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On Continuous Selections

In this paper "map" means "continuous mapping".

Definition 1. A set X is said to be S-linear if $S: X \times I \times X \to X$ is a mapping satisfying:

(1) $S_x(0,y) = y$, $S_x(1,y) = x$, $x,y \in X$

Then for any non-empty set $A \subset X$ let $\text{co}S\ A = \bigcap\{\emptyset \neq D \subset X : S_A(I,D) \subset D\}$ and $\text{co}S\ \emptyset = \emptyset$. If $\text{co}S\ A = A$, then A is S-convex.

Definition 2 (cf.[9]). A space X will be called locally S-contractible if X is S-linear and

(2) for any $x \in X$ there exists a neighbourhood U such that for any $z \in U$ $\{S_z(t,\cdot)_{|U}\}_{t \in I}$ is a homotopy joining the identity with a constant map.

If $U = X$, then X is S-contractible.

Let 2^X be the family of all non-empty subsets of any set $X \neq \emptyset$.

Definition 3 (cp.[8, Definition 2.8]). A space X is said to be (locally) of type 0 for the families of neighbourhoods $\mathcal{U}_A$ of all the sets of $\mathcal{A} \subset 2^X$ if $A \in \mathcal{A}$ implies $\overline{\text{co}S}\ A \in \mathcal{A}$ and X is (locally) S-contractible with S satisfying

(3) for any $A \in \mathcal{A}$ and any neighbourhood $V \in \mathcal{U}_{\text{co}S\ A}$ there exists a neighbourhood $U \in \mathcal{U}_A$ for which $\text{co}S\ U \subset V$.

For any metric space (Y,d) and $A \in 2^Y$, $r > 0$ we will write $B(A,r) = \{x \in Y : d(x,A) < r\}$.

Definition 4. A metric space (Y,d) will be called uniformly (locally) of type 0 for $\mathcal{A} \subset 2^Y$ and balls if it is (locally)

S-contractible for $\mathcal{S}$ satisfying (cf.[7, Def.1.1, p.558])

(4) for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any $A \in \mathcal{A}$ we have $\operatorname{coS} B(A,\delta) \subset B \operatorname{coS}(A,\varepsilon)$.

It is seen that any uniformly (locally) of type 0 space for $\mathcal{A}$ and balls is (locally) of type 0 for $\mathcal{A}$ and balls.

In the sequel the omission of the information about $\mathcal{A}$ will mean $\mathcal{A} = 2^Y$. The word "selection" replaces "continuous selection" and (Y,d) is always a metric space in this paper. The family of all complete, $\mathcal{S}$-convex sets of the S-contractible space (Y,d) will be denoted by $D(Y)$.

<u>Theorem</u> 1 (cp.[4, Theorem 3.2" p. 367]). Let X be a paracompact space and (Y,d) a space uniformly of type 0 for balls. Then any l.s.c. carrier $F: X \to D(Y)$ admits a selection.

<u>Proof</u>. From the lower semi-continuity of $F_0 := F$ it follows that the collection $W_1 = \{F_0^-(B(y,\delta_1))\}_{y\in Y} = \{x \in X : F(x) \cap B(y,\delta_1) \neq \emptyset\}_{y \in Y}$ [1, p.25] is an open cover of X. Let $A_1 = \{a_q\}_{q\in Q}$ be a locally finite partition of unity subordinated to W_1 [3, 5.1.9 p. 375] where Q is a well ordered set and $a_q^{-1}(0,1> \subset F^-(B(y_q,\delta_1))$ for $q \in Q$. Let us write $Q_x = \{q \in Q : a_q(x) \neq 0\}$. Then for $c_q(x) = a_q(x)/\sup\{a_q(x) : q \in Q\}$ the function $f_1(x) = S_{y_{q_1}}(c_{q_1}(x), S_{y_{q_2}}(c_{q_2}(x),\dots,S_{y_{q_n}}(c_{q_n}(x),y)\dots)$ for $Q_x = \{q_1,q_2,\dots,q_n\}$, $q_1 < q_2 < \dots < q_n$ is a map $f_1 : X \to Y$ (there exists $c_{q_i}(x)) = 1$ and $f_1(x)$ does not depend on y (cp.[9, p. 597])). From the definition of f_1 it follows that $f_1(x) \in \operatorname{coS} B(F_0(x),\delta_1)$ and in view of (4) we may assume $\delta_1 = \delta_1(\varkappa_1)$ and $f_1(x) \in B(\operatorname{coS} F(x),\varkappa_1) = B(F(x),\varkappa_1)$ ($F(x)$ is S-convex). Let us write $F_1(x) = B(f_1(x),\varkappa_1) \cap F(x) \neq \emptyset$. It can be seen that F_1 is l.s.c (and not necessarily S-convex). We repeat our considerations for $\delta_2(\varkappa_2)$, F_1, W_2, A_2 in order to obtain $f_2(x) \in \operatorname{coS} B(F_1(x),\delta_2) \subset B(\operatorname{coS} F_1(x),\varkappa_2) \subset B(F(x),\varkappa_2)$ and $F_2(x) = B(f_2(x),\varkappa_2) \cap F(x) \neq \emptyset$. Then by induction we define:

(5) $$f_n(x) \in \operatorname{coS} B(F_{n-1}(x),\delta_n) \subset B(\operatorname{coS} F_{n-1}(x),\varkappa_n) \subset B(F(x),\varkappa_n)$$

and:

(6) $$F_n(x) = B(f_n(x),\varkappa_n) \cap F(x).$$

It follows from (6) that $\operatorname{dia} F_n(x) < 2\varkappa_n$, $d(f_n(x),F_n(x)) < \varkappa_n$ and (5) implies:

$$d(f_n(x),f_{n+1}(x)) \leq d(f_n(x),F_n(x)) + \operatorname{dia} F_n(x) + d(F_n(x),f_{n+1}(x)) < \varkappa_n + 2\varkappa_n + \operatorname{dia} \operatorname{coS} F_n(x) + \varkappa_{n+1} \leq 3\varkappa_n + \operatorname{dia} \operatorname{coS} B(y,\varkappa_n) + \varkappa_{n+1}.$$

From (4) we obtain $g(\varkappa_n) := \sup\{\text{dia co}S\ B(y,\varkappa_n): y\in Y\} < \varkappa$ for any $\varkappa > 0$ and small enough $\varkappa_n$. Let us choose $\varkappa_n > 0$ for $n\in N$ in such a manner that $\Sigma\{\varkappa_n + g(\varkappa_n): n\in N\} < \infty$. Then for any $x\in X$ and big enough $m<n$ we have $d(f_m(x),f_n(x)) \leq d(f_m(x),f_{m+1}(x)) + \ldots + d(f_{n-1}(x),f_n(x)) < \varepsilon$. It means that $\{f_n\}_{n\in N}$ converges uniformly to a limit f that is continuous and $f(x)\in F(x)$ for all $x\in X$.

Definition 5. A metric space (Y,d) is said to be uniformly (locally) of type II for $\mathcal{A}\subset 2^Y$ and balls if it is (locally) S-contractible for S satisfying

(7) for any $\varepsilon>0$ there is $\delta>0$ such that for any $A\in\mathcal{A}$ we have $S_{B(A,\delta)}(I,B(A,\delta))\subset B(S_A(I,A),\varepsilon)$.

Theorem 2. Let X be a finite dimensional paracompact space and (Y,d) a metric space uniformly of type II for balls. Then any l.s.c. carrier $F: X\to D(Y)$ admits a selection.

Proof. Let us assume $\dim X \leq k-1$. In view of the Dowker theorem[3, 7.2.4 p.484] we may think (see proof of Theorem 1) that the partitions of unity $\{A_n\}_{n\in N}$ are of order $\leq k-1$. For any $A\in D(Y)$ we have $S_A(I,A) = A$ and in view of (7) for any $\varepsilon_1 > 0$ there is $\varepsilon > 0$ with $S_{B(A,\varepsilon)}(I,B(A,\varepsilon))\subset B(A,\varepsilon_1)$. There is $0<\delta<\varepsilon$ with $S_{B(A,\delta)}(I,B(A,\delta))\subset \subset B(S_A(I,A),\varepsilon) = B(A,\varepsilon)$ and hence $S_{B(A,\delta)}(I,S_{B(A,\delta)}(I,B(A,\delta)))\subset \subset S_{B(A,\delta)}(I,B(A,\varepsilon))\subset S_{B(A,\varepsilon)}(I,B(A,\varepsilon))\subset B(A,\varepsilon_1)$. By induction we prove that for k and any $\varkappa_n > 0$ there exists $\delta_n = \delta(\varkappa_n) > 0$ for which $S_{y_{q_1}}(I,S_{y_{q_2}}(I,\ldots,S_{y_{q_n}}(I,B(A,\delta_n)\ldots)\subset B(A,\varkappa_n)$. Now we follow the proof of Theorem 1 writing $S_{y_{q_1}}(I,\ldots,S_{y_{q_k}}(I,B(F(x),\delta)\ldots)$ in place of $\text{co}S\ B(F(x),\delta)$ and (7) in place of (4).

