(x) \leq \psi(x_0)$, $q(x, x_0) \leq 1$, and $\psi(x) - \epsilon q(y, x) < \psi(y)$, for each $y \in X \setminus \{x\}$.*

## 2 Fixed point theorems

The subsequent theorem extends the results of Caristi, and of Kada-Suzuki-Takahashi.

**Theorem 2.1** ([30, Theorem 24]) *Let $(X, d)$ be a complete metric space, $q$ a d-istance in $X$, and $\psi \colon X \to \mathbb{R}$ a lower semicontinuous mapping bounded below. Let $\preccurlyeq$ be defined by (2), and assume that $X \subset Y$, and $F \colon X \to 2^Y$ is a mapping satisfying:*

*for each $x \in X$ there exists a $y \in F(x)$ such that $y \preccurlyeq x$.*

*Then for any $x_0 \in X$, each maximal chain $A \subset X$ containing $x_0$ has a unique smallest element $x$, in addition satisfying conditions (i),...,(iv) of Theorem 1.13, and such that $x \in F(x)$.*

Another consequence of [30, Theorem 22] is the following one:

**Theorem 2.2** ([30, Theorem 23]) *Let $(X, d)$ be a complete metric space, $q$ a d-istance in $X$, and $\psi \colon X \to \mathbb{R}$ a lower semicontinuous mapping bounded below. Assume $X \subset Y$, and $F \colon X \to 2^Y$ is a mapping satisfying:*

*for each $x \in X \setminus F(x)$ there exists a $y \neq x$*
*such that $\psi(y) + q(y, x) - \psi(x) \leq 0$.*

*Then $F$ has a fixed point.*

Further generalizations of the above two results are [44, Theorems 8, 9].

The subsequent theorem is an extension of the Nadler theorem (see also [45, Lemma 1.3]).

**Theorem 2.3** ([45, Theorem 1.4]) *Let $(X, d)$ be a complete metric space and $F \colon X \to 2^X$ a closed-valued mapping satisfying:*

*for each $(x_n)_{n \in \mathbb{N}}$ in $X$ that converges, say to an $x \in X$,*
*if $\lim_{n \to \infty} d(F(x_n), x_n) = 0$, then $d(F(x), x) = 0$*

*(e.g. if the graph of $F$ is closed or $d(F(\cdot), \cdot)$ is lower semicontinuous). If, in addition,*

*for each $x \in X$ there exists a $y \in F(x)$ such*
*that $d(F(y), y) \leq \alpha d(y, x) \leq (\alpha + \epsilon) d(F(x), x)$*

*holds for some $\alpha + \epsilon < 1$, then $F$ has a fixed point.*

For the "compact" case we have the following (inward) result:

**Theorem 2.4** *Let $X$ be a compact set in a metric space $(Y, d)$, and let $F \colon X \to 2^Y$ be a mapping satisfying:*

*for each $x \in X \setminus F(x)$ there exists a $y \in X$*
*such that $d(y, F(y)) < d(x, F(x))$,*

*and such that $d(F(\cdot), \cdot)$ is lower semicontinuous. Then $F$ has a fixed point.*

Now, let us present some results for nonexpansive mappings.

**Definition 2.5** ([24, Definition 6]) *A metric space $(X, d)$ is a bead space if the following is satisfied:*

*for every $r > 0, \beta > 0$ there exists a $\delta > 0$ such that*
*for each $x, y \in X$ with $d(x, y) \geq \beta$ there exists a $z \in X$*
*such that $B(x, r + \delta) \cap B(y, r + \delta) \subset B(z, r - \delta)$.*

The above idea extends the notion of uniformly convex space as follows from the next theorem.

**Theorem 2.6** (cp. [26, Theorem 14]) *For any normed space $(X, \| \cdot \|)$ the following conditions are equivalent:*

> (i)    $X$ is a bead space,
> (iii)   $X$ is a uniformly convex space.

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If $(X, d)$ is a complete bead space, then each nonempty and bounded subset $A$ of $X$ has a unique central point ([24, Lemma 10]). The centre $c(F)$ of a mapping is presented in [24, Definition 3]. We have the following result:

**Theorem 2.7** ([24, Theorem 15]) *Let $(X, d)$ be a complete bead space, and let $f: X \to X$ be a mapping such that for a bounded nonempty set $Y \subset X$ we have $f(Y) \subset Y$. Then there exists a unique point $x \in c(f_{|Y})$; if condition*

$$d(f(x), f(y)) \leq d(x, y), \quad y \in Y$$

*is satisfied then $x$ is a fixed point for $f$.*

In particular, Theorem 2.7 yields the following one:

**Theorem 2.8** ([24, Theorem 17]) *Let $(X, d)$ be a complete bead space, and let $f: X \to X$ be a nonexpansive mapping. Assume that $Y \subset X$ is a bounded nonempty set such that $f(Y) \subset Y$. Then $c(f_{|Y})$ is a singleton consisting of a fixed point for $f$.*

In view of Theorem 2.6, the above two theorems (with short proofs) are extensions of the well known Browder-Göhde-Kirk theorem for uniformly convex spaces. More advanced results are presented in [25].

