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A generalization of Reich's fixed point theorem

Abstract. In the paper we give a generalization of Browder's fixed point theorem for inward maps which enables us to obtain results concerning generalized condensing maps.

DEFINITION 1 [4]. Let a set X and a function S be given that satisfy the following conditions:

- (1) $S: X \times I \times X \ni (x, t, y) \mapsto S_x(t, y) \in X;$
 (2) $S_x(0, y) = y, S_x(1, y) = x$ for any $x, y \in X$.

Then for any non-empty set $A \subset X$, let $\text{co}SA = \inf\{D \subset X: A \subset D$ and for any $x \in A, t \in I S_x(t, D) \subset D\}$. For $A = \emptyset$, let $\text{co}SA = \emptyset$. If $\text{co}SA = A$ then A is S -convex.

DEFINITION 2 (cf. [4]). A space X is S -contractible if S satisfies conditions (1), (2) and for any $x \in X$ $\{S_x(t, \cdot)\}$ is a homotopy joining the identity with a constant map ([1], p. 22).

DEFINITION 3. A space X is \bar{S} -contractible if it is S -contractible for S such that for any $A \subset X$ $\text{co}S\bar{\text{co}}SA = \bar{\text{co}}SA$.

DEFINITION 4 (cf. [4]). A space X is of *type I* (type \bar{I}) if there exists S such that X is S -contractible (\bar{S} -contractible) and

- (3) for any neighbourhood N of any $x \in X$ there exists a neighbourhood U such that $\text{co}SU \subset N$.

Let (M, d) be a metric space. For the non-empty sets $A, D \subset M$ and $r > 0$, let us write $d(A, D) = \inf\{d(x, y): x \in A, y \in D\}$, $B(A, r) = \{x \in M: d(A, x) < r\}$ and $\text{dia}A = \sup\{d(x, y): x, y \in A\}$.

The following theorem was proved in [4].

THEOREM 1. Let A be a compact type I subset of a metric space (M, d) and let $f: A \rightarrow M$ be a map. Let us write for $z \in M, s > 0$ $A(z, s) = \bar{\text{co}}S\{B(z, d(z, A) + s) \cap A\}$ and $A_z = \bigcap_{s>0} A(z, s)$. Then the condition

- (4) for every $z \in M \setminus A, f^{-1}(z) \cap A_z = \emptyset$

implies $\text{Fix } f \neq \emptyset$.

Ky Fan [2] has proved the following:

THEOREM. *Let A be a non-empty compact convex set in a locally convex linear Hausdorff topological space X . Then if a continuous mapping $f: A \rightarrow X$ satisfies*

(5) *for every $y \in A$ there exists $c \in \mathbb{C}$, $|c| < 1$, such that $cy + (1-c)f(y) \in A$,*

then $\text{Fix } f \neq \emptyset$.

PROPOSITION. *Let A be a convex subset of a linear normed space X . If $z \notin A$ and (5) holds with z in place of $f(y)$, then $y \notin A_z$.*

Proof. Suppose $y \in A_z$. Then $\|z - y\| = d(z, A)$, because for $S_x(t, y) := tw + (1-t)y$ we have $A_z = \bar{B}(z, d(z, A)) \cap A$. On the other hand, $cy + (1-c)z \in A$ and thus $\|z - cy - (1-c)z\| \leq |c| \|z - y\| + (1-c) \|z - z\| < d(z, A)$ is a contradiction.

The Proposition shows that Theorem 1 is more general than Ky Fan's theorem in the case where X is a linear normed space (simple examples show that these theorems are not equivalent).

LEMMA. *Let X be a \bar{S} -contractible space and $g: X \rightarrow 2^X$ a set valued mapping. If there is a compact set $D \subset X$ for which $\hat{g}(D) := \bigcup_{x \in D} g(x) \subset D$, then there also exists a set $C = \overline{\text{co}} S C$ with $\overline{\text{co}} S \hat{g}(C) = C$.*

Proof. Let $\mathcal{F} := \{F = \overline{\text{co}} S F \subset X: \text{co} S \hat{g}(F) \subset F\}$. Obviously, $\mathcal{F} \neq \emptyset$ as $X \in \mathcal{F}$. There exists a maximal chain $\mathcal{G} \subset \mathcal{F}$ consisting of sets $G \in \mathcal{F}$ with $G \cap D \neq \emptyset$. Then $C := \bigcap_{G \in \mathcal{G}} G$ is non-empty; moreover, $C = \overline{\text{co}} S C$ (cf. [4]) and $\overline{\text{co}} S \hat{g}(C) \subset C$. Besides we have $\overline{\text{co}} S \hat{g}(C) \in \mathcal{G}$ as $\emptyset \neq \hat{g}(D \cap C) \subset D$. Therefore $\overline{\text{co}} S \hat{g}(C) = C$ as $\overline{\text{co}} S \overline{\text{co}} S \hat{g}(C) = \overline{\text{co}} S \hat{g}(C)$.

THEOREM 2. *Let $A = \bar{A}$ be a type I subset of a metric space (M, d) and let a map $f: A \rightarrow M$ satisfy (4). If for any $C \subset A$ for which $A_{f(C)} \subset C(A_{f(C)} := \bigcup_{x \in f(C)} A_x)$, it follows from the compactness of $C \setminus \overline{\text{co}} S A_{f(C)}$ that C is compact, $\text{Fix } f \neq \emptyset$.*

Proof. Let $g(x) := A_{f(x)}$ for $x \in A$ and let $D \subset A$ be any compact set. We take a minimal set $\bar{E} = E$ containing D and such that $\hat{g}(E) \subset E$. We can see that $E \setminus \hat{g}(E) \subset D$, because $(E \setminus \hat{g}(E)) \cap (A \setminus D)$ is relatively open in E and would be rejected when nonempty. Therefore, E must be compact. According to Lemma there exists a set C such that $\overline{\text{co}} S \hat{g}(C) = C$ and so C must be compact. It can be seen that if $f(x) \notin C$, $f(x) \notin A$ and $x \notin C_{f(x)}$, and hence f has a fixed point in C .

THEOREM 3. *Let (M, d) be a S -contractible metric space and let $A = \overline{\text{co}} S A$ be a type I subset of M for S . Let $f: A \rightarrow M$ be a map. Suppose that for some $z \in A$ and for any $C \subset A$ satisfying $\hat{g}(C) := S_z(I, f(C)) \cap A \subset C$ it follows*

from the compactness of $\overline{C \setminus \overline{\text{co}} S \hat{g}(C)}$ that C is compact. If, in addition,

$$(6) \quad \text{for any } C = \overline{\text{co}} S \hat{g}(C) \text{ and } z \in f(C) \setminus A, \quad A_z \cap C \cap f^{-1}(z) = \emptyset$$

holds, f has a fixed point.

Proof. Let $g(x) := S_z(I, f(x)) \cap A$. Similarly as in the proof of Theorem 2 we obtain a compact set $C = \overline{\text{co}} S \hat{g}(C)$, and hence, by Theorem 1, $\text{Fix } f \neq \emptyset$.

Remark 1. Theorem 3 is more general than Reich's theorem ([5], Theorem C), because if a map is condensing for Kuratowski's measure of non-compactness in a Banach space, it is condensing for a class of measures [3] and is generalized condensing too. Besides, by [5], Lemma 3, it follows from (6) that the inwardness condition holds.

Remark 2. We can take condition (2) [4] (Leray-Schauder boundary condition) with $w = z$ instead of (6) in Theorem 3 for any open convex subset A of a Banach space where $S_x(t, y) = tx + (1-t)y$.

References

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