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GENERAL TOPOLOGY

Nonempty Intersection and Minimax Theorems

by

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Summary. Some results of Ky Fan [3] and Browder [1] are extended to a class of a nonlinear spaces.

DEFINITION 1 [7]. A set X is called S-linear if the following conditions are satisfied:

(1) $$S: X \times I \times X \ni (x, t, y) \mapsto S_x(t, y) \in X,$$

(2) $$S_x(0, y) = y, \ S_x(1, y) = x \quad \text{for every} \quad x, y \in X.$$

Then for any nonempty set $A \subset X$ let $\text{co}S\, A = \bigcap \{D \subset X : S_A(I, D) \subset D\}$. For $A = \emptyset$ let $\text{co}S\, A = \emptyset$. If $\text{co}S\, A = A$, A is S-convex. If $\{S_x : x \in X\}$ is a family of homotopies, space X is S-contractible [5].

For a set X let 2^X be the family of all nonempty subsets of X; if X is S-linear $T(X)$ is the family of all nonempty, S-convex and closed (X being a space) subsets of X.

THEOREM 1 (cf. [1, Theorem 1 p. 285]). *Let X be a paracompact space and Y an S-contractible space. Assume $F: X \to 2^Y$ has S-convex values and for every $y \in Y$ $F^-(y) = \{x \in X : y \in F(x)\}$ is open. Then there exists a continuous selection for F.*

Proof. The family $\mathscr{F} = \{F^-(y)\}_{y \in Y}$ is an open cover of X. Let $\mathscr{W} = \{W_t\}_{t \in T} = \{g_t^{-1}(0, 1\rangle\}_{t \in T}$ be a locally finite partition of unity being a barycentric refinement of $\mathscr{F}$ [2, 5.1.12 p. 377, 5.1.9 p. 375] for a well ordered set T. For y_t chosen in such a way that $St(W_t, \mathscr{W}) \subset F^-(y_t)$ we write $c_t(x) = g_t(x)/\sup\{g_t(x) : t \in T\}$ and $T_x = \{t \in T : c_t(x) \neq 0\} = \{t_1, \ldots, t_n\}$, $t_1 \prec t_2 \prec \ldots \prec t_n$ $(n = n(x))$. Then we define $f: X \to Y$ by

$$f(x) = S_{y_{t_1}}(c_{t_1}(x)), S_{y_{t_2}}(c_{t_2}(x), \ldots), S_{y_{t_n}}(c_{t_n}(x), y) \ldots),$$

where $y \in Y$ is arbitrary. Function f [5] is continuous (induction) [2, 2.3.6 p. 108, 3.1.10 p. 168, 3.4.8 p. 210]. If $c_{t_i}(x) \neq 0$, then $y_{t_i} \in F(x)$ and hence $f(x) \in F(x)$, $x \in X$, $F(x)$ being S-convex.

THEOREM 2 (cf. [1, Theorem 11, p. 295]). *Let K_i be a compact space of type I for S^i, $i = 1, \ldots, n$ (≥ 2) and let $K = \prod_{j=1}^{n} K_j$. Suppose $P_i \subset K$, $i = 1, \ldots, n$ satisfy:*
(a) *For each $x_j \in K_j$ the sets $P_j(x_j) = \{\hat{x}_j : \hat{x}_j \in \hat{K}_j, [x_j, \hat{x}_j] \in P_j\}$ are open in $\hat{K}_j$, $j = 1, \ldots, n$.*
(b) *For each $\hat{x}_j \in \hat{K}_j = \prod_{k \neq j} K_k$ $P_j(x_j) = \{x_j : x_j \in K_j, [x_j, \hat{x}_j] \in P_j\}$ are nonempty S^j-convex subsets of K_j, $j = 1, \ldots, n$.*
Then $\bigcap \{P_j : j = 1, \ldots, n\} \neq \emptyset$.

Proof. Let us define $F: K \to 2^K$ as follows: $F(x) = \{y \in K : [y_j, \hat{x}_j] \in P_j, j = 1, \ldots, n\}$ for $\hat{x}_j$ being the natural projection of x on $\hat{K}_j$. From (b) it follows that $F(x)$ is $\prod_{j=1}^{n} S^j$-convex (see [5]). It is seen that $F^-(y) = \bigcap \{\{P_j(y_j) \times K_j\} : j = 1, \ldots, n\}$ and in view of (a) $F^-(y)$ is open. Theorem 1 implies there is a continuous selection $f: X \to Y$ for F. Space K is of type I [5] and from [6, Theorem 1] f has a fixed point, say $x = f(x)$. By the definition of F this means $x = [x_j, \hat{x}_j] \in P_j$, $j = 1, \ldots, n$ i.e. $x \in \bigcap \{P_j : j = 1, \ldots, n\}$.

