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GENERAL TOPOLOGY

Three Fixed Point Theorems

by

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Summary. The paper contains three fixed point theorems for type I spaces defined below. Theorem 1 is a generalization of Tychonoff's theorem and the proof of it can be treated as a short proof of the Tychonoff theorem itself. The two other theorems are related to measures of noncompactness (Theorem 2) and the notion of generalized condensing maps (Theorem 3).

DEFINITION 1 (cf. [2, 11]). A space X is S contractible if for any $x \in X$ $\{S_x(t, \cdot)\}$ is a homotopy joining the identity with a constant map ([1] p. 22). Then for any $\emptyset \neq A \subset X$ $\mathrm{co}SA = \cap \{D \subset X: A \subset D$ and for any $x \in A$ $S_x(I, D) \subset D\}$. For $A = \emptyset$ let $\mathrm{co}S\,A = \emptyset$.

DEFINITION 2 [11]. A space X is of type I if there exists S such that X is S contractible and

(1) for any neighborhood N of any $x \in X$ there exists a neighborhood U such that $\mathrm{co}S\,U \subset N$.

THEOREM 1. *Let X be a type I space. Let $f: X \to X$ be a map such that $\overline{f(X)}$ is a compact set. Then f has a fixed point.*

Proof. Let us assume f has no fixed point. Thus in view of (1) there exists a family of open neighborhoods $\mathscr{W} = \{W_x\}_{x \in \overline{f(X)}}$ in $\overline{f(X)}$ such that

(2) for every $x \in \overline{f(X)}$ $\mathrm{co}S\,W_x \cap f^{-1}(W_x) = \emptyset$.

Let $\{U_i\}_{i=1,\ldots,n}$ be a finite cover consisting of the members of an open barycentric refinement of $\mathscr{W}$ (cf. [3] 5.1.12 p. 377). Define the mapping $g: I^n \to X$ as follows (cf. [4, 9]):

$$g(s_1, \ldots, s_n) = S_{x_1}\big(t_1, S_{x_2}\big(t_2, \ldots, S_{x_{n-1}}(t_{n-1}, x_n) \ldots\big)\big),$$

where $\sum_{i=1}^{n} s_i = 1$ $x_i \in U_i$ and $t_i = s_i / \max\{s_i: i = 1, \ldots, n\}$. From [3] (3.4.8 p. 210, 3.1.10 p. 168, 2.3.6 p. 108) it follows by induction that g is a map.

Besides let us write (cf. [9]):

$$K\{i_1, \ldots, i_k\} = \{g(s_1, \ldots, s_n): s_i = 0 \text{ for } i \neq i_j,\ j = 1, \ldots, k\}.$$

We see that $U_{i_1} \cap \ldots \cap U_{i_k} \neq \emptyset$ implies $\operatorname{co}S(U_{i_1} \cup \ldots \cup U_{i_k}) \subset \operatorname{co}S\, W_{i_j}$ and, on the other hand, $U_{i_1} \cap \ldots \cap U_{i_k} = \emptyset$ implies $f^{-1}(U_{i_1}) \cap \ldots \cap f^{-1}(U_{i_k}) = \emptyset$. Thus for $X_i = X \setminus f^{-1}(U_i) = \bar{X}_i$ by (2) we always have $K\{i_1, \ldots, i_k\} \subset X_{i_1} \cup \ldots \cup X_{i_k}$ which implies $\bigcap_{i=1}^{n} X_i \neq \emptyset$ as $g^{-1}(K\{i_1, \ldots, i_k\})$ is closed and contains a simplex (cf. [4, 5]). Now it is seen that

$$X \neq X \setminus \bigcap_{i=1}^{n} X_i = \bigcup_{i=1}^{n} f^{-1}(U_i) = X.$$

This contradiction proves that f has a fixed point.

Let now X be a space with a measure of noncompactness $b\colon 2^X \to R$ satisfying the following conditions for every $A, B \subset X$

(3) $b(A) = 0$ iff $\bar{A}$ is compact.

(4) if $A \subset B$ then $b(A) \leqslant b(B)$.

(5) $b(A) = b(A \cup \{x\})$ for every $x \in X$.

