ON THE CELLINA THEOREM OF NON-EMPTY INTERSECTION

BY

LECH PASICKI

The paper extends Cellina's theorem concerning intersection of closed convex sets in a Banach space to the sequences of sets in metric spaces, while Kuratowski's measure of noncompactness is substituted by a class of more general measures.

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Let (M, d) be a nonempty metric space with a mapping S.

$$S : M \times [0,1] \times M \to M,$$

$$S : (x, \alpha, y) \mapsto S_x(\alpha, y),$$

such that for all $x, y \in M S_x(0, y) = y$ and $S_x(1, y) = x$.

DEFINITION 1. We say that a nonempty set $A \subset M$ is S-convex i.e. $A = \mathrm{coS}\, A$ if for any $x \in A$, $\alpha \in [0, 1]$ $S_x(\alpha, A) \subset A$.

It is easily seen that an intersection of two S-convex sets is S-convex or empty.

Let $b : 2^M \to R$ be a measure of noncompactness that satisfies

1) $b(A) = 0$ iff $\bar{A}$ is compact.

2) $b(A \cup \{x\}) = b(A)$.

3) if $A \subset B$ then $b(A) \leqslant b(B)$.

LEMMA 1. *Let $\{A_n\}_{n \in N}$ be a decreasing sequence of closed sets in a complete metric space (M, d) with $\lim_{n\to\infty} b(A_n) = 0$. Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$* (cf. [2]).

Proof [3]. We choose a sequence $\{x_n\}_{n \in N}$ such that for all $n x_n \in A_n$. From 2), 3) it follows that $b(\{x_1, x_2, x_3, \ldots\}) = b(\{x_n, x_{n+1}, \ldots\}) \leqslant \leqslant b(A_n) \to 0$ as n tends to the infinity. Then $\{x_n\}_{n \in N}$ is compact and contains a convergent subsequence the limit of which belongs to all A_n.

REMARK. A measure that satisfies 1) and

4) for any $B \subset M$ $b(A \cup B) = \max\{b(A), b(B)\}$

satisfies 2) and 3).

THEOREM 1. *Let (M, d) be a complete metric space and let $\{A_n\}_{n\in N}$ be a decreasing sequence of closed S-convex sets with $\lim_{n\to\infty} b(A_n \setminus A_{n+1}) = 0$ for a measure of noncompactness that satisfies* 1) *and* 4).
Let, besides the following conditions, be satisfied:

5) for any $C \subset M$, $k \in [0,1]$, there exist $x \in C$ and $\alpha \in [0,1]$ such that $kb(C) = b(S_x(\alpha, C))$,

6) for any family of sets $\{C_n\}_{n\in N} \subset M$ $\lim_{n\to\infty} b(C_n) = 0$ implies that $\lim_{n\to\infty} b(\overline{co}S\, C_n) = 0$.

Then, we have $A : \bigcap_{n=1}^{\infty} A_n = \overline{co}S\, A \neq \emptyset$.

Proof (cf. [1]).

If there exists a subsequence $\{A_{n_k}\}$ with $\lim_{k\to\infty} b(A_{n_k}) = 0$, then in view of Lemma 1 $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

Suppose then, that for all $n \geqslant n_0$

7) $b(A_{n+1}) \geqslant b(A_n \setminus A_{n+1}) + a_n'$, $a_n' > 0$,

and $b(A_{n_0} \setminus A_{n_0+1}) > b(A_n \setminus A_{n+1})$ and let $\{a_n\}$ be a decreasing, convergent to zero sequence that satisfies $0 < a_n < a_n'$ for all $n \geqslant n_0$.

Let us choose $x \in A_{n_0}$ and $\alpha \in [0, 1]$ such that $b(S_x(\alpha, A_{n_0})| = b(A_n \setminus A_{n+1}) + a_n$. If we take $Z_1 := \overline{co}S\, S_x(\alpha, A_{n_0})$, then $Z_1 \subset A_{n_0}$ and $b(Z_1) = b(\{Z_1 \cap (A_{n_0} \setminus A_{n_0+1})\} \cup \{Z_1 \cap A_{n_0+1}\})$, but $b(\{Z_1 \cap \cap (A_{n_0} \setminus A_{n_0+1})\}) \leqslant b(A_n \setminus A_{n+1}) < b(A_{n_0} \setminus A_{n_0+1}) + a_{n_0} = b(Z_1)$. So in view of 4) it must be $b(Z_1) = b(Z_1 \cap A_{n_0+1})$.

