

Lech Pasicki

A Fixed Point Theorem

The paper contains a theorem being a generalization of a fixed point theorem of Ky Fan.

Definition 1. A set $X \neq \emptyset$ is S -linear if S is a mapping $S: X \times I \times X \ni (x, t, y) \mapsto S_x(t, y) \in X$ ($I = \langle 0, 1 \rangle$) such that:

$$(1) \quad S_x(0, y) = y, \quad S_x(1, y) = x \text{ for every } x, y \in X.$$

If A is a subset of X , then $\text{co}S A = \bigcap \{D \subset X: A \subset S_A(I, D) \subset D\}$.

For any metric space (X, d) and its non-empty subsets A, B we write $d(A, B) = \inf \{d(x, y): x \in A, y \in B\}$, $B(A, r) = \{x \in X: d(A, x) < r\}$ if $r > 0$. Let A be an S -linear subset of X , then for any $z \in X$, $\varepsilon > 0$ $A(z, \varepsilon) = \text{co}S[B(z, d(z, A) + \varepsilon) \cap A]$ and $A(z) = \bigcap \{A(z, \varepsilon): \varepsilon > 0\}$. If X is normed and A is convex, we have $A(z, \varepsilon) = \overline{B(z, d(z, A) + \varepsilon)} \cap \overline{A}$ and $A(z) = \{x \in \overline{A}: d(x, z) = d(z, A)\} = A(z, 0) = \overline{B(z, d(z, A))} \cap \overline{A}$.

In this paper we write "map" in place of "continuous mapping".

Definition 2 (cp. [6, Def. 3]). A metric space (X, d) is of type I if X is S -linear for an S such that $S_x: I \times X \rightarrow X$ is a map for each $x \in X$ and the following condition holds

$$(2) \quad \text{for every } x \in X, q > 0 \text{ there exists a } p > 0 \text{ such that} \\ \text{co}S B(x, p) \subset B(x, q).$$

Theorem 1. Let A be a closed set of type I for S in a metric space (X, d) . Assume $f: A \rightarrow X$ is a compact map as to satisfy

$$(3) \quad \text{there exists a lower semi-continuous mapping } p: C = f(A) \setminus A \rightarrow (0, \infty) \\ \text{with } 0 < d(f^{-1}(z), A(z, p(z))), z \in C.$$

Then f has a fixed point.

Proof. In view of the lower semi-continuity of p the set $p^{-1}((a, \infty))$

is open in C . From (3) it follows that to every $y \in C$ we can assign an $a > 0$ such that $y \in p^{-1}((a, \infty))$. Thus there exists a mapping $q: C \rightarrow (0, \infty)$ as to satisfy:

$$(4) \quad \text{for every } y \in C \quad B(y, q(y)) \cap C \subset \{z \in C : q(z) < p(z)/3\}.$$

We are going to define a retraction $r: A \cup f(A) \rightarrow A$ such that $r(z) \in A(z, p(z))$, $z \notin A$. Let $\{G_s\}_{s \in T}$ be a locally finite partition of unity such that $\mathcal{G} = \{G_s := \varepsilon_s^{-1}((0, 1))\}_{s \in T}$ is a star refinement of the open cover $\mathcal{B} = \{B_z := B(z, \min\{q(z), d(z, A)/2\}) \cap C\}_{z \in C}$ of C . Let $\text{St}(G_s, \mathcal{G}) \subset B_{z(s)}$ and $x(s) \in B(z(s), d(z(s), A) + \min\{d(z(s), A), q(z(s))\})$. If $z \in G_s$, then $d(z, x(s)) \leq d(z, z(s)) + d(z(s), x(s)) < \min\{q(z(s)), d(z(s), A)/2\} + d(z(s), A) + \min\{q(z(s)), d(z(s), A)\}$ as $z \in B_{z(s)}$. We have $d(z(s), A) \leq d(z(s), z) + d(z, A) \leq \min\{q(z(s)), d(z(s), A)/2\} + d(z, A)$. In view of (4) $q(z(s)) < p(z)/3$. Thus:

$$(5) \quad d(z, x(s)) < 3q(z(s)) + d(z, A) < p(z) + d(z, A)$$

holds. On the other hand we have $d(z, x(s)) < 5d(z(s), A)/2 \leq 5[d(z(s), z) + d(z, A)]/2 \leq 5[d(z(s), A)/2 + d(z, A)]/2 < 5(2d(z, A))/2$ which implies:

$$(6) \quad \text{dia}\{x(s) : z \in G_s\} \leq 10d(z, A)$$

We may assume that T is well ordered by " $<$ " and $T_z = \{s \in T : G_s(z) \neq 0\} = \{s_1, \dots, s_n\}$, $n = n(z)$ with $s_1 < \dots < s_n$. Then for $y \in A$, $c_s(z) = G_s(z)/\sup\{G_s(z) : s \in T\}$ we write:

$$r(z) = \begin{cases} z, & z \in A, \\ \varepsilon_{x(s_1)}(c_{s_1}(z), \varepsilon_{x(s_2)}(c_{s_2}(z), \dots, \varepsilon_{x(s_n)}(c_{s_n}(z), y) \dots)), & z \in C \end{cases}$$