Let us present a modification of an idea of Lasry and Robert.

**Definition 2.9** ([18, Definition 2]) *Let $Y$ be a topological space, and $M$ a linear topological space. We say that a mapping $F: Y \to 2^M$ is uhc (i.e. upper hemicontinuous) if for each continuous linear functional $x$ on $M$ (i.e. $x \in M^*$), mapping $\sup rx \circ F: Y \to \overline{\mathbb{R}}$ is upper semicontinuous, where $rx$ is the real part of $x$.*

If $M$ is a linear topological space, then any $F: Y \to 2^M$ which is usc is also uhc.

**Theorem 2.10** ([18, Theorem 2]) *Let $Y$ be a convex set in a locally convex space $M$, and $F: Y \to 2^M$ a uhc mapping with closed convex values. Assume that $Y \cap F(Y)$ is relatively compact in $Y$, and $Y \cap F(y) \neq \emptyset$, $y \in Y$. Then $F$ has a fixed point.*

An almost immediate consequence of Theorem 2.10 is the following one:

**Theorem 2.11** ([18, Theorem 4]) *Let $Y$ be a convex set in a locally convex space $M$, and $F: Y \to 2^M$ a uhc mapping with closed compact convex values and such that $Y \cap F(Y)$ is relatively compact in $Y$. If the following condition is satisfied:*

$$\inf rx(Y) \leq \sup rx(F(y)), \quad x \in M^*, y \in Y,$$

*then $F$ has a fixed point.*

For a $t \in \mathbb{N}$ let us adopt $t + + = 1$, and $i + + = i + 1$, $i < t$. These notations make the next theorem shorter.

**Theorem 2.12** ([18, Theorem 3]) *Assume the following conditions for all $i = 1, \ldots, t$: $Y_i$ is a convex set in a locally convex space $M_i$, $F_i: Y_i \to 2^{M_{i++}}$ is a uhc mapping with closed convex values, and such that $Y_{i++} \cap F_i(Y_i)$ is realtively compact, $Y_{i++} \cap F_i(y_i) \neq \emptyset$, $y_i \in Y_i$. Then $F_t \circ \cdots \circ F_1$ has a fixed point.*

A consequence of the above theorem is the subsequent one.

**Theorem 2.13** ([17, Theorem 3.1.8, p. 42]) *Assume the following conditions for all $i = 1, \ldots, t$: $X_i$ is a convex set in a locally convex space, and $F_i: X_i \to 2^{X_{i++}}$ is a compact mapping with closed convex values. Then $F_t \circ \cdots \circ F_1$ has a fixed point.*

Now, let us present an extension of a metric.

**Definition 2.14** ([47, Definition 1]) *A mapping $p: X \times X \to [0, \infty)$ is a felt metric for $X$ if the following system of conditions is satisfied:*

$$p(x, y) = 0 \ \textit{yields} \ x = y, \quad x, y \in X, \tag{3a}$$

$$p(x, y) = p(y, x), \quad x, y \in X, \tag{3b}$$

*for each $\epsilon > 0$ there exists a $\delta > 0$ such that*
$$p(z, y) < \delta \ \textit{yields} \ |p(z, x) - p(y, x)| < \epsilon, \quad x, y, z, \in X. \tag{3c}$$

*A felt metric space $(X, p)$ is 0-complete, if for each sequence $(x_n)_{n \in \mathbb{N}}$ in $X$, such that $\lim_{m,n \to \infty} p(x_n, x_m) = 0$, there exists an $x \in X$ for which $\lim_{n \to \infty} p(x_n, x) = 0$; a selfmapping $f$ on $X$ is 0-continuous at an $x \in X$, if $\lim_{n \to \infty} p(x_n, x) = 0$ yields $\lim_{n \to \infty} p(fx_n, fx) = 0$, for each sequence $(x_n)_{n \in \mathbb{N}}$ in $X$; $f$ is 0-continuous if it is 0-continuous at each $x \in X$.*

The subsequent corollary is a criterion for Cauchy sequences (in particular in metric spaces).

**Corollary 2.15** ([40, Corollary 6.9]) *Assume that $p$ is a felt metric for $X$, and $(x_n)_{n \in \mathbb{N}}$ is a sequence of points of $X$. Then $\lim_{m,n \to \infty} p(x_n, x_m) = 0$ iff $\lim_{n \to \infty} p(x_{n+1}, x_n) = 0$, and the following condition holds:*

*for each $\alpha > 0$ there exist $\epsilon > 0$, and $n_0 \in \mathbb{N}$ such that for all*
*$m, n > n_0$, $\alpha - \epsilon < p(x_n, x_m) < \alpha + \epsilon$ implies $p(x_{n+1}, x_{m+1}) \leq \alpha$.* $\quad$ (4)

The above corollary yields a fairy general fixed point theorem.