Remark. In the above theorems it is enough to assume $\mathcal{A} = \{F(x)\cap B(y,\varepsilon)\neq\emptyset : x\in X,\ y\in Y,\ 0<\varepsilon<\gamma\}$ in place of $\mathcal{A} = 2^Y$.

Theorem 3. Let (Y,d) be uniformly of type 0 (finite dimensional and uniformly of type II) for balls. Then any l.s.c. compact carrier $F: Y\to D(Y)$ (i.e. $\overline{F(Y)}$ is compact) has a fixed point.

Proof. Let f be a selection for F. In view of [10, Th.1] ([8, Th. 1.1]) f has a fixed point.

Theorem 4 (cp. [4, Bartle-Graves Theorem]). Let $f: Y\to X$ be an open mapping of a complete metric space (Y,d) uniformly of type 0 for balls in a paracompact space X. If $f^{-1}(x) = \overline{\text{co}S}\ f^{-1}(x)$, $x\in X$, then there is a map $g: X\to Y$ with $f\circ g = id_X$, i.e. $g(x)\in f^{-1}(x)$, $x\in X$.

Proof. It is enough to show that f^{-1} is l.s.c. Let U be an open subset of Y, then $\{x\in X: f^{-1}(x)\cap U\neq\emptyset\} = f(U)$ is open.

Definition 6. A mapping $f: Y \to X$ will be called (S^2, S^1)-linear if X, Y are respectively S^1-, S^2-contractible and $f(S^2_x(t,z)) = S^1_{f(x)}(t, f(z))$, $x, z \in Y$, $t \in I$.

Proposition 1. Let $f: Y \to X$ be (S^2, S^1)-linear open map of a complete metric space (Y,d) uniformly of type 0 for balls in a paracompact space X. Then there is a map $g: X \to Y$ with $f \circ g = id_X$.

Proof. For any $x \in X$ $f^{-1}(x)$ is closed and it is S^2-convex, x being S^1-convex. Hence Proposition 1 follows from Theorem 4.

Theorem 5 (cp. [4, Theorem 3.2" p.367]). If X is a T_1-space, then the following properties are equivalent

a) X is paracompact,

b) If (Y,d) is uniformly of type 0 then every l.s.c. carrier $F: X \to D(Y)$ admits a selection.

Proof. We have proved a) $\to$ b). Every Banach space satisfies (4) for $S_x(t,y) := tx + (1-t)y$ and hence b) $\to$ a).

Theorem 6 (cp. [4, Theorem 3.1" p.367]). The following properties of a T_1-space X are equivalent

a) X is normal and countably paracompact,

b) If (Y,d) is separable uniformly of type 0 then every l.s.c. carrier $F: X \to D(Y)$ admits a selection.

Proof. If we assume (Y,d) to be separable Banach space, we obtain b) $\to$ a). For a) $\to$ b) it is enough to consider a dense set $\{y_q\}_{q \in N}$ in the proof of Theorem 1.

Theorem 7 (cp. [4, Lemma 5.2 p. 373]). If X is perfectly normal, (Y,d) is complete space uniformly of type 0 for balls and for every $y \in Y$, $\varepsilon > 0$ $\overline{\cos}\, B(y,\varepsilon) = \bar{B}(y,\varepsilon)$, then for any l.s.c. carrier $F: X \to D(Y)$ there exists a countable collection $\mathcal{F}$ of selections for F such that for every $x \in X$ $\{f(x)\}_{f \in \mathcal{F}}$ is dense in $F(x)$.

Proof. We follow Michael's proof taking $y_j \pm v_k := B(y_j, 1/2^k)$.