In view of (6) $r: A \cup f(A) \rightarrow A$ is a continuous retraction [cp. [6, proof of Th. 1]]. The map $r \circ f$ has a fixed point [7, Th. 1], say x . If $f(x) = z \notin A$, then in view of (5) $r(z) \in A(z, p(z))$ which contradicts (3) as $x \in r^{-1}(z)$. Thus $f(x) \in A$ and $x = (r \circ f)(x) = f(x)$.

Proposition 1. Let X be a paracompact space and $p: X \rightarrow (0, \infty)$ a lower semi-continuous mapping. Then there exists a map $q: X \rightarrow (0, \infty)$ with $q(x) \leq p(x)$, $x \in X$.

Proof. Let $P(x) = (0, p(x))$, $x \in X$. We can see that the values of $P: X \rightarrow 2^{(0, \infty)}$ are convex. For any $y \in (0, \infty)$ the set $P^-(y) = \{x : y < p(x)\}$ is open, p being lower semi-continuous. Thus P admits a continuous selection [2, Th. 1 p.285].

From Proposition 1 it follows that, without the loss of generality, we may assume p to be continuous in Theorem 1.

In the remarks below we refer to the notations from Theorem 1.

Remark 1. The condition $0 < d(f^{-1}(z), A(z))$ implies $0 < d(x, A(z))$, $x \in f^{-1}(z)$ implies $f^{-1}(z) \cap A(z) = \emptyset$. If at least one of the sets $f^{-1}(z)$, $A(z)$ is compact, then above conditions are equivalent (we may insert also $A(z, \varepsilon)$ in place of $A(z)$ in this remark).

Remark 2. If there is a retraction $r: C \rightarrow A$ with $r(z) \in \bar{B}(z, d(z, A))$, $z \in C$, then the condition $d(x, f(x)) - d(f(x), A) > 0$, $x \in A$, $f(x) \notin A$ guarantees the existence of a fixed point of f . The same situation is for $r(z) \in A(z)$ and $0 < d(x, A(f(x)))$, $x \in A$, $f(x) \notin A$.

Lemma. If A is a compact S -linear set in a metric space (X, d) and $f: A \rightarrow X$ is a map satisfying:

$$(7) \quad 0 < d(f^{-1}(z), A(z)), \quad z \in C,$$

then (3) holds.

Proof. From the compactness of $f^{-1}(z)$ and $A(z, \varepsilon)$ follows the existence of $\varepsilon = \varepsilon(z) > 0$ with $f^{-1}(z) \cap A(z, \varepsilon) = \emptyset$ (we omit the routine proof of this fact). Suppose that for every $n \in \mathbb{N}$ there is $y_n \in B(z, 1/n) \cap C$ with $d(f^{-1}(y_n), A(y_n, 1/n)) \leq 1/n$. In view of the compactness of A we may assume $y_n = f(x_n)$, $x_n \in A(y_n, 1/n)$, $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} x_n = x$. Obviously $x \in f^{-1}(z)$ and $x \in A(z)$ for $z = f(x)$ (see [6, proof of Th. 4]) - a contradiction. Thus the mapping $p: C \rightarrow (0, \infty)$ $p(z) = \sup \{ \varepsilon < d(f^{-1}(y), A(y, \varepsilon)), y \in B(z, \varepsilon) \cap C \}$ is well defined. Suppose p is not lower semi-continuous. Then for an $\varepsilon > 0$ and a $z \in C$ each $B(z, 1/n)$, $n \in \mathbb{N}$ contains y_n with $p(y_n) \leq p(z) - \varepsilon$. This fact together with $A(y_n, p(y_n)) \subset A(z, p(z))$ for all $y_n \in B(z, \varepsilon)$ gives a contradiction.

Corollary 1. From Remark 1 and Lemma it follows that Theorem 1 is a generalization of Theorem 4 [6]. For normed spaces Theorem 4 [6] is equivalent to Ky Fan's theorem [5, Th. 2 p.235].

