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On S-Affine Mappings

Definition 1 [6, Def. 1]. Let a set X and a function S be given that satisfy the following conditions

(1) $S: X \times I \times X \ni (x,t,y) \to S_x(t,y) \in X$

(2) $S_x(0,y)=y,\ S_x(1,y) = x$ for any $x,y \in X$

Then for any set $A \subset X$ let $\operatorname{co}S\ A = \bigcap \{D \subset X:\ A \subset S_A(I,D) \subset D\}$. If $\operatorname{co}S\ A = A$, then A is S-convex.

Definition 2 [6, Def. 2]. A space X is S-contractible if S satisfies the conditions (1), (2) and for any $x \in X$ $\{S_x(t,\cdot)\}$ is a homotopy joining the identity with a constant map (cf. [2] p. 22).

Definition 3 [6, Def. 3]. A space X is of type I if there exists S for which X is S-contractible and

(3) for any neighbourhood V of $x \in X$ there exists a neighbourhood U of x such that $\operatorname{co}S\ U \subset V$.

Several fixed point theorems were proved in [5,6,7] for the type I spaces. Two examples below contain a simple application of one of these theorems. In Example 3 we give a non-trivial illustration to [7, Theorem 1].

Example 1. Let Y be the family of all real-valued continuous functions on I. For the family $K = \{\Phi_t:\ t \in I\} \subset R^Y$ of evaluation maps ($\Phi_t(g) = g(t)$, $g \in Y$) let X be the smallest linear space containing K; X equipped with the pointwise convergence topology is a ltvs and K is compact in X [8, 27(a) p. 72]. Any map $f: X \to K$ has a fixed point [7, Theorem 1] ($S_g(t,h) = tg+(1-t)h$, $g,h \in X$, $t \in I$), though Tychonoff's fixed point theorem cannot be applied if $f(X) = K$ as $\overline{\mathrm{co}}\ K$ is not compact [8, 27(c) p. 72].

Example 2. Let us define S: K×I×K→K as follows $S_{\Phi_p}(t,\Phi_q):= = \Phi_{tp+(1-t)q}$, $p,q,t\in I$ for K from Example 1. It is clear that S is a map as Y consists of continuous functions. Besides, K is S-contractible [4, 3.4.8 p. 210]. If $\Phi_t(g)$ is contained in an open set $U\subset R$, then there exists an $\varepsilon>0$ such that $\Phi_{(t-\varepsilon,t+\varepsilon)}(g)\subset U$ and hence (3) is satisfied. Thus K is of type I and any map f: K→K has a fixed point [7, Theorem 1]. Space K is metrizable by $d(\Phi_p,\Phi_q) = |p-q|$ (we omit the elementary proof of continuity of the inclusion map i: K→K for the both topologies in K) and f has a fixed point because K is an AR-space [2, (2.8) p. 101, 6, Prop. 2].

Example 3. Let $X = \{(x_1,x_2,x_3)\in\langle 0,\infty)^3 : x_ix_j = 0$ for $i\neq j$, $i,j= =1,2,3\}$ and $d(x,y)=|x_1-y_1| + |x_2-y_2| + |x_3-y_3|$ for $x,y\in X$. We define a map $S: X\times I\times X\to X$, $S_x(t,y) = (S_x(t,y)_1, S_x(t,y)_2, S_x(t,y)_3)$ as follows: if $x_iy_j = 0$, $i\neq j$, $i,j=1,2,3$, then $S_x(t,y)_i = tx_i+(1-t)y_i$, $i=1,2,3$; if $x_ix_j\neq 0$ for fixed $i\neq j$, then

$$S_x(t,y)_j = \begin{cases} t(-x_i)+(1-t)y_j, & t\in\langle 0, y_j/(x_i+y_j)\rangle =: J \\ 0, & t\in I\setminus J \end{cases}$$

$$S_x(t,y)_i = \begin{cases} 0, & t\in J \\ tx_i+(1-t)(-y_j), & t\in I\setminus J \end{cases}$$

and $S_x(t,y)_k = 0$ for $k\in\{1,2,3\}\setminus\{i,j\}$. If $A\subset X$ and A is such that $x\in A$ implies $x_j = 0$ for all $j\neq i$, then $(\mathrm{co}S\ A)_j = 0$ for $j\neq i$ and $(\mathrm{co}S\ A)_i = \mathrm{co}\ A_i$. In the contrary case $(\mathrm{co}S\ A)_j = \mathrm{co}(A_j\cup\{0\})$, $j=1,2,3$. It can be easily checked that all balls in X are S-convex and therefore X is of type I for S ((2) obviously holds). Let T be an uncountable set and $S^s = S$, $X_s = X$ for all $s\in T$. For each $x,y\in Z := \underset{s\in T}{P} X_s$, $t\in I$ $\mathcal{S}_x(t,y) := \underset{s\in T}{P} S^s_{x_s}(t,y_s)$. It is seen that $\mathcal{S}: Z\times I\times Z\to Z$ is a map for Z equipped with the Tychonoff topology. The more Z is of type I for $\mathcal{S}$ and Z is not metrizable T being uncountable. Any compact map f: Z→Z has a fixed point [7].

