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A short proof of the Caristi theorem

In 1975 Caristi [2] proved a fixed point theorem which was next used in proving some other results [2], [4]. The original proof used the transfinite induction method and was rather complicated. Then several other proofs were found [1], [3], [4], but all of them are more complicated than the new proof presented below, which uses Zorn's lemma, but in a simple way.

Theorem 1 (Wong [4]). *Let f be a self map on a non-empty complete metric space (X, d) and $V\colon X\to[0, \infty)$ a lower semicontinuous function. Let us presume that the following condition holds:*

(1) *For any $x \in X$, $x \neq f(x)$ there exists $y \in X-\{x\}$ such that:*

$$d(x, y) \leqslant V(x)-V(y).$$

By these assumptions f has a fixed point in X.

Proof. In view of Zorn's lemma there exists a maximal set $A \subset X$, $a \in A$ such that for all points $x, y \in A$

$$d(x, y) \leqslant |V(x)-V(y)|.$$

For $a = \inf\big(V(x)\colon x \in A\big)$ there exists a sequence of points $z_i \in A$ such that $\big(V(z_i)\big)_{i\in N}$ is non-increasing and $\lim\limits_{i\to\infty} V(z_i) = a$.

It follows from

$$d(z_i, z_j) \leqslant |V(z_i)-V(z_j)|$$

that there exists $b \in X$, $b = \lim\limits_{i\to\infty} z_i$.

For any $x \in A$, if $V(x) \neq a$, then for sufficiently large i we have

$$d(b, x) \leqslant d(b, z_i)+d(z_i, x) \leqslant d(b, z_i)+V(x)-V(z_i)$$

(if for $x_0 \in A\, V(x_0) = a$, then we obtain in a similar way $d(b, x_0) = 0$) and then by the lower semicontinuity of V

$$d(b, x) \leqslant V(x)-V(b).$$

This means that $b \in A$ and that there is no point $y \in X$ such that $b \neq y$ and

$$d(b, y) \leqslant V(b) - V(y)$$

because such y would belong to A. Then it must be so that:

$$d(b, f(b)) = 0 \qquad \text{Q.E.D.}$$

The theorem which follows, is a consequence of Theorem 1.

THEOREM 2 (Caristi [2]). *Let all assumptions of Theorem* 1 *except* (1) *be satisfied. Let us assume that for* $x \in X$

$$d(x, f(x)) \leqslant V(x) - V(f(x)).$$

Then f has a fixed point.

We can modify Wong's theorem as follows:

THEOREM 3. *Let (X, d) be a non-empty complete metric space and $V: X \to [0, \infty)$ a lower semicontinuous function. If a sentence formula g satisfies the following condition on X:*

from $\sim g(x)$ it follows that there exists $y \in X$ such that:

$$d(x, y) \leqslant V(x) - V(y),$$

then there exists $b \in X$ such that $g(b)$.

References

[1] F. E. Browder, *On theorem of Caristi and Kirk* (to appear).
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[3] W. A. Kirk, *Caristi's fixed point theorem and metric convexity* (to appear).
[4] C. S. Wong, *On a fixed point theorem of contractive type*, preprint.