Proposition 2. Let X, Y be topological spaces. If $F: X \to 2^Y$ is l.s.c. and for $A = \bar{A} \subset X$ g is a selection for $F|_A$ then G defined by:

$$(8) \qquad G(x) = \begin{cases} g(x) & \text{for } x \in A, \\ F(x) & \text{for } x \in X \setminus A \end{cases}$$

is l.s.c.

Proof. Let U be a neighbourhood in Y. Then we have $G^-(U) = (F^-(U) \cap X \setminus A) \cup (g^{-1}(U) \cap A)$. From the continuity of g it follows that $g^{-1}(U) \cap A = V \cap A$ V being open in X. Obviously $g^{-1}(U) \subset F^-(U)$ and hence $V \cap A = A \cap V \cap F^-(U)$. Thus $G^-(U) = [F^-(U) \cap (X \setminus A)] \cup [A \cap V \cap F^-(U)] = F^-(U) \cap [((X \setminus A) \cup A) \cap ((X \setminus A) \cup (V \cap F^-(U))] = F^-(U) \cap [(X \setminus A) \cup (V \cap F^-(U)]$ is open.

Remark. For $A = \{x_o\}$ we may take $g(x_o) = y$ for any $y \in F(x_o)$.

Theorem 8. Let $A=\bar{A}$ be a subset of a paracompact space X and let (Y,d) uniformly of type 0 for balls, Let $F: X \to D(Y)$ be l.s.c. Then any selection for $F|_A$ can be extended to a selection for F.

Proof. Let g be a selection for $F|_A$. From the Proposition 2 it follows that G given by (8) is l.s.c. and obviously $G(x) \in D(Y)$ for $x \in X$. In view of Theorem 1 G admits a selection.

Theorem 9 (cp.[5, Theorem 1.2, p.563]). Let $A=\bar{A}$ be a subset of finite dimensional paracompact space X and (Y,d) uniformly of type II for balls. Then any selection g for $F|_A$, where $F: X \to D(Y)$ is a l.s.c. carrier, can be extended to a selection for F.

Proof. Compare Proposition 2 with Theorem 2.

Corollary (cp. [6, Theorem 6.1 p.386]). Let X be a paracompact space and (Y,d) uniformly of type 0 for balls. Let $A \subset X$ be a retract of X and let $F: X \to D(Y)$ be al.s.c. carrier. Then any selection for $F|_A$ can be extended to a selection for F.

Proof. We have $A=\bar{A}$, A being a rectract.

Remark. It is not difficult to reformulate the above Corollary to suite Theorem 9.

Theorem 10 (cp. [2, Corollary 7.5, p.92]). Let $A=\bar{A}$ be a subset of a paracompact space X and (Y,d) a complete space uniformly of type 0 for balls. If $f: A \to Y$ is such a map that $\overline{\mathrm{co}}S\, f(A)$ is S-convex, then f can be extended to a map $h: X \to Y$ such that $h(X) \subset \overline{\mathrm{co}}S\, f(A)$.

Proof. Let us write:

$$G(x)=\begin{cases} f(x) & \text{for } x \in A, \\ Y & \text{for } x \in X \setminus A. \end{cases}$$

In view of Proposition 2 and Theorem 8 G admits a selection g. There is a retraction r of Y on $\overline{\mathrm{co}}S\, f(A)$ [9, Theorem 1] and $h := r \circ g$.

For X being finite dimensional we can obtain the uniformly of type II version of the above theorem.

Proposition 3. Let X be a topological space and (Y,d) uniformly of type 0 for balls. Then if $F: X \to 2^Y$ is l.s.c. and

(9) for any $\delta > 0$ and $x_0 \in X$ there exists a neighbourhood V of x_0 such that for any $x \in V$ $F(x_0) \subset B(F(x),\delta)$,

then $\mathrm{co}S\, F$ is l.s.c.

Proof. Let $\varepsilon > 0$, $x_0 \in X$ be arbitrary. Then there exists a neighbourhood V of x_0 such that for $x \in V$, $\mathrm{co}S\, F(x_0) \subset \mathrm{co}S\, B(F(x),\delta) \subseteq B(\mathrm{co}S\, F(x),\varepsilon)$ (see (4)).