Definition 4 [6, Def. 4]. A space X is of type II provided that there exists S such that X is S-contractible and

(4) for any neighbourhood V of $x\in X$ there exists a neighbourhood U of x for which $S_U(I,U)\subset V$.

Let 2^X be the family of all nonempty subsets of the set X. Besides, let C(X), T(X) denote the family of all closed and nonempty, closed, nonempty and S-convex subsets of X, respectively, for X being a topological space or S-linear topological space.

Definition 5 (cp. [9, p. 53]). Let X,S be as in the Definition 1. Then a multi-valued mapping $F\colon X \to 2^X$ is S-affine if for $x,y \in X$, $t \in I$

$$(5) \qquad S_{F(x)}(t,F(y)) := \cup \{S_z(t,F(y)) \colon z \in F(x)\} \subset F(S_x(t,y))$$

Now we give two examples of S-affine mappings.

Example 4. Let A be a convex subset of a normed space X. Assume $f\colon A \to A$ is affine and $F(x) = B(f(x),\delta) \cap A$, $x \in A$. If $w \in F(x)$ and $z \in F(y)$, then $u = tw + (1-t)z \in A$ (A is convex) and $\|f(tx+(1-t)y)-u\| = \|tf(x)+(1-t)f(y)-tw-(1-t)z\| \le t\,\|f(x)-w\| + (1-t)\,\|f(y)-z\| < \delta$ which implies $tF(x)+(1-t)F(y) \subset F(tx+(1-t)y)$, i.e. (5) holds. For $A = I \subset R = X$ and $F(x) = B(x,1/2) \cap I$ we have $F(1/2) = (0,1) \not\subset (1/4,3/4) = (1/2)F(0) + (1/2)F(1)$ which means that the inclusion $\subset$ in (5) cannot be substituted by the equality.

Example 5. Let X be as in the Example 3. If (i_1,i_2,i_3) is a permutation and $k \geq 0$, then $f\colon X \ni (x_1,x_2,x_3) \mapsto (kx_{i_1},kx_{i_2},kx_{i_3}) \in X$ is S-affine. The set $\{(x_1,x_2,x_3) \in X \colon x_i = 0\}$ is homeomorphic with R in a very special way. Assume that for example $x_3 = 0$. Let $h\colon R \to X$ be defined as follows:

$$h(x) = \begin{cases} (x,0) & \text{if } x \geq 0 \\ (0,-x) & \text{if } x < 0 \end{cases}$$

Then $h(tx+(1-t)y) = S_{h(x)}(t,h(y))$ for $x,y \in R$ and $|x| = \|h(x)\|$. Therefore (Example 4) for any S-convex subset A of X and any S-affine mapping $f\colon A \to A$ the mapping $F\colon A \to 2^A$, $F(x) = B(f(x),\delta) \cap A$ is S-affine.

Lemma 1 (cf. [9, Lemma 7.1.6 p. 54]). Let $F\colon X \to 2^X$ be S-affine and $\mathrm{Fix}\,F := \{x \in X \colon x \in F(x)\}$. Then Fix F is S-convex.

Proof. Let $\mathrm{Fix}\,F \neq \emptyset$. Then for $x,y \in \mathrm{Fix}\,F$, $t \in I$ $S_x(t,y) \subset S_x(t,F(y)) \subset S_{F(x)}(t,F(y)) \subset F(S_x(t,y))$.

Lemma 2 (cf. [9, Theorem 7.1.2]). Let $\mathcal{A}$ be a family of commuting mappings of X into 2^X such that

$$(6) \qquad \text{for each } F,G \in \mathcal{A} \quad F\colon \mathrm{Fix}\,G \to X$$

Then $F(\mathrm{Fix}\,G) \subset \mathrm{Fix}\,G$ and $F(G(X)) \subset G(X)$.

