BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
Mathematics
Vol. 31, No. 5–8, 1983

GENERAL TOPOLOGY

Some Fixed Point Theorems for Multi-Valued Mappings

by

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Presented by C. BESSAGA on April 6, 1981

Summary. The present paper contains further generalizations of the well-known theorems for the locally convex spaces. We operate here on the type I and type 0 spaces (see [5]). Some results are connected with the theory of S. Hahn [2].

We refer to Definitions 1, 2, 3 from [5] and repeat some notions from [4].

For a space X let 2^X, $C(X)$, $T(X)$ denote respectively the family of all nonempty, nonempty and closed, nonempty, closed and S-convex subsets of X.

Suppose X, Y, Z are nonempty and $G: X \to 2^Y$ is a multi-valued mapping. Then for $\emptyset \neq A \subset X$, $G(A) := \bigcup_{x \in A} G(x)$ and $G(\emptyset) = \emptyset$ ([1] p. 22). If $G_1: Y \to 2^Z$, $(G_1 \circ G)(x) := G_1(G(x))$ ([1] p. 24). If $H: 2^Y \to 2^Z$ is set-to-set function, $(H \circ G)(x) = H(G(x))$ $(G(x) \in 2^Y)$. Let it be in addition $(A \cap G)(x) := A \cap G(x)$ for $A \subset Y$.

DEFINITION 1 (cf. [1] pp. 114, 116). Let X, Y be spaces. A multi-valued mapping $G: X \to 2^Y$ is upper semi-continuous if for each neighborhood V of any $G(x)$ there exists a neighborhood U of x, for which $G(U) \subset V$; G is compact if it is upper semi-continuous and $\overline{G(X)}$ is compact.

In the sequel the multi-valued mappings will be called mappings.

Let us define for an S-contractible space X a special set-to-set function as follows:

(1) $F(A) = \bigcap_{U \in \mathcal{U}_A} \overline{\mathrm{co}} SU$, where $\mathcal{U}_A$ for $A \in 2^X$ are such families of neighborhoods as satisfy,

(2) if $V \in \mathcal{U}_A$, there exists $V_1 \in \mathcal{U}_A$ such that for any $\emptyset \neq C \subset V_1$ there exists $V_2 \in \mathcal{U}_C$, $V_2 \subset V$.

It can be seen that in particular $\mathcal{U}_A$ can be the family of all neighborhoods of A. Besides, from (2) follows

(3) $\emptyset \neq C \subset A$ implies $F(C) \subset F(A)$.

For an S-contractible subspace $D = \bar{D}$ of X $F|_{2^D}$ will denote the function obtained from F by taking $U \cap D$ in place of $U \in \mathcal{U}_A$ for $A \in 2^D$.

DEFINITION 2 (cf. [4] Def. 2.8). A space if of type 0 if it is S-contractible for S satisfying

(4) for any $A \subset X$ and any neighborhood V of $\overline{\text{co}}S\,A$ there exists a neighborhood U of A for which $\text{co}S\,U \subset V$.

DEFINITION 3. A space X is of compact type I (type 0) if X is of type I (type 0) and for any compact $A \subset X$, $\overline{\text{co}}S\,A$ is compact.

THEOREM 1 (cf. [4] Theorem 1.4). *Let X be a space of compact type I for which $G: X \to 2^X$ is compact. Then $F \circ G$ has a fixed point.*

Proof. Suppose $x \in (F \circ G)(x)$ for $x \in \overline{\text{co}}S$ $\overline{\text{co}}S\,G(X) = :C$. Thus a neighborhood $V \in \mathcal{U}_{G(x)}$ and such a neighborhood $U \subset C$ of x can be found for which $U \cap \overline{\text{co}}S\,V = \emptyset$ (C is compact). It follows that there exists a neighborhood $P \subset C$ of x with $G(P) \subset V_1 \subset V$ (for $A = G(x)$ V_1, V satisfy (2)). In view of (3) we have for $W := U \cap P$ $W \cap (F \circ G)(W) \subset U \cap F(G(P))$. We obtain from (2) $U \cap F(G(P)) \subset U \cap F(V_1) \subset U \cap \overline{\text{co}}S\,V = \emptyset$. Now we see that there exists an open cover $\mathcal{W} = \{W_x\}_{x \in \overline{\text{co}}S\,G(C)}$ of the set $\overline{\text{co}}S\,G(C)$

(5) $W_x \cap F(G(W_x)) = \emptyset$ for $x \in \overline{\text{co}}S\,G(C)$.