Let $n_1 > n_0$ and $b(A_{n_1} \setminus A_{n_1+1}) > b(A_n \setminus A_{n+1})$ for $n > n_1$. Repeating the above consideration several times, we obtain that $b(Z_1 \cap A_{n_1}) = b(Z_1)$. Now we choose $x \in Z_1 \cap A_{n_1}$ and $\alpha \in [0,1]$ such that $b(S_x(\alpha, Z_1 \cap A_{n_1})) = b(A_{n_1} \setminus A_{n_1+1}) + a_{n_1}$, $Z_2 = \overline{co}S\, S_x(\alpha, Z_1 \cap A_{n_1})$ etc.

We see that $Z_1 \supset Z_2 \supset \ldots$ and $\lim_{n\to\infty} b(Z_n) = 0$. If we choose $\{x_n\}_{n\in N}$ such that for any $n \in N$ $x_n \in Z_n$, then by reasoning as in the proof of Lemma 1 we state that $\bigcap_{n=1}^{\infty} \bar{Z}_n \neq \emptyset$, an therefore $A := \bigcap_{n=1}^{\infty} A_n \neq \emptyset$. It is obvious that $\overline{co}S\, A = A$.

DEFINITION 2. A contraction S is uniform if

8) for any $x \in A$, $\varepsilon > 0$, there exists $\delta > 0$ such that for any $\alpha_1, \alpha_2 \in [0,1]$, $y \in A$ $|\alpha_1 - \alpha_2| < \delta$ implies $d(S_x(\alpha_1, y), S_x(\alpha_2, y)) < \varepsilon$.

LEMMA 2. *If S is uniform, then it satisfies*

9) *for any $\emptyset \neq C \subset M$, $\{\alpha_n\}_{n\in N} \subset [0,1]$, $x \in M$ from $\lim_{n\to\infty} \alpha_n = \alpha_0$ it follows that $\lim_{n\to\infty} D(S_x(\alpha_n, C), S_x(\alpha_0, C)) = 0$, where D denotes the Hausdorff distance of sets for metric d.*

Proof. For any $z \in S_x(\alpha, C)$ $\alpha \in [0,1]$, there exists $z' \in C$ such that $z = S_x(\alpha, z')$, and therefore 8) implies 9).

THEOREM 2. *Let (M, d) be a complete metric space and let $\{A_n\}_{n\in N}$ be a decreasing sequence of closed S-convex sets for an uniform contraction S with $\lim_{n\to\infty} b(A_n \setminus A_{n+1}) = 0$ for a measure of noncompactness satisfying 1), 4), 6) and*

10) $\lim_{n\to\infty} D(C_n, C) = 0$ *implies* $\lim_{n\to\infty} b(C_n) = b(C)$ *for any* $\{C_n\} \subset M$.

Then we have $A := \bigcap_{n=1}^{\infty} A_n = \overline{\mathrm{co}}\mathrm{S}\, A \neq \emptyset$.

Condition 10) shows that b is continuous in Hausdorff metric which with 9) implies 5).

PROPOSITION. *If a measure b satisfies*

11) *there exists $f : [0, \infty) \to R$ such that $\lim_{t\to 0^+} f(t) = 0$, and for any $A \subset M$ $b(B(A, \varepsilon)) \leqslant b(A) + f(\varepsilon)$, where for $\emptyset \neq C \subset M$ $B(C, \varepsilon) = \{x \in M : \inf\{d(x, y) : y \in C\} < \varepsilon\}$, then 10) is satisfied.*

Proof. $D(C_n, C) < \delta$ is equivalent to $C_n \subset B(C, \delta)$ and $C \subset B(C_n, \delta)$, and therefore

$$b(C_n) \leqslant b(C) + f(\delta), b(C) \leqslant b(C_n) + f(\delta),$$

and hence $b(C) - f(\delta) \leqslant b(C_n) \leqslant b(C) + f(\delta)$.

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ul. Krupnicza 12/9, 31—123 Kraków
Poland

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