ZESZYTY NAUKOWE
AKADEMII GÓRNICZO-HUTNICZEJ IM. STANISŁAWA STASZICA

Nr 1049 Opuscula Mathematica z. 2 Kraków 1986

Lech Pasicki

An Application of Ky Fan's Theorem

Ky Fan has proved a theorem [1, Theorem 2 p. 235] which can be expressed in the following form:

Theorem. Let A be a compact convex subset of a normed space X. Assume $f: A \to X$ is a map as to satisfy:

(1) $d(f(x),A) < d(x,f(x))$ for all $x \in A$ with $f(x) \notin A$.

Then f has a fixed point.

For any $k \in N \cup \{0\}$ let C_I^n be the set of all real valued maps $x: \langle 0,1 \rangle = I \to R$ having the continuous derivatives of the n-th order on I. Let $X = \{x \in C_I^n : x(0) = x'(0) = \ldots = x^{(n-1)}(0) = 0\}$ and $Y = X \cap C_I^{n+1}$ be two spaces normed by $\|x\| = \sup\{|x^{(n)}(t)| : t \in I\}$. Let $A \subset K$ consist of all functions x which satisfy the following conditions for all $s,t \in I$, $s \leq t$:

(2) $L_{-1}(t) - L_{-1}(s) \leq x^{(n)}(t) - x^{(n)}(s)$

(3) $x^{(n)}(t) - x^{(n)}(s) \leq L_1(t) - L_1(s)$

Similarly let $B \subset Y$ consist of all functions x as to satisfy:

(4) $L'_{-1}(t) \leq x^{(n+1)}(t)$

(5) $x^{(n+1)}(t) \leq L'_1(t)$

for all $t \in I$. We claim that $L_{-1}, L_1 \in C_I^0$ ($L_{-1}, L_1 \in C_I^1$ if the differentiation is mentioned). It can be seen that A,B are convex, $\mathrm{Int}\, A = \mathrm{Int}\, B = \emptyset$ and $A = \bar{B}$ is compact.

We are going to apply Fan's theorem to the equation $x = Tx$ for $T: A \to A$ or $T: A \to B$, $n \in N$ (the case $n = 0$ was considered in [2]).

Lemma 1. Let $x \in A$, $z \in X\setminus A$ be two functions as to satisfy

(6) if (2) or (3) does not hold for z, x satisfies
(2) or (3), respectively, with the strong inequality.

Then $d(z,A) < d(z,x)$.

Proof. (cp. [2]). Let $y = z^{(n)}$, $u = x^{(n)}$ and $a = \|y-u\| > 0$. We consider the family $\mathcal{K}$ of all maximal intervals K for which $|y-u|(K) = \langle /2, a \rangle$. The function $z-x$ is uniformly continuous and thus $\mathcal{K}$ is finite. Let $I_0 = \langle s_0, t_0 \rangle$ contain the smallest number p_0 with $|y-u|(p_0) = a$. If I_k, p_k are known, $I_{k+1} = \langle s_{k+1}, t_{k+1} \rangle$ contains the smallest number $p_{k+1} > p_k$ for which $(y-u)(p_k)(y-u)(p_{k+1}) = -a^2$. Assume we have obtained the family $\mathcal{J} = \{I_k\}_{k=0,\dots,m}$. From the continuity of $y-u$ it follows that $\sup\{|y-u|(t) : t \in I\setminus \bigcup \mathcal{J}\} =: b < a$. Assume $(y-u)(s_0) > 0$ and for $d_k \in (0,1\rangle$, $k=0,\dots,m$ let

$$\delta_k(t) = \begin{cases} 0 & \text{for } t \in \langle 0, s_k \\ \left\{L_{(-1)^k}(t) - L_{(-1)^k}(s_k) - [u(t) - u(s_k)]\right\} d_k & \text{for } t \in \langle s_k, t_k \rangle \\ \delta_k(p_k) & \text{for } t \in (p_k, 1\rangle \end{cases}$$

Then for $r \leqslant s_0 \leqslant q \leqslant p_0$, $w := u + \delta_0$ and "small" d_0 we have

$$w(q) - w(r) = w(q) - w(s_0) + w(s_0) - w(r) = u(q) + \delta_0(q) - u(s_0) +$$
$$+ u(s_0) - u(r) \leqslant L_1(q) - L_1(s_0) + u(s_0) - u(r) \leqslant L_1(q) - L_1(r)$$

The condition (6) ensures the existence of d_0 for z not satisfying (3) in I_0. We obviously have $L_{-1}(q) - L_{-1}(r) \leqslant w(q) - w(r)$. Now it is easily seen that by integrating n times $u + \sum\{\delta_k : k = 0,\dots,m\}$ we obtain a function $v \in A$. Besides $\delta_k(1) \neq 0$, $k=0,\dots,m$ if does not satisfy (2) or (3) (see (6)). The numbers d_k can be established in such a manner that the sequence $((-1)^l \sum\{\delta_k(1) : k = 0,\dots,l\})$ increases and is bounded by $(a-b)/2$. Now we can see that $\|z-v\| < a$.

