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An application of a fixed point theorem

Abstract. In this paper some conditions are given that ensure the equation $Tx = x$ to have a solution with a given modulus of continuity (T operates on subsets of function spaces).

I. In [2] a theorem is given (Theorem 4) which in case of normed spaces can be expressed in the following form (see the proof).

THEOREM. *Let A be a compact convex subset of a normed space X. Suppose that $f\colon A \to X$ is a map such that for $f(x) \notin A$, $d(f(x), x) > d(f(x), A) := \inf\{d(f(x), y)\colon y \in A\}$. Then f has a fixed point.*

This theorem can be applied to cases when $\operatorname{Int} A = \emptyset$ and the Leray–Schauder theorem cannot be used.

Let C_I^0 (C_I^1) denote the set of all real continuous (continuously differentiable) functions on the domain $I = \langle 0, 1\rangle$. Let $X := \{x \in C_I^0\colon x(0) = 0\}$, $Y := X \cap C_I^1$ be two spaces equipped with the norm $\|x\| = \sup\{|x(t)|\colon t \in I\}$. Besides, let $A \subset X$ consist of all functions x that satisfy the following conditions for $t_1, t_2 \in I$, $t_1 \leqslant t_2$:

$$L_{-1}(t_2) - L_{-1}(t_1) \leqslant x(t_2) - x(t_1), \tag{1}$$

$$x(t_2) - x(t_1) \leqslant L_1(t_2) - L_1(t_1); \tag{2}$$

similarly let $B \subset Y$ consist of all functions x such that they satisfy for $t \in I$

$$L'_{-1}(t) \leqslant x'(t), \tag{3}$$

$$x'(t) \leqslant L'_1(t), \tag{4}$$

where $L_{-1}, L_1 \in C_I^0$ ($L_{-1}, L_1 \in C_I^1$ if the differentiation is mentioned) and for $t_1 \leqslant t$, $t_1, t \in I$, $L_{-1}(t) - L_{-1}(t_1) \leqslant L_1(t) - L_1(t_1)$ ($L'_{-1}(t) \leqslant L'_1(t)$ in the "differentiable" case).

It can be seen that A, B are convex, the closure of B in X equals to A, A is compact and $\operatorname{Int} A = \emptyset$.

LEMMA 1. *Let $x \in A$, $z \in X \setminus A$ be two functions satisfying for $t_1, t_2 \in I$, $t_1 \leqslant t_2$*

(5) *if* (1) *or* (2) *does not hold for* z, *then* x *satisfies* (1) *or* (2), *respectively, with the strong inequality.*

Then $d(z, A) < d(z, x)$.

Proof. Let us denote $a = \|z-x\|$ $(a > 0)$. Let us consider the set $\mathscr{K}$ of all maximal intervals K for which $|z-x|(K) = \langle a/2, a\rangle$. The function $z-x$ is uniformly continuous, which means that $\mathscr{K}$ is finite. Now we construct a special family of intervals. Let $I_0 = \langle t_0, t_1\rangle \in \mathscr{K}$ contain the smallest number p_0 with $|z-x|(p_0) = a$. If I_k, p_k are known, then $I_{k+1} = \langle t_{2k+2}, t_{2k+3}\rangle \in \mathscr{K}$ contains the smallest number $p_{k+1} > p_k$ for which $(z-x)(p_k)(z-x)(p_{k+1}) = -a^2$. Suppose we have obtained the family $\mathscr{I} = \{I_k\}_{k=0,1...,n}$. From the continuity of $z-x$ it follows that $\sup\{|z-x|(t)\colon t \in I\setminus \bigcup_{k=0}^{n} I_k\} =: b < a$. Let us assume $(z-x)(t_0) > 0$ and let for $d_k \in (0, 1\rangle$, $k = 0, 1, \ldots, n$,

$$\delta_k(t) = \begin{cases} 0 & \text{for } t \in \langle 0, t_{2k}), \\ \big(L_{(-1)^k}(t) - L_{(-1)^k}(t_{2k}) - (x(t) - x(t_{2k}))\big) d_k & \text{for } t \in \langle t_{2k}, p_k\rangle, \\ \delta_k(p_k) & \text{for } t \in (p_k, 1\rangle. \end{cases}$$

Suppose $q_1 \leqslant t_0 \leqslant q \leqslant p_0$, then for $w := x + \delta_0$ we have $w(q) - w(q_1) = w(q) - w(t_0) + w(t_0) - w(q_1) = x(q) - x(t_0) + \delta_0(q) + x(t_0) - x(q_1) \leqslant L_1(q) - L_1(t_0) + x(t_0) - x(q_1) \leqslant L_1(q) - L_1(q_1)$ and obviously, $w(q) - w(q_1) \geqslant L_{-1}(q) - L_{-1}(q_1)$. Now it is easily seen that $y := x + \sum_{k=0}^{n} \delta_k \in A$. Besides, $\delta_k(1) \neq 0$, $k = 0, \ldots, n$ (see (5)) if x does not satisfy (1) or (2). The numbers d_k can be established in such a manner that the sequence $\big((-1)^l \sum_{k=0}^{l} \delta_k(1)\big)_{l=0,\ldots,n}$ increases and is bounded by $(a-b)/2$. Now it can be seen that $\|z-y\| < a$.