THEOREM 3 (cf. [1, Theorem 12 p. 296]). *Let K_i be a compact space of type I for S^i, $i = 1, \ldots, n$ (≥ 2) and let $K = \prod_{j=1}^{n} K_j$. Let $f_1, \ldots, f_n$ be real valued functions defined on K having the following properties:*
(a) *For each $x_j \in K_j$, $f_j(x_j, \cdot): \hat{K}_j \to R$ is lower semicontinuous, $j = 1, \ldots, n$.*
(b) *For each $\hat{x}_j \in \hat{K}_j$, $f_j(\cdot, \hat{x}_j): K_j \to R$ is a quasi-concave function (i.e. $\{x_j \in K_j : f_j(x_j, \hat{x}_j) > t\}$ is S^j-convex, $t \in R$), $j = 1, \ldots, n$.*

If for each $\hat{x}_j \in \hat{K}_j$ there exists $y_j \in K_j$ such that $f_j(y_j, \hat{x}_j) > t_j$, $j = 1, \ldots, n$, then there is an $x \in K$ such that $f_j(x) > t_j$, $j = 1, \ldots, n$.

Proof. It can be seen that the sets $P_j = \{x \in K : f_j(x) > t_j\}$ satisfy conditions (a), (b) of Theorem 2 and hence $\bigcap \{P_j : j = 1, \ldots, n\} \neq \emptyset$.

THEOREM 4 (cf. [1, Theorem 14, p. 297]). *Let K_i be a compact space of type I for S^i, $i = 1, \ldots, n$ (≥ 2). Suppose that for each $j = 1, \ldots, n$ and each $\hat{x}_j \in \hat{K}_j$, $f_j(\cdot, \hat{x}_j): K_j \to R$ is a quasi-concave function and $f_j: K \to R$ is continuous. Then there exists a point $x \in K$ such that for $j = 1, \ldots, n$ $f_j(x) = \max \{f_j(y_j, \hat{x}_j) : y_j \in K_j\}$.*

Proof. Let for each $\hat{x}_j$ in K_j $g_j(\hat{x}_j) = \max \{f_j(y_j, \hat{x}_j) : y_j \in K_j\}$. By the uniform continuity of f_j on the compact space K, g_j, $j = 1, \ldots, n$ are continuous. For fixed $\varepsilon > 0$, let $P_j = \{x \in K : f_j(x_j, \hat{x}_j) > g_j(\hat{x}_j) - \varepsilon\}$. Condition (b) of Theorem 2 obviously holds for $P_j(\hat{x}_j) = \{y_j \in K_j : f_j(y_j, \hat{x}_j) > g_j(\hat{x}_j) - \varepsilon\}$. For each $y_j \in K_j$ $P_j(y_j) = \{\hat{x}_j \in \hat{K}_j : f_j(y_j, \hat{x}_j) > g(\hat{x}_j) - \varepsilon\}$ is open f_j, g_j being continuous. Hence, by Theorem 2 $G_\varepsilon := \bigcap \{P_j : j = 1, \ldots, n\} \neq \emptyset$. It is obvious that $G_\varepsilon \subset H_\varepsilon := \{x \in K : f_j(x) \geq g_j(\hat{x}_j) - \varepsilon, j = 1, \ldots, n\}$. The family $\{H_\varepsilon\}_{\varepsilon \in (0, \infty)}$ has the finite intersection property and hence $\bigcap \{H_\varepsilon : \varepsilon \in R^+\} \neq \emptyset$, H_ε being compact.

THEOREM 5 (cf. [1, Theorem 16 p. 299]). *Let $X \times Y$ be a type I compact space for $S^1 \times S^2$. Suppose $f: X \times Y \to R$ is such that $f(\cdot, y): X \to R$, $y \in Y$ are upper semi-continuous quasi-concave functions and $f(x, \cdot): Y \to R$, $x \in X$ are lower semi-continuous quasi-convex functions. Then* $\min\{\max\{f(x, y): :x \in X\}: y \in Y\} = \max\{\min\{f(x, y): y \in Y\}: x \in X\}$.

Proof. Let us write $f_1(x, y) = f(x, y)$, $f_2(x, y) = -f(x, y)$ and $t_1 = = \min\{\max\{f(x, y) - \varepsilon: x \in X\}: y \in Y\}$, $t_2 = -\max\{\min\{f(x, y) - \varepsilon: y \in Y\}: x \in X\}$ for a constant $\varepsilon > 0$. It can be seen that f_1, f_2 satisfy the semicontinuity and concavity assumptions of Theorem 3. Hence $H_\varepsilon := \{(x, y) \in X \times Y: :f(x, y) \geq \min\{\max\{f(x, y) - \varepsilon: x \in X\}: y \in Y\}, f(x, y) \leqslant \max\{\min\{f(x, y) + +\varepsilon: y \in X\} x \in X\}$ is nonempty for each $\varepsilon > 0$. The semi-continuities for f imply $H_\varepsilon = \bar{H}_\varepsilon$ and in the sequel $\min\{\max\{f(x, y): x \in X\}: y \in Y\} \leqslant \max\{\min\{f(x, y): y \in Y\}: x \in X\}$. The inequality for "$\geqslant$" always holds.