Proposition 2. Let A be an S -linear set in a metric space (X, d) . Assume that:

$$(8) \quad A(z, \varepsilon) = \bar{B}(z, d(z, A) + \varepsilon) \cap \bar{A}$$

holds. Then the following condition is satisfied:

$$(9) \quad d(f^{-1}(z), A(z, \varepsilon)) \geq d(f^{-1}(z), z) - d(z, A) - \varepsilon.$$

Proof. We have $A(z, \varepsilon) \subset \bar{B}(z, d(z, A) + \varepsilon)$ and hence $d(x, A(z, \varepsilon)) \geq d(x, \bar{B}(z, d(z, A) + \varepsilon))$ which gives $d(x, A(z, \varepsilon)) \geq d(x, z) - d(z, A) - \varepsilon$. Now we easily obtain (9).

Corollary 2. If there is a sequence of numbers ε , satisfying (8), which tends to the zero, then $A(z)$ satisfies (8) and is S -convex

[6, Prop. 1]. If $A(z)$ satisfies (8), then:

$$(10) \quad d(f^{-1}(z), A(z)) \geq d(f^{-1}(z), z) - d(z, A)$$

Remark 3. If $A(z)$ satisfies (8), then $d(z, A(z)) < d(z, f^{-1}(z))$ implies $d(z, A(z)) < d(z, x)$ for each $x \in f^{-1}(z)$ and $0 < d(f^{-1}(z), A(z))$. If at least one of the sets $f^{-1}(z)$, $A(z)$ is compact, the above conditions are equivalent.

The assumptions of Proposition 2 and Corollary 2 are satisfied for every convex set A in any linear metric space (X, d) which has only convex balls, if A is closed and non-empty.

Theorem 2 (cp. [1], [3, p.61]). Let A be a closed convex set in a normed space $(X, \|\cdot\|)$. If $0 \in \text{Int } A$ and $f: A \rightarrow X$ is a compact map satisfying one of the following conditions for all $x \in \text{Fr } A$:

- (11) $\|x\| \geq \|f(x)\|$ (Rothe)
 (12) $\|f(x) - x\| \geq \|f(x)\|$ (Altman)
 (13) $\|f(x) - x\|^2 \geq \|f(x)\|^2 - \|x\|^2$ (Altman)

then f has a fixed point.

Proposition 3. Let $A = \bar{B}(0, 1)$ in a normed space $(X, \|\cdot\|)$. If a map $f: A \rightarrow X$ satisfies (13), then (3) holds.

Proof. In view of (13) we have $\|x - z\| \geq (\|z\|^2 - \|x\|^2)^{1/2}$ for each $x \in f^{-1}(z)$. Hence $d(f^{-1}(z), z) \geq (\|z\|^2 - 1)^{1/2}$ and (10) implies:

$$\begin{aligned} d(f^{-1}(z), A(z)) &\geq (\|z\|^2 - 1)^{1/2} - (\|z\| - 1) = (\|z\| - 1)^{1/2} [(\|z\| + 1)^{1/2} - \\ &- (\|z\| - 1)^{1/2}] = 2(\|z\| - 1)^{1/2} / [(\|z\| + 1)^{1/2} + (\|z\| - 1)^{1/2}] > \\ &> [(\|z\| - 1) / (\|z\| + 1)]^{1/2} =: p(z). \end{aligned}$$

It is seen that $p: C \rightarrow (0, \infty)$ is continuous. The above inequality and (9) give $d(f^{-1}(z), A(z, p(z))) > p(z) - p(z) = 0$

One can compute p for (11) and for (12), but it is known that each of these conditions implies (13) (see [1]).

Remark 4. The assumption $0 \in \text{Int } A$ cannot be disregarded in the Altman theorems as it is seen that $f: \langle 0, 1 \rangle \rightarrow \mathbb{R}$, $f: x \mapsto x - 1$ has no fixed point and satisfies (12), (13). Rothe's theorem is obvious for $0 \in \text{Fr } A$ and it is false for $0 \notin A$ ($f(x) = 0$, $x \in A$).

Ky Fan applied the following inwardness condition

- (14) for each $x \in A$ such that $x \neq f(x)$ the line segment $[x, f(x)]$ contains at least two points of A

to any compact convex subset A of a normed space $(X, \|\cdot\|)$ and any map $f: A \rightarrow X$ (see [3, Th. 2.2., p.74]).

Proposition 4. Let $A = \text{co } A$ be a subset of a normed space $(X, \|\cdot\|)$. Assume that $x \in A$, $B = \text{co } B \subset X$ are such that $d(A, B) > 0$ and:

(15) there exists a $t \in (-\infty, 1)$ and a point $y \in B$ for which
 $tx + (1-t)y \in A$,

Then $d(A, B) < d(x, B)$.

Proof. Suppose $d(A, B) = d(x, B)$ and let $t \in (0, 1)$. Then $tz + (1-t)y \in B$, $z \in B$ and hence $d(A, B) = d(x, B) \leq \|tx + (1-t)y - tz - (1-t)y\| = t\|x - z\|$. Thus $d(A, B) = d(x, B) \leq \inf \{t\|x - z\| : z \in B\} = td(x, B)$ - a contradiction. If $t \leq 0$, then one can check that $y \in [x, tx + (1-t)y]$ which means $y \in A$ - a contradiction.