Corollary 1. *If* $T\colon A \to X$ *is a continuous operator satisfying* (5) *for* $z := Tx$, *the equation* $Tx = x$ *has a solution.*

Lemma 2. *Suppose* $x \in A$, $z \in Y\setminus B$ *are two functions such that*

(6) *if* (3) *or* (4) *does not hold for* z *and a* $t \in I$, *then* (1) *or* (2) *is satisfied with strong inequality for all* $t_1 < t < t_2$, $t_1, t_2 \in I$, *respectively.*

Then $d(z, A) < d(z, x)$.

Proof. Suppose $z(t_2) - z(t_1) > L_1(t_2) - L_1(t_1)$. Then we have $z'(t) > L_1'(t)$ on a subinterval of (t_1, t_2) and therefore $x(t_2) - x(t_1) < L_1(t_2) - L_1(t_1)$, i.e., (5) holds.

Corollary 2. *If* $T\colon A \to Y$ *is a continuous operator satisfying* (6) *for* $z := Tx$, *then the equation* $Tx = x$ *has a solution.*

Lemma 3. *Let* $x \in B$, $z \in X\setminus A$ *be such that*

(7) *if* (1) *or* (2) *does not hold for* z *and* $t_1, t_2 \in I$, $t_1 \leqslant t_2$, *then there exists* $t \in (t_1, t_2)$ *such that* (3) *or* (4) *is satisfied with strong inequality, respectively.*

Then $d(z, B) = d(z, A) < d(z, x)$.

Proof. Suppose $z(t_2)-z(t_1) > L_1(t_2)-L_1(t_1)$ and $x'(t) < L_1'(t)$ for a $t \in (t_1, t_2)$. Then $x(t_2)-x(t_1) = \int_{t_1}^{t_2} x'(t)\,dt < \int_{t_1}^{t_2} L_1'(t)\,dt = L_1(t_2)-L_1(t_1)$, i.e., (5) holds.

For an operator $T\colon B \to X$, let $\bar{T}\colon A \to X$ denote the continuous extension of T.

Lemma 4. *Let* $T\colon B \to X$ *be a continuous operator satisfying* (7) $(z := Tx)$. *Then* (5) *holds for* $\bar{T}x$.

Proof. Suppose $x \in A \backslash B$, $(\bar{T}x)(t_2)-(\bar{T}x)(t_1) > L_1(t_2)-L_1(t_1)$ and $x(t_2) - x(t_1) = L_1(t_2)-L_1(t_1)$ which means that $x(t) = x(t_1)+L_1(t)-L_1(t_1)$ on $\langle t_1, t_2\rangle$. It is seen that x can be approximated by the functions $x_n \in B$ with $x_n = x$ on $\langle t_1+\varepsilon_n, t_2-\varepsilon_n\rangle$, where ε_n decreases to zero when $n \to \infty$. So we have $(Tx_n)(t_2)-(Tx_n)(t_1) > L_1(t_2)-L_1(t_1)$, which implies $x(t_2)-x(t_1)$ $= \lim \int_{t_1+\varepsilon_n}^{t_2-\varepsilon_n} x_n'(t)\,dt < \int_{t_1}^{t_2} L_1'(t)\,dt = L_1(t_2)-L_1(t_1)$.

Corollary 3. *If* $T\colon B \to X$ *is a continuous operator satisfying* (7) $(z := Tx)$, *then the equation* $\bar{T}x = x$ *has a solution in* A.

Lemma 5. *Let* $z \in Y \backslash B$, $x \in B$ *be such that*

(8) *if* (3) *or* (4) *does not hold for* z *and a* $t \in I \backslash J$ $(\overline{I \backslash J} = I)$, *then* (3) *or* (4) *is satisfied with strong inequality, respectively.*

Then $d(z, A) < d(z, x)$.

Proof. Suppose $z(t_2)-z(t_1) > L_1(t_2)-L_1(t_1)$. Then there exists a subinterval of (t_1, t_2) on which $z'(t) > L_1'(t)$ and automatically, $x'(t) < L_1'(t)$. Hence $x(t_2)-x(t_1) = \int_{t_1}^{t_2} x'(t)\,dt < \int_{t_1}^{t_2} L_1'(t)\,dt = L_1(t_2)-L_1(t_1)$, i.e., (5) holds.