Proposition 4. Let X, (Y,d) be as in Proposition 3. If $F: X \to 2^Y$ is l.s.c. and $F(x)$ is compact for every $x \in X$, then (9) holds.

Proof. Let us consider $\{B(y,\delta)\}_{y\in F(x_0)}$. We can choose a finite cover $\{B(y_i,\delta)\}_{i=1,\dots,n}$ of $F(x_0)$. Then $V := \bigcap\{F^-(B(y_i,\delta)): i=1,\dots,n\}$ is open and for $x\in V$ $F(x_0)\subset B(F(x),\delta)$.

Definition 7. A space X is (locally) of type $\bar{0}$ for the families $\mathcal{U}_A$ of neighbourhoods of the sets $A\in\mathcal{A}\subset 2^X$ if it is (locally) of type 0 and $\operatorname{co}S\,U$ is S-convex for every $U\in\mathcal{U}_A$ and $A\in\mathcal{A}$.

Theorem 11. Let X be a paracompact space and (Y,d) a complete space uniformly of type $\bar{0}$ for balls; then if $F: X\to 2^Y$ is l.s.c. and satisfies (9), $\overline{\operatorname{co}S}\,F$ admits a selection.

Proof. From Proposition 3 it follows that $\operatorname{co}S\,F$ is l.s.c. and hence [4, Proposition 2.3, p.366] $\overline{\operatorname{co}S}\,F$ is l.s.c. The completness of (Y,d) implies $\overline{\operatorname{co}S}\,F(x)\in D(Y)$ for $x\in X$.

Example. Let us consider the hedgehog space of spininess $\mathfrak{m}$, $J(\mathfrak{m}) =: X$ [3, p.314]. For any $[(x,s_1)],[(y,s_2)]\in X$ let:

$$S_{[(x,s_1)]}[t,[(y,s_2)]] = \begin{cases} [tx+(1-t)y,s_2], & t\in I,\ \text{if}\ s_1=s_2\ \text{or}\ x=0\\ [(-tx+(1-t)y,s_2)], & t\in\langle 0,y/(x+y)\rangle,\ s_1\neq s_2,\ x\neq 0,\\ [(tx-(1-t)y,s_1)], & t\in(y/(x+y),1\rangle,\ s_1\neq s_2,\ x\neq 0.\end{cases}$$

It is seen that $S: X\times I\times X\to X$ is a map and (1) is satisfied. For any set $A\in 2^X$ we have $\operatorname{co}S\,A = [(\operatorname{conv} A,s)]$ if $A=[(B,s)]$ for a $s\in T$ (T - set of cardinality $\mathfrak{m}$) and $B\subset I$ and $\operatorname{co}S\,A = \bigcup\{[(\operatorname{conv}(A\cup\{0\}),s)]: s\in T,\ [(0,1\rangle,s)]\cap A\neq\emptyset\}$ otherwise. It is obvious that $\operatorname{co}S\,B(A,\delta) = B(\operatorname{co}S\,A,\delta)$. Moreover X is of type $\bar{0}$ (i.e. X is of type 0 and $\overline{\operatorname{co}S}\,A$ is S-convex, $A\in 2^X$).

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Streszczenie

O ciągłych selekcjach

Praca poświęcona jest badaniu uogólnień klasycznych twierdzeń Michaela o ciągłych selekcjach dla przestrzeni Banacha. Twierdzenie 1 wydaje się być niezależne od znanego twierdzenia Michaela dla struktur wypukłych. Przestrzenie, którymi operuje się w tej pracy, pozwalają na uzyskanie naturalnego uogólnienia twierdzenia Bartle'a-Gravesa o odwracalności, jak również związku z teorią punktów stałych przy zachowaniu pewnych formalnych podobieństw z wypukłością w przestrzeniach liniowych.

Лех Пасицки

Резюме

О непрерывных селекциях

Работа посвящается исследованию обобщений классических теорем Мишела о непрерывных селекциях для банахового пространства. Теорема 1 кажется быть независимой от известной теоремы Мишела для выпуклых структур. Пространства, которыми оперируется в этой работе, разрешают получить натуральное обобщение теоремы Бартла-Грейвса об обратимости, как и о связи с теорией постоянных точек при сохранении некоторых формальных подобий с выпуклостью в линейных пространствах.