Proof. Let $x \in \mathrm{Fix}\,G$. Then $y := F(x) = F(G(x)) = G(F(x)) = G(y)$. Besides, $F(G(X)) = (F\circ G)(X) = (G\circ F)(X) = G(F(X)) \subset G(X)$ (see [1, p. 24]).

Theorem 1. Let X be a normal space of type I and let $\mathcal{A}$ be a family of mappings of X into 2^X for which $\{\bar{o}\circ S\circ F\}_{F\in\mathcal{A}}$ is a commuting family of compact S-affine mappings satisfying (6). Then $\{\overline{co}S \circ F\}_{F \in \mathcal{A}}$ has a common fixed point.

Proof. By [5, Theorem 1.5] every $\overline{co}S\ F$ has a fixed point ($F \in \mathcal{A}$). The sets of fixed points are compact and S-convex for $F \in \mathcal{A}$. Thus in view of Lemma 2 $H := \bigcap \{\mathrm{Fix}\ coS \circ F:\ F \in \mathcal{A}\} = coS\ H \neq \emptyset$.

Example 6. If $h: I \to I$ is affine, then $f_h: K \to K$ with $f_h(\Phi_p) = \Phi_{h(p)}$, $p \in I$ is S-affine (cp. Example 2). If $\{f_h\}_{h \in \mathcal{A}}$ is a commuting family (it is enough to $\mathcal{A}$ to commute), then the mappings f_h, $h \in \mathcal{A}$ have a common fixed point. For example $\mathcal{A} = \{a \cdot id:\ a \in I,\ id: I \to I\}$ is a commuting family of maps. If in Example 5 we consider a commuting family of permutations, then the mappings $f: X \ni (x_1, x_2, x_3) \mapsto k(x_{i_1}, x_{i_2}, x_{i_3}) \in X$, $k \geqslant 0$ commute.

Theorem 2. Let X be a normal space of type I and let $\mathcal{A}$ be a commuting family of compact S-affine mappings of X to T(X) such that (6) is satisfied. Then $\mathcal{A}$ has a common fixed point.

Definition 6 [5, Def. 3.4]. Let X be S-contractible and $Z \subset X$. Then $G: Z \to 2^X$ is a generalized condensing mapping if it is upper semicontinuous for any compact set Q with $G(Q) \subset Q$ and

(7) for any $Q \subset Z$ with $G(Q) \subset Q$, card $[Q \setminus \overline{G(Q)}] \leqslant 1$ implies $\overline{G(Q)}$ is compact,

(8) if $Q \subset Z$ and $Q = \overline{co}S\ G(Q)$, then Q is compact.

Definition 7 [5]. A space X is of type $\bar{I}$ ($\overline{II}$) if it is of type I (II) for S satisfying

(9) for any $A \subset X$ $\overline{co}S\ A$ is S-convex

Remark. In Example 3 X (Z) is of type $\bar{I}$ for S ($\mathcal{S}$).

Theorem 3. Let X be a space of type $\bar{I}$ and let $\mathcal{A}$ be a commuting family of generalized condensing S-affine mappings of X into T(X) satisfying (6). Then $\mathcal{A}$ has a common fixed point.

Proof. It is known ([5], cf. [3]) that for every $G \in \mathcal{A}$ Fix $G \neq \emptyset$ and by (7) it must be compact. Now we follow the proof of Theorem 1.

Remark 1. Theorem 3 can be rearranged to suite the case of the commuting families of quasicompact mappings.

Remark 2. If we assume family $\mathcal{A}$ to consist of mappings F such that $(\overline{co}S \circ F)(X)$ are contained in the finite dimensional sets, then we obtain automatically the theorems for the type II spaces (see [5]).

Remark 3. In view of Lemma 2 one of the mappings from $\mathcal{A}$ may be not S-affine in the above theorems. Similarly it is enough to assume for only one of the mappings from $\mathcal{A}$ to be compact or generalized condensing (the S-affine one).

Remark 4. In the above theorems we may insert

(6') $F, G \in \mathfrak{A}$ implies $F(\mathrm{Fix}\ G) \subset \mathrm{Fix}\ G$

in place of (6). In this case $\mathfrak{A}$ need not to commute.

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Streszczenie

O odwzorowaniach S-afinicznych

Praca zawiera kilka twierdzeń o punktach stałych dla odwzorowań wielowartościowych. Twierdzenia te są uogólnieniami dobrze znanych wyników dla rodzin odwzorowań afinicznych.

Лех Пасицки

Резюме

Об S-афинных отображениях

Работа содержает несколько теорем о неподвижных точках для многозначных отображений. Эти теоремы являются обобщением хорошо известных результатов для семейства аффинных отображений.