Corollary 4. *If a continuous operator* $T\colon B \to Y$ *satisfies* (8) $(z := Tx)$, *then the equation* $\bar{T}x = x$ *has a solution in* A.

Corollary 5. *Let* $T\colon A \to X$ *be a continuous operator such that* $T|_B$ *satisfies* (7) *or* (8). *Then the equation* $Tx = x$ *has a solution.*

Example. Let $-L_{-1}(t) = L_1(t) = Lt$, $L > 0$ and $\mathscr{P}_i(x) = \{P = \langle a, b\rangle \subset I\colon\ x(t_2)-x(t_1) = L_i(t_2-t_1)$ for $t_1, t_2 \in P$, $t_1 < t_2\}$, $Q_i(x) = \bigcup \{P \in \mathscr{P}_i(x)\}$, $Q_i'(x) = I \backslash Q_i(x)$ for $i = -1, 1$. Assume $f\colon I \times R \to R$ is continuous and $f(0, 0) = 0$. For $f(t, x(t))$, $t \in I$, we define $g_x(t) = h(t, x(t))$ as follows: $g_x\colon I \to R$ is continuous, if $\langle a, b\rangle$ is an element of $\mathscr{P}_i(x)$, then the graphs of $h(\cdot, x(\cdot))$ and p_x are isometric on $\langle a, b\rangle$ for

$$p_x(t) = \begin{cases} \max\{-L(t-a), f(t, x(t))-f(a, x(a))\} & \text{for } i = -1, \\ \min\{L(t-a), f(t, x(t))-f(a, x(a))\} & \text{for } i = 1. \end{cases}$$

If (a, b) is any maximal open interval contained in $Q_i'(x)$, then $h(\cdot, x(\cdot))$ and $f(\cdot, x(\cdot))k(p_x(\cdot))+p_x(\cdot)$ have isometric graphs on (a, b) (k is continuous, $k(0) = 0$). From the continuity of f it follows that $h\colon I\times R\to R$ is continuous. It is obvious that $(Tx)(t) := h(t, x(t))$, $t\in I$, satisfies the assumptions of Corollary 1 and the assumptions of Corollary 2 hold for $(Tx)(t): = \int_0^t h(t, x(t))dt$. For $x\in B$, there are obtained examples for Corollaries 3, 4.

Let us consider $f(t, x(t)) = -(x(t))^{1/3} = h(t, x(t))$ ($k\equiv 0$). We can see that $f(A)\not\subset A$ as $-(Lt)^{1/3}$ does not satisfy the Lipschitz condition and for $x\neq 0$ we obviously have $\|x-Tx\| > \|-x-Tx\|$ which implies (1), (2).

COROLLARY 5. *Let $T\colon A\to X$ be a continuous operator such that $T|_B\colon B\to Y$ and $T|_B$ satisfies* (7) *or* (8). *Then the equation $Tx = x$ has a solution in A.*

The previous considerations can be extended to the multi-dimensional case. Let $x = (x^1, \ldots, x^k)\in A^1\times A^2\times \ldots \times A^k$ be a function, where A^i, i $1, \ldots, k$, are analogs of A with the functions L_{-1}^i, L_1^i, and suppose $T\colon \mathop{\mathbf{P}}_{i=1}^{k} A^i\to \mathop{\mathbf{P}}_{i=1}^{k} X$ is continuous. The domain of T may be the carthesian product of the sets of type A^i or B^i and similarly the range may contain X and Y. Then if for every $i = 1, \ldots, k$, $(Tx)^i$ and x^i satisfy one of conditions (5)–(8) (according to the case), then $Tx = x$ has a solution.

II. Suppose now $\mathcal{D}\colon I\to 2^{E^n}$ (E^n euclidean n-space) to be an upper semicontinuous multi-valued mapping and $D: = \bigcup\{(t, \mathcal{D}(t))\colon t\in I\}$. Let $X = \{x\in C_D^0\colon x(0, \cdot)\equiv 0\}$ and suppose $A\subset X$ consists of all functions which satisfy the following conditions for all (t_1, ξ_1), $(t_2, \xi_2)\in D$, $(t_1, \xi_1)\leqslant (t_2, \xi_2)$ (i.e. $t_1\leqslant t_2$, $\xi_1^i\leqslant \xi_2^i$ for $i = 1, \ldots, n$)

$$x(t_2, \xi_1)-x(t_1, \xi_1)\leqslant L_1(t_2, \xi_1)-L_1(t_1, \xi_1), \tag{9}$$

$$L_{-1}(t_2, \xi_1)-L_{-1}(t_1, \xi_1)\leqslant x(t_2, \xi_1)-x(t_1, \xi_1), \tag{10}$$

$$x(t_1, \xi_2)-x(t_1, \xi_1)\leqslant w_1(t_1, \xi_2-\xi_1), \tag{11}$$

$$w_{-1}(t_1, \xi_2-\xi_1)\leqslant x(t_1, \xi_2)-x(t_1, \xi_1), \tag{12}$$

where L_{-1}, L_1, w_{-1}, w_1 are continuous on D, $w_{-1}(t_1, 0) = w_1(t_1, 0) = 0$, (9) holds for L_{-1} in place of x, and (11) – for w_{-1} in place of x.