It follows ([3] 5.1.12 p. 377, 5.1.9 p. 375) that there exists a star finite partition of unity $\mathcal{U}$ subordinated to $\mathcal{W}$. Let us choose from $\mathcal{U}$ a cover $\mathcal{V} = \{f_i^{-1}(0, 1\rangle\}_{i=1,\dots,n}$ of $\overline{\text{co}}S\,G(C)$. Assume $x_i \in V_i \in \mathcal{V}$, $St(V_i, \mathcal{V}) \subset W_i \in \mathcal{W}$ for $i = 1, \dots, n$. In view of Tietze's theorem we may think f_i map C into I (obviously $\overline{\text{co}}S\,G(C) \subset \overline{\text{co}}S\,G(X) \subset C$).

Let us write $p_i(x) = \min\{1, |1 - \sum_{i=1}^{n} f_i(x)|\}$, $t_i(x) = (f_i(x) + p_i(x))/$ $/\max\{f_i(x) + p_i(x) : i = 1, \dots, n\}$. It can be seen that $p_i: C \to I$ are maps, $p_i(x) \neq 0$ for $\sum_{i=1}^{n} f_i(x) = 0$ and $p_i(x) = 0$ for $x \in \overline{\text{co}}S\,G(C)$. Besides for any $x \in C$ there exists an index i for which $t_i(x) = 1$. Now let for $x \in C$ and $y_i \in G(x_i)$

(6) $h(x) = S_{y_1}\big(t_1(x), S_{y_2}(t_2(x), \dots), S_{y_{n-1}}(t_{n-1}(x), y_n) \dots\big)$.

Function $H: C \to C$ is continuous (cf. [4], [3] 3.4.8 p. 210).

We see that $h(C) \subset \overline{\text{co}}S\,G(C) \subset \overline{\text{co}}S\,G(X)$ and thus $\overline{\text{co}}S\,h(C) \subset C$. Therefore h has a fixed point (in the proof of Theorem 1 [6], X can be replaced by any set $C \neq \emptyset$ with $\overline{\text{co}}S\,f(C) \subset C$). Suppose $x_0 = h(x_0)$. There exists a neighborhood W_i containing $D := \overline{\text{co}}S\,G(C) \cap \bigcup f_i^{-1}\{(0, 1) : f_i(x_0) \neq 0\}$ ($x_0 \in$ $\in D$). On the other hand $x_0 \in h(D) \subset \overline{\text{co}}S\,G(W_i) \subset (F \circ G)(W_i)$ which contradicts (5).

THEOREM 2 (cf. [4] Theorem 1.5). *Let* $\overline{\mathrm{co}}S \circ G: X \to C(X)$ *be a compact mapping for a space* X *of compact type I. Then* $\overline{\mathrm{co}}S \circ G$ *has a fixed point.*

Proof. It is enough to compare the proof of Theorem 1 and the proof of Theorem 1.5 [4] (X to be replaced by $\overline{\mathrm{co}}S\, \overline{\mathrm{co}}S\, G(X)$).

THEOREM 3 (cf. [4] Theorem 2.5). *Let* $X = \bar{X} \subset Y$ *be a subspace of compact type I for* S, *while* Y *is S-contractible. Suppose* $G: X \to 2^Y$ *is such a mapping that* $\overline{X \cap G}$ *is compact. Then* $F \circ G$ *has a fixed point.*

Proof. See Theorem 1 and the proof of [4] Theorem 2.5.

THEOREM 4 (cf. [4] Theorem 2.7). *Let* X *be a space of compact type I and let* $G: X \to T(X)$ *be a compact mapping. Then* G *has a fixed point.*

Proof. It follows from Theorem 2 as $\overline{\mathrm{co}}S\, G(x) = G(x)$.