LEMMA 1 (cf. [6, 10]). *Let X be a topological space equipped with a measure of noncompactness defined above and let $\{G_t\}_{t \in T}$ be a family of closed sets such that $b(G_t) > 0$ for all $t \in T$, $\inf\{b(G_t)\colon t \in T\} = 0$ and besides $b(G_t) < b(G_k)$ implies $G_t \subset G_k$ for every $k, t \in T$. Then $\bigcap_{t \in T} G_t \neq \emptyset$.*

Proof. Let $(t_i)_{i \in N}$ be a sequence of indexes for which $(b(G_{t_i}))_{i \in N}$ is a decreasing sequence convergent to zero. Let $x_n \in G_{t_n}$ for every $n \in N$. Then we have

$$b\Big(\bigcup_{n=1}^{\infty} x_n\Big) = b\Big(\bigcup_{n=k}^{\infty} x_n\Big) \leqslant b\Big(\bigcup_{n=k}^{\infty} G_{t_n}\Big) = b(G_{t_k}) \to 0 \text{ as } k \to \infty.$$

Thus $\overline{\bigcup_{n=1}^{\infty} x_n}$ is compact and $\emptyset \neq E = \bigcap_{k=1}^{\infty} \Big(\overline{\bigcup_{n=k}^{\infty} x_n}\Big) \subset \bigcap_{k=1}^{\infty} G_{t_k}$. We have $b\Big(\bigcap_{k=1}^{\infty} G_{t_k}\Big) = 0$ and therefore $E \subset \bigcap_{t \in T} G_t$.

DEFINITION 3. A space X is of type $\bar{I}$ if it is of type I for S satisfying

(6) for any $A \subset X$ $\overline{\operatorname{co}}S\, A$ is S-convex.

THEOREM 2. *Let f be a self-map for a type $\bar{I}$ space X equipped with a measure of noncompactness b satisfying conditions (3), (4), (5) such that $b(X) < \infty$ and for any $A \subset X$ such that $f(A) \subset A$ and $b(\overline{\operatorname{co}}S\, A) \leqslant b(\overline{\operatorname{co}}S\, f(A))$ implies $\overline{\operatorname{co}}S\, f(A)$ is compact. Then f has a fixed point* (cf. [12]).

Proof. Let $G_1 = X$, $G_n = \overline{\operatorname{co}}S\, f(G_{n-1})$ for $n > 1$. If for all n we have $b(G_n) > 0$, $\{G_i\}_{i \in N}$ satisfies the assumptions of Lemma 1 and therefore $G := \bigcap_{i=1}^{\infty} G_i \neq \emptyset$. Besides G is of type I ([11] Prop. 1) and obviously $f(G) \subset G$. Therefore f has a fixed point.

LEMMA 2 ([8] Lemma). *Let X be a convexly S-contractible space (i.e. (6) is satisfied) and $g\colon X \to 2^X$ a set-valued mapping. If there is a compact set $D \subset f$ for which $g(D) \subset D$), there exists a set $C = \overline{\operatorname{co}}S\, C$ with $\overline{\operatorname{co}}S\, g(C) = C$.*

THEOREM 3. *Let X be a type 1 space and $f: X \to X$ a map such that for any $C \subset X$ for which $f(C) \subset C$, from the compactness of $\overline{C \setminus \overline{\mathrm{co}} S f(C)}$ it follows that $\bar{C}$ is compact. Then f has a fixed point* (cf. [7]).

Proof (cf. [10]). In view of Lemma 2 it is enough to show that there exists a compact set $B \subset X$ such that $f(B) \subset B$. Let $B \subset X$ be any minimal closed set containing a compact set D and such that $f(B) \subset B$. We see that $B \setminus \overline{f(B)} \subset D$ because $(B \setminus \overline{f(B)}) \cap$ $\cap (X \setminus D)$ is relatively open in B and would be rejected when nonempty. Therefore B must be compact.

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Л. Пасицки, **Три теоремы о неподвижной точке**

Содержание. Статья содержит три теоремы о неподвижной точке отображении так называемых пространств типа 1. Теорема 1 является обобщением теоремы Тихонова; ее доказательство представляет собой короткое доказательство теоремы Тихонова. В двух других теоремах рассматриваются меры некомпактности (Теорема 2) и обобщенные накапливающие отображения (Теорема 3).