If we set $B = f(x) = y$ in (15), then it becomes clear that (7) is a generalization of (14). The fact that (7) is more general than (14) is known for A being convex and compact [5, p.235]:

Simple examples show that there is no dependence between (14) and (11), though if A is a ball, then every map satisfying (11) is inward as A is convex $A = \bar{B}(0, r)$.

The examples below show that there is no dependence between (14) and (12), (13) even if A is a ball.

Example 1. Let us consider R^2 normed by $\|(x, y)\| = \max\{|x|, |y|\}$ and $A = \bar{B}(0, 1)$. Then for $f: A \rightarrow X$, $f: (x, y) \rightarrow (x, y + \sin(10\pi y/3))$ we can see that $f(x, \pm 1) \in A$, $x \in \langle 0, 1 \rangle$ and hence f is inward. For $z = (1, 3/4)$ we have $\|z\|^2 + \|z - f(z)\|^2 = 2 < 49/16 = \|f(z)\|^2$ and (13) does not hold which implies (11), (12) are not satisfied.

Example 2. Let R^2 be normed by $\|(x, y)\| = |x| + |y|$. We consider $f: \bar{B}(0, 1) \rightarrow R^2$, $f: z = (x, y) \mapsto \|(x, y)\|(\max\{0, 8y^2/3 - 2/3\}, \max\{0, 8x^2/3 - 2/3\})$. For $z = (1, 0)$ we have $f(z) = (0, 2)$ and therefore (14) does not hold. We will show that (12) holds. It is enough to consider $z \in S(0, 1)$ for $1/2 \leq x$, $0 \leq y \leq 1/2$, i.e. $y = 1 - x$. Then $f(z) = (0, 8x^2/3 - 2/3)$, $f(z) - z = (-x, 8x^2/3 + x - 5/3)$. We have $8x^2/3 + x - 5/3 = 0$ for $x = 5/8$ and hence $\|f(z) - z\| - \|f(z)\| = x - 8x^2/3 - x + 5/3 - 8x^2/3 + 2/3 = -16x^2/3 + 7/3 - 25/12 + 7/3 > 0$ for $x \in \langle 1/2, 5/8 \rangle$. For $x \in \langle 5/8, 1 \rangle$ we have $\|f(z) - z\| - \|f(z)\| = x + 8x^2/3 + x - 5/3 - 8x^2/3 + 2/3 = 2x - 1 > 0$. Hence (12) holds and (13) too.

References

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Review by:

Prof. dr hab. Lech Górniewicz

Streszczenie

Twierdzenie o punkcie stałym

Głównym wynikiem pracy jest twierdzenie 1. Stanowi ono uogólnienie twierdzenia Ky Fana o punkcie stałym (por. 5 [Th. 2 str. 235]), orzekającego, iż każde ciągłe odwzorowanie $f: X \rightarrow E$, gdzie X jest zwartym, wypukłym podzbiorem przestrzeni unormowanej E , spełniające warunek $d(x, f(x)) > d(X, f(x))$ dla każdego $x \in X$ takiego, że $f(x) \in X$ ma punkt stały. W twierdzeniu 1 przestrzeń unormowana została zastąpiona przestrzenią metryczną z ogólniejszą strukturą liniową zadaną przez rodzinę homotopii (zob. Definicje 1, 2). O odwzorowaniu f zakłada się, że jest zwarte. W dalszym ciągu pracy rozważane są zależności między warunkami Altmana, Ky Fana i szczególną postacią warunku (3) z Tw. 1, tj. warunkiem (7), który jest dość ogólny mimo prostoty zapisu.

Лех Пасицкий

Резюме

Теорема о постоянной точке

Главным результатом работы является Теорема 1. Она становится обобщением теоремы Ky Fana о постоянной точке (ср. [5], т. 2, с. 235), говорящей о том, что любое непрерывное отображение $f: X \rightarrow E$, где X является замкнутым выпуклым подмножеством нормализованного пространства E , выполняющим условие $d(x, f(x)) > d(X, f(x))$ для любого $x \in X$, такого, что $f(x) \in X$, имеет постоянную точку. В Теореме 1 нормализованное пространство заменяется метрическим пространством с обобщенной заданной линейной структурой, обусловленной семейством гомотопий (см. Определения 1, 2). Предполагается, что отображение f является замкнутым. В дальнейшей части работы рассуждаются зависимости между условиями Алтмана, Ky Fana и особенным видом условия (3) Теоремы 1, т. е. условием (7), которое является несколько обобщенным, несмотря на простую запись.