Corollary 1. If $T: A \to X$ is a continuous operator satisfying (6) for $z := Tx$, the equation $x = Tx$ has a solution.

Lemma 2. Let $x \in A$, $z \in Y\setminus B$ be two functions as to satisfy:

(7) if (4) or (5) does not hold for z and a $r \in I$,
(2) or (3), respectively, holds for x with the strong
inequality for all $s \leqslant r \leqslant t$, $s,t \in I, s < t$.

Then $d(z,A) < d(z,x)$.

Proof (cp. [2]). Suppose $z^{(n)}(t) - z^{(n)}(s) > L_1(t) - L_1(s)$. Then we have $z^{(n+1)}(r) > L_1'(r)$ on an open subinterval of (s,t) and therefore $x^{(n)}(t) - x^{(n)}(s) < L_1(t) - L_1(s)$ i.e.
(6) holds.

Corollary 2. If $T: A \to Y$ is a continuous operator satisfying (7) for $z := Tx$, the equation $Tx=x$ has a solution.

Lemma 3. Let $x \in B$, $z \in Y \setminus B$ satisfy

(8) if (4) or (5) does not hold for z and a $r \in I$, (4) or (5), respectively, is satisfied for r with the strong inequality.

Then $d(z,A) < d(z,x)$.

Proof. Suppose $z^{(n)}(t) - z^{(n)}(s) > L_1(t) - L_1(s)$. Then there exists a subinterval of (s,t) on which $z^{(n+1)}(r) > L_1'(r)$ and $x^{(n+1)}(r) \leq L_1'(r)$. Hence $x^{(n)}(t) - x^{(n)}(s) = \int_s^t x^{(n+1)}(r)dr < \int_s^t L_1'(r)dr = L_1(t) - L_1(s)$, i.e. (6) holds.

Corollary 3. If a continuous operator $T: B \to Y$ satisfies (8) ($z := Tx$) and the extension $\bar{T}: A \to X$ of T is continuous, then $\bar{T}x = x$ has a solution.

The above corollary can be applied as follows: we have a continuous operator $T: A \to Y$ ($T: A \to X$), if $T|_B$ ($T|_B : B \to Y$) satisfies (8), the equation $Tx = x$ has a solution.

Example. Let us consider the Cauchy problem

$$(9) \quad \begin{cases} x^{(n)}(t) = f(t,x(t),x'(t),\dots,x^{(n)}(t)) \\ x(0) = \dots = x^{(n-1)}(0) = 0. \end{cases}$$

If f is continuous and defined on $I \times R^{n+1}$ we can assign to it an integral operator $T: A \to X$ and ask if $Tx = x$ has a solution in A. Assume $g(t,q)$, $h(t,q)$ are the lower and the upper bound respectively of $f(t,R^n,q)$ and g,h are continuous. If $g(t,u(t)) - g(s,u(s)) < L_{-1}(t) - L_{-1}(s)$ implies $u(t) - u(s) > L_{-1}(t) - L_{-1}(s)$ and $L_1(t) - L_1(s) < h(t,u(t)) - h(s,u(s))$ implies $u(t) - u(s) < L_1(t) - L_1(s)$ then in view of Corollary 1 (9) has a solution in A. The condition (6) can be written in the following form (for our example) $u(t) - u(s) \geq L_1(t) - L_1(s)$ implies $h(t,u,(t)) - h(s,u(s)) \leq L_1(t) - L_1(s)$ (the form of the conditions for L_{-1} is obvious).

There are operators of the form $(Tx)(t) = h(t,x(t))$ which are continuous on A, $n = 0$ and such that $TA \not\subset A$ and (6) holds (see [2]).

References

1. Fan K.: Extensions of Two Fixed Point Theorems ef F.E. Browder, Math. Z. 112 (1969), 234-240.

2. Pasioki L.: An Application of a Fixed Point Theorem, Comment. Math. (to appear).

Recenzent

Dr Józef Bryszewski

Streszczenie

Zastosowanie twierdzenia Ky Fana

W pracy stosuje się twierdzenie Ky Fana o punkcie stałym do rozstrzygania problemu istnienia rozwiązania równania x = Tx w zbiorze A tych funkcji z C^n_I, które spełniają nierówności (2) i (3) lub w zbiorze B, określonym przez (4) i (5).

Лех Пасицки

Резюме

Применение теоремы Ки Фана

В работе применено теорему Ки Фана о неподвижной точке, целью раскрытия проблемы существования решения уравнения x = T_x в множестве A тех функций с S_I^n, которые выполняют неравенства (2) и (3) или в множестве B, определённым неравенствами (4) и (5).