THEOREM 6 (cf. [1, Theorem 13 p. 296]). *Let $\{K_t\}_{t \in T}$ be a family of compact spaces of type I for $\{S^t\}_{t \in T}$. Let $K = \prod_{t \in T} K_t$, $H_t = \prod_{k \neq t} H_k$, $t \in T$. Let $\{P_t\}_{t \in T}$ be a family of closed subsets of K. Suppose that for each $t \in T$, $x \in H_t$ $R_t(x) = \{y_t \in K_t: [y_t, \hat{x}_t] \in P_t\}$ is nonempty and S^t-convex. Then $\bigcap\{P_t: :t \in T\} \neq \emptyset$.*

Proof. Let $F: K \to 2^K$ be defined as follows: $F(x) = \{y \in K: y_t \in R_t(x), t \in T\}$. From the definition of $R_t(x)$ it follows that $F: K \to T(K)$. Now we are going to prove that F is upper semi-continuous in the Tychonoff topology. It is enough to prove that the graph $G(F)$ of F is closed. Assume $[y_t, \hat{x}_t] \notin P_t$ for a $t \in T$. From the compactness of P_t it follows that there is a neighborhood $N_1 \times N_2 \subset K_t \times H_t$ of $[y_t, \hat{x}_t]$ for which $N_1 \times N_2 \cap \cap P_t = \emptyset$. Then $(N_1 \times H_t) \cap (K_t \times N_2)$ is the neighborhood in K we need. The space K is of type I [4, 2.9] and F has a fixed point [4, 2.7] which is equivalent to the intersection being nonempty.

THEOREM 7 (cf. [1, Theorem 15 p. 298]). *Let $\{K_t\}_{t \in T}$ be a family of the compact type I spaces for $\{S^t\}_{t \in T}$. Let for each $t \in T$, $f_t: K \to R$ be a continuous function and $\{y_t \in K_t: f_t(y_t, \hat{x}_t) \geqslant q_t\}$ an S^t-convex subset of K_t for each $\hat{x}_t \in \hat{K}_t$, $t \in T$. Then there exists $x \in K$ such that for each $t \in T$, $f_t(x) = = \max\{f_t(y_t, \hat{x}_t): y_t \in K_t\}$.*

THEOREM 8 (cf. [1, Theorem 17 p. 300]). *Let K be a compact space of type I for S. Suppose C is a closed subset of a space Y and $g: K \times \times K \to Y$ is a map such that for each $x \in K$ $\{y \in K: g(x, y) \in C\} \neq \emptyset$ is* S-convex. *Then there is $x \in K$ for which $g(x, x) \in C$.*

Proof. We define $F: K \to T(K)$ as follows: $F(x) = \{y \in K: g(x, y) \in C\}$. For each $x \in K$ $F(x)$ is closed because g is continuous and C is closed. The graph of F equals to $g^{-1}(C)$ and therefore it is compact and F is upper semi-continuous. In view of [4, 2.7] F has a fixed point.

THEOREM 9 (cf. [1, Theorem 18 p. 300]). *Let K be a compact space of type I for S and let Y be a space. Suppose $g: K \times K \to Y$ is such a map* *that for all* $x, y_1, y_2 \in K$ *and* $t \in I$ $S^1_{g(x,y_1)}(t, g(x, y_2)) := g(x, S_{y_1}(t, y_2))$ *satisfies* $S^1_{z_0}(I, z_0) = z_0$ *for a point* $z_0 \in Y$. *Suppose further that for every* $x \in K$ *there exists* $y \in K$ *for which* $g(x, y) = z_0$. *Then there is* $x \in K$ *such that* $g(x, x) = z_0$.

Proof. Let us write $C = \{z_0\}$ in Theorem 8. Suppose $g(x, y_1) = g(x, y_2) = z_0$. Then we have $g(x, S_{y_1}(t, y_2)) = S^1_{z_0}(t, z_0)$ and it is seen that $\{y: K: g(x, y) = z_0\} \neq \emptyset$ is S-convex for each $x \in X$. Thus we may apply Theorem 8.

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Л. Пасицки, **Не пустые пересечения и монимаксные теоремы**

Обобщены некоторые теоремы Фона (3) и Феликса Е. Бровдера для класса нелинейных пространств.