**Theorem 2.16** ([40, Theorem 6.4]) *Assume that $p$ is a felt metric for $X$, $f$ is a selfmapping on $X$, and $x_n = f^n x_0$, $n \in \mathbb{N}$. If $\lim_{n \to \infty} p(x_{n+1}, x_n) = 0$, (4) is satisfied, and $(X, p)$ is 0-complete, then $\lim_{n \to \infty} p(x_n, x) = p(x, x) = 0$ for some $x \in X$; if $\lim_{n \to \infty} p(fx_n, fx) = 0$, then $fx = x$.*

In particular, from Theorem 2.16 we get the following extension of the Banach fixed point theorem (a simplified version of [46, Theorem 2]):

**Theorem 2.17** ([47, Theorem 2]) *Assume that $p$ is a felt metric for $X$, $f$ is a selfmapping on $X$, $x_n = f^n x_0$, $n \in \mathbb{N}$, $\lim_{n \to \infty} p(x_{n+1}, x_n) = 0$, and let*

*for each $\alpha > 0$ there exists an $\epsilon > 0$ such that*
*$\alpha \leq p(y, x) < \alpha + \epsilon$ yields $p(fy, fx) \leq \alpha$, $\quad x, y \in X$* $\quad$ (5)

*be satisfied. If $(X, p)$ is 0-complete, then there exists an $x \in X$ such that $\lim_{n \to \infty} p(x_n, x) = p(x, x) = 0$ and $fx = x$.*

**Definition 2.18** ([48, Definition 2.1]) *A mapping $\varrho \colon X \times X \to [0, \infty)$ is a db-metric (dislocated b-metric) for $X$ if the following conditions are satisfied*

$$\varrho(x, y) = 0 \ yields \ x = y, \quad x, y \in X, \tag{6a}$$
$$\varrho(x, y) = \varrho(y, x), \quad x, y \in X, \tag{6b}$$
$$\varrho(x, z) \leq s[\varrho(x, y) + \varrho(y, z)], \quad x, y, z \in X, \ for \ an \ s \geq 1. \tag{6c}$$

*Then $(X, \varrho)$ is a db-metric space (dislocated b-metric space).*

The next lemma is a criterion for Cauchy sequences in db-metric spaces.

**Lemma 2.19** ([48, Lemma 2.3]) *For $X = \{x_n \colon n \in \mathbb{N}\}$ and $\varrho \colon X \times X \to [0, \infty)$ let (6b), (6c) be satisfied. Then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(X, \varrho)$ if and only if the following conditions hold*

$$\lim_{n \to \infty} \varrho(x_{n+1}, x_n) = 0, \tag{7a}$$

*for each $\delta > 0$ there exist $n_0, p \in \mathbb{N}$ such that*
$$0 < \varrho(x_n, x_m) < \delta \ yields \ \varrho(x_{n+p}, x_{m+p}) < \delta/2s, \quad m, n > n_0. \tag{7b}$$

The two (equivalent) theorems below extend the well known results of Czerwik for b-metric spaces, and of Matkowski for metric spaces.

**Theorem 2.20** ([48, Theorem 2.5]) *Assume that $(X, \varrho)$ is a 0-complete db-metric space, and $f$ is a selfmapping on $X$. If the following conditions are satisfied*

$$sup\{\varrho(fy, fx) \colon 0 < \varrho(y, x) \le \alpha\} < \alpha, \quad \alpha > 0, \tag{8a}$$

*for each $\alpha > 0$ there exists an $\epsilon > 0$ such that*
$$\alpha \le \varrho(y, x) \le \alpha + \epsilon \ yields \ \varrho(fy, fx) \le \alpha, \quad x, y \in X, \tag{8b}$$

*then $f$ has a unique fixed point $x$, and $\lim_{n \to \infty} \varrho(f^n x_0, x) = \varrho(fx, x) = 0$, $x_0 \in X$.*

**Theorem 2.21** ([48, Theorem 2.7]) *Assume that $(X, \varrho)$ is a 0-complete db-metric space, and $f$ is a selfmapping on $X$. If for a function $\varphi \colon (0, \infty) \to [0, \infty)$ the following conditions are satisfied*

$$\varrho(y, x) > 0 \ yields \ \varrho(fy, fx) \le \varphi(\varrho(y, x)), \quad x, y \in X, \tag{9a}$$

$$sup\{\varphi(t) \colon 0 < t \le \alpha\} < \alpha, \quad \alpha > 0, \tag{9b}$$

*for each $\alpha > 0$ there exists an $\epsilon > 0$ such that*
$$\alpha \le t \le \alpha + \epsilon \ \ yields \ \varphi(t) \le \alpha, \tag{9c}$$

*then $f$ has a unique fixed point $x$, and $\lim_{n \to \infty} \varrho(f^n x_0, x) = \varrho(fx, x) = 0$, $x_0 \in X$.*

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The list of all papers with me as a co-author is presented below.

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