THEOREM 5 (cf. [4] Theorem 2.12). *Let* $G: X \to 2^X$ *be a compact mapping for a space* X *of compact type* 0. *Then* $\overline{\mathrm{co}}S \circ G$ *has a fixed point.*

Proof. If $G: X \to 2^X$ is upper semi-continuous $\overline{\mathrm{co}}S \circ G$: $\overline{\mathrm{co}}S\, \overline{\mathrm{co}}S\, G(X) \to \to C(X)$ is upper semi-continuous for a space X of a compact type 0 (see [4] Lemma 2.10). The proof is finished by applying Theorem 2.

DEFINITION 4 ([4] Def. 3.1, cf. [2] pp. 12, 13). Let X be a space and for $\emptyset \neq Z \subset X$ let $G: Z \to 2^X$ be a mapping. Then S-contractible set $D = = \bar{D} \subset X$ is characteristic of G if $Z \cap D \neq \emptyset$, $G(Z \cap D) \subset D$ and $\overline{\mathrm{co}}S\, G(Z \cap D)$ is compact (in the case $G = \overline{\mathrm{co}}S \circ H$ we assume only the compactness of $\overline{G(Z \cap D)}$). Let X be an S-contractible space and $W = \overline{\mathrm{co}}S\, W \subset X$, $K = = \overline{\mathrm{co}}S\, K \subset X$; a mapping $G: W \cap K \to 2^K$ is quasicompact if it has a characteristic set on which G is upper semi-continuous.

THEOREM 6 (cf. [4] Theorem 3.2). *Let* X *be a space of compact type I for which* $\overline{\mathrm{co}}S \circ G: X \to C(X)$ *is quasicompact. Then* $\overline{\mathrm{co}}S \circ G$ *has a fixed point.*

THEOREM 7 (cf. [4] Theorem 3.3). *Let* X *be a space of compact type I for which* $G: X \to 2^X$ *is quasicompact. Then* $F \circ G$ *has a fixed point.*

The above Theorems follow from Theorems 1, 2 respectively.

DEFINITION 5. Let X be a space and for $\emptyset \neq Z \subset X$ let $G: Z \to 2^X$ be a mapping. Then a set $D \subset X$ is a weakly characteristic set of G if $Z \cap D \neq \emptyset$, $G(Z \cap D) \subset D$ and there exists S for which $\overline{\mathrm{co}}S\, G(Z \cap D)$ is compact, $\overline{\mathrm{co}}S\, \overline{\mathrm{co}}S\, G(Z \cap D) \subset D$ and S is of type I for the points of $\overline{\mathrm{co}}S\, G(Z \cap D)$. If X is a space $W \subset X$, $K \subset X$ $G: W \cap K \to 2^K$ ($\overline{\mathrm{co}}S \circ G$) is weakly quasicompact if G has a weak characteristic set on which G ($\overline{\mathrm{co}}S \circ G$) is upper semi-continuous.

THEOREM 8. *Let* X *be a normal type I space for which* $G: X \to 2^X$ *is quasicompact. Then* $F \circ G$ *has a fixed point.*

Proof. See the proofs of [4] Theorem 1.4 and Theorem 1.

THEOREM 9. *Let* X *be a normal type I space for which* $\mathrm{co}S\, G: X \to 2^X$ *is weakly quasicompact. Then* $\overline{\mathrm{co}}S \circ G$ *has a fixed point.*

Proof. See the proofs of [4] Theorem 1.5 and Theorem 2.

We can easily give versions of Theorems 1–5 for the weakly quasi-compact mappings.

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REFERENCES

[1] C. Berge, *Espaces topologiques, fonctions multivoques*, Dunod Paris, 1966.
[2] S. Hahn, *Zur Theorie nichtlinearer Operatorengleichungen in topologischen Vektorräumen*, Thesis Dresden, 1977.
[3] R. Engelking, *General Topology*, Warszawa 1977, PWN.
[4] L. Pasicki, *A fixed point theory for multi-valued mappings*, Proc. Amer. Math. Soc., **83** (1981), 781–789.
[5] L. Pasicki, *Retracts in metric spaces*, Proc. Amer. Math. Soc., **78** (1980), 595–600.
[6] L. Pasicki, *Three fixed point theorems*, Bull. Ac. Pol.: Math., **28** (1980), 173–175.

Л. Пасицки, **Некоторые теоремы о постоянным пункте для многозначных отображений**

Работа содержит обобщение теорем С. Хана.