It is easy to verify that A consists of uniformly continuous functions as D is compact [1], Theorem 3, p. 116.

LEMMA 6. *Let $z\in X\setminus A$, $x\in A$, be such that*

(13) *(11), (12) are satisfied with strong inequality for any* $t \in (0, 1\rangle$, $\xi_1 \leqslant \xi_2$ *and* (t, ξ_1), $(t, \xi_2) \in D$,

(14) *if for a fixed* ξ *and* $t_1 \leqslant t_2$, (t_1, ξ), $(t_2, \xi) \in D$, *(9) or (10) does not hold for* z, *then (9) or (10) holds for* x *(at the same points) with strong inequality.*

Then $d(z, A) < d(z, x)$.

Proof. Let $a = \|z-x\|$. Let us consider a family $\mathscr{K}$ of cylinders $\{K \times \times \Delta_K\}$ containing balls with the minimal diameters for which $|x(K \times \Delta_K)| = \langle a/2, a\rangle$. The family $\mathscr{K}$ is finite. Let $t' > 0$ be the smallest number for which $\{t'\} \times \mathscr{D}(t') \cap \bigcup \{K \times \Delta_K\} \neq \emptyset$ and let $L(q)$ denote $\min\{|w_1(t, \xi-\xi_1) - -x(t, \xi)+x(t, \xi_1)|, \quad |w_{-1}(t, \xi-\xi_1)-x(t, \xi)+x(t, \xi_1)|: \quad (t, \xi), \quad (t, \xi_1) \in D, \xi_1 \leqslant \xi, d(\xi_1, \xi) \geqslant q, t \geqslant t'\}$. It is seen that $L$is continuous, $L(q) > 0$ for $q > 0$ and $L(0) = 0$.

Let t_K be the left end of K and let p_K be the smallest number in K for which $|z-x|$ attains the value a in $K \times \Delta_K$. We write

$$L_K = \begin{cases} L_1 & \text{if } (z-x)(K \times \Delta_K) = \langle a/2, a\rangle, \\ L_{-1} & \text{if } (z-x)(K \times \Delta_K) = \langle -a, -a/2\rangle, \end{cases}$$

and for $d_K \in (0, 1\rangle$

$$\delta_K(t, \xi) = \begin{cases} 0 & \text{for } t < t_K, \\ \big(L_K(t, \xi)-L_K(t_K, \xi)-\big(x(t, \xi)-x(t_K, \xi)\big)\big)d_K & \text{for } t \in \langle t_K, p_K\rangle, \\ \delta(p_K, \xi) & \text{for } t \in (p_K, 1\rangle. \end{cases}$$

Besides, let

$$\beta_K(t, \xi) = \delta_K(t, \xi)\, 2L\big(d\big((t, \xi), D \backslash I \times \Delta_K\big)\big)/\mathrm{dia}\, \Delta_K.$$

For $p_{K_0} = \min\{p_K\colon K \times \Delta_K \in \mathscr{K}\}$ we can take any $d_{K_0} \in (0, 1\rangle$. If p_{K_1} is a minimal number for which d_{K_1} is not defined yet, then it is possible to multiply all known d_K by a small positive coefficient and choose the required d_{K_1} in such a manner that $y_{K_1} = x + \sum \beta_K \in A$ (the sum is extended on all K with known p_K) and y_{K_1} approximates z "better". It follows that there exists a function $y = x + \sum_{K \in \mathscr{K}} \beta_K \in A$ for which $d(z, y) < d(z, x)$.

Corollary 6. *Let* $T\colon A \to X$ *be a continuous operator such that* $Tx \notin A$ *implies (13), (14) are satisfied* $(z := Tx)$. *Then the equation* $Tx = x$ *has a solution.*

We can now generalize all considerations of part I in a natural way.

References

[1] C. Berge, *Espaces topologiques, fonctions multivoques*, Paris 1966, Dunod.

[2] L. Pasicki, *Retracts in metric spaces*, Proc. Amer. Math. Soc. 78 (1980), 595–600.